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On the stability of the critical p-Laplace equation

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For $1 < p < n$, it is well known that non-negative, energy weak solutions to $\Delta_p u + u^{p^*-1} = 0$ in $\mathbb{R}^n$ are completely classified. Moreover, due to a fundamental result by Struwe and its extensions, this classification is stable up to bubbling. In the present work, we investigate the stability of perturbations of the critical p-Laplace equation for any $1 < p < n$, under a condition that prevents bubbling. In particular, we show that any solution $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ to such a perturbed equation must be quantitatively close to a bubble. This result generalizes a recent work by the first author, together with Figalli and Maggi [15], in which a sharp quantitative estimate was established for $p = 2$. However, our analysis differs completely from theirs and is based on a quantitative P-function approach.

1. Introduction

For $n \in \mathbb{N}$ and $1 < p < n$, the critical p-Laplace equation

$$\Delta_p u + u^{p^*-1} = 0 \quad \text{in } \mathbb{R}^n \tag{1.1}$$

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arises as the Euler-Lagrange equation associated to the problem of finding the optimal constant in the Sobolev inequality, that is

$$S_p = S := \min_{u \in \mathcal{D}^{1,p}(\mathbb{R}^n)\setminus\{0\}} \frac{\|\nabla u\|_{L^p(\mathbb{R}^n)}}{\|u\|_{L^{p^*}(\mathbb{R}^n)}}. \tag{1.2}$$

This constant is attained by the functions belonging to the $(n+2)$-dimensional manifold of the *Talenti bubbles*

$$\mathcal{M} := \left\{ U_{a,b,z} \,\middle|\, U_{a,b,z}(x) := a\left(1 + b\,|x-z|^{\frac{p}{p-1}}\right)^{-\frac{n-p}{p}},\ a \in \mathbb{R}\setminus\{0\}, b>0, z \in \mathbb{R}^n \right\},$$

as shown by Aubin [4] and Talenti [43]. Furthermore, $\mathcal{M}$ coincides with the space of all weak solutions to

$$\Delta_p u + S^p \,\|u\|_{L^{p^*}(\mathbb{R}^n)}^{p-p^*} |u|^{p^*-2}u = 0 \quad \text{in } \mathbb{R}^n$$

which do not change sign. Among these functions, a significant role is played by the *p-bubbles*, defined as

$$U_p[z,\lambda](x) := \left(\frac{\lambda^{\frac{1}{p-1}}\, n^{\frac{1}{p}} \left(\frac{n-p}{p-1}\right)^{\frac{p-1}{p}}}{\lambda^{\frac{p}{p-1}} + |x-z|^{\frac{p}{p-1}}} \right)^{\frac{n-p}{p}}, \tag{1.3}$$

where $\lambda > 0$ and $z \in \mathbb{R}^n$ are the scaling and translation parameters, respectively, and they satisfy (1.1). Moreover, we notice that any p-bubble $U = U_p[z,\lambda]$ satisfies

$$\|U\|_{L^{p^*}(\mathbb{R}^n)}^{p^*} = \|\nabla U\|_{L^p(\mathbb{R}^n)}^p = S^n. \tag{1.4}$$

In the case $p = 2$, the seminal works of Gidas, Ni & Nirenberg [29], Caffarelli, Gidas & Spruck [9], and Chen & Li [11], established that these functions are the only positive classical solutions to (1.1).

Moreover, for $1 < p < n$, it is known from the more recent works of Damascelli, Merchán, Montoro & Sciunzi [19], Vétois [46], and Sciunzi [39] that the p-bubbles (1.3) are the only positive weak solutions to (1.1) in the energy space $\mathcal{D}^{1,p}(\mathbb{R}^n)$, where

$$\mathcal{D}^{1,p}(\mathbb{R}^n) := \{u \in L^{p^*}(\mathbb{R}^n) \,|\, \nabla u \in L^p(\mathbb{R}^n)\}.$$

See also [16] for the classification in an anisotropic setting. Finally, we also mention that this classification result has recently been extended without the energy assumption for certain values of p depending on n – see [10,37,47].

A related question to the study of the optimal constant in (1.2) is the stability of the Sobolev inequality

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq S^{-1}\|\nabla u\|_{L^p(\mathbb{R}^n)} \quad \text{for every } u \in \mathcal{D}^{1,p}(\mathbb{R}^n), \tag{1.5}$$

which is naturally associated with the *p-Sobolev deficit*

$$\delta_{Sob}(u) := \frac{\|\nabla u\|_{L^p(\mathbb{R}^n)}}{\|u\|_{L^{p^*}(\mathbb{R}^n)}} - S \quad \text{for } u \in \mathcal{D}^{1,p}(\mathbb{R}^n). \tag{1.6}$$

Clearly, $\delta_{Sob}(u) \geq 0$ and it vanishes only on the manifold of minimizers to the Sobolev quotient (1.2), i.e., on $\mathcal{M}$.

In Problem A of [7], Brezis & Lieb asked whether, for $p = 2$ and a function $u \in \mathcal{D}^{1,2}(\mathbb{R}^n)$, it is possible to control an appropriate distance between u and the manifold $\mathcal{M}$ in terms of the 2-Sobolev deficit (1.6). This question was affirmatively answered by Bianchi & Egnell [6], who proved that

$$c_n \inf_{v\in\mathcal{M}} \left(\frac{\|\nabla u - \nabla v\|_{L^2(\mathbb{R}^n)}}{\|\nabla u\|_{L^2(\mathbb{R}^n)}}\right)^2 \leq \delta_{Sob}(u), \tag{1.7}$$

for some dimensional constant $c_n > 0$. Moreover, the exponent at the left-hand side of (1.7) is sharp.

The technique developed in [6] strongly relies on the Hilbert structure of $\mathcal{D}^{1,2}(\mathbb{R}^n)$, making it unsuitable for addressing the problem in the general case $1 < p < n$. Consequently, the study of the stability issue for a general p has led to several significant contributions – see, for instance, [12,25,26,36]. The final and sharp resolution of the problem, both in terms of the strength of the distance from $\mathcal{M}$ and the optimal exponent, was provided by Figalli & Zhang [27]. Specifically, they showed that

$$c_{n,p} \inf_{v\in\mathcal{M}} \left(\frac{\|\nabla u - \nabla v\|_{L^p(\mathbb{R}^n)}}{\|\nabla u\|_{L^p(\mathbb{R}^n)}}\right)^{\max\{2,p\}} \leq \delta_{Sob}(u) \quad \text{for all } u \in \mathcal{D}^{1,p}(\mathbb{R}^n),$$

where the exponent is sharp, thereby fully settling the question of stability for (1.5) à la Bianchi-Egnell.

Another relevant line of research concerns the stability issue for critical points of the Sobolev quotient (1.2). In the case $p = 2$, Struwe [42] established a qualitative stability result for (1.1). In particular, he showed that if a non-negative u *almost* solves (1.1), then u is *close* to the sum of 2-bubbles in the $\mathcal{D}^{1,2}$-norm. This phenomenon is known as *bubbling*. We also mention that an analogous compactness result in $\mathbb{R}^n$ was obtained by Benci & Cerami [5].

From a PDEs perspective, this problem reduces to studying

$$\Delta u + \kappa(x)u^{2^*-1} = 0 \quad \text{in } \mathbb{R}^n, \tag{1.8}$$

for κ close to a constant, which can be regarded as a perturbation of (1.1) when $p = 2$.

A quantitative version of Struwe's theorem for (1.8) was studied by the first author, Figalli & Maggi [15] under the a priori energy assumption

$$\frac{1}{2}S^n \leq \int_{\mathbb{R}^n} |\nabla u|^2 \, dx \leq \frac{3}{2}S^n. \tag{1.9}$$

Recalling (1.4), we see that (1.9) ensures that the energy of u is nearly that of a single 2-bubble. Therefore, (1.9) is the main ingredient to prevent bubbling. A sharp study of this phenomenon was subsequently carried out by Figalli & Glaudo [24] and Deng, Sun & Wei [22] by assuming the counterpart of (1.9) for multiple bubbles.

In [15,22,24], the quantitative stability result is understood in the sense of $\mathcal{D}^{1,2}(\mathbb{R}^n)$, thus providing estimates on the L^2-norm of the gradient of the difference between the solution and a suitable sum of 2-bubbles in terms of some *deficit* $\mathrm{def}(u, \kappa)$, which measures how close is κ to being constant.

The analyses conducted in [15,22,24] mainly rely on the knowledge of the family (1.3), along with their spectral properties, and the Hilbert structure of the energy space $\mathcal{D}^{1,2}(\mathbb{R}^n)$. To the best of our knowledge, these aspects have so far hindered the extension of the techniques developed therein to the case $p \neq 2$.

For completeness, we also mention that the stability of the fractional Sobolev inequality in the space $\dot{W}^{s,2}(\mathbb{R}^n)$, following the approach of [15,22,24], has been studied by Aryan [3] and De Nitti & König [21]. Moreover, stability issues for more general semilinear equations and for non-energy solutions have been investigated by the authors, together with Cozzi, in [13].

1.1. Main results

We now return to the stability problem for (1.1) in the general case $1 < p < n$ and present our main results, along with a rough outline of the proof.

When $1 < p < n$, Alves [1] and Mercuri & Willem [35] extended Struwe's result to the p-Laplacian. Nevertheless, as mentioned above, its quantitative counterpart and the stability of critical points of the Sobolev inequality (1.5) remain mostly unexplored.

In this case, in order to obtain quantitative stability results, one has to face

$$\Delta_p u + \kappa(x) u^{p^*-1} = 0 \quad \text{in } \mathbb{R}^n, \tag{1.10}$$

for κ close to a constant, which can be seen as a perturbation of (1.1), with the a priori energy assumption

$$\frac{1}{2}S^n \leq \int_{\mathbb{R}^n} |\nabla u|^p \, dx \leq \frac{3}{2}S^n. \tag{1.11}$$

In particular, when $\kappa \equiv \kappa_0$, any solution $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ to (1.10) must be a Talenti bubble and, by testing the equation with u, it follows that

$$\kappa_0 = \kappa_0(u) := \frac{\int_{\mathbb{R}^n} \kappa(x) u^{p^*} dx}{\int_{\mathbb{R}^n} u^{p^*} dx} = \frac{\int_{\mathbb{R}^n} |\nabla u|^p \, dx}{\int_{\mathbb{R}^n} u^{p^*} dx}. \tag{1.12}$$

Thus, it is natural to measure the proximity of κ to κ_0 using the deficit

$$\operatorname{def}(u, \kappa) := \left\| (\kappa - \kappa_0)\, u^{p^*-1} \right\|_{L^{(p^*)'}(\mathbb{R}^n)}. \tag{1.13}$$

For further details regarding this choice, we refer to [15].

With this notation, the main result of [15] states that if $p = 2$ and $u \in \mathcal{D}^{1,2}(\mathbb{R}^n)$ is a non-negative function satisfying (1.10)–(1.11), then there exists a 2-bubble $\mathcal{U}_0$ of the form (1.3), i.e., $\mathcal{U}_0 = U_2[z, \lambda]$ for some $z \in \mathbb{R}^n$ and $\lambda > 0$, such that

$$\|u - \mathcal{U}_0\|_{\mathcal{D}^{1,2}(\mathbb{R}^n)} \le c_n \operatorname{def}(u, \kappa),$$

for a dimensional constant $c_n > 0$.

In the present paper, we aim to establish a quantitative stability result for positive weak solutions $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ to (1.10), under a condition that prevents bubbling. Our main contributions are the following.

Theorem 1.1. *Let $n \in \mathbb{N}$, $1 < p < n$, and $\kappa \in L^\infty(\mathbb{R}^n) \cap C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ a positive function. Let $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ be a positive weak solution to (1.10) satisfying (1.11). Moreover, suppose that*

$$\kappa_0(u) = 1,$$

where $\kappa_0(u)$ is defined in (1.12).

Then, there exist a large constant $C \ge 1$, a small $\vartheta \in (0, 1)$, and a p-bubble $\mathcal{U}_0$, of the form (1.3), such that

$$\|u - \mathcal{U}_0\|_{\mathcal{D}^{1,p}(\mathbb{R}^n)} \le C \operatorname{def}(u, \kappa)^{\vartheta}. \tag{1.14}$$

The constant C depends only on n, p, and $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$, while ϑ depends only on n and p.

The general case can be reduced to $\kappa_0(u) = 1$ by scaling, as already noticed in [15]. Thus, we have the following corollary.

Corollary 1.1.1. *Let $n \in \mathbb{N}$ be an integer, $1 < p < n$, and $\kappa \in L^\infty(\mathbb{R}^n) \cap C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ a positive function. Let $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ be a positive weak solution to (1.10) satisfying*

$$\frac{1}{2} \kappa_0(u)^{\frac{p}{p-p^*}} S^n \le \int_{\mathbb{R}^n} |\nabla u|^p \, dx \le \frac{3}{2} \kappa_0(u)^{\frac{p}{p-p^*}} S^n, \tag{1.15}$$

where $\kappa_0(u)$ is defined in (1.12).

Then, there exist a large constant $C \geq 1$*, a small* $\vartheta \in (0, 1)$*, and a function* $\mathcal{U} \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ *given by*

$$\mathcal{U} = \kappa_0(u)^{\frac{1}{p-p^*}}\, \mathcal{U}_0,$$

for some p-bubble $\mathcal{U}_0$ *of the form* (1.3)*, such that*

$$\|u - \mathcal{U}\|_{\mathcal{D}^{1,p}(\mathbb{R}^n)} \leq C\, \mathrm{def}(u, \kappa)^{\vartheta}.$$

The constant C *depends only on* n*,* p*,* $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$*, and* $\kappa_0(u)$*, while* ϑ *depends only on* n *and* p*.*

While writing this paper we learned that, independently of us, Liu & Zhang [34] have proved an alternative result, which will appear in a forthcoming paper. Their approach, inspired by [27], allows them to catch the sharp exponent in (1.14).

As far as we know, Theorem 1.1 and Corollary 1.1.1, as well as the main result in [34], are the first results establishing the quantitative closeness of a solution to (1.10), viewed as a perturbation of the critical equation (1.1), to a p-bubble. The exponent ϑ in (1.14) is far from optimal. However, we believe that our approach is suitable for being used in more general contexts – such as in the anisotropic setting.

Our strategy to proving Theorem 1.1 follows a more PDE-oriented strategy compared to that of [15]. Specifically, we employ a quantitative argument based on integral identities, similar to the one used in [16] as well as in [10,37,47], all of which are inspired by the seminal work of Serrin & Zou [41]. However, in our case, instead of directly applying an inequality for vector fields, we establish a different integral inequality tailored to our setting – this is the content of Proposition 3.3 below. This new inequality is more in the spirit of those used by the first author, Farina & Polvara [14] and Ou [37], and is obtained by exploiting a positive subsolution to a suitable PDE – see Subsection 1.2 below.

In the following subsection, we describe the proof strategy in more detail.

1.2. Outline of the proof

The proof begins by applying the result in [35] to identify the p-bubble close to u, denoted by U. We then reduce the problem to the case where $U = U_p[0, 1]$. This reduction is possible due to the symmetries of (1.1) and the family (1.3) – see Section 2 and Lemma 2.1 below. Once U is fixed, we establish upper and lower quantitative decay estimates for u and derive an upper bound for its gradient in Step 2. Additionally, we prove that there must be at least one point $x_0 \in \mathbb{R}^n$ where u attains its maximum, which will be further localized later in the proof.

Next, we introduce the fundamental auxiliary function v and the P-function, defined as

$$v := u^{-\frac{p}{n-p}} \quad \text{and} \quad P := n\,\frac{p-1}{p} v^{-1} \left|\nabla v\right|^{p} + \left(\frac{p}{n-p}\right)^{p-1} v^{-1}. \tag{1.16}$$

We also define the stress field associated with v by setting $a(\nabla v) := |\nabla v|^{p-2}\,\nabla v$, the tensor $W := \nabla a(\nabla v)$, and its traceless version $\mathring{W}$.

In Step 3, we show that v is trapped between two p-paraboloids, using terminology analogous to the case $p = 2$ – see also Fig. 1. Thus, one heuristically expects v to be close to a p-paraboloid. The key idea of the proof is to construct such an approximation for v and then translate this information back to u.

It is well established that v satisfies an elliptic equation involving P and a remainder term – see (3.32) below. Moreover, this equation can be rewritten in terms of W. Exploiting this formulation, we deduce an integral inequality involving P and W – the content of Proposition 3.3 below. This result is obtained through the rather technical Step 4 and Step 5. Notice that this is where the regularity of κ is required.

From Proposition 3.3, in Step 6 and Step 7, we infer an integral weighted estimate for $|\mathring{W}|$ – see (3.122). Notably, for a p-bubble, the P-function defined by (1.16) must be constant and

$$W = \frac{P}{n}\,\mathrm{Id}_n\,.$$

Therefore, when u is close to a p-bubble, we expect P to be approximately constant and, consequently, close to its mean on a small ball $B_{\mathsf{t}}(x_0)$ – recall that x_0 is the point where u attains its maximum and v its minimum – denoted by $\overline{P}$. This observation suggests that we should seek a weighted estimate for $|W - \mu\,\mathrm{Id}_n|$, where $\mu \in \mathbb{R}$ is properly chosen.

It turns out that such an estimate holds if we set $\mu = \overline{P}/n$ – see (3.121) below. However, we are only able to establish it within a large ball B_r, with r to be determined by the end of the proof.

Using (3.121), in Step 8, we construct a p-paraboloid Q centered at x_0, with $v(x_0)$ as its value at the center, such that both $v - \mathsf{Q}$ and $\nabla v - \nabla\mathsf{Q}$ are small – see estimates (3.145) and (3.151). Unfortunately, when going back to u, the p-paraboloid Q does not yield a p-bubble of the form (1.3).

To fix this issue, we introduce a refined approximation. Specifically, we define a new p-paraboloid $\mathcal{Q}$, also centered at x_0, ensuring that when inverted to recover u, it precisely yields a p-bubble of the form (1.3). In practice, this requires selecting a parameter λ, which represents the bubble's scaling factor, to uniquely determine $\mathcal{Q}$. We impose the condition $\nabla\mathsf{Q} = \nabla\mathcal{Q}$, thereby fixing λ. Moreover, it turns out that, up to a factor, $\lambda = 1/\overline{P}$. Finally, we aim to keep $v - \mathcal{Q}$ small.

At first glance, it may seem that we have no remaining degrees of freedom to adjust $\mathcal{Q}$. Nevertheless, this poses no issue since v and Q are already comparable. To estimate $v - \mathcal{Q}$, it therefore suffices to control $\mathsf{Q} - \mathcal{Q}$. Moreover, since Q and $\mathcal{Q}$ coincide at the first order, their difference is dictated by their zeroth-order terms. This reduces the problem to comparing $v(x_0)$ and λ, which, up to a factor, amounts to comparing $v(x_0)$ and $1/\overline{P}$.

Such a comparison is possible by choosing the radius $\mathtt{t}$ sufficiently small – see also Fig. 1 for the heuristics.

Once we establish that both $v-\mathcal{Q}$ and $\nabla v-\nabla\mathcal{Q}$ are small in the appropriate weighted norm, we return to u by defining the p-bubble

$$\mathcal{U} := \mathcal{Q}^{-\frac{n-p}{p}}.$$

This allows us to conclude that $\nabla u - \nabla\mathcal{U}$ is small in $L^p(B_r)$. Finally, we select r such that the decay estimates for ∇u and $\nabla\mathcal{U}$ ensure that $\|\nabla u - \nabla\mathcal{U}\|_{L^p(\mathbb{R}^n\setminus B_r)}$ is also small. The result then follows by reversing the initial reduction argument, with the aid of Lemma 2.1.

1.3. Structure of the paper

In Section 2, we recall some symmetry properties of the equation (1.1) and prove Corollary 1.1.1. Section 3 is dedicated to the proof of Theorem 1.1. In Section 4 we present an example which provides insights on the optimal exponent. Finally, as an application, we derive a quasi-symmetry result for solutions to (1.10) in Section 5.

2. Symmetries of the problem and proof of Corollary 1.1.1

This brief section is inspired by Section 2.1 in [24], where the analogous properties for the case $p=2$ are listed. Additionally, we provide the proof of Corollary 1.1.1.

Given $\lambda>0$ and $z\in\mathbb{R}^n$, let $T_{z,\lambda}: C_c^\infty(\mathbb{R}^n)\to C_c^\infty(\mathbb{R}^n)$ be the operator defined as

$$T_{z,\lambda}(\phi)(x) := \lambda^{\frac{n-p}{p}}\phi\left(\lambda(x-z)\right).$$

Clearly, this operator can be extended to functions which are non smooth with compact support in the same fashion.

The following lemma provides some fundamental properties of the family of transformations $T_{z,\lambda}$ which will be useful in the following.

Lemma 2.1. *The operator $T_{z,\lambda}$ enjoys the subsequent properties.*

(1) *For any $\phi\in C_c^\infty(\mathbb{R}^n)$, it holds that*

$$\int_{\mathbb{R}^n} |T_{z,\lambda}(\phi)|^{p^*}\,dx = \int_{\mathbb{R}^n} |\phi|^{p^*}\,dx.$$

(2) *For any $\phi\in C_c^\infty(\mathbb{R}^n)$, it holds that*

$$\int_{\mathbb{R}^n} |\nabla T_{z,\lambda}(\phi)|^{p}\,dx = \int_{\mathbb{R}^n} |\nabla\phi|^{p}\,dx.$$

(3) *For any* $\phi \in C_c^\infty(\mathbb{R}^n)$*, it holds that*

$$T_{z,\lambda}\left(T_{-z\lambda,1/\lambda}(\phi)\right) = T_{-z\lambda,1/\lambda}\left(T_{z,\lambda}(\phi)\right) = \phi.$$

(4) *For any p-bubble* U*,* $T_{z,\lambda}(U)$ *is still a p-bubble.*

(5) *The p-bubbles satisfy*

$$U_p[z,\lambda] = T_{z,1/\lambda}\left(U_p[0,1]\right) \quad \textit{and} \quad U_p[0,1] = T_{-z/\lambda,\lambda}\left(U_p[z,\lambda]\right).$$

Obviously all the properties of Lemma 2.1 hold also if the functions are non smooth with compact support, provided that the involved integrals are finite.

