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ORIGINAL RESEARCH

Inferential semantic contamination, harmony and realist pollution

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Abstract
This paper investigates the problem of semantic pollution for modal proof-systems. The author proposes that, when proof-systems are used to build antirealist theories of meaning, the real problems lurking behind semantic pollution are: inferential semantic contamination and realist pollution. The first notion deals with misbehavior of meaning as inferentially conceived, and is grounded on the idea of dependence of meaning between logical terms. The second notion deals with pollution with realist theories of meaning. The paper has two main sections. In the first section it is shown that both traditional proof-systems and labeled proof-systems for modal logics are problematic because of inferential semantic contamination. In the second section it is argued that, even if labeled systems were harmonious and semantically non-contaminated, they could still be polluted with realist meaning.

Keywords Modal logic · Semantic pollution · Labeled systems · Hypersequents · Harmony

1 Introduction

According to a well-established inferential tradition, theories of meaning should be developed using proof-theory as its core, without reference to model-theoretic elements.[1] This approach clearly rivals the traditional model-theoretic approach toward semantics, and as a consequence the Davidsonian treatment of meaning in natural

[1] See Dummett (1991), Brandom (2000) and Francez (2015).

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languages.[2] Not surprisingly, this inferential approach focuses mainly on logic, considered the central topic for testing the tools developed, and then, at a later time, moves toward non-logical language.[3] The work of Gentzen provides a good framework for developing this idea, since it offers both[4]:

- Proof-systems in which rules arguably define the meaning of logical constants;
- A series of internal *criteria* for correctness of systems that do not rely on truth preservation.

However, while Gentzen's original proof-systems proved to be well-suited for the treatment of propositional and first-order classical and intuitionistic logics, the treatment of modal logics has been more problematic.[5]

Among the systems proposed for modal logics, the ones developed by Sara Negri attracted the criticism of being semantically polluted, because of the deployment of labels referring to possible words and accessibility relations of Kripke's structures.[6] A calculus is usually considered semantically polluted if it relies on semantic (interpreted as model-theoretic) elements. Regarding modal systems, the argument is essentially that if we use the elements of a Kripke's structure to give meaning to modal terms, we are not really defining *inferentially* the meaning of these terms, even if the elements of the Kripke's structure occur inside some rules of inference.[7]

I argue that this approach toward semantic pollution has two main flaws:

1. it focuses on a model-theoretic interpretation of meaning, since it reserves the term *semantics* for this kind of pollution with elements of denotational models;
2. it takes for granted that the labels for possible worlds cannot be interpreted inferentially.

Hence, philosophers working in inferentialism should focus on two issues, lurking behind the usual approach toward semantic pollution.

About the first point, I claim that something like semantic pollution for inferential meaning should be investigated as well, especially when working inside a general inferential framework: a proof-system can lack purity even without any correlation with model-theoretic elements. In the framework of inferentialism, in order to avoid any confusion with the well-established concept of *semantic pollution*, I propose to use the term *semantic contamination* (opposed to *semantic non-contamination*) for

[2] See Davidson (1969).

[3] Both for mathematical theories (already in Gentzen (1969c) and Prawitz (1971)) and natural language (mainly through the work of Francez, see Francez (2015)).

[4] Gentzen (1969b).

[5] See Wansing (2002) for an overview of the problems faced in treating modal logic with standard sequent calculi.

[6] For Negri's systems, see Negri (2005) and Negri and von Plato (2011). For the criticism see Poggiolesi and Restall (2012).

[7] For an overview of different attitudes toward semantic pollution, see De Martin Polo (2024). I thank an anonymous referee for suggesting me this article.

these *phenomena* that regard only inferential meaning and treat elements inspired by model-theory as any other element. Not surprisingly, we will see that semantic contamination is heavily connected (although not in an obvious way) with purity of rules and separability.

As for the second point, I use the term *realist pollution* (opposed to *inferential purity*) to refer not only to the deployment of model-theoretic elements in the construction of a proof-system, but to the need to interpret these elements in a way that is incompatible with an antirealist conception of meaning.[8] The most obvious example of such an interpretation is one that assumes that labels refer to possible worlds intended as real entities, but as we will see there are other, less obvious, examples.[9]

I deal with inferential semantic contamination in Sect. 2. In Sect. 2.1, I characterize this notion as applied to both rules and derivations in natural deduction, and evaluate whether it is detrimental to the acceptability of a proof-system. Then, in Sect. 2.2 I extend this discussion to sequent calculi, by taking into account both the object-language and the metalinguistic interpretation of the sequent arrow. Section 2.3 applies the notion of inferential semantic contamination to Read's natural deduction systems and Negri's sequent calculi for modal logics, concluding that the problems of their systems go beyond their supposed pollution with realist labels.

