UNIVERSITÀ DEGLI STUDI DI MILANO

FACOLTÀ DI SCIENZE E TECNOLOGIE

Dipartimento di Matematica F. Enriques

Dottorato di Ricerca in Scienze Matematiche
XXXVIII ciclo

Tesi di Dottorato

Perfect fluids and sigma models:

a mathematical investigation between General Relativity and Cotton Gravity

Settore MATH-02/B

Tutor:
Prof. Marco Rigoli

Coordinatore del Dottorato:
Prof. Giulio Ciraolo

Candidato:
Filippo Mastropietro
Matricola R13989
Orcid 0009-0007-3832-5467

Anno Accademico 2024-2025

Dedicata a Timoti, Anna e Nicola.

Abstract

In this thesis, we deal with two physics-inspired structures on Riemannian manifolds, depending on a smooth map φ between Riemannian manifolds and several other functions and constants; they are called Cotton φ-Perfect Fluids (C-φ-PF) and φ-static perfect fluids space-times (φ-SPFST). The latter are special solutions of the Einstein field equations on a static space-time with matter sources modelled by a perfect fluid and a map φ, while the former are the natural generalization of φ-SPFSTs in the framework of a gravitational theory of recent introduction, called Cotton Gravity. We highlight the role of the Introduction in clarifying the physical background of this work and the expected contributions that such structures might give to Riemannian Geometry. In Chapter 1 we fix the notations and conventions and review some of the basics of Riemannian Geometry and General Relativity.

We begin Chapter 2 by introducing the theory of Cotton Gravity and showing how it preserves the solutions of General Relativity. We then derive the equations of C-φ-PF and show their variational origin; the class of variations that we need to consider is such that the connection, instead of the metric, is the fundamental entity. To obtain the main result of this chapter, a rigidity result for a C-φ-PF under some curvature conditions which are inspired by the theory of Ricci solitons, one needs to face a novel and specific property of Cotton Gravity, that of depending on third order derivatives of the metric tensor. In doing so, we will unveil the special relevance that Codazzi tensors hold in this theory, as well as the important relations between the equations of C-φ-PF and the first integrability condition of a φ-SPFST. Inspired by some works in conformal geometry, we then conclude this chapter by studying the obstruction to a C-φ-PF to be a φ-SPFST and relating it to some special algebraic properties of the Weyl tensor.

In Chapter 3 we deal with φ-SPFSTs. We derive other rigidity results under new curvature conditions, for both manifolds with and without boundary. The proofs in the two cases are surprisingly different, but they both rely on some special integral identities that make our assumptions effective. We then prove an Obata-type theorem that characterizes the standard spheres as the only compact manifolds supporting a closed, conformal vector field and a structure of generalized Harmonic-Einstein manifold; we apply it to deduce a rigidity result for a closed φ-SPFST for which a suitable modification of the Schouten tensor is Codazzi and has some elementary symmetric function in its eigenvalues which is positive and constant.

The research in this thesis resulted in the following works:

1. L. Branca, G. Colombo, P. Mastrolia, F. Mastropietro and M. Rigoli, *On the geometry of φ-Static Perfect Fluid Space-Times*, preprint, submitted for publication.

2. G. Colombo, F. Mastropietro and M. Rigoli, *On the geometry of Cotton Gravity*, preprint, submitted for publication.

Ringraziamenti

Ahimè, non sono un tipo di molte parole e l'imbarazzo di questa (più che dovuta) incombenza che sono i ringraziamenti sta già avendo la meglio su di me. Del resto, quando penso a tutte le persone che hanno attraversato la mia vita in questi tre anni e a quello che mi lega a loro, "grazie" inizia a sembrare una parola piccola piccola. Proviamoci comunque.
Grazie a Marco per avermi insegnato il valore del duro lavoro, della dedizione e della genuina passione per esso. Se la mia congenita cialtronaggine non ha intaccato il contenuto di questa tesi è soprattutto merito tuo.
Grazie ai miei mentori e collaboratori (Letizia, Giulio, Paolo), ai colleghi (Davide in testa) e ai compagni di studio (soprattutto chi, col passare degli anni, diventa sempre più difficile vedere). Chi ispirandomi con l'esempio, chi sopportandomi nei momenti più snervanti di questo percorso, avete tutti segnato la mia permanenza in via Saldini.
Alla mia famiglia e agli amici che rimangono va il più timido dei miei grazie: sbracciarsi in complimenti verso chi ha già la sfortuna di condividere gioie e dolori con me mi sembra una grandissima indecenza. Non me ne vogliate; senza dubbio volermi bene richiede una certa dose di pazienza.

Contents

Introduction

In this thesis, we will deal with some special structures on a Riemannian m-manifold (M,g) depending on several additional data: a smooth map $\varphi:(M,g)\to(N,h)$, where (N,h) is a second Riemannian manifold, a coupling constant $\alpha\in\mathbb{R},\alpha\neq 0$, a "potential" $U\in C^\infty(N)$ and three smooth functions $f,\mu,p\in C^\infty(M)$. The two most important systems that we are going to discuss are the φ-Static Perfect Fluid System:

$$\begin{cases}\mathrm{Ric}^\varphi+\operatorname{Hess}f-\mathrm{d}f\otimes\mathrm{d}f-\frac{1}{m-1}\big(\frac{S^\varphi}{2}-p+U(\varphi)\big)g=0,\\ \Delta f-|\nabla f|^2=-\frac{1}{m-1}\big[mp-mU(\varphi)+\frac{m-2}{2}S^\varphi\big],\\ 0=\tau(\varphi)-\mathrm{d}\varphi(\nabla f)-\frac{1}{\alpha}\nabla U(\varphi),\\ \mu+U(\varphi)=\frac{1}{2}S,\\ \nabla p-(\mu+p)\nabla f=0\end{cases}\tag{0.1}$$

and the Cotton-φ-Perfect Fluid system, which in local components read

$$\begin{cases}0= & C^\varphi_{ijk}+f_{ijk}-f_{ikj}-f_{ik}f_j+f_{ij}f_k\\ & -\frac{1}{m}[(f_{ttk}-2f_tf_{tk})\delta_{ij}-(f_{ttj}-2f_tf_{tj})\delta_{ik}]\\ & +\frac{1}{2m(m-1)}\big(S^\varphi_k\delta_{ij}-S^\varphi_j\delta_{ik}\big)\\ & +\frac{1}{m}U^a\varphi^a_j\delta_{ik}-\frac{1}{m}U^a\varphi^a_k\delta_{ij}+\frac{1}{m}(\mu_j\delta_{ik}-\mu_k\delta_{ij}),\\ 0= & -\frac{m-1}{m}f_{tti}+\frac{m-2}{m}f_tf_{ti}+(\Delta f)f_i+\frac{1}{2m}S^\varphi_i-f_tR^\varphi_{it}\\ & +\frac{1}{m}U^a\varphi^a_i-\frac{m-1}{m}\mu_i,\\ 0= & \nabla p-(\mu+p)\nabla f,\\ 0= & \tau(\varphi)-\mathrm{d}\varphi(\nabla f)-\frac{1}{\alpha}\nabla U(\varphi).\end{cases}\tag{0.2}$$

Here, $\mathrm{Ric}^\varphi, S^\varphi$ and C^φ stand for the φ-Ricci tensor, the φ-Scalar curvature and the φ-Cotton tensor. They are examples of what are called *φ-curvatures*. They are given by the expressions

$$\mathrm{Ric}^\varphi:=\mathrm{Ric}-\alpha\varphi^*h,\quad S^\varphi:=S-\alpha|\mathrm{d}\varphi|^2$$

and

$$C^\varphi_{ijk}:=A^\varphi_{ij,k}-A^\varphi_{ik,j},$$

where

$$A^\varphi := \mathrm{Ric}^\varphi - \frac{S^\varphi}{2(m-1)}g$$

is the φ-Schouten tensor. The expressions $|\mathrm{d}\varphi|^2$ and $\tau(\varphi)$ denote (twice of) the energy density and the tension field of the map φ, respectively, see Section 1.2 for more details. The genesis of (0.1) and (0.2) is rooted in concepts arising, partially independently, in mathematical relativity and differential geometry. These two tightly related fields have a long history of mutual contributions, where tools developed to treat specific problems in one area reveal to be crucial also in the other. To better highlight the importance of systems (0.1) and (0.2), we feel necessary to give a brief overview of (a part of) the story. We hope that even the experienced reader, to whom most of the following discussion will not be new, will find some interesting insights. At the starting point there are of course Einstein's field equations

$$\widehat{\mathrm{Ric}} - \frac{1}{2}\widehat{S}\widehat{g} = \widehat{T} \tag{0.3}$$

on a Lorentzian manifold $(\widehat{M}, \widehat{g})$. The meaning of (0.3) is well-known: the presence of mass, encoded in the tensor $\widehat{T}$, curves the space-time $\widehat{M}$ by prescribing its Einstein tensor $\widehat{E} := \widehat{\mathrm{Ric}} - \frac{1}{2}\widehat{S}\widehat{g}$. The gravitational field is just the effect of this curvature. If we imagine, as a toy model, that the cosmological constant be the only contribution to the field equations, then $\widehat{T}$ will have the form $\widehat{T} = -\Lambda\widehat{g}$, for some $\Lambda \in \mathbb{R}$. If we trace (0.3) we get

$$\widehat{S} = \frac{2n}{n-2}\Lambda,$$

n being the dimension of $\widehat{M}$, and plugging this information back in (0.3) we deduce

$$\widehat{\mathrm{Ric}} = \widehat{\Lambda}\widehat{g}, \tag{0.4}$$

where $\widehat{\Lambda} := \frac{2}{n-2}\Lambda$. In other words, the Ricci tensor is a constant multiple of the metric. Manifolds satisfying (0.4) are called *Einstein metrics*. While having the fundamental feature of representing the vacua in Einstein's theory of general relativity, Einstein metrics are also intensively studied in Riemannian geometry for totally independent reasons, the most important being that they are the fundamental example of a *canonical* metric on a manifold, that is, metrics whose existence or non-existence influences the topology of the manifold. See the excellent [10] for a monograph on Einstein manifolds and the more recent [19] for a discussion on the concept of canonical metric. Another important system of equations can be derived from (0.4): assume that $(\widehat{M}, \widehat{g})$ be a static space-time, that is, it holds $\widehat{M} = M \times \mathbb{R}, \widehat{g} = -e^{-2f}\mathrm{d}t \otimes \mathrm{d}t + g$ where (M, g) is a Riemannian manifold, $f \in C^\infty(M)$ and $t : M \times \mathbb{R} \to \mathbb{R}$ is the projection. Then (0.4) reduces on the Riemannian factor (M, g) giving rise to the system

$$\begin{cases} \mathrm{Ric} + \mathrm{Hess}\, f - \mathrm{d}f \otimes \mathrm{d}f - \frac{S}{m-1}g = 0, \\ \Delta f - |\nabla f|^2 = \frac{S}{m-1}. \end{cases} \tag{0.5}$$

A manifold (M, g) with a solution of (0.5) is called a *Vacuum Static Space*. It is customary to perform the substitution $u = e^{-f}$ so that (0.5) gives rise to the equation

$$\mathrm{Hess}\, u - u\mathrm{Ric} - \Delta u g = 0. \tag{0.6}$$

By construction, $u > 0$ on M, and one often adds a boundary component ∂M to M requiring that $u \equiv 0$ on M; ∂M can be used to model, for example, the horizon of one or more black holes, according to whether the boundary is connected or not. In the literature, triplets (M, g, u) satisfying (0.6) with u satisfying the aforementioned condition on its sign are usually referred to as vacuum static spaces; due to the possible presence of the boundary ∂M, such manifolds are in general different to those supporting a solution f of (0.5), but, in view of the substitution $u = e^{-f}$, these two notions clearly agree on $\mathrm{int}\, M$. In this thesis, apart from Section 3.2, we will only deal with manifolds without boundary, so that there will be no ambiguity in considering only solutions to (0.5). It is known from the work of Fischer and Marsden, [43], [44], that (0.6) is tightly related to the deformation of the scalar curvature: indeed, if we let $\mathcal{M}$ be the set of smooth Riemannian metrics on M and

$$\mathcal{S} : \mathcal{M} \to C^\infty(M)$$

be the scalar curvature operator, that is, the map that associates to every metric g its scalar curvature, then it turns out that the linearization $D\mathcal{S}^*_g(u)$ of the adjoint of $\mathcal{S}$ at the metric g evaluated at a function $u \in C^\infty(M)$ coincides with the left hand side of (0.6). Later on, Herzlich, [56], was able to prove that on every Einstein manifold (M, g) endowed with a conformal vector field X, that is, a smooth vector field satisfying

$$\mathcal{L}_X g = \frac{2}{m}\,\mathrm{div}\, X g,$$

we have that $u := \mathrm{div}\, X$ belongs to $\ker D\mathcal{S}^*_g$ so that it solves (0.6). Then, Miao and Tam, [75], by studying the critical points of a functional of the form

$$\mathbb{F}_X(g) = \int_{\partial M} G(X, \nu), \tag{0.7}$$

where G is a modification of the Einstein tensor and ν is the normal to the boundary, proved a partial converse to Herzlich's result, that is, they showed that every constant scalar curvature Riemannian manifold (M, g) admitting a conformal vector field X such that $\mathrm{div}\, X \in \ker D\mathcal{S}^*_g$ and satisfying some mild further assumptions must be an Einstein manifold. The introduction of the functional (0.7) was motivated by some earlier works on the notion of ADM mass.

Other interesting systems of equations can be derived from (0.3) by considering different stress energy tensors. The first type that we will deal with is called the *perfect fluid stress-energy tensor*. Its expression is

$$\widehat{T}^F := (\widehat{\mu} + \widehat{p})X^\flat \otimes X^\flat + \widehat{p}\widehat{g} \tag{0.8}$$

where $\widehat{\mu}, \widehat{p} \in C^\infty(\widehat{M})$ and X is a time-like, unitary vector field on $\widehat{M}$, i.e. $\widehat{g}(X,X) = -1$. Note that such an X might not exist in general, [88, Chapter 8], so that assuming its existence already imposes some (physically reasonable) constraints on the geometry of $(\widehat{M}, \widehat{g})$. The vector field X has to be interpreted as the space-time velocity of a fluid whose density and pressure are $\widehat{\mu}$ and $\widehat{p}$, respectively. The fluid being perfect means that we consider the contributions given by heat conduction and viscosity to be neglectable. See [88, pag. 62], and [54, pag. 69]. Despite a great number of exact solutions of (0.3) of perfect fluid type being known, [60], just a small fraction of them is considered physically acceptable, see [32]. Nevertheless, perfect fluid space-times have been vastly employed to build stellar models ([82], [63], [87]) and to describe the large-scale behaviour of the universe ([88, Chapter 5] and [78]). For a static space-time, the perfect fluid stress-energy tensors that we are going to consider have the form

$$\widehat{T}^F = (\widehat{\mu} + \widehat{p})e^{-2\widehat{f}}\mathrm{d}t \otimes \mathrm{d}t + \widehat{p}\widehat{g} \tag{0.9}$$

where $\widehat{\mu} = \mu \circ \widehat{\pi}_M, \widehat{p} = p \circ \widehat{\pi}_M$ for some smooth functions $\mu, p \in C^\infty(M)$. It is well-known, and we will prove it in Section 1.3, that equation

$$\widehat{\mathrm{Ric}} - \frac{1}{2}\widehat{S}\widehat{g} = \widehat{T}^F$$

on a static space time $M \times_f \mathbb{R}$ decomposes in the following system

$$\begin{cases} \mathrm{Ric} + \mathrm{Hess}\, f - \mathrm{d}f \otimes \mathrm{d}f - \frac{1}{m-1}\left(\frac{S}{2} - p\right)g = 0, \\ \Delta f - |\nabla f|^2 = -\frac{1}{m-1}\left[mp + \frac{m-2}{2}S\right], \\ \mu = \frac{1}{2}S, \\ \nabla p - (\mu + p)\nabla f = 0. \end{cases} \tag{0.10}$$

When the equation of state $p = -\mu$ is satisfied, it is easy to prove that μ is constant and (0.10) becomes the system of a vacuum static space. Performing again the substitution $u = e^{-f}$, we have that (0.10) takes the form

$$\begin{cases} \mathrm{Hess}\, u - u\left\{\mathrm{Ric} - \frac{1}{m-1}\left(\frac{S}{2} - p\right)g\right\} = 0, \\ \Delta u = \frac{u}{m-1}\left[mp + \frac{m-2}{2}S\right], \\ \mu = \frac{1}{2}S^\varphi, \\ (\mu + p)\nabla u = -u\nabla p, \end{cases} \tag{0.11}$$

which is a more common definition of static perfect fluid in the literature.
The second type of stress-energy tensor is described by a smooth map $\widehat{\varphi} : (\widehat{M}, \widehat{g}) \to (N, h)$, where (N, h) is a Riemannian manifold, a constant $\alpha \in \mathbb{R}, \alpha \neq 0$, and a potential $U \in C^\infty(N)$. Its precise form is

$$\widehat{T}^{\widehat{\varphi}} := \alpha\widehat{\varphi}^*h - \left(\frac{\alpha}{2}|\mathrm{d}\widehat{\varphi}|^2 + U(\widehat{\varphi})\right)\widehat{g}. \tag{0.12}$$

When $(N, h) = (\mathbb{R}, \mathrm{d}t^2)$, $\widehat{\varphi}$ is called a *scalar field*. When $\alpha > 0$ one speaks of a *quintessence*, while, if $\alpha < 0$, $\widehat{\varphi}$ is called a *phantom field*, see [31] and references therein. Tensors as (0.12) are often used to encode the (hypothetical) contribution of dark matter and dark energy on the field equations, [26], or to describe the gravitational effects of some elementary particles during the early life of the universe, [64]. The map $\widehat{\varphi}$ is sometimes called the *sigma model*. When $\widehat{T} = \widehat{T}^{\widehat{\varphi}}$, Einstein's field equations read

$$\widehat{\mathrm{Ric}}^{\widehat{\varphi}} - \frac{1}{2}\widehat{S}^{\widehat{\varphi}}\widehat{g} = U(\widehat{\varphi})\widehat{g} \tag{0.13}$$

with $\widehat{\mathrm{Ric}}^{\widehat{\varphi}}$ and $\widehat{S}^{\widehat{\varphi}}$ defined as before. It can easily be seen, [13, Section 2.4.1], that (0.13) are the Euler-Lagrange equations of the functional

$$\mathcal{F}(\widehat{g},\widehat{\varphi}) = \int_{\widehat{M}} \widehat{S} - \alpha|\mathrm{d}\widehat{\varphi}|^2_{\widehat{g}} - 2U(\widehat{\varphi})$$

with respect to compactly supported variations of the metric. Considering instead variations of the field $\widehat{\varphi}$, we get the further equation

$$\tau(\widehat{\varphi}) = \nabla U(\widehat{\varphi}),$$

where $\tau(\widehat{\varphi})$ is the tension field of $\widehat{\varphi}$ (see Section 1.2) and ∇U is the gradient of U with respect to the metric h. When U is constant, we get the system

$$\begin{cases} \widehat{\mathrm{Ric}}^{\widehat{\varphi}} = \Lambda\widehat{g}, \quad \Lambda\in\mathbb{R}, \\ \tau(\widehat{\varphi}) = 0 \end{cases} \tag{0.14}$$

which is called the *Harmonic-Einstein* system. Both static perfect fluid space-times and Harmonic-Einstein manifolds fit in a more general theoretical framework that mathematicians have been studying independently of their physical significance. Recall that a *Ricci soliton* is a Riemannian manifold (M,g) endowed with a smooth vector field X such that it holds

$$\mathrm{Ric} + \mathcal{L}_X g = \lambda g, \quad \lambda\in\mathbb{R}.$$

As it is well known, they often arise as limits of dilations of singularities of the Ricci flow. If $X = \nabla f$, for some smooth function $f \in C^\infty(M)$, then we say that (M,g,f) is a *gradient* Ricci soliton. After their introduction by Hamilton, [50], they raised an ever-growing interest in the mathematical community, leading to a variety of different works, a complete summary of which would be a too demanding task for us; the reader can consult the nice surveys [14] and [15]. They also attracted attention to several related structures, namely Ricci almost solitons [80], Yamabe solitons [23], [17], Yamabe quasi-solitons [57], [89], quasi Einstein manifolds [18], [55] and ρ-Einstein solitons [22]. All of those are particular instances of the more general notion of an Einstein-type structure, introduced in [21], given by the equation

$$\alpha\mathrm{Ric} + \frac{\beta}{2}\mathcal{L}_X g + \mu X^\flat\otimes X^\flat = (\rho S + \lambda)g,$$

where $\alpha, \beta, \mu, \rho \in \mathbb{R}, (\alpha, \beta, \mu) \neq (0,0,0), \lambda \in C^\infty(M)$ and $X \in \Gamma(TM)$. If $X = \nabla f$, $f \in C^\infty(M)$, then the latter becomes

$$\alpha \mathrm{Ric} + \beta \,\mathrm{Hess}\, f + \mu \mathrm{d}f \otimes \mathrm{d}f = (\rho S + \lambda) g, \tag{0.15}$$

that is, a *gradient* Einstein-type structures. By looking at equations (0.8) the reader will definitely be convinced that static perfect fluid space-times are particular instances of Einstein-type structures, at least when one neglects the presence of the boundary.
Vacuum static spaces and perfect fluids space-times have been vastly analyzed in the mathematical literature; what's more, their study contributed to the development of tools and ideas that revealed precious for more general classes of manifolds such as (0.15). It is the case, for example, of [59], where techniques introduced to deal with conformally flat vacuum static spaces have been rediscovered (maybe independently) in the context of Ricci solitons, see [16], and their generalizations, see [21], [6]. This circle of ideas, based on the detection of some "integrability conditions" for systems as (0.15), still finds as of today many applications and will indeed be crucial also in the present work. A similar situation, where ideas from the mathematical and physical communities influence each other, also arises in the case of Robinson-type identities, originally used to prove the spherical symmetry of static stellar models, [85], [63], and then employed in [86], [11] and [38]. We also cite on passing the very recent [30], [29] and [4].
The appearance of the extra contribution given by a smooth map $\varphi : (M, g) \to (N, h)$ in the present discussion is, from a purely mathematical point of view, more natural than one might think. Indeed, a coupled flow that depends both on a family of smooth metrics $\{g_t\}$ and smooth maps $\{\varphi_t\}$ is already considered in the seminal work of DeTurck, [35], on the short-time existence of the Ricci flow. Later on, List, [65], introduced the following flow:

$$\begin{cases} \partial_t g_t = -2\mathrm{Ric}_{g_t} + 2\alpha \varphi_t^* h, \\ \partial_t \varphi_t = \tau(\varphi_t), \end{cases} \tag{0.16}$$

for a fixed constant α and scalar maps $\{\varphi_t\}$, so that $(N, h) = (\mathbb{R}, g_{can})$ in this case. Quite surprisingly, List's original goal was to find good numerical approximations of vacuum static spaces. The link between vacuum static spaces, system (0.16) and Harmonic-Einstein manifolds is given by what is sometimes called the *harmonic map representation* of a static triple: given a three dimensional, scalar-flat solution of (0.6), it is easy to see that the conformal metric $\widetilde{g} = u^2 g$ satisfies

$$\begin{cases} \widetilde{\mathrm{Ric}} = 2\mathrm{d}\log u \otimes \mathrm{d}\log u, \\ \widetilde{\Delta} \log u = 0, \end{cases}$$

on int M, which is a Harmonic-Einstein structure; see also [84]. System (0.16) was later generalized by Buzano, [76], by considering a time-dependent α and a general target manifold (N, h); this is called the *Ricci-Harmonic flow*. It was Buzano's discovery that such a flow is sometimes better behaved than Ricci flow itself when it comes to the formation of singularities. These are now modelled by the *Ricci-Harmonic solitons*, that is, solutions of

$$\begin{cases} \mathrm{Ric}^{\varphi} + \mathcal{L}_X g = \lambda g, \quad \lambda \in \mathbb{R}, \\ \tau(\varphi) - \mathrm{d}\varphi(X) = 0 \end{cases} \tag{0.17}$$

for some smooth vector field X. These structures were later generalized, similarly to what has been done in [21], to the notion of a φ-Einstein type structure, which, in the gradient case, is given by

$$\begin{cases} \mathrm{Ric}^{\varphi} + \mathrm{Hess}\, f - \mu \mathrm{d}f \otimes \mathrm{d}f = \lambda g, \\ \tau(\varphi) - \mathrm{d}\varphi(\nabla f) = 0, \end{cases}$$

for $\mu \in \mathbb{R}, \lambda \in C^{\infty}(M)$. See [6]. Moreover, the Ricci-Harmonic flow appears naturally when one considers Ricci flow on warped product manifolds with a circle fiber S^1, [68], or on a special class of vector bundles, namely, $\mathbb{R}^n$-vector bundles over a Riemannian base with flat metrics on the fibers and such that the fiber-wise volume form si preserved by the flat connection, [66], [67], [90]. This type of bundles is particularly important for the study of immortal (that is, defined for all positive times) solutions of the Ricci flow, see [61].

We now introduce the first class of spaces that we want to study, namely the φ-static perfect fluid space-times (φ-SPFST for short). They are obtained from a static space-time satisfying

$$\widehat{\mathrm{Ric}} - \frac{1}{2}\widehat{S}\widehat{g} = \widehat{T}^F + \widehat{T}^{\widehat{\varphi}}, \tag{0.18}$$

that is, a static solution of Einstein's field equations with matter contributions given by both a perfect fluid and a field $\widehat{\varphi}$. We will see in Section 1.3 how to reduce (0.18) on the Riemannian factor (M, g) in order to obtain (0.1). It is interesting to note that similar stress-energy tensors have been recently considered in relations to the study of cosmological inflation, see [41] and [42].

The second class of manifolds that we want to deal with are named Cotton-φ-perfect fluids, C-φ-PF for short.

With the previous discussion, we hope to have convinced the reader that physically meaningful equations give rise to strong mathematical ideas. In recent times, the physical literature has seen a number of new modifications of General Relativity; this is mostly due to the general belief that dark matter and dark energy alone be no more a sufficient paradigm to meet the experimental evidences so that some deeper re-elaborations of Einstein's theory are needed. The central part of this thesis (Chapter 2) wants to be a first tentative approach to one of these theories, namely, *Cotton Gravity*, [51], from a strictly mathematical point of view, hoping that in the future it will successfully contribute to mathematical research. One of the advantages of Cotton Gravity with respect to General Relativity stems from the fact that the cosmological constant is no more a free parameter, but it is now an integration constant of a more general system of equations. This feature is considered desirable in that it should help us unveiling the true nature of the cosmological constant, especially in view of the discrepancy between its measured value and the expected gravitational effect of the quantum vacuum, see [37] and [24]. For this reason, it makes sense to consider also (covariant) derivatives of the curvature tensor and of the stress-energy tensor into the field equations, while possibly having all the solutions of Einstein's General Relativity to be solutions of this new theory. The equations that Harada proposed are the following

$$\widehat{C} = -(n-2)\,\mathrm{div}_1\,\widehat{\mathfrak{T}} \tag{0.19}$$

where $\widehat{C}$ is the Cotton tensor, $\widehat{\mathfrak{T}}$ is defined by

$$\widehat{\mathfrak{T}} := \frac{1}{n-2}\left(\widehat{T} \owedge \widehat{g} - \frac{1}{2(n-1)}(\mathrm{tr}_{\widehat{g}}\widehat{T})\widehat{g} \owedge \widehat{g}\right),$$

and $\mathrm{div}_1\,\widehat{\mathfrak{T}}$ stands for the divergence of $\widehat{\mathfrak{T}}$ with respect to the first index, which in components read

$$\left(\mathrm{div}_1\,\widehat{\mathfrak{T}}\right)_{\alpha\beta\gamma} := \widehat{\mathfrak{T}}^{\rho}{}_{\alpha\beta\gamma,\rho}.$$

The right hand side of (0.19) is chosen in a way to preserve all the solutions of General Relativity. Some skepticism regarding the physical validity of CG has been first raised by Bargueño, [9], but Harada was able to show that the galactic rotation curves of 84 far away galaxies can be explained by his theory without the need for dark matter contributions, [52]. Some other criticisms have been raised in [27], where it is argued that the higher number of solutions of Cotton gravity with respect to GR makes the theory unphysical, and in [3], where the authors claim that, due to the vanishing of some conserved charges in the simplest models of the theory, the formation of black holes without energy loss is likely to occur (see also [2], where a similar framework, called *Conformal Killing Gravity*, is addressed). These pessimistic conclusions were then confuted in [72], [24] and [73]. In any case, Cotton gravity is presently an active field of research and, besides these controversial disputes, it reveals to be certainly interesting from the purely geometric point of view. The purpose of Chapter 2 is to use its field equations as a stepping stone to derive meaningful generalizations of mathematical structures that, despite of their physical origin, have been vastly analysed by the mathematical community. We will therefore focus on the Cotton Gravity-version of φ-static perfect fluids, that is, the solutions of (0.19) for a stress-energy tensor $\widehat{T} = \widehat{T}^F + \widehat{T}^{\widehat{\varphi}}$.
Let us now discuss in details the content of this thesis. Chapter 1 is an introductory chapter, without original material. In Section 1.1 we discuss some of the basics of semi-Riemannian geometry, with a focus on the moving frame method. In Section 1.2 we introduce the formalism of φ-curvatures, that is, tensors that encode both the classical curvature tensors and terms depending on the field $\widehat{\varphi}$. They have already been discussed in [1] and [74]. Section 1.3 contains the derivations of the static perfect fluid system and the φ-static perfect fluid system.

In Chapter 2 we discuss the notion of C-φ-PF. Since every φ-SPFST is also a

C-φ-PF, the rigidity results of this chapter also apply to this class of spaces; we will delegate to Chapter 3 some rigidity results for φ-SPFSTs that we have not been able to generalize to the context of C-φ-PFs. The material we present in Chapter 2 is extracted from [46]. Firstly, we describe the field equations of Cotton gravity and, through some elementary computations, we show that their solutions include the solutions of Einstein's General Relativity. Moreover, as in Einstein's theory, the conservation equation $\operatorname{div} T = 0$ is implied by the field equations. We then show how, in this new theory, the field equations have a variational origin. What's important to note is that one needs to consider a specific class of variations, where the connection, instead of the metric, is the fundamental entity. Indeed, Harada, [51], already proved how Cotton flat metrics on a Lorentzian 4-manifolds are critical points of the action functional

$$\widehat{g} \to \int_{\widehat{M}} \left|\widehat{W}\right|^2$$

with respect to variations of the Levi-Civita connection that leave the metric g fixed. Performing similar computations, we will show that, in any dimension greater or equal than 3, the equation

$$\widehat{C} = -(n-2)\operatorname{div}_1 \widehat{\mathfrak{T}}^{\widehat{\varphi}}$$

has a variational origin. We remark that the situation is different for the stress-energy tensor $\widehat{T}^F$ since, even in Einstein's theory, it does not have a variational origin in the strict sense. We then proceed to derive (0.2) from the field equations. We also give an equivalent formulation of (0.2) which depends on two fundamental tensors, namely

$$\begin{aligned} D^A_{ijk} :=& \frac{1}{m-2}\Big[f_k R^\varphi_{ij} - f_j R^\varphi_{ik} + \frac{f_t}{m-1}\big(R^\varphi_{tk}\delta_{ij} - R^\varphi_{tj}\delta_{ik}\big) \\ &- \frac{S^\varphi}{m-1}(f_k\delta_{ij} - f_j\delta_{ik})\Big] \end{aligned}$$

and

$$\begin{aligned} D^B_{ijk} :=& \frac{1}{m-2}\Big[f_j f_{ik} - f_k f_{ij} + \frac{f_t}{m-1}(f_{tj}\delta_{ik} - f_{tk}\delta_{ij}) \\ &- \frac{\Delta f}{m-1}(f_j\delta_{ij} - f_k\delta_{ik})\Big]. \end{aligned}$$

These tensors were introduced in [16] for the study of Ricci solitons, where one has the validity of $D^A = D^B$, and, in this context, they appear in the first integrability condition of a Ricci soliton; as we will see, the same happens for a φ-SPFST. This leads to a first remarkable fact: the equations of C-φ-PF generalize the first integrability condition of a φ-SPFST, the main difference being that now D^A and D^B are, a priori, distinct. Therefore, one can hope to extend to this setting some rigidity results for φ-SPFSTs that rely on the integrability conditions of (0.1).

This calls for a generalization of [16]: there, making use of the *second* integrability condition of a Ricci soliton and of a suitable integral identity (in physicists' language, a Robinson-type identity) the authors were able to classify Bach-flat, complete shrinking Ricci solitons by proving the vanishing of D^A (and therefore of D^B). For this reason, in Section 2.3.1, we discuss how the vanishing of D^A and D^B affects the local geometry of a C-φ-PF and in Section 2.3.2 we derive the relevant Robinson-type identity. By doing so, we prove the main theorem of this Chapter, namely Theorem 2.36. As a by-product of the previous analysis, we show how a C-φ-PF is tightly related to the existence of a Codazzi tensor. Finally, in Section 2.4 we discuss some conditions under which a C-φ-PF reduces to a φ-SPFST. A fundamental concept in this direction is that of *Riemann compatibility*, introduced by Mantica and Molinari, [69], that, in this context, becomes a sort of integrability condition for system (0.2). Elaborating on some ideas of Mantica and Molinari, [69], [70] as well as of Gover, Nagy and Nurowski, [47], [48] we deduce some natural conditions on the Weyl tensor under which every solution of (0.2) gives rise to a solution of (0.1), giving a special emphasis to the case of dimension 4.

In Chapter 3, we deal with φ-static perfect fluid space-times. The material presented here is taken from [13]. Out of all the results of [13], we chose to present only those that are closer in spirit to those of Chapter 2; in particular, the main assumptions concern the φ-Cotton tensor, or the existence of a suitable Codazzi tensor. Firstly, note that using (0.1).iv) and (0.1).ii) into (0.1).i) we get that (0.1) implies the validity of

$$\begin{cases} \mathrm{Ric}^{\varphi} + \mathrm{Hess}\, f - \mathrm{d}f \otimes \mathrm{d}f = \lambda g, \quad \lambda \in C^{\infty}(M), \\ \tau(\varphi) = \mathrm{d}\varphi(\nabla f) + \frac{1}{\alpha}\nabla U(\varphi). \end{cases}$$

We add a parameter $\eta \in \mathbb{R}$ to obtain

$$\begin{cases} \mathrm{Ric}^{\varphi} + \mathrm{Hess}\, f - \eta \mathrm{d}f \otimes \mathrm{d}f = \lambda g, \quad \lambda \in C^{\infty}(M), \\ \tau(\varphi) = \mathrm{d}\varphi(\nabla f) + \frac{1}{\alpha}\nabla U(\varphi). \end{cases} \tag{0.20}$$

Chapter 3 actually deals with Riemannian manifolds supporting a solution of (0.20), φ-SPFSTs being a particular case of it. Equations (0.20) can be seen as a generalization of the φ-Einstein type manifold system, the new ingredient being the potential U. We begin by giving the first integrability condition of (0.20). Then, similarly to the previous chapter, we derive several rigidity results by proving the vanishing of the tensor D^A. Most of the results of this Chapter work under the assumption of vanishing of the total divergence of the φ-Cotton tensor, in components,

$$0 = \mathrm{div}^3\, C^{\varphi} := C^{\varphi}_{ijk,kji}. \tag{0.21}$$

When φ is constant, assumption (0.21) has been considered in [20]. There, a classification of complete, gradient, shrinking Ricci solitons satisfying (0.21) is given. We extend their result to the present setting. As in [20], our approach depends on a suitable integral formula together with a tricky integration by part. In Section 3.2, we also consider the case of manifolds with boundary. The presence of boundary terms in our integral formulas calls for a much more involved proof, which is actually entirely independent to the one of the previous section. Finally, in Section 3.3, we deal with compact manifolds supporting a solution of (0.20) for which a suitable modification of the φ-Cotton tensor vanishes. We first recall a classical Theorem of Obata, [77]: every compact Einstein manifold endowed with a non-Killing conformal vector field is isometric to a standard sphere. In [1], the authors were able to extend Obata's theorem to some class of φ-Einstein type manifolds. Using that result, they proved that every compact gradient φ-Einstein type manifold that's φ-Cotton flat and has some positive, constant higher order elementary symmetric polynomial in the eigenvalues of the φ-Schouten tensor is isometric to a standard sphere. In Section 3.3, we extend their result to solutions of (0.20). The main difference between our approach and theirs lie in the Obata-type theorem: in order to take into account the presence of the potential U, we need to perform some tricky computations. Unfortunately, to make our method work, we were forced to make the further assumption that X be a *closed* vector field. Luckily enough, this is always the case for our application, since we will always deal with *gradient* vector fields.

Chapter 1

Preliminaries

Throughout this thesis, every manifold is implicitly assumed to be smooth and connected. We denote the vector bundle of smooth vector fields on a manifold M by TM, while T^*M is be the co-tangent bundle. The vector bundle of tensor fields of type (r, s) is denoted by $T^{(r,s)}M$. Given a vector bundle E on M and an open subset U on M, we let $\Gamma(U, E)$ be the space of smooth sections of E supported on U, and we set $\Gamma(E) := \Gamma(M, E)$. Given a semi-Riemannian metric g on M, we have that g induces a linear isomorphism between TM and T^*M, called the *musical isomorphism*. Given a vector field $X \in \Gamma(TM)$, the associated 1-form is denoted by $X^\flat$, while, given a 1-form $\theta \in \Gamma(T^*M)$, its dual vector field is denoted by $\theta^\#$. The Lie derivative of a tensor T in the direction of a vector field X is denoted by $\mathcal{L}_X T$. Unless otherwise stated, Einstein summation convention over repeated indexes is on force. Integration over a semi-Riemannian manifold (M, g) will always be with respect to the standard volume measure, so that we will not make any reference to the volume element $\mathrm{d}V_g$.

1.1 Semi-Riemannian Geometry

In this section, we develop the basic tools of Riemannian and Lorentzian geometry. All the material presented here is classic, some references being [79] and [25]. See also the more recent [1] and [19], in particular for what concerns the moving frame formalism. Since at this level there is no substantial difference between the Riemannian and the Lorentzian cases, we will treat the more general case of a *semi-Riemannian manifold* (also called a *pseudo-Riemannian manifold*).

Definition 1.1. A semi-Riemannian manifold is a couple (M, g) where M is a smooth m-dimensional manifold, $m \geq 2$, and g is a symmetric, non-degenerate, $(0, 2)$-tensor field on M of constant signature.

Given a coordinate chart $(U, \{x^i\}_{i=1}^m)$, the metric tensor can be expressed by

$$g = g_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j.$$

The *Levi-Civita connection* ∇ of (M, g) is the unique linear connection on TM satisfying, for all vector fields $X, Y \in \Gamma(TM)$, the following conditions:

- ∇ is torsion-free, that is,

$$0 = \nabla_X Y - \nabla_Y X - [X, Y];$$

- ∇ is compatible with the metric, that is,

$$\nabla g = 0,$$

where ∇g denotes the covariant derivative of g, that will be defined in an instant.

In a coordinate chart $(U, \{x^i\})_{i=1}^m$, denoting $\partial_i := \frac{\partial}{\partial x^i}$, we have that ∇ is identified by the *Christoffel symbols*, i.e.,

$$\Gamma^i{}_{jk} \partial_i := \nabla_{\partial_j} \partial_k.$$

One can easily prove that they satisfy

$$\Gamma^i{}_{jk} = \frac{1}{2} g^{it} (\partial_j g_{tk} + \partial_k g_{tj} - \partial_t g_{jk}),$$

where the g^{ij}s are the components of the inverse g^{-1} of the metric. Given a tensor T of type (r, s), its covariant derivative ∇T is a tensor of type $(r, s+1)$ whose local components are given by

$$\begin{aligned} T^{j_1 \dots j_r}_{i_1 \dots i_s, k} = & \partial_k T^{j_1 \dots j_r}_{i_1 \dots i_s} - \Gamma^l{}_{k i_1} T^{j_1 \dots j_r}_{l i_2 \dots i_s} - \dots - \Gamma^l{}_{k i_s} T^{j_1 \dots j_r}_{i_1 \dots i_{s-1} l} \\ & + \Gamma^{j_1}{}_{kp} T^{p j_2 \dots j_r}_{i_1 \dots i_s} + \dots + \Gamma^{j_r}{}_{kp} T^{j_1 \dots j_{r-1} p}_{i_1 \dots i_s}. \end{aligned}$$

With these conventions in mind, we can define the $(1, 3)$-version of the *Riemann curvature tensor* by

$$R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \quad \forall X, Y, Z \in \Gamma(TM);$$

locally, it has components

$$R = R^t{}_{kij} \mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k \otimes \frac{\partial}{\partial x^t}$$

satisfying

$$R^t{}_{kij} = \partial_i \Gamma^t{}_{jk} - \partial_j \Gamma^t{}_{ik} + \Gamma^l{}_{jk} \Gamma^t{}_{il} - \Gamma^l{}_{ik} \Gamma^t{}_{jl}. \tag{1.1}$$

We will always use the metric tensor and its inverse to lower and raise indexes; in particular, the $(0, 4)$-version of the Riemann curvature tensor is the tensor

$$\mathrm{Riem} := R_{ijkt} \mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k \otimes \mathrm{d}x^t$$

of components

$$R_{ijkt} := g_{ip} R^p{}_{jkt}.$$

It satisfies the relations

$$R_{ijkt} = -R_{jikt} = R_{jitk} = R_{ktij}$$

as well as the *first and second Bianchi identities*:

$$R_{ijkt} + R_{iktj} + R_{itjk} = 0,$$
$$R_{ijkt,l} + R_{ijtl,k} + R_{ijlk,t} = 0.$$

Here and in the following, commas are used to denote covariant differentiation. In particular, the covariant derivative of Riem is written by

$$\nabla \mathrm{Riem} := R_{ijkt,l} \mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k \otimes \mathrm{d}x^t \otimes \mathrm{d}x^l.$$

Before going on, we want to present a method to derive the Levi-Civita connection and the curvature tensor of (M, g) which is alternative (but equivalent) to the one just introduced. This is called the *moving frame approach*. It has the advantage of considerably simplifying the computations, in some cases. Assume that the signature of the metric g is $(k, m-k)$. In a coordinate chart $(U, \{x^i\}_{i=1}^m)$, we can always apply the Gram-Schmidt algorithm to extract, from the 1-forms $\{\mathrm{d}x^i\}_{i=1}^m$, a family of *orthonormal* 1-forms $\{\theta^i\}_{i=1}^m$ on U such that the metric tensor g can be expressed by

$$g = -\sum_{i=1}^{k} \theta^i \otimes \theta^i + \sum_{j=k+1}^{m} \theta^j \otimes \theta^j;$$

therefore, the components g_{ij} of the metric on this co-frame form a diagonal matrix whose first k diagonal terms are -1 and the remaining diagonal terms are +1.

Remark 1.2. When g is a Riemannian metric, we deduce that $g_{ij} = \delta_{ij}$, the Kronecker delta. Due to this fact, the operation of lowering or raising indexes does not change the value of the components of a tensor. Therefore, when we will work with a Riemannian metric, we will consider every tensor to be purely covariant.

Let us denote the dual frame to $\{\theta^i\}$ with $\{e_i\}$. Differentiating the θ^is and using Cartan's Lemma (see [1], pag. 4), one can prove the existence, on U, of 1-forms $\{\theta^i{}_j\}_{i,j=1}^m$, called the *Levi-Civita connection forms*, satisfying

$$\begin{cases} \mathrm{d}\theta^i = -\theta^i{}_j \wedge \theta^j, & \text{(first structure equations)} \\ \theta_{ij} + \theta_{ji} = 0, \end{cases} \tag{1.2}$$

where $\theta_{ij} := g_{it}\theta^t{}_j$. Defining

$$\nabla e_j := \theta^i{}_j \otimes e_i,$$

one can see that ∇ coincides with the Levi-Civita connection: indeed, equation (1.2).ii) is equivalent to the compatibility of ∇ with the metric, while (1.2).i) is equivalent to ∇ being torsion-free; see [19, pag. 6] for more details. The connection satisfying these two conditions is unique, so that ∇ is indeed the Levi-Civita connection; in particular, in this co-frame we have

$$\Gamma^i{}_{kj} = \theta^i{}_j(e_k).$$

We can therefore re-write the definition of covariant derivative of a tensor T of type (r, s) as

$$\begin{aligned} T^{j_1\dots j_r}_{i_1\dots i_s,k}\theta^k =& \mathrm{d}T^{j_1\dots j_r}_{i_1\dots i_s} - T^{j_1\dots j_r}_{hi_2\dots i_s}\theta^k{}_{i_1} - \dots - T^{j_1\dots j_r}_{i_1\dots i_{s-1}h}\theta^h{}_{i_s} \\ &+ T^{hj_2\dots j_r}_{i_1\dots i_s}\theta^{j_1}{}_h + T^{j_1\dots j_{r-1}h}_{i_1\dots i_s}\theta^{j_r}{}_h, \end{aligned} \tag{1.3}$$

where d denotes the exterior differential. The Riemann curvature tensor can be expressed in terms of the $\theta^i{}_j$s in the following way: differentiate $\theta^i{}_j$ to *define* a 2-form $\Theta^i{}_j$ by the expression

$$\mathrm{d}\theta^i{}_j = -\theta^i{}_k \wedge \theta^k{}_j + \Theta^i{}_j, \quad \text{(second structure equations)}$$

which, as it is straightforward to check, also satisfies

$$\Theta_{ij} + \Theta_{ji} = 0,$$

where, again, $\Theta_{ij} := g_{it}\Theta^t{}_j$. Then, an easy computation gives the validity of

$$\Theta^i{}_j = \frac{1}{2} R^i{}_{jkt}\theta^k \wedge \theta^t.$$

A first, fundamental, application of the Riemann curvature tensor are the *Ricci commutation relations* for the second covariant derivatives of a tensor. In the next proposition, we present them for a $(0, 2)$-tensor, the general formula for a tensor of type (r, s) being deduced in a similar way.

Proposition 1.3. *Let T be a twice covariant tensor field on (M, g). Then we have*

$$T_{ij,kt} = T_{ij,tk} + T_{ip}R^p{}_{jkt} + T_{pj}R^p{}_{ikt} \tag{1.4}$$

Proof. By definition of covariant derivative we have

$$T_{ij,k}\theta^k = \mathrm{d}T_{ij} - T_{kj}\theta^k{}_i - T_{ik}\theta^k{}_j \tag{1.5}$$

and

$$T_{ij,kt}\theta^t = \mathrm{d}T_{ij,k} - T_{tj,k}\theta^t{}_i - T_{it,k}\theta^t{}_j - T_{ij,t}\theta^t{}_k. \tag{1.6}$$

Differentiating (1.5) we find

$$\begin{aligned} \mathrm{d}T_{ij,k} \wedge \theta^k + T_{ij,k}\mathrm{d}\theta^k =& -\mathrm{d}T_{kj} \wedge \theta^k{}_i - T_{kj}\mathrm{d}\theta^k{}_i - \mathrm{d}T_{ik} \wedge \theta^k{}_j \\ &- T_{ik}\mathrm{d}\theta^k{}_j. \end{aligned}$$

Using (1.6), (1.5) and the second structure equations we have

$$\begin{aligned}
T_{ij,kt}\theta^t\wedge\theta^k + T_{tj,k}\theta^t{}_i\wedge\theta^k + T_{it,k}\theta^t{}_j\wedge\theta^k + T_{ij,t}\theta^t{}_k\wedge\theta^k \\
+ T_{ij,k}\mathrm{d}\theta^k \\
= - T_{kj,t}\theta^t\wedge\theta^k{}_i - T_{tj}\theta^t{}_k\wedge\theta^k{}_i - T_{kt}\theta^t{}_j\wedge\theta^k{}_i \\
+ T_{kj}\theta^k{}_t\wedge\theta^t{}_i - T_{kj}\Theta^k{}_i \\
- T_{ik,t}\theta^t\wedge\theta^k{}_j - T_{tk}\theta^t{}_i\wedge\theta^k{}_j - T_{it}\theta^t{}_k\wedge\theta^k{}_j \\
+ T_{ik}\theta^k{}_t\wedge\theta^t{}_j - T_{ik}\Theta^k{}_j.
\end{aligned}$$

Using the first structure equations and simplifying we get

$$T_{ij,kt}\theta^t\wedge\theta^k = -T_{kj}\Theta^k{}_i - T_{ik}\Theta^k{}_j;$$

skew symmetrizing and recalling that $\Theta^k{}_i = \frac{1}{2}R^k{}_{ipq}\theta^p\wedge\theta^q$ we obtain

$$\frac{1}{2}(T_{ij.kt} - T_{ij,tk})\theta^t\wedge\theta^k = -\frac{1}{2}R^k{}_{ipq}T_{kj}\theta^p\wedge\theta^q - \frac{1}{2}R^k{}_{jpq}T_{ik}\theta^p\wedge\theta^q$$

and we are done. □

We define the Ricci tensor by

$$R_{ij} := R^k{}_{ikj}\,.$$

Note that $R_{ij} = R_{ji}$ as a consequence of the symmetries of Riem. Given a $(0,2)$-tensor P, we define its g-trace by

$$\mathrm{tr}_g P := g^{ij}P_{ij} = P^i{}_i\,.$$

The scalar curvature of (M,g) is

$$S := \mathrm{tr}_g\mathrm{Ric} = g^{ij}R_{ij}\,.$$

We have the following, classical

Definition 1.4. A semi-Riemannian manifold (M,g) of dimension $m\geq 2$ is called an Einstein manifold if it holds

$$\mathrm{Ric} = \Lambda g, \tag{1.7}$$

on M, for some constant $\Lambda\in\mathbb{R}$.

When $m\geq 3$, one can see that any $\Lambda\in C^\infty(M)$ satisfying (1.7) turns out to be constant. This is a consequence of the classical *Schur identity*:

$$R^j{}_{i,j} = \frac{1}{2}S_i,$$

that is obtained by taking twice the contraction of the second Bianchi identity with the metric. As such, it holds on any semi-Riemannian manifold. Given two $(0,2)$-tensors P and Q, their Kulkarni-Nomizu product $P \owedge Q$ is

$$(P \owedge Q)_{ijkt} := P_{ik}Q_{jt} + P_{jt}Q_{ik} - P_{it}Q_{jk} - P_{jk}Q_{it} \,.$$

With this notation in mind, the Weyl tensor of (M,g) is defined by

$$W_{ijkt} := R_{ijkt} - \frac{1}{m-2}(A \owedge g)_{ijkt}$$

where

$$A_{ij} := R_{ij} - \frac{1}{2(m-1)} S\, g_{ij}$$

is the Schouten tensor. Some simple computations reveal that the Weyl tensor has the same symmetries of the Riemann curvature tensor, and it has the additional feature of being totally trace-free, that is, every contraction of W with the metric is identically zero. The Cotton tensor of (M,g) is given by

$$C_{ijk} := A_{ij,k} - A_{ik,j} \,.$$

When $m \geq 4$, C can be alternatively, but equivalently, defined by

$$C_{ijk} = -\frac{m-3}{m-2} W^t{}_{ijk,t} \,.$$

We define the 1-divergence of a p times covariant tensor P by

$$(\operatorname{div}_1 P)_{i_1 \dots i_{p-1}} := g^{jk} P_{j i_1 \dots i_{p-1},k}$$

so that, when $m \geq 4$,

$$C = -\frac{m-3}{m-2} \operatorname{div}_1 W \,.$$

It is easily seen that the Cotton tensor is skew-symmetric in the last two indexes, i.e.

$$C_{ijk} = -C_{ikj}$$

and totally trace-free; moreover, it satisfies the "first Bianchi identity"

$$C_{ijk} + C_{jki} + C_{kij} = 0.$$

Its divergences satisfy the following properties:

$$C^i{}_{jk,i} = 0, \quad C_{ijk,}{}^k = C_{jik,}{}^k.$$

As a consequence of the second Bianchi identity of the Riemann tensor, one can prove the following "fake" second Bianchi identities:

$$\begin{aligned} W_{ijkl,t} + & W_{ijlt,k} + W_{ijtk,l} \\ & = \frac{1}{m-2}(C_{ikl}\delta_{tj} + C_{ilt}\delta_{kj} + C_{itk}\delta_{lj} - C_{jkl}\delta_{ti} - C_{jlt}\delta_{ki} - C_{jtk}\delta_{li}) \end{aligned} \tag{1.8}$$

and

$$C_{ijk,t} + C_{ikt,j} + C_{itj,k} = W^l{}_{ikt} R_{jl} + W^l{}_{itj} R_{kl} + W^l{}_{ijk} R_{tl}. \tag{1.9}$$

The geometric significance of W and C comes from *conformal geometry*.

Definition 1.5. Given a semi-Riemannian manifold (M, g) and another semi-Riemannian metric $\widetilde{g}$ on M of the same signature as of g, we will say that $\widetilde{g}$ is a *pointwise conformal deformation* of g if there exists a smooth function $u \in C^\infty(M)$ such that $\widetilde{g} = e^{2u}g$.

Recall that a *flat* manifold is, by definition, a semi-Riemannian manifold (M, g) with vanishing Riemann curvature tensor.

Definition 1.6. Let (M, g) be a semi-Riemannian manifold. We say that (M, g) is locally conformally flat if, for every $p \in M$, there exists an open neighbourhood U of p and a function $u \in C^\infty(U)$ such that $(U, e^{2u}g)$ is flat.

The $(0, 4)$-version of the Weyl tensor is conformally invariant when $m \geq 4$: this means that, for any pointwise conformal deformation $\widetilde{g}$ of g of Weyl tensor $\widetilde{W}$, we have

$$\widetilde{W} = W.$$

When $m = 3$, the algebraic symmetries of the Weyl tensor, together with the fact that it is totally trace-free, imply that it has to vanish everywhere, i.e. $W \equiv 0$. When $m = 3$, and only in this dimension, it can be proven that the Cotton tensor is conformally covariant of degree 3, that is, it satisfies

$$\widetilde{C} = e^{-3u}C,$$

where $\widetilde{g} = e^{2u}g$. Furthermore, we have the following deep theorem of Weyl and Schouten.