We observe that both the quantities $\kappa_0(u)$ and $\mathrm{def}(u,\kappa)$, given in (1.12) and (1.13), respectively, are invariant under the action of the operators $T_{z,\lambda}$.

More precisely, if $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ is a weak solution to (1.10) and $v \coloneqq T_{z,\lambda}(u)$, then $v \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ clearly is a weak solution to

$$\Delta_p v + \widehat{\kappa}(x) v^{p^*-1} = 0 \quad \text{in } \mathbb{R}^n$$

where $\widehat{\kappa}(x) = \kappa(\lambda(x-z))$, moreover

$$\widehat{\kappa} \in L^\infty(\mathbb{R}^n) \quad \text{with} \quad \|\widehat{\kappa}\|_{L^\infty(\mathbb{R}^n)} = \|\kappa\|_{L^\infty(\mathbb{R}^n)}, \tag{2.1}$$

and also

$$\kappa_0(u) = \kappa_0(v) \quad \text{and} \quad \mathrm{def}(u,\kappa) = \mathrm{def}(v,\widehat{\kappa}). \tag{2.2}$$

The transformations $T_{z,\lambda}$ play a central role in the study of the Sobolev inequality, as they preserve the two quantities $\|\cdot\|_{L^{p^*}(\mathbb{R}^n)}$ and $\|\nabla\cdot\|_{L^p(\mathbb{R}^n)}$. In particular, we will use these symmetries to reduce the problem to the case where, instead of considering a generic p-bubble, we can take the fundamental bubble $U_p[0,1]$.

We conclude this section by providing the proof of Corollary 1.1.1.

Proof of Corollary 1.1.1. The proof is a direct consequence of Theorem 1.1 and a scaling argument. Let us set $\kappa_0 = \kappa_0(u)$ and

$$w \coloneqq \kappa_0^{\frac{1}{p^*-p}}\, u.$$

It is clear that $w \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ is a positive weak solution to

$$\Delta_p w + \kappa_0^{-1}\kappa(x) w^{p^*-1} = 0 \quad \text{in } \mathbb{R}^n.$$

Furthermore, by (1.15), w satisfies (1.11) and, by (1.12), also

$$\kappa_0(w) = \frac{\int_{\mathbb{R}^n} |\nabla w|^p \, dx}{\int_{\mathbb{R}^n} w^{p^*} \, dx} = \kappa_0^{-1} \frac{\int_{\mathbb{R}^n} |\nabla u|^p \, dx}{\int_{\mathbb{R}^n} u^{p^*} \, dx} = 1.$$

By applying Theorem 1.1, we infer that there exist a large $C \geq 1$, a small $\vartheta \in (0,1)$, and a p-bubble $\mathcal{U}_0$, of the form (1.3), such that

$$\|w - \mathcal{U}_0\|_{\mathcal{D}^{1,p}(\mathbb{R}^n)} \leq C \operatorname{def}\left(w, \kappa_0^{-1}\kappa\right)^{\vartheta}, \tag{2.3}$$

where the constant C depends only on n, p, $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$, and κ_0, whereas ϑ depends only on n and p. Therefore, we define

$$\mathcal{U} := \kappa_0^{\frac{1}{p-p^*}} \mathcal{U}_0 \in \mathcal{D}^{1,p}(\mathbb{R}^n)$$

and observe that

$$\operatorname{def}\left(w, \kappa_0^{-1}\kappa\right) = \kappa_0^{\frac{p-1}{p^*-p}} \operatorname{def}(u, \kappa).$$

Hence, multiplying (2.3) by $\kappa_0^{\frac{1}{p-p^*}}$, we conclude that

$$\|u - \mathcal{U}\|_{\mathcal{D}^{1,p}(\mathbb{R}^n)} \leq C \kappa_0^{\frac{(p-1)\vartheta-1}{p^*-p}} \operatorname{def}(u, \kappa)^{\vartheta}. \quad \square$$

3. Proof of Theorem 1.1

Since $\kappa \in L^\infty(\mathbb{R}^n)$, it is well known, by the results of Peral [38], Serrin [40], DiBenedetto [23], and Tolksdorf [45], that any solution $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ of (1.10) actually satisfies

$$u \in W^{1,\infty}(\mathbb{R}^n) \cap C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n) \quad \text{for some } \alpha \in (0,1). \tag{3.1}$$

Moreover, by Vétois' Lemma 2.2 in [46], we also have that $u \in L^{p_*-1,\infty}(\mathbb{R}^n)$, where $p_* := \frac{p(n-1)}{n-p}$. Hence, by interpolation,

$$u \in L^q(\mathbb{R}^n) \quad \text{for every } q \in (p_* - 1, +\infty]\,. \tag{3.2}$$

Furthermore, Theorem 1.1 in [46] provides a priori estimate for u and its gradient, namely

$$u(x) \leq \frac{C_u}{1 + |x|^{\frac{n-p}{p-1}}} \quad \text{and} \quad |\nabla u(x)| \leq \frac{C_u}{1 + |x|^{\frac{n-1}{p-1}}} \quad \text{for every } x \in \mathbb{R}^n \tag{3.3}$$

for some constant $C_u > 0$ depending on u.

Concerning the regularity of the solution u to (1.10), further results have been established by Antonini, the first author & Farina [2] – see also the references therein. Let us define the critical set of u as

$$\mathcal{Z}_u := \{x \in \mathbb{R}^n \mid \nabla u(x) = 0\},$$

and the *stress field* associated with u by

$$a(\nabla u) := |\nabla u|^{p-2} \nabla u,$$

which is extended to zero on $\mathcal{Z}_u$. Then, the set $\mathcal{Z}_u$ is negligible, i.e., $|\mathcal{Z}_u| = 0$, and

$$\begin{gathered} a(\nabla u) \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n), \\ u \in W^{2,2}_{\mathrm{loc}}(\mathbb{R}^n \setminus \mathcal{Z}_u) \quad \text{and} \quad |\nabla u|^{p-2} \nabla^2 u \in L^2_{\mathrm{loc}}(\mathbb{R}^n \setminus \mathcal{Z}_u) \quad \text{for every } 1 < p < n, \\ u \in W^{2,2}_{\mathrm{loc}}(\mathbb{R}^n) \quad \text{and} \quad |\nabla u|^{p-2} \nabla^2 u \in L^2_{\mathrm{loc}}(\mathbb{R}^n) \quad \text{for every } 1 < p \leq 2. \end{gathered} \tag{3.4}$$

Finally, as $\kappa \in C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$, by Theorem 6.4 on page 284 of Ladyzhenskaya & Ural't-seva [32], we have

$$u \in C^{3,\alpha}_{\mathrm{loc}}(\mathbb{R}^n \setminus \mathcal{Z}_u). \tag{3.5}$$

Of course, it suffices to prove the theorem under the assumption that

$$\operatorname{def}(u,\kappa) \leq \gamma, \tag{3.6}$$

for some small $\gamma \in (0,1)$, depending only on n, p, and $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$. Indeed, if $\operatorname{def}(u,\kappa) > \gamma$, then it follows that

$$\|u - \mathcal{U}_0\|_{\mathcal{D}^{1,p}(\mathbb{R}^n)} \leq \|\nabla u\|_{L^p(\mathbb{R}^n)} + \|\nabla \mathcal{U}_0\|_{L^p(\mathbb{R}^n)} \leq 4S^{\frac{n}{p}} \leq \frac{4S^{\frac{n}{p}}}{\gamma} \operatorname{def}(u,\kappa)$$

for any p-bubble $\mathcal{U}_0$, using (1.4) and (1.11). Therefore, in what follows, we will assume that (3.6) is in force.

To improve readability, we divide the proof into several steps.

Step 1. Application of the Struwe-type result and reduction. For any $\epsilon > 0$, by the Struwe-type result of Mercuri & Willem [35] – see also [1, Theorem 2] for the case $p > 2$ – and considering (1.11), there exists a $\delta > 0$, depending on ϵ, and a p-bubble $U_\epsilon := U_p[z_\epsilon, \lambda_\epsilon]$, for a couple of parameters $z_\epsilon \in \mathbb{R}^n$ and $\lambda_\epsilon > 0$, such that if

$$\operatorname{def}(u,\kappa) \leq \delta, \tag{3.7}$$

then

$$\|\nabla(u - U_\epsilon)\|_{\mathcal{D}^{1,p}(\mathbb{R}^n)} \leq \epsilon. \tag{3.8}$$

We now scale and translate our functions in order to lead U_ϵ to be the p-bubble with center the origin and unit scale factor. Taking into account point (5) of Lemma 2.1, we set

$$U := U_p[0,1] = T_{-z_\epsilon/\lambda_\epsilon,\lambda_\epsilon}(U_\epsilon)$$

and define

$$u_\epsilon := T_{-z_\epsilon/\lambda_\epsilon,\lambda_\epsilon}(u)\,.$$

Therefore, by point (2) of Lemma 2.1 and (3.8), it follows that

$$\|\nabla(u_\epsilon - U)\|_{\mathcal{D}^{1,p}(\mathbb{R}^n)} \le \epsilon. \tag{3.9}$$

As noticed above, u_ϵ is a weak solution to (1.10) for a different κ satisfying (2.1)–(2.2) and u_ϵ enjoys the regularity properties listed in (3.4).

Since, according to points (1)–(2) of Lemma 2.1 and (2.1)–(2.2), all the relevant quantities are invariant, we will omit the subscript in what follows – writing therefore u instead of u_ϵ. The subscript will be restored when needed in Step 10.

Step 2. Sharp decay estimates in quantitative form. We claim that, if ϵ is properly selected, there exist two constants $c_0, C_0 > 0$, depending only on n, p, and $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$, such that

$$\frac{c_0}{1+|x|^{\frac{n-p}{p-1}}} \le u(x) \le \frac{C_0}{1+|x|^{\frac{n-p}{p-1}}} \quad \text{for every } x \in \mathbb{R}^n. \tag{3.10}$$

Furthermore, there exists a constant $C_1 \ge 1$, depending only on n, p, and $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$, such that

$$|\nabla u(x)| \le \frac{C_1}{1+|x|^{\frac{n-1}{p-1}}} \quad \text{for every } x \in \mathbb{R}^n. \tag{3.11}$$

Both the upper and lower bounds in (3.10) are essentially a consequence of (3.9) and the fact that u solves (1.10). The upper bound is obtained via a quantification the proofs of Lemma 3.1 and Theorem 1.1 in [46]. The lower bound is a consequence of this upper bound via the quantitative argument of Lemma 5.2 in [28].

We start by proving a weaker version of the upper bound in (3.10) which is needed to reach the final sharp estimate. In particular, we first show that

$$u(x) \le C|x|^{\frac{p-n}{p}} \quad \text{for every } x \in \mathbb{R}^n \setminus B_{r_1}, \tag{3.12}$$

for some $r_1 > 0$ and a constant $C \ge 1$, depending only on n, p, and $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$, and where, from now on, we denote $B_r := B_r(0)$ for $r > 0$. By (3.9) and Sobolev inequality, we immediately get that

$$\left| \|u\|_{L^{p^*}(\mathbb{R}^n\setminus B_r)} - \|U\|_{L^{p^*}(\mathbb{R}^n\setminus B_r)} \right| \leq \|u - U\|_{L^{p^*}(\mathbb{R}^n\setminus B_r)} \leq S^{-1}\epsilon,$$

which, in turn, implies

$$\int\limits_{\mathbb{R}^n\setminus B_r} u^{p^*}\,dx \leq 2^{p^*-1} \int\limits_{\mathbb{R}^n\setminus B_r} U^{p^*}\,dx + 2^{p^*-1} S^{-p^*} \epsilon^{p^*} \tag{3.13}$$

for every $r > 0$. We now arbitrarily fix some $\chi \in (0, 1)$. Since $U \in L^{p^*}(\mathbb{R}^n)$, there exists a sufficiently large $r_1 > 0$, depending only on n and p, such that

$$2^{p^*-1} \int\limits_{\mathbb{R}^n\setminus B_r} U^{p^*}\,dx \leq \chi\,\frac{S^n}{4} \quad \text{for every } r \geq r_1.$$

By further requiring that

$$\epsilon \leq \left(2^{1-p^*} \chi\,\frac{S^{n+p^*}}{4} \right)^{\frac{1}{p^*}} =: \epsilon_0,$$

we infer from (3.13) that

$$\int\limits_{\mathbb{R}^n\setminus B_r} u^{p^*}\,dx \leq \chi\,\frac{S^n}{2} \quad \text{for every } r \geq r_1. \tag{3.14}$$

We can now conclude that (3.12) holds true by Lemma 3.1 in [46].

As second purpose, we shall prove a universal upper bound for u. By (3.2), we know that $u \in L^\infty(\mathbb{R}^n)$. Moreover, we have already shown that u must decay at infinity, thus there exists a point $x_0 \in \mathbb{R}^n$ where u attains its maximum. We now claim that

$$\|u\|_{L^\infty(\mathbb{R}^n)} = u(x_0) \leq \mathcal{M} \tag{3.15}$$

for some $\mathcal{M} \geq 1$ depending only on n, p, and $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$. Our strategy to prove (3.15) is that of exploiting (3.9) and the proof of Theorem E.0.20 in [38] in order to derive a universal upper bound for a suitably large Lebesgue norm of u. Ultimately, we will deduce (3.15) by applying Theorem 1 in [40]. For the sake of completeness we reproduce the full argument in [38].

To this end, let us define, for $k \geq 0$, the functions $F_k, G_k : [0, +\infty) \to \mathbb{R}$ by

$$F_k(t) := \begin{cases} t^\beta & \text{if } t \leq k, \\ \beta k^{\beta-1}(t-k) + k^\beta & \text{if } t > k, \end{cases}$$

and

$$G_k(t) := \begin{cases} t^{(\beta-1)p+1} & \text{if } t \le k, \\ ((\beta-1)\,p+1)\,\beta\, k^{(\beta-1)p}\,(t-k) + k^{(\beta-1)p+1} & \text{if } t > k, \end{cases}$$

for some $\beta > 1$ to be determined soon. Note that $F_k, G_k \in C^{0,1}([0,+\infty))$, moreover one can check that the following inequalities hold true

$$G_k(t) \le t G_k'(t), \tag{3.16}$$

$$c_{\beta,p}\,(F_k'(t))^p \le G_k'(t), \tag{3.17}$$

$$t^{p-1} G_k(t) \le C_{\beta,p}\,(F_k(t))^p\,, \tag{3.18}$$

for a couple of constants $c_{\beta,p}, C_{\beta,p} > 0$ depending only on β and p. Fix $\beta > 1$, depending on n and p, such that $\beta p < p^*$, and consider

$$\xi_1 := \eta^p G_k(u),$$

where $\eta \in C_c^\infty(\mathbb{R}^n)$ will be chosen later. We can take ξ_1 as a test function in (1.10), so that

$$\begin{aligned}&\int_{\mathbb{R}^n} |\nabla u|^{p-2}\langle \nabla u, \nabla \xi_1\rangle\,dx \\ &\quad= \int_{\mathbb{R}^n} p\,|\nabla u|^{p-2}\eta^{p-1}G_k(u)\,\langle \nabla u, \nabla\eta\rangle + |\nabla u|^p \eta^p G_k'(u)\,dx = \int_{\mathbb{R}^n} \kappa u^{p^*-1}\eta^p G_k(u)\,dx.\end{aligned} \tag{3.19}$$

By exploiting (3.16) and Young's inequality we get

$$\begin{aligned}&\left|\int_{\mathbb{R}^n} p\,|\nabla u|^{p-2}\eta^{p-1}G_k(u)\,\langle \nabla u, \nabla\eta\rangle\,dx\right| \\ &\quad\le p\int_{\mathbb{R}^n} |\nabla u|^{p-1}\eta^{p-1}(G_k'(u))^{1/p'}\,\left(u^{p-1}G_k(u)\right)^{1/p}|\nabla\eta|\,dx \\ &\quad\le (p-1)\,\sigma\int_{\mathbb{R}^n} |\nabla u|^p\eta^p G_k'(u)\,dx + \frac{1}{\sigma}\int_{\mathbb{R}^n} u^{p-1}G_k(u)\,|\nabla\eta|^p\,dx.\end{aligned}$$

By taking $\sigma = (2\,(p-1))^{-1}$, the latter, together with (3.19), yields

$$\begin{aligned}
&\int_{\mathbb{R}^n} |\nabla u|^p \eta^p G_k'(u)\,dx \\
&\quad \le 4\,(p-1)\int_{\mathbb{R}^n} u^{p-1}G_k(u)|\nabla \eta|^p\,dx + 2\|\kappa\|_{L^\infty(\mathbb{R}^n)}\int_{\mathbb{R}^n} u^{p^*-1}G_k(u)\eta^p\,dx \\
&\quad \le 4C_{\beta,p}\,(p-1)\int_{\mathbb{R}^n} F_k^p(u)|\nabla \eta|^p\,dx + 2C_{\beta,p}\|\kappa\|_{L^\infty(\mathbb{R}^n)}\int_{\mathbb{R}^n} u^{p^*-p}F_k^p(u)\eta^p\,dx,
\end{aligned} \tag{3.20}$$

where we used (3.18) in the last line. Taking advantage of (3.17), we can estimate the left-hand side of (3.20) getting

$$\begin{aligned}
&c_{\beta,p}\int_{\mathbb{R}^n} |\nabla u|^p \eta^p \left(F_k'(u)\right)^p\,dx \\
&\qquad \le 4C_{\beta,p}\,(p-1)\int_{\mathbb{R}^n} F_k^p(u)|\nabla \eta|^p\,dx + 2C_{\beta,p}\|\kappa\|_{L^\infty(\mathbb{R}^n)}\int_{\mathbb{R}^n} u^{p^*-p}F_k^p(u)\eta^p\,dx,
\end{aligned}$$

which, in turn, implies

$$\int_{\mathbb{R}^n} |\nabla\left(\eta F_k(u)\right)|^p\,dx \le C_2\int_{\mathbb{R}^n} F_k^p(u)|\nabla \eta|^p\,dx + C_3\int_{\mathbb{R}^n} u^{p^*-p}F_k^p(u)\eta^p\,dx$$

for some $C_2 > 0$, depending only on n and p, and with $C_3 := 2\frac{C_{\beta,p}}{c_{\beta,p}}\|\kappa\|_{L^\infty(\mathbb{R}^n)}$. By Sobolev inequality we finally deduce

$$\left(\int_{\mathbb{R}^n} F_k^{p^*}(u)\eta^{p^*}\,dx\right)^{\frac{p}{p^*}} \le S^{-p}C_2\int_{\mathbb{R}^n} F_k^p(u)|\nabla \eta|^p\,dx + S^{-p}C_3\int_{\mathbb{R}^n} u^{p^*-p}F_k^p(u)\eta^p\,dx. \tag{3.21}$$

We now choose $\eta \in C_c^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\eta = \overline{B_{2r_2}(x_0)}$, for some universal $r_2 > 0$ to be determined in such a way that

$$\|u\|^{p^*-p}_{L^{p^*}(B_{2r_2}(x_0))} \le \frac{S^p}{2C_3}. \tag{3.22}$$

To prove the existence of an $r_2 > 0$ for which (3.22) holds true, we argue as above to deduce

$$\|u\|^{p^*-p}_{L^{p^*}(B_r(x_0))} \le 2^{\max\{0,p^*-p-1\}}\left(\|U\|^{p^*-p}_{L^{p^*}(B_r(x_0))} + S^{p-p^*}\epsilon^{p^*-p}\right) \quad \text{for every } r > 0.$$

We shall choose $r > 0$ in such a way that

$$\|u\|^{p^*-p}_{L^{p^*}(B_r(x_0))} \le 2^{\max\{0,p^*-p-1\}}\|U\|^{p^*-p}_{L^{p^*}(B_r(x_0))} \le \frac{S^p}{4C_3}, \tag{3.23}$$

and

$$\epsilon \leq \left(\frac{S^{p^*}}{2^{\max\{2,p^*-p+1\}}\, C_3} \right)^{\frac{1}{p^*-p}} =: \epsilon_1.$$

Since we can determine an $r_2 > 0$, depending only on n, p, and $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$, such that

$$\|U\|^{p^*-p}_{L^{p^*}(B_{2r_2}(x_0))} \leq \|U\|^{p^*-p}_{L^{p^*}(B_{2r_2})} \leq \frac{S^p}{2^{\max\{2,p^*-p+1\}}\, C_3},$$

the second inequality in (3.23) holds, and we conclude that (3.22) is verified. By Hölder inequality and (3.22), we finally infer form (3.21) that

$$\left(\int_{\mathbb{R}^n} F_k^{p^*}(u)\eta^{p^*}\, dx \right)^{\frac{p}{p^*}} \leq 2S^{-p}C_2 \int_{\mathbb{R}^n} F_k^p(u)|\nabla\eta|^p\, dx,$$

and, taking the limit as $k \to +\infty$, the monotone convergence theorem ensures that

$$\left(\int_{\mathbb{R}^n} u^{\beta p^*}\eta^{p^*}\, dx \right)^{\frac{p}{p^*}} \leq 2S^{-p}C_2 \int_{\mathbb{R}^n} u^{\beta p}|\nabla\eta|^p\, dx.$$

If we now suppose that $\eta = 1$ in $B_{r_2}(x_0)$, recalling that $\beta p < p^*$, we deduce

$$\|u\|_{L^{\beta p^*}(B_{r_2}(x_0))} \leq C,$$

for some $C > 0$ depending only on n, p, and $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$. With this estimate at hand, we can apply Theorem 1 in [40], from which (3.15) directly follows.