I deal with realist pollution of proof-systems for modal logic in Sect. 3. Stephen Read has argued that by giving a proof-system that satisfies the structural restrictions imposed by Gentzen (and then extended and strengthened by others) we provide an inferential interpretation of any of the elements that occur in the proof-system, and so that if Negri's systems respect this proviso then the labels for possible words can be interpreted inferentially.[10] Read's argument entails that, as opposed to what is generally agreed, *realist pollution* does not occur in Negri's calculus. As just stressed, in the previous section it is claimed that Negri's labeled system fails the proof-theoretic *criteria* independently of this accusation of realist pollution. In this section, I assume (for the purpose of discussion) that this system satisfies those *criteria*, and evaluate Read's argument that in this case a realist pollution would not be in place. In Sect. 3.1, I briefly explain the relation between formal systems and natural language, by focusing on a usually neglected example about the counterfactual conditional in Dummett (1991). Then, in Sect. 3.2, I present Dummett's rejection of the modal axiom B for metaphysical necessity and argue against Read's position: *realist pollution* can be a problem for proof-systems *formally* acceptable from an inferential point of view.

[8] For antirealism about meaning, see Dummett (1991).

[9] I thank an anonymous referee for stressing the need to avoid terminological confusion between semantic contamination and semantic pollution. I also thank him for suggesting the need to further clarify the term *realist pollution*

[10] Read (2015)

2 Inferential semantic contamination

In this section, we investigate semantic contamination, and so we focus on inferentialism and avoid any discussion about pollution with model-theoretic elements. Hence, if labels inspired from model-theoretic semantics occur in a calculus, they are checked with the usual *criteria*, as any other element of the system. We start by defining the notion of semantic contamination for natural deduction systems. Then, in the next subsection, we adapt this definition to traditional sequent calculi. We observe that the semantic interpretation of sequent calculi heavily relies on their relation with natural deduction, even though the explicit formulation of structural rules in sequent calculi poses new problems of inferential semantic contamination. Then, in the last subsection we deal with labeled systems, dealing with both labeled natural deduction and sequent systems.

2.1 Contamination in natural deduction

Inferential semantic contamination is a rather vague concept, which as we will see can be interpreted in different ways, depending on whether it is considered as a property of rules or of derivations. In this subsection, we will begin by focusing on contamination of the rules, starting with the meaning-conferring ones, and then move on to contamination of the derivations.

2.1.1 Impure I-rules

According to Dummett, a rule is pure iff only one term occurs in it, and impure otherwise.[11] I stay with this definition and evaluate the consequences of having pure or impure meaning-defining rules. If we endorse an inferential approach toward meaning, we should conclude that semantic contamination (as opposed to *semantic* non-contamination) follows from this syntactic conception of purity for the rules, and vice-versa. That is, if more than one term occurs schematically in a meaning-defining rule, we have a contamination between the meaning of these terms. A quite obvious example is given by the rules for intuitionistic negation in natural deduction (the meaning of $\neg$ and $\bot$ are intertwined):

$$\dfrac{\begin{array}{c}[A]\\ \bot\end{array}}{\neg A}\,\neg\mathrm{I} \qquad\qquad \dfrac{A \qquad \neg A}{\bot}\,\neg\mathrm{E}$$

Less obvious examples are given by disjunctive syllogism and Milne's rule for the introduction of the classical material conditional[12]:

[11] Of course, the requirement regards rules intended schematically, while any number of terms may occur in any particular application of a pure rule (I thank an anonymous referee for stressing this point). See Dummett (1991), pp. 256–7. For sequent calculi, this property is called *separation* in Wansing (2002) and Negri (2007).

[12] The first is a traditional inference rule already discussed by the stoics and famously rejected by relevant logicians, while for the second see Milne (2002).

$$\frac{A \vee B \qquad \neg A}{B}\,\mathrm{DS} \qquad\qquad \frac{\begin{matrix}[A]\\ B \vee C\end{matrix}}{(A \supset B) \vee C}\supset\mathrm{I}$$

Taking for granted that they are both meaning-conferring, the first suggests that the meaning of negation contaminates the meaning of disjunction (and/or *vice versa*), while the second suggests that the meaning of disjunction contaminates that of the classical conditional.

It should be clear that there are no reasons to impose purity on rules, and indeed, even though the rules for negation rely on the constant for absurdity, they are generally regarded as non-problematic.[13] Hence, this kind of semantic contamination for the rules should be accepted, at least *prima facie*. However, as we will see, accepting impurity does not mean that we should not impose any restrictions on the levels and ways of this *phenomenon*.

First of all, it is clear that purer rules should be preferred when available: if it is possible to characterize the meaning of a term without referring to that of others, this approach should be followed in the proof-system. Moreover, semantically contaminated *meaning-conferring rules* seem to impose a dependence of meaning between terms, and this suggests some restrictions on how this relation can be structured.[14] More specifically, it becomes reasonable to impose an ordering in the occurrence of terms in rules, to avoid circularity of the relation of dependence of meaning. As for natural deduction, this restriction applies to I-rules, since these are considered to be the meaning-conferring rules (both for verificationist and compositional reasons). More specifically, in natural deduction[15]:

- I-rules are meaning-conferring;
- E-rules are justified if they satisfy a formal *criterion* usually labeled “harmony”, which establishes that E-rules are somehow a consequence of their respective I-rules.

We will say something more about harmony in the next subsection and stay on I-rules here. Let us first of all give a formal definition for meaning-dependence:

Definition 1 (*Dependence of Meaning*) For every pair of logical terms $\ominus$ and $\oplus$, the meaning of $\oplus$ depends on the meaning of $\ominus$ ($\ominus \prec \oplus$) iff there is a sequence of logical terms $\circ_1, \dots, \circ_n$ such that $\circ_1 = \ominus$, $\circ_n = \oplus$ and for every $1 \leq i < n$, $\circ_i$ occurs in the premisses or in the discharged assumptions of an I-rule for $\circ_{i+1}$.