Theorem 1.7. *Let (M, g) be a semi-Riemannian manifold of dimension $m \geq 3$. Then, (M, g) is locally conformally flat if and only if*

- *$m = 3$ and $C \equiv 0$, or*
- *$m \geq 4$ and $W \equiv 0$.*

The last protagonist of this section is the *Bach tensor*, that first appeared in the physical literature, [7]. It has components

$$B_{ij} := \frac{1}{m-2}\big(C_{ijk,}{}^{k} + R^{tk}W_{tikj}\big).$$

This is easily seen to be symmetric and trace-free. Moreover, when $m = 4$, it has two important features, that account for its physical relevance: it is divergence free and conformally covariant of degree two, that is,

$$\widetilde{B} = e^{-2u}B,$$

for $\widetilde{g} = e^{2u}g$. When $m = 4$ and (M, g) is a a Riemannian manifold, the Bach tensor arises from the action functional

$$g \to \int_M |W|^2;$$

indeed, the critical points of the latter with respect to compactly supported variations of the metric are precisely the Riemannian metrics on M with vanishing Bach tensor. These are called *Bach-flat metrics*. Recall the Chern-Gauss-Bonnet formula: on a closed 4-dimensional Riemannian manifold it holds

$$8\pi^2\chi(M) = \frac{1}{4}\int_M |W|^2 + \int_M S_2(A) \tag{1.10}$$

where $\chi(M)$ is the Euler characteristic of M and $S_2(A) := \frac{1}{2}\left[(\mathrm{tr}_g A)^2 - |A|^2\right]$ is the second symmetric polynomial in the eigenvalues of the Schouten tensor A. Since $\chi(M)$ is a topological invariant, it is not detected by the variations of the metric, so that, by (1.10), Bach-flat metrics on a Riemannian 4-manifold are also the critical points of the functional

$$g \to \int_M S_2(A).$$

1.2 φ-curvatures

In this section, we will be interested in the data of a semi-Riemannian manifold (M, g), a Riemannian manifold (N, h) and a smooth map $\varphi : M \to N$. We fix indexes ranges

$$1 \leq i, j, k, ... \leq m, \quad 1 \leq a, b, c, ... \leq d = \dim N.$$

Given two open subsets $U \subset M, V \subset N$ such that $\varphi(U) \subset V$, we consider on them two local orthonormal bases $\{\theta^i\}_{i=1}^m$ and $\{E_a\}_{a=1}^d$ of T^*M and TN, respectively. The tangent map $\mathrm{d}\varphi : M \to N$ can be viewed as a section of $T^*M \otimes \varphi^*TN$ so that it has local components

$$\mathrm{d}\varphi = \varphi^a_i \theta^i \otimes \varphi^* E_a.$$

In what follows, we will suppress the pull-back notation, when clear from the context, and write $E_a = \varphi^* E_a$. The energy density of φ is defined to be

$$e(\varphi) := \frac{1}{2}|\mathrm{d}\varphi|^2 = \frac{1}{2} g^{ij} \varphi^a_i \varphi^a_j.$$

If we set $\{\omega^a\}_{a=1}^d$ to denote the 1-forms which are h-dual to $\{E_a\}_{a=1}^d$, then we will write $\{\omega^a{}_b\}_{a,b=1}^d$ for their connection forms. This allows us to define the covariant derivative of a section v of φ^*TN by the expression

$$\nabla v := v^a_i \theta^i \otimes E_a$$

where

$$v^a_i \theta^i := \mathrm{d}v^a + v^b \omega^a{}_b.$$

Moreover, the covariant derivative extends to any tensor bundle of the form $T^{(0,q)}M \otimes \varphi^* TN$. For the differential $\mathrm{d}\varphi$, its covariant derivative $\nabla \mathrm{d}\varphi$ has components

$$\nabla \mathrm{d}\varphi = \varphi^a_{ij} \theta^i \otimes \theta^j \otimes E_a$$

where

$$\varphi^a_{ij}\theta^j := \mathrm{d}\varphi^a_i - \varphi^a_j \theta^j{}_i + \varphi^b_i \omega^a{}_b.$$

The following commutation relation is proved in a similar way to Proposition 1.3 (see [1, Section 1.7] for a proof):

$$\varphi^a_{ijk} = \varphi^a_{ikj} + g^{tl}\varphi^a_t R_{lijk} - {}^N R^a{}_{bcd}\varphi^b_i \varphi^c_j \varphi^d_k, \tag{1.11}$$

where ${}^N R^a{}_{bcd}$ denote the components of the Riemann curvature tensor of (N, h). The tensor $\nabla \mathrm{d}\varphi$ is called the *generalized second fundamental form of* φ. It is easy to see that $\varphi^a_{ij} = \varphi^a_{ji}$. The *tension field* of φ is defined to be

$$\tau(\varphi) := \mathrm{tr}_g \nabla \mathrm{d}\varphi = g^{ij}\varphi^a_{ij} E_a.$$

This has been defined (in the Riemannian case) in [36]. It can be easily checked that, in a generic coordinate chart (U, x^α), the tension field has components

$$\tau(\varphi)^a = g^{\alpha\beta}\frac{\partial^2 \varphi^a}{\partial x^\alpha \partial x^\beta} - g^{\alpha\beta}\Gamma^\gamma{}_{\alpha\beta}\frac{\partial \varphi^a}{\partial x^\gamma} + {}^N\Gamma^a{}_{bc} g^{\alpha\beta}\frac{\partial \varphi^b}{\partial x^\alpha}\frac{\partial \varphi^c}{\partial x^\beta}$$

where ${}^N\Gamma^a{}_{bc}$ denote the Christoffel symbols of the Levi-Civita connection of (N, h).

Definition 1.8. A smooth map $\varphi : (M, g) \to (N, h)$ is said to be *harmonic* if it has vanishing tension field.

We underline that this terminology is somehow imprecise: it is more common, in the literature, to call a map with vanishing tension field harmonic only when (M, g) is a Riemannian manifold, while in Lorentzian signature such a map is usually called a *wave map*. In order to give a unified treatment of both cases, we will not make this distinction. It is possible to give the following variational characterization of harmonic maps: for any relatively compact domain $\Omega \subset M$, consider the functional

$$E_\Omega(\varphi) = \int_\Omega e(\varphi);$$

then φ is harmonic if and only if, for every such Ω, $E_\Omega(\varphi)$ is stationary with respect to variations of φ that agrees with φ on the boundary of Ω. The *stress-energy tensor* of φ is given by

$$T_\varphi := \varphi^* h - \frac{1}{2}|\mathrm{d}\varphi|^2 g;$$

we already commented its physical meaning in the Introduction. It can be proved with a simple computation that the divergence of T_φ is

$$\mathrm{div}_1 T_\varphi = h(\tau(\varphi), \mathrm{d}\varphi) = g^{ij}\varphi^a_{ij}\varphi^a_k\theta^k.$$

We say that φ is *conservative* if its stress-energy tensor is divergence-free. Therefore, every harmonic map is conservative.
One can generalize the notion of a harmonic map. Consider a function $G \in C^\infty(N)$ and a map $\varphi : (M, g) \to (N, h)$ satisfying

$$\tau(\varphi) = \nabla G(\varphi) \tag{1.12}$$

where $\nabla G(\varphi)$ denotes the pull-back on M via φ of the gradient of G in the h-metric. Note that we are using the same symbol to denote both the connection on (M, g) and that of (N, h); since they act on objects living on different manifolds, we hope that this will not create any confusion in the reader, and that it will always be clear from the context which one of the two we are using. In Riemannian signature, a smooth map φ satisfying (1.12) is called *G-harmonic*. With a slight abuse, we will use the same denomination also in arbitrary signature of the metric. Note that, for U constant, we obtain the case of harmonic maps. The previous definition goes back to the work of Fardoun, Ratto and Regbaoui ([39], [40], [83]); it comes from a variational setting that has been vastly analyzed by Lemaire in his Ph.D. Thesis, ([62]). Indeed, G-harmonic maps are the critical points of the functional

$$E(\varphi) = \frac{1}{2}\int \left[|d\varphi|^2 + 2G(\varphi)\right] dV_g,$$

(see [62] for the details). A map that satisfies

$$0 = h(\tau(\varphi) - \nabla G(\varphi), \mathrm{d}\varphi)$$

is called *G-conservative*.
As we shall see, it is useful to encode some informations on the non-linear field φ into the curvature tensors of (M, g) in order to see, at the same time, both the combined action of φ and that of the semi-Riemannian metric g of M. With these motivations (other can be found in [6], [28], [74]) we introduce modified curvature tensors depending on the map $\varphi : (M, g) \to (N, h)$. The first step in this direction, that is, the definition of the φ-Ricci tensor, is due to B. List, that merged the Ricci flow with the harmonic map flow (for details and background see [65]). For some fixed coupling constant $\alpha \neq 0$ we set

$$\mathrm{Ric}^\varphi := \mathrm{Ric} - \alpha\varphi^* h \tag{1.13}$$

for the φ-Ricci tensor and the φ-scalar curvature will be its contraction with the metric g, that is,

$$S^\varphi := S - \alpha|\mathrm{d}\varphi|^2. \tag{1.14}$$

In components, we have

$$R^{\varphi}_{ij} = R_{ij} - \alpha \varphi^a_i \varphi^a_j \tag{1.15}$$

(note that List uses the notation S_{ij} instead of R^{φ}_{ij}). The next formula (see [6] for a simple proof) should be called the φ-Schur's identity and it will be repeatedly used in the sequel

$$g^{ik} R^{\varphi}_{ij,k} = \frac{1}{2} S^{\varphi}_j - \alpha g^{ik} \varphi^a_{ik} \varphi^a_j. \tag{1.16}$$

From (1.16) we immediately infer that, for $m \geq 3$, if

$$\mathrm{Ric}^{\varphi} = \Lambda g, \qquad \Lambda \in C^{\infty}(M), \tag{1.17}$$

then the function Λ satisfies

$$(m-2)\nabla\Lambda = 2\alpha h(\tau(\varphi), \mathrm{d}\varphi)^{\sharp}. \tag{1.18}$$

In case φ is conservative we have

$$h(\tau(\varphi), \mathrm{d}\varphi) \equiv 0,$$

so that Λ is constant. This is, in particular, the case of a harmonic map, for which $\tau(\varphi) \equiv 0$. Thus, if (M, g) is harmonic-Einstein with respect to φ for some $\alpha \neq 0$, that is, the system

$$\begin{cases} \mathrm{Ric}^{\varphi} = \Lambda g, \\ \tau(\varphi) = 0 \end{cases} \tag{1.19}$$

holds on M, then Λ is constant, or in other terms the φ-scalar curvature S^{φ} is constant. This also suggests the use of the term φ-Einstein manifold for (M, g), but the previous terminology has been introduced before and we shall adhere to it. In analogy with the classical case, the φ-Schouten tensor is defined by setting

$$A^{\varphi} := \mathrm{Ric}^{\varphi} - \frac{S^{\varphi}}{2(m-1)} g \tag{1.20}$$

and the φ-Cotton tensor is given by

$$C^{\varphi}_{ijk} := A^{\varphi}_{ij,k} - A^{\varphi}_{ik,j}.$$

Recall the following

Definition 1.9. Let (M, g) be a semi-Riemannian manifold, and let P be a symmetric 2-covariant tensor on M. Then P is called a *Codazzi tensor* if it holds

$$0 = P_{ij,k} - P_{ik,j}.$$

Therefore, the φ-Cotton tensor measures the obstruction to A^φ being a Codazzi tensor. The φ-Weyl tensor is defined to formally respect the usual decomposition of the Riemann tensor, that is,

$$W^\varphi := \mathrm{Riem} - \frac{1}{m-2} A^\varphi \owedge g \tag{1.21}$$

where $\owedge$ is the Kulkarni-Nomizu product. Although W^φ has the same algebraic symmetries of the Riemann curvature tensor it is not totally trace free. Indeed,

$$g^{tk} W^\varphi_{tikj} = \alpha \varphi^a_i \varphi^a_j \tag{1.22}$$

while the remaining traces are determined by (1.22) and by the algebraic symmetries of W^φ. Note that, in terms of the classical counterparts, we have

$$A^\varphi = A - \alpha A(\varphi^* h), \tag{1.23}$$

and

$$W^\varphi = W + \frac{\alpha}{m-2} A(\varphi^* h) \owedge g, \tag{1.24}$$

where

$$A(\varphi^* h) := \varphi^* h - \frac{|\mathrm{d}\varphi|^2}{2(m-1)} g$$

is the "Schouten tensor" of the symmetric, 2-covariant tensor $\varphi^* h$. For the φ-Cotton tensor C^φ we have the symmetries

$$C^\varphi_{ijk} = -C^\varphi_{ikj} \quad \text{so that } g^{tk} C^\varphi_{itk} = 0, \tag{1.25}$$

however it is not totally trace free, since

$$g^{tk} C^\varphi_{tki} = \alpha g^{tk} \varphi^a_{tk} \varphi^a_i. \tag{1.26}$$

It is easy to prove that C^φ satisfies the Bianchi-type identity

$$C^\varphi_{ijk} + C^\varphi_{kij} + C^\varphi_{jki} = 0. \tag{1.27}$$

Again, from its definition it is immediate to see that

$$C^\varphi_{ijk} = C_{ijk} - \alpha \left[\varphi^a_{ik} \varphi^a_j - \varphi^a_{ij} \varphi^a_k - \frac{\varphi^a_t}{m-1} g^{tp} \left(\varphi^a_{pk} \delta_{ij} - \varphi^a_{pj} \delta_{ik} \right) \right]. \tag{1.28}$$

The next two formulas are "fake" second Bianchi identities for the φ-Weyl tensor and the φ-Cotton tensor. We have

$$\begin{aligned} W^\varphi_{ijkl,t} &+ W^\varphi_{ijlt,k} + W^\varphi_{ijtk,l} \\ &= \frac{1}{m-2} \left(C^\varphi_{ikl} \delta_{tj} + C^\varphi_{ilt} \delta_{kj} + C^\varphi_{itk} \delta_{lj} - C^\varphi_{jkl} \delta_{ti} - C^\varphi_{jlt} \delta_{ki} - C^\varphi_{jtk} \delta_{li} \right) \end{aligned} \tag{1.29}$$

and

$$C^{\varphi}_{ijk,t} + C^{\varphi}_{ikt,j} + C^{\varphi}_{itj,k} = g^{lp}\big(W^{\varphi}_{likt}R^{\varphi}_{jp} + W^{\varphi}_{litj}R^{\varphi}_{kp} + W^{\varphi}_{lijk}R^{\varphi}_{tp}\big). \tag{1.30}$$

The following alternative definition of the φ-Cotton tensor, for $m \geq 4$, points out at deep differences between the classical and the φ-curvatures (see Proposition 2.64 of [6])

$$-\frac{m-3}{m-2}C^{\varphi}_{jkt} = g^{sl}W^{\varphi}_{sjkt,l} - \alpha\big(\varphi^a_{jk}\varphi^a_t - \varphi^a_{jt}\varphi^a_k\big) - \frac{\alpha}{m-2}g^{ls}\varphi^a_{ls}(\varphi^a_k\delta_{jt} - \varphi^a_t\delta_{jk}). \tag{1.31}$$

The φ-Bach tensor, B^{φ}, is not defined in analogy with the classical one; indeed, for $m \geq 3$, we set

$$\begin{aligned}(m-2)B^{\varphi}_{ij} :=& g^{kt}C^{\varphi}_{ijk,t} + g^{kp}g^{tq}R^{\varphi}_{tk}W^{\varphi}_{piqj} - \alpha g^{tk}R^{\varphi}_{tj}\varphi^a_k\varphi^a_i \\ &+ \alpha\left(g^{kt}\varphi^a_{ij}\varphi^a_{kt} - g^{kt}\varphi^a_{ktj}\varphi^a_i - \frac{|\tau(\varphi)|^2}{m-2}\delta_{ij}\right).\end{aligned} \tag{1.32}$$

Note that B^{φ} is a symmetric, 2-covariant tensor; for a proof see [6]. Contrary to the Bach tensor, B^{φ} is not trace free in general. Indeed we have

$$g^{ij}B^{\varphi}_{ij} = \alpha\frac{m-4}{(m-3)^2}|\tau(\varphi)|^2. \tag{1.33}$$

For what concerns the divergence of B^{φ}, we have the next

Lemma 1.10. *Let (M,g) be a semi-Riemannian manifold of dimension $m \geq 3$ and let $\varphi : (M,g) \to (N,h)$ be a smooth map. Then*

$$\begin{aligned}(m-2)g^{tk}B^{\varphi}_{it,k} =& \left(\frac{m-4}{m-2}\right)\left[(R^{\varphi})^{jk}C^{\varphi}_{jki} + \alpha\tau(\varphi)^a\big(g^{tk}\varphi^a_{tki} + g^{tj}R^{\varphi}_{ij}\varphi^a_t\big)\right] \\ &+ \alpha\varphi^a_i\left[\frac{m}{(m-1)(m-2)}\tau(\varphi)^aS^{\varphi} + 2\alpha g^{jk}\tau(\varphi)^b\varphi^b_j\varphi^a_k\right] \\ &- \frac{\alpha}{2}\left(\frac{m-2}{m-1}\right)g^{jk}\varphi^a_i\varphi^a_jS^{\varphi}_k - 2\alpha\varphi^a_{jk}(R^{\varphi})^{jk}\varphi^a_i \\ &- \alpha\varphi^a_i\tau^a_2(\varphi),\end{aligned} \tag{1.34}$$

where

$$\tau^a_2(\varphi) := g^{tk}g^{sl}\varphi^a_{tksl} - {}^NR^a{}_{bcd}g^{sl}\varphi^b_s\varphi^c_l\tau(\varphi)^d$$

are the components of the bi-tension field *of the map φ and ${}^NR^a{}_{bcd}$ are the components of the curvature tensor of (N,h).*

Remark 1.11. We recall that

$$\tau_2(\varphi) = 0$$

is the Euler-Lagrange equation of the *bi-energy functional* E^{φ}_{τ} given, on a relatively compact domain Ω in M, by the prescription

$$E^{\varphi}_{\tau}(\Omega) = \frac{1}{2}\int_{\Omega}|\tau(\varphi)|^2.$$

Remark 1.12. Lemma 1.10 was first proved in [5], but note that the two formulas are slightly different due to some minor typos that are present in [5]. The proof that we give here is essentially the same as the one presented there, with some small modifications.

Proof. Define the tensors of components

$$\begin{aligned}
\mathcal{L}_{ij} &= \tau(\varphi)^a\varphi^a_{ij} - \frac{1}{m-2}|\tau(\varphi)|^2 g_{ij},\\
\mathcal{M}_{ij} &= (R^\varphi)^{tk}W^\varphi_{tikj},\\
\mathcal{N}_{ij} &= g^{kt}[C^\varphi_{ijk,t} - \alpha\varphi^a_i(\varphi^a_{ktj} + R^\varphi_{jk}\varphi^a_t)].
\end{aligned}$$

Then, using (1.32), it is immediate to see that

$$(m-2)B^\varphi_{ij} = \alpha\mathcal{L}_{ij} + \mathcal{M}_{ij} + \mathcal{N}_{ij}.$$

We compute separately the divergences of these three tensors. Using the commutation formula (1.11), we have

$$\begin{aligned}
g^{jk}\mathcal{L}_{ij,k} &= g^{tk}g^{jl}\left(\varphi^a_{tkj}\varphi^a_{il} + \varphi^a_{tk}\varphi^a_{ijl} - \frac{2}{m-2}\varphi^a_{tki}\varphi^a_{jl}\right)\\
&= g^{tk}g^{jl}\left[\varphi^a_{tkj}\varphi^a_{il} + \left(\frac{m-4}{m-2}\right)\varphi^a_{tk}\varphi^a_{jli} + \varphi^a_{tk}\varphi^a_jR_{li} - {}^NR^a{}_{bcd}\varphi^b_j\varphi^c_i\varphi^d_l\varphi^a_{tk}\right].
\end{aligned}$$

Using the definition of Ric^φ we deduce

$$\begin{aligned}
g^{jk}\mathcal{L}_{ij,k} =& g^{tk}g^{jl}\Bigg[\varphi^a_{tkj}\varphi^a_{il} + \left(\frac{m-4}{m-2}\right)\varphi^a_{tk}\varphi^a_{jli} + \varphi^a_{tk}\varphi^a_jR^\varphi_{li} + \alpha\varphi^a_{tk}\varphi^a_j\varphi^b_l\varphi^b_i\\
&- {}^NR^a{}_{bcd}\varphi^b_j\varphi^c_i\varphi^d_l\varphi^a_{tk}\Bigg].
\end{aligned}\tag{1.35}$$

Next, for the tensor $\mathcal{M}$, we use the symmetries of W^φ and formula (1.31) to get

$$\begin{aligned}
g^{jk}\mathcal{M}_{ij,k} =& g^{tk}g^{jl}g^{pq}\left[R^\varphi_{tj,p}W^\varphi_{kilq} + R^\varphi_{tj}W^\varphi_{kilp,q}\right]\\
=& g^{tk}g^{jl}\Bigg[\frac{1}{2}g^{pq}(R^\varphi_{tj,p} - R^\varphi_{tp,j})W^\varphi_{kilq} + \left(\frac{m-3}{m-2}\right)R^\varphi_{tj}C^\varphi_{kli} + \alpha R^\varphi_{tj}\varphi^a_{li}\varphi^a_k\\
&- \alpha R^\varphi_{tj}\varphi^a_{kl}\varphi^a_i\Bigg] + \frac{\alpha}{m-2}g^{tk}\varphi^a_{tk}(S^\varphi\varphi^a_i - g^{pq}R^\varphi_{pi}\varphi^a_q).
\end{aligned}$$

Using the definition of C^φ and equation (1.22) we obtain

$$\begin{aligned}
g^{jk}\mathcal{M}_{ij,k} =& g^{tk}g^{jl}g^{pq}\frac{1}{2}C^\varphi_{tjp}W^\varphi_{kilq} + \frac{\alpha}{2(m-1)}g^{jk}S^\varphi_j\varphi^a_k\varphi^a_i + \left(\frac{m-3}{m-2}\right)(R^\varphi)^{tk}C^\varphi_{tki}\\
&+ \alpha g^{tk}g^{jl}[R^\varphi_{tj}\varphi^a_{li}\varphi^a_k - R^\varphi_{tj}\varphi^a_{kl}\varphi^a_i] + \frac{\alpha}{m-2}g^{tk}\varphi^a_{tk}(S^\varphi\varphi^a_i - g^{pq}R^\varphi_{pi}\varphi^a_q).
\end{aligned}\tag{1.36}$$

For the tensor $\mathcal{N}$, we use the φ-Schur identity to compute

$$\begin{aligned} g^{jk}\mathcal{N}_{ij,k} =& g^{jk}g^{tl}\Big[C^{\varphi}_{ijt,lk} - \alpha\varphi^a_{ij}(\varphi^a_{tlk} + R^{\varphi}_{kt}\varphi^a_l)\Big] \\ &- \alpha\varphi^a_i\Big(g^{jk}g^{tl}\varphi^a_{jktl} + \frac{1}{2}g^{tl}S^{\varphi}_t\varphi^a_l - g^{tk}g^{jl}\alpha\varphi^b_{tk}\varphi^b_j\varphi^a_l + g^{jk}g^{tl}R^{\varphi}_{jt}\varphi^a_{kl}\Big). \end{aligned} \tag{1.37}$$

From the Ricci commutation relations and the symmetries of C^{φ} we get

$$\begin{aligned} g^{jk}g^{tl}C^{\varphi}_{ijt,lk} &= \frac{1}{2}g^{jk}g^{tl}\big(C^{\varphi}_{ijt,lk} - C^{\varphi}_{itj,lk}\big) = \frac{1}{2}g^{jk}g^{tl}\big(C^{\varphi}_{ijt,lk} - C^{\varphi}_{ijt,kl}\big) \\ &= \frac{1}{2}g^{jk}g^{tl}g^{pq}\big(R_{pijt}C^{\varphi}_{qkl} + R_{pjtk}C^{\varphi}_{iql} + R_{pjkt}C^{\varphi}_{ilq}\big) \\ &= \frac{1}{2}g^{jk}g^{pq}\big(g^{tl}R_{pikj}C^{\varphi}_{qjk} + R_{pj}C^{\varphi}_{iqk} - R_{pj}C^{\varphi}_{ikq}\big) \\ &= \frac{1}{2}g^{jk}g^{tl}g^{pq}R_{pijt}C^{\varphi}_{qkl}. \end{aligned}$$

Using the definition of W^{φ} we deduce

$$\begin{aligned} g^{jk}g^{tl}C^{\varphi}_{ijt,lk} =& g^{jk}g^{tl}\left[\frac{1}{2}g^{pq}W^{\varphi}_{pijt}C^{\varphi}_{qkl} + \frac{1}{m-2}R^{\varphi}_{jt}C^{\varphi}_{kil} - \frac{\alpha}{m-2}R^{\varphi}_{ij}\varphi^a_{tl}\varphi^a_k\right] \\ &+ \frac{\alpha S^{\varphi}}{(m-1)(m-2)}\tau(\varphi)^a\varphi^a_i. \end{aligned}$$

Inserting this information into (1.37) we get

$$\begin{aligned} g^{jk}\mathcal{N}_{ij,k} =& g^{jk}g^{tl}\left[g^{pq}\frac{1}{2}W^{\varphi}_{pijt}C^{\varphi}_{qkl} + \frac{1}{m-2}R^{\varphi}_{jt}C^{\varphi}_{kil} - \frac{\alpha}{m-2}R^{\varphi}_{ij}\varphi^a_{tl}\varphi^a_k\right] \\ &+ \frac{\alpha S^{\varphi}}{(m-1)(m-2)}\tau(\varphi)^a\varphi^a_i - \alpha g^{jk}g^{tl}\varphi^a_{ij}(\varphi^a_{tlk} + R^{\varphi}_{kt}\varphi^a_l) \\ &- \alpha\varphi^a_i g^{tk}\Big(g^{jl}\varphi^a_{tkjl} + \frac{1}{2}S^{\varphi}_t\varphi^a_k - \alpha g^{jl}\varphi^b_{tk}\varphi^b_j\varphi^a_l + g^{jl}R^{\varphi}_{jt}\varphi^a_{lk}\Big). \end{aligned} \tag{1.38}$$

Putting together equations (1.35), (1.36) and (1.38) we obtain equation (1.34).

□

Similarly to the classical case (that is, when φ is constant), when (M, g) is a 4-dimensional Riemannian manifold, the φ-Bach tensor has a variational origin. Consider the action functional

$$\mathcal{S}_{\varphi}(g) := \int S_2(A^{\varphi}) - |\tau(\varphi)|^2. \tag{1.39}$$

The functional $\mathcal{S}_{\varphi}$ has been introduced in [5] where it is shown that, in Riemannian signature and for $m = 4$, the critical points of $\mathcal{S}_{\varphi}$ with respect to compactly supported variations of the metric have vanishing φ-Bach tensor. Interestingly, in contrast with the classical case, the functional

$$(g, \varphi) \to \int_M |W^{\varphi}|^2 \tag{1.40}$$

does not reproduce the same result, that is, its critical points are not φ-Bach flat, in general.

1.3 Some concepts from General Relativity

In this section, we review the concepts from General Relativity that will be useful in the rest of this thesis. From now on, $(\widehat{M}, \widehat{g})$ will be a fixed Lorentzian manifold, that is, a semi-Riemannian manifold of signature $(-,+,...,+)$. By contrast with the previous sections, its dimension will be denoted with the letter n, while $m = n-1$ will be the dimension of its hypersurfaces. In general, expressions with a hat will refer to quantities defined on $(\widehat{M}, \widehat{g})$, so that, for example, $\widehat{\mathrm{Ric}}$ and $\widehat{S}$ will be the Ricci tensor and the scalar curvature of $(\widehat{M}, \widehat{g})$, respectively. We already talked about the relevant physical background in the introduction; for the sake of readability, we recall Einstein Field Equations:

$$\widehat{\mathrm{Ric}} - \frac{1}{2}\widehat{S}\widehat{g} = \widehat{T}, \tag{1.41}$$

for a stress-energy tensor $\widehat{T}$, as well as the following

Definition 1.13. Given a Lorentzian manifold $(\widehat{M}, \widehat{g})$ we say that $(\widehat{M}, \widehat{g})$ is a *static space-time* if it satisfies the following two conditions:

1. as a differentiable manifold, $\widehat{M}$ splits as $\widehat{M} = M \times \mathbb{R}$;
2. there exists a Riemannian metric g on M and a function $f \in C^\infty(M)$ such that
$$\widehat{g} = -e^{-2\widehat{f}}\mathrm{d}t \otimes \mathrm{d}t + g$$
where $t : \widehat{M} \to \mathbb{R}$ is the canonical projection onto the second factor and $\widehat{f} := f \circ \widehat{\pi}_M$.

We will denote the static space-time structure on $\widehat{M}$ by the expression $\widehat{M} = M \times_f \mathbb{R}$. Again, for a static space-time, we will consider perfect fluid stress-energy tensors of the form

$$\widehat{T}^F = (\widehat{\mu} + \widehat{p})e^{-2\widehat{f}}\mathrm{d}t \otimes \mathrm{d}t + \widehat{p}\widehat{g} \tag{1.42}$$

where $\widehat{\mu} = \mu \circ \widehat{\pi}_M, \widehat{p} = p \circ \widehat{\pi}_M$ for some smooth functions $\mu, p \in C^\infty(M)$. The second type of stress-energy tensor is described by a smooth map $\widehat{\varphi} : (\widehat{M}, \widehat{g}) \to (N, h)$, where (N, h) is a Riemannian manifold, a constant $\alpha \in \mathbb{R}, \alpha \neq 0$, and a potential $U \in C^\infty(N)$. Its precise form is

$$\widehat{T}^{\widehat{\varphi}} := \alpha\widehat{\varphi}^*h - \Big(\frac{\alpha}{2}|\mathrm{d}\widehat{\varphi}|^2 + U(\widehat{\varphi})\Big)\widehat{g}. \tag{1.43}$$

To study a static space-time, we will assume $\widehat{\varphi} = \varphi \circ \widehat{\pi}_M$, with $\varphi : (M, g) \to (N, h)$ a smooth map. The stress energy tensor that we will consider will be a combination of $\widehat{T}^F$ and $\widehat{T}^{\widehat{\varphi}}$, i.e.

$$\widehat{T} = \widehat{T}^F + \widehat{T}^{\widehat{\varphi}}. \tag{1.44}$$

We will study how the presence of such a tensor influences the geometry of a static space-time, both in Einstein's theory (Chapter 3), and in the more recent theory called *Cotton Gravity* (Chapter 2). We now compute, using the moving frame method, how the Levi-Civita connection and the curvature tensor of a static space-time reduce to its spatial and temporal components. In what follows, $(\widehat{M}, \widehat{g})$ will be an n-dimensional static space-time $\widehat{M} = M \times_f \mathbb{R}$ for an m-dimensional Riemannian factor (M, g) and $f \in C^\infty(M)$. As before, we have

$$\widehat{g} = -e^{-2f} \mathrm{d}t \otimes \mathrm{d}t + g$$

where t is the standard coordinate on $\mathbb{R}$ and f is identified with its lift to $\widehat{M}$. We fix indexes conventions

$$0 \leq \alpha, \beta, \gamma, ... \leq m, \text{ and } \quad 1 \leq i, j, k, ... \leq m.$$

Given a local orthonormal co-frame $\{\theta^i\}_{i=1}^m$ on (M, g), we consider a local orthonormal co-frame $\{\widehat{\theta}^\alpha\}_{\alpha=0}^m$ by the prescription

$$\widehat{\theta}^0 := e^{-f} \mathrm{d}t, \quad \widehat{\theta}^i := \theta^i$$

where, by abuse of notation, we identify the θ^is with their pull-back $\widehat{\pi}_M^* \theta^i$ on $\widehat{M}$. We let $\{\widehat{\theta}^\alpha{}_\beta\}_{\alpha,\beta=0}^m, \{\widehat{\Theta}^\alpha{}_\beta\}_{\alpha,\beta=0}^m$ be the connection and curvature forms associated to the $\widehat{\theta}^\alpha$s, while $\{\theta^i{}_j\}_{i,j=1}^m, \{\Theta^i{}_j\}_{i,j=1}^m$ will be the connection and curvature forms of the $\theta^i s$. The components of the Riemann curvature tensors of $(\widehat{M}, \widehat{g})$ and (M, g) are therefore respectively given by the expressions

$$\widehat{\Theta}^\alpha{}_\beta = \frac{1}{2} \widehat{R}^\alpha{}_{\beta\gamma\eta} \widehat{\theta}^\gamma \wedge \widehat{\theta}^\eta$$

and

$$\Theta^i{}_j = \frac{1}{2} R^i{}_{jkt} \theta^k \wedge \theta^t.$$

Proposition 1.14. *In the above setting, we have that*

- *the Levi-Civita connection forms* $\{\widehat{\theta}^\alpha{}_\beta\}$ *satisfy*

$$\begin{cases} \widehat{\theta}^0{}_i &= \widehat{\theta}^i{}_0 = -f_i \widehat{\theta}^0 = -f_i e^{-f} dt, \\ \widehat{\theta}^i{}_j &= \theta^i{}_j, \\ \widehat{\theta}^0{}_0 &= 0; \end{cases} \tag{1.45}$$

- *the non-vanishing components of the Riemann curvature tensor* $\widehat{\mathrm{Riem}}$ *satisfy*

$$\begin{cases} \widehat{R}^i{}_{jkl} = R^i{}_{jkl}, \\ \widehat{R}^0{}_{j0k} = f_{jk} - f_j f_k; \end{cases} \tag{1.46}$$

- *the Ricci curvature* $\widehat{\mathrm{Ric}}$ *and the scalar curvature* $\widehat{S}$ *satisfy*

$$\begin{cases} \widehat{R}_{ij} &= f_{ij} - f_i f_j + R_{ij} = \widehat{R}_{ji}, \\ \widehat{R}_{00} &= -\Delta f + |\nabla f|^2, \\ \widehat{R}_{j0} &= 0 = \widehat{R}_{0j} \end{cases} \tag{1.47}$$

and

$$\widehat{S} = S + 2\Delta f - 2|\nabla f|^2, \tag{1.48}$$

where Δf *and* $|\nabla f|^2$ *are computed with respect to the metric* g.

Proof. The validity of the first structure equations for g, (1.2), together with $\mathrm{d}f = f_k\theta^k$ yields

$$\begin{aligned} \mathrm{d}\widehat{\theta}^i &= \mathrm{d}\theta^i = -\theta^i{}_j \wedge \theta^j = -\widehat{\theta}^i{}_\alpha \wedge \widehat{\theta}^\alpha, \\ \mathrm{d}\widehat{\theta}^0 &= -e^{-f}\mathrm{d}f \wedge \mathrm{d}t = -e^{-f} f_k \theta^k \wedge \mathrm{d}t = -f_k \widehat{\theta}^k \wedge \widehat{\theta}^0. \end{aligned}$$

Thus, defining

$$\begin{aligned} &\widehat{\theta}^0{}_i = \widehat{\theta}^i{}_0 = -f_i\widehat{\theta}^0 = -f_i e^{-f}\mathrm{d}t, \\ &\widehat{\theta}^i{}_j = \theta^i{}_j, \\ &\widehat{\theta}^0{}_0 = 0, \end{aligned} \tag{1.49}$$

we have that (1.2) are satisfied. This gives (1.45). It follows that

$$\begin{aligned} &\widehat{\Theta}^i{}_j = \Theta^i{}_j, \\ &\widehat{\Theta}^0{}_j = (f_{jk} - f_j f_k)\widehat{\theta}^0 \wedge \widehat{\theta}^k; \end{aligned}$$

as a consequence, we have

$$\frac{1}{2}\widehat{R}^i{}_{j\sigma\mu}\widehat{\theta}^\sigma \wedge \widehat{\theta}^\mu = \frac{1}{2}R^i{}_{jkl}\theta^k \wedge \theta^l \tag{1.50}$$

and

$$\frac{1}{2}\widehat{R}^0{}_{j\sigma\mu}\widehat{\theta}^\sigma \wedge \widehat{\theta}^\mu = -(f_{jk} - f_j f_k)\widehat{\theta}^k \wedge \widehat{\theta}^0.$$

We let

$$\widetilde{a}_{j\sigma} = \begin{cases} 0, & \text{if } \sigma = 0; \\ -(f_{jk} - f_j f_k), & \text{if } \sigma = k. \end{cases}$$

Then

$$\begin{aligned} \frac{1}{2}\widehat{R}^0{}_{j\sigma\mu}\widehat{\theta}^\sigma \wedge \widehat{\theta}^\mu &= \widetilde{a}_{j\sigma}\widehat{\theta}^\sigma \wedge \widehat{\theta}^0 \\ &= \widetilde{a}_{j\sigma}\delta_{0\mu}\widehat{\theta}^\sigma \wedge \widehat{\theta}^\mu \\ &= \frac{1}{2}(\widetilde{a}_{j\sigma}\delta_{0\mu} - \widetilde{a}_{j\mu}\delta_{0\sigma})\widehat{\theta}^\sigma \wedge \widehat{\theta}^\mu, \end{aligned}$$

so that

$$\widehat{R}^0{}_{j\sigma\mu} = \widetilde{a}_{j\sigma}\delta_{0\mu} - \widetilde{a}_{j\mu}\delta_{0\sigma}. \tag{1.51}$$

In particular, we have

$$\widehat{R}^0{}_{j0\mu} = \widetilde{a}_{j0}\delta_{0\mu} - \widetilde{a}_{j\mu} = -\widetilde{a}_{j\mu}$$

and

$$\widehat{R}^0{}_{j0k} = -\widetilde{a}_{jk} = f_{jk} - f_j f_k. \tag{1.52}$$

From (1.51) it also follows

$$0 = \widehat{R}^0{}_{jkl} \tag{1.53}$$

and, from the symmetries of $\widehat{\mathrm{Riem}}$, one also gets,

$$0 = \widehat{R}^i{}_{j0k}. \tag{1.54}$$

From (1.50) and (1.54) we obtain

$$\begin{aligned} R^i{}_{jkl}\theta^k \wedge \theta^l =& \widehat{R}^i{}_{j\alpha\beta}\widehat{\theta}^\alpha \wedge \widehat{\theta}^\beta \\ =& \widehat{R}^i{}_{jkl}\widehat{\theta}^k \wedge \widehat{\theta}^l \\ =& \widehat{R}^i{}_{jkl}\theta^k \wedge \theta^l \end{aligned}$$

so that

$$\widehat{R}^i{}_{jkl} = R^i{}_{jkl}. \tag{1.55}$$

As far as the Ricci tensor is concerned, from equations (1.52) and (1.55) we have

$$\begin{aligned} &\widehat{R}_{ij} = f_{ij} - f_i f_j + R_{ij} = \widehat{R}_{ji}, \\ &\widehat{R}_{00} = -\Delta f + |\nabla f|^2, \\ &\widehat{R}_{j0} = 0 = \widehat{R}_{0j}. \end{aligned}$$

Since

$$\widehat{\mathrm{Ric}} = \widehat{R}_{00}\widehat{\theta}^0 \otimes \widehat{\theta}^0 + \widehat{R}_{ij}\widehat{\theta}^i \otimes \widehat{\theta}^j,$$

we deduce

$$\widehat{\mathrm{Ric}} = \mathrm{Ric} + \mathrm{Hess}(f) - \mathrm{d}f \otimes \mathrm{d}f - \big(\Delta f - |\nabla f|^2\big)e^{-2f}\mathrm{d}t \otimes \mathrm{d}t. \tag{1.56}$$

Taking the $\widehat{g}$-trace of the above equation, we conclude. □

As we previously announced, we derive the equations of a static perfect fluid space-time from the field equations.

Proposition 1.15. *In the above setting, assume that $(\widehat{M}, \widehat{g})$ solves*

$$\widehat{\mathrm{Ric}} - \frac{1}{2}\widehat{S}\widehat{g} = \widehat{T}^F \tag{1.57}$$

for $\widehat{T}^F$ given by (1.42). Then, on (M, g), we have the validity of the system

$$\begin{cases} \mathrm{Ric} + \mathrm{Hess}\, f - \mathrm{d}f \otimes \mathrm{d}f - \frac{1}{m-1}\left(\frac{S}{2} - p\right)g = 0, \\ \Delta f - |\nabla f|^2 = -\frac{1}{m-1}\left[mp + \frac{m-2}{2}S\right], \\ \mu = \frac{1}{2}S, \\ \nabla p - (\mu + p)\nabla f = 0. \end{cases} \tag{1.58}$$

Proof. From (1.42) we get

$$\widehat{T}^F_{ij} = p\delta_{ij}, \quad \widehat{T}^F_{00} = \mu, \quad \widehat{T}^F_{0i} = 0. \tag{1.59}$$

Using (1.57), (1.47) and (1.48) we deduce

$$R_{ij} + f_{ij} - f_i f_j - \frac{1}{2}\left(S + 2\Delta f - 2|\nabla f|^2\right)\delta_{ij} = p\delta_{ij}$$

so that

$$R_{ij} + f_{ij} - f_i f_j = \left(\frac{S}{2} + \Delta f - |\nabla f|^2 + p\right)\delta_{ij}. \tag{1.60}$$

Taking the trace, we find

$$S + \Delta f - |\nabla f|^2 = m\left(\frac{S}{2} + \Delta f - |\nabla f|^2 + p\right).$$

Simplifying, we get

$$\Delta f - |\nabla f|^2 = -\frac{1}{m-1}\left(mp + \frac{m-2}{2}S\right),$$

which is (1.42).ii). Inserting the latter in (1.60), we obtain (1.58).i). To deduce (1.58).iii), use (1.57) to get

$$\widehat{R}_{00} + \frac{1}{2}\widehat{S} = \widehat{T}^F_{00};$$

from (1.42), (1.47) and (1.48) we have

$$-\Delta f + |\nabla f|^2 + \frac{1}{2}\left(S + 2\Delta f - 2|\nabla f|^2\right) = \mu$$

so that

$$\frac{S}{2} = \mu.$$

To obtain the last equation, we compute the components of the covariant derivative of $\widehat{T}^F$

$$\widehat{T}^F_{\alpha\beta,\gamma}\widehat{\theta}^\gamma = \mathrm{d}\widehat{T}^F_{\alpha\beta} - \widehat{T}^F_{\gamma\beta}\widehat{\theta}^\gamma{}_\alpha - \widehat{T}^F_{\alpha\gamma}\widehat{\theta}^\gamma{}_\beta.$$

Equations (1.59) and (1.45) imply

$$\begin{aligned}\widehat{T}^F_{ij,\alpha}\widehat{\theta}^\alpha =& \mathrm{d}\widehat{T}^F_{ij} - \widehat{T}^F_{\gamma j}\widehat{\theta}^\gamma{}_i - \widehat{T}^F_{\gamma i}\widehat{\theta}^\gamma{}_j\\ =& \delta_{ij}\widehat{p}_\alpha\widehat{\theta}^\alpha - \widehat{p}\widehat{\theta}^j{}_i - \widehat{p}\widehat{\theta}^i{}_j\\ =& \delta_{ij}\widehat{p}_\alpha\widehat{\theta}^\alpha,\end{aligned}$$

that is,

$$\widehat{T}^F_{ij,\alpha} = \widehat{p}_\alpha\delta_{ij}. \tag{1.61}$$

Using (1.59) we have

$$\begin{aligned}\widehat{T}^F_{00,\alpha} =& \mathrm{d}\widehat{T}^F_{00} - \widehat{T}^F_{\gamma 0}\widehat{\theta}^\gamma{}_0 - \widehat{T}^F_{0\gamma}\widehat{\theta}^\gamma{}_0\\ =& \widehat{\mu}_\alpha\widehat{\theta}^\alpha,\end{aligned}$$

that is,

$$\widehat{T}^F_{00,\alpha} = \widehat{\mu}_\alpha. \tag{1.62}$$

Using (1.59) and (1.45) we get

$$\begin{aligned}\widehat{T}^F_{0i,\alpha}\widehat{\theta}^\alpha =& \mathrm{d}\widehat{T}^F_{0i} - \widehat{T}^F_{\gamma i}\widehat{\theta}^\gamma{}_0 - \widehat{T}^F_{0\gamma}\widehat{\theta}^\gamma{}_i\\ =& 0 - \widehat{p}\widehat{\theta}^i{}_0 - \widehat{\mu}\widehat{\theta}^0{}_i\\ =& -(\widehat{\mu}+\widehat{p})\widehat{\theta}^i{}_0 = (\widehat{\mu}+\widehat{p})f_i\widehat{\theta}^0\end{aligned}$$

so that

$$\widehat{T}^F_{0i,j} = 0, \quad \widehat{T}_{0i,0} = (\widehat{\mu}+\widehat{p})f_i. \tag{1.63}$$

Now, we compute the divergence of $\widehat{T}^F$. This is

$$g^{\alpha\beta}\widehat{T^F}_{\gamma\alpha,\beta} = \widehat{T}^F_{0i,i} - \widehat{T}^F_{\gamma 0,0}.$$

Using (1.63), (1.62) and (1.61) we get

$$\left(\mathrm{div}_1\,\widehat{T}^F\right)_0 = \widehat{T}^F_{0i,i} - \widehat{T}^F_{00,0} = 0 - \widehat{\mu}_0 = 0$$

and

$$\left(\mathrm{div}_1\,\widehat{T}^F\right)_j = \widehat{T}^F_{ji,i} - \widehat{T}^F_{j0,0} = p_j - (\mu+p)f_j$$

so that

$$\mathrm{div}_1\,\widehat{T}^F = \widehat{\nabla}\widehat{p} - (\widehat{\mu}+\widehat{p})\widehat{\nabla}\widehat{f}. \tag{1.64}$$

From the Schur's identity, we know that the left hand side of (1.57) has vanishing divergence; therefore, the same holds for the right hand side and from (1.64) we obtain (1.58).*iv*). □

***Remark* 1.16.** Equation (1.58).iv) is equivalent to the requirement that $\widehat{T}^F$ be divergence-free. This has the important physical meaning of an equation of motion. However, its presence in (1.58) is slightly redundant: indeed, one can prove that its validity follows from the first three equations of (1.58) alone, see [13, Proposition 3.3]. Moreover, using (1.58).ii) to eliminate p from (1.58).i), we find

$$\mathrm{Ric} + \mathrm{Hess}\, f - \mathrm{d}f \otimes \mathrm{d}f = \frac{1}{m}\big(S + \Delta f - |\nabla f|^2\big)g$$

and, setting $\lambda := \frac{1}{m}\big(S + \Delta f - |\nabla f|^2\big)$, the latter becomes

$$\mathrm{Ric} + \mathrm{Hess}\, f - \mathrm{d}f \otimes \mathrm{d}f = \lambda g.$$

The above equation is a special case of what is called a *gradient Einstein-type structure*, that is,

$$\alpha \mathrm{Ric} + \beta\, \mathrm{Hess}\, f + \mu \mathrm{d}f \otimes \mathrm{d}f = (\rho S + \lambda)g, \quad \alpha, \beta, \mu, \rho \in \mathbb{R}, \lambda \in C^\infty(M)$$

these have been defined in [21]. One can interpret equation (1.58).ii) and (1.58).iii) as being *definitions* for the functions μ and p. In this sense, the system of static perfect fluid space-time only depends on the derivatives of the metric g and of the function f. As a consequence, static perfect fluid space-times are *precisely* the Einstein-type structures with $(\alpha, \beta, \mu, \rho) = (1, 1, -1, 0)$.

For what concerns the geometry of the map $\widehat{\varphi} : (\widehat{M}, \widehat{g}) \to (N, h)$ when $(\widehat{M}, \widehat{g})$ is a static space-time, we have the next proposition.

Proposition 1.17. *Let $\varphi : (M, g) \to (N, h)$ be a smooth map of Riemannian manifolds. For some $f \in C^\infty(M)$, consider the static space-time $M \times_f \mathbb{R}$ of metric*

$$\widehat{g} = -e^{-2f}\mathrm{d}t \otimes \mathrm{d}t + g.$$

Let $\widehat{\varphi} = \varphi \circ \widehat{\pi}_M$. Then, in the previous co-frames, we have

$$\begin{cases} \widehat{\varphi}^a_{ij} = \varphi^a_{ij}, \\ \widehat{\varphi}^a_{i0} = 0, \\ \widehat{\varphi}^a_{00} = \varphi^a_i f_i \end{cases} \tag{1.65}$$

so that

$$\tau(\widehat{\varphi}) = \tau(\varphi) - \mathrm{d}\varphi(\nabla f). \tag{1.66}$$

Proof. We recall that,

$$\widehat{\varphi}^a_{\mu\nu}\widehat{\theta}^\nu = \mathrm{d}\widehat{\varphi}^a_\mu - \widehat{\varphi}^a_\nu \widehat{\theta}^\nu{}_\mu + \widehat{\varphi}^b_\mu \omega^a{}_b; \tag{1.67}$$

therefore, since $\widehat{\varphi}^a_0 = 0$,

$$\widehat{\varphi}^a_{i\nu}\widehat{\theta}^\nu = \varphi^a_{ij}\widehat{\theta}^j = \varphi^a_{ij}\theta^j,$$

which implies
$$\widehat{\varphi}^a_{ij} = \varphi^a_{ij} \quad \text{and} \quad \widehat{\varphi}^a_{i0} = 0. \tag{1.68}$$
To compute the coefficients $\widehat{\varphi}^a_{00}$, note that, again from (1.67) and (1.45), we have
$$\widehat{\varphi}^a_{0\nu}\widehat{\theta}^\nu = \mathrm{d}\widehat{\varphi}^a_0 - \widehat{\varphi}^a_\nu\widehat{\theta}^\nu{}_0 + \widehat{\varphi}^b_0\omega^a{}_b = -\varphi^a_k\widehat{\theta}^k_0 = \varphi^a_k f_k\widehat{\theta}^0,$$
and thus
$$\widehat{\varphi}^a_{00} = f_k\varphi^a_k, \tag{1.69}$$
which immediately implies
$$\tau(\widehat{\varphi}) = \tau(\varphi) - \mathrm{d}\varphi(\nabla f). \tag{1.70}$$
□

Putting together Propositions 1.15 and 1.17, it is easy to prove the following

Proposition 1.18. *In the above setting, assume that* $(\widehat{M}, \widehat{g})$ *solves*
$$\widehat{\mathrm{Ric}} - \frac{1}{2}\widehat{S}\widehat{g} = \widehat{T}^F + \widehat{T}^{\widehat{\varphi}} \tag{1.71}$$
for $\widehat{T}^F$ *and* $\widehat{T}^{\widehat{\varphi}}$ *given by* (1.42) *and* (1.43)*, respectively. Assume that* $\widehat{T}^F$ *and* $\widehat{T}^{\widehat{\varphi}}$ *are divergence-free. Then, on* (M, g)*, we have the validity of the system*
$$\begin{cases} \mathrm{Ric}^\varphi + \mathrm{Hess}\, f - \mathrm{d}f \otimes \mathrm{d}f - \frac{1}{m-1}\left(\frac{S^\varphi}{2} - p + U(\varphi)\right)g = 0, \\ \Delta f - |\nabla f|^2 = -\frac{1}{m-1}\left[mp - mU(\varphi) + \frac{m-2}{2}S^\varphi\right], \\ 0 = h\big(\tau(\varphi) - \mathrm{d}\varphi(\nabla f) - \frac{1}{\alpha}\nabla U(\varphi), \mathrm{d}\varphi\big), \\ \mu + U(\varphi) = \frac{1}{2}S, \\ \nabla p - (\mu + p)\nabla f = 0. \end{cases} \tag{1.72}$$

Chapter 2

Cotton-φ-Perfect Fluids

Following [51] we define

$$\widehat{\mathfrak{T}} := \frac{1}{n-2}\left(\widehat{T} \owedge \widehat{g} - \frac{1}{2(n-1)}(\mathrm{tr}_{\widehat{g}}\widehat{T})\widehat{g} \owedge \widehat{g}\right).$$

Then, the equations of Cotton gravity are

$$\widehat{C} = -(n-2)\,\mathrm{div}_1\,\widehat{\mathfrak{T}}\,. \tag{2.1}$$

We compute $\mathrm{div}_1\,\widehat{\mathfrak{T}}$ explicitly. By definition,

$$\widehat{\mathfrak{T}}_{\alpha\beta\gamma\delta} = \frac{1}{n-2}\left[\widehat{T}_{\alpha\gamma}g_{\beta\delta} + \widehat{T}_{\beta\delta}g_{\alpha\gamma} - \widehat{T}_{\alpha\delta}g_{\beta\gamma} - \widehat{T}_{\beta\gamma}g_{\alpha\delta} - \frac{1}{n-1}\widehat{T}^{\eta}{}_{\eta}(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma})\right]$$

so that

$$\begin{aligned}\mathrm{div}_1\,\widehat{\mathfrak{T}}_{\beta\gamma\delta} =& \frac{1}{n-2}g^{\alpha\rho}\left[\widehat{T}_{\alpha\gamma,\rho}g_{\beta\delta} + \widehat{T}_{\beta\delta,\rho}g_{\alpha\gamma} - \widehat{T}_{\alpha\delta,\rho}g_{\beta\gamma} - \widehat{T}_{\beta\gamma,\rho}g_{\alpha\delta} - \frac{1}{n-1}\widehat{T}^{\eta}{}_{\eta,\rho}(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma})\right]\\ =& \frac{1}{n-2}\Big[(\mathrm{div}_1\,\widehat{T})_{\gamma}g_{\beta\delta} - (\mathrm{div}_1\,\widehat{T})_{\delta}g_{\beta\gamma} + \widehat{T}_{\beta\delta,\gamma} - \widehat{T}_{\beta\gamma,\delta}\\ &- \frac{1}{n-1}(\widehat{T}^{\eta}{}_{\eta,\gamma}g_{\beta\delta} - \widehat{T}^{\eta}{}_{\eta,\delta}g_{\beta\gamma})\Big].\end{aligned} \tag{2.2}$$

Using (2.2) we can show that (2.1) implies $\mathrm{div}_1\,\widehat{T} = 0$. Indeed, substituting (2.2) into (2.1) we get

$$\widehat{C}_{\beta\gamma\delta} = (\mathrm{div}_1\,\widehat{T})_{\delta}g_{\beta\gamma} - (\mathrm{div}_1\,\widehat{T})_{\gamma}g_{\beta\delta} + \widehat{T}_{\beta\gamma,\delta} - \widehat{T}_{\beta\delta,\gamma} + \frac{1}{n-1}(\widehat{T}^{\eta}{}_{\eta,\gamma}g_{\beta\delta} - \widehat{T}^{\eta}{}_{\eta,\delta}g_{\beta\gamma})$$

and then contracting both sides of the resulting equality with $g^{\beta\gamma}$ we get

$$g^{\beta\gamma}\widehat{C}_{\beta\gamma\delta} = (n-2)(\mathrm{div}_1\,\widehat{T})_{\delta}\,.$$

Since $\widehat{C}$ is totally trace-free, we deduce $\mathrm{div}_1\,\widehat{T} = 0$. In view of this, the previous identity simplifies to

$$\widehat{C}_{\beta\gamma\delta} = \widehat{T}_{\beta\gamma,\delta} - \widehat{T}_{\beta\delta,\gamma} + \frac{1}{n-1}(\widehat{T}^{\eta}{}_{\eta,\gamma}g_{\beta\delta} - \widehat{T}^{\eta}{}_{\eta,\delta}g_{\beta\gamma})\,. \tag{2.3}$$

Using equation (2.3) it is also easy to show how the solutions of Einstein equations, (1.41), also solve (2.1). Indeed, tracing (1.41) one obtains $(n-2)\widehat{S} = -2\mathrm{tr}_{\widehat{g}}\widehat{T}$ and therefore

$$\begin{aligned}\widehat{\mathrm{Ric}} &= \widehat{T} + \frac{1}{2}\widehat{S}\widehat{g} = \widehat{T} - \frac{1}{n-2}(\mathrm{tr}_{\widehat{g}}\widehat{T})\widehat{g}\,,\\ \widehat{A} &= \widehat{\mathrm{Ric}} - \frac{1}{2(n-1)}\widehat{S}\widehat{g} = \widehat{T} - \frac{1}{n-1}(\mathrm{tr}_{\widehat{g}}\widehat{T})\widehat{g}\,,\end{aligned}$$

yielding

$$\widehat{C}_{\beta\gamma\delta} = \widehat{A}_{\beta\gamma,\delta} - \widehat{A}_{\beta\delta,\gamma} = \widehat{T}_{\beta\gamma,\delta} - \widehat{T}_{\beta\delta,\gamma} + \frac{1}{n-1}(\widehat{T}^{\eta}{}_{\eta,\gamma}g_{\beta\delta} - \widehat{T}^{\eta}{}_{\eta,\delta}g_{\beta\gamma}).$$

Our goal is to study, on a static space-time, the solutions of (2.1) for a stress-energy tensor that encodes both the informations of a perfect fluid and those of a nonlinear field $\widehat{\varphi}$ with a self-interacting potential U, that is, we have

$$\widehat{T} = \widehat{T}^{\widehat{\varphi}} + \widehat{T}^{F}$$

for tensors $\widehat{T}^{\widehat{\varphi}}$, $\widehat{T}^{F}$ defined by (1.43) and (1.42), respectively.