As in the proof of Theorem 1.1 in [46], we now show that the upper bound (3.12) can be promoted to the stronger one in (3.10). We first claim that

$$\|u\|_{L^{p^*-1}(\mathbb{R}^n)} \leq C, \tag{3.24}$$

for some $C > 0$ depending only on n, p, and $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$. We immediately note that, by (3.24) and Lemma 2.2 in [46], we have

$$\|u\|_{L^{p_*-1,\infty}(\mathbb{R}^n)} \leq C, \tag{3.25}$$

for some $C > 0$ depending only on n, p, and $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$.

We plan to achieve (3.24) by adapting to our case a procedure which originates from Brezis & Kato [8]. To this aim, let $\psi \in C_c^\infty(\mathbb{R}^n)$ be a cut-off function such that $\psi \equiv 1$ in $B_r(-z_\epsilon)$, $\psi \equiv 0$ in $\mathbb{R}^n \setminus B_{2r}(-z_\epsilon)$, and $|\nabla\psi| \leq 2/r$ in $B_{2r}(-z_\epsilon) \setminus B_r(-z_\epsilon)$. We now test (1.10) with

$$\xi_2 := u^\sigma \psi$$

for $\sigma > 0$, and get

$$\begin{aligned}&\int_{\mathbb{R}^n} |\nabla u|^{p-2} \langle \nabla u, \nabla \xi_2 \rangle \, dx \\ &\quad = \int_{\mathbb{R}^n} \sigma \, |\nabla u|^p u^{\sigma-1} \psi + |\nabla u|^{p-2} u^\sigma \, \langle \nabla u, \nabla \psi \rangle \, dx = \int_{\mathbb{R}^n} \kappa u^{p^*-1+\sigma} \psi \, dx.\end{aligned}$$

Thus, we deduce

$$\begin{aligned}\sigma \left(\frac{p}{p+\sigma-1} \right)^p \int_{\mathbb{R}^n} \left| \nabla \left(u^{\frac{p+\sigma-1}{p}} \right) \right|^p \psi \, dx + \int_{\mathbb{R}^n} |\nabla u|^{p-2} u^\sigma \, \langle \nabla u, \nabla \psi \rangle \, dx \\ = \int_{\mathbb{R}^n} \kappa u^{p^*-1+\sigma} \psi \, dx. \qquad (3.26)\end{aligned}$$

We observe that, for r large enough, we have

$$\begin{aligned}\left| \int_{\mathbb{R}^n} |\nabla u|^{p-2} u^\sigma \, \langle \nabla u, \nabla \psi \rangle \, dx \right| &\leq \int_{B_{2r}(-z_\epsilon) \setminus B_r(-z_\epsilon)} |\nabla u|^{p-1} |\nabla \psi| \, u^\sigma \, dx \\ &\leq \frac{C}{r} \int_{B_{2r}(0) \setminus B_r(0)} \frac{dx}{|x|^{n+\frac{n-p}{p-1}\sigma-1}} \leq C \mathcal{H}^{n-1}(\mathbb{S}^{n-1}) \, r^{-\frac{n-p}{p-1}\sigma},\end{aligned}$$

for some $C > 0$ depending on n, p, u, and ϵ, where we used a properly rescaled and translated version of (3.3). Therefore, letting $r \to +\infty$ in (3.26) and taking into account (3.2), the monotone convergence theorem ensures

$$\sigma \left(\frac{p}{p+\sigma-1} \right)^p \int_{\mathbb{R}^n} \left| \nabla \left(u^{\frac{p+\sigma-1}{p}} \right) \right|^p dx = \int_{\mathbb{R}^n} \kappa u^{p^*-1+\sigma} \, dx. \qquad (3.27)$$

Since $(p + \sigma - 1) \, p^* > (p_* - 1) \, p$ for every $\sigma > 0$, as a consequence of (3.2) and (3.27), we deduce that $u^{\frac{p+\sigma-1}{p}} \in \mathcal{D}^{1,p}(\mathbb{R}^n)$. Hence, Sobolev inequality and (3.27) yield

$$\sigma \left(\frac{p \, S}{p+\sigma-1} \right)^p \left(\int_{\mathbb{R}^n} u^{\frac{p+\sigma-1}{p} p^*} \, dx \right)^{\frac{p}{p^*}} \leq \int_{\mathbb{R}^n} \kappa u^{p^*-1+\sigma} \, dx.$$

Let $r > 0$ to be chosen shortly. By Hölder inequality, the previous estimate entails

$$\sigma\left(\frac{p\,S}{p+\sigma-1}\right)^p\left(\int\limits_{\mathbb{R}^n} u^{\frac{p+\sigma-1}{p}p^*}\,dx\right)^{\frac{p}{p^*}}$$

$$\leq \int\limits_{B_r} \kappa u^{p^*-1+\sigma}\,dx + \|\kappa\|_{L^\infty(\mathbb{R}^n)}\left(\int\limits_{\mathbb{R}^n\setminus B_r} u^{\frac{p+\sigma-1}{p}p^*}\,dx\right)^{\frac{p}{p^*}}\left(\int\limits_{\mathbb{R}^n\setminus B_r} u^{p^*}\,dx\right)^{\frac{p^*-p}{p^*}}. \quad (3.28)$$

By taking advantage of the bound (3.15), for the first integral on the right-hand side of (3.28) we deduce

$$\int\limits_{B_r} \kappa u^{p^*-1+\sigma}\,dx \leq \mathcal{M}^{p^*-1+\sigma}\|\kappa\|_{L^\infty(\mathbb{R}^n)}|B_1|\,r^n. \quad (3.29)$$

To handle the second summand on the right-hand side of (3.28) we start by noticing that

$$\|u\|^{p^*-p}_{L^{p^*}(\mathbb{R}^n\setminus B_r)} \leq 2^{\max\{0,p^*-p-1\}}\left(\|U\|^{p^*-p}_{L^{p^*}(\mathbb{R}^n\setminus B_r)} + S^{p-p^*}\epsilon^{p^*-p}\right).$$

We now choose $\epsilon > 0$ sufficiently small and $r > 0$ large enough in such a way that the second term on the right-hand side of (3.28) can be reabsorbed. Specifically, we require that

$$2^{\max\{0,p^*-p-1\}}\|\kappa\|_{L^\infty(\mathbb{R}^n)}\left(\|U\|^{p^*-p}_{L^{p^*}(\mathbb{R}^n\setminus B_r)} + S^{p-p^*}\epsilon^{p^*-p}\right) \leq \frac{\sigma}{2}\left(\frac{p\,S}{p+\sigma-1}\right)^p. \quad (3.30)$$

Note that we will choose $\sigma > 0$ shortly depending only on n and p. By assuming that

$$\epsilon \leq \left\{\frac{\sigma S^{p^*-p}}{2^{\max\{2,p^*-p+1\}}\|\kappa\|_{L^\infty(\mathbb{R}^n)}}\left(\frac{p\,S}{p+\sigma-1}\right)^p\right\}^{\frac{1}{p^*-p}} =: \epsilon_2,$$

which depends only on n, p, and $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$, and since there exists $r_3 > 0$, depending only on n, p, and $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$, such that

$$2^{\max\{0,p^*-p-1\}}\|\kappa\|_{L^\infty(\mathbb{R}^n)}\|U\|^{p^*-p}_{L^{p^*}(\mathbb{R}^n\setminus B_{r_3})} \leq \frac{\sigma}{4}\left(\frac{p\,S}{p+\sigma-1}\right)^p,$$

we infer that (3.30) holds true. Therefore, combining (3.28), (3.29), and (3.30) we get

$$\frac{\sigma}{2}\left(\frac{p\,S}{p+\sigma-1}\right)^p\left(\int\limits_{\mathbb{R}^n} u^{\frac{p+\sigma-1}{p}p^*}\,dx\right)^{\frac{p}{p^*}} \leq C,$$

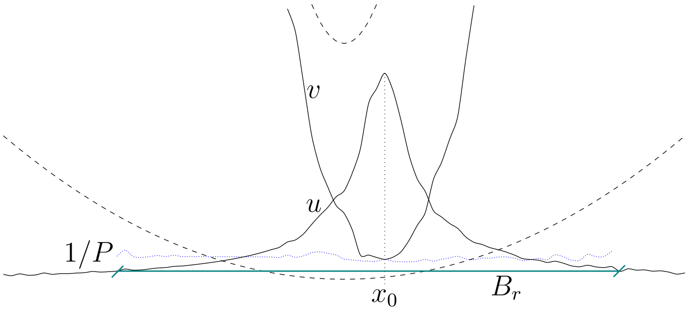

Fig. 1. The functions considered and the region B_r where the energy is concentrated. The dotted blue line is $1/P$ up to the correct factor. (For interpretation of the colors in the figure, the reader is referred to the web version of this article.)

for some $C > 0$ depending only on n, p, and $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$. Hence, (3.24) follows by choosing $\sigma = 1 - \frac{p}{p^*}$ in the previous estimate. We remark that the above approximation procedure used to obtain (3.27) is necessary.

As a consequence of (3.15), (3.25), and of the proof of Theorem 1.1 in [46], we deduce the upper bound in (3.10).

Finally, the lower bound in (3.10) is obtained by noticing that, from (1.11), (3.14), and the assumption $\kappa_0 = 1$, we get

$$\int_{B_{r_1}} u^{p^*}\, dx \geq (1-\chi)\, \frac{S^n}{2}$$

and arguing as in Lemma 5.2 of [28] – see also [41, Lemma 2.3].

Overall, by setting $\epsilon := \min\{\epsilon_0, \epsilon_1, \epsilon_2\}$, we deduce that (3.10) holds true. This also fixes the value of δ in (3.7). To prove the gradient bound, we observe that (3.15) and Theorem 1 in [23] yield

$$\|u\|_{C^1(\mathbb{R}^n)} \leq C_1,$$

for some $C_1 \geq 1$, depending only on n, p, and $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$. Therefore, the validity of (3.11) follows from the proof of Theorem 1.1 in [46], up to possibly enlarging C_1.

Step 3. Introduction of the P-function and of the relevant vector fields. As $u > 0$, we now introduce the function

$$v := u^{-\frac{p}{n-p}}. \tag{3.31}$$

A pictorial representation of the construction we consider is given in Fig. 1. From this definition and (1.10), it follows that v weakly solves

$$\Delta_p v = P + R \quad \text{in } \mathbb{R}^n, \tag{3.32}$$

where

$$P := n\,\frac{p-1}{p} v^{-1}\,|\nabla v|^{p} + \left(\frac{p}{n-p}\right)^{p-1} v^{-1}, \tag{3.33}$$

$$R := \left(\frac{p}{n-p}\right)^{p-1} (\kappa - 1)\, v^{-1} \tag{3.34}$$

The function defined in (3.33) is the so-called P-function exploited in [37,47] to obtain the classification result for local weak solutions of the p-Laplace equation and in [14] for the classification of solution to semilinear PDEs on Riemmanian manifolds. Whereas (3.34) represents the remainder term arising from the perturbation, i.e., due to the fact that κ is not necessarily constant.

We first observe that

$$u \in \mathcal{D}^{1,p}(\mathbb{R}^n) \quad \text{if and only if} \quad \int\limits_{\mathbb{R}^n} v^{-n}\, dx + \int\limits_{\mathbb{R}^n} v^{-n}\,|\nabla v|^{p}\, dx < +\infty. \tag{3.35}$$

Furthermore, v inherits some regularity properties from those of u listed in (3.1) and in (3.4)–(3.5), more precisely

$$\begin{gathered} \mathcal{Z}_v = \mathcal{Z}_u, \\ v \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n) \cap C^{3,\alpha}_{\mathrm{loc}}(\mathbb{R}^n \setminus \mathcal{Z}_v) \quad \text{and} \quad a\,(\nabla v) \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n), \\ v \in W^{2,2}_{\mathrm{loc}}(\mathbb{R}^n \setminus \mathcal{Z}_v) \quad \text{and} \quad |\nabla v|^{p-2}\,\nabla^2 v \in L^2_{\mathrm{loc}}(\mathbb{R}^n \setminus \mathcal{Z}_v) \quad \text{for every } 1<p<n, \\ v \in W^{2,2}_{\mathrm{loc}}(\mathbb{R}^n) \quad \text{and} \quad |\nabla v|^{p-2}\,\nabla^2 v \in L^2_{\mathrm{loc}}(\mathbb{R}^n) \quad \text{for every } 1<p\le 2. \end{gathered} \tag{3.36}$$

We claim that (3.10) and (3.11) give some pointwise bounds for v, its gradient, and the P-function. Indeed, from the sharp decay estimates (3.10), we deduce

$$\hat{c}_0 \left(1 + |x|^{\frac{p}{p-1}}\right) \le v(x) \le \widehat{C}_0 \left(1 + |x|^{\frac{p}{p-1}}\right) \quad \text{for every } x \in \mathbb{R}^n, \tag{3.37}$$

for a couple of constants $\hat{c}_0, \widehat{C}_0 > 0$ depending only on n, p, and $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$. Moreover, by (3.11) and (3.37), we get

$$|\nabla v(x)| \le \widehat{C}_1 \left(1 + |x|^{\frac{1}{p-1}}\right) \quad \text{for every } x \in \mathbb{R}^n, \tag{3.38}$$

for some $\widehat{C}_1 \ge 1$ depending only on n, p, and $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$. As a consequence of (3.37) and (3.38), it follows that $v \ge \hat{c}_0$ and

$$v^{-1}\,|\nabla v|^{p} \in L^\infty(\mathbb{R}^n) \quad \text{with} \quad v^{-1}\,|\nabla v|^{p} \le \widehat{C}_1^{p}\,\hat{c}_0^{-1}, \tag{3.39}$$

thus

$$P \in L^\infty(\mathbb{R}^n) \quad \text{with} \quad \|P\|_{L^\infty(\mathbb{R}^n)} \leq n\,\frac{p-1}{p}\,\widehat{C}_1^p\,\hat{c}_0^{-1} + \left(\frac{p}{n-p}\right)^{p-1} \hat{c}_0^{-1}, \tag{3.40}$$

$$R \in L^\infty(\mathbb{R}^n) \quad \text{with} \quad \|R\|_{L^\infty(\mathbb{R}^n)} \leq \left(\frac{p}{n-p}\right)^{p-1} \hat{c}_0^{-1}\,\|\kappa - 1\|_{L^\infty(\mathbb{R}^n)}. \tag{3.41}$$

For simplicity of notation, we also define the function $V : \mathbb{R}^n \to [0, +\infty)$ by

$$V(\xi) := \frac{|\xi|^p}{p}.$$

Note that $a(\xi) = \nabla V(\xi)$. The chain rule, (3.1), (3.4)–(3.5), (3.33), and (3.36) entail

$$\begin{aligned} V(\nabla v) &\in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n) \cap C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^n) \cap C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^n \setminus \mathcal{Z}_v), \\ P &\in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n) \cap C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^n) \cap C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^n \setminus \mathcal{Z}_v). \end{aligned} \tag{3.42}$$

Finally, we introduce the relevant vector field

$$W := \nabla a(\nabla v),$$

and extend it to zero on $\mathcal{Z}_v$. We also define the associated traceless version as

$$\mathring{W} = \mathring{\nabla} a(\nabla v) := \nabla a(\nabla v) - \frac{\operatorname{tr} \nabla a(\nabla v)}{n}\,\mathrm{Id}_n, \tag{3.43}$$

where Id_n is the n-dimensional identity matrix. This vector field will play a central role in our proof.

From (3.36), we deduce that

$$W \in L^2_{\mathrm{loc}}(\mathbb{R}^n) \cap C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n \setminus \mathcal{Z}_v).$$

From the chain rule and (3.36), we can also write

$$W(x) = \begin{cases} A(x) D^2 v(x) & \text{if } x \in \mathbb{R}^n \setminus \mathcal{Z}_v, \\ 0 & \text{if } x \in \mathcal{Z}_v, \end{cases} \tag{3.44}$$

where A is the matrix with components

$$[A(x)]_{ij} = \alpha_{ij}(x) := \partial_{\xi_i \xi_j} V(\nabla v)(x) \quad \text{for } x \in \mathbb{R}^n \setminus \mathcal{Z}_v,$$

that is

$$A = |\nabla v|^{p-2}\,\mathrm{Id}_n + (p-2)\,|\nabla v|^{p-4}\,\nabla v \otimes \nabla v \quad \text{on } \mathbb{R}^n \setminus \mathcal{Z}_v. \tag{3.45}$$

In particular, A is well defined and of class $C^{2,\alpha}_{\mathrm{loc}}$ in $\mathbb{R}^n \setminus \mathcal{Z}_v$, which has full measure, hence A is measurable and we extend it to zero on $\mathcal{Z}_v$.

With this notation at hand, we can rewrite (3.32) as

$$\operatorname{tr} W = \alpha_{ij} v_{ij} = P + R \quad \text{a.e. in } \mathbb{R}^n. \tag{3.46}$$

We conclude by noticing that a direct computation – see also [37, Lemma 2.1] – reveals

$$\nabla P = \frac{n}{v}\left(\nabla a\,(\nabla v)^{\mathsf{T}} - \frac{P}{n}\,\mathrm{Id}_n\right)\nabla v = \frac{n}{v}\left(W^{\mathsf{T}} - \frac{P}{n}\,\mathrm{Id}_n\right)\nabla v \quad \text{on } \mathbb{R}^n \setminus \mathcal{Z}_v. \tag{3.47}$$

Step 4. A differential identity. In this step, we prove a differential identity that holds for sufficiently smooth functions. This identity will be used in Step 5.

Proposition 3.1. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $w \in C^3(\Omega)$ be a positive function. Let $\mathcal{V} : \mathbb{R}^n \to [0, +\infty)$ be of class $C^3(\mathbb{R}^n \setminus \{0\})$, and define*

$$\mathsf{P} := \frac{n\,(p-1)}{w}\,\mathcal{V}\,(\nabla w) + \left(\frac{p}{n-p}\right)^{p-1}\frac{1}{w}$$

Then, by setting

$$\mathsf{a}\,(\nabla w) := \nabla_\xi \mathcal{V}\,(\nabla w)\,, \quad [\mathsf{A}]_{ij} = \tilde{\alpha}_{ij} := \partial_{\xi_i \xi_j}\mathcal{V}\,(\nabla w)\,, \quad \textit{and} \quad \mathcal{W} := \nabla \mathsf{a}\,(\nabla w)\,,$$

the following differential identity holds in $\Omega \setminus \{x \in \Omega \,|\, \nabla w(x) = 0\}$

$$\begin{aligned} \operatorname{div}\left(w^{2-n}\mathsf{A}\,\nabla \mathsf{P}\right) = w^{1-n}\Big\{&-n\,\langle \mathsf{A}\,\nabla \mathsf{P}, \nabla w\rangle - \mathsf{P}\operatorname{tr}\mathcal{W} \\ &+ n\,(p-1)\left[\operatorname{tr}\mathcal{W}^2 + \langle \nabla\,(\operatorname{tr}\mathcal{W})\,, \mathsf{a}\,(\nabla w)\rangle\right] - \mathsf{P} w_j \partial_{\xi_i\xi_j\xi_k}\mathcal{V}\,(\nabla w)\, w_{ki}\Big\}. \end{aligned} \tag{3.48}$$

Proof. In the proof all the calculations are done in the open set $\Omega \setminus \{x \in \Omega \,|\, \nabla w(x) = 0\}$, where P is of class C^2, thanks to our assumptions of w and $\mathcal{V}$.

By the chain rule, we have

$$\partial_j \mathcal{V}\,(\nabla w) = \mathsf{a}_k\,(\nabla w)\, w_{kj} \tag{3.49}$$

$$\mathcal{W}_{ik} = \partial_k \mathsf{a}_i(\nabla w) = \tilde{\alpha}_{ij} w_{jk}, \tag{3.50}$$

where $\mathsf{a}_k(\xi)$ denotes the k-th component of $\mathsf{a}(\xi)$.

Then, we compute

$$\partial_j \mathsf{P} = \frac{n\,(p-1)}{w}\,\partial_j \mathcal{V}\,(\nabla w) - \frac{\mathsf{P}}{w}\, w_j, \tag{3.51}$$

so that

$$
\begin{aligned}
\partial_{ij}\mathsf{P} &= -\frac{\partial_i \mathsf{P} w_j}{w} + \frac{\mathsf{P}}{w^2}\, w_i w_j - \frac{\mathsf{P}}{w}\, w_{ij} - \frac{n\,(p-1)}{w^2}\, w_i \partial_j \mathcal{V}(\nabla w) + \frac{n\,(p-1)}{w}\, \partial_{ij}\mathcal{V}(\nabla w) \\
&= -\frac{\partial_i \mathsf{P} w_j}{w} - \frac{\partial_j \mathsf{P} w_i}{w} - \frac{\mathsf{P}}{w}\, w_{ij} + \frac{n\,(p-1)}{w}\, \partial_{ij}\mathcal{V}(\nabla w),
\end{aligned}
\tag{3.52}
$$

where in the last equality we used (3.51).