[13] Indeed even Dummett pointed out that it is an “exorbitant” demand; see Dummett (1991), p. 257.

[14] Investigations of meaning-dependence in proof-theory are not well developed yet. See Cozzo (1994) (pp. 246–250) for an investigation on non-logical vocabulary, Milne (2002) for the first research regarding logical language and Ceragioli (2022) for a more recent development.

[15] Gentzen (1969b) and Prawitz (1965).

This relation is clearly semantic since grounded on meaning-conferring rules.[16] Moreover, it should be noted that the syntactic occurrences of logical terms in the I-rules for a given logical term do not render the whole picture of its meaning-dependence relations with other terms: we need to *trace back* those terms to the ones used to define them, and so on. Hence, this notion is a far from trivial supplementation of the usual tools of philosophical analysis for proof-systems. With this definition in place, a plausible requirement that we should impose on the ordering of the dependence of meaning seems to be the following:

Definition 2 (*Non-circularity*) For every pair of distinct logical constants $\oplus$ and $\ominus$, it does not hold that both $\ominus \prec \oplus$ and $\oplus \prec \ominus$.

Where $\ominus \prec \oplus$ (the meaning of $\oplus$ depends on the meaning of $\ominus$) in natural deduction means that there is a sequence of logical terms $\circ_1, \ldots, \circ_n$ such that $\circ_1 = \ominus$, $\circ_n = \oplus$ and for every $1 \leq i < n$, $\circ_i$ occurs in the premises or in the discharged assumptions of an I-rule for $\circ_{i+1}$. As an example, the occurrence of $\bot$ in $\neg$I can be seen as establishing a dependence relation between the meaning of absurdity and that of negation. Hence, non-circularity suggests that the meaning of absurdity should not be explained using negation, that is, $\neg$ should not occur neither in the premises nor in the discharged assumptions of any (alleged) I-rule for $\bot$.

This seems to be the only clause on the acceptability of I-rules that are impure. Since lack of purity for I-rules is acceptable in general, I will not use the term *semantic contamination* to refer to this phenomenon, but just *impurity*:

Definition 3 (*Impure I-rule*) An I-rule $\oplus$I is impure iff there is at least one logical term $\ominus$ that occurs in it (with $\oplus \neq \ominus$).

Impure I-rules impose dependence of meaning, but not contamination *stricto sensu*.

The charge of semantic pollution is usually raised against proof-systems in general, without specifying which one of its components is responsible. Moreover, it is generally regarded as a flaw in the system, that prevents it from properly characterizing the meaning of the terms that it governs. As for inferential semantic contamination, we have seen how meaning-conferring rules can be impure, argued that there is

[16] An anonymous referee suggested that the relation is *also* clearly syntactic or formal, since it depends only on the schematic forms of the rules. Surely, it can be interpreted as syntactic. Notice however that the syntactic and semantic aspects of the relation can be easily distinguished. As an example, a purely notational change would be semantically irrelevant, leaving intact the inferential roles, but change the syntactic relations between logical terms. The observation that the relation is formal is more interesting. Cozzo (1994) investigated the non-formal meaning-dependence established by atomic rules. On the contrary, we chose to investigate only logical rules and, as correctly observed by the referee, we assume that the focus on rule *schemata* is sufficient to detect meaning-dependence. This is not uncontroversial. Dummett seems to consider the *applications* of rules as relevant for meaning-dependence, as emerges for example from his 'complexity *criterion*' for I-rules: see Dummett (1991), p. 258. We assume that Dummett's reason for focusing on *applications* of rules regards atomic rules, such as Prawitz's harmonious and yet paradoxical pair of rules for set abstraction: see Prawitz (1965), pp. 94–95. Hence, since we focus only on logical rules, we can consider only schematic rules instead of their applications for the characterization of meaning-dependence. This remains true also when, in Sect. 2.3, we consider rules in which relational atoms occur.

nothing intrinsically wrong about it, and so decided to drop the label of *contamination* for this phenomenon. However, there are at least two other ways of interpreting the notion of inferential semantic contamination: as applied to E-rules and as applied to derivations in general. Let us check them one by one.

2.1.2 Semantically contaminated E-rules

Impure I-rules raise the issue of how their respective E-rules should be constituted. If we consider the full requirement of harmony, which certifies whether an E-rule matches its respective I-rule, the problem is quite complicated already for pure I-rules. Its extension to impure I-rules is even more controversial, especially if other restrictions are dropped.[17] There are different non-equivalent *criteria* for harmony in natural deduction, all inspired by Gentzen's original ideas that E-rules should somehow *follow* from I-rules. In general, it is agreed that Prawitz's Inversion Principle is at least a necessary condition for the justification of E-rules[18]:

> Let α be an application of an elimination rule that has B as consequence. Then, deductions that satisfy the sufficient condition $[\cdots]$ for deriving the major premiss of α, when combined with deductions of the minor premisses of α (if any), already "contain" a deduction of B; the deduction of B is thus obtainable directly from the given deductions without the addition of α.