2.1 Variational origin of the equations

As it is well-known, the stress energy tensor $\widehat{T}^{F}$ of a perfect fluid space-time is not derived from a Lagrangian so that it does not have a (clear) variational origin. A derivation of $\widehat{T}^{F}$ which is variational, but only in a relaxed sense, is given in [54, Chapter 3], where one varies an appropriate functional depending on μ with respect to the flow lines of the velocity vector field. By contrast, the situation is simpler for $\widehat{T}^{\widehat{\varphi}}$, since the equation

$$\widehat{\mathrm{Ric}} - \frac{1}{2}\widehat{S}\widehat{g} = \widehat{T}^{\widehat{\varphi}}$$

is the Euler-Lagrange equation of the functional

$$I[g] = \int \widehat{S} - \alpha|\mathrm{d}\widehat{\varphi}|^2 - 2U(\widehat{\varphi})\,.$$

Similarly, in the regime of Cotton gravity, the system

$$\widehat{C} = -(n-2)\,\mathrm{div}_1\,\widehat{\mathfrak{T}}^{\widehat{\varphi}} \tag{2.4}$$

has a variational origin. We need to set some notations first. Note that, in this section, we don't assume that $(\widehat{M},\widehat{g})$ be a static space-time. By equation (2.3), (2.4) implies

$$\widehat{C}_{\beta\gamma\delta} = -\left(\widehat{T}^{\widehat{\varphi}}_{\beta\delta,\gamma} - \widehat{T}^{\widehat{\varphi}}_{\beta\gamma,\delta} - \frac{1}{n-1}(\mathrm{tr}_{\widehat{g}}\widehat{T}^{\widehat{\varphi}})_{\gamma}g_{\beta\delta} + \frac{1}{n-1}(\mathrm{tr}_{\widehat{g}}\widehat{T}^{\widehat{\varphi}})_{\delta}g_{\beta\gamma}\right). \tag{2.5}$$

From the definition of $\widehat{T}^{\widehat{\varphi}}$ we get

$$\widehat{T}^{\widehat{\varphi}}_{\beta\delta,\gamma} = \alpha\left[\widehat{\varphi}^a_{\beta\gamma}\widehat{\varphi}^a_{\delta} + \widehat{\varphi}^a_{\beta}\widehat{\varphi}^a_{\delta\gamma} - \left(g^{\rho\eta}\widehat{\varphi}^a_{\rho\gamma}\widehat{\varphi}^a_{\eta} + \frac{1}{\alpha}U^a\widehat{\varphi}^a_{\gamma}\right)g_{\beta\delta}\right]$$

and

$$\mathrm{tr}_{\widehat{g}}\widehat{T}^{\widehat{\varphi}} = -\frac{n-2}{2}\alpha|\mathrm{d}\widehat{\varphi}|^2 - nU(\widehat{\varphi})$$

so that

$$(\mathrm{tr}_{\widehat{g}}\widehat{T}^{\widehat{\varphi}})_{\gamma} = -\alpha(n-2)g^{\rho\eta}\widehat{\varphi}^a_{\rho\gamma}\widehat{\varphi}^a_{\eta} - nU^a\widehat{\varphi}^a_{\gamma}\,.$$

Defining

$$\begin{aligned}(\widehat{C}^{U,\widehat{\varphi}})_{\beta\gamma\delta} :={}& \widehat{C}_{\beta\gamma\delta} - \alpha\left(\widehat{\varphi}^a_{\beta\delta}\widehat{\varphi}^a_{\gamma} - \widehat{\varphi}^a_{\beta\gamma}\widehat{\varphi}^a_{\delta}\right) + \frac{\alpha}{n-1}g^{\rho\eta}\widehat{\varphi}^a_{\rho}\left(\widehat{\varphi}^a_{\eta\delta}g_{\beta\gamma} - \widehat{\varphi}^a_{\eta\gamma}g_{\beta\delta}\right)\\ &- \frac{1}{n-1}U^a\widehat{\varphi}^a_{\delta}g_{\beta\gamma} + \frac{1}{n-1}U^a\widehat{\varphi}^a_{\gamma}g_{\beta\delta}\end{aligned}$$

we have that (2.5) is equivalent to

$$\widehat{C}^{U,\widehat{\varphi}} = 0\,. \tag{2.6}$$

Note that, setting

$$\widehat{A}^{U,\widehat{\varphi}} := \widehat{A} - \alpha\widehat{\varphi}^*h + \frac{\alpha}{2(n-1)}|\mathrm{d}\widehat{\varphi}|^2\widehat{g} - \frac{U(\widehat{\varphi})}{n-1}\widehat{g}, \tag{2.7}$$

then $\widehat{C}^{U,\widehat{\varphi}}$ measures the obstruction to $\widehat{A}^{U,\widehat{\varphi}}$ being a Codazzi tensor. Moreover, the conservation equation

$$0 = \mathrm{div}_1\,\widehat{T}^{\widehat{\varphi}}\,,$$

that is,

$$0 = \alpha g^{\gamma\eta}\widehat{\varphi}^a_{\gamma\eta}\widehat{\varphi}^a_{\beta} - U^a\widehat{\varphi}^a_{\beta},$$

is always implied by (2.5), as we already saw, and therefore also by (2.6), as it can be verified with a direct computation. Indeed, one has

$$g^{\beta\gamma}(\widehat{C}^{U,\widehat{\varphi}})_{\beta\gamma\delta} = \alpha g^{\beta\gamma}\widehat{\varphi}^a_{\beta\gamma}\widehat{\varphi}^a_{\delta} - U^a\widehat{\varphi}^a_{\delta}\,. \tag{2.8}$$

Now, to derive (2.6), following [51], we are interested in variations where the connection $\widehat{\nabla}$ is varying while the metric $\widehat{g}$ is fixed. To be more precise, recall that the difference of two connections on $T\widehat{M}$ is a tensor, in particular, a section of $T^{(1,2)}\widehat{M}$, the space of $(1,2)$-tensors

$$B = B^{\alpha}{}_{\beta\gamma}e_{\alpha}\otimes\theta^{\beta}\otimes\theta^{\gamma}.$$

Therefore, the space of connections on $T\widehat{M}$ is an affine space over $T^{(1,2)}\widehat{M}$. Having fixed a section B of $T^{(1,2)}\widehat{M}$, we define $\widehat{\nabla}^t = \widehat{\nabla} + tB$. In any arbitrary coordinate system $(U,\{x^{\alpha}\}_{\alpha=0}^{n-1})$ we have that the variation of the Christoffel symbols of $\widehat{\nabla}$ in the direction of B is

$$\delta\Gamma^{\alpha}{}_{\beta\gamma}\frac{\partial}{\partial x^{\alpha}} = \lim_{t\to 0}\frac{1}{t}\left(\widehat{\nabla}^t_{\partial x^{\beta}}\frac{\partial}{\partial x^{\alpha}} - \widehat{\nabla}_{\partial x^{\beta}}\frac{\partial}{\partial x^{\alpha}}\right) = B^{\alpha}{}_{\beta\gamma}\frac{\partial}{\partial x^{\alpha}}\,.$$

To derive (2.4), we modify the functional (1.39) of the Introduction in order to take into account the contribution of the potential U. From (2.7) we have

$$\widehat{A}^{U,\widehat{\varphi}} = \widehat{A}^{\widehat{\varphi}} - \frac{U(\widehat{\varphi})}{n-1}\widehat{g}\,.$$

We will study the functional

$$\mathcal{S}_{U,\widehat{\varphi}}(\widehat{g}) = \int S_2(\widehat{A}^{U,\widehat{\varphi}}) - \frac{\alpha}{2}\left|\tau(\widehat{\varphi}) - \frac{\nabla U}{\alpha}(\widehat{\varphi})\right|^2_h . \tag{2.9}$$

Proposition 2.1. *Let $(\widehat{M},\widehat{g})$ be an n-dimensional Lorentzian manifold. Then, equation (2.4) is satisfied by $(\widehat{M},\widehat{g})$ if and only if $(\widehat{M},\widehat{g})$ is a critical point of $\mathcal{S}_{U,\widehat{\varphi}}$ with respect to compactly supported variations of $\widehat{\nabla}$ that leave $\widehat{g}$ fixed.*

Proof. We know from (1.1) that the Riemann curvature tensor can be expressed using the Christoffel symbols by

$$\widehat{R}^{\alpha}{}_{\beta\gamma\rho} = \partial_\gamma \Gamma^{\alpha}{}_{\beta\rho} - \partial_\rho \Gamma^{\alpha}{}_{\beta\gamma} + \Gamma^{\eta}{}_{\beta\rho}\Gamma^{\alpha}{}_{\gamma\eta} - \Gamma^{\eta}{}_{\beta\gamma}\Gamma^{\alpha}{}_{\rho\eta}$$

so that its variation is

$$\delta\widehat{R}^{\alpha}{}_{\beta\gamma\rho} = \partial_\gamma B^{\alpha}{}_{\beta\rho} - \partial_\rho B^{\alpha}{}_{\beta\gamma} + B^{\eta}{}_{\beta\rho}\Gamma^{\alpha}{}_{\gamma\eta} + \Gamma^{\eta}{}_{\beta\rho}B^{\alpha}{}_{\gamma\eta} - B^{\eta}{}_{\beta\gamma}\Gamma^{\alpha}{}_{\rho\eta} - \Gamma^{\eta}{}_{\beta\gamma}B^{\alpha}{}_{\rho\eta}$$

where $B^{\alpha}{}_{\beta\gamma} = \delta\Gamma^{\alpha}{}_{\beta\gamma}$ is the variation of the Christoffel symbols. Recalling the expression for the covariant derivative of B

$$B^{\alpha}{}_{\beta\rho,\gamma} = \partial_\gamma B^{\alpha}{}_{\beta\rho} + B^{\eta}{}_{\beta\rho}\Gamma^{\alpha}{}_{\eta\gamma} - B^{\alpha}{}_{\eta\rho}\Gamma^{\eta}{}_{\beta\gamma} - B^{\alpha}{}_{\beta\eta}\Gamma^{\eta}{}_{\rho\gamma}$$

this yields

$$\delta\widehat{R}^{\alpha}{}_{\beta\gamma\rho} = B^{\alpha}{}_{\beta\rho,\gamma} - B^{\alpha}{}_{\beta\gamma,\rho}\,.$$

Lowering an index and skew-symmetrizing we get

$$\delta\widehat{R}_{\alpha\beta\gamma\rho} = \frac{1}{2}(B_{\alpha\beta\rho,\gamma} - B_{\beta\alpha\rho,\gamma} - B_{\alpha\beta\gamma,\rho} + B_{\beta\alpha\gamma,\rho}).$$

Therefore, the variation of $\widehat{\mathrm{Ric}}$ is

$$\delta\widehat{R}_{\alpha\beta} = \frac{1}{2}(B_{\alpha}{}^{\rho}{}_{\rho,\beta} - B^{\rho}{}_{\alpha\rho,\beta} - B_{\alpha}{}^{\rho}{}_{\beta,\rho} + B^{\rho}{}_{\alpha\beta,\rho}) \tag{2.10}$$

and that of $\widehat{S}$ is

$$\delta\widehat{S} = \frac{1}{2}(B^{\alpha\beta}{}_{\beta,\alpha} - B^{\alpha\beta}{}_{\alpha,\beta} - B^{\alpha\beta}{}_{\alpha,\beta} + B^{\alpha\beta}{}_{\beta,\alpha}) = B^{\alpha\beta}{}_{\beta,\alpha} - B^{\alpha\beta}{}_{\alpha,\beta}\,. \tag{2.11}$$

Next, note that, since $\widehat{\varphi}^*h$, $|\mathrm{d}\widehat{\varphi}|^2$ and $U(\widehat{\varphi})$ are independent from $\widehat{\nabla}$, their variation is zero, so that

$$\delta\widehat{A}^{U,\widehat{\varphi}} = \delta\widehat{A}\,. \tag{2.12}$$

By the definition of the tension field $\tau(\widehat{\varphi})$ one immediately gets

$$\delta\tau(\widehat{\varphi})^a = -g^{\alpha\beta}B^{\gamma}{}_{\alpha\beta}\frac{\partial\widehat{\varphi}^a}{\partial x^\gamma} = -B^{\alpha\beta}{}_{\beta}\frac{\partial\widehat{\varphi}^a}{\partial x^\alpha} \tag{2.13}$$

and therefore

$$\delta\left|\tau(\widehat{\varphi}) - \frac{\nabla U}{\alpha}\right|_h^2 = 2\left(\delta\tau(\widehat{\varphi})^a - \frac{1}{\alpha}\delta U^a\right)\left(\tau(\widehat{\varphi})^a - \frac{1}{\alpha}U^a\right) = -2B^{\alpha\beta}{}_{\beta}\frac{\partial\widehat{\varphi}^a}{\partial x^\alpha}\left(\tau(\widehat{\varphi})^a - \frac{1}{\alpha}U^a\right). \tag{2.14}$$

By the definition of $\widehat{A}^{U,\widehat{\varphi}}$ we compute

$$\mathrm{tr}_{\widehat{g}}\widehat{A}^{U,\widehat{\varphi}} = \mathrm{tr}_{\widehat{g}}\widehat{A}^{\widehat{\varphi}} - \frac{n}{n-1}U(\widehat{\varphi}) = \frac{n-2}{2(n-1)}\widehat{S}^{\widehat{\varphi}} - \frac{n}{n-1}U(\widehat{\varphi})\,. \tag{2.15}$$

Using (2.15) and (2.12) we get

$$\begin{aligned}
\delta S_2(\widehat{A}^{U,\widehat{\varphi}}) &= (\mathrm{tr}_{\widehat{g}}\widehat{A}^{U,\widehat{\varphi}})\delta(\mathrm{tr}_{\widehat{g}}\widehat{A}^{U,\widehat{\varphi}}) - (\widehat{A}^{U,\widehat{\varphi}})^{\beta\gamma}(\delta\widehat{A}^{U,\widehat{\varphi}})_{\beta\gamma} \\
&= (\mathrm{tr}_{\widehat{g}}\widehat{A}^{U,\widehat{\varphi}})\delta(\mathrm{tr}_{\widehat{g}}\widehat{A}) - (\widehat{A}^{U,\widehat{\varphi}})^{\beta\gamma}(\delta\widehat{A})_{\beta\gamma} \\
&= \frac{n-2}{2(n-1)}(\mathrm{tr}_{\widehat{g}}\widehat{A}^{U,\widehat{\varphi}})\delta\widehat{S} - (\widehat{A}^{U,\widehat{\varphi}})^{\beta\gamma}(\delta\widehat{\mathrm{Ric}})_{\beta\gamma} + \frac{1}{2(n-1)}(\widehat{A}^{U,\widehat{\varphi}})^{\beta\gamma}g_{\beta\gamma}\delta\widehat{S} \\
&= \frac{1}{2}(\mathrm{tr}_{\widehat{g}}\widehat{A}^{U,\widehat{\varphi}})\delta\widehat{S} - (\widehat{A}^{U,\widehat{\varphi}})^{\beta\gamma}\delta\widehat{R}_{\beta\gamma}\,.
\end{aligned}$$

Using (2.10) and (2.11) we get

$$\delta S_2(\widehat{A}^{U,\widehat{\varphi}}) = \frac{1}{2}(\mathrm{tr}_{\widehat{g}}\widehat{A}^{U,\widehat{\varphi}})(B^{\alpha\beta}{}_{\beta,\alpha} - B^{\alpha\beta}{}_{\alpha,\beta}) - \frac{1}{2}(\widehat{A}^{U,\widehat{\varphi}})^{\alpha\beta}(B_{\alpha}{}^{\rho}{}_{\rho,\beta} - B^{\rho}{}_{\alpha\rho,\beta} - B_{\alpha}{}^{\rho}{}_{\beta,\rho} + B^{\rho}{}_{\alpha\beta,\rho})\,.$$

Integrating and using the divergence theorem we get

$$\begin{aligned}
\int \delta S_2(\widehat{A}^{U,\widehat{\varphi}}) &= \int \frac{1}{2}(\mathrm{tr}_{\widehat{g}}\widehat{A}^{U,\widehat{\varphi}})_{\beta}B^{\alpha\beta}{}_{\alpha} - \frac{1}{2}(\mathrm{tr}_{\widehat{g}}\widehat{A}^{U,\widehat{\varphi}})_{\alpha}B^{\alpha\beta}{}_{\beta} \\
&\quad + \frac{1}{2}\int(\widehat{A}^{U,\widehat{\varphi}})_{\alpha}{}^{\beta}{}_{,\beta}(B^{\alpha\rho}{}_{\rho} - B^{\rho\alpha}{}_{\rho}) - \frac{1}{2}\int(\widehat{A}^{U,\widehat{\varphi}})^{\alpha\beta}{}_{,\rho}(B_{\alpha}{}^{\rho}{}_{\beta} - B^{\rho}{}_{\alpha\beta})\,.
\end{aligned}$$

Using the $\widehat{\varphi}$-Schur's identity and (2.15) we get

$$\begin{aligned}
(\widehat{A}^{U,\widehat{\varphi}})_{\alpha}{}^{\beta}{}_{,\beta} &= (\widehat{R}^{\widehat{\varphi}})_{\alpha}{}^{\beta}{}_{,\beta} - \frac{1}{2(n-1)}\widehat{S}^{\widehat{\varphi}}_{\alpha} - \frac{1}{n-1}U^a\widehat{\varphi}^a_{\alpha} \\
&= \frac{n-2}{2(n-1)}\widehat{S}^{\widehat{\varphi}}_{\alpha} - \alpha\tau(\widehat{\varphi})^a\widehat{\varphi}^a_{\alpha} - \frac{1}{n-1}U^a\widehat{\varphi}^a_{\alpha} \\
&= (\mathrm{tr}_{\widehat{g}}\widehat{A}^{U,\widehat{\varphi}})_{\alpha} - \alpha\left(\tau(\widehat{\varphi})^a - \frac{1}{\alpha}U^a\right)\widehat{\varphi}^a_{\alpha}
\end{aligned}$$

and substituting the latter into the above integral identity we obtain

$$\begin{aligned}
\int \delta S_2(\widehat{A}^{U,\widehat{\varphi}}) =& \frac{1}{2}\int -\left(\widehat{A}^{U,\widehat{\varphi}}\right)^{\alpha\beta}{}_{,\rho}(B_{\alpha}{}^{\rho}{}_{\beta} - B^{\rho}{}_{\alpha\beta}) \\
& - \alpha\left(\tau(\widehat{\varphi})^a - \frac{1}{\alpha}U^a\right)\widehat{\varphi}^a_{\alpha}(B^{\alpha\rho}{}_{\rho} - B^{\rho\alpha}{}_{\rho}).
\end{aligned}$$

Renaming indexes we get

$$\begin{aligned}\int \delta S_2(\widehat{A}^{U,\widehat{\varphi}}) =& \frac{1}{2}\int \left(\left(\widehat{A}^{U,\widehat{\varphi}}\right)_{\beta\alpha,\rho} - \left(\widehat{A}^{U,\widehat{\varphi}}\right)_{\beta\rho,\alpha}\right) B^{\rho\alpha\beta} \\ &+ \alpha\left[\left(\tau(\widehat{\varphi})^a - \frac{1}{\alpha}U^a\right)\left(\widehat{\varphi}^a_\alpha g_{\beta\rho} - \widehat{\varphi}^a_\rho g_{\beta\alpha}\right)\right] B^{\rho\alpha\beta}.\end{aligned}$$

Recalling that

$$\widehat{C}^{U,\widehat{\varphi}}_{\beta\alpha\rho} = \widehat{A}^{U,\widehat{\varphi}}_{\beta\alpha,\rho} - \widehat{A}^{U,\widehat{\varphi}}_{\beta\rho,\alpha}$$

we get

$$\int \delta S_2(\widehat{A}^{U,\widehat{\varphi}}) = \frac{1}{2}\int \left[\widehat{C}^{U,\widehat{\varphi}}_{\beta\alpha\rho} + \alpha\left(\tau(\widehat{\varphi})^a - \frac{1}{\alpha}U^a\right)\left(\widehat{\varphi}^a_\alpha g_{\beta\rho} - \widehat{\varphi}^a_\rho g_{\alpha\beta}\right)\right] B^{\rho\alpha\beta}. \tag{2.16}$$

Therefore, computing the variation of (2.9) and using (2.16) and (2.14) we find

$$\begin{aligned}\delta \mathcal{S}_{U,\widehat{\varphi}} =& \int \delta S_2(\widehat{A}^{U,\widehat{\varphi}}) - \frac{\alpha}{2}\delta\left(\left|\tau(\widehat{\varphi}) - \frac{1}{\alpha}\nabla U\right|^2_h\right) \\ =& \int \frac{1}{2}\left[\widehat{C}^{U,\widehat{\varphi}}_{\beta\alpha\rho} + \alpha\left(\tau(\widehat{\varphi})^a - \frac{1}{\alpha}U^a\right)\left(\widehat{\varphi}^a_\alpha g_{\beta\rho} - \widehat{\varphi}^a_\rho g_{\alpha\beta}\right)\right] B^{\rho\alpha\beta} \\ &+ \alpha\left(\tau(\widehat{\varphi})^a - \frac{1}{\alpha}U^a\right)\widehat{\varphi}^a_\rho g_{\alpha\beta} B^{\rho\alpha\beta} \\ =& \frac{1}{2}\int \left[\widehat{C}^{U,\widehat{\varphi}}_{\beta\alpha\rho} + \alpha\left(\tau(\widehat{\varphi})^a - \frac{1}{\alpha}U^a\right)\left(\widehat{\varphi}^a_\alpha g_{\beta\rho} + \widehat{\varphi}^a_\rho g_{\alpha\beta}\right)\right] B^{\rho\alpha\beta}.\end{aligned}$$

Therefore, the compactly supported variations of $\mathcal{S}_{U,\widehat{\varphi}}$ with respect to $\widehat{\nabla}$ satisfy

$$0 = \widehat{C}^{U,\widehat{\varphi}}_{\rho\beta\alpha} + \alpha\left(\tau(\widehat{\varphi})^a - \frac{1}{\alpha}U^a\right)\left(\widehat{\varphi}^a_\alpha g_{\beta\rho} + \widehat{\varphi}^a_\beta g_{\alpha\rho}\right). \tag{2.17}$$

Contracting (2.17) with respect to $g^{\rho\beta}$ and using (2.8) we deduce

$$0 = \alpha\left(\tau(\widehat{\varphi})^a - \frac{1}{\alpha}U^a\right)\widehat{\varphi}^a_\alpha + \alpha(n+1)\left(\tau(\widehat{\varphi})^a - \frac{1}{\alpha}U^a\right)\widehat{\varphi}^a_\alpha$$

which implies

$$\tau(\widehat{\varphi})^a\widehat{\varphi}^a_\alpha - \frac{1}{\alpha}U^a\widehat{\varphi}^a_\alpha = 0$$

so that (2.17) implies

$$0 = \widehat{C}^{U,\widehat{\varphi}}$$

which is (2.6). As we already discussed, this is equivalent to (2.5) and we are done. $\square$

2.2 Deduction of the system

In this section, we show how the equations of Cotton gravity, for the stress energy tensor $\widehat{T} = \widehat{T}^F + \widehat{T}^{\widehat{\varphi}}$, reduce on the components of a static space-time. Motivated by physical reasons (see Remark 2.3), we will split the natural condition $0 = \operatorname{div}_1 \widehat{T}$ into the two conditions $0 = \operatorname{div}_1 \widehat{T}^F = \operatorname{div}_1 \widehat{T}^{\widehat{\varphi}}$.

Theorem 2.2. *Let $\widehat{M} = M \times_f \mathbb{R}$ be the static space-time with metric $\widehat{g}$ given by*

$$\widehat{g} = -e^{-2\widehat{f}}\mathrm{d}t^2 + g$$

with $f \in C^\infty(M)$, $\widehat{f} := f \circ \widehat{\pi}_M$, (M, g) an m-dimensional Riemannian manifold. For $\mu, p \in C^\infty(M)$, $\varphi : (M, g) \to (N, h)$ smooth, define

$$\widehat{\mu} = \mu \circ \widehat{\pi}_M, \qquad \widehat{p} = p \circ \widehat{\pi}_M, \qquad \widehat{\varphi} = \varphi \circ \widehat{\pi}_M.$$

Define the stress-energy tensor $\widehat{T}$ by

$$\widehat{T} = \widehat{T}^{\widehat{\varphi}} + \widehat{T}^F$$

with

$$\begin{aligned}
\widehat{T}^{\widehat{\varphi}} &= \alpha\, \widehat{\varphi}^* h - \left[U(\widehat{\varphi}) + \frac{\alpha}{2}|\mathrm{d}\widehat{\varphi}|^2\right]\widehat{g} \\
\widehat{T}^F &= (\widehat{\mu} + \widehat{p})e^{-2\widehat{f}}\mathrm{d}t^2 + \widehat{p}\widehat{g}\,,
\end{aligned}$$

for some $\alpha \in \mathbb{R} \setminus \{0\}$, and assume that both $\widehat{T}^{\widehat{\varphi}}$ and $\widehat{T}^F$ are divergence free. In this setting, Harada's field equations yield, on M,

$$\begin{cases}
0 = & C^{\varphi}_{ijk} + f_{ijk} - f_{ikj} - f_{ik}f_j + f_{ij}f_k \\
& -\frac{1}{m}[(f_{ttk} - 2f_t f_{tk})\delta_{ij} - (f_{ttj} - 2f_t f_{tj})\delta_{ik}] \\
& +\frac{1}{2m(m-1)}\left(S^{\varphi}_k\delta_{ij} - S^{\varphi}_j\delta_{ik}\right) \\
& +\frac{1}{m}U^a\varphi^a_j\delta_{ik} - \frac{1}{m}U^a\varphi^a_k\delta_{ij} + \frac{1}{m}(\mu_j\delta_{ik} - \mu_k\delta_{ij}), \\
0 = & -\frac{m-1}{m}f_{tti} + \frac{m-2}{m}f_t f_{ti} + (\Delta f)f_i + \frac{1}{2m}S^{\varphi}_i - f_t R^{\varphi}_{it} \\
& +\frac{1}{m}U^a\varphi^a_i - \frac{m-1}{m}\mu_i, \\
0 = & \nabla p - (\mu + p)\nabla f, \\
0 = & h\big(\tau(\varphi) - \mathrm{d}\varphi(\nabla f) - \frac{1}{\alpha}\nabla U(\varphi), \mathrm{d}\varphi\big).
\end{cases} \tag{2.18}$$

Proof. Let $\{\theta^i\}_{i=1}^m$ be an orthonormal coframe on (M, g). Let $\{\widehat{\theta}^\alpha\}_{\alpha=0}^m$ be an orthonormal coframe on $(\widehat{M}, \widehat{g})$ such that $\widehat{\theta}^i = \widehat{\pi}_M^*\theta^i$ for $1 \le i \le m$ and $\widehat{\theta}^0 = e^{-\widehat{f}}\mathrm{d}t$. As in Section 1.3, $\{\theta^i{}_j\}_{i,j=1}^m$ and $\{\widehat{\theta}^\alpha{}_\beta\}_{\alpha,\beta=0}^m$ will be the Levi-Civita connection forms for the coframes $\{\theta^i\}_{i=0}^m$ and $\{\widehat{\theta}^\alpha\}_{\alpha=0}^m$, respectively. Then, by Proposition 1.14, we have

$$\begin{cases}
\widehat{\theta}^i{}_j = \widehat{\pi}_M^*\theta^i{}_j, \\
\widehat{\theta}^0{}_i = \widehat{\theta}^i{}_0 = -f_i\widehat{\theta}^0\,.
\end{cases} \tag{2.19}$$

The components of the Ricci tensor of $(M, \widehat{g})$ with respect to $\{\widehat{\theta}^\alpha\}$ are

$$\begin{cases} \widehat{R}_{ij} = R_{ij} + f_{ij} - f_i f_j, \\ \widehat{R}_{00} = -\Delta f + |\nabla f|^2, \\ \widehat{R}_{0i} = 0, \end{cases} \tag{2.20}$$

By the definition of covariant derivative, we have

$$\begin{aligned} R_{ij,k}\theta^k &= \mathrm{d}R_{ij} - R_{kj}\theta^k{}_i - R_{ik}\theta^k{}_j\,, \\ f_{ij}\theta^j &= \mathrm{d}f_i - f_k\theta^k{}_i\,, \\ f_{ijk}\theta^k &= \mathrm{d}f_{ij} - f_{kj}\theta^k{}_i - f_{ik}\theta^k{}_j\,, \\ \widehat{R}_{\alpha\beta,\gamma}\widehat{\theta}^\gamma &= \mathrm{d}\widehat{R}_{\alpha\beta} - \widehat{R}_{\gamma\beta}\widehat{\theta}^\gamma{}_\alpha - \widehat{R}_{\alpha\gamma}\widehat{\theta}^\gamma{}_\beta\,. \end{aligned}$$

Then, using (2.19) and (2.20) we get

$$\begin{aligned} \widehat{R}_{ij,\gamma}\widehat{\theta}^\gamma &= \mathrm{d}\widehat{R}_{ij} - \widehat{R}_{\gamma j}\widehat{\theta}^\gamma{}_i - \widehat{R}_{i\gamma}\widehat{\theta}^\gamma{}_i \\ &= \mathrm{d}\widehat{R}_{ij} - \widehat{R}_{kj}\theta^k{}_i - \widehat{R}_{ik}\theta^k{}_j \\ &= (R_{ij,k} + f_{ijk} - f_{ik}f_j - f_i f_{jk})\theta^k\,, \end{aligned}$$

that is,

$$\begin{cases} \widehat{R}_{ij,k} = R_{ij,k} + f_{ijk} - f_{ik}f_j - f_i f_{jk}, \\ \widehat{R}_{ij,0} = 0\,. \end{cases} \tag{2.21}$$

In the same fashion, we have

$$\begin{aligned} \widehat{R}_{0i,\gamma}\widehat{\theta}^\gamma &= \mathrm{d}\widehat{R}_{0i} - \widehat{R}_{\gamma i}\widehat{\theta}^\gamma{}_0 - \widehat{R}_{0\gamma}\widehat{\theta}^\gamma{}_i \\ &= -\widehat{R}_{ki}\widehat{\theta}^k{}_0 - \widehat{R}_{00}\widehat{\theta}^0{}_i \\ &= (f_k\widehat{R}_{ki} + f_i\widehat{R}_{00})\widehat{\theta}^0 \\ &= (f_k R_{ki} + f_k f_{ki} - |\nabla f|^2 f_i - f_i\Delta f + f_i|\nabla f|^2)\widehat{\theta}^0\,, \end{aligned}$$

that is,

$$\begin{cases} \widehat{R}_{0i,j} = 0, \\ \widehat{R}_{0i,0} = R_{ij}f_j + f_{ij}f_j - (\Delta f)f_i\,. \end{cases} \tag{2.22}$$

Lastly, from

$$\widehat{R}_{00,\gamma}\widehat{\theta}^\gamma = \mathrm{d}\widehat{R}_{00} - 2\widehat{R}_{0\gamma}\theta^\gamma{}_0 = \mathrm{d}\widehat{R}_{00} = -\mathrm{d}\Delta f + \mathrm{d}|\nabla f|^2$$

we get

$$\begin{cases} \widehat{R}_{00,i} = -f_{jji} + 2f_{ij}f_j, \\ \widehat{R}_{00,0} = 0\,. \end{cases} \tag{2.23}$$

Next, since

$$\widehat{S}_\alpha = g^{\gamma\beta}\widehat{R}_{\gamma\beta,\alpha} = \widehat{R}_{tt,\alpha} - \widehat{R}_{00,\alpha}$$

we deduce

$$\begin{cases} \widehat{S}_i = S_i + 2f_{jji} - 4f_{ij}f_j, \\ \widehat{S}_0 = 0\,. \end{cases} \tag{2.24}$$

We are ready to compute how the Cotton tensor of $(\widehat{M}, \widehat{g})$ splits on the spatial and the time components. Recalling that $n = \dim \widehat{M} = m+1$, $\widehat{C}$ reads, in components,

$$\widehat{C}_{\alpha\beta\gamma} = \widehat{R}_{\alpha\beta,\gamma} - \widehat{R}_{\alpha\gamma,\beta} - \frac{1}{2m}(\widehat{S}_\gamma g_{\alpha\beta} - \widehat{S}_\beta g_{\alpha\gamma})\,.$$

Therefore, from (2.21) and (2.24) we get

$$\begin{aligned} \widehat{C}_{ijk} &= R_{ij,k} - R_{ik,j} + f_{ijk} - f_{ikj} - f_{ik}f_j + f_{ij}f_k \\ &\quad - \frac{1}{2m}(S_k g_{ij} - S_j g_{ik}) - \frac{1}{m}[(\Delta f - |\nabla f|^2)_k g_{ij} - (\Delta f - |\nabla f|^2)_j g_{ik}] \\ &= C_{ijk} + f_{ijk} - f_{ikj} - f_{ik}f_j + f_{ij}f_k \\ &\quad + \frac{1}{2m(m-1)}(S_k g_{ij} - S_j g_{ik}) - \frac{1}{m}[(\Delta f - |\nabla f|^2)_k g_{ij} - (\Delta f - |\nabla f|^2)_j g_{ik}]\,. \end{aligned}$$

A direct computation yields $\widehat{C}_{ij0} = \widehat{C}_{0ij} = 0$. Lastly, we have, from (2.23), (2.22) and (2.24),

$$\begin{aligned} \widehat{C}_{00i} &= \widehat{R}_{00,i} - \widehat{R}_{0i,0} - \frac{1}{2m}(\widehat{S}_i g_{00} - \widehat{S}_0 g_{0i}) \\ &= -(\Delta f - |\nabla f|^2)_i - R_{ij}f_j - f_{ij}f_j + (\Delta f)f_i + \frac{1}{2m}S_i + \frac{1}{m}(\Delta f - |\nabla f|^2)_i \\ &= -R_{ij}f_j - \frac{m-1}{m}(\Delta f)_i + \frac{m-2}{m}f_{ij}f_j + (\Delta f)f_i + \frac{1}{2m}S_i\,. \end{aligned}$$

Putting together the above computations, we get

$$\begin{cases} \widehat{C}_{ijk} = & C_{ijk} + f_{ijk} - f_{ikj} - f_{ik}f_j + f_{ij}f_k \\ & -\frac{1}{m}[(f_{ttk} - 2f_t f_{tk})\delta_{ij} - (f_{ttj} - 2f_t f_{tj})\delta_{ik}] \\ & +\frac{1}{2m(m-1)}(S_k\delta_{ij} - S_j\delta_{ik}), \\ \widehat{C}_{00i} = & -\frac{m-1}{m}f_{tti} + \frac{m-2}{m}f_t f_{ti} + (\Delta f)f_i + \frac{1}{2m}S_i - f_t R_{it}, \\ \widehat{C}_{ij0} = & 0, \\ \widehat{C}_{0ij} = & 0. \end{cases} \tag{2.25}$$

We now study the field equations. Having defined

$$\widehat{\mathfrak{T}} := \frac{1}{m-1}\left(\widehat{T} \owedge \widehat{g} - \frac{\mathrm{tr}_{\widehat{g}}\widehat{T}}{2m}\widehat{g} \owedge \widehat{g}\right),$$

we have that the field equations of Cotton gravity are

$$\widehat{C} = -(m-1)\,\mathrm{div}_1\,\widehat{\mathfrak{T}}\,.$$

As we already discussed in the beginning of this Chapter, the field equations imply that $\widehat{T}$ is divergence free. From our assumptions, we have $\widehat{\varphi}^a_0 = 0$, $\widehat{\mu}_0 = \widehat{p}_0 = 0$. Furthermore, we are also assuming that $\widehat{T}^F$ and $\widehat{T}^{\widehat{\varphi}}$ are divergence-free, which, by (the proofs of) Propositions 1.15 and 1.17, can be seen to be equivalent to

$$\begin{cases} 0 & = (\widehat{\mu} + \widehat{p})\widehat{\nabla}\widehat{f} - \widehat{\nabla}\widehat{p}, \\ 0 & = \alpha h\left(\tau(\widehat{\varphi}) - \frac{1}{\alpha}(\nabla U)(\widehat{\varphi}), \mathrm{d}\widehat{\varphi}\right). \end{cases} \tag{2.26}$$

From equation (2.2) and since $\mathrm{div}_1\,\widehat{T} = 0$ we have

$$\begin{aligned} \mathrm{div}_1\,\widehat{\mathfrak{T}}_{\alpha\beta\gamma} =& \frac{1}{m-1}\left[\widehat{T}_{\alpha\gamma,\beta} - \widehat{T}_{\alpha\beta,\gamma}\right] - \frac{1}{m(m-1)}\left(\mathrm{tr}_{\widehat{g}}\widehat{T}\right)_{\beta} g_{\alpha\gamma} \\ &+ \frac{1}{m(m-1)}\left(\mathrm{tr}_{\widehat{g}}\widehat{T}\right)_{\gamma} g_{\alpha\beta}. \end{aligned} \tag{2.27}$$

Since $\widehat{T} = \widehat{T}^{\widehat{\varphi}} + \widehat{T}^F$ we have

$$\mathrm{div}_1\,\widehat{\mathfrak{T}} = \mathrm{div}_1\,\widehat{\mathfrak{T}}^{\widehat{\varphi}} + \mathrm{div}_1\,\widehat{\mathfrak{T}}^F$$

with an obvious meaning of notations. We compute the last two terms separately. From the definition (1.43) of $\widehat{T}^{\widehat{\varphi}}$ one easily sees that

$$\widehat{T}^{\widehat{\varphi}}_{\alpha\beta,\gamma} = \alpha\widehat{\varphi}^a_{\alpha\gamma}\widehat{\varphi}^a_{\beta} + \alpha\widehat{\varphi}^a_{\alpha}\widehat{\varphi}^a_{\beta\gamma} - \left(\alpha g^{\eta\rho}\widehat{\varphi}^a_{\eta\gamma}\widehat{\varphi}^a_{\rho} + U^a\widehat{\varphi}^a_{\gamma}\right)g_{\alpha\beta} \tag{2.28}$$

and

$$\mathrm{tr}_{\widehat{g}}\widehat{T}^{\widehat{\varphi}} = -\alpha\frac{m-1}{2}|\mathrm{d}\widehat{\varphi}|^2 - (m+1)U(\widehat{\varphi}). \tag{2.29}$$

Therefore, from (2.27) we get

$$
\begin{aligned}
(m-1)\,\mathrm{div}_1\,\widehat{\mathfrak{T}}^{\widehat{\varphi}}_{\alpha\beta\gamma} =&\widehat{T}^{\widehat{\varphi}}_{\alpha\gamma,\beta} - \widehat{T}^{\widehat{\varphi}}_{\alpha\beta,\gamma} - \frac{1}{m}\left[\left(\mathrm{tr}_{\widehat{g}}\widehat{T}^{\widehat{\varphi}}\right)_{\beta} g_{\alpha\gamma} - \left(\mathrm{tr}_{\widehat{g}}\widehat{T}^{\widehat{\varphi}}\right)_{\gamma} g_{\alpha\beta}\right] \\
=&\alpha\Bigg\{\widehat{\varphi}^a_{\alpha\beta}\widehat{\varphi}^a_{\gamma} - \widehat{\varphi}^a_{\alpha\gamma}\widehat{\varphi}^a_{\beta} - \frac{1}{\alpha}U^a\widehat{\varphi}^a_{\beta}g_{\alpha\gamma} + \frac{1}{\alpha}U^a\widehat{\varphi}^a_{\gamma}g_{\alpha\beta} \\
&- g^{\eta\rho}\widehat{\varphi}^a_{\eta\beta}\widehat{\varphi}^a_{\rho}g_{\alpha\gamma} + g^{\eta\rho}\widehat{\varphi}^a_{\eta\gamma}\widehat{\varphi}^a_{\rho}g_{\alpha\beta} \\
&+ \frac{1}{m}\left[(m-1)g^{\eta\rho}\widehat{\varphi}^a_{\eta\beta}\widehat{\varphi}^a_{\rho}g_{\alpha\gamma} + \frac{m+1}{\alpha}U^a\widehat{\varphi}^a_{\beta}g_{\alpha\gamma}\right] \\
&- \frac{1}{m}\left[(m-1)g^{\eta\rho}\widehat{\varphi}^a_{\eta\gamma}\widehat{\varphi}^a_{\rho}g_{\alpha\beta} + \frac{m+1}{\alpha}U^a\widehat{\varphi}^a_{\gamma}g_{\alpha\beta}\right]\Bigg\} \\
=&\alpha\Bigg\{\widehat{\varphi}^a_{\alpha\beta}\widehat{\varphi}^a_{\gamma} - \widehat{\varphi}^a_{\alpha\gamma}\widehat{\varphi}^a_{\beta} - \left(1-\frac{m+1}{m}\right)\frac{1}{\alpha}U^a\widehat{\varphi}^a_{\beta}g_{\alpha\gamma} \\
&+ \left(1-\frac{m+1}{m}\right)\frac{1}{\alpha}U^a\widehat{\varphi}^a_{\gamma}g_{\alpha\beta} - \left(1-\frac{m-1}{m}\right)g^{\eta\rho}\widehat{\varphi}^a_{\eta\beta}\widehat{\varphi}^a_{\rho}g_{\alpha\gamma} \\
&+ \left(1-\frac{m-1}{m}\right)g^{\eta\rho}\widehat{\varphi}^a_{\eta\gamma}\widehat{\varphi}^a_{\rho}g_{\alpha\beta}\Bigg\} \\
=&\alpha\Bigg\{\widehat{\varphi}^a_{\alpha\beta}\widehat{\varphi}^a_{\gamma} - \widehat{\varphi}^a_{\alpha\gamma}\widehat{\varphi}^a_{\beta} + \frac{1}{\alpha m}U^a\widehat{\varphi}^a_{\beta}g_{\alpha\gamma} \\
&- \frac{1}{\alpha m}U^a\widehat{\varphi}^a_{\gamma}g_{\alpha\beta} - \frac{1}{m}g^{\eta\rho}\widehat{\varphi}^a_{\eta\beta}\widehat{\varphi}^a_{\rho}g_{\alpha\gamma} + \frac{1}{m}g^{\eta\rho}\widehat{\varphi}^a_{\eta\gamma}\widehat{\varphi}^a_{\rho}g_{\alpha\beta}\Bigg\}.
\end{aligned}
$$

Set

$$
\begin{aligned}
\Phi_{\alpha\beta\gamma} :=& -(m-1)\,\mathrm{div}_1\,\widehat{\mathfrak{T}}^{\widehat{\varphi}}_{\alpha\beta\gamma} \\
=& -\alpha\Bigg\{\widehat{\varphi}^a_{\alpha\beta}\widehat{\varphi}^a_{\gamma} - \widehat{\varphi}^a_{\alpha\gamma}\widehat{\varphi}^a_{\beta} + \frac{1}{\alpha m}U^a\widehat{\varphi}^a_{\beta}g_{\alpha\gamma} - \frac{1}{\alpha m}U^a\widehat{\varphi}^a_{\gamma}g_{\alpha\beta} \\
&- \frac{1}{m}g^{\eta\rho}\widehat{\varphi}^a_{\eta\beta}\widehat{\varphi}^a_{\rho}g_{\alpha\gamma} + \frac{1}{m}g^{\eta\rho}\widehat{\varphi}^a_{\eta\gamma}\widehat{\varphi}^a_{\rho}g_{\alpha\beta}\Bigg\}.
\end{aligned}
$$

From Proposition 1.17 we have

$$
\begin{cases}
\widehat{\varphi}^a_{00} = \varphi^a_t f_t, \\
\widehat{\varphi}^a_{0i} = 0, \\
\widehat{\varphi}^a_{ij} = \varphi^a_{ij}.
\end{cases}
\tag{2.30}
$$

Using (2.30) and $\widehat{\varphi}^a_0 = 0$ the following are easily computed

$$
\begin{cases}
\Phi_{ijk} = & -\alpha\varphi^a_{ij}\varphi^a_k + \alpha\varphi^a_{ik}\varphi^a_j - \frac{1}{m}U^a\varphi^a_j\delta_{ik} \\
& +\frac{1}{m}U^a\varphi^a_k\delta_{ij} + \frac{\alpha}{m}\varphi^a_{tj}\varphi^a_t\delta_{ik} - \frac{\alpha}{m}\varphi^a_{tk}\varphi^a_t\delta_{ij}, \\
\Phi_{00i} = & -\alpha\varphi^a_i\varphi^a_t f_t - \frac{1}{m}U^a\varphi^a_i + \frac{\alpha}{m}\varphi^a_{ti}\varphi^a_t, \\
\Phi_{ij0} = & 0, \\
\Phi_{0ij} = & 0.
\end{cases}
\tag{2.31}
$$

We now study the perfect fluid stress-energy tensor

$$\widehat{T}^F = (\widehat{\mu} + \widehat{p})e^{-2\widehat{f}}\mathrm{d}t \otimes \mathrm{d}t + \widehat{p}\widehat{g}.$$

As we already discussed, $\widehat{\mu}_0 = \widehat{p}_0 = 0$ and

$$\begin{cases} \widehat{\theta}^i{}_j = -\widehat{\theta}^j{}_i = \theta^i{}_j, \\ \widehat{\theta}^0{}_i = \widehat{\theta}^i{}_0 = -f_i\widehat{\theta}^0. \end{cases} \tag{2.32}$$

We compute the components of the covariant derivative of $\widehat{T}^F$. Similar computations have been done in the proof of Proposition 1.15, but we repeat them for the ease of the reader. Since

$$\widehat{T}^F_{\alpha\beta,\gamma}\widehat{\theta}^\gamma = \mathrm{d}\widehat{T}^F_{\alpha\beta} - \widehat{T}^F_{\gamma\beta}\widehat{\theta}^\gamma{}_\alpha - \widehat{T}^F_{\alpha\gamma}\widehat{\theta}^\gamma{}_\beta,$$

we get, from the definition of $\widehat{T}^F$,

$$\begin{cases} \widehat{T}^F_{0\alpha} = \widehat{\mu}\delta_{0\alpha}, \\ \widehat{T}^F_{i\alpha} = \widehat{p}\delta_{i\alpha} \end{cases} \tag{2.33}$$

and

$$\mathrm{tr}_{\widehat{g}}\widehat{T}^F = m\widehat{p} - \widehat{\mu}. \tag{2.34}$$

Therefore (2.33) and (2.32) imply

$$\begin{aligned} \widehat{T}^F_{ij,\alpha}\widehat{\theta}^\alpha =& \mathrm{d}\widehat{T}^F_{ij} - \widehat{T}^F_{\gamma j}\widehat{\theta}^\gamma{}_j - \widehat{T}^F_{\gamma i}\widehat{\theta}^\gamma{}_j \\ =& \delta_{ij}\widehat{p}_\alpha\widehat{\theta}^\alpha - \widehat{p}\widehat{\theta}^j{}_i - \widehat{p}\widehat{\theta}^i{}_j \\ =& \delta_{ij}\widehat{p}_\alpha\widehat{\theta}^\alpha, \end{aligned}$$

that is,

$$\widehat{T}^F_{ij,\alpha} = \widehat{p}_\alpha\delta_{ij}. \tag{2.35}$$

Using (2.33) we have

$$\begin{aligned} \widehat{T}^F_{00,\alpha} =& \mathrm{d}\widehat{T}^F_{00} - \widehat{T}^F_{\gamma 0}\widehat{\theta}^\gamma{}_0 - \widehat{T}^F_{0\gamma}\widehat{\theta}^\gamma{}_0 \\ =& \widehat{\mu}_\alpha\widehat{\theta}^\alpha, \end{aligned}$$

that is,

$$\widehat{T}^F_{00,\alpha} = \widehat{\mu}_\alpha. \tag{2.36}$$

Using (2.33) and (2.32) we get

$$\begin{aligned} \widehat{T}^F_{0i,\alpha}\widehat{\theta}^\alpha =& \mathrm{d}\widehat{T}^F_{0i} - \widehat{T}^F_{\gamma i}\widehat{\theta}^\gamma{}_0 - \widehat{T}^F_{0\gamma}\widehat{\theta}^\gamma{}_i \\ =& 0 - \widehat{p}\widehat{\theta}^i{}_0 - \widehat{\mu}\widehat{\theta}^0{}_i \\ =& -(\widehat{\mu} + \widehat{p})\widehat{\theta}^i{}_0 = (\widehat{\mu} + \widehat{p})f_i\widehat{\theta}^0 \end{aligned}$$

so that

$$\widehat{T}^F_{0i,j} = 0, \quad \widehat{T}_{0i,0} = (\widehat{\mu} + \widehat{p}) f_i. \tag{2.37}$$

From (2.27), (2.34) and (2.35) we deduce

$$\begin{aligned}
\operatorname{div}_1 \widehat{\mathfrak{T}}^F_{ijk} =& \frac{1}{m-1}\left(\widehat{T}^F_{ik,j} - \widehat{T}^F_{ij,k}\right) - \frac{1}{m(m-1)}\left(\mathrm{tr}_{\widehat{g}} \widehat{T}^F\right)_j \delta_{ik} + \frac{1}{m(m-1)}\left(\mathrm{tr}_{\widehat{g}} \widehat{T}^F\right)_k \delta_{ij} \\
=& \frac{1}{m-1}(p_j\delta_{ik} - p_k\delta_{ij}) - \frac{1}{m(m-1)}[(mp_j - \mu_j)\delta_{ik} - (mp_k - \mu_k)\delta_{ij}] \\
=& \frac{1}{m(m-1)}[\mu_j\delta_{ik} - \mu_k\delta_{ij}],
\end{aligned}$$

that is,

$$\operatorname{div}_1 \widehat{\mathfrak{T}}^F_{ijk} = \frac{1}{m(m-1)}[\mu_j\delta_{ik} - \mu_k\delta_{ij}]. \tag{2.38}$$

From (2.35) and (2.37) we get

$$\begin{aligned}
\operatorname{div}_1 \widehat{\mathfrak{T}}^F_{ij0} =& \frac{1}{m-1}\left(\widehat{T}^F_{i0,j} - \widehat{T}^F_{ij,0}\right) - \frac{1}{m(m-1)}\left[\left(\mathrm{tr}_{\widehat{g}} \widehat{T}^F\right)_j g_{i0} - \left(\mathrm{tr}_{\widehat{g}} \widehat{T}^F\right)_0 g_{ij}\right] \\
=& 0
\end{aligned}$$

and, using (2.37),

$$\begin{aligned}
\operatorname{div}_1 \widehat{\mathfrak{T}}^F_{0ij} =& \frac{1}{m-1}\left(\widehat{T}^F_{0j,i} - \widehat{T}^F_{0i,j}\right) - \frac{1}{m(m-1)}\left[\left(\mathrm{tr}_{\widehat{g}} \widehat{T}^F\right)_i g_{0j} - \left(\mathrm{tr}_{\widehat{g}} \widehat{T}^F\right)_j g_{0i}\right] \\
=& 0.
\end{aligned}$$

Using (2.34), (2.36) and (2.37) we obtain

$$\begin{aligned}
\operatorname{div}_1 \widehat{\mathfrak{T}}^F_{0i,0} =& \frac{1}{m-1}\left[\widehat{T}^F_{00,i} - \widehat{T}^F_{0i,0}\right] - \frac{1}{m(m-1)}\left[\left(\mathrm{tr}_{\widehat{g}} \widehat{T}^F\right)_i g_{00} - \left(\mathrm{tr}_{\widehat{g}} \widehat{T}^F\right)_0 g_{0i}\right] \\
=& \frac{1}{m-1}[\mu_i - (\mu + p) f_i] + \frac{1}{m(m-1)}(mp_i - \mu_i) \\
=& \frac{1}{m}\mu_i - \frac{\mu + p}{m-1} f_i + \frac{1}{m-1} p_i.
\end{aligned}$$

Putting together these informations we get

$$\begin{cases}
\operatorname{div}_1 \widehat{\mathfrak{T}}^F_{ijk} = \frac{1}{m(m-1)}(\mu_j\delta_{ik} - \mu_k\delta_{ij}), \\
\operatorname{div}_1 \widehat{\mathfrak{T}}^F_{0i0} = \frac{1}{m}\mu_i + \frac{1}{m-1}[p_i - (\mu + p) f_i], \\
\operatorname{div}_1 \widehat{\mathfrak{T}}^F_{0ij} = 0, \\
\operatorname{div}_1 \widehat{\mathfrak{T}}^F_{ij0} = 0.
\end{cases} \tag{2.39}$$

Combining (2.25), (2.31) and (2.39) the field equations become

$$\begin{aligned}0 =& C_{ijk} + f_{ijk} - f_{ikj} - f_{ik}f_j + f_{ij}f_k \\ &- \frac{1}{m}[(f_{ttk} - 2f_t f_{tk})\delta_{ij} - (f_{ttj} - 2f_t f_{tj})\delta_{ik}] \\ &+ \frac{1}{2m(m-1)}(S_k\delta_{ij} - S_j\delta_{ik}) \\ &+ \alpha\varphi^a_{ij}\varphi^a_k - \alpha\varphi^a_{ik}\varphi^a_j + \frac{1}{m}U^a\varphi^a_j\delta_{ik} - \frac{1}{m}U^a\varphi^a_k\delta_{ij} \\ &- \frac{\alpha}{m}\varphi^a_{tj}\varphi^a_t\delta_{ik} + \frac{\alpha}{m}\varphi^a_{tk}\varphi^a_t\delta_{ij} + \frac{1}{m}(\mu_j\delta_{ik} - \mu_k\delta_{ij})\end{aligned}$$

and

$$\begin{aligned}0 =& -\frac{m-1}{m}f_{tti} + \frac{m-2}{m}f_t f_{ti} + (\Delta f)f_i + \frac{1}{2m}S_i - f_t R_{it} \\ &+ \alpha\varphi^a_i\varphi^a_t f_t + \frac{1}{m}U^a\varphi^a_i - \frac{\alpha}{m}\varphi^a_{ti}\varphi^a_t \\ &- \frac{m-1}{m}\mu_i + p_i - (\mu + p)f_i.\end{aligned}$$

Using the definitions of the φ-curvatures the above equations can be re-written as

$$\begin{cases} 0 = & C^{\varphi}_{ijk} + f_{ijk} - f_{ikj} - f_{ik}f_j + f_{ij}f_k \\ & -\frac{1}{m}[(f_{ttk} - 2f_t f_{tk})\delta_{ij} - (f_{ttj} - 2f_t f_{tj})\delta_{ik}] \\ & +\frac{1}{2m(m-1)}(S^{\varphi}_k\delta_{ij} - S^{\varphi}_j\delta_{ik}) \\ & +\frac{1}{m}U^a\varphi^a_j\delta_{ik} - \frac{1}{m}U^a\varphi^a_k\delta_{ij} + \frac{1}{m}(\mu_j\delta_{ik} - \mu_k\delta_{ij}), \\ 0 = & -\frac{m-1}{m}f_{tti} + \frac{m-2}{m}f_t f_{ti} + (\Delta f)f_i + \frac{1}{2m}S^{\varphi}_i - f_t R^{\varphi}_{it} \\ & +\frac{1}{m}U^a\varphi^a_i - \frac{m-1}{m}\mu_i + p_i - (\mu + p)f_i. \end{cases} \tag{2.40}$$

□

Remark 2.3. In what follows, we will replace condition (2.18).iv) with the stronger

$$\tau(\varphi) = \mathrm{d}\varphi(\nabla f) + \frac{1}{\alpha}\nabla U. \tag{2.41}$$

This is motivated by the following reasoning: the stress-energy tensor $\widehat{T}^{\widehat{\varphi}}$ is obtained from the matter action

$$I(\widehat{g}, \widehat{\varphi}) = \int \left(\alpha|\mathrm{d}\widehat{\varphi}|^2_{\widehat{g}} + 2U(\widehat{\varphi})\right)$$

by considering the critical points of its variations with respect to the metric.
As we discussed in the Introduction, the critical points for variations of the field $\widehat{\varphi}$ of this action are $\frac{1}{\alpha}U$-harmonic, that is, they satisfy

$$\tau(\widehat{\varphi}) = \frac{1}{\alpha}\nabla U(\widehat{\varphi}) \tag{2.42}$$

see [62] for a proof in Riemannian signature, the Lorentzian case being identical. As we just proved, (2.42) reduces to (2.41) on a static space-time, so that it is natural to consider equation (2.41) instead of (2.43). This point of view also motivates the splitting of the natural condition

$$0 = \mathrm{div}_1 \widehat{T}$$

into the two conditions

$$0 = \mathrm{div}_1 \widehat{T}^F = \mathrm{div}_1 \widehat{T}^{\widehat{\varphi}}.$$