Our goal is to compute

$$
\begin{aligned}
\operatorname{div}\left(w^{2-n}\tilde{\alpha}_{ij}\partial_j \mathsf{P}_j\right) &= w^{1-n}\left[w\,\tilde{\alpha}_{ij}\partial_{ij}\mathsf{P} + w\,\partial_i\tilde{\alpha}_{ij}\partial_j\mathsf{P} + (2-n)\,w_i\tilde{\alpha}_{ij}\partial_j\mathsf{P}\right] \\
&=: w^{1-n}\left[A_1 + A_2 + (2-n)\,w_i\tilde{\alpha}_{ij}\partial_j\mathsf{P}\right].
\end{aligned}
\tag{3.53}
$$

We now proceed with the computation of A_1 and A_2. From (3.52), and using the symmetry property $\tilde{\alpha}_{ij} = \tilde{\alpha}_{ji}$, we obtain

$$
\begin{aligned}
A_1 &= -\,2\tilde{\alpha}_{ij}\partial_i\mathsf{P}w_j - \mathsf{P}\tilde{\alpha}_{ij}w_{ij} + n\,(p-1)\,\tilde{\alpha}_{ij}\partial_{ij}\mathcal{V}(\nabla w) \\
&= -2\tilde{\alpha}_{ij}\partial_i\mathsf{P}w_j - \mathsf{P}\operatorname{tr}\mathcal{W} + n\,(p-1)\,\tilde{\alpha}_{ij}\partial_{ij}\mathcal{V}(\nabla w),
\end{aligned}
\tag{3.54}
$$

where in the last equality we used (3.50). For the last summand in (3.54), exploiting (3.49), we get

$$
\begin{aligned}
\tilde{\alpha}_{ij}\partial_{ij}\mathcal{V}(\nabla w) &= \tilde{\alpha}_{ij}\partial_i\left(\partial_j\mathcal{V}(\nabla w)\right) = \tilde{\alpha}_{ij}\partial_i\left(\mathsf{a}_k\left(\nabla w\right)w_{kj}\right) \\
&= \tilde{\alpha}_{ij}\tilde{\alpha}_{kl}w_{kj}w_{li} + \tilde{\alpha}_{ij}\mathsf{a}_k\left(\nabla w\right)w_{kji} = \operatorname{tr}\mathcal{W}^2 + \tilde{\alpha}_{ij}\mathsf{a}_k\left(\nabla w\right)w_{kji} \\
&= \operatorname{tr}\mathcal{W}^2 + \partial_k\left(\tilde{\alpha}_{ij}w_{ij}\right)\mathsf{a}_k(\nabla w) - \left(\partial_k\tilde{\alpha}_{ij}\right)\mathsf{a}_k\left(\nabla w\right)w_{ij}.
\end{aligned}
$$

Since

$$
\partial_k\tilde{\alpha}_{ij} = \partial_{\xi_i\xi_j\xi_l}\mathcal{V}\left(\nabla w\right)w_{lk},
\tag{3.55}
$$

from the previous equation, we obtain

$$
\begin{aligned}
\tilde{\alpha}_{ij}\partial_{ij}\mathcal{V}(\nabla w) &= \operatorname{tr}\mathcal{W}^2 + \partial_k\left(\operatorname{tr}\mathcal{W}\right)\mathsf{a}_k(\nabla w) - \partial_{\xi_i\xi_j\xi_l}\mathcal{V}\left(\nabla w\right)w_{lk}\mathsf{a}_k\left(\nabla w\right)w_{ij} \\
&= \operatorname{tr}\mathcal{W}^2 + \partial_k\left(\operatorname{tr}\mathcal{W}\right)\mathsf{a}_k(\nabla w) - \partial_{\xi_i\xi_j\xi_l}\mathcal{V}\left(\nabla w\right)\partial_l\mathcal{V}\left(\nabla w\right)w_{ij},
\end{aligned}
$$

where in the last equality we used (3.49). Substituting this into (3.54), we get

$$
\begin{aligned}
A_1 = &-2\tilde{\alpha}_{ij}\partial_i\mathsf{P}w_j - \mathsf{P}\operatorname{tr}\mathcal{W} \\
&+ n\,(p-1)\left\{\operatorname{tr}\mathcal{W}^2 + \partial_k\left(\operatorname{tr}\mathcal{W}\right)\mathsf{a}_k(\nabla w) - \partial_{\xi_i\xi_j\xi_l}\mathcal{V}\left(\nabla w\right)\partial_l\mathcal{V}\left(\nabla w\right)w_{ij}\right\}.
\end{aligned}
\tag{3.56}
$$

Now, from (3.51) and (3.55), we deduce that

$$
A_2 = \left(-\mathsf{P}w_j + n\,(p-1)\,\partial_j\mathcal{V}\left(\nabla w\right)\right)\partial_{\xi_i\xi_j\xi_k}\mathcal{V}\left(\nabla w\right)w_{ki},
\tag{3.57}
$$

therefore, inserting (3.56) and (3.57) into (3.53), we establish the validity of (3.48). $\square$

Step 5. The approximation procedure. In this step, we define an approximation for (3.32) to ensure that solutions enjoy the necessary regularity to apply Proposition 3.1 of Step 4. We aim to derive from this an integral identity valid for solutions of the approximate problem – namely (3.69) below –, and ultimately take the limit to establish an integral inequality for solutions of the original problem (3.32).

We begin by defining the approximation. First, let $r > 0$ be fixed. Given $\varepsilon \in (0,1)$ and $\xi \in \mathbb{R}^n$, we define

$$V^\varepsilon(\xi) := \frac{1}{p}\left[\varepsilon^2 + |\xi|^2\right]^{\frac{p}{2}} \quad \text{and} \quad a^\varepsilon(\xi) := \nabla V^\varepsilon(\xi).$$

Let $v^\varepsilon = v^{\varepsilon,r} \in W^{1,p}(B_r)$ be a weak solution to

$$\begin{cases} \operatorname{div}\left(a^\varepsilon\left(\nabla v^\varepsilon\right)\right) = P + R & \text{in } B_r, \\ v^\varepsilon = v & \text{on } \partial B_r. \end{cases} \tag{3.58}$$

The direct method of the calculus of variations – see, for instance, [17, Theorem 3.30] – applied to the functional

$$\mathcal{F}_\varepsilon(v) := \int_{B_r} V^\varepsilon(\nabla v^\varepsilon) + (P+R)\, v^\varepsilon \, dx$$

ensures the existence of a unique minimizer $v^\varepsilon \in W^{1,p}(B_r)$, with $v^\varepsilon - v \in W_0^{1,p}(B_r)$, satisfying (3.58) in the weak sense.

By the regularity of v, P, and κ, classical elliptic estimates – see, for instance, [32] – give that

$$v^\varepsilon \in W^{2,2}_{\mathrm{loc}}(B_r) \cap C^{1,\alpha}_{\mathrm{loc}}(B_r), \tag{3.59}$$

$$v^\varepsilon \to v \quad \text{in } W^{1,p}(B_r) \cap C^{1,\alpha}_{\mathrm{loc}}(B_r), \tag{3.60}$$

see also [2] for (3.60). Moreover, for any $B_\rho \Subset B_r$, there exists a constant $C > 0$, independent of ε, such that

$$\int_{B_\rho} \left[\varepsilon^2 + |\nabla v^\varepsilon|^2\right]^{p-2} \|D^2 v^\varepsilon\|^2 \, dx \le C, \tag{3.61}$$

$$\|v^\varepsilon\|_{C^{1,\alpha}(B_\rho)} \le C \quad \text{and} \quad \|a^\varepsilon\left(\nabla v^\varepsilon\right)\|_{W^{1,2}(B_\rho)} \le C. \tag{3.62}$$

Finally, by setting $\mathcal{Z}_{v^\varepsilon} := \{x \in B_r \mid \nabla v^\varepsilon(x) = 0\}$, we have

$$|\mathcal{Z}_{v^\varepsilon}| = 0 \quad \text{and} \quad v^\varepsilon \in C^3\left(B_r \setminus \left(\mathcal{Z}_{v^\varepsilon} \cup \mathcal{Z}_v\right)\right). \tag{3.63}$$

Taking advantage of the regularity of v and v^ε, by defining

$$W^\varepsilon \coloneqq \nabla a^\varepsilon \left(\nabla v^\varepsilon\right),$$

we have that $W^\varepsilon \in L^2_{\rm loc}(B_r)$ and we can write

$$W^\varepsilon(x) = \begin{cases} A^\varepsilon(x) D^2 v^\varepsilon(x) & \text{if } x \in B_r \setminus \left(\mathcal{Z}_{v^\varepsilon} \cup \mathcal{Z}_v\right), \\ 0 & \text{if } x \in \mathcal{Z}_{v^\varepsilon} \cup \mathcal{Z}_v. \end{cases} \tag{3.64}$$

Here, A^ε is the matrix with components

$$\left[A^\varepsilon(x)\right]_{ij} = \alpha^\varepsilon_{ij}(x) \coloneqq \partial_{\xi_i \xi_j} V^\varepsilon\left(\nabla v^\varepsilon\right)(x) \quad \text{for } x \in \mathbb{R}^n \setminus \mathcal{Z}_{v^\varepsilon}, \tag{3.65}$$

that is

$$A^\varepsilon = \left[\varepsilon^2 + |\nabla v^\varepsilon|^2\right]^{\frac{p-2}{2}} \mathrm{Id}_n + (p-2)\left[\varepsilon^2 + |\nabla v^\varepsilon|^2\right]^{\frac{p-4}{2}} \nabla v^\varepsilon \otimes \nabla v^\varepsilon \quad \text{on } B_r \setminus \mathcal{Z}_{v^\varepsilon}. \tag{3.66}$$

As for the matrix A defined in (3.45), we extend A^ε to zero on $\mathcal{Z}_{v^\varepsilon}$ and observe that it is of class $C^{1,\alpha}_{\rm loc}$ in $B_r \setminus (\mathcal{Z}_{v^\varepsilon} \cup \mathcal{Z}_v)$. Hence, A^ε is measurable on B_r. In addition, we also have

$$\operatorname{tr} W^\varepsilon = \alpha^\varepsilon_{ij} v^\varepsilon_{ij} = P + R \quad \text{a.e. in } B_r. \tag{3.67}$$

We are now in a position to state and prove the fundamental integral identity related to our approximation (3.58). This is exactly the content of the following result.

Proposition 3.2. *Let $v^\varepsilon = v^{\varepsilon,r}$ be the unique weak solution of (3.58) and set*

$$P_\varepsilon \coloneqq \frac{n\,(p-1)}{v^\varepsilon} V^\varepsilon\left(\nabla v^\varepsilon\right) + \left(\frac{p}{n-p}\right)^{p-1} \frac{1}{v^\varepsilon}. \tag{3.68}$$

Then, for any $\phi \in C^\infty_c(B_r)$ and any $t \in \mathbb{R}$, the following identity holds

$$\begin{aligned} -\int_{B_r} (v^\varepsilon)^{2-n} \left\langle A^\varepsilon \nabla P_\varepsilon, \nabla(P^t \phi)\right\rangle dx &= \int_{B_r} \operatorname{div}\left((v^\varepsilon)^{2-n} A^\varepsilon \nabla P_\varepsilon\right) P^t \phi \, dx \\ &= \int_{B_r} (v^\varepsilon)^{1-n} \Big\{ -n \left\langle A^\varepsilon \nabla P_\varepsilon, \nabla v^\varepsilon \right\rangle - P_\varepsilon \operatorname{tr} W^\varepsilon \\ &\quad + n\,(p-1)\left[\operatorname{tr}[W^\varepsilon]^2 + \left\langle \nabla\left(\operatorname{tr} W^\varepsilon\right), a^\varepsilon(\nabla v^\varepsilon)\right\rangle\right] - P_\varepsilon v^\varepsilon_j \, \partial_{\xi_i \xi_j \xi_k} V^\varepsilon(\nabla v^\varepsilon) v^\varepsilon_{ki} \Big\} P^t \phi \, dx, \end{aligned} \tag{3.69}$$

where P is the P-function defined in (3.33) and A^ε is the matrix given by (3.65).

Proof. From (3.59), we know that $v^\varepsilon \in W^{2,2}_{\mathrm{loc}}(B_r)$ is a strong solution to (3.67). Moreover, from the explicit expression of A^ε given by (3.66), we deduce that $\alpha^\varepsilon_{ij} \in C^{0,\alpha}_{\mathrm{loc}}(B_r)$. Since $P \in C^{0,\alpha}(B_r)$, by (3.42), and $R \in C^{0,\alpha}(B_r)$, by (3.36) and the regularity of κ, Theorem 9.19 in [30] yields

$$v^\varepsilon \in C^{2,\alpha}_{\mathrm{loc}}(B_r),$$

which, in turn, implies $\alpha^\varepsilon_{ij} \in C^{1,\alpha}_{\mathrm{loc}}(B_r)$ and $P_\varepsilon \in C^{1,\alpha}_{\mathrm{loc}}(B_r)$.

Again, by Theorem 9.19 in [30], we have that $v^\varepsilon \in W^{3,2}_{\mathrm{loc}}(B_r)$, which implies that $V^\varepsilon(\nabla v^\varepsilon) \in W^{2,2}_{\mathrm{loc}}(B_r)$, since, by the chain rule

$$\nabla V^\varepsilon(\nabla v^\varepsilon) = a^\varepsilon(\nabla v^\varepsilon)\, D^2 v^\varepsilon.$$

As a consequence, from (3.68), we infer

$$P_\varepsilon \in W^{2,2}_{\mathrm{loc}}(B_r),$$

and thus $(A^\varepsilon \nabla P_\varepsilon)_i = \alpha^\varepsilon_{ij}\partial_j P_\varepsilon \in W^{1,2}_{\mathrm{loc}}(B_r)$.

Therefore, for any test function $\phi \in C^\infty_c(B_r)$, integrating by parts, we deduce

$$-\int_{B_r} (v^\varepsilon)^{2-n} \left\langle A^\varepsilon \nabla P_\varepsilon, \nabla(P^t\phi)\right\rangle dx = \int_{B_r} \operatorname{div}\left((v^\varepsilon)^{2-n} A^\varepsilon \nabla P_\varepsilon\right) P^t\phi\, dx.$$

Finally, taking advantage of (3.63), we apply Proposition 3.1 with $w = v^\varepsilon$, $\mathcal{V} = V^\varepsilon$, and $\Omega = B_r \setminus (\mathcal{Z}_{v^\varepsilon} \cup \mathcal{Z}_v)$ to conclude that (3.48) holds in Ω. Since both $\mathcal{Z}_{v^\varepsilon}$ and $\mathcal{Z}_v$ are negligible sets in B_r, we deduce that (3.48) holds a.e. in B_r, and the conclusion follows. □

We now extend Proposition 3.2 to a solution of (3.32) in order to derive the fundamental inequality required for our proof. This extension relies on a subtle argument to obtain a meaningful inequality in its full strength. The key difficulty is that the tensor W is not necessarily symmetric, except when $p = 2$.

Proposition 3.3. *Let v be a weak solution of (3.32). Then, we have*

$$A\nabla P \in L^2_{\mathrm{loc}}(\mathbb{R}^n).$$

In addition, for every non-negative $\phi \in C^\infty_c(\mathbb{R}^n)$ such that $\operatorname{supp}\phi \subseteq \overline{B_r}$ and any $t \in \mathbb{R}$, it follows that

$$\begin{aligned}-\int_{B_r} v^{2-n}\left\langle A\nabla P, \nabla(P^t\phi)\right\rangle dx &\geq n\,(p-1)\,(1-c_p)\int_{B_r} v^{1-n}\,|\mathring{W}|^2\, P^t\phi\, dx \\ &\quad + (p-1)\int_{B_r} v^{1-n}\left\{n\,\langle\nabla R, a(\nabla v)\rangle + R\,(P+R)\right\} P^t\phi\, dx,\end{aligned} \tag{3.70}$$

where $c_p \in [0,1)$ is defined as

$$c_p := \frac{(1-\varrho_p)^2}{1+\varrho_p^2} \quad \text{with} \quad \varrho_p := (p-1)^{\operatorname{sgn}(2-p)}.$$

We first recall an elementary yet fundamental observation from Lemma 3.1 in [31], due to Guarnotta & Mosconi, whose proof can be found therein.

Lemma 3.1. *Let $X = PS$, where $P, S \in \operatorname{Sym}(n)$, with P being positive definite. Moreover, let $\lambda_{\min}$ and $\lambda_{\max}$ denote the minimum and maximum eigenvalues of P, respectively, and define the ratio $\varrho := \lambda_{\min}/\lambda_{\max} \in (0,1]$. Then, we have*

$$\left|X - X^{\mathsf{T}}\right|^2 \leq 2c\,|X|^2$$

with

$$c := \frac{(1-\varrho)^2}{1+\varrho^2} \in [0,1). \tag{3.71}$$

Additionally, the map $\varrho \mapsto c = c(\varrho)$ is decreasing on $(0,1]$.

Building on this, we establish the following algebraic lemma. Although this result is also elementary, we provide a detailed proof due to its significance.

Lemma 3.2. *Let $X = PS$, where $P, S \in \operatorname{Sym}(n)$, with P being positive definite. Then, the following inequality holds*

$$\operatorname{tr} X^2 - |X|^2 \geq -c\,|\mathring{X}|^2,$$

where $c \in [0,1)$ is the constant defined in (3.71).

Proof. We begin by computing

$$\begin{aligned} \operatorname{tr} X^2 - |X|^2 &= \operatorname{tr} X^2 - \operatorname{tr}\left[XX^{\mathsf{T}}\right] = \operatorname{tr}\left[X\left(X - X^{\mathsf{T}}\right)\right] = 2\operatorname{tr}\left[XX_{\mathrm{A}}\right] \\ &= 2\operatorname{tr}\left[\left(X_{\mathrm{S}} + X_{\mathrm{A}}\right)X_{\mathrm{A}}\right] = 2\operatorname{tr}\left[X_{\mathrm{S}}X_{\mathrm{A}} + X_{\mathrm{A}}^2\right] \end{aligned} \tag{3.72}$$

where X_{S} and X_{A} denote the symmetric and the antisymmetric part of X, respectively. Moreover, we have

$$4X_{\mathrm{S}}X_{\mathrm{A}} = \left(X + X^{\mathsf{T}}\right)\left(X - X^{\mathsf{T}}\right) = X^2 - XX^{\mathsf{T}} + X^{\mathsf{T}}X - \left(X^{\mathsf{T}}\right)^2,$$

whence, by the cyclic property of the trace, we have

$$\operatorname{tr}\left[X_{\mathrm{S}}X_{\mathrm{A}}\right] = 0. \tag{3.73}$$

Therefore, using (3.72), (3.73), and the antisymmetry of X_{A}, we deduce that

$$\operatorname{tr} X^2 - |X|^2 = -2 \operatorname{tr}\left[X_{\mathrm{A}} X_{\mathrm{A}}^{\mathsf{T}}\right] = -2\,|X_{\mathrm{A}}|^2.$$

We observe that $\mathring{X}_{\mathrm{A}} = X_{\mathrm{A}}$, thus the latter implies

$$\operatorname{tr} X^2 - |X|^2 = -2\left|\mathring{X}_{\mathrm{A}}\right|^2. \tag{3.74}$$

Since $P \in \mathrm{Sym}(n)$ is positive definite, it admits an inverse $P^{-1} \in \mathrm{Sym}(n)$, allowing us to write

$$\mathring{X} = PS - \frac{\operatorname{tr} X}{n}\,\mathrm{Id}_n = P\left(S - \frac{\operatorname{tr} X}{n} P^{-1}\right),$$

which is the product of two symmetric matrices with the first one being positive definite. By applying Lemma 3.1, we obtain

$$\left|\mathring{X} - \mathring{X}^{\mathsf{T}}\right|^2 \leq 2c\,|\mathring{X}|^2,$$

that is

$$2\left|\mathring{X}_{\mathrm{A}}\right|^2 \leq c\,|\mathring{X}|^2.$$

Combining this with (3.74), the conclusion easily follows. $\square$

We are now in a position to proceed with the proof of Proposition 3.3.

Proof of Proposition 3.3. Let $r > 0$ be fixed and $\phi \in C_c^\infty(\mathbb{R}^n)$ be non-negative and such that $\operatorname{supp} \phi \subseteq \overline{B_r}$.

First, by (3.67), we observe that

$$\nabla\left(\operatorname{tr} W^\varepsilon\right) = \nabla P + \nabla R \quad \text{a.e. in } B_r.$$

Therefore, from this, (3.69), and taking into account (3.67) once more, we deduce

$$\begin{aligned}
&-\int_{B_r} (v^\varepsilon)^{2-n} \left\langle A^\varepsilon \nabla P_\varepsilon, \nabla(P^t \phi)\right\rangle dx \\
&\qquad = \int_{B_r} (v^\varepsilon)^{1-n} \left\{-n \left\langle A^\varepsilon \nabla P_\varepsilon, \nabla v^\varepsilon\right\rangle - P_\varepsilon P - P_\varepsilon R + n\,(p-1)\left[\operatorname{tr}[W^\varepsilon]^2\right.\right. \\
&\qquad\quad \left.\left. + \left\langle \nabla P, a^\varepsilon(\nabla v^\varepsilon)\right\rangle + \left\langle \nabla R, a^\varepsilon(\nabla v^\varepsilon)\right\rangle\right] - P_\varepsilon v^\varepsilon_j\, \partial_{\xi_i \xi_j \xi_k} V^\varepsilon(\nabla v^\varepsilon) v^\varepsilon_{ki}\right\} P^t \phi \, dx.
\end{aligned} \tag{3.75}$$

We now shall let $\varepsilon \to 0$ in (3.75). To this aim we study the convergence of the terms on both sides of (3.75).

We first prove that

$$V^\varepsilon(\nabla v^\varepsilon) \to V(\nabla v) \text{ in } C^{0,\alpha}_{\mathrm{loc}}(B_r) \text{ and } V^\varepsilon(\nabla v^\varepsilon) \rightharpoonup V(\nabla v) \text{ weakly in } W^{1,2}_{\mathrm{loc}}(B_r), \tag{3.76}$$

$$a^\varepsilon(\nabla v^\varepsilon) \to a(\nabla v) \text{ in } C^{0}_{\mathrm{loc}}(B_r) \text{ and } a^\varepsilon(\nabla v^\varepsilon) \rightharpoonup a(\nabla v) \text{ weakly in } W^{1,2}_{\mathrm{loc}}(B_r), \tag{3.77}$$

$$P_\varepsilon \to P \text{ in } C^{0,\alpha}_{\mathrm{loc}}(B_r) \text{ and } P_\varepsilon \rightharpoonup P \text{ weakly in } W^{1,2}_{\mathrm{loc}}(B_r). \tag{3.78}$$

The Hölder and uniform convergences in (3.76) and (3.77) follow from (3.60) and since $V^\varepsilon \in C^1(\mathbb{R}^n)$.