Formally, this principle requires the availability of a reduction procedure for each maximal-formula, that is for any formula that is the conclusion of an I-rule and the major premise of an E-rule. More specifically, there must be an algorithmic rewriting procedure such that, for every derivation in which occurs a formula that is the conclusion of an I-rule and the major premise of an E-rule, the procedure rewrites the derivation removing this formula. Sometimes the Inversion Principle is complemented with the stronger requirement of normalization, which states the termination of the reduction procedure.[19] Luckily, in what follows we will only need to assume that the Inversion Principle is a necessary condition for harmony, without discussing whether it is also sufficient (as claimed by Read (2010)) or it must be complemented with normalization (as claimed by Prawitz (1971)).[20] Moreover, absence of semantic contamination for E-rules can in general be discussed as an independent necessary *criterion* for E-rules, so we will avoid discussing harmony when it is not strictly necessary in the economy of the paper. We will see that in most cases this *prima facie* evaluation is already sufficient to reject a proof-system, without discussing harmony.

Let me, first of all, propose a definition of semantic contamination for elimination rules, and then defend this proposal. The characterization that I propose is the following:

[17] See Ceragioli (2022).

[18] Prawitz (1965), p. 33.

[19] See Prawitz (1965), Negri and von Plato (2001) and Murzi and Steinberger (2017).

[20] This assumption will be used in Sect. 2.3 to evaluate Read's labeled natural deduction systems.

Definition 4 (*Semantic contamination for E-rules*) A rule $\otimes$E is semantically contaminated iff $\oplus$ occurs in it and it is such that $\oplus \not\prec \otimes$.

As for its justification, just consider that since E-rules should be nothing more than a consequence of their I-rules, there cannot be any justifications for the occurrence in it of logical terms that do not occur in the respective I-rules or in the I-rules for terms on which the one under analysis depends. So, when this happens I claim that the elimination rule is semantically contaminated by the meaning of those terms that unjustifiably occur in them.

It goes without saying that semantically contaminated E-rules should be avoided, if we want logical theorems to be analytic truths, that is true in virtue of the meaning of the logical terms that occur in them.[21] Hence, only logical terms that depend on its main connective can occur in an E-rule, in order for the proof-system to be inferentially acceptable. However, even though semantic non-contamination qualifies as a necessary *criterion* for proof-theoretic acceptability, it cannot be sufficient as well. Indeed, while Prior's rules for the infamous connective *tonk* are clearly semantically non-contaminated, they cannot be regarded as proof-theoretically acceptable. Granted this limitation, we will see that by relying on semantic contamination it is already possible to draw interesting conclusions about many modal systems.

Admittedly, some clarifications should be given about how to discern a connective that occurs as the main connective of an E-rule and one that occurs as part of the context. That is, in definition 4 we take for granted that the rule is an E-rule for $\otimes$ and not for $\oplus$, although both these terms occur in it, and this interpretation certainly needs some discussion. As an example, assuming the standard set of I-rules for conjunction and implication, the rules

$$\frac{A \supset B \qquad A \wedge C}{B} \qquad\qquad \frac{A \supset B \qquad A \wedge C}{C}$$

are clearly contaminated E-rules (since neither $\wedge \prec \supset$ nor $\supset \prec \wedge$), but it is less clear whether they count as E-rules for conjunction or implication. We will postpone this issue until the investigation of semantic contamination in the framework of sequent calculus, when a more precise characterization of L-rules (which, as we will see, correspond to E-rules in sequent calculus) will be provided.

2.1.3 Semantically contaminated derivations

The derivations of a proof-system can be semantically contaminated as well, and this notion should not be conflated with that of impurity of I-rules or semantic contamination for E-rules. A proof is semantically contaminated if it deploys semantic elements that are not needed to understand the meaning of the statement of the theorem. More generally, a derivation is semantically contaminated if it deploys semantic elements

[21] Even though we take both of them for granted, the analytic/synthetic distinction and the analytic status of logic are, of course, controversial. Moreover, even the meaning of 'analytic' is contentious, since Boghossian (1996) proposed to characterize it epistemically, without requiring truth in virtue of meaning. For a defense of the analytic/synthetic distinction that rejects Boghossian's reformulation, see Russell (2008).

that are needed to understand the meaning of neither its open assumptions nor its conclusion. Since one of the grounding ideas of inferentialism is that meaning is conferred by the rules, the natural formal counterpart of semantic non-contamination for derivations is separability, defined as:

Definition 5 (*Separability*) To prove a logical consequence $\Gamma \vdash C$ we only need to use the rules for the logical constants that occur in Γ or C, together with the rules for the constants on which those depend. That is, in order to prove a logical consequence $\Gamma \vdash C$, it is enough to use the rules for the constants $\circ_1, \ldots, \circ_n$ such that for every $1 \leq i \leq n$:

- $\circ_i$ occurs in Γ or C; or
- for some $j \neq i$ such that $1 \leq j \leq n$, $\circ_j$ occurs in Γ or C and $\circ_i \prec \circ_j$.

Hence, semantic contamination corresponds to a lack of separability.