Definition 2.4. Given a Riemannian manifold (M, g) and a smooth map $\varphi : (M, g) \to (N, h)$ that targets another Riemannian manifold, and given $\alpha \in \mathbb{R}, \alpha \neq 0, U \in C^\infty(N), \mu, p \in C^\infty(M)$ and a function $f \in C^\infty(M)$, we will say that (M, g, f) is a C-φ-PF if it satisfies

$$\begin{cases} 0 = & C^{\varphi}_{ijk} + f_{ijk} - f_{ikj} - f_{ik}f_j + f_{ij}f_k \\ & -\frac{1}{m}[(f_{ttk} - 2f_t f_{tk})\delta_{ij} - (f_{ttj} - 2f_t f_{tj})\delta_{ik}] \\ & +\frac{1}{2m(m-1)}\left(S^{\varphi}_k\delta_{ij} - S^{\varphi}_j\delta_{ik}\right) \\ & +\frac{1}{m}U^a\varphi^a_j\delta_{ik} - \frac{1}{m}U^a\varphi^a_k\delta_{ij} + \frac{1}{m}(\mu_j\delta_{ik} - \mu_k\delta_{ij}), \\ 0 = & -\frac{m-1}{m}f_{tti} + \frac{m-2}{m}f_t f_{ti} + (\Delta f)f_i + \frac{1}{2m}S^{\varphi}_i - f_t R^{\varphi}_{it} \\ & +\frac{1}{m}U^a\varphi^a_i - \frac{m-1}{m}\mu_i, \\ 0 = & \nabla p - (\mu + p)\nabla f, \\ 0 = & \tau(\varphi) - \mathrm{d}\varphi(\nabla f) - \frac{1}{\alpha}\nabla U(\varphi). \end{cases} \tag{2.43}$$

We want to write system (2.43) in an equivalent form, that will be useful in the next section. From equation (2.43).ii) we get

$$\begin{aligned} \frac{m-1}{m}f_{tti} - 2\frac{m-1}{m}f_t f_{ti} = &\left(\frac{m-2}{m} - 2\frac{m-1}{m}\right)f_t f_{ti} + (\Delta f)f_i + \frac{1}{2m}S^{\varphi}_i \\ &- f_t R^{\varphi}_{ti} + \frac{1}{m}U^a\varphi^a_i - \frac{m-1}{m}\mu_i. \end{aligned}$$

Simplifying and dividing by $m-1$ we deduce

$$\begin{aligned} \frac{1}{m}(f_{tti} - 2f_t f_{ti}) = &-\frac{1}{m-1}[f_t f_{ti} - (\Delta f)f_i] + \frac{1}{2m(m-1)}S^{\varphi}_i \\ &- \frac{1}{m-1}f_t R^{\varphi}_{ti} + \frac{1}{m(m-1)}U^a\varphi^a_i - \frac{1}{m}\mu_i. \end{aligned}$$

Using the latter into (2.43).i) we obtain

$$\begin{aligned}
0 =& C^{\varphi}_{ijk} + f_{ijk} - f_{ikj} - f_{ik}f_j + f_{ij}f_k \\
&+ \frac{1}{m-1}(f_t f_{tk} - (\Delta f)f_k)\delta_{ij} - \frac{1}{2m(m-1)}S^{\varphi}_k\delta_{ij} + \frac{1}{m-1}f_t R^{\varphi}_{tk}\delta_{ij} \\
&- \frac{1}{m(m-1)}U^a\varphi^a_k\delta_{ij} + \frac{1}{m}\mu_k\delta_{ij} \\
&- \frac{1}{m-1}(f_t f_{tj} - (\Delta f)f_j)\delta_{ik} + \frac{1}{2m(m-1)}S^{\varphi}_j\delta_{ik} - \frac{1}{m-1}f_t R^{\varphi}_{tj}\delta_{ik} \\
&+ \frac{1}{m(m-1)}U^a\varphi^a_j\delta_{ik} - \frac{1}{m}\mu_j\delta_{ik} \\
&+ \frac{1}{2m(m-1)}\left(S^{\varphi}_k\delta_{ij} - S^{\varphi}_j\delta_{ik}\right) + \frac{1}{m}U^a\left(\varphi^a_j\delta_{ik} - \varphi^a_k\delta_{ij}\right) \\
&+ \frac{1}{m}(\mu_j\delta_{ik} - \mu_k\delta_{ij}).
\end{aligned}$$

Simplifying, the latter becomes

$$\begin{aligned}
0 =& C^{\varphi}_{ijk} + f_{ijk} - f_{ikj} + f_{ij}f_k - f_{ik}f_j + \frac{1}{m-1}[f_t f_{tk} - (\Delta f)f_k]\delta_{ij} \\
&- \frac{1}{m-1}[f_t f_{tj} - (\Delta f)f_j]\delta_{ik} + \frac{1}{m-1}f_t\left(R^{\varphi}_{tk}\delta_{ij} - R^{\varphi}_{tj}\delta_{ik}\right) \\
&- \frac{1}{m-1}U^a\varphi^a_k\delta_{ij} + \frac{1}{m-1}U^a\varphi^a_j\delta_{ik}.
\end{aligned} \tag{2.44}$$

Using the Ricci commutation relations and the definition of W^{φ} we get

$$\begin{aligned}
f_{ijk} - f_{ikj} &= f_t R_{tijk} \\
&= f_t W^{\varphi}_{tijk} + \frac{1}{m-2}\left[f_j R^{\varphi}_{ik} - f_k R^{\varphi}_{ij} + f_t R^{\varphi}_{tj}\delta_{ik} - f_t R^{\varphi}_{tk}\delta_{ij}\right] \\
&- \frac{S^{\varphi}}{(m-1)(m-2)}(f_j\delta_{ik} - f_k\delta_{ij}).
\end{aligned} \tag{2.45}$$

Plugging (2.45) into (2.44) we deduce

$$\begin{aligned}
0 =& C^{\varphi}_{ijk} + f_t W^{\varphi}_{tijk} + \frac{1}{m-2}\left[f_j R^{\varphi}_{ik} - f_k R^{\varphi}_{ij} + f_t R^{\varphi}_{tj}\delta_{ik} - f_t R^{\varphi}_{tk}\delta_{ij}\right] \\
&- \frac{S^{\varphi}}{(m-1)(m-2)}(f_j\delta_{ik} - f_k\delta_{ij}) + \frac{1}{m-1}f_t\left(R^{\varphi}_{tk}\delta_{ij} - R^{\varphi}_{tj}\delta_{ik}\right) \\
&+ f_{ij}f_k - f_{ik}f_j - \frac{1}{m-1}[f_t f_{tj} - (\Delta f)f_j]\delta_{ik} + \frac{1}{m-1}[f_t f_{tk} - (\Delta f)f_k]\delta_{ij} \\
&- \frac{1}{m-1}U^a\varphi^a_k\delta_{ij} + \frac{1}{m-1}U^a\varphi^a_j\delta_{ik}.
\end{aligned}$$

Simplifying

$$\begin{aligned}0 =& C^{\varphi}_{ijk} + f_t W^{\varphi}_{tijk} + \frac{1}{m-2}\Big[f_j R^{\varphi}_{ik} - f_k R^{\varphi}_{ij} + \frac{f_t}{m-1}\big(R^{\varphi}_{tj}\delta_{ik} - R^{\varphi}_{tk}\delta_{ij}\big) \\ &- \frac{S^{\varphi}}{m-1}(f_j\delta_{ik} - f_k\delta_{ij})\Big] \\ &+ f_{ij}f_k - f_{ik}f_j + \frac{1}{m-1}f_t(f_{tk}\delta_{ij} - f_{tj}\delta_{ik}) - \frac{\Delta f}{m-1}(f_k\delta_{ij} - f_j\delta_{ik}) \\ &- \frac{1}{m-1}U^a\big(\varphi^a_k\delta_{ij} - \varphi^a_j\delta_{ik}\big).\end{aligned} \tag{2.46}$$

We define

$$\begin{aligned}D^A_{ijk} :=& \frac{1}{m-2}\Big[f_k R^{\varphi}_{ij} - f_j R^{\varphi}_{ik} + \frac{f_t}{m-1}\big(R^{\varphi}_{tk}\delta_{ij} - R^{\varphi}_{tj}\delta_{ik}\big) \\ &- \frac{S^{\varphi}}{m-1}(f_k\delta_{ij} - f_j\delta_{ik})\Big]\end{aligned} \tag{2.47}$$

and

$$\begin{aligned}D^B_{ijk} :=& \frac{1}{m-2}\Big[f_j f_{ik} - f_k f_{ij} + \frac{f_t}{m-1}(f_{tj}\delta_{ik} - f_{tk}\delta_{ij}) \\ &- \frac{\Delta f}{m-1}(f_j\delta_{ij} - f_k\delta_{ik})\Big]\end{aligned} \tag{2.48}$$

so that (2.46) becomes

$$0 = C^{\varphi}_{ijk} + f_t W^{\varphi}_{tijk} - D^A_{ijk} - (m-2)D^B_{ijk} - \frac{1}{m-1}U^a\big(\varphi^a_k\delta_{ij} - \varphi^a_j\delta_{ik}\big). \tag{2.49}$$

System (2.43) is therefore equivalent to

$$\begin{cases} 0 = & C^{\varphi}_{ijk} + f_t W^{\varphi}_{tijk} - D^A_{ijk} - (m-2)D^B_{ijk} - \frac{1}{m-1}U^a\big(\varphi^a_k\delta_{ij} - \varphi^a_j\delta_{ik}\big), \\ 0 = & -\frac{m-1}{m}f_{tti} + \frac{m-2}{m}f_t f_{ti} + (\Delta f)f_i + \frac{1}{2m}S^{\varphi}_i \\ & -f_t R^{\varphi}_{ti} + \frac{1}{m}U^a\varphi^a_i - \frac{m-1}{m}\mu_i, \\ 0 = & \nabla p - (\mu + p)\nabla f, \\ 0 = & \tau(\varphi) - \mathrm{d}\varphi(\nabla f) - \frac{1}{\alpha}\nabla U(\varphi). \end{cases} \tag{2.50}$$

Equation (2.50).i) has two important features, that are not seen in (2.43).i): on the one hand, the tensors D^A and D^B appear explicitly in it; as we will see in Section 2.3.1, these tensors are fundamental in the study of the geometry of the regular level sets of f. On the other hand, (2.50).i) share some striking formal similarities with the first integrability condition of a *φ-static perfect fluid space-time*. These structures, introduced in [13], arise as solutions, on a static space-time $M \times_f \mathbb{R}$, of the Einstein field equations

$$\widehat{\mathrm{Ric}} - \frac{1}{2}\widehat{S}\widehat{g} = \widehat{T}^F + \widehat{T}^{\widehat{\varphi}} \tag{2.51}$$

with source given by the combination of a perfect fluid and a non-linear field, subject to the further condition

$$\begin{cases} \operatorname{div} \widehat{T}^F = 0, \\ \tau(\widehat{\varphi}) = \frac{\nabla U}{\alpha}. \end{cases} \tag{2.52}$$

From Proposition 1.18, we know that equations (2.51) and (2.52) gives, on (M, g),

$$\begin{cases} \mathrm{Ric}^\varphi + \operatorname{Hess} f - \mathrm{d}f \otimes \mathrm{d}f - \frac{1}{m-1}\left(\frac{S^\varphi}{2} - p + U(\varphi)\right)g = 0, \\ \Delta_f f = -\frac{1}{m-1}\left[mp - mU(\varphi) + \frac{m-2}{2}S^\varphi\right], \\ \tau(\varphi) - \mathrm{d}\varphi(\nabla f) = \frac{\nabla U}{\alpha}, \\ \mu + U(\varphi) = \frac{1}{2}S^\varphi, \\ \nabla p - (\mu + p)\nabla f = 0. \end{cases} \tag{2.53}$$

Here, we are using the notation

$$\Delta_f h := \Delta h - g(\nabla h, \nabla f),$$

for any $h \in C^\infty(M)$. The operator Δ_f is often called the f-*laplacian.*

Performing the substitution $u = e^{-f}$, system (2.53) becomes

$$\begin{cases} \operatorname{Hess} u - u\left\{\mathrm{Ric}^\varphi - \frac{1}{m-1}\left(\frac{S^\varphi}{2} - p + U(\varphi)\right)g\right\} = 0, \\ \Delta u = \frac{u}{m-1}\left[mp - mU(\varphi) + \frac{m-2}{2}S^\varphi\right], \\ u\tau(\varphi) = -\mathrm{d}\varphi(\nabla u) + \frac{u}{\alpha}\nabla U, \\ \mu + U(\varphi) = \frac{1}{2}S^\varphi, \\ (\mu + p)\nabla u = -u\nabla p. \end{cases} \tag{2.54}$$

From (2.54) one can see that, when φ is constant and $U \equiv 0$, the system reduces to that of a classical perfect fluid. Working as in Remark 1.16, and using (2.53).*iv*) in (2.53).*ii*), one easily sees that system (2.53) is equivalent to

$$\begin{cases} \mathrm{Ric}^\varphi + \operatorname{Hess} f - \mathrm{d}f \otimes \mathrm{d}f = \lambda g, \\ \Delta_f f = \frac{1}{m-1}[S^\varphi - m(\mu + p)], \\ \tau(\varphi) = \mathrm{d}\varphi(\nabla f) + \frac{\nabla U}{\alpha}, \\ \frac{1}{2}S^\varphi = U(\varphi) + \mu, \\ (\mu + p)\nabla f - \nabla p = 0. \end{cases} \tag{2.55}$$

Definition 2.5. Let (M, g) be an m-dimensional Riemannian manifold and let $\varphi : (M, g) \to (N, h)$ be a smooth map that targets another Riemannian manifold. Given $\alpha \in \mathbb{R}, \alpha \neq 0, U \in C^\infty(N), \mu, p, \lambda \in C^\infty(M)$ and $f \in C^\infty(M)$, we will say that (M, g) is a φ-static perfect fluid space-time (φ-SPFST for short) if f is a solution of (2.55).

***Remark* 2.6.** Recall that every solution of Einstein's field equations for a stress energy tensor T is also a solution of the Cotton gravity equations, for the same stress-energy tensor. Therefore, we have that every φ-SPFST is also a C-φ-PF.

The next result, which corresponds to the case $\eta = 1$ of Proposition 4.1 of [13], gives some integrability conditions for φ-SPFSTs. We will prove a generalization of its first part in Proposition 3.1.

Proposition 2.7. *Let (M, g) be a manifold of dimension $m \geq 3$. Let $\varphi : (M, g) \to (N, h)$, $U : (N, h) \to \mathbb{R}$, $\lambda : (M, g) \to \mathbb{R}$ be smooth maps, $\alpha \in \mathbb{R} \setminus \{0\}$ and let $f \in C^\infty(M)$; assume that equations (2.55).i) and (2.55).iii) hold on M. We then have*

$$(m-1)D^A_{ijk} = C^\varphi_{ijk} + f_t W^\varphi_{tijk} + \frac{U^a}{m-1}\left(\varphi^a_j \delta_{ik} - \varphi^a_k \delta_{ij}\right) \tag{2.56}$$

and

$$\begin{aligned}(m-2)B^\varphi_{ij} =&(m-1)\left[D^\varphi_{ijk,k} - \frac{\alpha}{m-2}\varphi^a_{ss}\varphi^a_i f_j\right] + \frac{m-3}{m-2} f_k C^\varphi_{jik} - W^\varphi_{tijk} f_t f_k \\ &+ \frac{U^a}{m-1}\left[(m-2)\varphi^a_{ij} - \frac{1}{m-2}\varphi^a_{ss}\delta_{ij}\right] \\ &+ \frac{U^{ab}}{m-1}\left[\varphi^a_k \varphi^b_k \delta_{ij} - m\varphi^a_i \varphi^b_j\right] \\ &+ f_j U^a \varphi^a_i .\end{aligned} \tag{2.57}$$

***Remark* 2.8.** It is easy to prove how, assuming the validity of (2.55).*i*), one has $D^A = D^B$. Therefore, under this assumption, (2.50).*i*) becomes (3.3).
Equation (2.57) is important in order to obtain rigidity results for φ-SPFSTs, see Theorem 4.17 of [13]. A generalization of (2.57), with applications to C-φ-PFs, is given by Proposition 2.20 below.

2.3 Local structure and a rigidity result

In this section, we prove a rigidity result for a C-φ-PF (in fact, for a slightly more general system). In Section 2.3.1 we study how the vanishing of D^A and D^B, as defined in (2.47) and (2.48), affects the geometry of regular level sets of f. Then we find conditions that grant the validity of $D^A \equiv D^B \equiv 0$. In doing so, we also show how C-φ-PFs are related to the existence of suitable Codazzi tensors.

2.3.1 The local structure of a C-φ-PF

We are going to prove the next

Theorem 2.9. *Let (M, g) be a smooth, m-dimensional Riemannian manifold and let $\varphi : (M, g) \to (N, h)$ be a smooth map that targets another Riemannian manifold*

(N,h). For a constant $0\neq\alpha\in\mathbb{R}$ and some functions $U\in C^\infty(N)$, $f\in C^\infty(M)$, assume that the following system holds:

$$\begin{cases} 0 = C^\varphi_{ijk} + f_t W^\varphi_{tijk} - \frac{1}{m-1}U^a\varphi^a_k\delta_{ij} + \frac{1}{m-1}U^a\varphi^a_j\delta_{ik} - D^A_{ijk} - (m-2)D^B_{ijk}, \\ \tau(\varphi) = \mathrm{d}\varphi(\nabla f) + \frac{\nabla U}{\alpha}. \end{cases} \tag{2.58}$$

Assume that $D^A\equiv D^B\equiv 0$ and that φ is $\frac{1}{\alpha}U$-harmonic. Let $c\in\mathbb{R}$ be a regular value of f and let $\Sigma=\Sigma_c$ be the corresponding level set. Then, $\forall p\in\Sigma$, there exists in M an open neighbourhood A of p such that $g_{|A}$ is a warped product $(I\times_\rho(\Sigma\cap A), dr^2+\rho^2(r)g_\Sigma)$ where I is an open interval, r is the signed distance function from Σ and $\rho: I\to\mathbb{R}$ is the warping factor. Moreover $U(\varphi)$ is locally constant on Σ and (Σ, g_Σ) satisfies

$$\begin{cases} \mathrm{Ric}^{\varphi_{|\Sigma}} = \frac{S^{\varphi_{|\Sigma}}}{m-1}g_\Sigma, \\ h\big(\tau(\varphi_{|\Sigma}), \mathrm{d}\varphi_{|\Sigma}\big) = 0, \end{cases} \tag{2.59}$$

where $\mathrm{Ric}^{\varphi_{|\Sigma}}$ and $S^{\varphi_{|\Sigma}}$ denote the φ-Ricci curvature and the φ-scalar curvature of (Σ, g_Σ).

Remark 2.10. Theorem 2.9 is similar to Theorem 4.14 of [13]. There, (M,g) satisfied

$$\begin{cases} \mathrm{Ric}^\varphi + \mathrm{Hess}\, f - \eta \mathrm{d}f\otimes\mathrm{d}f = \lambda g \\ \tau(\varphi) = \mathrm{d}\varphi(\nabla f) + \frac{\nabla U}{\alpha}. \end{cases} \tag{2.60}$$

As we saw in Section 2.2, such a manifold satisfies (2.58) when $\eta=1$, but this is just a necessary condition, so that the present setting is more general. The proof of Theorem 2.9 follows the same lines of that of Theorem 4.14 of [13] with some small modifications. We will split the proof into several propositions.

From here on, c will be a regular value of f, Σ will be the corresponding level set and $p\in\Sigma$. At such a p, $\{e_i\}_{i=1}^m$ will be an orthonormal frame such that

$$e_m = \frac{\nabla f}{|\nabla f|}$$

and upper case letters $A,B,C,\ldots$, will denote indexes ranging from 1 to $m-1$. The dual co-frame of $\{e_i\}_{i=1}^m$ will be denoted by $\{\theta^i\}_{i=1}^m$. For such a co-frame we have

$$\theta^m_A(e_B) = \Pi_{AB} = -\frac{f_{AB}}{|\nabla f|},$$

see Proposition 8.1 of [21].

Proposition 2.11. *Let (M, g) be a Riemannian manifold and let P be a 2-covariant symmetric tensor field on M. For $f \in C^\infty(M)$, define*

$$D(P)_{ijk} := \frac{1}{m-2}\Big[f_k P_{ij} - f_j P_{ik} + \frac{f_t}{m-1}(P_{tk}\delta_{ij} - P_{tj}\delta_{ik}) - \frac{P_{tt}}{m-2}(f_k\delta_{ij} - f_j\delta_{ik})\Big]. \tag{2.61}$$

Then $D(P) \equiv 0$ if and only if we have that the two following conditions are satisfied:

i) at any regular point $p \in M$ of f, ∇f is an eigenvector of P;

ii) at any regular point $p \in M$ of f and for any orthonormal frame $\{e_i\}_{i=1}^m$ at p such that $e_m = \frac{\nabla f}{|\nabla f|}$ we have

$$P_{AB} = \frac{P_{CC}}{m-1}\delta_{AB}.$$

***Remark** 2.12.* Proposition 2.11 implies that, when $D(P) \equiv 0$, P has at most two eigenvalues at any regular point p of f, with eigenspaces $\langle\nabla f\rangle$ and $\langle\nabla f\rangle^\perp$.

Proof of Proposition 2.11. For the time of this proof, we will set $D = D(P)$ for simplicity. The condition that ∇f be an eigenvector of P at p is equivalent to

$$f_j f_t P_{tk} = f_k f_t P_{tj}. \tag{2.62}$$

Indeed, in the dual co-frame $\{\theta^i\}_{i=1}^m$ of $\{e_i\}$, (2.62) is equivalent to

$$|\nabla f| f_j P_{mk} = |\nabla f| f_k P_{mj};$$

therefore, since $f_A = 0, \forall A \in \{1, ..., m-1\}$ and $f_m = |\nabla f| \neq 0$, we get

$$0 = f_A P_{mm} = f_m P_{mA} = |\nabla f| P_{mA}$$

so that $P_{mA} = 0, \forall A \in \{1, ..., m-1\}$. Now, contracting (2.61) with f_i we get

$$\begin{aligned} f_i D_{ijk} =& \frac{1}{m-2}\Big[f_t f_k P_{tj} - f_t f_j P_{tk} + \frac{1}{m-1} f_t (P_{tk} f_j - P_{tj} f_k) \\ &- \frac{P_{tt}}{m-1}(f_j f_k - f_k f_j)\Big] \\ =& \frac{1}{m-1}(f_t f_k P_{tj} - f_t f_j P_{tk}) \end{aligned}$$

implying that (2.62) is equivalent to

$$D_{mjk} = 0. \tag{2.63}$$

Having the validity of (2.62), we also get

$$\begin{aligned} D_{ABC} =& \frac{1}{m-1}\bigg[f_C P_{AB} - f_B P_{AC} + \frac{|\nabla f|}{m-1}(P_{mC}\delta_{AB} - P_{mB}\delta_{AC}) \\ & - \frac{P_{tt}}{m-1}(f_C\delta_{AB} - f_B\delta_{AC})\bigg] \\ =& 0 \end{aligned}$$

since $f_A = f_B = f_C = 0$. We are only left to prove that, assuming (2.62), the condition

$$D_{ABm} = 0$$

is equivalent to

$$P_{AB} = \frac{P_{CC}}{m-1}\delta_{AB}.$$

From (2.61), using $f_A = f_B = 0, f_m = |\nabla f|$ and $P_{Am} = 0, \forall A$, we deduce

$$\begin{aligned} D_{ABm} =& \frac{1}{m-2}\bigg[f_m P_{AB} - f_B P_{Am} + \frac{|\nabla f|}{m-1}(P_{mm}\delta_{AB} - P_{mB}\delta_{Am}) \\ & - \frac{P_{tt}}{m-1}(f_m\delta_{AB} - f_B\delta_{Am})\bigg] \\ =& \frac{|\nabla f|}{m-2}\bigg[P_{AB} + \frac{P_{mm}}{m-1}\delta_{AB} - \frac{1}{m-1}(P_{CC} + P_{mm})\delta_{AB}\bigg] \\ =& \frac{|\nabla f|}{m-2}\bigg[P_{AB} - \frac{P_{CC}}{m-1}\delta_{AB}\bigg] \end{aligned}$$

and we are done. □

Corollary 2.13. *Let (M, g) be a Riemannian manifold and let $f \in C^\infty(M)$. Assume $D^B \equiv 0$. Then:*

i) for every regular point p of f, ∇f is an eigenvector of Hess f*;*

ii) for any regular level set Σ of f, $i : \Sigma \hookrightarrow M$ is totally umbilical.

Proof. Since $D^B = D(-\operatorname{Hess} f)$, the conclusion follows from Proposition 2.11 and $f_{AB} = -|\nabla f|\Pi_{AB}$. □

Proposition 2.14. *Let (M, g) be a Riemannian manifold and let $f \in C^\infty(M)$. Assume that, at any regular point p of f,*

$$D^B = 0, \quad R^\varphi_{Am} = 0, \quad \forall A, \quad \mathrm{d}\varphi(\nabla f) = 0. \tag{2.64}$$

For a regular value $c \in \mathbb{R}$ of f, let $\Sigma = f^{-1}(c)$ be the corresponding level set of f. Then, $\forall p \in \Sigma$, $\exists A$, $p \in A \subset M$ open in M such that $g_{|A}$ is a warped product

$$\big(I \times_\rho (\Sigma \cap A), \mathrm{d}r^2 + \rho^2(r) g_\Sigma\big) \tag{2.65}$$

where I is an open interval, r is the signed distance function from $\Sigma \cap A$ and $\rho : I \to \mathbb{R}$ is the warping factor.

Proof. First, we prove that the mean curvature H of Σ is constant on it. Since $D^B = 0$ we have, by Corollary 2.13,

$$\Pi_{AB} = \frac{H}{m-1}\delta_{AB}.$$

Using Codazzi equations and $R^\varphi_{Am} = 0$ we get

$$\begin{aligned} H_B = \Pi_{AA,B} &= \Pi_{AB,A} - R_{mAAB} \\ &= \frac{H_B}{m-1} + R^\varphi_{mB} + \alpha\varphi^a_m\varphi^a_B \\ &= \frac{H_B}{m-1} + \alpha\varphi^a_m\varphi^a_B \end{aligned}$$

and therefore

$$\frac{m-2}{m-1}H_B = \alpha\varphi^a_m\varphi^a_B.$$

From $\mathrm{d}\varphi(\nabla f) = 0$ we get

$$\varphi^a_m = 0$$

so that $H_B = 0$ and H is constant on Σ.
Since $D^B = 0$, we have, see Corollary 2.13,

$$\frac{1}{2}|\nabla f|^2_A = f_t f_{At} = |\nabla f| f_{Am} = 0$$

so that $|\nabla f|$ is locally constant on each regular level set of f. As it is well-known, this implies that, $\forall p \in \Sigma$, there exists an open neighbourhood A of p in M such that $f_{|A}$ only depends on r. See the proof of the equivalence of item (ii) and item (iii) of Lemma 4.4 of [13] for more details. By restricting A, we can assume that it is made only of regular points of f, and it is a tubular neighbourhood of $A \cap \Sigma$. We can choose r so that

$$\nabla r = \frac{\nabla f}{|\nabla f|}.$$

Expressing g in Fermi coordinates $(x^1, \dots, x^{m-1}, r)$ we get, on A,

$$g = \mathrm{d}r \otimes \mathrm{d}r + g_{AB}(\underline{x}, r)\,\mathrm{d}x^A \otimes \mathrm{d}x^B$$

and

$$\operatorname{Hess} f = f''\,\mathrm{d}r \otimes \mathrm{d}r + \frac{f'}{2}\,\partial_r g_{AB}(\underline{x}, r)\,\mathrm{d}x^A \otimes \mathrm{d}x^B$$

where $\underline{x} = (x^1, \dots, x^{m-1})$. Therefore

$$f_{AB} = \frac{f'}{2}\,\partial_r g_{AB}. \tag{2.66}$$

Since $i : \Sigma_{\bar{c}} \hookrightarrow M$ is totally umbilical, for every $\bar{c}$ sufficiently close to c, we get

$$\frac{f_{AB}}{f'} = -\Pi_{AB} = -\frac{H}{m-1}\,g_{AB} \tag{2.67}$$

Comparing (2.66) and (2.67) we have

$$-\frac{H}{m-1}\,g_{AB} = \frac{1}{2}\,\partial_r g_{AB}$$

and integrating the above expression

$$g_{AB}(\underline{x}, r) = e^{-2(m-1)^{-1}\int_0^r H(s)\,ds}\,g_{AB}(\underline{x}, 0);$$

since $g_{AB}(\underline{x}, 0)\,\mathrm{d}x^A \otimes \mathrm{d}x^B = g_\Sigma$, we are done. $\square$

Proposition 2.15. *Let* (M, g) *satisfy*

$$\begin{cases} 0 = C^\varphi_{ijk} + f_t W^\varphi_{tijk} - \frac{1}{m-1}U^a\varphi^a_k\delta_{ij} + \frac{1}{m-1}U^a\varphi^a_j\delta_{ik} - D^A_{ijk} - (m-2)D^B_{ijk}, \\ \tau(\varphi) = \mathrm{d}\varphi(\nabla f) + \frac{\nabla U}{\alpha}. \end{cases} \tag{2.68}$$

Assume that

$$D^A \equiv D^B \equiv 0$$

and

$$\tau(\varphi) = \frac{\nabla U}{\alpha}$$

and let Σ *be as before. Then,*

$$\begin{cases} R^{\varphi_{|\Sigma}} = \frac{S^{\varphi_{|\Sigma}}}{m-1} g_\Sigma, \\ h(\tau(\varphi_{|\Sigma}), \mathrm{d}\varphi_{|\Sigma}) = 0. \end{cases} \tag{2.69}$$

Proof. From (2.68).i) and $D^A \equiv D^B \equiv 0$ we get

$$0 = C^\varphi_{AmB} + |\nabla f| W^\varphi_{mAmB} - \frac{1}{m-1}U^a\varphi^a_B\delta_{Am} + \frac{1}{m-1}U^a\varphi^a_m\delta_{AB}. \tag{2.70}$$

We need to prove

$$W^\varphi_{mAmB} = 0.$$

Since $D^A = 0$, Proposition 2.11 implies

$$R^\varphi_{Am} = 0, \quad \forall A$$

and therefore

$$\begin{aligned}
0 = \mathrm{d}R^{\varphi}_{Am} &= R^{\varphi}_{Am,k}\theta^k + R^{\varphi}_{km}\theta^k{}_A + R^{\varphi}_{Ak}\theta^k{}_m \\
&= R^{\varphi}_{Am,k}\theta^k + R^{\varphi}_{Bm}\theta^B{}_A + R^{\varphi}_{mm}\theta^m{}_A + R^{\varphi}_{AB}\theta^B{}_m + R^{\varphi}_{Am}\theta^m_m \\
&= R^{\varphi}_{Am,k}\theta^k + R^{\varphi}_{mm}\theta^m{}_A + R^{\varphi}_{AB}\theta^B{}_m.
\end{aligned}$$

Since $D^A = 0$, Proposition 2.11 gives

$$R^{\varphi}_{AB} = \frac{S^{\varphi} - R^{\varphi}_{mm}}{m-1}\delta_{AB} \tag{2.71}$$

so that

$$\begin{aligned}
R^{\varphi}_{Am,k}\theta^k &= -R^{\varphi}_{mm}\theta^m{}_A - \frac{1}{m-1}(S^{\varphi} - R^{\varphi}_{mm})\theta^A{}_m \\
&= \frac{1}{m-1}(S^{\varphi} - mR^{\varphi}_{mm})\theta^m{}_A.
\end{aligned}$$

Recalling that $\theta^m{}_A(e_B) = \Pi_{AB} = -\frac{f_{AB}}{|\nabla f|}$, we get

$$R^{\varphi}_{Am,B} = -\frac{1}{m-1}(S^{\varphi} - mR^{\varphi}_{mm})\frac{f_{AB}}{|\nabla f|}\,. \tag{2.72}$$

By the definition of the φ-Cotton tensor, using (2.71) and (2.72) we obtain

$$\begin{aligned}
C^{\varphi}_{ABm} =& R^{\varphi}_{AB,m} - R^{\varphi}_{Am,B} - \frac{1}{2(m-1)}\left(S^{\varphi}_m\delta_{AB} - S^{\varphi}_B\delta_{Am}\right) \\
=& \frac{1}{m-1}\left(S^{\varphi} - R^{\varphi}_{mm}\right)_m\delta_{AB} + \frac{1}{m-1}\left(S^{\varphi} - mR^{\varphi}_{mm}\right)\frac{f_{AB}}{|\nabla f|} \\
&- \frac{S^{\varphi}_m}{2(m-1)}\delta_{AB} \\
=& \frac{1}{2(m-1)}S^{\varphi}_m\delta_{AB} - \frac{1}{m-1}R^{\varphi}_{mm,m}\delta_{AB} + \frac{1}{m-1}\left(S^{\varphi} - mR^{\varphi}_{mm}\right)\frac{f_{AB}}{|\nabla f|}.
\end{aligned}$$

By the φ-Schur identity

$$\begin{aligned}
\frac{1}{2}S^{\varphi}_m =& \alpha\varphi^a_{ll}\varphi^a_m + R^{\varphi}_{lm,l} \\
=& \alpha\varphi^a_{ll}\varphi^a_m + R^{\varphi}_{Am,A} + R^{\varphi}_{mm,m}
\end{aligned}$$

and, since $\varphi^a_{ll} = \frac{1}{\alpha}U^a$, we deduce

$$C^{\varphi}_{ABm} = \frac{1}{m-1}U^a\varphi^a_m\delta_{AB} + \frac{1}{m-1}R^{\varphi}_{Cm,C}\delta_{AB} + \frac{1}{m-1}\left(S^{\varphi} - mR^{\varphi}_{mm}\right)\frac{f_{AB}}{|\nabla f|}.$$

Taking the trace of (2.72) we have

$$R^{\varphi}_{Cm,C} = -\frac{1}{m-1}\left(S^{\varphi} - mR^{\varphi}_{mm}\right)\frac{f_{BB}}{|\nabla f|},$$

and therefore

$$C^{\varphi}_{ABm} = \frac{1}{m-1}U^a\varphi^a_m\delta_{AB} + \frac{1}{m-1}\left(S^{\varphi} - mR^{\varphi}_{mm}\right)\left(\frac{f_{AB}}{|\nabla f|} - \frac{f_{CC}}{(m-1)|\nabla f|}\delta_{AB}\right).$$

Since $i : \Sigma \hookrightarrow M$ is totally umbilical it holds

$$f_{AB} = \frac{f_{CC}}{m-1}\delta_{AB}$$

so that

$$C^{\varphi}_{ABm} = \frac{U^a\varphi^a_m}{m-1}\delta_{AB};$$

from (2.70) we get

$$0 = W^{\varphi}_{mAmB} \tag{2.73}$$

as we wanted to prove.
Using (2.73) and the definition of W^{φ} we deduce

$$\begin{aligned} R_{mAmB} =& W^{\varphi}_{mAmB} + \frac{1}{m-2}(R^{\varphi}_{mm}\delta_{AB} + R^{\varphi}_{AB}\delta_{mm} - R^{\varphi}_{mB}\delta_{Am} - R^{\varphi}_{Am}\delta_{mB}) \\ &- \frac{S^{\varphi}}{(m-1)(m-2)}(\delta_{mm}\delta_{AB} - \delta_{mB}\delta_{mA}) \\ =& \frac{1}{m-2}(R^{\varphi}_{mm}\delta_{AB} + R^{\varphi}_{AB} - \frac{S^{\varphi}}{m-1}\delta_{AB}), \end{aligned}$$

so that (2.71) implies

$$R_{mAmB} = \frac{1}{m-1}R^{\varphi}_{mm}\delta_{AB}. \tag{2.74}$$

From the Gauss equation and the fact that $i : \Sigma \hookrightarrow M$ is totally umbilical we deduce

$${}^{\Sigma}R_{AC} = R_{AC} - R_{AmCm} + (m-2)H^2\delta_{AC}$$

so that, by the definition of $\mathrm{Ric}^{\varphi_{|\Sigma}}$, we obtain

$$R^{\varphi_{|\Sigma}}_{AC} = {}^{\Sigma}R_{AC} - \alpha\varphi^a_A\varphi^a_C = R^{\varphi}_{AC} - R_{AmCm} + (m-2)H^2\delta_{AC}.$$

From (2.71) and (2.74) we have

$$\begin{aligned} R^{\varphi_{|\Sigma}}_{AC} &= \frac{S^{\varphi} - R^{\varphi}_{mm}}{m-1}\delta_{AC} - \frac{1}{m-1}R^{\varphi}_{mm}\delta_{AC} + (m-2)H^2\delta_{AC} \\ &= \left[\frac{1}{m-1}(S^{\varphi} - 2R^{\varphi}_{mm}) + (m-2)H^2\right]\delta_{AC}. \end{aligned}$$

Tracing the above equation, it holds

$$S^{\varphi_{|\Sigma}} = S^{\varphi} - 2R^{\varphi}_{mm} + (m-2)(m-1)H^2 \tag{2.75}$$

so that

$$R^{\varphi|_\Sigma}_{AC} = \frac{S^{\varphi|_\Sigma}}{m-1}\delta_{AC},$$

which is (2.69).i).

To prove (2.69).ii), we first need to prove that $S^{\varphi|_\Sigma}$ is constant. Since $D^A \equiv 0$, by its definition, (2.47), we get

$$0 = R^\varphi_{ij} f_k - R^\varphi_{ik} f_j + \frac{1}{m-1} f_t (R^\varphi_{tk}\delta_{ij} - R^\varphi_{tj}\delta_{ik}) - \frac{S^\varphi}{m-1}(f_k\delta_{ij} - f_j\delta_{ik}).$$

Taking the divergence of the latter with respect to i and using the φ-Schur identity we obtain

$$\begin{aligned}
0 =& R^\varphi_{ij,i} f_k + R^\varphi_{ij} f_{ik} - R^\varphi_{ik,i} f_j - R^\varphi_{ik} f_{ij} \\
&+ \frac{1}{m-1} f_t (R^\varphi_{tk,j} - R^\varphi_{tj,k}) + \frac{1}{m-1}(f_{tj} R^\varphi_{tk} - f_{tk} R^\varphi_{tj}) \\
&- \frac{1}{m-1}(S^\varphi_j f_k - S^\varphi_k f_j) - \frac{1}{m-1} S^\varphi (f_{jk} - f_{kj}) \\
=& \frac{1}{2}\left(S^\varphi_j f_k - S^\varphi_k f_j\right) - \alpha \varphi^a_{tt}\left(\varphi^a_j f_k - \varphi^a_k f_j\right) + \frac{m-2}{m-1}(R^\varphi_{ij} f_{ik} - R^\varphi_{ik} f_{ij}) \\
&+ \frac{1}{m-1} f_t (R^\varphi_{tk,j} - R^\varphi_{tj,k}) - \frac{1}{m-1}(S^\varphi_j f_k - S^\varphi_k f_j).
\end{aligned}$$

Using the definition of C^φ and

$$\tau(\varphi) = \frac{\nabla U}{\alpha}$$

we get

$$\begin{aligned}
0 =& \left[\frac{1}{2} + \frac{1}{2(m-1)^2} - \frac{1}{m-1}\right]\left(S^\varphi_j f_k - S^\varphi_k f_j\right) + \frac{m-2}{m-1}\left(R^\varphi_{tj} f_{tk} - R^\varphi_{tk} f_{tj}\right) \\
&+ \frac{1}{m-1} f_t C^\varphi_{tkj} - U^a\left(\varphi^a_j f_k - \varphi^a_k f_j\right).
\end{aligned}$$

From (2.58) and $D^A \equiv D^B \equiv 0$, we have

$$f_t C^\varphi_{tkj} = \frac{1}{m-1} U^a \varphi^a_j f_k - \frac{1}{m-1} U^a \varphi^a_k f_j,$$

so that

$$\begin{aligned}
0 =& \frac{(m-2)^2}{2(m-1)^2}\left(S^\varphi_j f_k - S^\varphi_k f_j\right) + \frac{m-2}{m-1}\left(R^\varphi_{tj} f_{tk} - R^\varphi_{tk} f_{tj}\right) \\
&+ \frac{1-(m-1)^2}{(m-1)^2} U^a\left(\varphi^a_j f_k - \varphi^a_k f_j\right).
\end{aligned}$$

From Proposition 2.11, and since $D^A \equiv D^B \equiv 0$, Ric^φ and $\mathrm{Hess}\, f$ have the same eigenspaces so that they commute, i.e.

$$R^\varphi_{tj} f_{tk} - R^\varphi_{tk} f_{tj} = 0.$$

Therefore,

$$0=\Big[\frac{(m-2)^2}{2}S^{\varphi}_j+(1-(m-1)^2)U^a\varphi^a_j\Big]f_k-\Big[\frac{(m-2)^2}{2}S^{\varphi}_k+(1-(m-1)^2)U^a\varphi^a_k\Big]f_j.$$

Choosing $j=A$ and $k=B$ we get

$$\begin{aligned}0=&\Big[\frac{(m-2)^2}{2}S^{\varphi}_A+(1-(m-1)^2)U^a\varphi^a_A\Big]f_m-\Big[\frac{(m-2)^2}{2}S^{\varphi}_m+(1-(m-1)^2)U^a\varphi^a_m\Big]f_A\\=&|\nabla f|\Big[\frac{(m-2)^2}{2}S^{\varphi}_A+(1-(m-1)^2)U^a\varphi^a_A\Big],\end{aligned}$$

so that

$$0=\frac{(m-2)^2}{2}S^{\varphi}_A+(1-(m-1)^2)U^a\varphi^a_A. \tag{2.76}$$

Recall equation (2.75), that is,

$$S^{\varphi_{|\Sigma}}=S^{\varphi}-2R^{\varphi}_{mm}+(m-2)(m-1)H^2$$

and differentiate

$$\begin{aligned}S^{\varphi_{|\Sigma}}_A=&S^{\varphi}_A-2R^{\varphi}_{mm,A}+2(m-2)(m-1)HH_A\\=&S^{\varphi}_A-2R^{\varphi}_{mm,A},\end{aligned}$$

where the last equality follows because H is constant on Σ. Now,

$$\begin{aligned}R^{\varphi}_{mm,A}=&R_{mm,A}-2\alpha\varphi^a_m\varphi^a_{Am}\\=&R_{mm,A}\end{aligned}$$

since $\varphi^a_m=0$. Since g is a warped product metric, R_{mm} is a function of r alone so that it is locally constant on Σ, see chapter 1.7 of [1]. Therefore $R_{mm,A}=0$, and

$$S^{\varphi_{|\Sigma}}_A=S^{\varphi}_A$$

so that (2.76) becomes

$$0=\frac{(m-2)^2}{2}S^{\varphi_{|\Sigma}}_A+(1-(m-1)^2)U^a\varphi^a_A. \tag{2.77}$$

By a direct computation, we have

$$\begin{aligned}\tau^a(\varphi_{|\Sigma})=&\tau^a(\varphi)-\varphi^a_{mm}+(m-1)\varphi^a_mH\\=&\tau^a(\varphi)-\varphi^a_{mm}.\end{aligned}$$

We want to prove that $\varphi^a_{mm}=0$. Since $\tau(\varphi)=\frac{\nabla U}{\alpha}$, (2.68).$ii$) gives

$$\varphi^a_tf_t=0.$$

Differentiating, since $f_{Am} = 0 = \varphi^a_m$

$$\begin{aligned}0 =& \varphi^a_{tm} f_t + \varphi^a_t f_{tm} \\ =& \varphi^a_{mm}|\nabla f| + \varphi^a_m f_{mm} + \varphi^a_A f_A m \\ =& \varphi^a_{mm}|\nabla f|\end{aligned}$$

so that

$$\tau^a(\varphi_{|\Sigma}) = \tau(\varphi) = \frac{\nabla U}{\alpha}. \tag{2.78}$$

Taking the divergence of (2.71) and using the φ-Schur identity we get

$$R^{\varphi_{|\Sigma}}_{AC,C} = \frac{1}{2} S^{\varphi_{|\Sigma}}_A - \alpha\tau^a(\varphi_{|\Sigma})\varphi^a_A = \frac{S^{\varphi_{|\Sigma}}_A}{m-1}$$

so that

$$\frac{m-2}{2(m-1)} S^{\varphi_{|\Sigma}}_A = \tau^a(\varphi_{|\Sigma})\varphi^a_A = U^a \varphi^a_A \tag{2.79}$$

Combining with (2.77) we get

$$\frac{(m-2)^2}{2} S^{\varphi_{|\Sigma}}_A + (1-(m-1)^2)\frac{m-2}{2(m-1)} S^{\varphi_{|\Sigma}}_A = 0;$$

since

$$\frac{(m-2)^2}{2} + (1-(m-1)^2)\frac{m-2}{2(m-1)} = -\frac{(m-2)^2}{2(m-1)} < 0$$

we obtain

$$S^{\varphi_{|\Sigma}}_A = 0.$$

From (2.77) we also have

$$U^a \varphi^a_A = 0 \tag{2.80}$$

and from (2.78)

$$0 = U^a \varphi^a_A = \alpha\tau^a(\varphi_{|\Sigma})\varphi^a_A$$

which is (2.69).ii). □

Proof of Theorem 2.9. This follows from Propositions 2.11 and 2.15 combined. □

2.3.2 Cotton Gravity and Codazzi Tensors

It has been first pointed out by Mantica and Molinari, [71], how, on an n-dimensional Lorentzian manifold $(\widehat{M}, \widehat{g})$, Cotton Gravity's field equations

$$\widehat{C} = -(n-2)\,\mathrm{div}_1\,\widehat{\mathfrak{T}} \tag{2.81}$$

be equivalent to the requirement that the tensor

$$\widehat{Z}_{\alpha\beta} := \widehat{R}_{\alpha\beta} - \widehat{T}_{\alpha\beta} - \frac{\widehat{S} - 2\mathrm{tr}_{\widehat{g}}\widehat{T}}{2(n-1)} g_{\alpha\beta} \tag{2.82}$$

be Codazzi. Indeed, by (2.82)

$$\begin{aligned}\widehat{Z}_{\alpha\beta,\gamma} - \widehat{Z}_{\alpha\gamma,\beta} =& \widehat{R}_{\alpha\beta,\gamma} - \widehat{R}_{\alpha\gamma,\beta} - \widehat{T}_{\alpha\beta,\gamma} + \widehat{T}_{\alpha\gamma,\beta} \\ & - \frac{\widehat{S}_\gamma}{2(n-1)} g_{\alpha\beta} + \frac{\widehat{S}_\beta}{2(n-1)} g_{\alpha\gamma} + \frac{\left(\mathrm{tr}_{\widehat{g}}\widehat{T}\right)_\gamma}{n-1} g_{\alpha\beta} - \frac{\left(\mathrm{tr}_{\widehat{g}}\widehat{T}\right)_\beta}{n-1} g_{\alpha\gamma},\end{aligned}$$

that is,

$$\widehat{Z}_{\alpha\beta,\gamma} - \widehat{Z}_{\alpha\gamma,\beta} = \widehat{C}_{\alpha\beta\gamma} - \widehat{T}_{\alpha\beta,\gamma} + \widehat{T}_{\alpha\gamma,\beta} + \frac{1}{n-1}\left[\left(\mathrm{tr}_{\widehat{g}}\widehat{T}\right)_\gamma g_{\alpha\beta} - \left(\mathrm{tr}_{\widehat{g}}\widehat{T}\right)_\beta g_{\alpha\gamma}\right]. \tag{2.83}$$

By equation (2.3) we see that the right hand side of (2.83) vanishes if and only if (2.81) holds, as we wanted to show. As we saw in the previous section, choosing, on a static space-time $\widehat{M} = M \times_f \mathbb{R}$, the stress-energy tensor

$$\widehat{T} = \widehat{T}^F + \widehat{T}^{\widehat{\varphi}} = \alpha\widehat{\varphi}^* h - \left(\frac{\alpha}{2}|\mathrm{d}\widehat{\varphi}|^2 + U(\widehat{\varphi})\right)\widehat{g} + (\widehat{\mu} + \widehat{p})\, e^{-2\widehat{f}}\mathrm{d}t \otimes \mathrm{d}t + \widehat{p}\widehat{g}, \tag{2.84}$$

and assuming that $\mathrm{div}_1\,\widehat{T}^F = \mathrm{div}_1\,\widehat{T}^{\widehat{\varphi}} = 0$ holds, we have that (2.81) breaks down into system

$$\begin{cases} 0 = & C^\varphi_{ijk} + f_{ijk} - f_{ikj} - f_{ik}f_j + f_{ij}f_k \\ & -\frac{1}{m}[(f_{ttk} - 2f_tf_{tk})\delta_{ij} - (f_{ttj} - 2f_tf_{tj})\delta_{ik}] \\ & -\frac{1}{m}U^a\left(\varphi^a_k\delta_{ij} - \varphi^a_j\delta_{ik}\right) + \frac{1}{2m(m-1)}\left(S^\varphi_k\delta_{ij} - S^\varphi_j\delta_{ik}\right) \\ & -\frac{1}{m}(\mu_k\delta_{ij} - \mu_j\delta_{ik}), \\ 0 = & -\frac{m-1}{m}f_{tti} + \frac{m-2}{m}f_tf_{ti} + (\Delta f)f_i + \frac{1}{2m}S^\varphi_i - f_tR^\varphi_{ti} \\ & +\frac{1}{m}U^a\varphi^a_i - \frac{m-1}{m}\mu_i, \\ 0 = & h\left(\tau(\varphi) - \mathrm{d}\varphi(\nabla f) - \frac{\nabla U}{\alpha}, \mathrm{d}\varphi\right), \\ 0 = & \nabla p - (\mu + p)\nabla f. \end{cases} \tag{2.85}$$

Define a 2-covariant symmetric tensor on M

$$Z := \mathrm{Ric}^\varphi + \mathrm{Hess}\, f - \mathrm{d}f \otimes \mathrm{d}f - \lambda g, \tag{2.86}$$

for some $\lambda \in C^\infty(M)$. In the following we will make different choices of λ. If

$$\lambda = \frac{S^\varphi + 2\Delta_f f + 2U(\varphi) + 2\mu}{2m},$$

then Z is the projection of $\widehat{Z}$ on (M, g). Indeed, tracing (2.84) with respect to $\widehat{g}$ we find

$$\begin{aligned}\mathrm{tr}_{\widehat{g}}\widehat{T} &= \alpha|\mathrm{d}\widehat{\varphi}|^2 - \alpha\frac{n}{2}|\mathrm{d}\widehat{\varphi}|^2 - nU(\widehat{\varphi}) - (\widehat{\mu} + \widehat{p}) + n\widehat{p} \\ &= \alpha\frac{2-n}{2}|\mathrm{d}\widehat{\varphi}|^2 - nU(\widehat{\varphi}) - \widehat{\mu} + (n-1)\widehat{p},\end{aligned}$$

so that (2.82) becomes

$$\begin{aligned}\widehat{Z}_{\alpha\beta} =& \widehat{R}_{\alpha\beta} - \alpha\widehat{\varphi}^* h + \frac{\alpha}{2}|\mathrm{d}\widehat{\varphi}|^2\widehat{g} + U(\widehat{\varphi})\widehat{g} - (\widehat{\mu}+\widehat{p})e^{-2\widehat{f}}\mathrm{d}t\otimes\mathrm{d}t \\ &- \widehat{p}g_{\alpha\beta} - \frac{\widehat{S} - \alpha(2-n)|\mathrm{d}\widehat{\varphi}|^2 + 2nU(\widehat{\varphi}) + 2\widehat{\mu} - 2(n-1)\widehat{p}}{2(n-1)}g_{\alpha\beta}\end{aligned}$$

and, simplifying,

$$\widehat{Z}_{\alpha\beta} = \widehat{R}^{\widehat{\varphi}}_{\alpha\beta} - (\widehat{\mu}+\widehat{p})e^{-2\widehat{f}}\mathrm{d}t\otimes\mathrm{d}t - \frac{\widehat{S}^{\widehat{\varphi}} + 2U(\widehat{\varphi}) + 2\widehat{\mu}}{2(n-1)}g_{\alpha\beta}.$$

Recall from Proposition 1.14 that

$$\widehat{R}_{ij} = R_{ij} + f_{ij} - f_i f_j$$

and

$$\widehat{S} = S + 2\Delta_f f;$$

since $n = m+1$ we then have

$$\widehat{Z}_{ij} = R^{\varphi}_{ij} + f_{ij} - f_i f_j - \frac{S^{\varphi} + 2\Delta_f f + 2U(\varphi) + 2\mu}{2m}\delta_{ij}$$

as we claimed. A straightforward computation reveals that (2.85).i) is equivalent to the requirement that Z be Codazzi for the choice $\lambda = \frac{S^{\varphi}+2\Delta_f f+2U(\varphi)+2\mu}{2m}$. Recall that, using (2.85).ii) into (2.85).i), and elaborating a bit, the latter becomes

$$\begin{aligned}0 =& C^{\varphi}_{ijk} + f_t W^{\varphi}_{tijk} - \frac{1}{m-1}U^a\varphi^a_k\delta_{ij} + \frac{1}{m-1}U^a\varphi^a_j\delta_{ik} \\ &- D^A_{ijk} - (m-2)D^B_{ijk}.\end{aligned} \tag{2.87}$$

We want to characterize (2.87) in terms of the existence of a suitable Codazzi tensor.

Proposition 2.16. *Let (M,g) be an m-dimensional Riemannian manifold and $\varphi : (M,g) \to (N,h)$ a smooth map with (N,h) a second Riemannian manifold. Let $\alpha \in \mathbb{R}, \alpha \neq 0, f \in C^\infty(M)$. Assume that, for some $\lambda \in C^\infty(M)$, the tensor*

$$Z = \mathrm{Ric}^{\varphi} + \mathrm{Hess}\, f - \mathrm{d}f\otimes\mathrm{d}f - \lambda g$$

is a Codazzi tensor. Then we have the validity of

$$\begin{aligned}0 =& C^{\varphi}_{ijk} + f_t W^{\varphi}_{tijk} - \frac{\alpha}{m-1}(\varphi^a_{tt} - \varphi^a_t f_t)\big(\varphi^a_k\delta_{ij} - \varphi^a_j\delta_{ik}\big) \\ &- D^A_{ijk} - (m-2)D^B_{ijk}\end{aligned} \tag{2.88}$$

on (M,g).

The proof of Proposition 2.16 relies on the following

Lemma 2.17. *Let Z be defined as in (2.86). Set*

$$\overline{D}_{ijk} = Z_{ij,k} - Z_{ik,j} - \frac{1}{m-1}[(Z_{tt,k} - Z_{tk,t})\delta_{ij} - (Z_{tt,j} - Z_{tj,t})\delta_{ik}]. \tag{2.89}$$

Then we have

$$\begin{aligned} &D^A_{ijk} + (m-2)D^B_{ijk} + \overline{D}_{ijk} \\ &= C^{\varphi}_{ijk} + f_t W^{\varphi}_{tijk} - \frac{\alpha}{m-1}(\varphi^a_{tt} - \varphi^a_t f_t)\big(\varphi^a_k \delta_{ij} - \varphi^a_j \delta_{ik}\big). \end{aligned} \tag{2.90}$$

Remark 2.18. From (2.90) we see that $\overline{D}$ is independent from the choice of λ.