Next, taking advantage of the uniform bound (3.62), we can extract a subsequence, relabeled as $a^\varepsilon(\nabla v^\varepsilon)$, such that

$$a^\varepsilon(\nabla v^\varepsilon) \rightharpoonup a(\nabla v) \quad \text{weakly in } W^{1,2}_{\mathrm{loc}}(B_r).$$

Then, we have

$$a^\varepsilon_k(\nabla v^\varepsilon) = \partial_{\xi_k} V^\varepsilon(\nabla v^\varepsilon) = \left[\varepsilon^2 + |\nabla v^\varepsilon|^2\right]^{\frac{p-2}{2}} v^\varepsilon_k,$$

where $a^\varepsilon_k(\xi)$ denotes the k-th component of $a_k(\xi)$. By the chain rule

$$\partial_j V^\varepsilon(\nabla v^\varepsilon) = a^\varepsilon_k(\nabla v^\varepsilon)\, v^\varepsilon_{kj} \quad \text{a.e. in } B_r, \tag{3.79}$$

so that, using (3.61) and (3.62), we deduce

$$\|\nabla V^\varepsilon(\nabla v^\varepsilon)\|_{L^2(B_\rho)} \le C \quad \text{for any } \rho \in (0,r),$$

and for some $C > 0$ depending on ρ. Thus, the weak convergence in (3.76) follows, up to a subsequence. Since the argument holds for any subsequence, it follows that the weak convergences hold for the whole sequence.

Finally, the convergences in (3.78) are a consequence of the definition of P_ε in (3.68), (3.60), and (3.76).

We now deal with the left-hand side of (3.75) and show that

$$A^\varepsilon \nabla P_\varepsilon \rightharpoonup A\nabla P \quad \text{weakly in } L^2_{\mathrm{loc}}(B_r). \tag{3.80}$$

Recall that the matrices A and A^ε, given in (3.45) and (3.66) respectively, are well defined and of class C^1 in the set

$$\mathcal{B}^\varepsilon_r := B_r \setminus (\mathcal{Z}_{v^\varepsilon} \cup \mathcal{Z}_v),$$

which has full measure in B_r. By (3.51), we have

$$\left(A^{\varepsilon}\nabla P_{\varepsilon}\right)_i = \alpha^{\varepsilon}_{ij}\partial_j P_{\varepsilon} = -\frac{P_{\varepsilon}}{v^{\varepsilon}}\,\alpha^{\varepsilon}_{ij}v^{\varepsilon}_j + \frac{n\,(p-1)}{v^{\varepsilon}}\,\alpha^{\varepsilon}_{ij}\partial_j V^{\varepsilon}\left(\nabla v^{\varepsilon}\right) \quad \text{in } \mathcal{B}^{\varepsilon}_r, \tag{3.81}$$

where, thanks to (3.66), we have

$$\alpha^{\varepsilon}_{ij} = \left[\varepsilon^2 + |\nabla v^{\varepsilon}|^2\right]^{\frac{p-2}{2}}\delta_{ij} + (p-2)\left[\varepsilon^2 + |\nabla v^{\varepsilon}|^2\right]^{\frac{p-4}{2}} v^{\varepsilon}_i v^{\varepsilon}_j \quad \text{in } \mathcal{B}^{\varepsilon}_r.$$

Let $\rho \in (0, r)$ be fixed. The latter, together with (3.62), implies

$$\alpha^{\varepsilon}_{ij}v^{\varepsilon}_j \le C \quad \text{in } \mathcal{B}^{\varepsilon}_{\rho}, \tag{3.82}$$

for some $C > 0$ independent of ε.

We now study the pointwise convergence of $A^{\varepsilon}\nabla P_{\varepsilon}$. To this end, we extract a subsequence $\{\varepsilon_m\}_{m\in\mathbb{N}}$ and define

$$\mathcal{B}_r := \bigcap_{m\in\mathbb{N}} \mathcal{B}^{\varepsilon_m}_r,$$

which is a Borel set of full measure in B_r.

Since $\nabla^2_{\xi}V^{\varepsilon} \to \nabla^2_{\xi}V$ in $\mathbb{R}^n \setminus \{0\}$, form (3.60), we deduce

$$\alpha^{\varepsilon_m}_{ij}v^{\varepsilon_m}_j \to \alpha_{ij}v_j \quad \text{in } \mathcal{B}_r$$

as $m \to +\infty$. Taking advantage of (3.82), the dominated convergence theorem entails

$$\alpha^{\varepsilon_m}_{ij}v^{\varepsilon_m}_j \to \alpha_{ij}v_j \quad \text{in } L^2(B_{\rho}). \tag{3.83}$$

By using (3.64) and (3.79), we obtain

$$\alpha^{\varepsilon}_{ij}\partial_j V^{\varepsilon}\left(\nabla v^{\varepsilon}\right) = \alpha^{\varepsilon}_{ij}a^{\varepsilon}_k\left(\nabla v^{\varepsilon}\right)v^{\varepsilon}_{kj} = a^{\varepsilon}_k\left(\nabla v^{\varepsilon}\right)\left(W^{\varepsilon}\right)_{ik} \quad \text{in } \mathcal{B}_r, \tag{3.84}$$

and, thanks to (3.77), that

$$a^{\varepsilon}_k\left(\nabla v^{\varepsilon}\right)\left(W^{\varepsilon}\right)_{ik} \rightharpoonup a_k\left(\nabla v\right)W_{ik} \quad \text{weakly in } L^2(B_{\rho}). \tag{3.85}$$

Furthermore, from (3.44) and the chain rule $\partial_j V\left(\nabla v\right) = a_k\left(\nabla v\right)v_{kj}$, we also get

$$a_k\left(\nabla v\right)W_{ik} = a_k\left(\nabla v\right)\alpha_{ij}v_{jk} = \alpha_{ij}\partial_j V\left(\nabla v\right) \quad \text{in } \mathbb{R}^n \setminus \mathcal{Z}_v,$$

where we also used (3.36). The latter, together with (3.84) and (3.85), yields

$$\alpha^{\varepsilon}_{ij}\partial_j V^{\varepsilon} \rightharpoonup \alpha_{ij}\partial_j V\left(\nabla v\right) \quad \text{weakly in } L^2(B_{\rho}). \tag{3.86}$$

Piecing (3.60), (3.78), (3.83), and (3.86) with (3.81), we are led to

$$\alpha_{ij}^{\varepsilon_m}\partial_j P_{\varepsilon_m} \rightharpoonup -\frac{P}{v}\,\alpha_{ij}v_j + \frac{n\,(p-1)}{v}\,\alpha_{ij}\partial_j V\,(\nabla v) = \alpha_{ij}\partial_j P \quad \text{weakly in } L^2(B_\rho),$$

where the last identity follows from (3.51). As a consequence, $A\nabla P \in L^2(B_\rho)$ and (3.80) follows since the previous argument hold for every subsequence $\{\varepsilon_m\}_{m\in\mathbb{N}}$.

We now handle the right-hand side of (3.75). First, by exploiting (3.60), (3.77), (3.78), and (3.80), we infer

$$-n\,\langle A^\varepsilon \nabla P_\varepsilon, \nabla v^\varepsilon\rangle \rightharpoonup -n\,\langle A\nabla P, \nabla v\rangle \quad \text{weakly in } L^2_{\mathrm{loc}}(B_r), \tag{3.87}$$

and

$$\begin{aligned} &-P_\varepsilon P - P_\varepsilon R + n\,(p-1)\left[\langle \nabla P, a^\varepsilon(\nabla v^\varepsilon)\rangle + \langle \nabla R, a^\varepsilon(\nabla v^\varepsilon)\rangle\right] \to \\ &\qquad -P^2 - PR + n\,(p-1)\left[\langle \nabla P, a\,(\nabla v)\rangle + \langle \nabla R, a\,(\nabla v)\rangle\right] \quad \text{in } C^0_{\mathrm{loc}}(B_r). \end{aligned} \tag{3.88}$$

We shall now show that

$$v_j^\varepsilon\, \partial_{\xi_i\xi_j\xi_k} V^\varepsilon(\nabla v^\varepsilon) v_{ki}^\varepsilon \rightharpoonup (p-2)\,(P+R) \quad \text{weakly in } L^2_{\mathrm{loc}}(B_r). \tag{3.89}$$

We start by proving that

$$\begin{aligned} v_j^\varepsilon\, \partial_{\xi_i\xi_j\xi_k} V^\varepsilon(\nabla v^\varepsilon) v_{ki}^\varepsilon &= (p-2)\left[1 - \frac{\varepsilon^2}{\varepsilon^2 + |\nabla v^\varepsilon|^2}\right](P+R) \\ &\quad + 2\,(p-2)\,\frac{\varepsilon^2\, v_i^\varepsilon\, v_k^\varepsilon}{\left[\varepsilon^2 + |\nabla v^\varepsilon|^2\right]^2}\left[\varepsilon^2 + |\nabla v^\varepsilon|^2\right]^{\frac{p-2}{2}} v_{ik}^\varepsilon \quad \text{in } B_r \setminus \mathcal{Z}_{v^\varepsilon}. \end{aligned} \tag{3.90}$$

In $\mathbb{R}^n \setminus \{0\}$, a direct computation reveals

$$\begin{aligned} a_i^\varepsilon(\xi) = \partial_i V^\varepsilon(\xi) &= \left[\varepsilon^2 + |\xi|^2\right]^{\frac{p-2}{2}} \xi_i, \\ \partial_{ij} V^\varepsilon(\xi) &= \left[\varepsilon^2 + |\xi|^2\right]^{\frac{p-2}{2}} \delta_{ij} + (p-2)\left[\varepsilon^2 + |\xi|^2\right]^{\frac{p-4}{2}} \xi_i\,\xi_j, \end{aligned} \tag{3.91}$$

and

$$\begin{aligned} \partial_{ijk} V^\varepsilon(\xi) = (p-2)&\left[\varepsilon^2 + |\xi|^2\right]^{\frac{p-4}{2}} (\xi_k\delta_{ij} + \xi_i\delta_{jk} + \xi_j\delta_{ik}) \\ &+ (p-2)\,(p-4)\left[\varepsilon^2 + |\xi|^2\right]^{\frac{p-6}{2}} \xi_i\xi_j\xi_k. \end{aligned}$$

The latter implies

$$\begin{aligned}\partial_{ijk}V^\varepsilon(\xi)\xi_j = (p-2)\left[\varepsilon^2+|\xi|^2\right]^{\frac{p-4}{2}}\left(2\xi_i\xi_k+|\xi|^2\delta_{ik}\right) \\ +(p-2)(p-4)\left[\varepsilon^2+|\xi|^2\right]^{\frac{p-6}{2}}|\xi|^2\,\xi_i\xi_k \quad \text{in } \mathbb{R}^n\setminus\{0\}.\end{aligned} \tag{3.92}$$

Finally, by (3.91), we can rewrite (3.92) in the form

$$\partial_{ijk}V^\varepsilon(\xi)\xi_j = (p-2)\left[1-\frac{\varepsilon^2}{\varepsilon^2+|\xi|^2}\right]\partial_{ik}V^\varepsilon(\xi)+\frac{2\,(p-2)\,\varepsilon^2\,\xi_i\xi_k}{\left[\varepsilon^2+\xi^2\right]^2}\left[\varepsilon^2+|\xi|^2\right]^{\frac{p-2}{2}}.$$

Therefore, by (3.65), we conclude that

$$\begin{aligned}v^\varepsilon_j\,\partial_{\xi_i\xi_j\xi_k}V^\varepsilon(\nabla v^\varepsilon)v^\varepsilon_{ki} = (p-2)\left[1-\frac{\varepsilon^2}{\varepsilon^2+|\nabla v^\varepsilon|^2}\right]\alpha^\varepsilon_{ik}v^\varepsilon_{ik} \\ +2\,(p-2)\,\frac{\varepsilon^2\,v^\varepsilon_i\,v^\varepsilon_k}{\left[\varepsilon^2+|\nabla v^\varepsilon|^2\right]^2}\left[\varepsilon^2+|\nabla v^\varepsilon|^2\right]^{\frac{p-2}{2}}v^\varepsilon_{ik} \quad \text{in } B_r\setminus\mathcal{Z}_{v^\varepsilon},\end{aligned}$$

from which (3.90) follows by (3.67).

Since $P,R\in L^\infty(\mathbb{R}^n)$, by the dominated convergence theorem

$$(p-2)\left[1-\frac{\varepsilon^2}{\varepsilon^2+|\nabla v^\varepsilon|^2}\right](P+R)\to(p-2)\,(P+R) \quad \text{in } L^2_{\mathrm{loc}}(B_r). \tag{3.93}$$

Then, we simply have

$$\frac{\varepsilon^2\,|v^\varepsilon_i\,v^\varepsilon_k|}{\left[\varepsilon^2+|\nabla v^\varepsilon|^2\right]^2}\le\frac{\varepsilon^2\,|\nabla v^\varepsilon|^2}{\left[\varepsilon^2+|\nabla v^\varepsilon|^2\right]^2}\le\frac{1}{4}.$$

Thus, taking advantage of (3.61), we deduce

$$\begin{aligned}\int_{B_\rho}\left\{\varepsilon^2\,v^\varepsilon_i\,v^\varepsilon_k\left[\varepsilon^2+|\nabla v^\varepsilon|^2\right]^{-2}\left[\varepsilon^2+|\nabla v^\varepsilon|^2\right]^{\frac{p-2}{2}}v^\varepsilon_{ik}\right\}^2dx \\ \le\frac{1}{16}\left\|\left[\varepsilon^2+|\nabla v^\varepsilon|^2\right]^{\frac{p-2}{2}}D^2v^\varepsilon\right\|^2_{L^2(B_\rho)}\le C.\end{aligned} \tag{3.94}$$

As above, we extract a subsequence $\{\varepsilon_m\}_{m\in\mathbb{N}}$ and notice that

$$\frac{\varepsilon_m^2\,v^{\varepsilon_m}_i\,v^{\varepsilon_m}_k}{\left[\varepsilon_m^2+|\nabla v^{\varepsilon_m}|^2\right]^2}\left[\varepsilon^2+|\nabla v^\varepsilon|^2\right]^{\frac{p-2}{2}}v^{\varepsilon_m}_{ik}\to 0 \quad \text{in } \mathcal{B}_r. \tag{3.95}$$

From (3.94), (3.95), and since the subsequence is arbitrary, we conclude that

$$2\,(p-2)\,\frac{\varepsilon_m^2\,v_i^{\varepsilon_m}\,v_k^{\varepsilon_m}}{\left[\varepsilon_m^2+|\nabla v^{\varepsilon_m}|^2\right]^2}\left[\varepsilon^2+|\nabla v^{\varepsilon}|^2\right]^{\frac{p-2}{2}}\,v_{ik}^{\varepsilon_m}\rightharpoonup 0\quad\text{weakly in } L^2_{\mathrm{loc}}(B_r). \tag{3.96}$$

Hence, (3.89) immediately follows from (3.93) and (3.96).

We now handle the remaining term in (3.75), namely that involving $\mathrm{tr}[W^\varepsilon]^2$, and notice that we cannot directly conclude by (3.77). To this end, we write

$$\mathrm{tr}[W^\varepsilon]^2=|W^\varepsilon|^2+\mathrm{tr}[W^\varepsilon]^2-|W^\varepsilon|^2$$

and apply Lemma 3.2 to $W^\varepsilon=A^\varepsilon D^2 v^\varepsilon$. Note that all these computations hold on $B_r\setminus(\mathcal{Z}_{v^\varepsilon}\cup\mathcal{Z}_v)$ which has full measure in B_r. Therefore, we deduce that

$$\mathrm{tr}[W^\varepsilon]^2\geq|W^\varepsilon|^2-c_{p,\varepsilon}\,|\mathring{W}^\varepsilon|^2. \tag{3.97}$$

In particular, the constant $c_{p,\varepsilon}$ given in (3.71) is determined by the ratio

$$\varrho_{p,\varepsilon}:=\frac{\lambda_{\min}(A^\varepsilon)}{\lambda_{\max}(A^\varepsilon)}.$$

Moreover, if ϱ_p denotes the same ratio referred to A, which is

$$\varrho_p:=\frac{\min\{1,p-1\}}{\max\{1,p-1\}}=(p-1)^{\mathrm{sgn}(2-p)}\,,$$

a straightforward computation reveals that $\varrho_{p,\varepsilon}\geq\varrho_p$. As a consequence, Lemma 3.1 yields $c_{p,\varepsilon}\leq c_p$, where c_p is defined in (3.71) and referred to A, which depends only on p. Combining this estimate with (3.97), we obtain

$$\mathrm{tr}[W^\varepsilon]^2\geq|W^\varepsilon|^2-c_p\,|\mathring{W}^\varepsilon|^2. \tag{3.98}$$

Now, we notice that

$$|\mathring{W}^\varepsilon|^2=|W^\varepsilon|^2-\frac{1}{n}\,(\mathrm{tr}\,W^\varepsilon)^2=|W^\varepsilon|^2-\frac{1}{n}\,(P+R)^2\quad\text{a.e. in } B_r, \tag{3.99}$$

where we used (3.67). From (3.98) and (3.99), we conclude that

$$\mathrm{tr}[W^\varepsilon]^2\geq(1-c_p)\,|\mathring{W}^\varepsilon|^2+\frac{1}{n}\,(P+R)^2\,. \tag{3.100}$$

Finally, we shall show that

$$\liminf_{\varepsilon\to0}\int_{B_r}(v^\varepsilon)^{1-n}\,|\mathring{W}^\varepsilon|^2P^t\phi\,dx\geq\int_{B_r}v^{1-n}\,|\mathring{W}|^2P^t\phi\,dx. \tag{3.101}$$

We first prove that

$$\liminf_{\varepsilon\to 0}\int_{B_r}(v^\varepsilon)^{1-n}\left|\nabla a^\varepsilon\left(\nabla v^\varepsilon\right)\right|^2 P^t\phi\,dx \geq \int_{B_r} v^{1-n}\left|\nabla a\left(\nabla v\right)\right|^2 P^t\phi\,dx. \tag{3.102}$$

From this, (3.60), and (3.99), we infer the validity of (3.101).

We start by writing

$$\int_{B_r}(v^\varepsilon)^{1-n}\left|\nabla a^\varepsilon\left(\nabla v^\varepsilon\right)\right|^2 P^t\phi\,dx - \int_{B_r} v^{1-n}\left|\nabla a\left(\nabla v\right)\right|^2 P^t\phi\,dx = I_1^\varepsilon + I_2^\varepsilon,$$

where

$$\begin{aligned} I_1^\varepsilon &:= \int_{B_r}\left[(v^\varepsilon)^{1-n} - v^{1-n}\right]\left|\nabla a^\varepsilon\left(\nabla v^\varepsilon\right)\right|^2 P^t\phi\,dx,\\ I_2^\varepsilon &:= \int_{B_r} v^{1-n}\left[\left|\nabla a^\varepsilon\left(\nabla v^\varepsilon\right)\right|^2 - \left|\nabla a\left(\nabla v\right)\right|^2\right] P^t\phi\,dx. \end{aligned}$$

Since v is bounded below, by (3.40), (3.60), and (3.62), we easily deduce that

$$\lim_{\varepsilon\to 0} I_1^\varepsilon = 0. \tag{3.103}$$

Then, the weak lower semicontinuity of the L^2-norm, together with (3.77), yields

$$\liminf_{\varepsilon\to 0}\int_{B_r}\left|\nabla a^\varepsilon\left(\nabla v^\varepsilon\right) v^{\frac{1-n}{2}}\left(P^t\phi\right)^{\frac{1}{2}}\right|^2 dx \geq \int_{B_r} v^{1-n}\left|\nabla a\left(\nabla v\right)\right|^2 P^t\phi\,dx.$$

As a consequence, we infer that

$$\liminf_{\varepsilon\to 0} I_2^\varepsilon \geq 0.$$

This, together with (3.103), implies the validity of (3.102).