Although contamination for E-rules concerns occurrences of connectives and contamination for derivations concerns occurrences of rules, it seems obvious that contaminated E-rules can lead to contaminated derivations. Indeed, when applied in a derivation, the logical terms that contaminate the E-rule can be the main connective in the conclusion of an I-rule that contaminates the derivation. However, since as already claimed the rules for *tonk* are non-contaminated, we have clear examples of semantically contaminated derivations in which only semantically non-contaminated rules occur, such as:

$$\dfrac{\dfrac{\dfrac{[A]}{A \supset A}\,{\supset}\mathrm{I}}{(A \supset A) tonk (B \wedge C)}\,tonk\mathrm{I}}{B \wedge C}\,tonk\mathrm{E}$$

Given its conclusion, the only rules that could occur in this derivation are $\wedge$-rules, so both $\supset$I and the rules for *tonk* contaminate the derivation, even though they are all semantically non-contaminated rules. Hence, the notions of semantic contamination for rules and for derivations should not be conflated.

It is in these two detrimental senses of the term that logicians should avoid semantic contamination: a proof-system should not have semantically contaminated E-rules and should be separable. Let me be clear, as for semantic non-contamination of E-rules, separability is not the full harmony constraint, and so constitutes a necessary but not sufficient *criterion* of inferential acceptability. However, it can be used to obtain relevant results. As an example, it is sufficient to show the inadequacy of the natural deduction system for classical logic that results from substituting classical *reductio ad absurdum*

$$\dfrac{\begin{array}{c}[\neg A]\\ \bot\end{array}}{A}\,\mathrm{RAA}$$

in place of *ex falso quodlibet*. Indeed, since the implicational fragment of classical logic is a proper extension of its intuitionistic correspondent, this extension contaminates any derivation of a purely implicational classical theorem that is not intuitionistically valid with an application of a rule for negation. Hence, we can reject this proof-system for classical logic without even discussing the full requirement of harmony.

2.2 Contamination in sequent calculi

We have introduced semantic contamination in natural deduction because these proof-systems have a less controversial individuation of the meaning-conferring rules, and this makes it easier to characterize meaning-dependence between logical terms. Moreover, Gentzen's inferential theory of meaning and its generalizations are based on natural deduction, of which sequent calculus is presented just as an evolution, needed to prove some results more easily. However, most of the modal proof-systems have been developed using (generalizations of) sequent calculi, so we need to discuss semantic contamination for them as well.

At first glance, in sequent calculus it is not clear which rules count as meaning-conferring and which ones as "derived". In order to settle this issue, we have to consider that sequent calculus has two natural readings[22]:

Metalinguistic The sequent arrow is interpreted as a (eventually multiple-conclusion) derivability relation between the premises on the left and the conclusions on the right, and so the calculus in itself is interpreted as a proof-system for these metalinguistic facts;
Conditional The sequent arrow is interpreted as an object-language conditional, so that the calculus is a proof-system for object-language sentences.

The same sequent $\Gamma \Rightarrow \Delta$ is interpreted as $\Gamma \vdash \Delta$ according to the first reading, and as $\bigwedge \Gamma \supset \bigvee \Delta$ according to the second. When Gentzen first proposed his sequent calculus LK in Gentzen (1969a), he explicitly interpreted it in the second way and proved its adequacy according to this reading. On the other hand, Prawitz interpreted sequent calculus as a meta-calculus for natural deduction, as clearly results by his proof of their equivalence.[23]

According to the first reading, sequent rules explain how to construct derivations *from the outside*, since in it "the sentences [...] are mentioned rather than used".[24] This suggests a correspondence between I-rules and R-rules, and respectively between E-rules and L-rules. More specifically, R-rules describe modifications of the corresponding derivation by I-rules applied to their conclusion

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B}\,\mathrm{R}\vee \quad \rightsquigarrow \quad \dfrac{\begin{array}{c}\Gamma\\ \vdots\\ A\end{array}}{A \vee B}\,\vee\mathrm{I}$$

[22] See Shoesmith and Smiley (1978) (p. 33) for this distinction between conditional and metalinguistic interpretation of sequent calculus.

[23] See Prawitz (1965), Appendix A.

[24] Rumfitt (2000), p. 795.

while L-rules describe modifications of their open assumptions by E-rules

$$\frac{\Gamma, A \Rightarrow C}{\Gamma, A \wedge B \Rightarrow C}\,\mathrm{R}\wedge \quad \rightsquigarrow \quad \begin{array}{cc} \Gamma & \dfrac{A \wedge B}{A}\,\wedge\mathrm{E} \\ & \vdots \\ & C \end{array}$$

Of course, when more than one formula can occur in the succedent, then metalinguistic reading of the calculus refers to a multiple-conclusion reformulation of natural deduction. Nonetheless, this should not discourage our endorsement of this reading.[25]

According to the second reading, the rules of sequent calculus work more or less as natural deduction rules, since they modify object language sentences. However, interpreted this way, all the rules of sequent calculus are I-rules (since they always introduce new connectives in the conclusion) and they are not only impure, but even complex, that is some logical terms occur in subordinate position in their premises and conclusion[26]:

$$\frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \supset B, \Delta}\,\mathrm{R}\supset \quad \rightsquigarrow \quad \frac{\bigwedge(\Gamma \cup \{A\}) \supset \bigvee(\Delta \cup \{B\})}{\bigwedge\Gamma \supset \bigvee(\Delta \cup \{A \supset B\})}\,\supset\mathrm{I}$$