Proof of Lemma 2.17. We expand (2.89) using (2.86). From (2.86)

$$Z_{ij,k} = R^{\varphi}_{ij,k} + f_{ijk} - f_{ik}f_j - f_i f_{jk} - \lambda_k \delta_{ij},$$

so that

$$Z_{tt,k} = S^{\varphi}_k + (\Delta_f f)_k - m\lambda_k \tag{2.91}$$

and

$$Z_{tk,t} = R^{\varphi}_{tk,t} + f_{tkt} - (\Delta f)f_k - f_t f_{tk} - \lambda_k.$$

Using the φ-Schur identity and the Ricci commutation relations

$$\begin{aligned} Z_{tk,t} =& \frac{1}{2}S^{\varphi}_k - \alpha\varphi^a_{tt}\varphi^a_k + (\Delta f)_k + f_t R_{tk} - (\Delta f)f_k \\ &- f_t f_{tk} - \lambda_k; \end{aligned}$$

From the definition of Ric^{φ} we get

$$\begin{aligned} Z_{tk,t} =& \frac{1}{2}S^{\varphi}_k - \alpha\varphi^a_k(\varphi^a_{tt} - \varphi^a_t f_t) + (\Delta f)_k + f_t R^{\varphi}_{tk} - (\Delta f)f_k \\ &- f_t f_{tk} - \lambda_k. \end{aligned} \tag{2.92}$$

Taking the difference of (2.91) and (2.92) we have

$$\begin{aligned} Z_{tt,k} - Z_{tk,t} =& \frac{1}{2}S^{\varphi}_k - 2f_t f_{tk} - (m-1)\lambda_k - f_t R^{\varphi}_{tk} \\ &+ \alpha\varphi^a_k(\varphi^a_{tt} - \varphi^a_t f_t) + (\Delta f)f_k + f_t f_{tk} \end{aligned}$$

so that

$$\begin{aligned} Z_{tt,k} - Z_{tk,t} =& \frac{1}{2}S^{\varphi}_k - f_t f_{tk} - (m-1)\lambda_k - f_t R^{\varphi}_{tk} \\ &+ \alpha\varphi^a_k(\varphi^a_{tt} - \varphi^a_t f_t) + (\Delta f)f_k; \end{aligned} \tag{2.93}$$

then (2.89) becomes

$$
\begin{aligned}
\overline{D}_{ijk} =& R^{\varphi}_{ij,k} - R^{\varphi}_{ik,j} + f_{ijk} - f_{ikj} - f_{ik}f_j + f_{ij}f_k \\
& - \lambda_k\delta_{ij} + \lambda_j\delta_{ik} - \frac{1}{m-1}\left(\frac{1}{2}S^{\varphi}_k - f_t f_{tk} + (\Delta f)f_k\right)\delta_{ij} \\
& - \frac{1}{m-1}(-f_t R^{\varphi}_{tk} - (m-1)\lambda_k + \alpha\varphi^a_k(\varphi^a_{tt} - \varphi^a_t f_t))\delta_{ij} \\
& + \frac{1}{m-1}\left(\frac{1}{2}S^{\varphi}_j - f_t f_{tj} + (\Delta f)f_j - f_t R^{\varphi}_{tj} - (m-1)\lambda_j + \alpha\varphi^a_j(\varphi^a_{tt} - \varphi^a_t f_t)\right)\delta_{ik},
\end{aligned}
$$

that is,

$$
\begin{aligned}
\overline{D}_{ijk} =& C^{\varphi}_{ijk} + f_{ijk} - f_{ikj} - f_{ik}f_j + f_{ij}f_k \\
& - \frac{1}{m-1}[(\Delta f)f_k - f_t f_{tk} - f_t R^{\varphi}_{tk} + \alpha\varphi^a_k(\varphi^a_{tt} - \varphi^a_t f_t)]\delta_{ij} \\
& + \frac{1}{m-1}\big[(\Delta f)f_j - f_t f_{tj} - f_t R^{\varphi}_{tj} + \alpha\varphi^a_j(\varphi^a_{tt} - \varphi^a_t f_t)\big]\delta_{ik}.
\end{aligned}
\tag{2.94}
$$

Using the Ricci commutation relations and the definition of W^{φ} we get

$$
\begin{aligned}
f_{ijk} - f_{ikj} =& f_t R_{tijk} \\
=& f_t W^{\varphi}_{tijk} + \frac{1}{m-2}\big(f_j R^{\varphi}_{ik} - f_k R^{\varphi}_{ij} + f_t R^{\varphi}_{tj}\delta_{ik} - f_t R^{\varphi}_{tk}\delta_{ij}\big) \\
& - \frac{S^{\varphi}}{(m-1)(m-2)}(f_j\delta_{ik} - f_k\delta_{ij}).
\end{aligned}
\tag{2.95}
$$

Inserting (2.95) in (2.94) we find

$$
\begin{aligned}
\overline{D}_{ijk} =& C^{\varphi}_{ijk} + f_t W^{\varphi}_{tijk} + \frac{1}{m-2}\big(f_j R^{\varphi}_{ik} - f_k R^{\varphi}_{ij} + f_t R^{\varphi}_{tj}\delta_{ik} - f_t R^{\varphi}_{tk}\delta_{ij}\big) \\
& - \frac{S^{\varphi}}{(m-1)(m-2)}(f_j\delta_{ik} - f_k\delta_{ij}) - f_{ik}f_k + f_{ij}f_k \\
& - \frac{1}{m-1}[(\Delta f)f_k - f_t f_{tk} - f_t R^{\varphi}_{tk} + \alpha\varphi^a_k(\varphi^a_{tt} - \varphi^a_t f_t)]\delta_{ij} \\
& + \frac{1}{m-1}\big[(\Delta f)f_j - f_t f_{tj} - f_t R^{\varphi}_{tj} + \alpha\varphi^a_j(\varphi^a_{tt} - \varphi^a_t f_t)\big]\delta_{ik}.
\end{aligned}
$$

Simplifying and rearranging

$$
\begin{aligned}
\overline{D}_{ijk} =& C^{\varphi}_{ijk} + f_t W^{\varphi}_{tijk} + \frac{1}{m-2}\left[f_j R^{\varphi}_{ik} - f_k R^{\varphi}_{ij} - \frac{f_t}{m-1}\big(R^{\varphi}_{tk}\delta_{ij} - R^{\varphi}_{tj}\delta_{ik}\big)\right] \\
& - \frac{S^{\varphi}}{(m-1)(m-2)}(f_j\delta_{ik} - f_k\delta_{ij}) - f_{ik}f_j + f_{ij}f_k \\
& - \frac{1}{m-1}(-f_t f_{tk}\delta_{ij} + f_t f_{tj}\delta_{ik} + (\Delta f)f_k\delta_{ij} - (\Delta f)f_j\delta_{ik}) \\
& - \frac{\alpha}{m-1}(\varphi^a_{tt} - \varphi^a_t f_t)\big(\varphi^a_k\delta_{ij} - \varphi^a_j\delta_{ik}\big).
\end{aligned}
$$

From the definitions of D^A and D^B we deduce

$$\begin{aligned}&D^A_{ijk}+(m-2)D^B_{ijk}+\overline{D}_{ijk}\\&=C^{\varphi}_{ijk}+f_tW^{\varphi}_{tijk}-\frac{\alpha}{m-1}(\varphi^a_{tt}-\varphi^a_tf_t)\big(\varphi^a_k\delta_{ij}-\varphi^a_j\delta_{ik}\big)\end{aligned}$$

which is (2.90). □

Proof of Proposition 2.16. From the definition of $\overline{D}$ it is clear that $\overline{D}\equiv 0$ if Z is Codazzi so that (2.90) becomes (2.88). □

It is now clear how (2.87) generalizes (2.85).i): the former holds when there exists a function $\lambda\in C^\infty(M)$ such that Z is Codazzi, while the latter holds only when $\lambda=\frac{S^\varphi+2\Delta_f f+2U(\varphi)+2\mu}{2m}$ makes Z Codazzi.
The next result has (2.87) as the most important curvature assumption.

Theorem 2.19. *Let (M,g) be a complete m-dimensional Riemannian manifold and let $\varphi:(M,g)\to(N,h)$ be a smooth map of Riemannian manifolds. Let $\alpha\in\mathbb{R},\alpha>0,f\in C^\infty(M),U\in C^\infty(N)$. Assume that (M,g) satisfies*

$$\begin{cases}0=C^{\varphi}_{ijk}+f_tW^{\varphi}_{tijk}-\frac{1}{m-1}U^a\big(\varphi^a_k\delta_{ij}-\varphi^a_j\delta_{ik}\big)-D^A_{ijk}-(m-2)D^B_{ijk},\\ \tau(\varphi)-\mathrm{d}\varphi(\nabla f)=\frac{\nabla U}{\alpha}.\end{cases}\tag{2.96}$$

Let $S^2(M)$ be the space of 2-covariant symmetric tensors on M and define a linear map $F:S^2(M)\to C^\infty(M)$ by setting, for $\beta\in\Gamma(S^2(M)),\beta=\beta_{tk}\theta^t\otimes\theta^k$ locally,

$$F(\beta):=\left(\frac{m}{m-1}U^a\varphi^a_tf_k+D^A_{tik}f_i-W^{\varphi}_{tikj}f_if_j\right)\beta_{tk}.\tag{2.97}$$

Assume that

- *f is proper;*
- *$B^\varphi(\nabla f,\nabla f)=0$;*
- *φ is $\frac{1}{\alpha}U$-harmonic;*
- *$\forall p\in M$ regular for f, we have that ∇f is an eigenvector of* Ric^φ *at p;*
- $\mathrm{Ric}^\varphi+\mathrm{Hess}\,f\in\ker(F)$.

Then, for each regular level set Σ of f and for every $p\in\Sigma$, there exists $A\in M$ open such that $p\in A$ and $g_{|A}$ is a warped product metric. Moreover, (Σ,g_Σ) satisfies

$$\begin{cases}\mathrm{Ric}^{\varphi_{|\Sigma}}=\frac{S^{\varphi_{|\Sigma}}}{m-1}g_\Sigma,\\ \tau(\varphi_{|\Sigma})=0.\end{cases}$$

We divide the proof of Theorem 2.19 in several propositions.

Proposition 2.20. *Assume the validity of*

$$\tau(\varphi) - \mathrm{d}\varphi(\nabla f) = \frac{\nabla U}{\alpha}. \tag{2.98}$$

Then we have

$$\begin{aligned}
&D^A_{ijk,k} + (m-2)D^B_{ijk,k} + \overline{D}_{ijk,k} \\
=&(m-2)B^\varphi_{ij} - (R^\varphi_{tk} + f_{tk})W^\varphi_{tikj} + \alpha(R^\varphi_{tj} + f_{tj})\varphi^a_t\varphi^a_i \\
&- \frac{m-2}{m-1}U^a\varphi^a_{ij} + \frac{m}{m-1}U^{ab}\varphi^a_i\varphi^b_j + \frac{1}{(m-1)(m-2)}U^a\varphi^a_{tt}\delta_{ij} \\
&+ \frac{m-3}{m-2}f_tC^\varphi_{jti} + \frac{\alpha}{m-2}f_j\varphi^a_i\varphi^a_{tt} - \frac{1}{m-1}U^{ab}\varphi^a_k\varphi^b_k\delta_{ij}.
\end{aligned} \tag{2.99}$$

Proof. Take the divergence of (2.90)

$$\begin{aligned}
&D^A_{ijk,k} + (m-2)D^B_{ijk,k} + \overline{D}_{ijk,k} \\
=&C^\varphi_{ijk,k} + f_{tk}W^\varphi_{tijk} + f_tW^\varphi_{tijk,k} - \frac{1}{m-1}U^{ab}\varphi^b_k\left(\varphi^a_k\delta_{ij} - \varphi^a_j\delta_{ik}\right) \\
&- \frac{1}{m-1}U^a\left(\varphi^a_{kk}\delta_{ij} - \varphi^a_{ij}\right).
\end{aligned} \tag{2.100}$$

We will use equations (1.31) and (1.32), that is,

$$\begin{aligned}
W^\varphi_{tijk,t} =&\frac{m-3}{m-2}C^\varphi_{ikj} + \alpha\left(\varphi^a_{ij}\varphi^a_k - \varphi^a_{ik}\varphi^a_j\right) \\
&+ \frac{\alpha}{m-2}\varphi^a_{tt}\left(\varphi^a_j\delta_{ik} - \varphi^a_k\delta_{ij}\right)
\end{aligned}$$

and

$$\begin{aligned}
C^\varphi_{ijk,k} =&(m-2)B^\varphi_{ij} - R^\varphi_{tk}W^\varphi_{tikj} + \alpha R^\varphi_{tj}\varphi^a_t\varphi^a_i \\
&- \alpha\left(\varphi^a_{ij}\varphi^a_{tt} - \varphi^a_i\varphi^a_{ttj} - \frac{1}{m-2}|\tau(\varphi)|^2\delta_{ij}\right)
\end{aligned}$$

into (2.100) to get

$$\begin{aligned}
&D^A_{ijk,k} + (m-2)D^B_{ijk,k} + \overline{D}_{ijk,k} \\
=&(m-2)B^\varphi_{ij} - R^\varphi_{tk}W^\varphi_{tikj} + \alpha R^\varphi_{tj}\varphi^a_t\varphi^a_i \\
&- \alpha\left(\varphi^a_{ij}\varphi^a_{tt} - \varphi^a_i\varphi^a_{ttj} - \frac{1}{m-2}|\tau(\varphi)|\delta_{ij}\right) - f_{tk}W^\varphi_{tikj} \\
&+ \frac{m-3}{m-2}f_tC^\varphi_{jti} + \alpha f_t\left(\varphi^a_{ij}\varphi^a_t - \varphi^a_{jt}\varphi^a_i\right) \\
&+ \frac{\alpha}{m-2}f_t\varphi^a_{pp}(\varphi^a_i\delta_{tj} - \varphi^a_t\delta_{ij}) \\
&- \frac{1}{m-1}U^{ab}\varphi^b_k\left(\varphi^a_k\delta_{ij} - \varphi^a_j\delta_{ik}\right) - \frac{1}{m-1}U^a\left(\varphi^a_{kk}\delta_{ij} - \varphi^a_{ij}\right).
\end{aligned} \tag{2.101}$$

Taking the covariant derivative of (2.98) we get

$$\varphi^a_{ttj} - \varphi^a_{tj} f_t - \varphi^a_t f_{tj} = \frac{1}{\alpha} U^{ab} \varphi^b_j. \tag{2.102}$$

Using (2.102) and (2.98) into (2.101) we obtain

$$\begin{aligned}
&D^A_{ijk,k} + (m-2) D^B_{ijk,k} + \overline{D}_{ijk,k} \\
&\quad =(m-2) B^\varphi_{ij} - (R^\varphi_{tk} + f_{tk}) W^\varphi_{tikj} + \alpha R^\varphi_{tj} \varphi^a_t \varphi^a_i \\
&\qquad - U^a \varphi^a_{ij} + U^{ab} \varphi^a_i \varphi^b_j + \alpha f_{tj} \varphi^a_t \varphi^a_i + \frac{\alpha}{m-2} |\tau(\varphi)|^2 \delta_{ij} \\
&\qquad + \frac{m-3}{m-2} f_t C^\varphi_{jti} + \frac{\alpha}{m-2} \varphi^a_{tt} \varphi^a_i f_j - \frac{\alpha}{m-2} f_t \varphi^a_t \varphi^a_{pp} \delta_{ij} \\
&\qquad - \frac{1}{m-1} U^{ab} \varphi^b_k \big(\varphi^a_k \delta_{ij} - \varphi^a_j \delta_{ik} \big) - \frac{1}{m-1} U^a \big(\varphi^a_{kk} \delta_{ij} - \varphi^a_{ij} \big) \\
&\quad =(m-2) B^\varphi_{ij} - (R^\varphi_{tk} + f_{tk}) W^\varphi_{tikj} + \alpha \big(R^\varphi_{tj} + f_{tj} \big) \varphi^a_t \varphi^a_i \\
&\qquad - \frac{m-2}{m-1} U^a \varphi^a_{ij} + \frac{m}{m-1} U^{ab} \varphi^a_i \varphi^b_j + \frac{1}{m-2} U^a \varphi^a_{tt} \delta_{ij} \\
&\qquad + \frac{m-3}{m-2} f_t C^\varphi_{jti} + \frac{\alpha}{m-2} \varphi^a_{tt} \varphi^a_i f_j - \frac{1}{m-1} U^{ab} \varphi^a_k \varphi^b_k \delta_{ij} - \frac{1}{m-1} U^a \varphi^a_{tt} \delta_{ij}.
\end{aligned}$$

After some simplifications, the latter becomes

$$\begin{aligned}
&D^A_{ijk,k} + (m-2) D^B_{ijk,k} + \overline{D}_{ijk,k} \\
&\quad =(m-2) B^\varphi_{ij} - (R^\varphi_{tk} + f_{tk}) W^\varphi_{tikj} + \alpha (R^\varphi_{tj} + f_{tj}) \varphi^a_t \varphi^a_i \\
&\qquad - \frac{m-2}{m-1} U^a \varphi^a_{ij} + \frac{m}{m-1} U^{ab} \varphi^a_i \varphi^a_j + \frac{1}{(m-1)(m-2)} U^a \varphi^a_{tt} \delta_{ij} \\
&\qquad + \frac{m-3}{m-2} f_t C^\varphi_{jti} + \frac{\alpha}{m-2} f_j \varphi^a_i \varphi^a_{tt} - \frac{1}{m-1} U^{ab} \varphi^a_k \varphi^b_k \delta_{ij}.
\end{aligned}$$

that is, (2.20). □

Let

$$Y_k := f_i f_j \big(D^A_{ijk} + (m-2) D^B_{ijk} + \overline{D}_{ijk} \big) - \frac{1}{m-1} U^a \big(\varphi^a_j f_j f_k - |\nabla f|^2 \varphi^a_k \big); \tag{2.103}$$

then we have

Proposition 2.21. *In the assumptions of Proposition 2.20 and for Y defined in (2.103), we have*

$$\begin{aligned}
\operatorname{div} Y =& \frac{m-2}{2} \Big(\big| D^A \big|^2 + (m-2) \big| D^B \big|^2 \Big) + D^A_{ijk} (f_{ik} f_j + R^\varphi_{ik} f_j) \\
&+ \overline{D}_{ijk} f_j f_{ik} + (m-2) B^\varphi_{ij} f_i f_j - (R^\varphi_{tk} + f_{tk}) W^\varphi_{tikj} f_i f_j \\
&+ \alpha (R^\varphi_{tj} + f_{tj}) \varphi^a_t \varphi^a_i f_i f_j - U^a \varphi^a_{ij} f_i f_j + U^{ab} \varphi^a_i \varphi^b_j f_i f_j \\
&+ \frac{|\nabla f|^2}{m-2} U^a \varphi^a_{tt} + \frac{\alpha}{m-2} |\nabla f|^2 \varphi^a_{tt} \varphi^a_i f_i \\
&+ \frac{1}{m-1} U^a \varphi^a_j f_k f_{jk} - \frac{1}{m-1} U^a \varphi^a_j f_j \Delta f.
\end{aligned} \tag{2.104}$$

Proof. Taking the divergence of Y

$$\begin{aligned}\operatorname{div} Y =& f_{ik}f_j\big(D^A_{ijk} + (m-2)D^B_{ijk} + \overline{D}_{ijk}\big) + f_i f_{jk}\big(D^A_{ijk} + (m-2)D^B_{ijk} + \overline{D}_{ijk}\big)\\ &+ f_i f_j\big(D^A_{ijk,k} + (m-2)D^B_{ijk,k} + \overline{D}_{ijk,k}\big) - \frac{1}{m-1}U^{ab}\varphi^b_k\big(\varphi^a_j f_j f_k - |\nabla f|^2\varphi^a_k\big)\\ &- \frac{1}{m-1}U^a\big(\varphi^a_{jk}f_j f_k + \varphi^a_j f_{jk} f_k + \varphi^a_j f_j \Delta f - |\nabla f|^2\varphi^a_{kk} - 2\varphi^a_k f_{jk} f_j\big);\end{aligned}$$

using the symmetries of D^A, D^B and $\overline{D}$ we get

$$\begin{aligned}\operatorname{div} Y =& f_{ik}f_j\big(D^A_{ijk} + (m-2)D^B_{ijk} + \overline{D}_{ijk}\big) + f_i f_j\big(D^A_{ijk,k} + (m-2)D^B_{ijk,k} + \overline{D}_{ijk,k}\big)\\ &- \frac{1}{m-1}U^{ab}\varphi^b_k\big(\varphi^a_j f_j f_k - |\nabla f|^2\varphi^a_k\big)\\ &- \frac{1}{m-1}U^a\big(\varphi^a_{jk}f_j f_k - \varphi^a_j f_{jk} f_k + \varphi^a_j f_j \Delta f - |\nabla f|^2\varphi^a_{kk}\big).\end{aligned} \tag{2.105}$$

Using the definition of D^B, its symmetries and the fact that it is totally trace-free we get

$$f_{ik}f_j D^B_{ijk} = \frac{1}{2}(f_{ik}f_j - f_{ij}f_k)D^B_{ijk} = \frac{m-2}{2}|D^B|^2. \tag{2.106}$$

In the same fashion we have

$$R^\varphi_{ij} f_k D^A_{ijk} = \frac{m-2}{2}|D^A|^2,$$

so that

$$\begin{aligned}f_{ik}f_j D^A_{ijk} &= (f_{ik}f_j + R^\varphi_{ik}f_j)D^A_{ijk} - R^\varphi_{ik}f_j D^A_{ijk}\\ &= (f_{ik}f_j + R^\varphi_{ik}f_j)D^A_{ijk} + R^\varphi_{ik}f_j D^A_{ikj}\\ &= \frac{m-2}{2}|D^A|^2 + (f_{ik}f_j + R^\varphi_{ik}f_j)D^A_{ijk}.\end{aligned} \tag{2.107}$$

Using (2.107), (2.106) and (2.99) in (2.105) we get

$$\begin{aligned}\operatorname{div} Y =& \frac{m-2}{2}|D^A|^2 + \frac{(m-2)^2}{2}|D^B|^2 + D^A_{ijk}(f_{ik}f_j + R^\varphi_{ik}f_j)\\ &+ f_{ik}f_j\overline{D}_{ijk} + f_i f_j\big[(m-2)B^\varphi_{ij} - (R^\varphi_{tk} + f_{tk})W^\varphi_{tikj}\big]\\ &+ f_i f_j\left[\alpha\big(R^\varphi_{tj} + f_{tj}\big)\varphi^a_t\varphi^a_i - \frac{m-2}{m-1}U^a\varphi^a_{ij} + \frac{m}{m-1}U^{ab}\varphi^a_i\varphi^b_j\right]\\ &+ f_i f_j\left[\frac{1}{(m-1)(m-2)}U^a\varphi^a_{tt}\delta_{ij} + \frac{m-3}{m-2}f_t C^\varphi_{jti} + \frac{\alpha}{m-2}\varphi^a_{tt}\varphi^a_i f_j\right]\\ &- \frac{|\nabla f|^2}{m-1}U^{ab}\varphi^a_k\varphi^b_k - \frac{1}{m-1}U^{ab}\varphi^b_k\big(\varphi^a_j f_j f_k - |\nabla f|^2\varphi^a_k\big)\\ &- \frac{1}{m-1}U^a\big(\varphi^a_{jk}f_j f_k - \varphi^a_j f_{jk} f_k + \varphi^a_j f_j \Delta f - |\nabla f|^2\varphi^a_{kk}\big).\end{aligned}$$

Rearranging we get

$$\begin{aligned}\operatorname{div} Y =&\frac{m-2}{2}\left(\left|D^A\right|^2+(m-2)\left|D^B\right|^2\right)+D^A_{ijk}(f_{ik}f_j+R^{\varphi}_{ik}f_j)\\&+f_{ik}f_j\overline{D}_{ijk}+(m-2)B^{\varphi}_{ij}f_if_j-(R^{\varphi}_{tk}+f_{tk})W^{\varphi}_{tikj}\\&+\alpha\big(R^{\varphi}_{tj}+f_{tj}\big)\varphi^a_t\varphi^a_if_if_j-U^a\varphi^a_{ij}f_if_j+U^{ab}\varphi^a_if_i\varphi^a_jf_j\\&+\frac{|\nabla f|^2}{m-2}U^a\varphi^a_{tt}+\frac{\alpha}{m-2}|\nabla f|^2\varphi^a_{tt}\varphi^a_if_i+\frac{1}{m-1}U^a\varphi^a_jf_kf_{jk}\\&-\frac{1}{m-1}U^a\varphi^a_jf_j\Delta f\end{aligned}$$

which is (2.104). □

Proposition 2.22. *In the assumptions of Proposition 2.20, assume also that $\overline{D}\equiv 0$ and that φ is $\frac{1}{\alpha}U$-harmonic, that is,*

$$\tau(\varphi)=\frac{\nabla U}{\alpha}. \tag{2.108}$$

Then we have

$$\operatorname{div} Y =(m-2)B^{\varphi}_{ij}f_if_j+\frac{|\nabla f|^2}{\alpha(m-2)}|\nabla U|^2+\frac{m-2}{2}\left(\left|D^A\right|^2+(m-2)\left|D^B\right|^2\right) \tag{2.109}$$

$$\begin{aligned}&-\frac{m}{m-1}U^a\varphi^a_tf_kR^{\varphi}_{kt}\\&+\left\{\frac{m}{m-1}U^a\varphi^a_tf_k+D^A_{tik}f_i-W^{\varphi}_{tikj}f_if_j\right\}(R^{\varphi}_{tk}+f_{tk}).\end{aligned}$$

Proof. From (2.108) and (2.98) we get

$$\mathrm{d}\varphi(\nabla f)=0,$$

that is, in components

$$\varphi^a_if_i=0. \tag{2.110}$$

Therefore, from (2.104) and $\overline{D}\equiv 0$,

$$\begin{aligned}\operatorname{div} Y =&\frac{m-2}{2}\left(\left|D^A\right|^2+(m-2)\left|D^B\right|^2\right)+D^A_{ijk}(f_{ik}f_j+R^{\varphi}_{ik}f_j)\\&+(m-2)B^{\varphi}_{ij}f_if_j-(R^{\varphi}_{tk}+f_{tk})W^{\varphi}_{tikj}-U^a\varphi^a_{ij}f_if_j\\&+\frac{|\nabla f|^2}{m-2}U^a\varphi^a_{tt}+\frac{1}{m-1}U^a\varphi^a_jf_kf_{jk}.\end{aligned}$$

Taking the covariant derivative of (2.110) we obtain

$$\varphi^a_{ij}f_j+\varphi^a_jf_{ij}=0$$

so that

$$\begin{aligned}\frac{1}{m-1}U^a\varphi^a_j f_{jk}f_k - U^a\varphi^a_{ij}f_if_j =& \frac{1}{m-1}U^a\varphi^a_j f_{jk}f_k + U^a\varphi^a_j f_{ij}f_i\\ =& \frac{m}{m-1}U^a\varphi^a_j f_{jk}f_k;\end{aligned}$$

using also (2.108) we get

$$\begin{aligned}\operatorname{div} Y =& (m-2)B^\varphi_{ij}f_if_j + \frac{|\nabla f|^2}{\alpha(m-2)}|\nabla U|^2 + \frac{m-2}{2}\left(\left|D^A\right|^2 + (m-2)\left|D^B\right|^2\right)\\ &- \frac{m}{m-1}U^a\varphi^a_t f_k R^\varphi_{kt}\\ &+ \left\{\frac{m}{m-1}U^a\varphi^a_t f_k + D^A_{tik}f_i - W^\varphi_{tikj}f_if_j\right\}(R^\varphi_{tk} + f_{tk}).\end{aligned}$$

which is (2.109). $\square$

Proof of Theorem 2.19. By assumption we have

$$B^\varphi(\nabla f, \nabla f) = 0 \tag{2.111}$$

and

$$0 = \left\{\frac{m}{m-1}U^a\varphi^a_t f_k + D^A_{tik}f_i - W^\varphi_{tikj}f_if_j\right\}(R^\varphi_{tk} + f_{tk}) \tag{2.112}$$

so that (2.109) becomes

$$\begin{aligned}\operatorname{div} Y =& \frac{m-2}{2}\left(\left|D^A\right|^2 + (m-2)\left|D^B\right|^2\right) - \frac{m}{m-1}U^a\varphi^a_j R^\varphi_{jk}f_k\\ &+ \frac{|\nabla f|^2}{\alpha(m-2)}|\nabla U|^2.\end{aligned}$$

Since ∇f is an eigenvector of $\operatorname{Ric}^\varphi$ we get, at any regular point $p \in M$ of f,

$$U^a\varphi^a_j R^\varphi_{jk}f_k = \Lambda U^a\varphi^a_j f_j, \quad \text{for some } \Lambda \in \mathbb{R}.$$

Since φ is $\frac{U}{\alpha}$-harmonic, we get

$$\varphi^a_i f_i = 0$$

and therefore

$$U^a\varphi^a_j R^\varphi_{jk}f_k = 0$$

so that

$$\operatorname{div} Y = \frac{m-2}{2}\left(\left|D^A\right|^2 + (m-2)\left|D^B\right|^2\right) + \frac{|\nabla f|^2}{\alpha(m-2)}|\nabla U|^2. \tag{2.113}$$

Note that, by the definition (2.103) of Y, we get

$$Y_k f_k = 0. \tag{2.114}$$

Let $\delta, \eta \in \mathbb{R}$, $\delta < \eta$ be two regular values of f. Set

$$\Omega_{\delta,\eta} = \{x \in M : \delta \leq f(x) \leq \eta\};$$

then $\Omega_{\delta,\eta}$ is compact since f is proper. Integrating (2.113) on $\Omega_{\delta,\eta}$, using the divergence theorem and (2.114) we obtain

$$0 = \int_{\partial\Omega_{\delta,\eta}} Y_k \nu_k = \int_{\Omega_{\delta,\eta}} \frac{m-2}{2}\left(\left|D^A\right|^2 + (m-2)\left|D^B\right|^2\right) + \frac{|\nabla f|^2}{\alpha(m-2)}|\nabla U|^2$$

where $\nu = -\frac{\nabla f}{|\nabla f|}$ is the inward unit normal to $\partial\Omega_{\delta,\eta}$. Letting $\delta \to -\infty, \eta \to +\infty$, we get $D^A = D^B = 0$ on M and $\nabla U = 0$ at any regular point. Since φ is $\frac{U}{\alpha}$-harmonic, we deduce that φ is harmonic at every point and since $D^A \equiv D^B \equiv 0$ we can use Theorem 2.9 to conclude. □

Remark. When $U \equiv 0$, some of the assumptions of Theorem 2.19 are unnecessary. Indeed, when $U \equiv 0$, Proposition 2.21 gives the validity of

$$\begin{aligned}
\operatorname{div} Y =& \frac{m-2}{2}\left(\left|D^A\right|^2 + (m-2)\left|D^B\right|^2\right) + \overline{D}_{ijk} f_j f_{ik} \\
&+ (m-2) B^{\varphi}_{ij} f_i f_j + \frac{\alpha}{m-2}|\nabla f|^2 |\tau(\varphi)|^2 \\
&+ (R^{\varphi}_{ik} + f_{ik})\left[f_j D^A_{ijk} - f_t f_j W^{\varphi}_{itkj} + \alpha \varphi^a_i \varphi^a_t f_t f_k\right]
\end{aligned}$$

for the same vector field Y. If, instead of studying the map F given by (2.97), one asks that $\mathrm{Ric}^{\varphi} + \mathrm{Hess}\, f \in \ker G$, where G is the linear map $G : S^2(M) \to C^{\infty}(M)$ given by

$$G(\beta) := \left[f_j D^A_{ijk} - f_t f_j W^{\varphi}_{itkj} + \alpha \varphi^a_i \varphi^a_t f_t f_k\right]\beta_{ik}, \quad \beta \in \Gamma(S^2(M)),$$

then the same conclusion of Theorem 2.19 is obtained following the same procedure, without the need to assume that φ be $\frac{1}{\alpha}U$-harmonic and that ∇f be an eigenvector of Ric^{φ}.

2.4 Riemann compatibility

The following definition has been given by Mantica and Molinari, [69], [70] :

Definition 2.23. Let (M, g) be a semi-Riemannian manifold and P be a symmetric 2-covariant tensor. We say that P is *Riemann compatible* if it holds

$$R^l{}_{ikt} P_{jl} + R^l{}_{itj} P_{kl} + R^l{}_{ijk} P_{tl} = 0. \tag{2.115}$$

Contracting (2.115) with the metric and using the orthogonal decomposition of Riem one easily sees that (2.115) is equivalent to the system

$$\begin{cases} W^l{}_{ikt}P_{jl} + W^l{}_{itj}P_{kl} + W^l{}_{ijk}P_{tl} = 0, \\ P_{jl}R^l{}_i = P_{il}R^l{}_j, \end{cases} \tag{2.116}$$

in particular P is Riemann compatible if and only if it is Weyl compatible and it commutes with the Ricci tensor of (M, g). Every Codazzi tensor is Riemann compatible as the next proposition shows.

Proposition 2.24 (Proposition 2.2 of [69])**.** *Let (M, g) be a semi-Riemannian manifold and P be a 2-covariant symmetric tensor on M. Set*

$$C(P)_{ijk} := P_{ij,k} - P_{ik,j}.$$

Then we have

$$\begin{aligned} C(P)_{ijk,t} +& C(P)_{ikt,j} + C(P)_{itj,k} \\ &= R^l{}_{ikt}P_{jl} + R^l{}_{itj}P_{kl} + R^l{}_{ijk}P_{tl}. \end{aligned} \tag{2.117}$$

Proof. By the definition of $C(P)$, the left hand side of (2.117) is

$$P_{ij,kt} - P_{ik,jt} + P_{ik,tj} - P_{it,kj} + P_{it,jk} - P_{ij,tk}.$$

Rearranging terms and using the Ricci commutation relations, we get

$$\begin{aligned} &P_{ij,kt} - P_{ik,jt} + P_{ik,tj} - P_{it,kj} + P_{it,jk} - P_{ij,tk} \\ &\quad = P_{ij,kt} - P_{ij,tk} + P_{ik,tj} - P_{ik,jt} + P_{it,jk} - P_{it,kj} \\ &\quad = R^l{}_{ikt}P_{jl} + R^l{}_{itj}P_{kl} + R^l{}_{ijk}P_{tl} \\ &\qquad + R^l{}_{jkt}P_{il} + R^l{}_{ktj}P_{il} + R^l{}_{tjk}P_{il}. \end{aligned}$$

Using the first Bianchi identity we conclude. □

Corollary 2.25. *Every Codazzi tensor is Riemann compatible.*

Remark 2.26. Note that being Riemann compatible is a more general condition than being Codazzi: for example, assuming that P is Riemann compatible, then the tensor

$$P - \lambda g,$$

where λ is an arbitrary smooth function on M, is still Riemann compatible, as a consequence of the first Bianchi identity. However, if P is a Codazzi tensor, $P - \lambda g$ will in general not be.

As we saw in Section 2.3.2, every n-dimensional Lorentzian manifold $(\widehat{M}, \widehat{g})$ that solves Cotton gravity's field equations for a stress-energy tensor $\widehat{T}$ admits a Codazzi tensor, namely

$$\widehat{Z} := \widehat{\mathrm{Ric}} - \widehat{T} - \frac{\widehat{S} - 2\mathrm{tr}_{\widehat{g}}\widehat{T}}{2(n-1)}\widehat{g}.$$

Moreover, every m-dimensional C-φ-PF admits a Codazzi tensor, that is,

$$\mathscr{C} := \mathrm{Ric}^{\varphi} + \mathrm{Hess}\, f - \mathrm{d}f \otimes \mathrm{d}f - \frac{S^{\varphi} + 2\Delta_f f + 2U(\varphi) + 2\mu}{2m} g.$$

Therefore, from Remark 2.26, both $\mathscr{C}$ and $\mathrm{Ric}^{\varphi} + \mathrm{Hess}\, f - \mathrm{d}f \otimes \mathrm{d}f$ are Riemann compatible.

Proposition 2.27. *Let (M, g) be an m-dimensional Riemannian manifold. Assume that there exists $f \in C^{\infty}(M)$ that solves the system of C-φ-PF. Then*

$$\mathrm{Ric}^{\varphi} + \mathrm{Hess}\, f - \mathrm{d}f \otimes \mathrm{d}f$$

is Riemann compatible.

We already saw in (2.116) that a Riemann compatible tensor is also Weyl compatible and it commutes with the Ricci tensor. We want to show that, on a C-φ-PF, the same result holds for the φ-curvatures. First, we have the following

Proposition 2.28. *Let (M, g) be an m-dimensional Riemannian manifold and $\varphi : (M, g) \to (N, h)$ a smooth map that targets another Riemannian manifold. For some $f \in C^{\infty}(M), \alpha \in \mathbb{R}, \alpha \neq 0$, set*

$$P := \mathrm{Ric}^{\varphi} + \mathrm{Hess}\, f - \mathrm{d}f \otimes \mathrm{d}f.$$

Assume the validity, on M, of

$$\begin{cases} 0 = & C^{\varphi}_{ijk} + f_t W^{\varphi}_{tijk} - \frac{1}{m-1} U^a \varphi^a_k \delta_{ij} + \frac{1}{m-1} U^a \varphi^a_j \delta_{ik} - D^A_{ijk} - (m-2) D^B_{ijk}, \\ 0 = & -\frac{m-1}{m} f_{tti} + \frac{m-2}{m} f_t f_{ti} + (\Delta f) f_i + \frac{1}{2m} S^{\varphi}_i - f_t R^{\varphi}_{ti} + \frac{1}{m} U^a \varphi^a_i - \frac{m-1}{m} \mu_i, \\ 0 = & (\mu + p) f_i - p_i, \\ 0 = & \tau(\varphi) - \mathrm{d}\varphi(\nabla f) - \frac{1}{\alpha} \nabla U. \end{cases} \tag{2.118}$$

for some $\mu, p \in C^{\infty}(M), U \in C^{\infty}(N)$. Then $\varphi^* h$ commutes with P.

Proof. From the definition (2.47) of D^A we immediately get

$$f_t D^A_{tjk} = \frac{1}{m-1} \big(f_t R^{\varphi}_{tj} f_k - f_t R^{\varphi}_{tk} f_j \big). \tag{2.119}$$

Indeed,

$$\begin{aligned} f_t D^A_{tjk} =& \frac{1}{m-2} \Bigg[f_t R^{\varphi}_{tj} f_k - f_t R^{\varphi}_{tk} f_j + \frac{1}{m-1} \big(f_t R^{\varphi}_{tk} f_j - f_t R^{\varphi}_{tj} f_k \big) \\ & - \frac{S^{\varphi}}{m-1} (f_j f_k - f_k f_j) \Bigg] \\ =& \frac{1}{m-2} \left[\frac{m-2}{m-1} \big(f_t R^{\varphi}_{tj} f_k - f_t R^{\varphi}_{tk} f_j \big) \right] \\ =& \frac{1}{m-1} \big(f_t R^{\varphi}_{tj} f_k - f_t R^{\varphi}_{tk} f_j \big). \end{aligned}$$

In the same fashion, we have

$$f_t D^B_{tjk} = \frac{1}{m-1}(f_t f_{tk} f_j - f_t f_{tj} f_k). \tag{2.120}$$

We contract (2.118).i) with f_i and we use (2.119) and (2.120) to get

$$\begin{aligned}
0 =& f_t C^\varphi_{tjk} + f_s f_t W^\varphi_{stjk} - \frac{1}{m-1} U^a \varphi^a_k f_j + \frac{1}{m-1} U^a \varphi^a_j f_k - f_t D^A_{tjk} - (m-2) f_t D^B_{tjk} \\
=& f_t C^\varphi_{tjk} - \frac{1}{m-1} U^a \varphi^a_k f_j + \frac{1}{m-1} U^a \varphi^a_j f_k - \frac{1}{m-1}\big(f_t R^\varphi_{tj} f_k - f_t R^\varphi_{tk} f_j\big) \\
&- \frac{m-2}{m-1}(f_t f_{tk} f_j - f_t f_{tj} f_k),
\end{aligned}$$

that is,

$$\begin{aligned}
f_t C^\varphi_{tjk} =& \frac{1}{m-1}\Big[U^a \varphi^a_k f_j - U^a \varphi^a_j f_k + f_t\big(R^\varphi_{tj} f_k - R^\varphi_{tk} f_j\big) \\
&+ (m-2) f_t (f_{tk} f_j - f_{tj} f_k)\Big].
\end{aligned} \tag{2.121}$$

We will also need the following:

$$\mu_j f_i - \mu_i f_j = 0. \tag{2.122}$$

Take the covariant derivative of (2.118).iii)

$$\mu_j f_i + p_j f_i + (\mu + p) f_{ij} - p_{ij} = 0$$

and use (2.118).iii) into the latter to obtain

$$\mu_j f_i = -(\mu + p) f_i f_j - (\mu + p) f_{ij} + p_{ij}.$$

Since the right hand side of the equation above is symmetric in i and j, the left hand side has to be so, and we obtain (2.122). Now, take the covariant derivative of (2.118)

$$\begin{aligned}
0 =& -\frac{m-1}{m} f_{ttij} + \frac{m-2}{m} f_t f_{tij} + \frac{m-2}{m} f_{ti} f_{tj} + (\Delta f)_j f_i \\
&+ (\Delta f) f_{ij} + \frac{1}{2m} S^\varphi_{ij} - f_t R^\varphi_{ti,j} - R^\varphi_{ti} f_{tj} + \frac{1}{m} U^a \varphi^a_{ij} \\
&+ \frac{1}{m} U^{ab} \varphi^a_i \varphi^b_j - \frac{m-1}{m} \mu_{ij};
\end{aligned}$$

skew symmetrize the equation above, observing that $f_t f_{tij}$ is skew symmetric by the Ricci commutation identities,

$$0 = (\Delta f)_j f_i - (\Delta f)_i f_j - f_t\big(R^\varphi_{ti,j} - R^\varphi_{tj,i}\big) - R^\varphi_{ti} f_{tj} + R^\varphi_{tj} f_{ti}.$$

From the definition of C^φ we get

$$0 =(\Delta f)_j f_i - (\Delta f)_i f_j - f_t C^\varphi_{tij} - \frac{1}{2(m-1)}\big(S^\varphi_j f_i - S^\varphi_i f_j\big) \tag{2.123}$$
$$- R^\varphi_{ti} f_{tj} + R^\varphi_{tj} f_{ti}.$$

Insert (2.121) in (2.123)

$$0 =(\Delta f)_j f_i - (\Delta f)_i f_j - \frac{1}{m-1}\Big[U^a\varphi^a_j f_i - U^a\varphi^a_i f_j + f_t\big(R^\varphi_{ti} f_j - R^\varphi_{tj} f_i\big) \tag{2.124}$$
$$+ (m-2) f_t (f_{tj} f_i - f_{ti} f_j)\Big] - \frac{1}{2(m-1)}\big(S^\varphi_j f_i - S^\varphi_i f_j\big)$$
$$- R^\varphi_{ti} f_{tj} + R^\varphi_{tj} f_{ti}.$$

Multiply (2.118).$ii)$ by $\frac{m}{m-1}$ and rearrange to deduce

$$(\Delta f)_i =\frac{m-2}{m-1} f_t f_{ti} + \frac{m}{m-1}(\Delta f) f_i + \frac{1}{2(m-1)} S^\varphi_i - \frac{m}{m-1} f_t R^\varphi_{ti}$$
$$+ \frac{1}{m-1} U^a \varphi^a_i - \mu_i.$$

Use the latter in (2.124) to obtain

$$0 =\frac{m-2}{m-1} f_t f_{tj} f_i + \frac{m}{m-1}(\Delta f) f_j f_i + \frac{1}{2(m-1)} S^\varphi_j f_i - \frac{m}{m-1} f_t R^\varphi_{tj} f_i$$
$$+ \frac{1}{m-1} U^a \varphi^a_j f_i - \mu_j f_i$$
$$- \frac{m-2}{m-1} f_t f_{ti} f_j - \frac{m}{m-1}(\Delta f) f_i f_j - \frac{1}{2(m-1)} S^\varphi_i f_j + \frac{m}{m-1} f_t R^\varphi_{ti} f_j$$
$$- \frac{1}{m-1} U^a \varphi^a_i f_j + \mu_i f_j$$
$$- \frac{1}{m-1}\Big[U^a\varphi^a_j f_i - U^a\varphi^a_i f_j + f_t\big(R^\varphi_{ti} f_j - R^\varphi_{tj} f_i\big)$$
$$+ (m-2) f_t (f_{tj} f_i - f_{ti} f_j)\Big] - \frac{1}{2(m-1)}\big(S^\varphi_j f_i - S^\varphi_i f_j\big)$$
$$- R^\varphi_{ti} f_{tj} + R^\varphi_{tj} f_{ti}.$$

Simplifying and using (2.122) we get

$$0 = -f_t R^\varphi_{tj} f_i + f_t R^\varphi_{ti} f_j - R^\varphi_{ti} f_{tj} + R^\varphi_{tj} f_{ti}.$$

Since Ric^φ clearly commutes with itself, the latter gives the commutativity of Ric^φ and P. From Proposition 2.27, we have that P is Riemann compatible so that in particular it commutes with the Ricci tensor. Combining these two informations we deduce that P and $\varphi^* h$ commute and we are done. □

Proposition 2.29. *Let (M, g) be an m-dimensional C-φ-PF. Set*

$$P := \mathrm{Ric}^{\varphi} + \mathrm{Hess}\, f - \mathrm{d}f \otimes \mathrm{d}f.$$

Then we have

$$\begin{cases} W^{\varphi}_{likt} P_{jl} + W^{\varphi}_{litj} P_{kl} + W^{\varphi}_{lijk} P_{tl} = 0, \\ P_{jl} R^{\varphi}_{li} = P_{il} R^{\varphi}_{lj}, \end{cases} \tag{2.125}$$

that is, P is φ-Weyl compatible and it commutes with Ric^{φ}.

Proof. From Proposition 2.27 we get that P commutes with Ric, while Proposition 2.28 tells us that P commutes with Ric^{φ} so that we have (2.125).ii). By Proposition 2.27, we obtain

$$0 = P_{tl} W_{lijk} + P_{kl} W_{litj} + P_{jl} W_{likt}. \tag{2.126}$$

From (1.24) we have

$$\begin{aligned} W_{lijk} =& W^{\varphi}_{lijk} - \frac{\alpha}{m-2}\left(\varphi^a_l \varphi^a_j \delta_{ik} - \varphi^a_l \varphi^a_k \delta_{ij} + \varphi^a_i \varphi^a_k \delta_{lj} - \varphi^a_i \varphi^a_j \delta_{lk}\right) \\ &+ \frac{\alpha}{(m-2)(m-1)} |\mathrm{d}\varphi|^2 (\delta_{lj}\delta_{ik} - \delta_{lk}\delta_{ij}). \end{aligned}$$

Using the latter in (2.126) we deduce

$$\begin{aligned} 0 =& P_{tl} W^{\varphi}_{lijk} - \frac{\alpha}{m-2}\left(P_{tl}\varphi^a_l \varphi^a_j \delta_{ik} - P_{tl}\varphi^a_l \varphi^a_k \delta_{ij} + P_{tj}\varphi^a_i \varphi^a_k - P_{tk}\varphi^a_i \varphi^a_j\right) \\ &+ \frac{\alpha}{(m-2)(m-1)} |\mathrm{d}\varphi|^2 (P_{jt}\delta_{ik} - P_{kt}\delta_{ij}) \\ &+ P_{kl} W^{\varphi}_{litj} - \frac{\alpha}{m-2}\left(P_{kl}\varphi^a_l \varphi^a_t \delta_{ij} - P_{kl}\varphi^a_l \varphi^a_j \delta_{it} + P_{tk}\varphi^a_i \varphi^a_j - P_{jk}\varphi^a_i \varphi^a_t\right) \\ &+ \frac{\alpha}{(m-2)(m-1)} |\mathrm{d}\varphi|^2 (P_{kt}\delta_{ij} - P_{kj}\delta_{it}) \\ &+ P_{jl} W^{\varphi}_{likt} - \frac{\alpha}{m-2}(P_{jl}\varphi^a_l \varphi^a_k \delta_{it} - P_{jl}\varphi^a_l \varphi^a_t \delta_{ik} + P_{kj}\varphi^a_i \varphi^a_t - P_{tj}\varphi^a_i \varphi^a_k) \\ &+ \frac{\alpha}{(m-2)(m-1)} |\mathrm{d}\varphi|^2 (P_{jk}\delta_{it} - P_{jt}\delta_{ik}) \end{aligned}$$

.

Using that P and $\varphi^* h$ commute and simplifying we obtain (2.125).i). □

Before going on, let us set some notations. Consider the homomorphism

$$R : \Lambda^2 TM \to \mathrm{End}\, TM$$

that sends $X \wedge Y \in \Gamma(\Lambda^2 TM)$ to the endomorphism of TM of local components

$$R(X \wedge Y)^i{}_j = X^k Y^t R^i{}_{jkt}.$$

Then, for any positive integer k, we can define a tensor field ω_k of type $(0, 4k)$ on M by

$$\omega_k(X_1, ..., X_{4k}) = \mathrm{Trace}[R(X_1 \wedge X_2) \circ R(X_3 \wedge X_4) \circ ... \circ R(X_{4k-1} \wedge X_{4k})]$$

where the composition and the trace operation are taken with respect to End TM. For $4k \leq m$, the k-th *Pontryagin form*, $\Omega_k(R)$, of M is defined to be the total anti-symmetrization of ω_k.
The next theorem shows how the existence of a Riemann compatible tensor can constrain the geometry of (M, g).

Theorem 2.30 (Theorem 5.3 of [69])**.** *Let (M, g) be an m-dimensional Riemannian manifold. Suppose that there exist on M a Riemann compatible symmetric tensor P and a point $x \in M$ such that P has m distinct eigenvalues at x. Then every Pontryagin form of M vanishes at x.*

As we will see, Theorem 2.30 can be strengthened when $m = 4$. Indeed, we have

Theorem 2.31. *Let (M, g) be a compact, orientable Riemannian manifold of dimension 4 and let P be a Weyl compatible tensor on M. If, on a dense open subset of M, it holds*

$$P \neq \frac{\mathrm{tr}_g P}{4} g,$$

then the signature $\tau(M)$ of M is zero.

Theorems 2.30 and 2.31 generalize two results of Derdzinski and Shen, see [34], valid under the assumption that P be Codazzi. In the case of a φ-SPFST, the tensor

$$\mathrm{Ric}^{\varphi} + \mathrm{Hess}\, f - \mathrm{d}f \otimes \mathrm{d}f$$

is proportional to the metric so that it is clearly Riemann compatible. This suggested to us that, in some way, one might try to use Riemann compatibility (and Weyl compatibility) to study how far solutions of the Cotton gravity equations are from classical solutions (that is, of Einstein field equations). In this direction, in the next subsection, inspired by works of Gover, Nurowski and Nagy, [48],[47], we show how the existence of a non-trivial Weyl-compatible tensor constrains the algebraic structure of the Weyl tensor, giving a special emphasis to the case $m = 4$. This will gives us natural conditions on the Weyl tensor under which any solution of the C-φ-PF equations gives rise to a solution of the φ-SPFST system.

2.4.1 Weyl compatibility and the case of dimension 4

Let us introduce some notations. Let (M, g) be an m-dimensional Riemannian manifold and assume that it is orientable; let ϵ be its volume form. In an orthonormal co-frame, its components are given by the so called *Levi-Civita symbol.* Define an m-times covariant tensor

$$W^*_{i_1 i_2 \dots i_{m-2} jk} := \epsilon_{i_1 i_2 \dots i_{m-2} pq} W^{pq}{}_{jk}.$$

Let $S^2_0(M)$ be the space of traceless symmetric 2-covariant tensors on M, while $T^{(0,m-2)}M$ will be the space of $(m-2)$-covariant tensors on M. Then W^* induces a linear map

$$W^*_{|_{S^2_0}} : S^2_0(M) \to T^{(0,m-2)}M$$

by sending $P \in \Gamma(S^2_0(M))$ of components P_{ij} to the tensor $W^*_{|_{S^2_0}}(P)$ of components

$$W^*_{|_{S^2_0}}(P)_{i_2 \dots i_{m-2} j} = W^*_{t i_2 .. i_{m-2} k j} P_{tk}.$$

When $m = 4$, as it is well-known, W^* is an algebraic Weyl tensors and $W^*_{|_{S^2_0}}$ is an endomorphism of $S^2_0(M)$. It is clear how, in a similar way, the Weyl tensor also gives rise to a linear map $W_{|_{S^2_0}}$ which, in any dimension $m \geq 4$, is actually an endomorphism of $S^2_0(M)$. The following simple lemma is probably well-known among experts.

Lemma 2.32. *Let (M, g) be an m-dimensional orientable Riemannian manifold and let P belong to $\Gamma(S^2_0(M))$. Then P lies in the kernel of $W^*_{|_{S^2_0}}$ if and only if P is Weyl compatible.*

Proof. Let $p \in M$ be fixed and let $\{\theta^i\}_{i=1}^m$ be a positively oriented orthonormal co-frame at p. Then, at p, we have

$$\epsilon = \theta^1 \wedge \theta^2 \wedge \dots \wedge \theta^m.$$

First, note that, by the obvious symmetries of ϵ and by the definition of W^* we have

$$W^*_{i_1 i_2 \dots i_{m-2} j q} = 0$$

whenever there are two coinciding indices among $i_1, i_2, \dots, i_{m-2}$. Assume now that $i_1, i_2, \dots, i_m$ are distinct indices. Suppressing the Einstein summation convention for the time of this proof, we have

$$\begin{aligned}
&\sum_{t,q} P_{tq} W^*_{t i_2 \dots i_{m-2} j q} \\
&\qquad = \sum_q \left(P_{i_1 q} W^*_{i_1 i_2 \dots i_{m-2} j q} + P_{i_{m-1} q} W^*_{i_{m-1} i_2 \dots i_{m-2} j q} + P_{i_m q} W^*_{i_m i_2 \dots i_{m-2} j q} \right) \\
&\qquad = \pm \sum_q \left(P_{i_1 q} W_{i_{m-2} i_m j q} + P_{i_{m-1} q} W_{i_m i_1 j q} + P_{i_m q} W_{i_1 i_{m-1} j q} \right)
\end{aligned}$$

where the last equality follows from the definition of W^* and the sign $\pm$ coincides with the determinant of $\theta^{i_1} \wedge \theta^{i_2} \wedge \dots \wedge \theta^{i_m}$. Using the symmetries of the Weyl tensor we get

$$\begin{aligned}
&\sum_{t,q} P_{tq} W^*_{t i_2 \dots i_{m-2} j q} \\
&\qquad = \mp \sum_q \left(P_{i_1 q} W_{q j i_{m-2} i_m} + P_{i_{m-1} q} W_{q j i_m i_1} + P_{i_m q} W_{q j i_1 i_{m-1}} \right)
\end{aligned}$$

which gives the desired result. $\square$

From Lemma 2.32 we deduce that if $W^*_{|_{S^2_0}}$ is injective, then there exists no non-zero trace-free Weyl compatible tensor. Before proving the main result of this section, that is, Proposition 2.36 below, we provide, for $m = 4$, simple conditions on the Weyl tensor under which $W^*_{|_{S^2_0}}$ is injective. When $m = 4$ it is well-known that the space of 2-forms on M decomposes in the spaces of self-dual and anti-self dual forms, i.e.

$$\Lambda^2(M) = \Lambda^2_+(M) \oplus \Lambda^2_-(M).$$

Moreover, the Weyl tensor also decomposes in self-dual and anti-self dual parts, which take the form

$$W^+ := \frac{1}{2}(W + W^*), \quad W^- := \frac{1}{2}(W - W^*).$$

These define endomorphisms

$$W^+_{|_{\Lambda^2_+}} : \Lambda^2_+ \to \Lambda^2_+, \quad W^-_{|_{\Lambda^2_-}} : \Lambda^2_- \to \Lambda^2_-$$

which assign to a 2-form $\omega \in \Gamma(\Lambda^2_\pm)$ of components ω_{ij} the two form $W^\pm_{|_{\Lambda^2_\pm}}(\omega)$ of components

$$W^\pm_{|_{\Lambda^2_\pm}}(\omega)_{ij} = W^\pm_{ijkt}\omega_{kt}.$$

With these notations in mind, it is proved in [47] that the injectivity of $W^*_{|_{S^2_0}}$ holds under the assumptions

$$W^+ = 0, \quad \det W^-_{|_{\Lambda^2_-}} \neq 0, \tag{2.127}$$

where $\det W^-_{|_{\Lambda^2_-}}$ is the determinant of $W^-_{|_{\Lambda^2_-}}$. More in depth, it is proved there, see also [49, Section 4], that the endomorphism $W^*_{|_{S^2_0}}$ is completely determined by the endomorphisms $W^+_{|_{\Lambda^2_+}}$ and $W^-_{|_{\Lambda^2_-}}$. Indeed, consider at a given point $p \in M$, an orthonormal basis $\omega^1, \omega^2, \omega^3$ of $\Lambda^2_+(T^*_pM)$ made of eigenforms of $W^+_{|_{\Lambda^2_+}}$ of respective eigenvalues $\lambda^+_1, \lambda^+_2, \lambda^+_3$. Similarly, consider at p an orthonormal basis η^1, η^2, η^3 of $\Lambda^2_-(T^*_pM)$ made of eigenforms of $W^-_{|_{\Lambda^2_-}}$ of respective eigenvalues $\lambda^-_1, \lambda^-_2, \lambda^-_3$. Then it is proved in Item iv) of Proposition 4.1 of [49] that the nine 2-covariant tensors

$$h^{\alpha,\beta}_{ij} := \omega^\alpha_{ip}\eta^\beta_{pj}, \quad \alpha, \beta \in \{1, 2, 3\}$$

are symmetric, trace-free and they form a basis of $S^2_0(M)$ at p. Moreover, $W_{|_{S^2_0}}$ is diagonalized on this basis and the eigenvalue corresponding to $h^{\alpha,\beta}$ is

$$\lambda^+_\alpha + \lambda^-_\beta,$$

see Proposition 4.3 of [49]. Similarly, $W^*_{|_{S^2_0}}$ is diagonalized by $\{h^{\alpha,\beta}\}_{\alpha,\beta=1,2,3}$ and $h^{\alpha,\beta}$ has eigenvalue

$$\lambda^+_\alpha - \lambda^-_\beta.$$

From this we see that $W^*_{|_{S^2_0}}$ fails to be injective at p if and only if, for some $\alpha, \beta \in \{1, 2, 3\}$ we have

$$\lambda^+_\alpha = \lambda^-_\beta.$$

In particular, this cannot happen if (2.127) holds.