By using (3.100), we can rewrite (3.75) as

$$\begin{aligned} -\int_{B_r}(v^\varepsilon)^{2-n}\left\langle A^\varepsilon\nabla P_\varepsilon, \nabla(P^t\phi)\right\rangle dx \geq \int_{B_r}(v^\varepsilon)^{1-n}\Big\{&-n\left\langle A^\varepsilon\nabla P_\varepsilon, \nabla v^\varepsilon\right\rangle - P_\varepsilon P - P_\varepsilon R\\ &+ n\left(p-1\right)\left[\left(1-c_p\right)|\mathring{W}^\varepsilon|^2 + \frac{1}{n}\left(P+R\right)^2 + \langle\nabla P, a^\varepsilon(\nabla v^\varepsilon)\rangle + \langle\nabla R, a^\varepsilon(\nabla v^\varepsilon)\rangle\right]\\ &- P_\varepsilon v_j^\varepsilon\,\partial_{\xi_i\xi_j\xi_k}V^\varepsilon(\nabla v^\varepsilon)v_{ki}^\varepsilon\Big\}P^t\phi\,dx. \end{aligned} \tag{3.104}$$

Thus, by (3.60), (3.80), (3.87), (3.88), (3.89), and (3.101), we let $\varepsilon\to 0$ in (3.104) and deduce

$$-\int_{B_r} v^{2-n}\left\langle A\nabla P,\nabla(P^t\phi)\right\rangle dx \geq n\,(p-1)\,(1-c_p)\int_{B_r} v^{1-n}\,|\mathring{W}|^2 P^t\phi\,dx$$
$$+\int_{B_r} v^{1-n}\left\{-n\left\langle A\nabla P,\nabla v\right\rangle + n\,(p-1)\left[\left\langle\nabla P,a\left(\nabla v\right)\right\rangle + \left\langle\nabla R,a\left(\nabla v\right)\right\rangle\right]\right. \tag{3.105}$$
$$\left. +\,(p-1)\,(P+R)^2-(p-1)\,P\,(P+R)\right\}P^t\phi\,dx.$$

To conclude, we observe that by p-homogeneity of V, we have

$$\alpha_{ij}v_i=\partial_{\xi_i\xi_j}V\left(\nabla v\right)v_i=(p-1)\,\partial_{\xi_j}V\left(\nabla v\right)=(p-1)\,a_j\left(\nabla v\right)\quad\text{in }\mathbb{R}^n\setminus\mathcal{Z}_v,$$

therefore

$$\begin{aligned}-n\left\langle A\nabla P,\nabla v\right\rangle &= -n\,\alpha_{ij}v_i\partial_jP=-n\,(p-1)\,\partial_jP\,a_j\left(\nabla v\right)\\ &= -n\,(p-1)\left\langle\nabla P,a\left(\nabla v\right)\right\rangle\quad\text{a.e. in }\mathbb{R}^n.\end{aligned}\tag{3.106}$$

From (3.105) and (3.106), we easily deduce (3.70). □

Step 6. Quantitative integral estimate. We now aim to derive a quantitative estimate for an integral involving $|\mathring{W}|$, the traceless tensor defined in (3.43), by exploiting Proposition 3.3. Specifically, for any $t\geq 1$, we shall prove that

$$\int_{\mathbb{R}^n} v^{1-n}\,|\mathring{W}|^2P^t\,dx+\int_{\mathbb{R}^n} v^{2-n}\,|\nabla v|^{p-2}P^{t-1}\,|\nabla P|^2\,dx\leq C_\sharp\,\mathrm{def}(u,\kappa),\tag{3.107}$$

for some constant $C_\sharp>0$ depending only on n, p, $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$, and t.

First, recalling (3.45), note that

$$A\nabla P=|\nabla v|^{p-2}\,\nabla P+(p-2)\,|\nabla v|^{p-4}\left\langle\nabla v,\nabla P\right\rangle\nabla v\quad\text{a.e. in }\mathbb{R}^n,\tag{3.108}$$

therefore

$$\left\langle A\nabla P,\nabla P\right\rangle=|\nabla v|^{p-2}\,|\nabla P|^2+(p-2)\,|\nabla v|^{p-4}\,|\left\langle\nabla v,\nabla P\right\rangle|^2\quad\text{a.e. in }\mathbb{R}^n.$$

Let us now define the annulus $A_r\coloneqq B_{2r}\setminus B_r$. Moreover, let $\phi\in C_c^\infty(\mathbb{R}^n)$ be a cut-off function such that $0\leq\phi\leq 1$, with $\phi=1$ in B_r, $\phi=0$ in $\mathbb{R}^n\setminus B_{2r}$, and $|\nabla\phi|\leq 2/r$ in A_r. By using ϕ^2 in (3.70) as a test function, we obtain

$$n\,(p-1)\,(1-c_p)\int_{B_{2r}} v^{1-n}\,|\mathring{W}|^2P^t\phi^2\,dx+t\int_{B_{2r}} v^{2-n}\,|\nabla v|^{p-2}P^{t-1}\,|\nabla P|^2\phi^2\,dx$$
$$\leq I_1^r+I_2^r+I_3^r+I_4^r,\tag{3.109}$$

where

$$I_1^r \coloneqq (2-p)\,t \int\limits_{B_{2r}} v^{2-n}\,|\nabla v|^{p-4} P^{t-1}\,|\langle \nabla v, \nabla P\rangle|^2\,\phi^2\,dx,$$

$$I_2^r \coloneqq -2 \int\limits_{A_r} v^{2-n}\,\langle A\nabla P, \nabla\phi\rangle\,P^t\phi\,dx,$$

$$I_3^r \coloneqq (1-p) \int\limits_{B_{2r}} v^{1-n}\,\left(R^2+PR\right)P^t\phi^2\,dx,$$

$$I_4^r \coloneqq n\,(1-p) \int\limits_{B_{2r}} v^{1-n}\,\langle \nabla R, a\,(\nabla v)\rangle\,P^t\phi^2\,dx.$$

Now, we shall estimate each of these integrals. In the following calculations C and C' will be positive constants, possibly differing from line to line, which do not depend on r, but only on n, p, $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$, and t.

For the first integral, we trivially have

$$I_1^r \le (2-p)_+\,t \int\limits_{B_{2r}} v^{2-n}\,|\nabla v|^{p-2} P^{t-1}\,|\nabla P|^2\,\phi^2\,dx, \tag{3.110}$$

and notice that $1-(2-p)_+ = \min\{p-1, 1\}$.

For I_2^r, taking advantage of (3.46) and (3.47), we deduce

$$|\nabla P| \le C\left(v^{-1}\,|\mathring{W}||\nabla v| + v^{-1}\,|\nabla v||R|\right).$$

Therefore, by exploiting (3.108) and applying Young's inequality, we get

$$\begin{aligned}
|I_2^r| &\le C \int\limits_{A_r} v^{2-n}\,|\nabla v|^{p-2}\,|\nabla P||\nabla\phi|P^t\phi\,dx \\
&\le C \int\limits_{A_r} v^{1-n}\,|\nabla v|^{p-1}\,|\mathring{W}||\nabla\phi|P^t\phi\,dx + C' \int\limits_{A_r} v^{1-n}\,|\nabla v|^{p-1}\,|\nabla\phi||R|P^t\phi\,dx \\
&\le \frac{n}{2}\,(p-1)\,(1-c_p) \int\limits_{B_{2r}} v^{1-n}\,|\mathring{W}|^2 P^t\phi^2\,dx + C \int\limits_{A_r} v^{1-n}\,|\nabla v|^{2(p-1)}\,|\nabla\phi|^2 P^t\,dx \\
&\quad + C' \int\limits_{A_r} v^{1-n}\,|\nabla v|^{p-1}\,|\nabla\phi||R|P^t\phi\,dx.
\end{aligned} \tag{3.111}$$

We then observe that, for r large enough, by exploiting (3.33), (3.35), (3.38), (3.40) and since $t \ge 1$, we have

$$\int_{A_r} v^{1-n} |\nabla v|^{2(p-1)} |\nabla \phi|^2 P^t \, dx \le C \int_{A_r} v^{-n} + v^{-n} |\nabla v|^p \, dx \to 0, \tag{3.112}$$

as $r \to +\infty$, and analogously, by exploiting (3.34), (3.35), (3.38), and (3.40), we get

$$\int_{A_r} v^{1-n} |\nabla v|^{p-1} |\nabla \phi| |R| P^t \phi \, dx \le C \int_{A_r} v^{-n} \, dx \to 0, \tag{3.113}$$

as $r \to +\infty$.

For the third integral I_3^r, since P satisfies (3.40), we start by writing

$$I_3^r \le C \int_{B_{2r}} v^{1-n} R^2 \, \phi^2 \, dx + C' \int_{B_{2r}} v^{1-n} |R| P \, \phi^2 \, dx.$$

Then, by (3.34), (3.40), and Hölder inequality, we have

$$\begin{aligned} \int_{B_{2r}} v^{1-n} |R| P \, \phi^2 \, dx &\le C \int_{B_{2r}} v^{-n} |\kappa - 1| \, dx \\ &\le C \left(\int_{B_{2r}} v^{-n} \, dx \right)^{\frac{1}{p^*}} \left(\int_{B_{2r}} v^{-n} |\kappa - 1|^{(p^*)'} \, dx \right)^{\frac{1}{(p^*)'}} \le C \operatorname{def}(u, \kappa). \end{aligned} \tag{3.114}$$

Analogously, exploiting also (3.41), we deduce

$$\int_{B_{2r}} v^{1-n} R^2 \, \phi^2 \, dx \le C \operatorname{def}(u, \kappa).$$

As a result,

$$|I_3^r| \le C \operatorname{def}(u, \kappa). \tag{3.115}$$

For I_4^r we perform an integration by parts to obtain

$$I_4^r = n \, (1 - p) \, (J_1^r + J_2^r + J_3^r + J_4^r) \, , \tag{3.116}$$

where

$$J_1^r := (1 - n) \int_{B_{2r}} v^{-n} \langle \nabla v, a(\nabla v) \rangle \, R P^t \phi^2 \, dx,$$

$$J_2^r := t \int_{B_{2r}} v^{1-n} \langle \nabla P, a(\nabla v) \rangle \, R P^{t-1} \phi^2 \, dx,$$

$$J_3^r := 2 \int\limits_{A_r} v^{1-n} \langle \nabla\phi, a(\nabla v)\rangle RP^t \phi \, dx,$$

$$J_4^r := \int\limits_{B_{2r}} v^{1-n} [\operatorname{tr} W] RP^t \phi^2 \, dx = \int\limits_{B_{2r}} v^{1-n} (P+R) RP^t \phi^2 \, dx.$$

For J_1^r we exploit (3.34), (3.39), and (3.40) to deduce

$$|J_1^r| \le C \int\limits_{B_{2r}} v^{-n} |\kappa - 1| \, v^{-1} |\nabla v|^p \, dx \le C \operatorname{def}(u,\kappa), \tag{3.117}$$

arguing as for (3.114).

For the second integral we have

$$\begin{aligned}
|J_2^r| &\le t \int\limits_{B_{2r}} v^{1-n} |\nabla P| |\nabla v|^{p-1} |R| P^{t-1} \phi^2 \, dx \le C \int\limits_{B_{2r}} v^{-n} |\nabla P| |\nabla v|^{p-1} |\kappa-1| P^{t-1} \phi^2 \, dx \\
&= C \int\limits_{B_{2r}} v^{\frac{2-n}{2}} |\nabla v|^{\frac{p-2}{2}} P^{\frac{t-1}{2}} |\nabla P| \, \phi \, v^{-\frac{n}{2}-1} |\nabla v|^{\frac{p}{2}} |\kappa - 1| P^{\frac{t-1}{2}} \phi \, dx \\
&\le \frac{t}{2} \min\{p-1, 1\} \int\limits_{B_{2r}} v^{2-n} |\nabla v|^{p-2} P^{t-1} |\nabla P|^2 \phi^2 \, dx \\
&\quad + C \int\limits_{B_{2r}} v^{-n-2} |\nabla v|^p |\kappa-1|^2 P^{t-1} \phi^2 \, dx \\
&\le \frac{t}{2} \min\{p-1, 1\} \int\limits_{B_{2r}} v^{2-n} |\nabla v|^{p-2} P^{t-1} |\nabla P|^2 \phi^2 \, dx \\
&\quad + C \|\kappa - 1\|_{L^\infty(\mathbb{R}^n)} \int\limits_{B_{2r}} v^{-2} |\nabla v|^p v^{-n} |\kappa - 1| P^{t-1} \phi^2 \, dx.
\end{aligned}$$

Hence, using (3.39) and that v is bounded below in the last inequality, we obtain

$$|J_2^r| \le \frac{t}{2} \min\{p-1, 1\} \int\limits_{B_{2r}} v^{2-n} |\nabla v|^{p-2} P^{t-1} |\nabla P|^2 \, dx + C \operatorname{def}(u,\kappa). \tag{3.118}$$

For J_3^r we exploit (3.34), (3.35), (3.38), and (3.40) to deduce that, for r sufficiently large,

$$|J_3^r| \le C \int\limits_{A_r} v^{-n} |\nabla v|^{p-1} |\nabla \phi| \, dx \le C \int\limits_{A_r} v^{-n} \, dx \to 0, \tag{3.119}$$

as $r \to +\infty$.

The fourth term can be handled in a completely analogous way to I_3^r, yielding

$$|J_4^r| \leq C\,\mathrm{def}(u,\kappa). \tag{3.120}$$

By collecting (3.109)–(3.113) with (3.115)–(3.120), letting $r \to +\infty$, and applying Fatou's lemma, we infer (3.107).

Step 7. Fundamental estimate for W. Let $r > 1$ to be determined later in dependence of $\mathrm{def}(u,\kappa)$. We will choose r in such a way that $r \to +\infty$ as $\mathrm{def}(u,\kappa) \to 0$.

We aim to derive an estimate for $\nabla a(\nabla v)$ in $L^q_w(B_r)$ with $q \in [1,2]$ and weight $w := v^{-n}$. Specifically, we will show that, by appropriately selecting a constant μ, it follows that

$$\int_{B_r} v^{-n}\,|\nabla a(\nabla v) - \mu\,\mathrm{Id}_n|^q\,dx \leq C_\flat\,\mathcal{F}_q \quad \text{for every } q \in [1,2], \tag{3.121}$$

for some $C_\flat > 0$ and some function of the deficit $\mathcal{F}_q$.

Notice that, by definition of the P-function (3.33), (3.107) with $t = 1$ easily yields

$$\int_{\mathbb{R}^n} v^{-n}\,|\mathring{W}|^2\,dx \leq C_4\,\mathrm{def}(u,\kappa), \tag{3.122}$$

for some $C_4 > 0$ depending only on n, p, and $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$. Therefore, by Hölder inequality, we deduce

$$\int_{\mathbb{R}^n} v^{-n}\,|\mathring{W}|^q\,dx \leq \left(\int_{\mathbb{R}^n} v^{-n}\,|\mathring{W}|^2\,dx\right)^{\frac{q}{2}} \left(\int_{\mathbb{R}^n} v^{-n}\,dx\right)^{\frac{2-q}{2}} \leq C_5\,\mathrm{def}(u,\kappa)^{\frac{q}{2}}, \tag{3.123}$$

for some $C_5 > 0$ depending only on n, p, $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$, and q.

Recall that in Step 2 we have identified a point $x_0 \in \mathbb{R}^n$ where u attains its maximum. Consequently, v attains its minimum at x_0. By taking advantage of (3.37), we can quantitatively locate this point, indeed we must have

$$v(x_0) \leq v(0) \leq \widehat{C}_0. \tag{3.124}$$

Hence, by setting

$$\mathcal{R} := \left(\frac{\widehat{C}_0 - \hat{c}_0}{\hat{c}_0}\right)^{\frac{p-1}{p}}, \tag{3.125}$$

which depends only on n, p, and $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$, we can conclude that $x_0 \in B_{\mathcal{R}}$. Let $\mathsf{t} \in (0,1)$ to be determined later on. From this, we have $B_{\mathsf{t}}(x_0) \subseteq B_r$ provided that

$$r \geq \mathcal{R} + 2. \tag{3.126}$$

We now define

$$\overline{P} := \fint_{B_{\mathtt{t}}(x_0)} P \, dx$$

and observe that, by exploiting (3.46), we can write

$$\begin{aligned}\int_{B_r} v^{-n} \left| W - \frac{\overline{P}}{n} \mathrm{Id}_n \right|^q dx & \\ \leq 4^{q-1} \int_{B_r} v^{-n} \, |\mathring{W}|^q \, dx + 4^{q-1} \int_{B_r} v^{-n} \, |P - \overline{P}|^q \, dx + 4^{q-1} \int_{B_r} v^{-n} \, |R|^q \, dx & \\ =: 4^{q-1} \left\{ \int_{B_r} v^{-n} \, |\mathring{W}|^q \, dx + \mathcal{I}_1 + \mathcal{I}_2 \right\}. & \end{aligned} \tag{3.127}$$

We shall proceed with the estimate of both $\mathcal{I}_1$ and $\mathcal{I}_2$. In the subsequent calculations C will denote a positive constant, possibly differing from line to line, which depends only on n, p, $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$, and q.

By (3.42), the Poincaré inequality of Equation (7.45) in [30], (3.37), and since v is bounded below, we get

$$\mathcal{I}_1 \leq C \, r^{\frac{np}{p-1}} \, \mathcal{C}_P \, (r, \mathtt{t})^q \int_{B_r} v^{-n} \, |\nabla P|^q \, dx, \tag{3.128}$$

where

$$\mathcal{C}_P \, (r, \mathtt{t}) := 2^n r^n \, \mathtt{t}^{1-n}. \tag{3.129}$$

From (3.46) and (3.47), we deduce

$$\begin{aligned}\int_{B_r} v^{-n} \, |\nabla P|^q \, dx \leq C \int_{B_r} v^{-n-q} \, |\mathring{W}|^q |\nabla v|^q \, dx + C \int_{B_r} v^{-n-q} \, |\nabla v|^q |R|^q \, dx & \\ =: \mathcal{J}_1 + \mathcal{J}_2. & \end{aligned} \tag{3.130}$$

Exploiting the definition of the remainder (3.34), (3.39), and that v is bounded below, we have

$$\mathcal{J}_2 \leq C \int_{B_r} v^{-n} v^{-2q + \frac{q}{p}} \, v^{-\frac{q}{p}} \, |\nabla v|^q |\kappa - 1|^q \, dx \leq C \int_{B_r} v^{-n} \, |\kappa - 1| \, dx$$

$$\leq C \operatorname{def}(u, \kappa), \tag{3.131}$$

where for the last inequality one argues as for (3.114). In a similar fashion, we get

$$\mathcal{J}_1 \leq C \int_{B_r} v^{-n} v^{-q+\frac{q}{p}} v^{-\frac{q}{p}} |\nabla v|^q |\mathring{W}|^q \, dx \leq C \operatorname{def}(u, \kappa)^{\frac{q}{2}}, \tag{3.132}$$

where we also used (3.123).

For the second integral, by (3.34) and since v is bounded below, we have

$$\mathcal{I}_2 \leq C \int_{B_r} v^{-n-q} |\kappa - 1|^q \, dx \leq C \int_{B_r} v^{-n} |\kappa - 1| \, dx \leq C \operatorname{def}(u, \kappa). \tag{3.133}$$

Since (3.6) is in force and we may assume $r > 1$, combining (3.127) with (3.123), (3.128), and (3.130)–(3.133), we deduce that

$$\int_{B_r} v^{-n} \left| \nabla a(\nabla v) - \frac{\overline{P}}{n} \operatorname{Id}_n \right|^q dx \leq C_\flat \, r^{\frac{np}{p-1}} \left[1 + \mathcal{C}_P(r, \mathsf{t})^q\right] \operatorname{def}(u, \kappa)^{\frac{q}{2}},$$

for some $C_\flat > 0$, depending only on n, p, $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$, and q. This is precisely (3.121) with

$$\mathfrak{F}_q := r^{\frac{np}{p-1}} \left[1 + \mathcal{C}_P(r, \mathsf{t})^q\right] \operatorname{def}(u, \kappa)^{\frac{q}{2}} \quad \text{and} \quad \mu = \frac{\overline{P}}{n},$$

where $\mathcal{C}_P(r, \mathsf{t})$ is given by (3.129).

Step 8. Construction of the approximating functions. Our first goal here is to construct a function Q that approximates v in $L^p_\omega(B_r)$, for $\omega := v^{-n-p+1}$, and whose gradient approximates ∇v in $L^p_w(B_r)$, with $w = v^{-n}$. This will mainly follow from (3.121). Indeed, it captures proximity information between W and a suitable multiple of Id_n, which is arguably the strongest information one might hope to obtain.

We start by defining the function

$$\mathsf{Q}(x) := v(x_0) + \frac{p-1}{p} \left(\frac{\overline{P}}{n}\right)^{\frac{1}{p-1}} |x - x_0|^{\frac{p}{p-1}},$$

which is of class $C^\infty(\mathbb{R}^n \setminus \{x_0\})$ with $\mathcal{Z}_{\mathsf{Q}} = \{x_0\}$. Moreover, up to uniquely extending the stress field on $\mathcal{Z}_{\mathsf{Q}}$, we notice that

$$a(\nabla \mathsf{Q})(x) = \frac{\overline{P}}{n}(x - x_0) \quad \text{for } x \in \mathbb{R}^n, \tag{3.134}$$

$$\nabla a(\nabla \mathsf{Q})(x) = \frac{\overline{P}}{n} \operatorname{Id}_n \quad \text{for } x \in \mathbb{R}^n.$$

Therefore, we have $a(\nabla \mathsf{Q}) \in C^\infty(\mathbb{R}^n)$ and, from (3.121), we deduce that

$$\int_{B_r} v^{-n} \left|\nabla a(\nabla v) - \nabla a(\nabla \mathsf{Q})\right|^q dx \leq C_\flat \, \mathcal{F}_q \quad \text{for every } q \in [1,2]. \tag{3.135}$$

For convenience of notation, we set

$$\zeta := a(\nabla v) - a(\nabla \mathsf{Q}),$$

which is well defined in $\mathbb{R}^n$ since $a(\nabla v)$ has been extended to zero on $\mathcal{Z}_v$, and notice that we can rewrite (3.135) as

$$\int_{B_r} v^{-n} \left|\nabla \zeta\right|^q dx \leq C_\flat \, \mathcal{F}_q \quad \text{for every } q \in [1,2], \tag{3.136}$$

with $\zeta \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$ by (3.36).

We now introduce the small parameter $\tau \in (0,1)$, to be chosen later, and define, for each component of ζ,

$$\overline{\zeta_i} := \fint_{B_\tau(x_0)} \zeta_i \, dx = \fint_{B_\tau(x_0)} a_i(\nabla v) \, dx,$$

where we used that, by (3.134), each component of $a(\nabla \mathsf{Q})$ is harmonic in $\mathbb{R}^n$. Therefore, we get

$$|\overline{\zeta_i}| \leq \fint_{B_\tau(x_0)} |a_i(\nabla v)| \, dx. \tag{3.137}$$

Furthermore, we also have

$$\begin{aligned} \|a(\nabla v)\|_{L^1(B_\tau(x_0))} &\leq \|\nabla v\|^{p-1}_{L^{p-1}(B_\tau(x_0))} = \|\nabla v - \nabla v(x_0)\|^{p-1}_{L^{p-1}(B_\tau(x_0))} \\ &\leq C\tau^{n+\alpha(p-1)}, \end{aligned} \tag{3.138}$$

for some $C > 0$ depending only on n, p, and $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$.