We will not discuss this interpretation of sequent calculus in detail, since a proper inferentialist study of it would lead us too far astray. Anyway, neither harmony nor avoidance of semantic contamination are clearly provable for sequent calculi, when they are interpreted in this way. As an example, the rule of Cut corresponds to a generalization of the transitivity of the conditional, which in itself has nothing to do with maximal formulas or harmony. Hence, Cut-elimination does not entail harmony according to this interpretation of the sequent arrow.[27]

Let us come back to the first reading. If R-rules correspond to I-rules, each of them defines the meaning of the logical term that gets introduced in its conclusion, where we impose that each R-rule can introduce only one such connective in the right-hand side of the conclusion and none in its left-hand side, alongside those that occur both in the premises and in the conclusion. If, apart from the one introduced in the conclusion, other logical terms occur in the R-rule, the rule is *impure* and establishes a meaning-dependence between the main logical term and the other terms occurring in it. Like the E-rules that they describe, L-rules can be *semantically contaminated*. However, as opposed to E-rules, which can have plenty of different structures, L-rules are local and quite homogeneous, and so it is easier to detect their main connective.[28] The main logical term of an L-rule is the one that gets introduced in the

[25] See Boričić (1985) for an early development of multiple-conclusion natural deduction and Read (2000) for a recent reappraisal. Moreover, see Milne (2002) for a different interpretation of multiple-succedent sequent calculus into natural deduction.

[26] For complex rules, see Dummett (1991) (p. 257) and Milne (2002).

[27] Poggiolesi (2010), p. 31 argues that Gentzen's calculus LK is syntactically pure because the sequents can be read as material conditionals. Even though she clearly refers to semantic *pollution* (or at most to what we have labeled *realist pollution*) and not to what we call *semantic contamination*, a proof that LK *so interpreted* is still harmonious should be given, as we will see.

[28] This is not such a minor issue. The objection that the I-rule for the Liar sentence (•I) should be interpreted as an E-rule for negation has been raised against Read's proposal of a harmonious inferential ver-

left-hand side of its conclusion, where we impose that each L-rule can introduce only one such connective in the left-hand side of the conclusion and none in its right-hand side, alongside those that occur both in the premises and in the conclusion. If other logical terms occur in the rule, the *criterion* of semantic contamination is the same holding for E-rules explained in Definition 4.

As an example, let us now evaluate the modal rules for S4 in Troelstra and Schwichtenberg (2000). The $\Box$-rules

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta}\ \mathrm{L}\Box \qquad\qquad \frac{\Box\Gamma \Rightarrow A, \Diamond\Delta}{\Box\Gamma \Rightarrow \Box A, \Diamond\Delta}\ \mathrm{R}\Box$$

seem acceptable, even though a little strange. Indeed, $\Box$R establishes a dependence relation of its main term $\Box$ on $\Diamond$, which $\Box$L (being pure) does not use. Admittedly, even though $\Box$L is clearly acceptable, it could be objected that $\Box$R *is* problematic, since it characterizes necessity using necessity itself, circularly presupposing what it should explain. This situation forces us to reconsider our requirement that dependence of meaning should not be circular. Indeed, even if we regard global meaning holism as unacceptable, circular dependence of meaning can still be plausible for some fragments of the language, such as for the color terms.[29] Hence, possibility and necessity could constitute such a holistic fragment of the (non-holistic) language, and this would explain the evident circularity of $\Box$R. However, this liberalization of the non-circularity *criterion* leads nowhere, as becomes clear as soon as the rules for possibility are checked:

$$\frac{\Box\Gamma, A \Rightarrow \Diamond\Delta}{\Box\Gamma, \Diamond A \Rightarrow \Diamond\Delta}\ \mathrm{L}\Diamond \qquad\qquad \frac{\Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow \Diamond A, \Delta}\ \mathrm{R}\Diamond$$

As opposed to those for necessity, $\Diamond$R is pure and clearly acceptable, while $\Diamond$L not only characterizes the meaning of possibility in a circular way, but is even semantically contaminated, relying on the meaning of necessity, even though $\Box \not\prec \Diamond$. Hence, this system for S4 is clearly problematic because of semantic contamination of its L-rule for possibility, regardless of whether the circularity in its R-rule for necessity is acceptable or not.

Let us evaluate a possible recovery strategy for this system. The rules for necessity of S4 in Troelstra and Schwichtenberg (2000) are a modification of those proposed in Ohinishi and Matsumoto (1957):

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta}\ \mathrm{L}\Box \qquad\qquad \frac{\Box\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}\ \mathrm{R}\Box$$

Apart from the problem of the circular characterization of necessity, for which we decide to suspend judgment, the rules are pure. Unluckily, they are not complete when complemented with their counterpart for possibility, since they cannot prove

sion of the Liar's paradox, see Gabbay (2017), p. S113. See Francez (2018) for a reply.