Remark 2.33. It is possible to give a more precise characterization of the relation between W^+ and W^- on a four-manifold admitting a non-trivial Weyl compatible tensor. Indeed, we have the validity of the following

Lemma 2.34. *Let (M, g) be a Riemannian four-manifold and let P be a Weyl compatible tensor on M. Then, at any point $x \in M$ such that*

$$P \neq \frac{\mathrm{tr}_g P}{4} g$$

holds at x, we have that the spectras of $W^+_{|_{\Lambda^2_+}}$ and $W^-_{|_{\Lambda^2_-}}$ coincide, with equal multiplicities.

The above result has been proved in Lemma 2 of [33] under the stronger assumption that P be Codazzi. The same argument used there works in this situation if one applies Proposition 2.4 of [70] instead of Theorem 1 of [34].

Remark 2.35. From Lemma 2.34 one can also give a simple proof of Theorem 2.31. Indeed, it is well known, see equation (2.21) of [12], that, on an orientable 4-manifold, the first Pontryagin form is

$$\Omega_1(R) = \left(\left|W^+\right|^2 - \left|W^-\right|^2\right)\epsilon,$$

where ϵ is again the volume form of (M, g). From the Hirzebruch signature formula one gets

$$\tau(M) = \int_M \left(\left|W^+\right|^2 - \left|W^-\right|^2\right).$$

Under the assumptions of Theorem 2.31 and using Lemma 2.34 we obtain $|W^+|^2 = |W^-|^2$ on a dense open subset of M, so that $\tau(M) = 0$ and we are done.

In the next Theorem, we apply the discussion above to the setting of C-φ-PF, showing how, under suitable conditions on the Weyl tensor, every solution of the C-φ-PF system also solves that of φ-SPFST.

Theorem 2.36. *Let (M, g) be an orientable Riemannian manifold; assume that there exists $f \in C^\infty(M)$ that solves the C-φ-PF system. Assume that $W^*_{|_{S^2_0}}$ is injective. Then, there exist two constants $\Lambda_1, \Lambda_2 \in \mathbb{R}$ such that (M, g, f) satisfies*

$$\begin{cases} \mathrm{Ric}^\varphi + \mathrm{Hess}\, f - \mathrm{d}f \otimes \mathrm{d}f = \lambda g, \\ \Delta_f f = \frac{1}{m-1}[S^\varphi - m(\mu + p)] + \Lambda_1 e^f, \\ \tau(\varphi) = \mathrm{d}\varphi(\nabla f) + \frac{\nabla U}{\alpha}, \\ \frac{1}{2} S^\varphi = U(\varphi) + \mu + \Lambda_2, \\ (\mu + p)\nabla f - \nabla p = 0. \end{cases} \tag{2.128}$$

Remark 2.37. Assume that (2.128) holds. Consider the substitution

$$\begin{cases} \widetilde{\mu} = \mu + \Lambda_2, \\ \widetilde{p} = p - \frac{m-1}{m}\Lambda_1 e^f - \Lambda_2; \end{cases} \tag{2.129}$$

then, we have

$$\begin{cases} \mathrm{Ric}^\varphi + \mathrm{Hess}\, f - \mathrm{d}f \otimes \mathrm{d}f = \lambda g, \\ \Delta_f f = \frac{1}{m-1}[S^\varphi - m(\widetilde{\mu} + \widetilde{p})], \\ \tau(\varphi) = \mathrm{d}\varphi(\nabla f) + \frac{\nabla U}{\alpha}, \\ \frac{1}{2} S^\varphi = U(\varphi) + \widetilde{\mu}, \\ (\widetilde{\mu} + \widetilde{p})\nabla f - \nabla \widetilde{p} = 0. \end{cases} \tag{2.130}$$

Indeed, a quick computation reveals that equations (2.128)$.ii), iv), v)$ give equations (2.130)$.ii), iv), v)$ respectively. From this we see that every solution of (2.128) is a φ-SPFST, for a different choice of pressure and density.

Proof of Theorem 2.36. From Proposition 2.27, we have that

$$P := \mathrm{Ric}^\varphi + \mathrm{Hess}\, f - \mathrm{d}f \otimes \mathrm{d}f$$

is a Riemann compatible tensor, so that it is also Weyl compatible. From Lemma 2.32 we deduce that

$$W^*_{|_{S^2_0}}\left(\mathring{P}\right) = 0$$

where $\mathring{P}$ denotes the trace-less part of P. Therefore, since $W^*_{|_{S^2_0}}$ is injective, there exists $\lambda \in C^\infty(M)$ such that

$$\mathrm{Ric}^\varphi + \mathrm{Hess}\, f - \mathrm{d}f \otimes \mathrm{d}f = \lambda g.$$

From the equation above and the equations of C-φ-PF we deduce the validity of

$$\begin{cases} \mathrm{Ric}^\varphi + \mathrm{Hess}\, f - \mathrm{d}f \otimes \mathrm{d}f = \lambda g, \\ 0 = -\frac{m-1}{m} f_{tti} + \frac{m-2}{m} f_t f_{ti} + \Delta f f_i + \frac{1}{2m} S^\varphi_i \\ \qquad -f_t R^\varphi_{ti} + \frac{1}{m} U^a \varphi^a_i - \frac{m-1}{m}\mu_i, \\ \tau(\varphi) = \mathrm{d}\varphi(\nabla f) + \frac{\nabla U}{\alpha}(\varphi), \\ (\mu + p)\nabla p - \nabla f = 0. \end{cases} \tag{2.131}$$

We are going to prove that (2.131) implies (2.128). We first show the validity of

$$\frac{1}{2}S_i^\varphi - (m-1)\lambda_i = f_t R_{ti}^\varphi + f_t f_{ti} - (\Delta f) f_i - U^a \varphi_i^a. \tag{2.132}$$

Tracing (2.131).i) we get

$$m\lambda = S^\varphi + \Delta_f f; \tag{2.133}$$

taking the covariant derivative of (2.133) we deduce

$$S_i^\varphi - m\lambda_i = -f_{tti} + 2f_t f_{ti}. \tag{2.134}$$

Computing the divergence of (2.131).i)

$$R_{ij,j}^\varphi + f_{ijj} - f_j f_{ij} - f_i f_{jj} = \lambda_i$$

and using the φ-Schur identity and the Ricci commutation relations we obtain

$$\frac{1}{2}S_i^\varphi - \alpha\varphi_{tt}^a\varphi_i^a + f_{tti} + f_t R_{ti} - f_t f_{ti} - f_i f_{tt} = \lambda_i;$$

from the definition of Ric^φ and (2.131).iii) we deduce

$$\begin{aligned}
-\alpha\varphi_{tt}^a\varphi_i^a + f_t R_{ti} &= -\,\alpha\varphi_{tt}^a\varphi_i^a + f_t R_{ti}^\varphi + \alpha\varphi_t^a f_t \varphi_i^a \\
&= -\,U^a\varphi_i^a + f_t R_{ti}^\varphi
\end{aligned}$$

and therefore

$$\frac{1}{2}S_i^\varphi - U^a\varphi_i^a + f_t R_{ti}^\varphi + f_{tti} - f_t f_{ti} - f_i f_{tt} = \lambda_i.$$

Rearranging we get

$$\frac{1}{2}S_i^\varphi - \lambda_i = -f_t R_{ti}^\varphi - f_{tti} + f_t f_{ti} + f_i \Delta f + U^a\varphi_i^a. \tag{2.135}$$

Subtracting (2.135) from (2.134) we obtain (2.132).
From (2.132) we obtain the validity of (2.128).iv) in the following way: first, rearrange (2.132) to deduce

$$f_t R_{ti}^\varphi - (\Delta f) f_i = \frac{1}{2}S_i^\varphi - (m-1)\lambda_i - f_t f_{ti} + U^a\varphi_i^a.$$

Insert the latter into (2.131).ii):

$$\begin{aligned}
0 = &-\frac{m-1}{m}f_{tti} + \frac{m-2}{m}f_t f_{ti} - \frac{1}{2}S_i^\varphi + (m-1)\lambda_i + f_t f_{ti} \\
&- U^a\varphi_i^a + \frac{1}{2m}S_i^\varphi + \frac{1}{m}U^a\varphi_i^a - \frac{m-1}{m}\mu_i \\
= &-\frac{m-1}{m}(\Delta_f f)_i - \frac{m-1}{2m}S_i^\varphi + (m-1)\lambda_i - \frac{m-1}{m}U^a\varphi_i^a \\
&- \frac{m-1}{m}\mu_i,
\end{aligned}$$

that is,

$$0 = (\Delta_f f)_i + \frac{1}{2}S^\varphi_i - m\lambda_i + U^a\varphi^a_i + \mu_i.$$

Integrating the latter, we get

$$0 = \Delta_f f + \frac{1}{2}S^\varphi - m\lambda + U^\varphi + \mu + \Lambda_2$$

for some $\Lambda_2 \in \mathbb{R}$. Use (2.133) to obtain

$$\begin{aligned}
-\Lambda_2 =& \Delta_f f - \frac{1}{2}S^\varphi - \Delta_f f + U(\varphi) + \mu \\
=& -\frac{1}{2}S^\varphi + U(\varphi) + \mu
\end{aligned}$$

and therefore (2.128).iv).
Take the covariant derivative of (2.128).iv) and rearrange

$$U^a\varphi^a_i = \frac{1}{2}S^\varphi_i - \mu_i;$$

use the latter in (2.131).ii)

$$\begin{aligned}
0 =& -\frac{m-1}{m}f_{tti} + \frac{m-2}{m}f_t f_{ti} + (\Delta f)f_i \\
&+ \frac{1}{2m}S^\varphi_i - f_t R^\varphi_{ti} + \frac{1}{m}U^a\varphi^a_i - \frac{m-1}{m}\mu_i \\
=& -\frac{m-1}{m}f_{tti} + \frac{m-2}{m}f_t f_{ti} + (\Delta f)f_i + \frac{1}{m}S^\varphi_i - R^\varphi_{it}f_t - \mu_i.
\end{aligned}$$

From (2.131).i) we obtain

$$\begin{aligned}
0 =& -\frac{m-1}{m}f_{tti} + \frac{m-2}{m}f_t f_{ti} + (\Delta f)f_i + \frac{1}{m}S^\varphi_i - \mu_i \\
&+ f_t f_{ti} - |\nabla f|^2 f_i - \lambda f_i \\
=& -\frac{m-1}{m}f_{tti} + 2\frac{m-1}{m}f_t f_{ti} + (\Delta_f f)f_i + \frac{1}{m}S^\varphi_i \\
&- \mu_i - \lambda f_i \\
=& -\frac{m-1}{m}(\Delta_f f)_i + (\Delta_f f)f_i + \frac{1}{m}S^\varphi_i - \mu_i - \lambda f_i.
\end{aligned}$$

Use (2.133) to get

$$\begin{aligned}
0 =& -\frac{m-1}{m}(\Delta_f f)_i + (\Delta_f f)f_i + \frac{1}{m}S^\varphi_i - \mu_i \\
&- \frac{1}{m}S^\varphi f_i - \frac{1}{m}(\Delta_f f)f_i,
\end{aligned}$$

that is,

$$0 = -\frac{m-1}{m}(\Delta_f f)_i + \frac{m-1}{m}(\Delta_f f) f_i + \frac{1}{m}S^\varphi_i - \frac{1}{m}S^\varphi f_i - \mu_i.$$

Using (2.131).v), the latter can be re-written

$$\begin{aligned} 0 = &- \frac{m-1}{m}(\Delta_f f)_i + \frac{m-1}{m}(\Delta_f f) f_i + \frac{1}{m}S^\varphi_i - \frac{1}{m}S^\varphi f_i \\ &- \mu_i - p_i + (\mu + p) f_i. \end{aligned}$$

Multiply by $\frac{m}{m-1}$ to deduce

$$\begin{aligned} 0 = &- (\Delta_f f)_i + (\Delta_f f) f_i + \frac{1}{m-1}S^\varphi_i - \frac{1}{m-1}S^\varphi f_i \\ &- \frac{m}{m-1}(\mu_i + p_i) + \frac{m}{m-1}(\mu + p) f_i \end{aligned}$$

which is equivalent to

$$0 = -\left\{ e^{-f}\left[\Delta_f f - \frac{1}{m-1}S^\varphi + \frac{m}{m-1}(\mu + p) \right] \right\}_i.$$

Integrating the latter we deduce, for some $\Lambda_1 \in \mathbb{R}$,

$$\Lambda_1 e^f = \Delta_f f - \frac{1}{m-1}[S^\varphi - m(\mu + p)]$$

which is (2.128).ii). □

In the proof of Theorem 2.36, the validity of (2.128) is obtained as a consequence of system (2.131). In the next proposition we are going to show that they are actually equivalent.

Proposition 2.38. *Let (M, g) be an m-dimensional Riemannian manifold and let $f \in C^\infty(M)$. Then f solves*

$$\begin{cases} \mathrm{Ric}^\varphi + \mathrm{Hess}\, f - \mathrm{d}f \otimes \mathrm{d}f = \lambda g, \\ 0 = -\frac{m-1}{m} f_{tti} + \frac{m-2}{m} f_t f_{ti} + (\Delta f) f_i + \frac{1}{2m}S^\varphi_i - R^\varphi_{ti} f_t + \frac{1}{m}U^a \varphi^a_i - \frac{m-1}{m}\mu_i, \\ \tau(\varphi) = \mathrm{d}\varphi(\nabla f) + \frac{\nabla U}{\alpha}, \\ (\mu + p)\nabla f - \nabla p = 0, \end{cases} \tag{2.136}$$

if and only if it solves

$$\begin{cases} \mathrm{Ric}^\varphi + \mathrm{Hess}\, f - \mathrm{d}f \otimes \mathrm{d}f = \lambda g, \\ \Delta_f f = \frac{1}{m-1}[S^\varphi - m(\mu + p)] + \Lambda_1 e^f, \\ \tau(\varphi) = \mathrm{d}\varphi(\nabla f) + \frac{\nabla U}{\alpha}, \\ \frac{1}{2}S^\varphi = U(\varphi) + \mu + \Lambda_2, \\ (\mu + p)\nabla f - \nabla p = 0, \end{cases} \tag{2.137}$$

for some constants Λ_1, Λ_2.

Proof. That (2.137) be implied by (2.136) follows from the proof of Theorem 2.36 . For the converse, assume that (2.137) holds. We first prove, as an intermediate step, that (2.18).i) holds; that is, we prove the validity of

$$\begin{aligned} 0 =& C^{\varphi}_{ijk} + f_{ijk} - f_{ikj} - f_{ik}f_j + f_{ij}f_k \\ &- \frac{1}{m}[(f_{ttk} - 2f_t f_{tk})\delta_{ij} - (f_{ttj} - 2f_t f_{tj})\delta_{ik}] \\ &- \frac{1}{m}U^a\left(\varphi^a_k\delta_{ij} - \varphi^a_j\delta_{ik}\right) + \frac{1}{2m(m-1)}\left(S^{\varphi}_k\delta_{ij} - S^{\varphi}_j\delta_{ik}\right) \\ &- \frac{1}{m}(\mu_k\delta_{ij} - \mu_j\delta_{ik}). \end{aligned} \tag{2.138}$$

Tracing (2.137).i) we get

$$\lambda = \frac{1}{m}(S^{\varphi} + \Delta_f f),$$

so that (2.137).i) can be re-written as

$$\mathrm{Ric}^{\varphi} + \operatorname{Hess} f - \mathrm{d}f \otimes \mathrm{d}f - \frac{1}{m}(S^{\varphi} + \Delta_f f) = 0. \tag{2.139}$$

Using (2.137).iv) we rewrite (2.139):

$$\mathrm{Ric}^{\varphi} + \operatorname{Hess} f - \mathrm{d}f \otimes \mathrm{d}f - \frac{1}{m}\left(\frac{1}{2}S^{\varphi} + \Delta_f f + \mu + U(\varphi) + \Lambda_2\right) = 0. \tag{2.140}$$

Equation (2.138) is obtained from (2.140) taking its covariant derivative

$$\begin{aligned} &R^{\varphi}_{ij,k} + f_{ijk} - f_{ik}f_j - f_i f_{jk} \\ &\qquad - \frac{1}{m}\left(\frac{1}{2}S^{\varphi}_k + (\Delta_f f)_k + \mu_k + U^a\varphi^a_k\right)\delta_{ij} = 0 \end{aligned} \tag{2.141}$$

and skew symmetrizing with respect to j and k. Indeed, we have

$$\begin{aligned} 0 =& R^{\varphi}_{ij,k} - R^{\varphi}_{ik,j} + f_{ijk} - f_{ikj} - f_{ik}f_j + f_{ij}f_k \\ &- \frac{1}{m}[(f_{ttk} - 2f_t f_{tk})\delta_{ij} - (f_{ttj} - 2f_t f_{tj})\delta_{ik}] \\ &- \frac{1}{2m}\left(S^{\varphi}_k\delta_{ij} - S^{\varphi}_j\delta_{ik}\right) - \frac{1}{m}(\mu_k\delta_{ij} - \mu_j\delta_{ik}) - \frac{1}{m}\left(U^a\varphi^a_k\delta_{ij} - U^a\varphi^a_j\delta_{ik}\right). \end{aligned} \tag{2.142}$$

Using the definition of C^{φ} in (2.142) we get

$$\begin{aligned} 0 =& C^{\varphi}_{ijk} + f_{ijk} - f_{ikj} - f_{ik}f_j + f_{ij}f_k \\ &- \frac{1}{m}[(f_{ttk} - 2f_t f_{tk})\delta_{ij} - (f_{ttj} - 2f_t f_{tj})\delta_{ik}] \\ &+ \frac{1}{2m(m-1)}\left(S^{\varphi}_k\delta_{ij} - S^{\varphi}_j\delta_{ik}\right) - \frac{1}{m}(\mu_k\delta_{ij} - \mu_j\delta_{ik}) \\ &- \frac{1}{m}\left(U^a\varphi^a_k\delta_{ij} - U^a\varphi^a_j\delta_{ik}\right) \end{aligned}$$

which is (2.138). Tracing (2.138) with respect to i and j and using (1.26) we get

$$\begin{aligned} 0 =& \alpha\varphi^a_{tt}\varphi^a_k + f_{ttk} - f_{tkt} - f_t f_{tk} + (\Delta f) f_k \\ & - \frac{m-1}{m}(f_{ttk} - 2f_t f_{tk}) + \frac{1}{2m}S^\varphi_k - \frac{m-1}{m}\mu_k - \frac{m-1}{m}U^a\varphi^a_k. \end{aligned} \tag{2.143}$$

Using the Ricci commutation relations and the definition of Ric^φ we get

$$f_{ttk} - f_{tkt} = -R_{tk} f_t = -R^\varphi_{tk} f_t - \alpha\varphi^a_t\varphi^a_k f_t$$

so that (2.143) becomes

$$\begin{aligned} 0 =& \alpha\varphi^a_k(\varphi^a_{tt} - \varphi^a_t f_t) - R^\varphi_{tk} f_t - f_t f_{tk} + (\Delta f) f_k \\ & - \frac{m-1}{m} f_{ttk} + 2\frac{m-1}{m} f_{tk} f_t + \frac{1}{2m}S^\varphi_k - \frac{m-1}{m}\mu_k - \frac{m-1}{m}U^a\varphi^a_k. \end{aligned}$$

Using (2.137).iii) and simplifying we get

$$\begin{aligned} 0 =& U^a\varphi^a_k - R^\varphi_{tk} f_t - f_t f_{tk} + (\Delta f) f_k - \frac{m-1}{m} f_{ttk} \\ & + 2\frac{m-1}{m} f_{tk} f_t + \frac{1}{2m}S^\varphi_k - \frac{m-1}{m}\mu_k - \frac{m-1}{m}U^a\varphi^a_k \\ =& \frac{1}{m}U^a\varphi^a_k - R^\varphi_{tk} f_t + \frac{m-2}{m} f_{tk} f_t + (\Delta f) f_k \\ & - \frac{m-1}{m} f_{ttk} + \frac{1}{2m}S^\varphi_k - \frac{m-1}{m}\mu_k \\ =& - \frac{m-1}{m} f_{ttk} + \frac{m-2}{m} f_{tk} f_t + (\Delta f) f_k + \frac{1}{2m}S^\varphi_k - R^\varphi_{tk} f_t \\ & + \frac{1}{m}U^a\varphi^a_k - \frac{m-1}{m}\mu_k \end{aligned}$$

which, up to reordering, is equivalent to (2.136).ii). Since (2.137).iii) is (2.136).iii) and (2.137).v) is (2.136).iv) we conclude. □

Chapter 3

φ-SPFST

In this chapter, we deal with φ-static perfect fluid space-times, that is, solutions to

$$
\begin{cases}
\mathrm{Ric}^{\varphi} + \mathrm{Hess}\, f - \mathrm{d}f \otimes \mathrm{d}f - \frac{1}{m-1}\left(\frac{S^{\varphi}}{2} - p + U(\varphi)\right)g = 0, \\
\Delta f - |\nabla f|^2 = -\frac{1}{m-1}\left[mp - mU(\varphi) + \frac{m-2}{2}S^{\varphi}\right], \\
\tau(\varphi) = \mathrm{d}\varphi(\nabla f) + \frac{1}{\alpha}\nabla U(\varphi), \\
\mu + U(\varphi) = \frac{1}{2}S, \\
\nabla p - (\mu + p)\nabla f = 0.
\end{cases}
\tag{3.1}
$$

We already met them in (2.53). Working as in Remark 1.16, we can see that (3.1).i), .ii) and (3.1).iii) combined imply the validity of

$$
\begin{cases}
i)\, \mathrm{Ric}^{\varphi} + \mathrm{Hess}(f) - \mathrm{d}f \otimes \mathrm{d}f = \lambda g, \\
ii)\, \tau(\varphi) = \mathrm{d}\varphi(\nabla f) + \frac{1}{\alpha}(\nabla U)(\varphi),
\end{cases}
$$

with $\lambda := \frac{1}{m}\left(S^{\varphi} + \Delta f - |\nabla f|^2\right)$. We add a parameter $\eta \in \mathbb{R}$ to obtain

$$
\begin{cases}
i)\, \mathrm{Ric}^{\varphi} + \mathrm{Hess}(f) - \eta \mathrm{d}f \otimes \mathrm{d}f = \lambda g, \\
ii)\, \tau(\varphi) = \mathrm{d}\varphi(\nabla f) + \frac{1}{\alpha}(\nabla U)(\varphi).
\end{cases}
\tag{3.2}
$$

When $U \equiv 0$, structures as (3.2) are called *φ-Einstein-type structures*, see [5]. We will be interested in rigidity results for manifolds supporting a solution $f \in C^{\infty}(M)$ of (3.2). The first fundamental tool for this investigation is the following first integrability condition, that we already met in Proposition 2.7 for $\eta = 1$, and that we now prove.

Proposition 3.1. *Let (M, g) be a Riemannian manifold of dimension $m \geq 3$. Let $\varphi : (M, g) \to (N, h)$, $U : (N, h) \to \mathbb{R}$, $\lambda : (M, g) \to \mathbb{R}$ be smooth maps, $\alpha \in \mathbb{R}\setminus\{0\}$ and let $f \in C^{\infty}(M)$ be a solution of (3.2) on M. We then have*

$$
[1 + \eta(m-2)]D^{A}_{ijk} = C^{\varphi}_{ijk} + f_t W^{\varphi}_{tijk} + \frac{U^a}{m-1}\left(\varphi^a_j \delta_{ik} - \varphi^a_k \delta_{ij}\right).
\tag{3.3}
$$

Proof. To obtain equation (3.3), we need to find a "good" expression for C^φ. We use the first equation of system (3.2) and the Ricci commutation relations to deduce

$$\begin{aligned}
C^\varphi_{ijk} =& R^\varphi_{ij,k} - R^\varphi_{ik,j} - \frac{1}{2(m-1)}\left(S^\varphi_k\delta_{ij} - S^\varphi_j\delta_{ik}\right)\\
=& - f_{ijk} + f_{ikj} + \eta(f_{ik}f_j + f_i f_{jk} - f_{ij}f_k - f_i f_{jk})\\
&+ \lambda_k\delta_{ij} - \lambda_j\delta_{ik} - \frac{1}{2(m-1)}\left(S^\varphi_k\delta_{ij} - S^\varphi_j\delta_{ik}\right)\\
=& - f_p R_{pijk} + \eta(f_{ik}f_j - f_{ij}f_k) + \left(\lambda_k - \frac{1}{2(m-1)}S^\varphi_k\right)\delta_{ij}\\
&- \left(\lambda_j - \frac{1}{2(m-1)}S^\varphi_j\right)\delta_{ik},
\end{aligned}$$

that is,

$$\begin{aligned}
C^\varphi_{ijk} =& - f_p R_{pijk} + \eta(f_{ik}f_j - f_{ij}f_k) + \left(\lambda_k - \frac{1}{2(m-1)}S^\varphi_k\right)\delta_{ij}\\
&- \left(\lambda_j - \frac{1}{2(m-1)}S^\varphi_j\right)\delta_{ik}.
\end{aligned}\tag{3.4}$$

Now we contract (3.4) with respect to i and j and recall that $C^\varphi_{ttk} = \alpha\varphi^a_{tt}\varphi^a_k$, to obtain

$$\alpha\varphi^a_{tt}\varphi^a_k = f_t R_{tk} + \eta(f_{tk}f_t - \Delta f f_k) + (m-1)\lambda_k - \frac{1}{2}S^\varphi_k.$$

Using the definition of Ric^φ and rearranging the terms we get

$$\frac{1}{2}S^\varphi_k - (m-1)\lambda_k = f_t R^\varphi_{tk} + \eta(f_{tk}f_t - \Delta f f_k) - \alpha\varphi^a_k(\varphi^a_{tt} - \varphi^a_t f_t).$$

Inserting the second equation of (3.2) into the above formula we deduce

$$\frac{1}{2}S^\varphi_k - (m-1)\lambda_k = f_t R^\varphi_{tk} + \eta(f_{tk}f_t - \Delta f f_k) - U^a\varphi^a_k,\tag{3.5}$$

while using (3.5) into (3.4) we obtain

$$\begin{aligned}
C^\varphi_{ijk} =& - f_p R_{pijk} + \eta(f_{ik}f_j - f_{ij}f_k)\\
&- \frac{\delta_{ij}}{m-1}[f_t R^\varphi_{tk} + \eta(f_{tk}f_t - \Delta f f_k) - U^a\varphi^a_k]\\
&+ \frac{\delta_{ik}}{m-1}[f_t R^\varphi_{tj} + \eta(f_{tj}f_t - \Delta f f_j) - U^a\varphi^a_j].
\end{aligned}\tag{3.6}$$

We now multiply the first equation of system (3.2) by ηf_k, that is

$$\eta f_{ij}f_k = -\eta R^\varphi_{ij}f_k + \eta f_i f_j f_k + \eta\lambda f_k\delta_{ij},$$

and skew-symmetrize the latter equation, in order to deduce

$$\eta(f_{ik}f_j - f_{ij}f_k) = \eta(R^\varphi_{ij}f_k - R^\varphi_{ik}f_j + \lambda f_j\delta_{ik} - \lambda f_k\delta_{ij}); \tag{3.7}$$

contracting (3.7) with respect to i and j we also have

$$\eta(f_{tk}f_t - \Delta f f_k) = \eta(S^\varphi f_k - f_t R^\varphi_{tk} - (m-1)\lambda f_k). \tag{3.8}$$

Using (3.7) and (3.8) into (3.6) we obtain

$$\begin{aligned} C^\varphi_{ijk} = &- f_p R_{pijk} + \eta(R^\varphi_{ij}f_k - R^\varphi_{ik}f_j + \lambda f_j\delta_{ik} - \lambda f_k\delta_{ij}) \\ &- \frac{\delta_{ij}}{m-1}[(1-\eta)f_t R^\varphi_{tk} + \eta S^\varphi f_k - (m-1)\eta\lambda f_k - U^a\varphi^a_k] \\ &+ \frac{\delta_{ik}}{m-1}\big[(1-\eta)f_t R^\varphi_{tj} + \eta S^\varphi f_j - (m-1)\eta\lambda f_j - U^a\varphi^a_j\big], \end{aligned}$$

which can be written as

$$\begin{aligned} C^\varphi_{ijk} = &- f_p R_{pijk} + \eta(R^\varphi_{ij}f_k - R^\varphi_{ik}f_j) \\ &- \frac{\delta_{ij}}{m-1}[(1-\eta)f_t R^\varphi_{tk} + \eta S^\varphi f_k - U^a\varphi^a_k] \\ &+ \frac{\delta_{ik}}{m-1}\big[(1-\eta)f_t R^\varphi_{tj} + \eta S^\varphi f_j - U^a\varphi^a_j\big]. \end{aligned} \tag{3.9}$$

Now we use the definition of W^φ to get

$$\begin{aligned} f_p R_{pijk} = &f_p W^\varphi_{pijk} + \frac{f_j}{m-2}R^\varphi_{ik} + \frac{f_p R^\varphi_{pj}}{m-2}\delta_{ik} - \frac{f_k}{m-2}R^\varphi_{ij} \\ &- \frac{f_p R^\varphi_{pk}}{m-2}\delta_{ij} - \frac{S^\varphi}{(m-1)(m-2)}(f_j\delta_{ik} - f_k\delta_{ij}). \end{aligned}$$

Inserting the previous relation into (3.9) we obtain, after some simplifications,

$$\begin{aligned} C^\varphi_{ijk} = &\frac{1+\eta(m-2)}{m-2}\left[R^\varphi_{ij}f_k - R^\varphi_{ik}f_j - \frac{f_t}{m-1}\big(R^\varphi_{tj}\delta_{ik} - R^\varphi_{tk}\delta_{ij}\big)\right] \\ &+ \frac{1+\eta(m-2)}{(m-1)(m-2)}S^\varphi(f_j\delta_{ik} - f_k\delta_{ij}) - f_p W^\varphi_{pijk} + \frac{U^a}{m-1}\varphi^a_k\delta_{ij} \\ &- \frac{U^a}{m-1}\varphi^a_j\delta_{ik}. \end{aligned}$$

Using the definition (2.138) of D^A we get

$$C^\varphi_{ijk} = [1+\eta(m-2)]D^A_{ijk} - f_p W^\varphi_{pijk} + \frac{U^a}{m-1}\varphi^a_k\delta_{ij} - \frac{U^a}{m-1}\varphi^a_j\delta_{ik},$$

and therefore (3.3).

□

It is easy to prove that, when (3.2).i) holds, then we have $D^A \equiv D^B$. Therefore, from (3.3), one can see that, if (3.2) holds, if $D^A \equiv 0$ and if φ is $\frac{1}{\alpha}U$-harmonic, then the assumption of Theorem 2.9 are satisfied, so that we have local rigidity of (M, g). In this spirit, similarly to Theorem 2.19, one can prove the next result

Theorem 3.2 (Theorem 4.17 of [13]). *Let (M, g) be a complete Riemannian manifold of dimension $m \geq 3$. Let $\varphi : (M, g) \to (N, h)$, $U : N \to \mathbb{R}$ be smooth maps, $\lambda \in C^\infty(M)$ and let $f \in C^\infty(M)$ be a solution of (3.2). Let Σ be a regular level set of f. Assume that:*

1. *f is proper;*
2. *either $\alpha > 0$ and $\eta > -\frac{1}{m-2}$ or $\alpha < 0$ and $\eta < -\frac{1}{m-2}$;*
3. *we have the validity of*
$$B^\varphi(\nabla f, \nabla f) = 0; \tag{3.10}$$
4. *for each regular $p \in M$, ∇f_p is an eigenvector of Ric^φ_p;*
5. *φ is $\frac{1}{\alpha}U$-harmonic*

Then, for each $p \in \Sigma$, there exists $p \in A \subset M$, A open in M such that $g_{|_A}$ is a warped product metric. Moreover, $U(\varphi)$ is constant on Σ and (Σ, g_Σ) satisfies

$$\begin{cases} \mathrm{Ric}^{\varphi_{|_\Sigma}} = \frac{S^{\varphi_{|_\Sigma}}}{m-1} g_\Sigma, \\ \tau(\varphi_{|_\Sigma}) = 0 \end{cases}$$

and $S^{\varphi_{|_\Sigma}}$ is constant.

One can easily show that, when $\eta = 1$, Theorem 2.19 generalizes Theorem 3.2. Indeed, under the validity of (3.2) and from (3.3), we see that (2.96) is satisfied. Moreover, when (3.2) holds and φ is $\frac{1}{\alpha}U$-harmonic, we immediately deduce that $\mathrm{d}\varphi(\nabla f) = 0$; this, and equation $\mathrm{Ric}^\varphi + \mathrm{Hess}\, f - \mathrm{d}f \otimes \mathrm{d}f = \lambda g$, imply the condition on the operator F given in Theorem 2.19. Therefore, when $\eta = 1$, the assumptions of Theorem 3.2 imply the validity of those of Theorem 2.19. When $\eta \neq 1$, the proof of Theorem 3.2 is still pretty similar to that of Theorem 2.19, so that we have not felt it necessary to include it in the present discussion. Instead, we refer to [13, Theorem 4.17] for the full proof.
In the next Section, we deal with two further rigidity results for system (3.2), the conclusion being again the vanishing of the tensor D^A. We remark here that we have not been successful in providing meaningful generalizations of them for C-φ-PFs.

3.1 Other Rigidity Results

In this Section, we deal with the next two theorems, that are inspired by [20].

Theorem 3.3. *Let (M, g) be a complete Riemannian manifold satisfying system (3.2), where f is a proper function and $\eta \neq -\frac{1}{m-2}$. If M is non-compact, we will also require*

$$f(x) \to +\infty \qquad \text{as } x \to +\infty.$$

Assume that

$$\operatorname{div}^3 C^{\varphi} = 0, \tag{3.11}$$
$$h(\tau(\varphi), d\varphi) = 0. \tag{3.12}$$

Then we have two possibilities:

i) if $\eta \neq 0$ and we further assume

$$W^{\varphi}(\nabla f, \cdot, \cdot, \cdot) = 0, \tag{3.13}$$

then we have

$$C^{\varphi} = 0 \tag{3.14}$$

and

$$D^A = 0; \tag{3.15}$$

ii) if $\eta = 0$ we have

$$C^{\varphi} = 0$$

and, if we further assume (3.13), also

$$D^A = 0.$$

Theorem 3.4. *Let (M, g) be a complete Riemannian manifold satisfying system (3.2), where f is a proper function and $\eta \neq -\frac{1}{m-2}$. If M is non-compact, we also require*

$$f(x) \to +\infty \qquad \text{as } x \to +\infty.$$

Assume that

$$\operatorname{div}^3 C^{\varphi} = 0, \tag{3.16}$$
$$\varphi \text{ is } \frac{1}{\alpha}U\text{-harmonic}, \tag{3.17}$$
$$\nabla f_p \text{ is an eigenvector of } \operatorname{Ric}^{\varphi}_p \text{ for every regular point } p \text{ of } f. \tag{3.18}$$

Then we have two possibilities:

i) if $\eta \neq 0$ and we further assume

$$W^{\varphi}(\nabla f, \cdot, \cdot, \cdot) = 0, \tag{3.19}$$

then we have

$$C^{\varphi} = -\frac{1}{2(m-1)} \operatorname{div}_1 \left(U(\varphi) g \owedge g \right) \tag{3.20}$$

and

$$D^A = 0; \tag{3.21}$$

ii) if $\eta = 0$ we have

$$C^{\varphi} = -\frac{1}{2(m-1)} \operatorname{div}_1 \left(U(\varphi) g \owedge g \right)$$

and, if we further assume (3.19), also

$$D^A = 0.$$

Remark 3.5. Condition (3.18) is necessary for (3.21), as observed in Proposition 2.11.

Although the assumptions of the two Theorems that we have presented are different, the techniques of their proofs are similar and we get the same conclusion (further comments are given, after their proofs, in Remark 3.10).
To prove Theorems 3.3 and 3.4 we will need some preliminary results.
We begin with a very general lemma.

Lemma 3.6. *If $\omega \in \Lambda^2(M)$ is a 2-form, its* total divergence $\operatorname{div}^2 \omega$ *vanishes. In components,*

$$\omega_{tk,kt} = 0.$$

Proof. The proof is a simple application of the Ricci commutation formulas. Indeed

$$\begin{aligned}
\omega_{tk,kt} &= \frac{1}{2}(\omega_{tk,kt} - \omega_{kt,kt}) \\
&= \frac{1}{2}(\omega_{tk,kt} - \omega_{tk,tk}) \\
&= \frac{1}{2}(R_{ptkt}\omega_{pk} + R_{pkkt}\omega_{tp}) \\
&= R_{pk}\omega_{pk} = 0.
\end{aligned}$$

□

The core of the proof of Theorem 3.4 lies in the next Proposition.

Proposition 3.7. *Let (M, g) be a Riemannian manifold admitting a solution $f \in C^\infty(M)$ of*

$$\mathrm{Ric}^\varphi + \mathrm{Hess}(f) - \eta \mathrm{d}f \otimes \mathrm{d}f = \lambda g, \tag{3.22}$$

for $\eta \neq -\frac{1}{m-2}$. Then we have the validity of the following formula:

$$\begin{aligned}\frac{1}{2[(m-2)\eta+1]}|C^\varphi|^2 = f_t C^\varphi_{tjk,jk} &- \frac{\eta(m-2)}{2(m-2)\eta+2} f_t W^\varphi_{ptjk} C^\varphi_{pjk} \\ &+ \frac{1}{(m-2)\eta+1}\left[\lambda_k - \frac{1}{2(m-1)} S^\varphi_k - \eta\lambda f_k\right] C^\varphi_{ttk} \\ &- \frac{\eta}{(m-2)\eta+1}\left[f_t R^\varphi_{tk} - \frac{S^\varphi}{(m-1)} f_k\right] C^\varphi_{ttk}.\end{aligned} \tag{3.23}$$

Proof. We apply Lemma 3.6 to the 2-form of components

$$f_t C^\varphi_{tjk}$$

to obtain

$$\begin{aligned}0 = \left(f_t C^\varphi_{tjk}\right)_{jk} &= f_{tjk} C^\varphi_{tjk} + f_{tj} C^\varphi_{tjk,k} + f_{tk} C^\varphi_{tjk,j} + f_t C^\varphi_{tjk,jk} \\ &= f_{tjk} C^\varphi_{tjk} + f_t C^\varphi_{tjk,jk},\end{aligned}$$

that is,

$$0 = f_{tjk} C^\varphi_{tjk} + f_t C^\varphi_{tjk,jk}. \tag{3.24}$$

We claim the validity of the equation

$$\begin{aligned}0 =& f_t C^\varphi_{tjk,jk} - \frac{1}{2}|C^\varphi|^2 + \eta f_k R^\varphi_{tj} C^\varphi_{tjk} \\ &+ C^\varphi_{ttk}\left(\lambda_k - \frac{1}{2(m-1)} S^\varphi_k - \eta\lambda f_k\right).\end{aligned} \tag{3.25}$$

To prove it, take covariant derivative of equation (3.22) and skew symmetrize with respect to the last two indexes to deduce

$$f_{tjk} - f_{tkj} = -R^\varphi_{tj,k} + R^\varphi_{tk,j} + \eta f_{tk} f_j - \eta f_{tj} f_k + \lambda_k \delta_{tj} - \lambda_j \delta_{tk}.$$

Using the definition of C^φ we get

$$\begin{aligned}f_{tjk} - f_{tkj} =& - C^\varphi_{tjk} - \frac{1}{2(m-1)}\left(S^\varphi_k \delta_{tj} - S^\varphi_j \delta_{tk}\right) + \eta f_{tk} f_j - \eta f_{tj} f_k \\ &+ \lambda_k \delta_{tj} - \lambda_j \delta_{tk}.\end{aligned} \tag{3.26}$$

From the skew-symmetry of the last two indexes of C^φ and equation (3.24) we obtain

$$0 = f_t C^\varphi_{tjk,jk} + \frac{1}{2}(f_{tjk} - f_{tkj}) C^\varphi_{tjk}.$$

Inserting (3.26) into the above formula we have

$$\begin{aligned}0 =& f_t C^{\varphi}_{tjk,jk} - \frac{1}{2}|C^{\varphi}|^2 - \frac{1}{2(m-1)} S^{\varphi}_k C^{\varphi}_{ttk} + \lambda_k C^{\varphi}_{ttk} \\ &+ \eta f_{tk} f_j C^{\varphi}_{tjk}.\end{aligned}$$

Using again equation (3.22), which we rewrite as

$$f_{tk} = -R^{\varphi}_{tk} + \eta f_t f_k + \lambda \delta_{tk},$$

into the latter, we obtain (3.25).
The last formula that we will need to conclude is

$$\begin{aligned}\frac{(m-2)\eta+1}{(m-2)} f_k R^{\varphi}_{pj} C^{\varphi}_{pjk} =& \frac{1}{2}|C^{\varphi}|^2 - \frac{1}{2} f_t W^{\varphi}_{ptjk} C^{\varphi}_{pjk} \\ - C^{\varphi}_{ttk}\Big(\lambda_k &- \frac{1}{2(m-1)} S^{\varphi}_k - \eta\lambda f_k + \frac{1}{m-2} f_t R^{\varphi}_{tk} - \frac{S^{\varphi}}{(m-1)(m-2)} f_k\Big).\end{aligned} \tag{3.27}$$

To deduce it, we apply the Ricci commutation rules to obtain

$$\begin{aligned}f_t C^{\varphi}_{tjk,jk} &= \frac{1}{2} f_t \big(C^{\varphi}_{tjk,jk} - C^{\varphi}_{tjk,kj}\big) \\ &= \frac{1}{2} f_t R_{ptjk} C^{\varphi}_{pjk} + \frac{1}{2} f_t R_{pjjk} C^{\varphi}_{tpk} + \frac{1}{2} f_t R_{pkjk} C^{\varphi}_{tjp} \\ &= \frac{1}{2} f_t R_{ptjk} C^{\varphi}_{pjk} + f_t R^{\varphi}_{pk} C^{\varphi}_{tpk} \\ &= \frac{1}{2} f_t R_{ptjk} C^{\varphi}_{pjk}.\end{aligned}$$

Using the definition of W^{φ} into the above equation we get

$$\begin{aligned}f_t C^{\varphi}_{tjk,jk} =& \frac{1}{2} f_t W^{\varphi}_{ptjk} C^{\varphi}_{pjk} + \frac{1}{2(m-2)}\big(f_t R^{\varphi}_{tk} C^{\varphi}_{ppk} - f_t R^{\varphi}_{tj} C^{\varphi}_{pjp}\big) \\ &+ \frac{1}{2(m-2)}\big(f_k R^{\varphi}_{pj} C^{\varphi}_{pjk} - f_j R^{\varphi}_{pk} C^{\varphi}_{pjk}\big) \\ &- \frac{S^{\varphi}}{2(m-1)(m-2)}\big(f_k C^{\varphi}_{ttk} - f_j C^{\varphi}_{tjt}\big) \\ =& \frac{1}{2} f_t W^{\varphi}_{ptjk} C^{\varphi}_{pjk} + \frac{1}{(m-2)} f_t R^{\varphi}_{tk} C^{\varphi}_{ppk} \\ &+ \frac{1}{(m-2)} f_k R^{\varphi}_{pj} C^{\varphi}_{pjk} - \frac{S^{\varphi}}{(m-1)(m-2)} f_k C^{\varphi}_{ttk},\end{aligned}$$

that is,

$$\begin{aligned}f_t C^{\varphi}_{tjk,jk} =& \frac{1}{2} f_t W^{\varphi}_{ptjk} C^{\varphi}_{pjk} + \frac{1}{(m-2)} f_t R^{\varphi}_{tk} C^{\varphi}_{ppk} \\ &+ \frac{1}{(m-2)} f_k R^{\varphi}_{pj} C^{\varphi}_{pjk} - \frac{S^{\varphi}}{(m-1)(m-2)} f_k C^{\varphi}_{ttk}.\end{aligned} \tag{3.28}$$

Inserting (3.28) into (3.25) and rearranging we get (3.27). Using (3.27) into (3.25) we obtain

$$
\begin{aligned}
0 =& f_t C^{\varphi}_{tjk,jk} - \frac{1}{2}|C^{\varphi}|^2 + C^{\varphi}_{ttk}\left(\lambda_k - \frac{1}{2(m-1)}S^{\varphi}_k - \eta\lambda f_k\right) \\
&+ \frac{\eta(m-2)}{(m-2)\eta+1}\left(\frac{1}{2}|C^{\varphi}|^2 - \frac{1}{2}f_t W^{\varphi}_{ptjk}C^{\varphi}_{pjk}\right) \\
&- \frac{\eta(m-2)}{(m-2)\eta+1}C^{\varphi}_{ttk}\left(\lambda_k - \frac{1}{2(m-1)}S^{\varphi}_k\right. \\
&\left. -\eta\lambda f_k + \frac{1}{m-2}f_t R^{\varphi}_{tk} - \frac{S^{\varphi}}{(m-1)(m-2)}f_k\right),
\end{aligned}
$$

which, after some simplifications, becomes (3.23). □

Proposition 3.8. *Let (M, g) be a Riemannian manifold admitting a solution of*

$$\mathrm{Ric}^{\varphi} + \mathrm{Hess}(f) - \eta \mathrm{d}f \otimes \mathrm{d}f = \lambda g,$$

for $\eta \neq -\frac{1}{m-2}$. Let $z : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $z(f)$ is compactly supported on M. Then we have the validity of the following integral formula:

$$
\begin{aligned}
\frac{1}{2[(m-2)\eta+1]}\int_M |C^{\varphi}|^2[z(f) - z'(f)]e^{-f} =& -\int_M C^{\varphi}_{tjk,kjt}z(f)e^{-f} \qquad (3.29)\\
&- \frac{\eta(m-2)}{2(m-2)\eta+2}\int_M f_t W^{\varphi}_{ptjk}C^{\varphi}_{pjk}[z(f) - z'(f)]e^{-f} \\
&+ \frac{1}{(m-2)\eta+1}\int_M \left(\lambda_k - \frac{1}{2(m-1)}S^{\varphi}_k - \eta\lambda f_k\right)C^{\varphi}_{ttk}[z(f) - z'(f)]e^{-f} \\
&- \frac{1}{(m-2)\eta+1}\int_M \left(f_t R^{\varphi}_{tk} - \frac{S^{\varphi}}{(m-1)}f_k\right)C^{\varphi}_{ttk}[z(f) - z'(f)]e^{-f}.
\end{aligned}
$$

Proof. Integrating by parts, we have

$$
\begin{aligned}
\int_M f_t C^{\varphi}_{tik,ki}z(f)e^{-f} =& \int_M C^{\varphi}_{tik,kit}z(f)e^{-f} \\
&+ \int_M f_t C^{\varphi}_{tik,ki}z'(f)e^{-f},
\end{aligned}
$$

which implies

$$\int_M f_t C^{\varphi}_{tik,ki}[z(f) - z'(f)]e^{-f} = \int_M \mathrm{div}^3\, C^{\varphi} z(f)e^{-f}. \qquad (3.30)$$

Multiplying (3.23) by $[z(f) - z'(f)]e^{-f}$, integrating on M and using (3.30) we immediately obtain (3.29).

□

As a computational short-hand, we define

$$F_{ijk} = C^{\varphi}_{ijk} - \frac{1}{m-1}U^a\varphi^a_k\delta_{ij} + \frac{1}{m-1}U^a\varphi^a_j\delta_{ik}. \tag{3.31}$$

Then, if we assume the validity of (3.17), we have

$$\begin{aligned}
|F|^2 &= |C^{\varphi}|^2 + \frac{2m}{(m-1)^2}|\nabla(U(\varphi))|^2 - \frac{2}{(m-1)^2}|\nabla(U(\varphi))|^2 - \frac{4}{m-1}C^{\varphi}_{ijk}(U^a\varphi^a_k\delta_{ij})\\
&= |C^{\varphi}|^2 + \frac{2}{m-1}|\nabla(U(\varphi))|^2 - \frac{4\alpha}{m-1}\varphi^a_{tt}\varphi^a_k U^b\varphi^b_k\\
&= |C^{\varphi}|^2 - \frac{2}{m-1}|\nabla(U(\varphi))|^2
\end{aligned}$$

where we have used

$$C^{\varphi}_{ttk} = \alpha\varphi^a_{tt}\varphi^a_k$$

and (3.17). Thus we have

$$|F|^2 = |C^{\varphi}|^2 - \frac{2}{m-1}|\nabla(U(\varphi))|^2. \tag{3.32}$$

Corollary 3.9. *Let (M, g) satisfy system (3.2) with $\eta \neq -\frac{1}{m-2}$. Assume that*

$$\varphi \text{ is } \frac{1}{\alpha}U\text{-harmonic}, \tag{3.33}$$

$$\nabla f_p \text{ is an eigenvector of } \mathrm{Ric}^{\varphi}_p, \text{ for every regular point } p \text{ of } f. \tag{3.34}$$

Moreover, when $\eta \neq 0$, also assume

$$W^{\varphi}(\nabla f, \cdot, \cdot, \cdot) = 0.$$

Let $z : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $z(f)$ is compactly supported on M. Then

$$\frac{1}{2[1+(m-2)\eta]}\int_M |F|^2[z(f) - z'(f)]e^{-f} = -\int_M \operatorname{div}^3 C^{\varphi} z(f)e^{-f}, \tag{3.35}$$

where the components of F have been defined in (3.31).

Proof. From the second equation of (3.2) and since φ is $\frac{1}{\alpha}U$-harmonic we get

$$\varphi^a_t f_t = 0.$$

Since

$$C^{\varphi}_{ttk} = \alpha\varphi^a_{tt}\varphi^a_k,$$

we obtain that, from assumptions (3.33) and (3.34), equation (3.29) reduces to

$$\frac{1}{2[(m-2)\eta+1]}\int_M |C^\varphi|^2[z(f)-z'(f)]e^{-f} = -\int_M \operatorname{div}^3 C^\varphi z(f)e^{-f} \\ + \int_M \alpha\varphi^a_{tt}\varphi^a_k\left[\frac{1}{(m-2)\eta+1}\left(\lambda_k - \frac{1}{2(m-1)}S^\varphi_k\right)\right][z(f)-z'(f)]e^{-f}.$$

To deal with the last term, we apply (3.5), that is,

$$\frac{1}{2}S^\varphi_k = R^\varphi_{ik}f_i + \eta(f_{ik}f_i - \Delta f f_k) + (m-1)\lambda_k - \alpha\varphi^a_k(\varphi^a_{tt} - \varphi^a_t f_t). \tag{3.36}$$

This, together with assumptions (3.33) and (3.34), implies

$$\alpha\varphi^a_{tt}\varphi^a_k\left[\frac{1}{(m-2)\eta+1}\left(\lambda_k - \frac{1}{2(m-1)}S^\varphi_k\right)\right] \\ = \frac{\alpha^2}{(m-1)[(m-2)\eta+1]}\varphi^a_{tt}\varphi^a_k\varphi^b_{pp}\varphi^b_k \\ = \frac{1}{(m-1)[(m-2)\eta+1]}|\nabla U(\varphi)|^2,$$

and therefore

$$\frac{1}{2[(m-2)\eta+1]}\int_M |C^\varphi|^2[z(f)-z'(f)]e^{-f} = -\int_M \operatorname{div}^3 C^\varphi z(f)e^{-f} \\ + \frac{1}{(m-1)[(m-2)\eta+1]}\int_M |\nabla U(\varphi)|^2[z(f)-z'(f)]e^{-f}.$$

Rearranging and using equation (3.32) we conclude. □

We are ready for the proof of Theorem 3.4. We will use equation (3.35) together with a trick taken from [20].