Note that this is the only one of two points in the proof where we exploit $C^{1,\alpha}$ estimates in a quantitative form. We remark that this is possible since, by taking advantage of (3.36)–(3.37), of the fact that $x_0 \in B_{\mathcal{R}}$, and of Theorem 1.1 in [23], there exist $C > 0$ and $\alpha \in (0,1)$, depending at most on n, p, and $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$, such that

$$|\nabla v(x) - \nabla v(x_0)| \leq C\,|x - x_0|^\alpha \quad \text{for every } x \in B_2(x_0).$$

Additionally, by Theorem 3 in [44], we can assume that $\alpha \in (0, \alpha_M)$, where $\alpha_M \in (0,1)$ is the maximal exponent of Hölder regularity for the gradient of p-harmonic functions, which depends only on n and p. As a consequence, α depends only on n and p.

As a result, (3.137) simplifies to

$$|\overline{\zeta_i}| \leq C\tau^{\alpha(p-1)}, \tag{3.139}$$

for some $C > 0$ depending only on n, p, and $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$.

Observe that (3.126) yields that $B_\tau(x_0) \subseteq B_r$, hence, by applying once more the Poincaré inequality of Equation (7.45) in [30], we get

$$\|\zeta_i - \overline{\zeta_i}\|^q_{L^q_w(B_r)} \leq C\, r^{\frac{np}{p-1}}\, \mathcal{C}_P\,(r,\tau)^q \int\limits_{B_r} v^{-n}\, |\nabla\zeta|^q\, dx \leq C\, r^{\frac{np}{p-1}}\, \mathcal{C}_P\,(r,\tau)^q\, \mathcal{F}_q, \tag{3.140}$$

where $\mathcal{C}_P\,(r,\tau)$ is given by (3.129) and we used (3.136).

Since for each component of ζ we have

$$\|\zeta_i\|^q_{L^q_w(B_r)} \leq 2^{q-1}\|\zeta_i - \overline{\zeta_i}\|^q_{L^q_w(B_r)} + 2^{q-1}|\overline{\zeta_i}|^q\, \|v^{-n}\|_{L^1(\mathbb{R}^n)},$$

from (3.139) and (3.140), we conclude that

$$\|\zeta\|^q_{L^q_w(B_r)} \leq C_6 \left[r^{\frac{np}{p-1}}\, \mathcal{C}_P\,(r,\tau)^q\, \mathcal{F}_q + \tau^{\alpha q(p-1)}\right] =: C_6\, \mathcal{G}_q, \tag{3.141}$$

for some $C_6 > 0$ depending only on n, p, $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$, and q.

Note that (3.141) encloses a smallness information at the level of the stress field. We aim to transfer it at the level of the gradient first and of the function $v - \mathsf{Q}$ later.

It is well known – see, for instance, [33, (I), Chapter 12] – that for $p \geq 2$ it holds

$$|b-a|^p \leq 2^{p-2}\left\langle |b|^{p-2}\, b - |a|^{p-2}\, a, b-a\right\rangle \quad \text{for every } a, b \in \mathbb{R}^n,$$

from which

$$|b-a|^p \leq 2^{\frac{p(p-2)}{p-1}}\left||b|^{p-2}\, b - |a|^{p-2}\, a\right|^{\frac{p}{p-1}} \quad \text{for every } a, b \in \mathbb{R}^n.$$

Thus, for $p \geq 2$, (3.141) entails

$$\int\limits_{B_r} v^{-n}\, |\nabla\,(v - \mathsf{Q})|^p\, dx \leq 2^{\frac{p(p-2)}{p-1}}\, C_6\, \mathcal{G}_{\frac{p}{p-1}}. \tag{3.142}$$

For $1 < p \leq 2$, it is well established – see, for instance, [33, (VII), Chapter 12] – that

$$(p-1)\left(1 + |a|^2 + |b|^2\right)^{\frac{p-2}{2}}\, |b-a|^2 \leq \left\langle |b|^{p-2}\, b - |a|^{p-2}\, a, b-a\right\rangle$$
$$\text{for all } a, b \in \mathbb{R}^n \setminus \{0\}\,,$$

from which

$$|b-a|^p \le \frac{1}{(p-1)^p}\left(1+|a|^2+|b|^2\right)^{\frac{p(2-p)}{2}}\left||b|^{p-2}\,b-|a|^{p-2}\,a\right|^p \quad \text{for all } a,b\in\mathbb{R}^n\setminus\{0\}\,.$$

We also notice that, taking advantage of (3.38) and of the definition of Q, it holds

$$|\nabla v| \le 2C_1\, r^{\frac{1}{p-1}} \quad \text{and} \quad |\nabla \mathsf{Q}| \le C\, r^{\frac{1}{p-1}} \quad \text{in } B_r, \tag{3.143}$$

for some $C>0$ depending only on n, p, and $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$. As a consequence, by exploiting (3.141) and (3.143), we get

$$\begin{aligned}\int_{B_r} v^{-n}\,|\nabla(v-\mathsf{Q})|^p\,dx &\le \frac{1}{(p-1)^p}\int_{B_r} v^{-n}\left(1+|\nabla v|^2+|\nabla\mathsf{Q}|^2\right)^{\frac{p(2-p)}{2}}|\zeta|^p\,dx\\ &\le C\,r^{\frac{p(2-p)}{p-1}}\int_{B_r} v^{-n}\,|\zeta|^p\,dx \le C\,C_6\,r^{\frac{p(2-p)}{p-1}}\,\mathcal{G}_p,\end{aligned} \tag{3.144}$$

for some $C>0$ depending only on n, p, and $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$. Thus, from (3.142) and (3.144), we read

$$\int_{B_r} v^{-n}\,|\nabla(v-\mathsf{Q})|^p\,dx \le C_7\,r^{\frac{p(2-p)_+}{p-1}}\,\mathcal{G}_{\min\left\{p,\frac{p}{p-1}\right\}}, \tag{3.145}$$

for some $C_7>0$ depending only on n, p, and $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$.

By (3.37) and the fact that $x_0\in B_{\mathcal{R}}$, we easily deduce that

$$\hat{c}_1\,(1+|x-x_0|)^{\frac{p}{p-1}} \le v(x) \le \widehat{C}_1\,(1+|x-x_0|)^{\frac{p}{p-1}} \quad \text{for every } x\in\mathbb{R}^n, \tag{3.146}$$

$$\hat{c}_1\,(1+|x|)^{\frac{p}{p-1}} \le v(x) \le \widehat{C}_1\,(1+|x|)^{\frac{p}{p-1}} \quad \text{for every } x\in\mathbb{R}^n, \tag{3.147}$$

for a couple of constants $\hat{c}_1, \widehat{C}_1>0$ depending only on n, p, and $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$.

Let $\sigma>p-1$ to be determined soon. Since $v(x_0)=\mathsf{Q}(x_0)$, the fundamental theorem of calculus and Jensen inequality yield

$$\begin{aligned}\int_{B_r} v^{-\sigma}(x)\,|(v-\mathsf{Q})(x)|^p\,dx &= \int_{B_r} v^{-\sigma}(x)\,|(v-\mathsf{Q})(x)-(v-\mathsf{Q})(x_0)|^p\,dx\\ &= \int_{B_r} v^{-\sigma}(x)\left|\int_0^1 \frac{d}{dt}(v-\mathsf{Q})(tx+(1-t)x_0)\,dt\right|^p dx\\ &\le \int_{B_r} v^{-\sigma}(x)\,|x-x_0|^p\int_0^1 |\nabla(v-\mathsf{Q})(tx+(1-t)x_0)|^p\,dt\,dx.\end{aligned} \tag{3.148}$$

By (3.146), we also have

$$
\begin{aligned}
v^{-\sigma}(x)\,|x-x_0|^p &\le C\,(1+|x-x_0|)^{-\sigma\frac{p}{p-1}+p}\,(1+|x-x_0|)^{-p}\,|x-x_0|^p\\
&\le C\,(1+|x-x_0|)^{-\sigma\frac{p}{p-1}+p},
\end{aligned}
$$

for some $C>0$ depending only on n, p, and $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$, therefore

$$
\begin{aligned}
&\int_{B_r} v^{-\sigma}(x)\,|x-x_0|^p \int_0^1 |\nabla(v-\mathsf{Q})(tx+(1-t)x_0)|^p\,dt\,dx \qquad (3.149)\\
&\le C\int_{B_r}\int_0^1 (1+|x-x_0|)^{-\sigma\frac{p}{p-1}+p}\,|\nabla(v-\mathsf{Q})(tx+(1-t)x_0)|^p\,dt\,dx.
\end{aligned}
$$

On the other hand, by taking advantage of the fact that $x_0\in B_{\mathcal{R}}$, one can directly verify that

$$
\frac{1+|x-x_0|}{1+|tx+(1-t)x_0|}\ge\frac{1}{1+|x_0|}\ge\frac{1}{1+\mathcal{R}}\quad\text{for every } t\in[0,1].
$$

This, together with (3.147), (3.148), (3.149), and Tonelli's theorem, gives

$$
\begin{aligned}
&\int_{B_r} v^{-\sigma}(x)\,|(v-\mathsf{Q})(x)|^p\,dx \qquad (3.150)\\
&\le C\int_{B_r}\int_0^1 (1+|tx+(1-t)x_0|)^{-\sigma\frac{p}{p-1}+p}\,|\nabla(v-\mathsf{Q})(tx+(1-t)x_0)|^p\,dt\,dx\\
&\le C\int_0^1\int_{B_r} (1+|y|)^{-\frac{p}{p-1}(\sigma-p+1)}\,|\nabla(v-\mathsf{Q})(y)|^p\,dy\,dt\\
&\le C\int_{B_r} v^{-(\sigma-p+1)}(y)\,|\nabla(v-\mathsf{Q})(y)|^p\,dy,
\end{aligned}
$$

for some $C>0$ depending only on n, p, $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$, and σ. By choosing $\sigma=n+p-1$ in (3.150) and using (3.145), we deduce

$$
\int_{B_r} v^{-n-p+1}\,|(v-\mathsf{Q})|^p\,dx\le C_8\,r^{\frac{p(2-p)_+}{p-1}}\,\mathcal{G}_{\min\left\{p,\frac{p}{p-1}\right\}}, \qquad (3.151)
$$

for some $C_8>0$ depending only on n, p, and $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$.

Note that (3.145) and (3.151) imply that Q approximates v both at the zero and first order in an appropriate sense. However, when inverting Q through the transformation in (3.31), to go back to u, the result is not a p-bubble of the form (1.3), but only a function belonging to the manifold of Talenti bubbles. This suggests that a further approximation is required.

To this purpose, we define

$$\mathcal{Q}(x) := \frac{\lambda^{\frac{p}{p-1}} + |x-x_0|^{\frac{p}{p-1}}}{\lambda^{\frac{1}{p-1}}\, n^{\frac{1}{p}} \left(\frac{n-p}{p-1}\right)^{\frac{p-1}{p}}} \quad \text{with} \quad \lambda = \frac{1}{\overline{P}} \left(\frac{p}{p-1}\right)^{p-1} n^{\frac{1}{p}} \left(\frac{n-p}{p-1}\right)^{-\frac{(p-1)^2}{p}}.$$

A direct computation reveals that

$$\nabla \mathcal{Q} = \nabla \mathsf{Q} \quad \text{in } \mathbb{R}^n, \tag{3.152}$$

therefore (3.145) holds with $\mathcal{Q}$ in place of Q, that is

$$\int_{B_r} v^{-n} \left|\nabla\left(v-\mathcal{Q}\right)\right|^p dx \le C_7\, r^{\frac{p(2-p)_+}{p-1}}\, \mathcal{G}_{\min\left\{p, \frac{p}{p-1}\right\}}. \tag{3.153}$$

Moreover, we have

$$\mathsf{Q}(x) - \mathcal{Q}(x) = v(x_0) - \frac{1}{\overline{P}}\left(\frac{p}{n-p}\right)^{p-1} \qquad \text{for every } x \in \mathbb{R}^n, \tag{3.154}$$

and we shall now show that the right-hand side is quantitatively small. To this aim, we compute

$$\begin{aligned} &v(x_0)\overline{P} - \left(\frac{p}{n-p}\right)^{p-1} \\ &\qquad = \fint_{B_{\mathsf{t}}(x_0)} \frac{n(p-1)}{p}\frac{v(x_0)}{v}\left|\nabla v - \nabla v(x_0)\right|^p + \left(\frac{p}{n-p}\right)^{p-1} \frac{v(x_0)-v}{v}\, dx. \end{aligned}$$

Thus, by exploiting (3.36), (3.38), as well as the fact that v is bounded below, and arguing as in the deduction of (3.138), we obtain

$$\left| v(x_0)\overline{P} - \left(\frac{p}{n-p}\right)^{p-1} \right| \le C\, \mathsf{t}^{\min\{\alpha p, 1+\alpha\}}, \tag{3.155}$$

for some $C > 0$ depending only on n, p, and $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$. See, in particular, Theorem 3 in [44]. Moreover, since we have already proven that $x_0 \in B_{\mathcal{R}}$, with $\mathcal{R}$ defined in (3.125), the bounds (3.38) and (3.124) give that $v \le C$ in $B_{\mathsf{t}}(x_0)$, for some $C > 0$ depending only on n, p, and $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$. As a consequence,

$$\overline{P} \geq \left(\frac{p}{n-p}\right)^{p-1} \fint_{B_{\mathrm{t}}(x_0)} v^{-1}\,dx \geq C, \tag{3.156}$$

for some $C>0$ depending only on n, p, and $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$. This, together with (3.40), also implies that

$$\Lambda_1 \leq \lambda \leq \Lambda_2, \tag{3.157}$$

for some $\Lambda_1, \Lambda_2 > 0$ depending only on n, p, and $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$. In addition, dividing both sides of (3.155) by $\overline{P}$, and exploiting (3.156), we conclude that

$$\left| v(x_0) - \frac{1}{\overline{P}}\left(\frac{p}{n-p}\right)^{p-1} \right| \leq C\,\mathrm{t}^{\min\{\alpha p, 1+\alpha\}},$$

for some $C>0$ depending only on n, p, and $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$. From the latter and (3.154), we read off

$$\|\mathsf{Q} - \mathcal{Q}\|_{L^\infty(\mathbb{R}^n)} \leq C\,\mathrm{t}^{\min\{\alpha p, 1+\alpha\}},$$

which, in turn, implies

$$\|\mathsf{Q} - \mathcal{Q}\|^p_{L^p_\omega(\mathbb{R}^n)} \leq \|\mathsf{Q} - \mathcal{Q}\|^p_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n} v^{-n-p+1}\,dx \leq C_9\,\mathrm{t}^{p\min\{\alpha p, 1+\alpha\}}, \tag{3.158}$$

for some $C_9>0$ depending only on n, p, and $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$.

Taking advantage of (3.151) and (3.158), we deduce

$$\begin{aligned}\int_{B_r} v^{-n-p+1}\,|v-\mathcal{Q}|^p\,dx &\leq 2^{p-1}C_8\,r^{\frac{p(2-p)_+}{2(p-1)}}\,\mathcal{G}_{\min\left\{p,\frac{p}{p-1}\right\}} + 2^{p-1}C_9\,\mathrm{t}^{p\min\{\alpha p,1+\alpha\}} \\ &\leq C_{10}\left[r^{\frac{p(2-p)_+}{p-1}}\,\mathcal{G}_{\min\left\{p,\frac{p}{p-1}\right\}} + \mathrm{t}^{p\min\{\alpha p,1+\alpha\}}\right],\end{aligned} \tag{3.159}$$

for some $C_{10}>0$ depending only on n, p, and $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$.

Step 9. Going back to u. We are now ready to return to the level of u by defining the approximation for our original function and showing the smallness of the $\mathcal{D}^{1,p}$-norm. Finally, we will adjust the parameter introduced so far in terms of $\operatorname{def}(u,\kappa)$.

For this purpose, we define the function which approximates u by

$$\mathcal{U} := \mathcal{Q}^{-\frac{n-p}{p}},$$

inverting $\mathcal{Q}$ through (3.31). In such a way, $\mathcal{U}$ is a p-bubble of the form (1.3), in particular $\mathcal{U} = U_p[x_0,\lambda]$. Moreover, a direct computation reveals that

$$\nabla u = \frac{p-n}{p}\, v^{-\frac{n}{p}}\, \nabla v \quad \text{and} \quad \nabla \mathcal{U} = \frac{p-n}{p}\, \mathcal{Q}^{-\frac{n}{p}}\, \nabla \mathcal{Q} \quad \text{in } \mathbb{R}^n.$$

Therefore, we see that

$$\begin{aligned} \|\nabla (u-\mathcal{U})\|^p_{L^p(B_r)} &= \left(\frac{n-p}{p}\right)^p \int_{B_r} \left| v^{-\frac{n}{p}}\, \nabla v - \mathcal{Q}^{-\frac{n}{p}}\, \nabla \mathcal{Q}\right|^p dx \\ &\le 2^{p-1} \left(\frac{n-p}{p}\right)^p \left\{ \int_{B_r} v^{-n} \left|\nabla (v-\mathcal{Q})\right|^p dx + \int_{B_r} \left| v^{-\frac{n}{p}} - \mathcal{Q}^{-\frac{n}{p}}\right|^p \left|\nabla \mathcal{Q}\right|^p dx \right\}. \end{aligned} \tag{3.160}$$

We now aim to estimate both terms on the right-hand side of (3.160).

By taking advantage of (3.152) and (3.157), we easily see that

$$\mathcal{Q}^{-1}\left|\nabla \mathcal{Q}\right|^p \le C,$$

for some $C > 0$ depending only on n, p, and $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$, from which

$$\begin{aligned} \left| v^{-\frac{n}{p}} - \mathcal{Q}^{-\frac{n}{p}}\right|^p \left|\nabla \mathcal{Q}\right|^p &= v^{-n}\mathcal{Q}^{-n} \left| v^{\frac{n}{p}} - \mathcal{Q}^{\frac{n}{p}}\right|^p \left|\nabla \mathcal{Q}\right|^p \\ &\le C v^{-n}\mathcal{Q}^{-n+1} \left| v^{\frac{n}{p}} - \mathcal{Q}^{\frac{n}{p}}\right|^p. \end{aligned} \tag{3.161}$$

Then, the fundamental theorem of calculus and Jensen inequality entail

$$\begin{aligned} \left| v^{\frac{n}{p}} - \mathcal{Q}^{\frac{n}{p}}\right|^p &= \left| \int_0^1 \frac{d}{dt}\left(tv + (1-t)\mathcal{Q}\right)^{\frac{n}{p}} dt \right|^p \\ &\le \left(\frac{n}{p}\right)^p \int_0^1 \left(tv + (1-t)\mathcal{Q}\right)^{n-p} dt\, |v-\mathcal{Q}|^p. \end{aligned} \tag{3.162}$$

In light of (3.146) and (3.157), we have

$$\hat{c}_2\, v \le \mathcal{Q} \le \widehat{C}_2\, v \quad \text{in } \mathbb{R}^n,$$

for a couple of constants $\hat{c}_2, \widehat{C}_2 > 0$ depending only on n, p, and $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$. Therefore,

$$\begin{aligned} v^{-n}\mathcal{Q}^{-n+1} \int_0^1 \left(tv + (1-t)\mathcal{Q}\right)^{n-p} dt\, |v-\mathcal{Q}|^p &\le v^{-n}\mathcal{Q}^{1-p}\, |v-\mathcal{Q}|^p \\ &\le v^{-n-p+1}\, |v-\mathcal{Q}|^p \quad \text{on } \{v \le \mathcal{Q}\}, \end{aligned} \tag{3.163}$$

and

$$v^{-n}\mathcal{Q}^{-n+1}\int_0^1 (tv+(1-t)\mathcal{Q})^{n-p}\,dt\,|v-\mathcal{Q}|^p \le v^{-p}\mathcal{Q}^{-n+1}\,|v-\mathcal{Q}|^p \le \hat{c}_2^{1-n}\,v^{-n-p+1}\,|v-\mathcal{Q}|^p \quad \text{on } \{\mathcal{Q}\le v\}\,. \tag{3.164}$$

Piecing (3.161)–(3.164), it follows that

$$\int_{B_r} \left|v^{-\frac{n}{p}} - \mathcal{Q}^{-\frac{n}{p}}\right|^p |\nabla \mathcal{Q}|^p\,dx \le C\int_{B_r} v^{-n-p+1}\,|v-\mathcal{Q}|^p\,dx, \tag{3.165}$$

for some constant $C>0$ depending only on n, p, and $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$.

Combining (3.160) with (3.153), (3.159), and (3.165), we conclude that

$$\|\nabla(u-\mathcal{U})\|^p_{L^p(B_r)} \le C_{11}\left[r^{\frac{p(2-p)_+}{p-1}}\,\mathcal{G}_{\min\left\{p,\frac{p}{p-1}\right\}} + \mathsf{t}^{p\min\{\alpha p,1+\alpha\}}\right], \tag{3.166}$$

for some $C_{11}>0$ depending only on n, p, and $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$, with

$$\mathcal{G}_q = r^{\frac{2np}{p-1}}\,\mathcal{C}_P\,(r,\tau)^q\,[1+\mathcal{C}_P\,(r,\mathsf{t})^q]\,\mathrm{def}(u,\kappa)^{\frac{q}{2}} + \tau^{\alpha q(p-1)}$$

and where $\mathcal{C}_P\,(r,\cdot)$ is defined in (3.129).