[29] See Weiss and Kürbis (2024) for a recent discussion about circularity and molecular theories of meaning. Allowing circular dependence of meaning seems necessary also for bilateral proof-systems: see Ceragioli (2022).

the equivalences between $\Box A$ and $\neg\Diamond\neg A$, and between $\Diamond A$ and $\neg\Box\neg A$.[30] Anyway, the system seems fine for characterizing the $\Diamond$-free fragment of S4, being semantically non-contaminated, and so it deserves to be discussed in this regard.

Before discussing this issue, some clarifications about structural rules is needed. Let us first of all define the following properties:

Definition 6 (*Admissibility*) A rule is admissible in a proof-system iff whenever an instance of all its premises is derivable, the corresponding instance of the conclusion is derivable too.

Definition 7 (*Invertibility*) A rule is invertible in a proof-system iff from the derivability of an instance of its conclusion, the derivability of the corresponding instance of the premise (or premises) follows.

Let us moreover define the height of a derivation as the greatest number of successive applications of rules in it. With this notion, we can define:

Definition 8 (*Height preserving Admissibility*) A rule is height-preserving admissible in a proof-system iff whenever an instance of all its premises is derivable with a derivation of height *n*, the corresponding instance of the conclusion is derivable too with a derivation of height *m* such that $m \leq n$.

Definition 9 (*Height preserving Invertibility*) A rule is height-preserving invertible in a proof-system iff, if there is a derivation of height *n* of an instance of its conclusion, then there is a derivation of height *m* of the corresponding instance of its premise (or premises), such that $m \leq n$.

As an example, the rule of Weakening is height-preserving admissible in those proof-systems in which the Axiom can have formulas in the context:

$$\frac{}{\Gamma, A \Rightarrow A, \Delta}\ \text{Axiom}$$

Indeed, when the premise of an application of Weakening is derivable, its conclusion can be derived with a derivation of the same height, just by adding the formula introduced by the application of Weakening to the context of the applications of Axiom. Moreover, in some proof-systems (usually called *à la* Kleene) all the operational rules are height-preserving invertible and both Weakening and Contraction are height-preserving admissible. This result follows from the formulation of the Axiom with context and the context-sharing formulation of the operational rules.[31] However, even though the admissibility of Cut is considered a key property for sequent calcu-

[30] Kripke (1963) and Routley (1975).

[31] Moreover, in some cases (such as for the intuitionistic calculus) a repetition of the main formula of a rule in one of the premises can be needed. See Troelstra and Schwichtenberg (2000) (especially section 3.5) for the details.

lus, its height-preserving admissibility is usually not required, and indeed holds for almost no system.

Height-preserving admissibility of the structural rules and height-preserving invertibility of the operational ones are usually considered desirable properties. However, while their desirability for the efficiency of proof-search is obvious, their link to meaning-theory is much less clear. The invertibility of the operational rules follows from the admissibility of the structural rules, so we will just discuss the reasons for this requirement.

The main reason for requiring the admissibility of structural rules is that when they are not admissible they seem to somehow interfere with the meaning of the logical terms.[32] Indeed, if some logical consequence is provable only using structural rules (which is the standard situation, in Gentzen's original systems), then we have only two alternatives:

1. structural rules play a role in determining the meaning of the consequence, and so the derivation is analytic;
2. the derivation is semantically contaminated by the occurrence of structural rules.

The problem with the second alternative is obvious, since we are trying to avoid semantically contaminated derivations. The first alternative is nonetheless much harder to reject, since structural rules clearly should have a role in proof-theory. It is a standard assumption of sequent calculus that operational rules alone define the meaning of logical terms, while structural rules characterize logical consequence in general.[33] This position seems convincing, at least *prima facie*, but it hardly demands the admissibility of structural rules, even less their height-preserving admissibility. Indeed, there is nothing wrong in relying on the properties of the logical consequence in order to prove that something logically follows from something, and so there are no reasons to require the admissibility of the structural rules that characterize logical consequence.

However, a less stringent property should still be required: structural rules should be restricted to atomic sentences, while their applications to non-atomic formulas should be dispensable.[34] Note that this is already the usual requirement for the rule of Axiom, for which full admissibility is generally not provable.[35] To see the reason for this requirement, consider the extension of a standard system with the left and right rules for some $\ominus$, a new logical term. If the structural rules for non-atomic sentences are not admissible, then we need to extend the system with their application to sentences containing this new term. That is, we are not really extending the system only with $\ominus$R and $\ominus$L, but with some new applications of structural rules as well.[36] Hence, either the meaning of $\ominus$ itself (not only that of consequence) is character-

[32] Poggiolesi (2010), pp. 24–26.

[33] For a recent debate on this issue, see Paoli (2014) and Hjortland (2014).

[34] See Hacking (1979).

[35] See as an example (Poggiolesi, 2010) p. 26.

[36] The situation is analogous to the extension of the Induction Schema with instances containing the truth predicate when the rules

ized and so contaminated by this extension of the structural rules, or otherwise each derivation in which these rules occur is contaminated by these applications of rules that are not meaning-theoretically justified. In both cases, the extension of structural rules with non-admissible non-atomic applications leads to semantic contamination and so should be rejected.