Proof (of Theorem 3.4). From equation (3.35) and assumption (3.16) we get

$$\frac{1}{2[1+(m-2)\eta]}\int_M |F|^2 e^{-f}(z(f)-z'(f)) = 0. \tag{3.37}$$

If M is compact, set $z(f) \equiv 1$ to obtain $F \equiv 0$ and therefore (3.20). If M is non-compact, let

$$z_k(t) = \begin{cases} 1 & t \in [-k,k], \\ \frac{2k+t}{k} & t \in [-2k,-k], \\ \frac{2k-t}{k} & t \in [k,2k], \\ 0 & t \in (-\infty,-2k] \cup [2k,+\infty) \end{cases}$$

for a positive natural number k. Since $f(x) \to +\infty$ as $x \to +\infty$ by assumptions, there exists $A \in \mathbb{R}$ such that

$$f(x) \geq -2A \quad \text{on } M.$$

Let $k \geq 2A$. Then, for $f(x) \in [-2k, -k]$, we have

$$z_k(f(x)) - z_k'(f(x)) = 2 + \frac{f(x)-1}{k} \geq 2 - \frac{2A+1}{k} \geq 1 - \frac{1}{k}$$

and, for $f(x) \in [k, 2k]$, we have

$$z_k(f(x)) - z_k'(f(x)) = 2 - \frac{f(x)-1}{k} \geq \frac{1}{k}.$$

Therefore, from (3.37) we deduce

$$0 = \int_{\overline{\Omega}_{2k}} |F|^2 e^{-f}(z_k(f) - z_k'(f)) \geq \frac{1}{k}\int_{\overline{\Omega}_{2k}} |F|^2 e^{-f}$$

where

$$\overline{\Omega}_{2k} = \overline{\{x \in M : f(x) \in [-2k, 2k]\}}.$$

Letting $k \to +\infty$ we get $F \equiv 0$ and therefore equation (3.20). To conclude, if $F \equiv 0$ and $f_t W^{\varphi}_{tijk} = 0$, the first integrability condition (3.3) of system (3.2) implies $D^A \equiv 0$. □

Proof (of Theorem 3.3). Assumption (3.12) reads, in components,

$$\varphi^a_{tt}\varphi^a_k = 0, \quad \forall i = 1, ..., m.$$

Since $C^{\varphi}_{ttk} = \alpha\varphi^a_{tt}\varphi^a_k$ we deduce

$$C^{\varphi}_{ttk} = 0.$$

Therefore, when either $\eta = 0$ or the zero radial Weyl curvature condition (3.13) holds, using (3.12) and (3.29), we deduce that, for any smooth, compactly supported function $z : \mathbb{R} \to \mathbb{R}$, we have

$$\frac{1}{2[(m-2)\eta+1]}\int_M |C^{\varphi}|^2[z(f) - z'(f)]e^{-f} = -\int_M C^{\varphi}_{tjk,kjt} z(f)e^{-f}.$$

The proof now continues as that of Theorem 3.4, replacing F with C^{φ}, to give

$$C^{\varphi} \equiv 0.$$

From the first integrability condition (3.3), we deduce

$$[1 + \eta(m-2)]D^A_{ijk} = f_t W^{\varphi}_{tijk} - \frac{U^a\varphi^a_k}{m-1}\delta_{ij} + \frac{U^a\varphi^a_j}{m-1}\delta_{ik}.$$

If we are assuming (3.13), then we have

$$[1 + \eta(m-2)]D^A_{ijk} = -\frac{U^a\varphi^a_k}{m-1}\delta_{ij} + \frac{U^a\varphi^a_j}{m-1}\delta_{ik}. \tag{3.38}$$

Since D^A is totally trace-free, tracing the above equation with respect to i and j gives

$$0 = [1 + \eta(m-2)]D^A_{ttk} = -\frac{m}{m-1}U^a\varphi^a_k + \frac{1}{m-1}U^a\varphi^a_k = -U^a\varphi^a_k.$$

From equation (3.38) and assumption $\eta \neq -\frac{1}{m-2}$ we deduce $D^A \equiv 0$, and this concludes the proof. □

Remark 3.10. Before concluding this section, we want to further discuss the differences and the similarities of Theorems 3.3 and 3.4; indeed, we will see in Proposition 3.13 below, that, assuming that (3.2) holds, then assumptions (3.17) and (3.18) together imply (3.12), that is, φ is conservative, on the set of regular points of f. However, this is not enough to deduce that Theorem 3.4 is a consequence of Theorem 3.3; indeed, when one runs the argument used in the proof of Theorem 3.3, the validity of (3.12) on the set of regular points of f is not enough to conclude, unless one assumes that this set is dense in M. This can be seen directly from formula (3.29): assuming the existence of an open set at which f is constant, we have that φ need not be conservative there, so that C^φ need not be totally traceless (see equation (1.26)); therefore, the last two lines of (3.29) will in general be non-zero.

Proposition 3.11. *Let (M, g) be an m-dimensional Riemannian manifold, $m \geq 3$. Assume that $f \in C^\infty(M)$ solves*

$$\begin{cases} \mathrm{Ric}^\varphi + \mathrm{Hess}\, f - \eta \mathrm{d}f \otimes \mathrm{d}f = \lambda g, \\ \tau(\varphi) = \mathrm{d}\varphi(\nabla f) + \frac{\nabla U}{\alpha} \end{cases} \tag{3.39}$$

for some $\lambda \in C^\infty(M)$, $\eta \in \mathbb{R}$, $\varphi : (M, g) \to (N, h)$ smooth and $U \in C^\infty(N)$. Then:

- *if $\eta = -\frac{1}{m-2}$, then $U(\varphi)$ is constant on the regular level sets of f;*
- *if $\eta = 1$, then $\frac{S^\varphi}{2} - U(\varphi)$ is constant on the regular level sets of f;*
- *if $\eta \neq 1, -\frac{1}{m-2}$, then*

$$\frac{1 + \eta(m-2)}{2}S^\varphi + (1 - \eta m)U(\varphi)$$

  is constant on the regular level sets of f if and only if ∇f is an eigenvector of Ric^φ at every regular point of f.

Proof. From the definition (2.47) of D^A, one immediately gets

$$f_t D^A_{tjk} = \frac{1}{m-1} f_t\big(R^\varphi_{tj} f_k - R^\varphi_{tk} f_j\big), \tag{3.40}$$

see also the proof of Proposition 2.11. From the definition of C^φ, one easily proves that

$$C^\varphi_{tij,t} = \alpha\big\{R^\varphi_{it}\varphi^a_t\varphi^a_j - R^\varphi_{jt}\varphi^a_t\varphi^a_i + \varphi^a_{tti}\varphi^a_j - \varphi^a_{ttj}\varphi^a_i\big\} \tag{3.41}$$

see also equation (2.27) of [5]. We compute the 1-divergence of $\overline{D}$:

$$
\begin{aligned}
D^A_{ijk,i} =& \frac{1}{m-2}\Big[R^\varphi_{ij,i} f_k + R^\varphi_{ij} f_{ik} - R^\varphi_{ik,i} f_j - R^\varphi_{ik} f_{ij} \\
&+ \frac{1}{m-1}\big(f_{tj} R^\varphi_{tk} - f_{tk} R^\varphi_{tj}\big) + \frac{1}{m-1} f_t \big(R^\varphi_{tk,j} - R^\varphi_{tj,k}\big) \\
&- \frac{1}{m-1}\big(S^\varphi_j f_k - S^\varphi_k f_j\big) - \frac{S^\varphi}{m-1}\big(f_{kj} - f_{jk}\big)\Big].
\end{aligned}
$$

Using the φ-Schur's identity, the definition of C^φ and (3.39).i) we get

$$
\begin{aligned}
D^A_{ijk,i} =& \frac{1}{m-2}\Big[\frac{1}{2} S^\varphi_j f_k - \alpha \varphi^a_{tt} \varphi^a_j f_k - \frac{1}{2} S^\varphi_k f_j + \alpha \varphi^a_{tt} \varphi^a_k f_j \\
&+ \frac{m-2}{m-1}\big(R^\varphi_{tj} f_{kt} - R^\varphi_{tk} f_{tj}\big) + \frac{1}{m-1} f_t C^\varphi_{tkj} \\
&+ \frac{1}{2(m-1)^2}\big(f_k S^\varphi_j - f_j S^\varphi_k\big) - \frac{1}{m-1}\big(S^\varphi_j f_k - S^\varphi_k f_j\big)\Big] \\
=& \frac{1}{m-2}\Big[\Big(\frac{1}{2(m-1)^2} + \frac{1}{2} - \frac{1}{m-1}\Big)\big(S^\varphi_j f_k - S^\varphi_k f_j\big) \\
&- \alpha\big(\varphi^a_{tt}\varphi^a_j f_k - \varphi^a_{tt}\varphi^a_k f_j\big) + \frac{1}{m-1} f_t C^\varphi_{tkj} \\
&+ \eta \frac{m-2}{m-1}\big(R^\varphi_{tj} f_t f_k - R^\varphi_{tk} f_t f_j\big)\Big],
\end{aligned}
$$

that is,

$$
\begin{aligned}
D^A_{ijk,i} =& \frac{1}{m-2}\Big[\frac{(m-2)^2}{2(m-1)^2}\big(S^\varphi_j f_k - S^\varphi_k f_j\big) - \alpha \varphi^a_{tt}\big(\varphi^a_j f_k - \varphi^a_k f_j\big) \\
&+ \frac{1}{m-1} f_t C^\varphi_{tkj} + \eta \frac{m-2}{m-1}\big(R^\varphi_{tj} f_t f_k - R^\varphi_{tk} f_t f_j\big)\Big].
\end{aligned} \tag{3.42}
$$

We compute the 1-divergence of the first integrability condition (3.3) and use

(3.42), (3.41) and (1.31):

$$
\begin{aligned}
0 =&C^{\varphi}_{ijk,i} + f_t W^{\varphi}_{tijk,i} + f_{ti} W^{\varphi}_{tijk} - (1+\eta(m-2)) D^{A}_{ijk,i}\\
&- \frac{U^a}{m-1}\varphi^a_{kj} - \frac{U^{ab}}{m-1}\varphi^a_k\varphi^b_j + \frac{U^a}{m-1}\varphi^a_{jk} + \frac{U^{ab}}{m-1}\varphi^a_k\varphi^b_j\\
=&\alpha\big[R^{\varphi}_{jt}\varphi^a_t\varphi^a_k - R^{\varphi}_{kt}\varphi^a_t\varphi^a_j + \varphi^a_{ttj}\varphi^a_k - \varphi^a_{ttk}\varphi^a_j\big]\\
&+ \frac{m-3}{m-2} f_t C^{\varphi}_{tjk} + \alpha f_t\big(\varphi^a_{tk}\varphi^a_j - \varphi^a_{tj}\varphi^a_k\big) + \frac{\alpha}{m-2}\varphi^a_{tt}\big(\varphi^a_k f_j - \varphi^a_j f_k\big)\\
&- \frac{1+\eta(m-2)}{m-2}\left[\frac{(m-2)^2}{2(m-1)^2}\big(S^{\varphi}_j f_k - S^{\varphi}_k f_j\big) - \alpha\varphi^a_{tt}\big(\varphi^a_j f_k - \varphi^a_k f_j\big)\right.\\
&\left.+ \frac{1}{m-1} f_t C^{\varphi}_{tkj} + \eta\frac{m-2}{m-1}\big(R^{\varphi}_{tj} f_t f_k - R^{\varphi}_{tk} f_t f_j\big)\right]\\
=&\alpha\big[R^{\varphi}_{jt}\varphi^a_t\varphi^a_k - R^{\varphi}_{kt}\varphi^a_t\varphi^a_j + \varphi^a_{ttj}\varphi^a_k - \varphi^a_{ttk}\varphi^a_j + f_t\big(\varphi^a_{tk}\varphi^a_j - \varphi^a_{tj}\varphi^a_k\big)\big]\\
&+ \left[\frac{m-3}{m-2} + \frac{1+\eta(m-2)}{(m-1)(m-2)}\right] f_t C^{\varphi}_{tjk} + \alpha\left[\frac{1}{m-2} - \frac{1+\eta(m-2)}{m-2}\right]\varphi^a_{tt}\big(\varphi^a_k f_j - \varphi^a_j f_k\big)\\
&- [1+\eta(m-2)]\frac{m-2}{2(m-1)^2}\big(S^{\varphi}_j f_k - S^{\varphi}_k f_j\big)\\
&- \frac{\eta}{m-1}(1+\eta(m-2))\big(R^{\varphi}_{tj} f_t f_k - R^{\varphi}_{tk} f_t f_j\big),
\end{aligned}
$$

that is,

$$
\begin{aligned}
0 =&\alpha\big[R^{\varphi}_{jt}\varphi^a_t\varphi^a_k - R^{\varphi}_{kt}\varphi^a_t\varphi^a_j + \varphi^a_{ttj}\varphi^a_k - \varphi^a_{ttk}\varphi^a_j + f_t\big(\varphi^a_{tk}\varphi^a_j - \varphi^a_{tj}\varphi^a_k\big)\big]\\
&+ \frac{m-2+\eta}{m-1} f_t C^{\varphi}_{tjk} - \alpha\eta\varphi^a_{tt}\big(\varphi^a_k f_j - \varphi^a_j f_k\big)\\
&- [1+\eta(m-2)]\frac{m-2}{2(m-1)^2}\big(S^{\varphi}_j f_k - S^{\varphi}_k f_j\big)\\
&- \frac{\eta}{m-1}(1+\eta(m-2))\big(R^{\varphi}_{tj} f_t f_k - R^{\varphi}_{tk} f_t f_j\big).
\end{aligned}
\tag{3.43}
$$

We take the covariant derivative of (3.39).$ii)$

$$
0 = \varphi^a_{tti} - \varphi^a_{ti} f_t - \varphi^a_t f_{ti} - \frac{U^{ab}}{\alpha}\varphi^b_i;
$$

using this identity and (3.39).$i)$ we find

$$
\begin{aligned}
&R^{\varphi}_{jt}\varphi^a_t\varphi^a_k - R^{\varphi}_{kt}\varphi^a_t\varphi^a_j + \varphi^a_{ttj}\varphi^a_k - \varphi^a_{ttk}\varphi^a_j + f_t\big(\varphi^a_{tk}\varphi^a_j - \varphi^a_{tj}\varphi^a_k\big)\\
&= R^{\varphi}_{jt}\varphi^a_t\varphi^a_k - R^{\varphi}_{kt}\varphi^a_t\varphi^a_j + f_{tj}\varphi^a_t\varphi^a_k - f_{tk}\varphi^a_t\varphi^a_j\\
&+ \frac{U^{ab}}{\alpha}\varphi^a_j\varphi^b_k - \frac{U^{ab}}{\alpha}\varphi^a_k\varphi^b_j\\
&= \eta\big(f_t f_j\varphi^a_t\varphi^a_k - f_t f_k\varphi^a_t\varphi^a_j\big);
\end{aligned}
$$

therefore, using (3.39).ii) again, (3.43) becomes

$$
\begin{aligned}
0 =&\eta\alpha\big(f_t\varphi^a_t\varphi^a_k f_j - f_t\varphi^a_t\varphi^a_j f_k\big) + \frac{m-2+\eta}{m-1}f_tC^{\varphi}_{tjk}\\
&-\alpha\eta\varphi^a_{tt}\big(\varphi^a_k f_j - \varphi^a_j f_k\big)\\
&-\frac{1+\eta(m-2)}{2(m-1)^2}(m-2)\big(S^{\varphi}_j f_k - S^{\varphi}_k f_j\big) - \frac{\eta}{m-1}(1+\eta(m-2))\big(R^{\varphi}_{tj}f_tf_k - R^{\varphi}_{tk}f_tf_j\big)\\
=&\eta U^a\big(\varphi^a_j f_k - \varphi^a_k f_j\big) + \frac{m-2+\eta}{m-1}f_tC^{\varphi}_{tjk}\\
&-\frac{1+\eta(m-2)}{2(m-1)^2}(m-2)\big(S^{\varphi}_j f_k - S^{\varphi}_k f_j\big) - \frac{\eta}{m-1}(1+\eta(m-2))\big(R^{\varphi}_{tj}f_tf_k - R^{\varphi}_{tk}f_tf_j\big)
\end{aligned}
$$

that is,

$$
\begin{aligned}
0 =&\eta U^a\big(\varphi^a_j f_k - \varphi^a_k f_j\big) + \frac{m-2+\eta}{m-1}f_tC^{\varphi}_{tjk}\\
&-\frac{1+\eta(m-2)}{2(m-1)^2}(m-2)\big(S^{\varphi}_j f_k - S^{\varphi}_k f_j\big) - \frac{\eta}{m-1}(1+\eta(m-2))\big(R^{\varphi}_{tj}f_tf_k - R^{\varphi}_{tk}f_tf_j\big).
\end{aligned}
\tag{3.44}
$$

From (3.3) we get

$$
f_tC^{\varphi}_{tjk} = (1+\eta(m-2))f_tD^A_{tjk} + \frac{1}{m-1}U^a\big(\varphi^a_k f_j - \varphi^a_j f_k\big);
$$

using this relation and (3.40) in (3.44) we have

$$
\begin{aligned}
0 =&\eta U^a\big(\varphi^a_j f_k - \varphi^a_k f_j\big) + \frac{m-2+\eta}{m-1}(1+\eta(m-2))f_tD^A_{tjk}\\
&+\frac{m-2+\eta}{(m-1)^2}U^a\big(\varphi^a_k f_j - \varphi^a_j f_k\big) - \eta(1+\eta(m-2))f_tD^A_{tjk}\\
&-\frac{1+\eta(m-2)}{2(m-1)^2}(m-2)\big(S^{\varphi}_j f_k - S^{\varphi}_k f_j\big)\\
=&\frac{m-2+\eta-\eta(m-1)^2}{(m-1)^2}U^a\big(\varphi^a_k f_j - \varphi^a_j f_k\big)\\
&+(1+\eta(m-2))\left[\frac{m-2+\eta}{m-1} - \eta\right]f_tD^A_{tjk}\\
&-\frac{1+\eta(m-2)}{2(m-1)^2}(m-2)\big(S^{\varphi}_j f_k - S^{\varphi}_k f_j\big)
\end{aligned}
$$

that is,

$$
\begin{aligned}
0 =&\frac{m-2}{(m-1)^2}(1-\eta m)U^a\big(\varphi^a_k f_j - \varphi^a_j f_k\big)\\
&+(1+\eta(m-2))\frac{m-2}{m-1}(1-\eta)f_tD^A_{tjk}\\
&-\frac{1+\eta(m-2)}{2(m-1)^2}(m-2)\big(S^{\varphi}_j f_k - S^{\varphi}_k f_j\big)
\end{aligned}
$$

and therefore

$$\begin{aligned}(1+\eta(m-2))\frac{m-2}{m-1}(1-\eta)f_tD^A_{tjk}&\\ =\frac{m-2}{(m-1)^2}\Bigg[f_k\bigg(&\frac{1+\eta(m-2)}{2}S^\varphi_j+(1-\eta m)U^a\varphi^a_j\bigg)\\ -f_j\bigg(&\frac{1+\eta(m-2)}{2}S^\varphi_k+(1-\eta m)U^a\varphi^a_k\bigg)\Bigg].\end{aligned}$$

- if $\eta=-\frac{1}{m-2}$, the equation above becomes

$$0=\frac{m-2}{(m-1)^2}\left(1+\frac{m}{m-2}\right)U^a\left(\varphi^a_jf_k-\varphi^a_kf_j\right);$$

  therefore $\nabla(U(\varphi))$ and ∇f are proportional and if $\nabla f\neq 0$, we get that $U(\varphi)$ is constant on the regular level set of f.

- If $\eta=1$, we get

$$0=\frac{m-2}{m-1}\left(\frac{1}{2}S^\varphi_jf_k-U^a\varphi^a_jf_k-\frac{1}{2}S^\varphi_kf_j+U^a\varphi^a_kf_j\right)$$

  so that $\frac{S^\varphi}{2}-U(\varphi)$ is constant on the regular level sets of f.

- If $\eta\neq 1,-\frac{1}{m-2}$, we deduce that

$$\frac{1+\eta(m-2)}{2}S^\varphi+(1-\eta m)U(\varphi)$$

  is constant on the regular level sets of f if and only if we have $f_tD^A_{tjk}=0$; by Proposition 2.11, this happens if and only if ∇f is an eigenvector of Ric^φ.

$\square$

Proposition 3.12. *Let (M,g) be an m-dimensional Riemannian manifold, $m\geq 3$. Assume that $f\in C^\infty(M)$ solves*

$$\begin{cases}\mathrm{Ric}^\varphi+\operatorname{Hess}f-\eta\,\mathrm{d}f\otimes\mathrm{d}f=\lambda g,\\ \tau(\varphi)=\mathrm{d}\varphi(\nabla f)+\frac{\nabla U}{\alpha}\end{cases}\tag{3.45}$$

for some $\lambda\in C^\infty(M)$, $\eta\in\mathbb{R}$, $\varphi:(M,g)\to(N,h)$ smooth and $U\in C^\infty(N)$. Assume that, at any regular point of f, ∇f be an eigenvector of Ric^φ. Then $U(\varphi)$ is constant on the regular level sets of f.

Proof. From equation (3.36) we get

$$\frac{1}{2}S^\varphi_k-(m-1)\lambda_k=f_tR^\varphi_{tk}+\eta(f_tf_{tk}-\Delta ff_k)-U^a\varphi^a_k.$$

Tracing (3.45).i) we have

$$\lambda = \frac{1}{m}\left(S^\varphi + \Delta f - \eta|\nabla f|^2\right),$$

so that

$$\begin{aligned}\frac{1}{2}S^\varphi_k - \frac{m-1}{m}S^\varphi_k - \frac{m-1}{m}(\Delta f)_k + 2\eta\frac{m-1}{m}f_t f_{tk}\\ = f_t R^\varphi_{tk} + \eta(f_t f_{tk} - \Delta f f_k) - U^a \varphi^a_k\end{aligned}$$

that we re-write

$$\begin{aligned}-\frac{m-2}{2m}S^\varphi_k + U^a\varphi^a_k \\ = f_t R^\varphi_{tk} + \frac{m-1}{m}(\Delta f)_k - 2\eta\frac{m-1}{m}f_t f_{tk} + \eta(f_t f_{tk} - \Delta f f_k).\end{aligned} \tag{3.46}$$

Let $\{e_i\}_{i=1}^m$ be an orthonormal frame such that $e_m = \frac{\nabla f}{|\nabla f|}$ and let $\{\theta^i\}_{i=1}^m$ be its dual co-frame. Then

$$f_A = 0, \quad \forall A \in \{1, ..., m-1\}.$$

From Proposition 2.11, we deduce that ∇f is an eigenvector of Ric^φ if and only if $|\nabla f|^2$ is constant on the regular level sets of f; this happens if and only if f locally depends only on the distance function r. This implies that

$$R^\varphi_{Am} = f_{Am} = (\Delta f)_A = 0.$$

Therefore, (3.46) becomes

$$-\frac{m-2}{2m}S^\varphi_A + U^a\varphi^a_A = 0; \tag{3.47}$$

the previous proposition also gives

$$\frac{1+\eta(m-2)}{2}S^\varphi_A + (1-\eta m)U^a\varphi^a_A = 0. \tag{3.48}$$

Solving the system given by (3.47) and (3.48) we deduce $U^a\varphi^a_A = S^\varphi_A = 0$. $\square$

Proposition 3.13. *Let (M, g) be an m-dimensional Riemannian manifold, $m \geq 3$. Assume that $f \in C^\infty(M)$ solves*

$$\begin{cases}\mathrm{Ric}^\varphi + \mathrm{Hess}\, f - \eta\mathrm{d}f \otimes \mathrm{d}f = \lambda g, \\ \tau(\varphi) = \mathrm{d}\varphi(\nabla f) + \frac{\nabla U}{\alpha}\end{cases} \tag{3.49}$$

for some $\lambda \in C^\infty(M)$, $\eta \in \mathbb{R}$, $\varphi : (M, g) \to (N, h)$ smooth and $U \in C^\infty(N)$. Assume that the following hold:

- *φ is $\frac{1}{\alpha}U$-harmonic:*

- *at every regular point of f, we have that ∇f is an eigenvector of* Ric^{φ}.

Then, $U(\varphi)$ is locally constant on the set of regular values of f. In particular, φ is conservative there.

Proof. By the previous proposition

$$U^a \varphi^a_A = 0, \quad \forall A \in \{1, ..., m-1\}.$$

Since (3.49).$ii)$ holds and since φ is $\frac{1}{\alpha}U$-harmonic, we have

$$0 = \varphi^a_i f_i = |\nabla f| \varphi^a_m$$

so that $U^a \varphi^a_m = 0$. Therefore $\nabla(U(\varphi))$ vanishes at every regular point of f; form this fact and (3.49).$ii)$ we deduce $\varphi^a_{tt}\varphi^a_i = 0, \forall i$, so that φ is conservative. $\square$

3.2 The Boundary Case

In this subsection we prove a rigidity result for Riemannian manifolds with non-empty boundary.
Before focusing on manifolds with boundary, we highlight some facts concerning system (3.2), that is,

$$\begin{cases} i)\, \mathrm{Ric}^{\varphi} + \mathrm{Hess}(f) - \eta df \otimes df = \lambda g, \\ ii)\, \tau(\varphi) = d\varphi(\nabla f) + \frac{1}{\alpha}(\nabla U)(\varphi), \end{cases}$$

where

$$m\lambda = S^{\varphi} + \Delta f - \eta |\nabla f|^2.$$

If we let

$$u = e^{-\eta f}, \tag{3.50}$$

then system (3.2) transforms into

$$\begin{cases} i)\, u\eta \mathrm{Ric}^{\varphi} - \mathrm{Hess}(u) = \frac{1}{m}(\eta u S^{\varphi} - \Delta u) g \\ ii)\, \eta u \tau(\varphi) = -\mathrm{d}\varphi(\nabla u) + \frac{\eta u}{\alpha}(\nabla U)(\varphi). \end{cases} \tag{3.51}$$

Note that looking for solutions of system (3.2) is equivalent to finding positive solutions of system (3.51); however, the latter system can be considered an extension of (3.2) if one looks for possibly changing-sign solutions u. In our setting, system (3.51) will play a fundamental role in the study of manifolds with boundary, $\partial M = u^{-1}(\{0\})$.

From now on, let (M, g) be a connected Riemannian manifold of dimension $m \geq 3$ with non-empty boundary; let $\varphi : (M, g) \to (N, h)$, where (N, h) is a Riemannian manifold of dimension n, be a smooth map, let $U : (N, h) \to \mathbb{R}$ be a smooth function and let $u \in C^{\infty}(M)$ be a solution of (3.51) such that $u > 0$ on $\mathrm{int}(M)$ and $\partial M = u^{-1}(\{0\})$. In this setting, we have the following

Theorem 3.14. *Let (M,g) be a connected Riemannian manifold of dimension $m \geq 3$ and let $u \in C^\infty(M)$ be a solution of system (3.51), with $\alpha \in \mathbb{R} \setminus \{0\}$ and $\eta \neq -\frac{1}{m-2}$, 0. Assume that*

$$\operatorname{div}^3 C^\varphi = 0; \tag{3.52}$$

$$\varphi \text{ is } \frac{1}{\alpha} U\text{-harmonic}; \tag{3.53}$$

$$\nabla f_p \text{ is an eigenvector of } \operatorname{Ric}^\varphi_p, \text{ for every regular point } p \text{ of } f; \tag{3.54}$$

$$W^\varphi(\nabla f, \cdot, \cdot, \cdot) = 0. \tag{3.55}$$

Then

$$C^\varphi = -\frac{1}{2(m-1)} \operatorname{div}_1 \left(U(\varphi) g \owedge g\right) \tag{3.56}$$

and

$$D^A \equiv 0.$$

For the proof of Theorem 3.14 we need some preliminary results. First we introduce the $(0,3)$-tensor D^φ, whose components are

$$(m-2)D^\varphi_{ijk} := u_{ik}u_j - u_{ij}u_k + \frac{u_t}{m-1}(u_{tj}\delta_{ik} - u_{tk}\delta_{ij}) + \frac{\Delta u}{m-1}(u_k\delta_{ij} - u_j\delta_{ik}). \tag{3.57}$$

Lemma 3.15. *Let (M,g) be a connected Riemannian manifold of dimension $m \geq 3$, let $u \in C^\infty(M)$ be a solution of system (3.51), with $\alpha \in \mathbb{R}\setminus\{0\}$ and $\eta \neq -\frac{1}{m-2}, 0$ and let D^φ be the tensor field on M defined in (3.57). Then,*

$$\frac{[1+\eta(m-2)]}{\eta u} D^\varphi = \eta u C^\varphi - W^\varphi(\nabla u, \cdot, \cdot, \cdot) + \eta u \frac{1}{2(m-1)} \operatorname{div}_1 \left(U(\varphi) g \owedge g\right). \tag{3.58}$$

on $\operatorname{int}(M)$.

Proof. Note that, when we consider the change of variable (3.50), the first integrability condition (3.58) of system (3.51) can be easily obtained by (3.3). Indeed, by (2.47), we have

$$D^A_{ijk} = (u\eta)^{-2} D^\varphi_{ijk};$$

hence, by (3.3), we deduce (3.58). □

Lemma 3.16. *Let (M,g) be a connected Riemannian manifold of dimension $m \geq 3$ and let $u \in C^\infty(M)$ be a solution of system (3.51), with $\alpha \in \mathbb{R} \setminus \{0\}$ and $\eta \neq -\frac{1}{m-2}$, 0. Define the vector field Z on M of components*

$$\begin{aligned} Z_i =& u\{R^\varphi_{tk}W^\varphi_{tikj}\}_j - \left(\frac{m-4}{m-2}\right) uR^\varphi_{tk}C^\varphi_{tki} + 2\alpha u \varphi^a_i \varphi^a_{jk} R^\varphi_{jk} - \alpha\{uR^\varphi_{tk}(\varphi^a_t\varphi^a_i)_k\} \\ & - \alpha\left\{u\left(\frac{1}{2}S^\varphi_t \varphi^a_t \varphi^a_i - \alpha\varphi^b_{ss}\varphi^b_t\varphi^a_t\varphi^a_i\right)\right\} \\ & + \left[u\left(\varphi^a_{ik}U^a - U^{ab}\varphi^b_k\varphi^a_i - \frac{\alpha}{m-2}|\tau(\varphi)|^2\delta_{ik}\right)_k\right], \end{aligned} \tag{3.59}$$

and assume that φ *is* $\frac{1}{\alpha}U$*-harmonic and* ∇u *is an eigenvector of* Ric^φ*, for every regular point of* u *(that are assumptions* (3.53) *and* (3.54)*, respectively). Then, on* $\mathrm{int}(M)$*, we have*

$$\begin{aligned}\operatorname{div} Z = &- u_t W^\varphi_{tijk}\left(\frac{1}{2(\eta u)^2} D^\varphi_{ijk} + R^\varphi_{ij,k}\right) - \frac{[1+\eta(m-2)]}{(\eta u)^3}|D^\varphi|^2 \\ &- u C^\varphi_{ikj,jki} + \left(\frac{m-4}{m-2}\right) u\left[U^a\left(\frac{1}{\alpha}U^{ab}\varphi^b_i + R^\varphi_{ij}\varphi^a_j\right)\right]_i \\ &+ \left\{\varphi^a_i\left(\frac{m}{(m-1)(m-2)}U^a S^\varphi + 2\alpha U^b \varphi^b_j \varphi^a_i\right) - \frac{\alpha}{2}\left(\frac{m-2}{m-1}\right)\varphi^a_i\varphi^a_j S^\varphi_j \right.\\ &\left. - 2\alpha \varphi^a_{jk} R^\varphi_{jk}\varphi^a_i - \alpha\varphi^a_i \tau^a_2(\varphi)\right\}_i u.\end{aligned} \tag{3.60}$$

If we further assume

$$u_t W^\varphi_{tijk} = 0,$$

equation (3.60) *becomes*

$$\begin{aligned}\operatorname{div} Z = &- \frac{[1+\eta(m-2)]}{(\eta u)^3}|D^\varphi|^2 - u C^\varphi_{ikj,jki} + \left(\frac{m-4}{m-2}\right) u\left[U^a\left(U^{ab}\varphi^b_i + R^\varphi_{ij}\varphi^a_j\right)\right]_i \\ &+ \left\{\varphi^a_i\left(\frac{m}{(m-1)(m-2)}U^a S^\varphi + 2\alpha U^b \varphi^b_j \varphi^a_i\right) - \frac{\alpha}{2}\frac{m-2}{m-1}\varphi^a_i\varphi^a_j S^\varphi_j \right.\\ &\left. - 2\alpha \varphi^a_{jk} R^\varphi_{jk}\varphi^a_i - \alpha\varphi^a_i \tau^a_2(\varphi)\right\}_i u.\end{aligned} \tag{3.61}$$

Proof. Let X be the vector field of components

$$X_i = u\{R^\varphi_{tk}W^\varphi_{tikj}\}_k - \left(\frac{m-4}{m-2}\right) u R^\varphi_{tk} C^\varphi_{tki} + 2\alpha u \varphi^a_i \varphi^a_{jk} R^\varphi_{jk};$$

computing its divergence and using $0 = \tau(\varphi) - \frac{1}{\alpha}(\nabla U)(\varphi)$ and hence $d\varphi(\nabla u) = 0$, we obtain

$$\begin{aligned}X_{ii} = &\left\{u\left[R^\varphi_{tk}W^\varphi_{tikj}\right]_k\right\}_i - \left(\frac{m-4}{m-2}\right) u_i R^\varphi_{tk} C^\varphi_{tki} - \left(\frac{m-4}{m-2}\right) u R^\varphi_{tk,i} C^\varphi_{tki} - \left(\frac{m-4}{m-2}\right) u R^\varphi_{tk} C^\varphi_{tki,i} \\ &+ 2\alpha u_i \varphi^a_i \varphi^a_{jk} R^\varphi_{jk} + 2\alpha u \varphi^a_{ii}\varphi^a_{jk} R^\varphi_{jk} + 2\alpha u \varphi^a_i \varphi^a_{jki} R^\varphi_{jk} + 2\alpha u \varphi^a_i \varphi^a_{jk} R^\varphi_{jk,i} \\ = &\left\{u\left[R^\varphi_{tk}W^\varphi_{tikj}\right]_k\right\}_i - \left(\frac{m-4}{m-2}\right) u_i R^\varphi_{tk} C^\varphi_{tki} - \left(\frac{m-4}{m-2}\right) u R^\varphi_{tk,i} C^\varphi_{tki} - \left(\frac{m-4}{m-2}\right) u R^\varphi_{tk} C^\varphi_{tki,i} \\ &+ 2\alpha u \varphi^a_{ii}\varphi^a_{jk} R^\varphi_{jk} + 2\alpha u \varphi^a_i \varphi^a_{jki} R^\varphi_{jk} + 2\alpha u \varphi^a_i \varphi^a_{jk} R^\varphi_{jk,i} \\ = &\left\{u\left[R^\varphi_{tk}W^\varphi_{tikj}\right]_k\right\}_i - \left(\frac{m-4}{m-2}\right) u_i R^\varphi_{tk} C^\varphi_{tki} - \left(\frac{m-4}{m-2}\right) u\{R^\varphi_{tk} C^\varphi_{tki}\}_i + 2\alpha u\{\varphi^a_i \varphi^a_{jk} R^\varphi_{jk}\}_i.\end{aligned} \tag{3.62}$$

Since φ is $\frac{1}{\alpha}U$-harmonic, equation (1.34) rewrites as

$$
\begin{aligned}
(m-2)B^{\varphi}_{ik,k} =& \frac{m-4}{m-2}\left[R^{\varphi}_{jk}C^{\varphi}_{jki} + U^a\left(\frac{1}{\alpha}U^{ab}\varphi^b_i + R^{\varphi}_{ij}\varphi^a_j\right)\right] \\
&+ \varphi^a_i\left(\frac{m}{(m-1)(m-2)}U^aS^{\varphi} + 2\alpha U^b\varphi^b_j\varphi^a_i\right) \\
&- \frac{\alpha}{2}\frac{m-2}{m-1}\varphi^a_i\varphi^a_jS^{\varphi}_j - 2\alpha\varphi^a_{jk}R^{\varphi}_{jk}\varphi^a_i - \alpha\varphi^a_i\tau^a_2(\varphi);
\end{aligned} \tag{3.63}
$$

therefore, taking its divergence, we obtain

$$
\begin{aligned}
(m-2)B^{\varphi}_{ik,ki} =& \left(\frac{m-4}{m-2}\right)\left(R^{\varphi}_{jk}C^{\varphi}_{jki}\right)_i + \left(\frac{m-4}{m-2}\right)\left(U^a(\frac{1}{\alpha}U^{ab}\varphi^b_i + R^{\varphi}_{ij}\varphi^a_j)\right)_i \\
&+ \left[\varphi^a_i\left(\frac{m}{(m-1)(m-2)}U^aS^{\varphi} + 2\alpha U^b\varphi^b_j\varphi^a_i\right)\right. \\
&\left.- \frac{\alpha}{2}\frac{m-2}{m-1}\varphi^a_i\varphi^a_jS^{\varphi}_j - 2\alpha\varphi^a_{jk}R^{\varphi}_{jk}\varphi^a_i - \alpha\varphi^a_i\tau^a_2(\varphi)\right]_i .
\end{aligned} \tag{3.64}
$$

To simplify the writing we set

$$
\begin{aligned}
\mathcal{A} = -&\left[\varphi^a_i\left(\frac{m}{(m-1)(m-2)}U^aS^{\varphi} + 2\alpha U^b\varphi^b_j\varphi^a_i\right)\right. \\
&\left.- \frac{\alpha}{2}\frac{m-2}{m-1}\varphi^a_i\varphi^a_jS^{\varphi}_j - \alpha\varphi^a_i\tau^a_2(\varphi)\right]_i .
\end{aligned}
$$

As a consequence, using (3.64), we have

$$
\begin{aligned}
&u\frac{m-4}{m-2}(R^{\varphi}_{jk}C^{\varphi}_{jki})_i - 2\alpha u(\varphi^a_{jk}R^{\varphi}_{jk}\varphi^a_i)_i \\
&= u(m-2)B^{\varphi}_{ik,ki} - \left(\frac{m-4}{m-2}\right)u\left(\frac{1}{\alpha}U^a(U^{ab}\varphi^b_i + R^{\varphi}_{ij}\varphi^a_j)\right)_i + \mathcal{A}u.
\end{aligned} \tag{3.65}
$$

Using the first equation of (3.51) into (3.57), we get

$$
\begin{aligned}
(m-2)D^{\varphi}_{ijk} =& \eta uR^{\varphi}_{ik}u_j - \frac{1}{m}(\eta uS^{\varphi} - \Delta u)\delta_{ik}u_j - \eta uR^{\varphi}_{ij}u_k + \frac{1}{m}(\eta uS^{\varphi} - \Delta u)\delta_{ij}u_k \\
&+ \frac{u_t}{m-1}\left[R^{\varphi}_{tj}\delta_{ik}\eta u - \frac{1}{m}(\eta uS^{\varphi} - \Delta u)\delta_{ik}\delta_{tj} - R^{\varphi}_{tk}\delta_{ij}\eta u\right. \\
&\left.+ \frac{1}{m}(\eta uS^{\varphi} - \Delta u)\delta_{ij}\delta_{tk}\right] + \frac{1}{m-1}\Delta u(u_k\delta_{ij} - u_j\delta_{ik}) \\
=& \eta u\left\{R^{\varphi}_{ik}u_j - R^{\varphi}_{ij}u_k + \frac{u_t}{m-1}(R^{\varphi}_{tj}\delta_{ik} - R^{\varphi}_{tk}\delta_{ij})\right. \\
&\left.+ \frac{S^{\varphi}}{m-1}(u_k\delta_{ij} - u_j\delta_{ik})\right\}.
\end{aligned} \tag{3.66}
$$

Hence, by (3.53), (3.54) and (3.66), we deduce

$$(m-2)D^{\varphi}_{ijk}C^{\varphi}_{ijk} = -2\eta u u_k R^{\varphi}_{ij}C^{\varphi}_{ijk}. \tag{3.67}$$

Thus, inserting (3.67) and (3.65) into (3.62), we get the validity of

$$\begin{aligned} X_{ii} =& \left\{u\left[R^{\varphi}_{tk}W^{\varphi}_{tikj}\right]_j\right\}_i + \frac{m-4}{2\eta u}D^{\varphi}_{ijk}C^{\varphi}_{ijk} - u(m-2)B^{\varphi}_{ik,ki} \\ &+ \left(\frac{m-4}{m-2}\right)u\left[U^a\left(\frac{1}{\alpha}U^{ab}\varphi^b_i + R^{\varphi}_{ij}\varphi^a_j\right)\right]_i - \mathcal{A}u \end{aligned} \tag{3.68}$$

on $\operatorname{int}(M)$.

Next, using the definition of B^{φ} and that φ is $\frac{1}{\alpha}U$-harmonic, we obtain

$$\begin{aligned} u(m-2)B^{\varphi}_{ik,ki} =& uC^{\varphi}_{ikj,jki} + u(R^{\varphi}_{st}W^{\varphi}_{sitk})_{ki} - \alpha u(R^{\varphi}_{tk}\varphi^a_t\varphi^a_i)_{ki} \\ &+ u\left(\varphi^a_{ij}U^a - U^{ab}\varphi^b_k\varphi^a_i - \frac{\alpha}{m-2}|\tau(\varphi)|^2\delta_{ik}\right)_{ki} \\ =& uC^{\varphi}_{ikj,jki} + [u(R^{\varphi}_{st}W^{\varphi}_{sitk})_k]_i - u_i(R^{\varphi}_{st}W^{\varphi}_{sitk})_k \\ &- \alpha[u(R^{\varphi}_{tk}\varphi^a_t\varphi^a_i)_k]_i + \alpha u_i(R^{\varphi}_{tk}\varphi^a_t\varphi^a_i)_k \\ &+ \left[u\left(\varphi^a_{ik}U^a - U^{ab}\varphi^b_k\varphi^a_i - \frac{\alpha}{m-2}|\tau(\varphi)|^2\delta_{ik}\right)_k\right]_i \\ &- u_i\left(\varphi^a_{ik}U^a - U^{ab}\varphi^b_k\varphi^a_i - \frac{\alpha}{m-2}|\tau(\varphi)|^2\delta_{ik}\right)_k. \end{aligned}$$

Inserting the latter into (3.68), we obtain, on $\operatorname{int}(M)$,

$$\begin{aligned} X_{ii} =& \frac{m-4}{2\eta u}D^{\varphi}_{ijk}C^{\varphi}_{ijk} - uC^{\varphi}_{ikj,jki} + u_i\{R^{\varphi}_{st}W^{\varphi}_{sitk}\}_k + \alpha[u(R^{\varphi}_{tk}\varphi^a_t\varphi^a_i)_k]_i \\ &- \alpha u_i(R^{\varphi}_{tk}\varphi^a_t\varphi^a_i)_k - \left[u\left(\varphi^a_{ik}U^a - U^{ab}\varphi^b_k\varphi^a_i - \frac{\alpha}{m-2}|\tau(\varphi)|^2\delta_{ik}\right)_k\right]_i \\ &+ u_i\left(\varphi^a_{ik}U^a - U^{ab}\varphi^b_k\varphi^a_i - \frac{\alpha}{m-2}|\tau(\varphi)|^2\delta_{ik}\right)_k \\ &+ \left(\frac{m-4}{m-2}\right)u\left[U^a\left(\frac{1}{\alpha}U^{ab}\varphi^b_i + R^{\varphi}_{ij}\varphi^a_j\right)\right]_i - \mathcal{A}u. \end{aligned} \tag{3.69}$$

Observe that by (1.31), since φ is $\frac{1}{\alpha}U$-harmonic and ∇u is an eigenvector of $\operatorname{Ric}^{\varphi}$ for every regular point of u, we get

$$\begin{aligned} u_i\{R^{\varphi}_{st}W^{\varphi}_{sitk}\}_k &= u_iR^{\varphi}_{st,k}W^{\varphi}_{sitk} + u_iR^{\varphi}_{st}W^{\varphi}_{sitk,k} \\ &= u_iR^{\varphi}_{st,k}W^{\varphi}_{sitk} + \left(\frac{m-3}{m-2}\right)u_iR^{\varphi}_{st}C^{\varphi}_{tsi} + \alpha u_iR^{\varphi}_{ts}\varphi^a_{ti}\varphi^a_s \end{aligned}$$

which by (3.67), on $\operatorname{int}(M)$, can be rewritten as

$$\begin{aligned} u_i\{R^{\varphi}_{st}W^{\varphi}_{sitk}\}_k &= u_iR^{\varphi}_{st,k}W^{\varphi}_{sitk} + \left(\frac{m-3}{m-2}\right)u_iR^{\varphi}_{st}C^{\varphi}_{tsi} + \alpha u_iR^{\varphi}_{ts}(\varphi^a_i\varphi^a_s)_t \\ &= u_iR^{\varphi}_{st,k}W^{\varphi}_{sitk} - \frac{m-3}{2\eta u}D^{\varphi}_{tsi}C^{\varphi}_{tsi} + \alpha u_iR^{\varphi}_{ts}(\varphi^a_i\varphi^a_s)_t. \end{aligned}$$

Therefore, on $\mathrm{int}(M)$, we have

$$\begin{aligned}
X_{ii} =& -\frac{1}{2\eta u}D^{\varphi}_{ijk}C^{\varphi}_{ijk} - uC^{\varphi}_{ikj,jki} + u_iR^{\varphi}_{st,k}W^{\varphi}_{sitk} + \alpha u_iR^{\varphi}_{ts}(\varphi^a_i\varphi^a_s)_t \\
&+ \alpha[u(R^{\varphi}_{tk}\varphi^a_t\varphi^a_i)_k]_i - \alpha u_i(R^{\varphi}_{tk}\varphi^a_t\varphi^a_i)_k \\
&- \left[u\left(\varphi^a_{ik}U^a - U^{ab}\varphi^b_k\varphi^a_i - \frac{\alpha}{m-2}|\tau(\varphi)|^2\delta_{ik}\right)_k\right]_i \\
&+ u_i\left(\varphi^a_{ik}U^a - U^{ab}\varphi^b_k\varphi^a_i - \frac{\alpha}{m-2}|\tau(\varphi)|^2\delta_{ik}\right)_k \\
&+ \left(\frac{m-4}{m-2}\right)u\left[U^a\left(\frac{1}{\alpha}U^{ab}\varphi^b_i + R^{\varphi}_{ij}\varphi^a_j\right)\right]_i - \mathcal{A}u;
\end{aligned}$$

using the φ-Schur's identity, we have

$$\begin{aligned}
X_{ii} =& -\frac{1}{2\eta u}D^{\varphi}_{ijk}C^{\varphi}_{ijk} - uC^{\varphi}_{ikj,jki} + u_iR^{\varphi}_{st,k}W^{\varphi}_{sitk} + \alpha u_iR^{\varphi}_{ts}(\varphi^a_i\varphi^a_s)_t \\
&+ \alpha\left\{u\left(\frac{1}{2}S^{\varphi}_t\varphi^a_t\varphi^a_i - \alpha\varphi^b_{ss}\varphi^b_t\varphi^a_t\varphi^a_i\right)\right\}_i + \alpha\{uR^{\varphi}_{tk}(\varphi^a_t\varphi^a_i)_k\}_i \\
&- \alpha u_i\left(\frac{1}{2}S^{\varphi}_t\varphi^a_t\varphi^a_i - \alpha\varphi^b_{ss}\varphi^b_t\varphi^a_t\varphi^a_i + R^{\varphi}_{tk}(\varphi^a_t\varphi^a_i)_k\right) \\
&- \left[u\left(\varphi^a_{ik}U^a - U^{ab}\varphi^b_k\varphi^a_i - \frac{\alpha}{m-2}|\tau(\varphi)|^2\delta_{ik}\right)_k\right]_i \\
&+ u_i\left(\varphi^a_{ik}U^a - U^{ab}\varphi^b_k\varphi^a_i - \frac{\alpha}{m-2}|\tau(\varphi)|^2\delta_{ik}\right)_k \\
&+ \left(\frac{m-4}{m-2}\right)u\left[U^a\left(\frac{1}{\alpha}U^{ab}\varphi^b_i + R^{\varphi}_{ij}\varphi^a_j\right)\right]_i - \mathcal{A}u;
\end{aligned}$$

since $\varphi^a_iu_i = 0$ we deduce

$$\begin{aligned}
X_{ii} =& -\frac{1}{2\eta u}D^{\varphi}_{ijk}C^{\varphi}_{ijk} - uC^{\varphi}_{ikj,jki} + u_iR^{\varphi}_{st,k}W^{\varphi}_{sitk} + \alpha\left\{u\left(\frac{1}{2}S^{\varphi}_t\varphi^a_t\varphi^a_i - \alpha\varphi^b_{ss}\varphi^b_t\varphi^a_t\varphi^a_i\right)\right\}_i \qquad (3.70) \\
&+ \alpha\{uR^{\varphi}_{tk}(\varphi^a_t\varphi^a_i)_k\}_i - \left[u\left(\varphi^a_{ik}U^a - U^{ab}\varphi^b_k\varphi^a_i - \frac{\alpha}{m-2}|\tau(\varphi)|^2\delta_{ik}\right)_k\right]_i \\
&+ u_i\left(\varphi^a_{ik}U^a - U^{ab}\varphi^b_k\varphi^a_i - \frac{\alpha}{m-2}|\tau(\varphi)|^2\delta_{ik}\right)_k \\
&+ \frac{m-4}{m-2}u\left[U^a\left(\frac{1}{\alpha}U^{ab}\varphi^b_i + R^{\varphi}_{ij}\varphi^a_j\right)\right]_i - \mathcal{A}u
\end{aligned}$$

on $\operatorname{int}(M)$ and

$$
\begin{aligned}
&u_i\left(\varphi^a_{ik}U^a - U^{ab}\varphi^b_k\varphi^a_i - \frac{\alpha}{m-2}|\tau(\varphi)|^2\delta_{ik}\right)_k \\
&\quad = u_i\Bigg(\varphi^a_{ikk}U^a + \varphi^a_{ik}U^{ab}\varphi^b_k - U^{abc}\varphi^c_k\varphi^b_k\varphi^a_i - U^{ab}\varphi^b_{kk}\varphi^a_i \\
&\qquad - U^{ab}\varphi^b_k\varphi^a_{ik} - \alpha\frac{2}{m-2}\varphi^a_{tti}\varphi^a_{ss}\Bigg) \\
&\quad = u_i\left(\varphi^a_{ikk}U^a - \alpha\frac{2}{m-2}\varphi^a_{tti}\varphi^a_{ss}\right).
\end{aligned}
$$

Using (1.11), that is, the commutation rule

$$
\varphi^a_{ijk} = \varphi^a_{ikj} + R_{tijk}\varphi^a_t - {}^N R^a_{bcd}\varphi^b_i\varphi^c_j\varphi^d_k,
$$

we get

$$
\begin{aligned}
&u_i\left(\varphi^a_{ik}U^a - U^{ab}\varphi^b_k\varphi^a_i - \frac{\alpha}{m-2}|\tau(\varphi)|^2\delta_{ik}\right)_k \\
&\quad = u_i\left[\left(\frac{m-4}{m-2}\right)\varphi^a_{kki}U^a + U^a R_{si}\varphi^a_s - {}^N R^a_{bcd}\varphi^b_k\varphi^c_i\varphi^d_k U^a\right] \\
&\quad = u_i\left(\frac{m-4}{\alpha(m-2)}U^{ab}\varphi^b_i U^a + U^a R^{\varphi}_{si}\varphi^a_s - {}^N R^a_{bcd}\varphi^b_k\varphi^c_i\varphi^d_k U^a\right) \\
&\quad = u_i U^a R^{\varphi}_{si}\varphi^a_s = 0,
\end{aligned}
$$

where the last equality follows by (3.54). As a consequence, (3.70) becomes

$$
\begin{aligned}
X_{ii} = &-\frac{1}{2\eta u}D^{\varphi}_{ijk}C^{\varphi}_{ijk} - uC^{\varphi}_{ikj,jki} + u_i R^{\varphi}_{st,k}W^{\varphi}_{sitk} \\
&+ \alpha\left\{u\left(\frac{1}{2}S^{\varphi}_t\varphi^a_t\varphi^a_i - \alpha\varphi^b_{ss}\varphi^b_t\varphi^a_t\varphi^a_i\right)\right\}_i + \alpha\{uR^{\varphi}_{tk}(\varphi^a_t\varphi^a_i)_k\}_i \\
&- \left[u\left(\varphi^a_{ik}U^a - U^{ab}\varphi^b_k\varphi^a_i - \frac{\alpha}{m-2}|\tau(\varphi)|^2\delta_{ik}\right)_k\right]_i \\
&+ \frac{m-4}{m-2}u\left[U^a\left(U^{ab}\varphi^b_i + R^{\varphi}_{ij}\varphi^a_j\right)\right]_i - \mathcal{A}u.
\end{aligned}
$$

On $\operatorname{int}(M)$, since $u > 0$, by (3.58), we obtain

$$
\begin{aligned}
C^{\varphi}_{ijk}D^{\varphi}_{ijk} &= \frac{1}{\eta u}D^{\varphi}_{ijk}\left\{u_t W^{\varphi}_{tijk} - \eta u\frac{U^a}{m-1}\left(\varphi^a_j\delta_{ik} - \varphi^a_k\delta_{ik}\right) + \frac{[1+\eta(m-2)]}{\eta u}D^{\varphi}_{ijk}\right\} \\
&= \frac{1}{\eta u}u_t W^{\varphi}_{tijk}D^{\varphi}_{ijk} + \frac{[1+\eta(m-2)]}{(\eta u)^2}|D^{\varphi}|^2,
\end{aligned}
\tag{3.71}
$$

where the last equality is a consequence of the fact that D^φ is totally trace-free (since $D^\varphi = (u\eta)^2 D^A$). It follows that, on $\mathrm{int}(M)$, we have the validity of

$$\begin{aligned} X_{ii} =& -\frac{1}{2(\eta u)^2} u_t W^\varphi_{tijk} D^\varphi_{ijk} - \frac{[1+\eta(m-2)]}{(\eta u)^3}|D^\varphi|^2 - uC^\varphi_{ikj,jki} + u_i R^\varphi_{st,k} W^\varphi_{sitk} \\ &+ \alpha\left\{u\left(\frac{1}{2}S^\varphi_t \varphi^a_t \varphi^a_i - \alpha\varphi^b_{ss}\varphi^b_t\varphi^a_t\varphi^a_i\right)\right\}_i + \alpha\{uR^\varphi_{tk}(\varphi^a_t\varphi^a_i)_k\}_i \\ &- \left[u\left(\varphi^a_{ik}U^a - U^{ab}\varphi^b_k\varphi^a_i - \frac{\alpha}{m-2}|\tau(\varphi)|^2\delta_{ik}\right)_k\right]_i \\ &+ \frac{m-4}{m-2}u\left[U^a\left(\frac{1}{\alpha}U^{ab}\varphi^b_i + R^\varphi_{ij}\varphi^a_j\right)\right]_i - \mathcal{A}u. \end{aligned}$$

Therefore, we have

$$\begin{aligned} &X_{ii} - \alpha\left\{u\left(\frac{1}{2}S^\varphi_t \varphi^a_t \varphi^a_i - \alpha\varphi^b_{ss}\varphi^b_t\varphi^a_t\varphi^a_i\right)\right\}_i - \alpha\{uR^\varphi_{tk}(\varphi^a_t\varphi^a_i)_k\}_i \\ &+ \left[u\left(\varphi^a_{ik}U^a - U^{ab}\varphi^b_k\varphi^a_i - \frac{\alpha}{m-2}|\tau(\varphi)|^2\delta_{ik}\right)_k\right]_i = -\frac{1}{2(\eta u)^2} u_t W^\varphi_{tijk} D^\varphi_{ijk} \\ &- \frac{[1+\eta(m-2)]}{(\eta u)^3}|D^\varphi|^2 - uC^\varphi_{ikj,jki} + u_i R^\varphi_{st,k} W^\varphi_{sitk} \\ &+ \frac{m-4}{m-2}u\left[U^a\left(\frac{1}{\alpha}U^{ab}\varphi^b_i + R^\varphi_{ij}\varphi^a_j\right)\right]_i - \mathcal{A}u, \end{aligned}$$

and substituting the expression of $\mathcal{A}$ into the latter, we get (3.60). □

Proof (of Theorem 3.14). Let Z be the vector field defined in (3.59); from Lemma 3.16 and assumptions (3.52) and (3.55) we get

$$\begin{aligned} \mathrm{div}\, Z =& -\frac{[1+\eta(m-2)]}{(\eta u)^3}|D^\varphi|^2 + \left(\frac{m-4}{m-2}\right)\left[U^a\left(\frac{1}{\alpha}U^{ab}\varphi^b_i - R^\varphi_{ij}\varphi^a_j\right)\right]_i u \\ &+ \left\{\varphi^a_i\left(\frac{m}{(m-1)(m-2)}U^a S^\varphi + 2\alpha U^b\varphi^b_j\varphi^a_i\right) - \frac{\alpha}{2}\left(\frac{m-2}{m-1}\right)\varphi^a_i\varphi^a_j S^\varphi_j \right. \\ &\left. - 2\alpha\varphi^a_{jk}R^\varphi_{jk}\varphi^a_i - \alpha\varphi^a_i\tau^a_2(\varphi)\right\}_i u. \end{aligned} \tag{3.72}$$

Let

$$M_\varepsilon := \{x \in M \,:\, u(x) \geq \varepsilon\}, \quad \partial M_\varepsilon := \{x \in M \,:\, u(x) = \varepsilon\}.$$

Then, using the divergence theorem, we deduce

$$\int_{\partial M_\varepsilon} -g(Z,\nu) = \int_{M_\varepsilon} \mathrm{div}\, Z,$$

where

$$\nu = \frac{\nabla u}{|\nabla u|}$$

is the inward unit normal. For the boundary part, we have, from the definition (3.59) of the vector field Z, that

$$\begin{aligned}\int_{\partial M_\varepsilon} g(Z,\nu) =& \varepsilon \int_{\partial M_\varepsilon} \left\{ \left(R^\varphi_{tk} W^\varphi_{tikj}\right)_j - \frac{m-4}{m-2} R^\varphi_{tk} C^\varphi_{tki} + 2\alpha \varphi^a_i \varphi^a_{jk} R^\varphi_{jk} \right\} \nu_i \\ & - \varepsilon \int_{\partial M_\varepsilon} \left\{ R^\varphi_{tk} (\varphi^a_i \varphi^a_t)_k + \alpha \left(\frac{1}{2} S^\varphi_t \varphi^a_t \varphi^a_i - \alpha \varphi^b_{tt} \varphi^b_s \varphi^a_s \varphi^a_i \right) \right\} \nu_i \\ & + \varepsilon \int_{\partial M_\varepsilon} \left\{ \left(\varphi^a_{ik} U^a - U^{ab} \varphi^b_k \varphi^a_i - \frac{\alpha}{m-2} |\tau(\varphi)|^2 \delta_{ik} \right)_k \right\} \nu_i\end{aligned}$$

and this term vanishes as $\varepsilon \to 0^+$. For the left hand side, we integrate by parts equation (3.72) to obtain

$$\begin{aligned}\int_{M_\varepsilon} \operatorname{div} Z =& -[1+\eta(m-2)] \int_{M_\varepsilon} \frac{1}{(\eta u)^3} |D^\varphi|^2 \\ & - \frac{m-4}{m-2} \int_{M_\varepsilon} u_i \left[U^a \left(\frac{1}{\alpha} U^{ab} \varphi^b_i + R^\varphi_{ij} \varphi^a_j \right) \right] \\ & - \frac{m-4}{m-2} \varepsilon \int_{\partial M_\varepsilon} \left[U^a \left(\frac{1}{\alpha} U^{ab} \varphi^b_i + R^\varphi_{ij} \varphi^a_j \right) \right] \nu_i \\ & - \int_{M_\varepsilon} u_i \left(\frac{m}{(m-1)(m-2)} S^\varphi U^a + 2\alpha U^b \varphi^b_j \varphi^a_j \right) \varphi^a_i \\ & - \varepsilon \int_{\partial M_\varepsilon} \left(\frac{m}{(m-1)(m-2)} S^\varphi U^a + 2\alpha U^b \varphi^b_j \varphi^a_j \right) \varphi^a_i \nu_i \\ & + \int_{M_\varepsilon} u_i \left(\frac{\alpha}{2} \frac{m-2}{m-1} S^\varphi_j \varphi^a_j \varphi^a_i + 2\alpha \varphi^a_{jk} R^\varphi_{jk} \varphi^a_i \right) \\ & + \varepsilon \int_{\partial M_\varepsilon} \left(\frac{\alpha}{2} \frac{m-2}{m-1} S^\varphi_j \varphi^a_j \varphi^a_i + 2\alpha \varphi^a_{jk} R^\varphi_{jk} \varphi^a_i \right) \nu_i \\ & + \int_{M_\varepsilon} \alpha \varphi^a_i \tau^a_2(\varphi) u_i + \varepsilon \int_{\partial M_\varepsilon} \alpha \varphi^a_i \tau^a_2(\varphi) \nu_i .\end{aligned}$$

Using (3.54) and (3.53) we deduce

$$\int_{M_\varepsilon} \operatorname{div} Z = -[1+\eta(m-2)] \int_{M_\varepsilon} \frac{1}{(\eta u)^3} |D^\varphi|^2$$

and therefore, letting ε tend to zero, we conclude

$$D^\varphi \equiv 0.$$

□

3.3 Some Conditions on the Cotton Tensor

For the ease of readability, we recall here system (3.2):

$$\begin{cases} i)\ \mathrm{Ric}^\varphi + \mathrm{Hess}(f) - \eta \mathrm{d}f \otimes \mathrm{d}f = \lambda g, \\ ii)\ \tau(\varphi) = d\varphi(\nabla f) + \frac{1}{\alpha} (\nabla U)(\varphi). \end{cases} \tag{3.73}$$

As we have seen in Theorems 3.4 and 3.14, a condition that is often met for a system of the type (3.73) is

$$C^{\varphi}_{ijk} = \frac{U^a \varphi^a_k}{m-1}\delta_{ij} - \frac{U^a \varphi^a_j}{m-1}\delta_{ik}. \tag{3.74}$$

It is immediate to see that condition (3.74) is satisfied if and only if the modification of the φ-Schouten tensor given by

$$A^{\varphi} - \frac{U(\varphi)}{m-1} g \tag{3.75}$$

is a Codazzi tensor. We will prove the following

Theorem 3.17. *Let (M, g) be compact with empty boundary and satisfy system (3.73), for some non-constant function f and $\alpha > 0$. Assume that (3.74) holds and that the (normalized) k-th elementary symmetric polynomial σ_k in the eigenvalues of $A^{\varphi} - \frac{U(\varphi)}{m-1} g$ is constant. If $k \geq 2$, assume, moreover, that σ_k is positive and that there exists a point of M at which all the eigenvalues of $A^{\varphi} - \frac{U(\varphi)}{m-1} g$ are positive. Then φ is constant and (M, g) is isometric to a Euclidean sphere.*

Before proceeding, we recall some results on the elementary symmetric polynomials, then we will prove an Obata-type result which will be instrumental in the proof of Theorem 3.17.