Moreover, by (3.126), it holds that $B_1(x_0)\subseteq B_r$, therefore, exploiting also (3.157), we have

$$|\nabla\mathcal{U}(x)| \le C\,|x|^{\frac{1-n}{p-1}} \quad \text{for } x\in\mathbb{R}^n\setminus B_r,$$

for some $C>0$ depending only on n, p, and $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$. The latter and (3.11) imply

$$\|\nabla(u-\mathcal{U})\|^p_{L^p(\mathbb{R}^n\setminus B_r)} \le C\int_{\mathbb{R}^n\setminus B_r} |x|^{-\frac{p}{p-1}(n-1)}\,dx \le C\,r^{-\frac{n-p}{p-1}}, \tag{3.167}$$

for some $C>0$ depending only on n, p, and $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$.

We finish this step by adjusting the parameters. We choose

$$\begin{aligned} r &:= \mathrm{def}(u,\kappa)^{-\min\left\{\frac{q}{4\mathfrak{p}},\frac{\alpha q(p-1)^2}{16(n-1)p(2-p)_+}\right\}} = \mathrm{def}(u,\kappa)^{-\mathsf{m}},\\ \mathsf{t} = \tau &:= \mathrm{def}(u,\kappa)^{\frac{1}{8(n-1)}}, \end{aligned}$$

where

$$q := \min\left\{p,\frac{p}{p-1}\right\} \quad \text{and} \quad \mathfrak{p} := 2nq + \frac{p}{p-1}\left[2n+(2-p)_+\right].$$

Thus, by summing (3.166) with (3.167), we deduce

$$\|\nabla (u - \mathcal{U})\|_{L^p(\mathbb{R}^n)} \le \mathcal{C}\, \mathrm{def}(u,\kappa)^{\vartheta},$$

for some $\mathcal{C} > 0$, depending only on n, p, and $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$, and some $\vartheta \in (0,1)$, depending only on n and p. In addition, we observe that the above requirements on the parameters, in particular (3.126), are verified provided that

$$\gamma \le \min\left\{\frac{1}{2}, \delta, (\mathcal{R}+2)^{-\frac{1}{\mathfrak{m}}}\right\},$$

where γ and δ verifies (3.6) and (3.7), respectively.

Step 10. Conclusion. Here, we need to restore the subscript, as announced at the end of Step 1, and therefore write u_ϵ instead of u.

In Step 9, we have proven that

$$\|\nabla (u_\epsilon - \mathcal{U})\|_{L^p(\mathbb{R}^n)} \le \mathcal{C}\, \mathrm{def}(u,\kappa)^{\vartheta}. \tag{3.168}$$

Recall that we set $u_\epsilon = T_{-z_\epsilon/\lambda_\epsilon,\lambda_\epsilon}(u)$. Hence, point (3) of Lemma 2.1 implies that $u = T_{z_\epsilon,\lambda_\epsilon}(u_\epsilon)$. Defining $\mathcal{U}_0 \coloneqq T_{z_\epsilon,\lambda_\epsilon}(\mathcal{U})$, which remains a p-bubble by point (4) of Lemma 2.1, (3.168) reads as

$$\|\nabla (u - \mathcal{U}_0)\|_{L^p(\mathbb{R}^n)} \le \mathcal{C}\, \mathrm{def}(u,\kappa)^{\vartheta},$$

which is the desired conclusion.

4. On the optimal stability result

In the paper [34], the authors prove that for any non-negative $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ satisfying (1.11), there exist a large constant $C \ge 1$ and a p-bubble $\mathcal{U}_0$, of the form (1.3), such that the estimate

$$\|u - \mathcal{U}_0\|_{\mathcal{D}^{1,p}(\mathbb{R}^n)} \le C \left\|\Delta_p u + u^{p^*-1}\right\|_{\mathcal{D}^{-1,p'}(\mathbb{R}^n)}^{\min\left\{1,\frac{1}{p-1}\right\}} \tag{4.1}$$

holds. Hereafter, $\mathcal{D}^{-1,p'}(\mathbb{R}^n)$ denotes the dual space of $\mathcal{D}^{1,p}(\mathbb{R}^n)$.

Truthfully, the estimate in (4.1) is stronger than that in (1.14). Indeed, by Sobolev embedding and duality, we have

$$L^{(p^*)'}(\mathbb{R}^n) \hookrightarrow \mathcal{D}^{-1,p'}(\mathbb{R}^n),$$

which allows us to control the norm on the right-hand side of (4.1) via $\mathrm{def}(u,\kappa)$ for a solution to (1.10). Moreover, the proof of Theorem 1.1 fundamentally relies on the fact that u is a solution to (1.10).

In the remainder of this section, we provide an example which shows the optimality of (4.1) for $1 < p < 2$. To this aim, we adapt to our setting the approach used in [15] to show the optimality of the result therein – see [15, Remark 1.2]. Essentially, we construct a suitable perturbation of a p-bubble by adding a small cut-off function, which does not significantly perturb the energy and ensures that the new function remains close to being a solution of (1.1).

Let us fix $U \coloneqq U_p[0,1]$ and consider a non-negative, radially symmetric, and radially decreasing cut-off function $\phi \in C_c^\infty(B_1)$. For $\varepsilon \in (0,1)$, define

$$u_\varepsilon \coloneqq U + \varepsilon\phi,$$

which is a small, non-negative perturbation of U. Since ϕ is radial with respect to the origin, it easily follows that $\mathcal{Z}_{u_\varepsilon} = \{0\}$ and that $u_\varepsilon \in C^\infty(\mathbb{R}^n \setminus \{0\}) \cap C^1(\mathbb{R}^n)$.

We shall show that $u_\varepsilon \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ verifies (1.11) provided ε is sufficiently small, and we will provide precise estimates for both sides of (4.1).

We start by noticing that

$$\nabla U(x) = -|\nabla U(x)|\,\frac{x}{|x|} \quad \text{and} \quad \nabla\phi(x) = -|\nabla\phi(x)|\,\frac{x}{|x|} \quad \text{for every } x \in \mathbb{R}^n \setminus \{0\},$$

therefore

$$\begin{aligned} |\nabla u_\varepsilon|^2 &= |\nabla U|^2 + 2\varepsilon\,\langle\nabla U, \nabla\phi\rangle + \varepsilon^2\,|\nabla\phi|^2 = |\nabla U|^2 + 2\varepsilon\,|\nabla U||\nabla\phi| + \varepsilon^2\,|\nabla\phi|^2 \\ &= (|\nabla U| + \varepsilon\,|\nabla\phi|)^2 \quad \text{in } \mathbb{R}^n, \end{aligned} \tag{4.2}$$

since the latter holds trivially at the origin. Notice that (4.2) is geometrically obvious, as ∇U and $\nabla\phi$ are parallel at each point and both vanish at the origin, thanks to our definition of ϕ. By Hölder inequality

$$\int\limits_{\mathbb{R}^n} |\nabla U|^{p-1}|\nabla\phi|\,dx \le \|\nabla U\|_{L^p(\mathbb{R}^n)}^{p-1}\|\nabla\phi\|_{L^p(\mathbb{R}^n)},$$

hence a Taylor expansion, together with (1.4) and (4.2), yields

$$S^n = \int\limits_{\mathbb{R}^n} |\nabla U|^p\,dx \le \int\limits_{\mathbb{R}^n} |\nabla u_\varepsilon|^p\,dx = \int\limits_{\mathbb{R}^n} |\nabla U|^p\,dx + O(\varepsilon) = S^n + O(\varepsilon).$$

As a result, u_ε satisfies (1.11) provided ε is small enough.

By taking $\mathcal{U}_0 = U$, for the left-hand side of (4.1) we trivially have

$$\|u_\varepsilon - U\|_{\mathcal{D}^{1,p}(\mathbb{R}^n)} = \varepsilon\,\|\nabla\phi\|_{L^p(\mathbb{R}^n)}. \tag{4.3}$$

We then claim that

$$\left\|\Delta_p u_\varepsilon + u_\varepsilon^{p^*-1}\right\|_{\mathcal{D}^{-1,p'}(\mathbb{R}^n)} = O(\varepsilon). \tag{4.4}$$

By duality and density, we have

$$\begin{aligned}\left\|\Delta_p u_\varepsilon + u_\varepsilon^{p^*-1}\right\|_{\mathcal{D}^{-1,p'}(\mathbb{R}^n)} &= \sup_{\substack{\eta\in\mathcal{D}^{1,p}(\mathbb{R}^n)\\ \|\nabla\eta\|_{L^p(\mathbb{R}^n)}=1}} \left|\left\langle \Delta_p u_\varepsilon + u_\varepsilon^{p^*-1}, \eta\right\rangle\right| \\ &= \sup_{\substack{\eta\in C_c^\infty(\mathbb{R}^n)\\ \|\nabla\eta\|_{L^p(\mathbb{R}^n)}=1}} \left|\int_{\mathbb{R}^n} -\left\langle |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon, \nabla\eta\right\rangle + u_\varepsilon^{p^*-1}\eta\, dx\right|,\end{aligned} \tag{4.5}$$

and we notice that, by testing (1.1) with $\eta \in C_c^\infty(\mathbb{R}^n)$, we can write

$$\begin{aligned}&\int_{\mathbb{R}^n} -\left\langle |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon, \nabla\eta\right\rangle + u_\varepsilon^{p^*-1}\eta\, dx \\ &\qquad = \int_{\mathbb{R}^n} \left\langle |\nabla U|^{p-2}\nabla U - |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon, \nabla\eta\right\rangle + \left(u_\varepsilon^{p^*-1} - U^{p^*-1}\right)\eta\, dx.\end{aligned} \tag{4.6}$$

We shall now estimate both terms on the right-hand side of (4.6).

For the second term, Hölder and Sobolev inequalities entail that

$$\int_{\mathbb{R}^n} U^{p^*-2}\phi\,|\eta|\, dx \le \|U\|_{L^{p^*}(\mathbb{R}^n)}^{p^*-2}\|\phi\|_{L^{p^*}(\mathbb{R}^n)}\|\eta\|_{L^{p^*}(\mathbb{R}^n)} \le S^{-2}\|U\|_{L^{p^*}(\mathbb{R}^n)}^{p^*-2}\|\nabla\phi\|_{L^p(\mathbb{R}^n)}$$

for every $\eta \in C_c^\infty(\mathbb{R}^n)$ with $\|\nabla\eta\|_{L^p(\mathbb{R}^n)} = 1$. Consequently, a Taylor expansion gives that

$$\sup_{\substack{\eta\in C_c^\infty(\mathbb{R}^n)\\ \|\nabla\eta\|_{L^p(\mathbb{R}^n)}=1}} \left|\int_{\mathbb{R}^n} \left(u_\varepsilon^{p^*-1} - U^{p^*-1}\right)\eta\, dx\right| = O(\varepsilon). \tag{4.7}$$

Exploiting the well-known inequality – see, for instance, [18, Lemma 2.1] –

$$\left||b|^{p-2}b - |a|^{p-2}a\right| \le C_p\left(|b|+|a|\right)^{p-2}|a-b| \quad \text{for every } a, b\in\mathbb{R}^n \text{ with } |b|+|a|>0,$$

we can estimate the first term on the right-hand side of (4.6) deducing

$$\left|\int_{\mathbb{R}^n} \left\langle |\nabla U|^{p-2}\nabla U - |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon, \nabla\eta\right\rangle dx\right| \le C_p\int_{\mathbb{R}^n} \left(|\nabla u_\varepsilon| + |\nabla U|\right)^{p-2}\varepsilon\,|\nabla\phi||\nabla\eta|\, dx$$

whence, by means of Hölder inequality, we get

$$\int_{\mathbb{R}^n} (|\nabla u_\varepsilon| + |\nabla U|)^{p-2}\, \varepsilon\, |\nabla\phi||\nabla\eta|\, dx \le \varepsilon \|\nabla\phi\|_{L^p(\mathbb{R}^n)} \||\nabla u_\varepsilon| + |\nabla U|\|^{p-2}_{L^p(\mathbb{R}^n)}$$

for every $\eta \in C_c^\infty(\mathbb{R}^n)$ with $\|\nabla\eta\|_{L^p(\mathbb{R}^n)} = 1$. Taking into account that both u_ε and U satisfy (1.11) for ε small enough, we conclude that

$$\sup_{\substack{\eta\in C_c^\infty(\mathbb{R}^n)\\ \|\nabla\eta\|_{L^p(\mathbb{R}^n)}=1}} \left| \int_{\mathbb{R}^n} \left\langle |\nabla U|^{p-2}\, \nabla U - |\nabla u_\varepsilon|^{p-2}\, \nabla u_\varepsilon, \nabla\eta \right\rangle dx \right| = O(\varepsilon). \tag{4.8}$$

As a result, (4.5)–(4.8) imply the validity of (4.4).

Finally, assuming that (4.1) holds for some exponent ϑ, the estimates (4.3) and (4.4) lead to a contradiction unless $0 < \vartheta \le 1$.

5. An application to quasi-symmetry

In the recent paper [28], extending the work in [13], the second author proved quasi-symmetry for non-negative, non-trivial weak solutions to (1.10) using a quantitative version of the moving plane method. Unfortunately, the dependence on the deficit, which is precisely (1.13), is of logarithmic-type due to technical issues in the propagation of the weak Harnack inequality – see also [13].

Here, we show that by exploiting Corollary 1.1.1 and the analytic tools developed in [20] and further analyzed in [28], we can derive an approximate symmetry result for (1.10) under perhaps more natural assumptions than those of Theorem 1.3 in [28] and with a power-like dependence on the deficit (1.13). Specifically, we will prove the following result.

Theorem 5.1. *Let $n \in \mathbb{N}$ be an integer, $2 < p < n$, and let $\kappa \in L^\infty(\mathbb{R}^n) \cap C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ be a positive function. Let $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ be a positive weak solution to (1.10) satisfying (1.15), where $\kappa_0(u)$ is defined in (1.12), and the bound*

$$\|u\|_{L^\infty(\mathbb{R}^n)} \le C_0, \tag{5.1}$$

for some $C_0 \ge 1$.

Then, there exist a point $\mathcal{O} \in \mathbb{R}^n$, a large constant $C > 0$, and a constant $\vartheta' \in (0, 1)$ such that

$$|u(x) - u(y)| \le C \operatorname{def}(u, \kappa)^{\vartheta'} \tag{5.2}$$

for every $x, y \in \mathbb{R}^n$ satisfying $|x - \mathcal{O}| = |y - \mathcal{O}|$. Moreover, if u_Θ denotes any rotation of u centered at $\mathcal{O}$, we also have

$$\|u - u_\Theta\|_{\mathcal{D}^{1,p}(\mathbb{R}^n)} \le C \operatorname{def}(u, \kappa)^{\vartheta'}. \tag{5.3}$$

The constant C depends only on n, p, C_0, $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$, and $\kappa_0(u)$, while ϑ' depends only on n and p.

As mentioned, Theorem 5.1 should be compared with Theorem 1.3 in [28]. Clearly, condition (5.1) is a necessary assumption to fix the scale of u and is present in both results – explicitly here and implicitly in [28] through the global decay assumption. The decay assumption in [28] is replaced here by the energy bound (1.11). Both serve to prevent the occurrence of bubbling phenomena and are therefore needed in some form. Finally, apart from the improvement in the estimates, another novelty in Theorem 5.1 is that the lower bound for the gradient of u is no longer required. This condition was derived in [39] as part of the proof of the symmetry result and imposed a priori in [28] for quantitative reasons. Here, however, this assumption is unnecessary since (5.2) is obtained by comparing u with a multiple of a p-bubble, whose gradient is explicitly known.

Proof of Theorem 5.1. Let us set $\kappa_0 = \kappa_0(u)$. By applying Corollary 1.1.1, there exists a function $\mathcal{U} \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ of the form

$$\mathcal{U} = \kappa_0(u)^{\frac{1}{p-p^*}}\,\mathcal{U}_0,$$

for some p-bubble $\mathcal{U}_0 := U_p[\mathcal{O}, \lambda]$, such that

$$\|u - \mathcal{U}\|_{\mathcal{D}^{1,p}(\mathbb{R}^n)} \le C\,\mathrm{def}(u, \kappa)^{\vartheta}. \tag{5.4}$$

The constant C depends only on n, p, $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$, and κ_0, while ϑ depends only on n and p. Moreover, as follows from the proof of Theorem 1.1, the scaling factor $\lambda > 0$ depends only on n, p, and $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$.

From now on, the constant C will denote a quantity that may vary from line to line, but depends only on n, p, $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$, and κ_0.

Let Θ be any rotation centered at $\mathcal{O}$ and define $u_\Theta(x) := u(\Theta x)$ and $\mathcal{U}_\Theta(x) := \mathcal{U}(\Theta x)$ for every $x \in \mathbb{R}^n$. Since $\mathcal{U}$ is radially symmetric with respect to $\mathcal{O}$ we clearly have that $\mathcal{U}_\Theta = \mathcal{U}$. Therefore, a change of variables gives

$$\|u_\Theta - \mathcal{U}\|_{\mathcal{D}^{1,p}(\mathbb{R}^n)} = \|u_\Theta - \mathcal{U}_\Theta\|_{\mathcal{D}^{1,p}(\mathbb{R}^n)} \le C\,\mathrm{def}(u, \kappa)^{\vartheta}.$$

Combining the latter with (5.4) via the triangle inequality, we infer (5.3).

We plan to prove (5.2) by applying the local boundedness comparison inequality of Theorem 2.3 in [28] – see also [20, Theorem 3.2] for the original result. More precisely, we will show that

$$\sup_{B_1(x)} (u - \mathcal{U})_+ \le C\,\mathrm{def}(u, \kappa)^{\vartheta'} \quad \text{for every } x \in \mathbb{R}^n, \tag{5.5}$$

for some $\vartheta' \in (0, 1)$ depending only on n and p.

To this end, we observe that

$$\Delta_p u + \kappa u^{p^*-1} = \Delta_p \mathcal{U} + \kappa_0 \mathcal{U}^{p^*-1} \quad \text{in } \mathbb{R}^n. \tag{5.6}$$

Defining

$$c(x) := \begin{cases} \dfrac{\mathcal{U}^{p^*-1}(x) - u^{p^*-1}(x)}{\mathcal{U}(x) - u(x)} & \text{if } \mathcal{U}(x) \neq u(x), \\ 0 & \text{if } \mathcal{U}(x) = u(x), \end{cases}$$

and setting $g := (\kappa - \kappa_0)\, u^{p^*-1} \in L^{(p^*)'}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, we can rewrite (5.6) as

$$\Delta_p u + \kappa_0\, c\, u + g = \Delta_p \mathcal{U} + \kappa_0\, c\, \mathcal{U} \quad \text{in } \mathbb{R}^n. \tag{5.7}$$

Now, we note that $\|\mathcal{U}\|_{L^\infty(\mathbb{R}^n)} \le C$ and also $\|\nabla \mathcal{U}\|_{L^\infty(\mathbb{R}^n)} \le C$. Moreover, by Theorem 1 in [23] and assumption (5.1), we deduce that $\|\nabla u\|_{L^\infty(\mathbb{R}^n)} \le C$. Consequently, we have

$$c \in L^\infty(\mathbb{R}^n) \quad \text{with } 0 \le c \le (p^* - 1) \max\{u, \mathcal{U}\}^{p^*-2} \le C. \tag{5.8}$$

By (5.1), we can establish (5.2) assuming that $\operatorname{def}(u, \kappa) \le \gamma$ for some $\gamma \in (0, 1)$ depending only on n, p, $\|\kappa\|_{L^\infty(\mathbb{R}^n)}$, and κ_0. Moreover, up to a translation, we may also assume that $\mathcal{O}$ is the origin. Thus, by repeating the argument in Step 2 of the proof of Theorem 1.1, we deduce that

$$u(x) \le \frac{C}{1 + |x|^{\frac{n-p}{p-1}}} \quad \text{for every } x \in \mathbb{R}^n. \tag{5.9}$$

Now, we observe that

$$|\nabla \mathcal{U}(x)| = C \left(\lambda^{\frac{p}{p-1}} + |x|^{\frac{p}{p-1}} \right)^{-\frac{n}{p}} |x|^{\frac{1}{p-1}} \quad \text{for every } x \in \mathbb{R}^n.$$

Fixing $R_0 = \lambda$ and $R > 0$, and setting $B_R := B_R(0)$, this enables us to conclude that

$$\begin{aligned} &\int\limits_{B_{R_0}} \frac{1}{|\nabla \mathcal{U}|^{(p-1)r}}\, dx \le C, \\ &\int\limits_{B_R \setminus B_{R_0}} \frac{1}{|\nabla \mathcal{U}|^{(p-1)r}}\, dx \le C R^{n+(n-1)r} \quad \text{for every } r \in \left(\frac{p-2}{p-1}, 1 \right), \end{aligned} \tag{5.10}$$

where C now depends on r as well.

Since (5.7)–(5.10) suffice to repeat the argument in Step 3 of the proof of Theorem 1.3 in [28], we infer the validity of (5.5). Furthermore, applying the same reasoning to the function $\mathcal{U} - u$, we obtain

$$\sup_{B_1(x)} (\mathcal{U} - u)_+ \leq C \operatorname{def}(u, \kappa)^{\vartheta'} \quad \text{for every } x \in \mathbb{R}^n,$$

possibly for a smaller ϑ'. The latter, together with (5.5), entails

$$\|u - \mathcal{U}\|_{L^\infty(\mathbb{R}^n)} \leq C \operatorname{def}(u, \kappa)^{\vartheta'}. \tag{5.11}$$

Since this argument also applies to the function $u_\Theta - \mathcal{U}_\Theta$, we similarly get

$$\|u_\Theta - \mathcal{U}\|_{L^\infty(\mathbb{R}^n)} = \|u_\Theta - \mathcal{U}_\Theta\|_{L^\infty(\mathbb{R}^n)} \leq C \operatorname{def}(u, \kappa)^{\vartheta'},$$

possibly for a smaller ϑ'. Combining this with (5.11) via the triangle inequality, we establish (5.2). This concludes the proof. □

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Data availability

No data was used for the research described in the article.

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