3.3.1 On the elementary symmetric polynomials

Here we collect some well-known facts about elementary symmetric polynomials and we fix the notation. Given m real numbers $\lambda_1, \lambda_2, ..., \lambda_m$, we set

$$S_k := \sum_{i_1 < ... < i_k} \lambda_{i_1} \cdot .. \cdot \lambda_{i_k}$$

for the k-th elementary symmetric polynomial and

$$\sigma_k := \binom{m}{k}^{-1} S_k$$

for its normalization. Then we have the validity of Newton's inequality,

$$\sigma_{k-1}\sigma_{k+1} \leq \sigma_k^2$$

where, provided $\sigma_{k-1} \neq 0$, the equality is obtained if and only if all the λ_i's coincide (see [53] for a proof). Moreover, we have Gårding's inequalities ([45])

$$\sigma_1 \geq \sigma_2^{\frac{1}{2}} \geq ... \geq \sigma_k^{\frac{1}{k}},$$

which hold when all of the σ_j's, $j = 1, .., k$, are positive. Again, equality is obtained if and only if all the λ_i's are equal. Combining them, one can prove the validity of the next lemma, which is Lemma 5.33 of [6].

Lemma 3.18. *Let A be a 2-covariant symmetric tensor on (M, g) with $m = \dim M \geq 3$. Let σ_k be the k-th normalized symmetric function in the eigenvalues of A; if $k \geq 2$, assume that it is positive on M and assume that there exists a point $p \in M$ at which all the eigenvalues of A are positive. Then*

$$\sigma_1 \sigma_k - \sigma_{k+1} \geq 0, \tag{3.76}$$

with equality holding at a point $x \in M$ if and only if the eigenvalues of A at x coincide.

Proof of Lemma 3.18. For $k = 1$, (3.76) coincides with the Newton's inequality so that the conclusion immediately follows.
For $k \geq 2$, note that, by assumptions, the set

$$\{(\lambda_1(x), \lambda_2(x), ..., \lambda_m(x)) \in \mathbb{R}^m : x \in M\}$$

is contained in the connected component of $\{x \in \mathbb{R}^m : \sigma_k(x) > 0\}$ that contains the positive orthant

$$\{(x_1, x_2, .., x_m) \in \mathbb{R}^m : x_i > 0, \forall i = 1, .., m\}.$$

As it is well-known, this connected component coincides with the set

$$\{x \in \mathbb{R}^m : \sigma_i(x) > 0, \forall i = 1, .., k\}$$

so that Gårding's inequalities can be applied. Moreover, since $\sigma_{k-1} > 0$, we obtain from Newton's inequalities that, for $k \geq 2$,

$$\sigma_{k+1} = \frac{\sigma_{k+1}\sigma_{k-1}}{\sigma_{k-1}} \leq \frac{\sigma_k^2}{\sigma_{k-1}} = \sigma_k \frac{\sigma_k}{\sigma_{k-1}}.$$

We claim

$$\frac{\sigma_k}{\sigma_{k-1}} \leq \sigma_1$$

which, since $\sigma_k > 0$, would imply (3.76). To prove the claim we use Gårding's inequalities and the positivity of σ_1, σ_k and σ_{k-1} to deduce

$$\sigma_k = \sigma_k^{1/k} \sigma_k^{(k-1)/k} \leq \sigma_1 \sigma_k^{(k-1)/k} \leq \sigma_1 \sigma_{k-1}.$$

This proves the claim. Moreover, equality at a point $x \in M$ of (3.76) would imply equality in the Newton's inequalities and in Gårding's inequalities and this happens if and only if all of the eigenvalues of A at x coincide. $\square$

In the following, we will fix a symmetric 2-covariant tensor A on M. The σ_i's will be intended as functions of the eigenvalues of A. The *Newton endomorphisms*

$$P_k = P_k(A) : TM \to TM$$

associated to A are inductively defined by

$$P_0 = \mathrm{id}, \quad P_k = S_k \,\mathrm{id} - A \circ P_{k-1}, \qquad 1 \leq k \leq m. \tag{3.77}$$

Here A is identified with the corresponding endomorphism $A : TM \to TM$. Note that, by Cayley-Hamilton's Theorem, we have $P_m \equiv 0$. It is well known that the P_k's satisfy the following formulas (see [8]):

$$\mathrm{tr} P_k = (m-k) S_k, \tag{3.78}$$
$$\mathrm{tr}(A \circ P_k) = (k+1) S_{k+1}. \tag{3.79}$$

Setting

$$c_k = (m-k) \binom{m}{k}, \tag{3.80}$$

a simple computation shows that equations (3.78) and (3.79) become

$$\mathrm{tr} P_k = c_k \sigma_k, \tag{3.81}$$
$$\mathrm{tr}(A \circ P_k) = c_k \sigma_{k+1}. \tag{3.82}$$

As a last well-known fact, we recall that if A is a Codazzi tensor, then all of the P_i's are divergence-free; this is a consequence of the following

Lemma 3.19. *Let A be a 2-covariant symmetric tensor and let P_k denote the k-th symmetric Newton operator given by (3.77). Then the following formula holds:*

$$\mathrm{div}(P_k)_i = -A_{ij} \,\mathrm{div}(P_{k-1})_j - C(A)_{jit}(P_{k-1})_{jt}, \tag{3.83}$$

where

$$C(A)_{jit} = A_{ji,t} - A_{jt,i}$$

is the obstruction to A being a Codazzi tensor.

Proof. From (3.77) we deduce

$$(P_k)_{ij,j} = (S_k)_i - A_{ip,j}(P_{k-1})_{pj} - A_{ip}(P_{k-1})_{pj,j}.$$

To obtain (3.83) from it, we only need to prove the validity of

$$(S_k)_i = A_{pj,i}(P_{k-1})_{pj}. \tag{3.84}$$

To deduce it, we fix $p \in M$ such that each eigenvalue of A is smooth at p. As it is well-known (see Paragraph 16.10 of [10]), the set of points at which all the eigenvalues of A are smooth is dense in M, so that it is enough to prove the validity of (3.84) at p.

We choose a local reference frame in a neighbourhood of p that diagonalizes A: from the definition of S_k, we immediately obtain

$$(S_k)_i = (\lambda_1)_i S_{k-1}|_{\lambda_1=0} + (\lambda_2)_i S_{k-1}|_{\lambda_2=0} + ... + (\lambda_m)_i S_{k-1}|_{\lambda_m=0}$$

where $S_{k-1}|_{\lambda_j=0}$ denotes the sum of those summands of S_{k-1} in which λ_j does not appear. With this notations in mind, a simple mathematical induction that uses only the definition (3.77) of P_k gives that, in the chosen frame, P_{k-1} takes the form

$$P_{k-1} = \mathrm{diag}(S_{k-1}|_{\lambda_1=0}, S_{k-1}|_{\lambda_2=0}, ..., S_{k-1}|_{\lambda_m=0}).$$

Therefore we have

$$A_{pj,i}(P_{k-1})_{pj} = (\lambda_1)_i S_{k-1}|_{\lambda_1=0} + (\lambda_2)_i S_{k-1}|_{\lambda_2=0} + ... + (\lambda_m)_i S_{k-1}|_{\lambda_m=0}$$

and (3.84) follows. □

Corollary 3.20. *Let A be a 2-covariant symmetric Codazzi tensor. Then all of the Newton operators in the eigenvalues of A are divergence-free. Moreover, we also have*

$$\begin{aligned}[(P_{k-1}\overset{\circ}{)} \circ A]_{ij,i} &= \frac{m-k}{m}(S_k)_j \\ &= \frac{c_k}{m}(\sigma_k)_j\end{aligned} \tag{3.85}$$

for $k = 1, ..., m-1$, where

$$[(P_{k-1}\overset{\circ}{)} \circ A] := (P_{k-1}) \circ A - \frac{1}{m}\mathrm{tr}[(P_{k-1}) \circ A]g.$$

Proof. Since A is Codazzi, we have

$$\begin{aligned}\mathrm{div}(P_1)_i &= (S_1)_i - A_{ij,j} \\ &= A_{jj,i} - A_{ij,j} = 0,\end{aligned}$$

and therefore the first part of the statement follows by induction on k, using (3.83). To prove (3.85), we use (3.79) to deduce

$$[(P_{k-1}\overset{\circ}{)} \circ A] = (P_{k-1}) \circ A - \frac{k}{m}S_k g.$$

Taking the divergence of the above expression we get

$$[(P_{k-1}\overset{\circ}{)} \circ A]_{ij,i} = (P_{k-1})_{it,i}A_{tj} + (P_{k-1})_{it}A_{tj,i} - \frac{k}{m}(S_k)_j.$$

Using div $(P_k) = 0$, the fact that A is Codazzi and (3.84) we deduce

$$[(P_{k-1}\overset{\circ}{)}_{it}A_{tj}]_i = \frac{m-k}{k}(S_k)_j,$$

that is, (3.85). □

Remark 3.21. Symmetrizing the last term of (3.83) we get

$$C(A)_{jit}(P_{k-1})_{jt} = \frac{1}{2}(P_{k-1})_{jt}[C(A)_{jit} + C(A)_{tij}].$$

Reasoning as in the first part of the proof of Corollary 3.20, we deduce that P_k is divergence-free assuming only

$$C(A)_{jit} + C(A)_{tij} = 0.$$

The Newton endomorphisms give rise to a family of second order differential operators, $L_k = L_k(A)$, defined as follows: let $u \in C^2(M)$ and set Hess(u) to denote both the 2-covariant tensor and the corresponding endomorphism; we set

$$L_k u = \mathrm{tr}(P_k \circ \mathrm{Hess}(u)), \tag{3.86}$$

that we also rewrite in the useful form

$$L_k u = \sum_{i=0}^{k} (-1)^i S_{k-i} \mathrm{tr}\big(A^i \circ \mathrm{Hess}(u)\big),$$

where $A^0 = \mathrm{id}$ and, for $i \geq 1$, A^i is the i-th iterated composition of A with itself. A computation shows that $L_k u$ can also be expressed in the form

$$L_k u = \mathrm{div}\,(P_k(\nabla u)) - g(\mathrm{div}\, P_k, \nabla u).$$

From the above formula we see that L_k is semielliptic whenever P_k is positive semidefinite and it is in divergence form when $\mathrm{div}\, P_k = 0$. From Corollary 3.20 we see that the second condition is satisfied when A is a Codazzi tensor.

Suppose now that, for some

$$p(x), q(x), l(x) \in C^\infty(M),$$

we can express Hess(u) in the form

$$\mathrm{Hess}(u) = p(x)g + q(x) du \otimes du - l(x)A. \tag{3.87}$$

Then from equations (3.82), (3.81) and (3.86) we obtain, after some algebraic manipulations,

$$L_k u = c_k[p(x)\sigma_k - l(x)\sigma_{k+1}] + q(x) g(P_k(\nabla u), \nabla u) \tag{3.88}$$

for $1 \leq k \leq m-1$.

3.3.2 An Obata-type result

Recall that a vector field X on (M, g) is said to be *conformal* if

$$\mathcal{L}_X g = \frac{2\,\mathrm{div}\, X}{m} g, \tag{3.89}$$

where $\mathcal{L}_X g$ is the Lie derivative of the metric in the direction of X, while X is *Killing* when it is conformal and $\mathrm{div}\, X \equiv 0$. Moreover, X is *closed* when, in an orthonormal coframe,

$$X_{ij} = X_{ji}.$$

We now prove the next result, which is inspired by a classical theorem of Obata ([77]).

Proposition 3.22. *Let (M,g) be a closed, connected Riemannian manifold supporting the structure*

$$\begin{cases} \mathrm{Ric}^{\varphi} = \frac{S^{\varphi}}{m} g, \\ \tau(\varphi) = \frac{1}{\alpha} \nabla U(\varphi). \end{cases} \tag{3.90}$$

Assume that there exists a closed, conformal, non-Killing vector field X on (M,g) and assume

$$d\varphi(X) = 0. \tag{3.91}$$

Then φ is constant and (M,g) is isometric to a Euclidean sphere.

When U is constant, the above result has been proved, without assuming that X is closed, in Lemma 5.2 of [6]. We first prove the following

Lemma 3.23. *Let (M,g) be a Riemannian manifold of dimension $m \geq 3$ and X a conformal vector field on (M,g). Let $\varphi : (M,g) \to (N,h)$ be a smooth map and $\alpha \in \mathbb{R} \backslash \{0\}$. Set*

$$\gamma = \operatorname{div} X.$$

Then

$$\begin{aligned} \mathrm{Hess}(\gamma) =& \frac{S^{\varphi}}{(m-1)(m-2)} \gamma g + \frac{m}{2(m-1)(m-2)} g(\nabla S^{\varphi}, X) g - \frac{m}{m-2} \mathcal{L}_X \mathrm{Ric}^{\varphi} \\ & - \frac{m}{m-2} \alpha \left[\mathcal{L}_X(\varphi^* h) - \frac{1}{2(m-1)} \mathrm{tr}(\mathcal{L}_X(\varphi^* h)) g \right]. \end{aligned} \tag{3.92}$$

In particular,

$$\Delta \gamma = -\frac{S^{\varphi}}{m-1} \gamma - \frac{m}{2(m-1)} g(\nabla S^{\varphi}, X) - \frac{m}{2(m-1)} \alpha \mathrm{tr}(\mathcal{L}_X(\varphi^* h)). \tag{3.93}$$

Proof. We first prove

$$\mathrm{Hess}(\gamma) = \frac{S}{(m-1)(m-2)} \gamma g + \frac{m}{2(m-1)(m-2)} g(\nabla S, X) g - \frac{m}{m-2} \mathcal{L}_X \mathrm{Ric}. \tag{3.94}$$

We apply Ricci commutations rules twice to deduce

$$\begin{aligned} \gamma_{ij} = X_{ttij} &= X_{titj} + X_{pj} R_{ptti} + X_p R_{ptti,j} \\ &= X_{tijt} + X_{pi} R_{pttj} + X_{tp} R_{pitj} - X_{pj} R_{pi} - X_p R_{pi,j} \\ &= X_{tijt} - X_{pi} R_{pj} + X_{tp} R_{pitj} - X_{pj} R_{pi} - X_p R_{pi,j}. \end{aligned}$$

Since X is conformal, we have

$$X_{ij} + X_{ji} = \frac{2}{m} \gamma \delta_{ij}. \tag{3.95}$$

From (3.95) and the Ricci commutation rules we deduce

$$\begin{aligned}\gamma_{ij} =& - X_{itjt} + \frac{2}{m}X_{ttji} - X_{pi}R_{pj} + X_{tp}R_{pitj} - X_{pj}R_{pi} - X_pR_{pi,j} \\ =& - X_{ijtt} - X_{pt}R_{pitj} - X_pR_{pitj,t} + \frac{2}{m}X_{ttji} - X_{pi}R_{pj} + X_{tp}R_{pitj} \\ & - X_{pj}R_{pi} - X_pR_{pi,j}.\end{aligned}$$

Using the first and second Bianchi identities we obtain

$$\begin{aligned}\gamma_{ij} =& - X_{ijtt} + \frac{2}{m}X_{ttji} - X_{pt}R_{ptij} - X_pR_{ij,p} + X_pR_{pj,i} - X_pR_{pi,j} \\ & - X_{pi}R_{pj} - X_{pj}R_{pi}.\end{aligned}$$

Recalling that

$$(\mathcal{L}_X\mathrm{Ric})_{ij} = X_tR_{ij,t} + X_{ti}R_{tj} + X_{tj}R_{it}$$

we obtain, rearranging some terms,

$$\gamma_{ij} = -X_{ijtt} + \frac{2}{m}X_{ttji} - X_{pt}R_{ptij} + X_t(R_{tj,i} - R_{ti,j}) - (\mathcal{L}_X\mathrm{Ric})_{ij}.$$

Symmetrizing the above expression and using (3.95) we deduce

$$\begin{aligned}\gamma_{ij} =& - \frac{1}{2}(X_{ij} + X_{ji})_{tt} + \frac{2}{m}\gamma_{ij} - (\mathcal{L}_X\mathrm{Ric})_{ij} \\ =& - \frac{1}{m}\gamma_{tt}\delta_{ij} + \frac{2}{m}\gamma_{ij} - (\mathcal{L}_X\mathrm{Ric})_{ij},\end{aligned}$$

which, after some simplifications, becomes, in global notation,

$$\mathrm{Hess}(\gamma) = -\frac{\Delta\gamma}{m-2}g - \frac{m}{m-2}\mathcal{L}_X\mathrm{Ric}. \tag{3.96}$$

Tracing (3.96) and simplifying we get

$$\Delta\gamma = -\frac{S}{m-1}\gamma - \frac{m}{2(m-1)}g(\nabla S, X).$$

Substituting it into (3.96) we obtain (3.94). From the definitions of Ric^φ and S^φ we immediately get

$$\mathcal{L}_X\mathrm{Ric} = \mathcal{L}_X\mathrm{Ric}^\varphi + \alpha\mathcal{L}_X(\varphi^*h) \tag{3.97}$$

and

$$\begin{aligned}S\gamma g + \frac{m}{2}g(\nabla S, X) =& S^\varphi\gamma g + \frac{m}{2}g(\nabla S^\varphi, X)g \\ & + \alpha\frac{m}{2}\mathrm{tr}(\mathcal{L}_X(\varphi^*h))g.\end{aligned} \tag{3.98}$$

Using (3.97) and (3.98) into (3.94) we obtain (3.92), while tracing (3.92) we obtain (3.93). □

Remark 3.24. Assume now that $X \in \mathrm{Ker}(d\varphi)$, that is,

$$\varphi^a_i X_i = 0.$$

Taking the covariant derivative of the above equation we get

$$\varphi^a_{ij} X_i + \varphi^a_i X_{ij} = 0,$$

which implies $\mathcal{L}_X(\varphi^* h) = 0$. Therefore, (3.92) and (3.93) reduce, respectively, to

$$\mathrm{Hess}(\gamma) = \frac{S^\varphi}{(m-1)(m-2)}\gamma g + \frac{m}{2(m-1)(m-2)} g(\nabla S^\varphi, X) g - \frac{m}{m-2}\mathcal{L}_X \mathrm{Ric}^\varphi \tag{3.99}$$

and

$$\Delta\gamma = -\frac{S^\varphi}{m-1} - \frac{m}{2(m-1)} g(X, \nabla S^\varphi). \tag{3.100}$$

Suppose now

$$\begin{cases} \mathrm{Ric}^\varphi = \frac{S^\varphi}{m} g \\ X \in \mathrm{Ker}(d\varphi). \end{cases}$$

Then, taking the divergence of the first of the above equations and using the φ-Schur's identity, we get

$$\frac{1}{2} S^\varphi_k - \alpha \varphi^a_{tt} \varphi^a_k = R^\varphi_{tk,t} = \frac{S^\varphi_k}{m}$$

and

$$\frac{m-2}{2m} S^\varphi_k = \varphi^a_{tt}\varphi^a_k.$$

Since $X \in \mathrm{Ker}(d\varphi)$, it follows that

$$\frac{m-2}{2m} S^\varphi_k X_k = \varphi^a_{tt}\varphi^a_k X_k = 0,$$

that is, since $m \geq 3$,

$$S^\varphi_k X_k = 0. \tag{3.101}$$

Furthermore, using $\mathrm{Ric}^\varphi = \frac{S^\varphi}{m} g$ and (3.101),

$$\begin{aligned} (\mathcal{L}_X \mathrm{Ric}^\varphi)_{ij} =& X_t R^\varphi_{ij,k} + X_{ti} R^\varphi_{tj} + X_{tj} R^\varphi_{ti} \\ =& \frac{1}{m} X_t S^\varphi_t \delta_{ij} + \frac{S^\varphi}{m} X_{ti}\delta_{tj} + \frac{S^\varphi}{m} X_{tj}\delta_{ti} \\ =& \frac{1}{m}(S^\varphi_t X_t \delta_{ij} + S^\varphi X_{ji} + S^\varphi X_{ij}) \\ =& \frac{S^\varphi}{m}(\mathcal{L}_X g)_{ij} = \frac{2}{m^2} S^\varphi \gamma \delta_{ij}, \end{aligned}$$

that is,

$$(\mathcal{L}_X \mathrm{Ric}^{\varphi})_{ij} = \frac{2}{m^2} S^{\varphi} \gamma \delta_{ij}. \tag{3.102}$$

Hence, (3.99) and (3.100) become, respectively,

$$\mathrm{Hess}\,(\gamma) = -\frac{1}{m(m-1)} S^{\varphi} \gamma g \tag{3.103}$$

and

$$\Delta \gamma = -\frac{1}{m-1} S^{\varphi} \gamma. \tag{3.104}$$

Remark 3.25. From equation (3.104) we deduce (see Appendix A of [81] or [58]) that γ satisfies the *unique continuation property*. In particular, it vanishes on some open subset of M if and only if it vanishes on the connected components of M containing it.

Remark 3.26. Since we do not know wether S^{φ} is constant or not, it seems hard to prove Proposition 3.22 from equation (3.103) using the original ideas of Obata, or those used in [6]. Instead, to prove Proposition 3.22, we will prove the constancy of U under the further assumption that X is closed and then apply Lemma 5.2 of [6] to conclude. In this direction, the next Lemma will prove to be useful.

Lemma 3.27. *Let (M, g) be a Riemannian manifold, let $\varphi : (M, g) \to (N, h)$ be a smooth map with target another Riemannian manifold (N, h) and let X be a smooth, closed, conformal vector field on M. Assume that, for some $\eta \in \mathbb{R}$, (M, g) supports the structure*

$$\begin{cases} \mathrm{Ric}^{\varphi} + \frac{1}{2}\mathcal{L}_X g - \eta X^{\flat} \otimes X^{\flat} = \lambda g, \\ \tau(\varphi) = \frac{1}{\alpha} \nabla U(\varphi), \\ d\varphi(X) = 0, \end{cases} \tag{3.105}$$

where $\lambda = \frac{1}{m}\left(S^{\varphi} + \mathrm{div}\, X - \eta |X|^2\right)$. Then $\tau(\varphi)$ vanishes identically on the set

$$\{x \in M : (\mathrm{div}\, X)(x) \neq 0\}.$$

Proof. The third equation of (3.105) reads, in components,

$$\varphi^a_i X_i = 0. \tag{3.106}$$

Setting

$$\gamma = \mathrm{div}\, X,$$

since X is conformal and closed we have

$$X_{ij} = \frac{\gamma}{m} \delta_{ij}. \tag{3.107}$$

Take the covariant derivative of (3.106) and use (3.107) to deduce

$$0 = \varphi^a_{ij} X_i + \varphi^a_i X_{ij} = \varphi^a_{ij} X_i + \frac{\gamma}{m} \varphi^a_j.$$

Compute the divergence of the expression above and use (3.107) to get

$$\begin{aligned} 0 =& \varphi^a_{ijj} X_i + \varphi^a_{ij} X_{ij} + \frac{\gamma}{m} \varphi^a_{jj} + \frac{1}{m} \varphi^a_j \gamma_j \\ =& \varphi^a_{ijj} X_i + 2\frac{\gamma}{m} \varphi^a_{jj} + \frac{1}{m} \varphi^a_j \gamma_j, \end{aligned}$$

that is,

$$0 = \varphi^a_{ijj} X_i + 2\frac{\gamma}{m} \varphi^a_{jj} + \frac{1}{m} \varphi^a_j \gamma_j. \tag{3.108}$$

Since X is conformal and $\lambda = \frac{1}{m}(S^\varphi + \operatorname{div} X - \eta |X|^2)$, the first equation of (3.105) becomes

$$\mathrm{Ric}^\varphi - \eta X^\flat \otimes X^\flat = \frac{1}{m}\big(S^\varphi - \eta |X|^2\big) g. \tag{3.109}$$

Taking the divergence of (3.107) we have

$$X_{jij} = \frac{\gamma_i}{m}.$$

Applying the Ricci commutation relations and using $d\varphi(X) = 0$ we deduce

$$\frac{\gamma_i}{m} = X_t R_{ti} + X_{jji} = X_t R^\varphi_{ti} + \gamma_i.$$

Rearranging and using (3.109) we get

$$\frac{m-1}{m} \gamma_i = -R^\varphi_{ti} X_t = -\frac{S^\varphi}{m} X_i - \eta \frac{m-1}{m} |X|^2 X_i. \tag{3.110}$$

From equations (3.110) and (3.106) we deduce

$$\varphi^a_j \gamma_j = 0. \tag{3.111}$$

We want to prove

$$\varphi^a_{ijj} X_i = 0. \tag{3.112}$$

We exploit the commutation relation (1.11), i.e.

$$\varphi^a_{ijk} = \varphi^a_{ikj} + \varphi^a_t R_{tijk} - {}^N\! R^a_{bcd} \varphi^b_i \varphi^c_j \varphi^d_k,$$

which implies

$$\varphi^a_{ijj} = \varphi^a_{jij} = \varphi^a_{jji} + \varphi^a_t R_{ti} - {}^N\! R^a_{bcd} \varphi^b_j \varphi^c_i \varphi^d_j.$$

Contracting the above with X_i we get

$$\varphi^a_{ijj}X_i = \varphi^a_{jji}X_i + \varphi^a_t R_{ti}X_i - {}^N R^a_{bcd}\varphi^b_j\varphi^c_i\varphi^d_j X_i. \tag{3.113}$$

From (3.106) we deduce

$${}^N R^a_{bcd}\varphi^b_j\varphi^c_i\varphi^d_j X_i = 0,$$

and using (3.106) and (3.109) we obtain

$$\varphi^a_t R_{ti}X_i = \varphi^a_t R^{\varphi}_{ti}X_i = \frac{S^{\varphi}}{m}\varphi^a_t X_t + \eta\frac{m-1}{m}|X|^2\varphi^a_t X_t = 0.$$

On the other hand, differentiating the second equation of (3.105) we have

$$\varphi^a_{tti} = \frac{1}{\alpha}U^{ab}\varphi^b_i,$$

and contracting with X_i and using (3.106) we deduce

$$\varphi^a_{tti}X_i = \frac{1}{\alpha}U^{ab}\varphi^b_i X_i = 0.$$

Inserting in (3.113) we obtain

$$\varphi^a_{ijj}X_i = 0,$$

that is, equation (3.112). Using (3.112) and (3.111) into (3.108) we deduce

$$0 = 2\frac{\gamma}{m}\varphi^a_{tt}. \tag{3.114}$$

Therefore, on the set where $\gamma = \operatorname{div} X$ does not vanish, we have that $\tau(\varphi)$ vanishes. □

Proof (of Proposition 3.22). Since X is conformal and closed, we are in the assumptions of Lemma 3.27, that is, we are in the case $\eta = 0$ of that Lemma. Applying it, we deduce that $\tau(\varphi)$ vanishes on the set

$$A := \{x \in M : \gamma(x) \neq 0\}.$$

From Remark 3.25, we deduce that γ satisfies the unique continuation property; therefore, it vanishes on an open subset of M if and only if it vanishes identically. Since X is non-Killing by assumption, this cannot happen so that the zero set of γ does not contain any open subset of M. As a consequence, A is dense in M. Since $\tau(\varphi)$ vanishes there, we deduce that, by continuity, $\tau(\varphi) = 0$ on all of M. Therefore (M, g) is a harmonic-Einstein manifold and we are in the conditions to apply Lemma 5.2 of [6] to conclude. □

3.3.3 Proof of the Main Theorem

From our assumptions, we know that

$$A^{\varphi} - \frac{U(\varphi)}{m-1} g$$

is a Codazzi tensor; in the following, P_k will be the k-th Newton operator "in its eigenvalues", which is divergence free by Corollary 3.20. Therefore

$$[(P_k)_{ij} f_i]_j = (P_k)_{ij} f_{ij},$$

that is

$$\operatorname{div}(P_k(\nabla f)) = \operatorname{tr}(P_k \circ \operatorname{Hess}(f)). \tag{3.115}$$

We first prove the validity of the following identity:

$$\begin{aligned} \operatorname{tr}\Big\{\Big(P_k - \frac{c_k}{m}\sigma_k \operatorname{id}\Big) \circ \operatorname{Hess}(u)\Big\} - g\Big(\nabla \log u^{1-\frac{\eta}{\beta}}, \Big(P_k - \frac{c_k}{m}\sigma_k \operatorname{id}\Big)(\nabla u)\Big) \\ = c_k \beta u(\sigma_{k+1} - \sigma_1 \sigma_k), \end{aligned} \tag{3.116}$$

where $u = e^{-\beta f}$, for some $\beta \in \mathbb{R}$, $\beta \neq 0$. Note that, to prove it, we will only use the constancy of σ_k, equation (3.74) and the first equation of (3.73).
Rewriting (3.73) i) we get

$$f_{ij} = \eta f_i f_j - A^{\varphi}_{ij} + \frac{U(\varphi)}{m-1}\delta_{ij} - \frac{1}{2(m-1)} S^{\varphi} \delta_{ij} - \frac{U(\varphi)}{m-1}\delta_{ij} + \lambda \delta_{ij}. \tag{3.117}$$

We perform the substitution $u = e^{-\beta f}$ to obtain

$$\begin{aligned} \operatorname{Hess}(u) = &\beta u\Big(\frac{S^{\varphi}}{2(m-1)} - \lambda(x) + \frac{U(\varphi)}{m-1}\Big) g + \Big(1 - \frac{\eta}{\beta}\Big)\frac{1}{u} du \otimes du \\ &+ \beta u\Big(A^{\varphi} - \frac{U(\varphi)}{m-1} g\Big), \end{aligned} \tag{3.118}$$

which is a structure of the form (3.87) for the choices

$$\begin{aligned} &p(x) = \beta u\Big(\frac{S^{\varphi}}{2(m-1)} - \lambda + \frac{U(\varphi)}{m-1}\Big)(x), \quad q(x) = \Big(1 - \frac{\eta}{\beta}\Big)\frac{1}{u}(x), \\ &l(x) = -\beta u(x) \end{aligned}$$

and $A = A^{\varphi} - \frac{U(\varphi)}{m-1} g$. From equation (3.88) we deduce

$$\begin{aligned} L_k u = &c_k \beta u\Big[\Big(\frac{S^{\varphi}}{2(m-1)} - \lambda(x) + \frac{U(\varphi)}{m-1}\Big)\sigma_k + \sigma_{k+1}\Big] \\ &+ \Big(1 - \frac{\eta}{\beta}\Big)\frac{1}{u} g(P_k(\nabla u), \nabla u). \end{aligned}$$

A simple computation gives the validity of

$$\frac{U(\varphi)}{m-1} = \frac{m-2}{2m(m-1)} S^{\varphi} - \sigma_1$$

and, therefore,

$$\begin{aligned} L_k u =& c_k \beta u(-\sigma_1\sigma_k + \sigma_{k+1}) + g\Big(P_k(\nabla u), \nabla \log u^{1-\frac{\eta}{\beta}}\Big) \\ &+ c_k\beta u \sigma_k \frac{S^{\varphi}}{m} - c_k \beta u \sigma_k \lambda. \end{aligned}$$

Tracing equation (3.118) we obtain

$$\begin{aligned} \Delta u =& \frac{m}{2(m-1)}\beta u S^{\varphi} + \frac{m}{m-1}\beta u U(\varphi) - m\beta u\lambda(x) + \left(1 - \frac{\eta}{\beta}\right)\frac{1}{u}|\nabla u|^2 \\ &+ \frac{m-2}{2(m-1)}\beta u S^{\varphi} - \frac{m}{m-1}\beta u U(\varphi) \\ =& S^{\varphi}\beta u - m\beta u \lambda(x) + \left(1 - \frac{\eta}{\beta}\right)\frac{1}{u}|\nabla u|^2, \end{aligned}$$

so that

$$\begin{aligned} &L_k u - \frac{c_k}{m}\sigma_k \Delta u - g\Big(P_k(\nabla u), \nabla \log u^{1-\frac{\eta}{\beta}}\Big) + \frac{c_k}{m}\sigma_k g\Big(\nabla u, \nabla \log u^{1-\frac{\eta}{\beta}}\Big) \\ &= c_k \beta u(\sigma_{k+1} - \sigma_1\sigma_k); \end{aligned}$$

since σ_k is constant, the above equation can be rewritten as

$$\begin{aligned} &\operatorname{tr}\Big\{\Big(P_k - \frac{c_k}{m}\sigma_k \operatorname{id}\Big) \circ \operatorname{Hess}(u)\Big\} - g\Big(\nabla \log u^{1-\frac{\eta}{\beta}}, \Big(P_k - \frac{c_k}{m}\sigma_k \operatorname{id}\Big)(\nabla u)\Big) \\ =& c_k\beta u(\sigma_{k+1} - \sigma_1\sigma_k), \end{aligned}$$

which is (3.116). The importance of (3.116) stems from the fact that, when P_k is divergence-free, its left-hand side becomes a weighted divergence; indeed we have

$$\begin{aligned} &\operatorname{tr}\Big\{\Big(P_k - \frac{c_k}{m}\sigma_k \operatorname{id}\Big) \circ \operatorname{Hess}(u)\Big\} - g\Big(\nabla \log u^{1-\frac{\eta}{\beta}}, \Big(P_k - \frac{c_k}{m}\sigma_k \operatorname{id}\Big)(\nabla u)\Big) \\ =& u^{1-\frac{\eta}{\beta}} \operatorname{div}\Big(u^{\frac{\eta}{\beta}-1}\Big(P_k - \frac{c_k}{m}\sigma_k \operatorname{id}\Big)(\nabla u)\Big). \end{aligned}$$

Therefore, multiplying (3.116) by $u^{\frac{\eta}{\beta}-1}$, integrating the result over M, using the divergence theorem and $\beta \neq 0$ we obtain

$$\int_M u^{\frac{\eta}{\beta}}(\sigma_{k+1} - \sigma_1\sigma_k) = 0.$$

From $u > 0$ and (3.76) we get

$$\sigma_{k+1} - \sigma_1\sigma_k = 0,$$

so that the equality case in (3.76) is satisfied. Therefore, all the eigenvalues of $A^{\varphi} - \frac{U(\varphi)}{m-1}g$ coincide so that A^{φ} is proportional to the metric, and hence Ric^{φ} is too. We then have that

$$\mathrm{Ric}^{\varphi} = \gamma g$$

holds, for some $\gamma \in C^{\infty}(M)$. Tracing the equation above, we deduce

$$m\gamma = S^{\varphi}$$

and

$$\mathrm{Ric}^{\varphi} = \frac{S^{\varphi}}{m}g$$

so that the first equation of a harmonic-Einstein manifold is satisfied. From equation

$$C^{\varphi}_{iik} = \alpha\varphi^a_k\varphi^a_{ii}$$

and (3.74) we deduce

$$\alpha\varphi^a_k\varphi^a_{ii} = \frac{1}{\alpha}U^a\varphi^a_k.$$

Contracting the above equation with f_k and using the second equation of (3.73) we deduce

$$0 = \alpha\varphi^a_k f_k\left(\varphi^a_{tt} - \frac{1}{\alpha}U^a\right) = \alpha\left|\tau(\varphi) - \frac{1}{\alpha}(\nabla U)(\varphi)\right|^2.$$

In particular, we have that φ is $\frac{U}{\alpha}$-harmonic and, again from the second equation of (3.73), $\nabla f \in \mathrm{Ker}(d\varphi)$. In other words, we have proved the validity of

$$\begin{cases} \mathrm{Ric}^{\varphi} = \frac{S^{\varphi}}{m}g \\ \tau(\varphi) = \frac{1}{\alpha}\nabla U(\varphi) \end{cases} \tag{3.119}$$

and

$$\mathrm{d}\varphi(\nabla f) = 0 = \mathrm{d}\varphi(\nabla u).$$

Moreover, when $\eta = 0$, since A^{φ} is proportional to the metric, (3.117) tells us that also $\mathrm{Hess}(f)$ is, so that ∇f is a closed conformal vector field. When $\eta \neq 0$, choosing $\beta = \eta$ in (3.118) shows us that ∇u is a closed conformal vector field. In both cases, we are in the assumptions of Proposition 3.22, and we can use it to conclude.

Bibliography

[1] L. J. Alías, P. Mastrolia, and M. Rigoli. *Maximum principles and geometric applications*. Springer Monographs in Mathematics. Springer, Cham, 2016.

[2] E. Altas and B. Tekin. Consistency problems of conformal killing gravity. *Phys. Rev. D*, 111(6):064083, 2025.

[3] E. Altas and B. Tekin. Vanishing of conserved charges in cotton gravity. *Phys. Rev. D*, 111(2):L021503, 2025.

[4] M. Andrade, B. Leandro, and T. Policarpo. Mass and topology of hypersurfaces in static perfect fluid spaces. *General Relativity and Gravitation*, 57:37, 01 2025.

[5] A. Anselli. Bach and Einstein's equations in presence of a field. *Int. J. Geom. Methods Mod. Phys.*, 18(5):Paper No. 2150077, 68, 2021.

[6] A. Anselli, G. Colombo, and M. Rigoli. On the geometry of Einstein-type structures. *Nonlinear Anal.*, 204:Paper No. 112198, 84, 2021.

[7] R. Bach. Zur weylschen relativit¨atstheorie und der weylschen erweiterung des kr¨ummungstensorbegriffs. *Mathematische Zeitschrift*, 9:110–135, 1921.

[8] J. L. M. Barbosa and A. G. Colares. Stability of hypersurfaces with constant r-mean curvature. *Ann. Global Anal. Geom.*, 15(3):277–297, 1997.

[9] P. Bargueño. Comment on "emergence of the cotton tensor for describing gravity. *Phys. Rev. D*, 104(8), 2021.

[10] A. L. Besse. *Einstein manifolds*. Classics in Mathematics. Springer-Verlag, Berlin, 2008. Reprint of the 1987 edition.

[11] S. Borghini and L. Mazzieri. On the mass of static metrics with positive cosmological constant: I. *Classical Quantum Gravity*, 35(12):125001, 43, 2018.

[12] J. P. Bourguignon. Les variétés de dimension 4 à signature non nulle dont la courbure est harmonique sont d'einstein. *Invent. Math.*, 63(2):263–286, 1981.

[13] L. Branca, G.Colombo, P. Mastrolia, F. Mastropietro, and M. Rigoli. On the geometry of φ-static perfect fluid space-times. *preprint, submitted for publication.*

[14] H. D. Cao. Geometry of complete gradient shrinking ricci solitons. *arXiv.org*, 2009.

[15] H. D. Cao. Recent progress on ricci solitons. *arXiv.org*, 2009.

[16] H. D. Cao and Qiang Chen. On Bach-flat gradient shrinking Ricci solitons. *Duke Math. J.*, 162(6):1149–1169, 2013.

[17] H. D. Cao, X. Sun, and Y. Zhang. On the structure of gradient yamabe solitons. *Mathematical research letters*, 19(4):767–774, 2012.

[18] J. Case, Y. J. Shu, and G. Wei. Rigidity of quasi-einstein metrics. *Differential geometry and its applications*, 29(1):93–100, 2011.

[19] G. Catino and P. Mastrolia. *A perspective on canonical Riemannian metrics*, volume 336 of *Progress in Mathematics*. Birkhäuser/Springer, Cham, [2020] ©2020.

[20] G. Catino, P. Mastrolia, and D. D. Monticelli. Gradient Ricci solitons with vanishing conditions on Weyl. *J. Math. Pures Appl. (9)*, 108(1):1–13, 2017.

[21] G. Catino, P. Mastrolia, D. D. Monticelli, and Marco Rigoli. On the geometry of gradient einstein-type manifolds. *Pacific J. Math.*, 286(1):39–67, 2017.

[22] G. Catino and L. Mazzieri. Gradient einstein solitons. *Nonlinear analysis*, 132:66–94, 2016.

[23] L. Cerbo and M. Disconzi. Yamabe solitons, determinant of the laplacian and the uniformization theorem for riemann surfaces. *Letters in Mathematical Physics*, 83:13–18, 01 2008.

[24] P. Chen and J. Feng. Cosmological constant as an integration constant. *Eur. Phys. J. C*, 84(12), 2024.

[25] S. S. Chern, W. H. Chen, and K. S. Lam. *Lectures on differential geometry.* 1999.

[26] S. Chervon, V. Shchigolev, and V. Zhuravlev. Nonlinear fields in models of cosmological inflation. *Russian Physics Journal*, 39:139–145, 02 1996.

[27] G. Clément and K. Nouicer. Cotton gravity is not predictive. *Phys. Lett. B*, 856:138947, 2024.

[28] G. Colombo, L. Mari, and M. Rigoli. Einstein-type structures, Besse's conjecture, and a uniqueness result for a φ-CPE metric in its conformal class. *J. Geom. Anal.*, 32(11):Paper No. 267, 32, 2022.

[29] J. Costa, R. Diógenes, N. Pinheiro, and E. Ribeiro Jr. Geometry of static perfect fluid space-time. *Classical and Quantum Gravity*, 40(20):205012, sep 2023.

[30] F. Coutinho, R. Diógenes, B. Leandro, and E. Ribeiro. Static perfect fluid space-time on compact manifolds. *Class. Quantum Grav.*, 37(1):015003, 2020.

[31] F. Cremona, L. Pizzocchero, and O. Sarbach. Gauge-invariant spherical linear perturbations of wormholes in einstein gravity minimally coupled to a self-interacting phantom scalar field. *Phys. Rev. D*, 101:104061, 2020.

[32] M. S. R. Delgaty and K. Lake. Physical acceptability of isolated, static, spherically symmetric, perfect fluid solutions of einstein's equations. *Comput. Phys. Commun.*, 395(115), 1998.

[33] A. Derdzinski. Riemannian metrics with harmonic curvature on 2-sphere bundles over compact surfaces. *Bull. Soc. Math. France*, 116:133–156, 1988.

[34] A. Derdzinski and C. L. Shen. Codazzi tensor fields, curvature and pontryagin forms. *Proc. London Math. Soc.*, 47(3):15–26, 1983.

[35] D. DeTurck. Deforming metrics in the direction of their ricci tensors. *Journal of Differential Geometry*, 18:157–162, 1983.

[36] J. Eells and L. Lemaire. A report on harmonic maps [Bull. London Math. Soc. **10** (1978), no. 1, 1–68; MR0495450 (82b:58033)]. In *Two reports on harmonic maps*, pages 1–68. World Sci. Publ., River Edge, NJ, 1995.

[37] G. F. R. Ellis, H. van Elst, J. Murugan, and J. P. Uzan. On the trace-free einstein equations as a viable alternative to general relativity. *Classical and Quantum Gravity*, 28(22):225007, October 2011.

[38] B. Leandro F. Coutinho and H. Reis. On the fluid ball conjecture. *Ann. Glob. Anal. Geom.*, 60:455–468, 2021.

[39] A. Fardoun and A. Ratto. Harmonic maps with potential. *Calc. Var. Partial Differential Equations*, 5(2):183–197, 1997.

[40] A. Fardoun, A. Ratto, and R. Regbaoui. On the heat flow for harmonic maps with potential. *Ann. Global Anal. Geom.*, 18(6):555–567, 2000.

[41] D. Fermi, M. Gengo, and L. Pizzocchero. On the necessity of phantom fields for solving the horizon problem in scalar cosmologies. *Universe*, 5(3):76, 2019.

[42] D. Fermi, M. Gengo, and L. Pizzocchero. Integrable scalar cosmologies with matter and curvature. *Nuclear Physics B*, 957:115095, 2020.

[43] A. E. Fischer and J. E. Marsden. Manifolds of riemannian metrics with prescribed scalar curvature. *Bulletin (new series) of the American Mathematical Society*, 80(3):479–484, 1974.

[44] A. E. Fischer and J. E. Marsden. Deformations of the scalar curvature. *Duke mathematical journal*, 42(3):519–547, 1975.

[45] L. Gårding. An inequality for hyperbolic polynomials. *J. Math. Mech.*, 8:957–965, 1959.

[46] G.Colombo, F. Mastropietro, and M. Rigoli. On the geometry of cotton gravity. *preprint, submitted for publication.*

[47] A. R. Gover and P.A. Nagy. Four-dimensional conformal c-spaces. *Quarterly journal of mathematics*, 58(4):443–462, 2007.

[48] A. R. Gover and P. Nurowski. Obstructions to conformally einstein metrics in n dimensions. *J. Geom. Phys.*, 56(3):450–484, 2006.

[49] M. J. Gursky, X. Cao, and H. Tran. Curvature of the second kind and a conjecture of nishikawa. *Comm. Math. Helvetici*, 2021.

[50] R. S. Hamilton. The ricci flow on surfaces. *Contemporary Mathematics*, (71):237–261, 1988.

[51] J. Harada. Emergence of the Cotton tensor for describing gravity. *Phys. Rev. D*, 103(11):Paper No. L121502, 6, 2021.

[52] J. Harada. Cotton gravity and 84 galaxy rotation curves. *Phys. Rev. D*, 106(6):064044, 2022.

[53] G. H. Hardy, J. E. Littlewood, and G. Pólya. *Inequalities*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1988. Reprint of the 1952 edition.

[54] S. W. Hawking and G. F. R. Ellis. *The large scale structure of space-time*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, anniversary edition, 2023. With a foreword by Abhay Ashtekar.

[55] Chenxu He, Peter Petersen, and William Wylie. On the classification of warped product einstein metrics. *Communications in Analysis and Geometry*, 20:271–311, 2010.

[56] M. Herzlich. Computing asymptotic invariants with the ricci tensor on asymptotically flat and asymptotically hyperbolic manifolds. *Annales Henri Poincaré*, 17(12):3605–3617, May 2016.

[57] G. Huang and H. Li. On a classification of the quasi yamabe gradient solitons. *Methods and applications of analysis*, 21(3):379–390, 2014.

[58] J. L. Kazdan. Unique continuation in geometry. *Comm. Pure Appl. Math.*, 41(5):667–681, 1988.

[59] O. Kobayashi and M. Obata. Conformally-flatness and static space-time. In *Manifolds and Lie groups (Notre Dame, Ind., 1980)*, volume 14 of *Progr. Math.*, pages 197–206. Birkhäuser, Boston, MA, 1981.

[60] D. Kramer, H. Stephani N. D. E. Herlt, and M. MacCallum. *Exact solutions of Einstein's field equations.* Cambridge University Press, England, 1980.

[61] R. A. Lafuente and A. Thompson. Expanding ricci solitons and higgs bundles, 2025.

[62] L. Lemaire. On the existence of harmonic maps. 1977.

[63] L. Lindblom. Some properties of static general relativistic stellar models. *J. Math. Phys.*, 21(6):1455–1459, 1980.

[64] A. Linde. *Particle Physics and Inflationary Cosmology.* Gordon and Breach, 1990.

[65] B. List. Evolution of an extended Ricci flow system. *Comm. Anal. Geom.*, 16(5):1007–1048, 2008.

[66] J. Lott. On the long-time behavior of type-iii ricci flow solutions. *Mathematische annalen*, 339(3):627–666, 2007.

[67] J. Lott. Dimensional reduction and the long-time behavior of ricci flow. *Commentarii mathematici Helvetici*, 85(3):485–534, 2010.

[68] J. Lott and N. Sesum. Ricci flow on three-dimensional manifolds with symmetry. *Commentarii mathematici Helvetici*, 89(1):1–32, 2014.

[69] C. A. Mantica and L. G. Molinari. Riemann compatible tensors. *Colloq. Math*, 128:197–210, 2012.

[70] C. A. Mantica and L. G. Molinari. Weyl compatible tensors. *Int. J. Geom. Meth. Mod. Phys.*, 11:1450070 (15 pp), 2014.

[71] C. A. Mantica and L. G. Molinari. Codazzi tensors and their spacetimes, and cotton gravity. *Gen. Relativ. Gravit.*, 55(62), 2023.

[72] C. A. Mantica and L. G. Molinari. Note on conserved currents in static conformal killing gravity. *Phys. Rev. D*, 112:024070, Jul 2025.

[73] C. A. Mantica, L. G. Molinari, S. Nájera, and R. A Sussman. Response to a critique of "cotton gravity". *preprint*, 2024.

[74] L. Marini and M. Rigoli. On the geometry of φ-curvatures. *J. Math. Anal. Appl.*, 483(2):22 pp, 2020.

[75] P. Miao and L. F. Tam. Some functionals on compact manifolds with boundary. *Mathematische Zeitschrift*, 286(3-4):1525–1537, 2017.

[76] R. Müller. Ricci flow coupled with harmonic map flow. *Ann. Sci. Éc. Norm. Supér. (4)*, 45(1):101–142, 2012.

[77] M. Obata. Certain conditions for a Riemannian manifold to be isometric with a sphere. *J. Math. Soc. Japan*, 14:333–340, 1962.

[78] T. Oliynyk. Future stability of the flrw fluid solutions in the presence of a positive cosmological constant. *Commun. Math. Phys.*, 346:293–312, 2016.

[79] B. O'Neill. *Semi-Riemannian Geometry With Applications to Relativity, 103, Volume 103 (Pure and Applied Mathematics)*. Academic Press, 1983.

[80] S. Pigola, M. Rigoli, M. Rimoldi, and A. Setti. Ricci almost solitons. *Annali Scuola Normale - Classe di Scienze*, page 757–799, January 2012.

[81] S. Pigola, M. Rigoli, and A. G. Setti. *Vanishing and finiteness results in geometric analysis*, volume 266 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2008. A generalization of the Bochner technique.

[82] S. Rahman and M. Visser. Spacetime geometry of static fluid spheres. *Class. Quant. Grav.*, 935(19), 2002.

[83] A. Ratto. Harmonic maps with potential. *Rend. Circ. Mat. Palermo (2) Suppl.*, 49:229–242, 1997.

[84] M. Reiris. Static solutions from the point of view of comparison geometry. *Journal of Mathematical Physics*, 53(1), January 2012.

[85] D. C. Robinson. A simple proof of the generalization of israel's theorem. *General Relativity and Gravitation*, 8(8):695–698, 1977.

[86] Y. Shen. A note on fischer–marsden's conjecture. *Proc. Amer. Math. Soc.*, 125:901–905, 1997.

[87] A. Masood ul Alam. Proof that static stellar models are spherical. *General Relativity and Gravitation*, 39:55–85, 2007.

[88] R. Wald. *General Relativity*. Chicago University Press, 2010.

[89] L. F. Wang. On noncompact quasi yamabe gradient solitons. *Differential Geometry and its Applications*, 31(3):337–348, 2013.

[90] M. B. Williams. Results on coupled ricci and harmonic map flows. *Advances in geometry*, 15(1):7–26, 2015.