Let us now return to Ohinishi and Matsumoto's system for the $\Diamond$-free fragment of S4. Even though Cut-elimination holds for it, Weakening cannot be restricted to its atomic applications. Indeed, since in $\Box$R the succedent must contain only one formula (the one that is active in the inference) the sequent $\Box A \Rightarrow \Box A, \Box B$ cannot be proved using atomic Weakening, and so non-atomic Weakening is needed to prove the validity of the classical theorem $\Box A \vee (\Box A \supset \Box B)$.[37] Hence, we can conclude that even dropping $\Diamond$ Ohinishi and Matsumoto's system still remains problematic, since its derivations can be contaminated by non-atomic applications or structural rules.

2.3 Negri-Read modal systems

While with ordinary proof-systems for modal logics the problem is proving the desired structural properties such as harmony for natural deduction, and Cut-elimination or admissibility of the structural rules for sequent calculi, with more recent generalizations of these proof-systems (and in particular with labeled systems and tree-hypersequents) these properties are "easily" provable.[38] According to the received wisdom, what becomes problematic is showing that these systems are not semantically polluted: that the occurrence of labels inspired from model-theoretic elements is acceptable. As we have seen, in inferentialist theories of meaning, when proof-systems are interpreted as giving meaning to logical terms, we should focus on two distinct problems: *inferential* semantic contamination and *realist* pollution. In this section we will discuss inferential semantic contamination for these generalized proof-systems, focusing mainly on Read's labeled natural deduction and Negri's labeled sequent calculi, while in the next section we will deal with realist pollution.

The choice of labeled systems has two main reasons: first of all, they are usually the main and more uncontroversial target of the accusation of semantic pollution, and so discussing their status regarding sematic contamination is particularly interesting; secondarily, Negri's sequent calculi can be easily read as meta-systems that describe derivations in Read's natural deduction systems for modal logic. This second point is relevant because our justification of sequent calculus is based on the metalinguistic reading of the sequent arrow, analyzed in Sect. 2.2, which establishes a connection between sequent calculus and natural deduction. In this subsection, we will elaborate further on this connection, by discussing in parallel the flaws of both Read's and

$$\frac{A}{\mathscr{T}(\ulcorner A \urcorner)}\ \mathscr{T}\mathrm{I} \qquad \frac{\mathscr{T}(\ulcorner A \urcorner)}{A}\ \mathscr{T}\mathrm{E}$$

are added to an arithmetical system. See Shapiro (1998).

[37] This fact has been already observed in Hacking (1979) (see also Wansing (2002)). What is new here is its discussion alongside other phenomena of semantic contamination.

[38] See Viganò (2000) for labeled systems and Poggiolesi (2010) for tree-hypersequents.

Negri's systems. Even though we endorse a flavor of proof-theoretic semantics that is based on natural deduction, we have decided to deal also with Negri's sequent calculi, alongside Read's natural deduction systems, because the debate about proof-theory for modal logic seems to focus mainly on sequent calculus.[39]

As starting point, Sara Negri proposes the system G3K, corresponding to the modal logic K. The system has two rules of Axiom, one for labeled sentences (formulas of the form $w : A$) and the other for relational atoms (formulas of the form $w \rhd u$):

$$\Gamma, w \rhd o \Rightarrow w \rhd o, \Delta \qquad\qquad \Gamma, w : p \Rightarrow w : p, \Delta$$

Analogously, in Read's natural deduction systems both labeled sentences and relational atoms can occur as open assumptions. The classical basis of Negri's system is constituted by the operational and structural rules of the system G3c,[40] which are easily reformulated as rules for labeled formulas. The same labeling process gives Read's natural deduction rules for the propositional operators. As an example, Read's I-rules and Negri's R-rule for conjunction are:

$$\frac{w : A \qquad w : B}{w : A \wedge B}\ \wedge\mathrm{I} \qquad\qquad \frac{\Gamma \ \Rightarrow\ w : A, \Delta \qquad \Gamma \ \Rightarrow\ w : B, \Delta}{\Gamma \ \Rightarrow\ w : A \wedge B, \Delta}\ \mathrm{R}\wedge$$

More importantly, in both systems, the rules for necessity and possibility are an inferential reformulation of Kripke's clauses. In particular, Negri proposes the following rules:

$$\frac{\Gamma, o : A, w : \Box A, w \rhd o \ \Rightarrow\ \Delta}{\Gamma, w : \Box A, w \rhd o \ \Rightarrow\ \Delta}\ \mathrm{L}\Box \qquad\qquad \frac{\Gamma, w \rhd o \ \Rightarrow\ o : A, \Delta}{\Gamma \ \Rightarrow\ w : \Box A, \Delta}\ \mathrm{R}\Box$$

$$\frac{\Gamma, o : A, w \rhd o \ \Rightarrow\ \Delta}{\Gamma, w : \Diamond A \ \Rightarrow\ \Delta}\ \mathrm{L}\Diamond \qquad\qquad \frac{\Gamma, w \rhd o \ \Rightarrow\ w : \Diamond A, o : A, \Delta}{\Gamma, w \rhd o \ \Rightarrow\ w : \Diamond A, \Delta}\ \mathrm{R}\Diamond$$

To emulate the universal nature of Kripke's clauses for the truth of a necessary formula and for the falsity of a possible formula, we add the requirement that in $\Box$R and $\Diamond$L o must be a new label, that is it cannot occur in the conclusion of the rule. The same strategy is endorsed by Read, who proposes the following rules:

$$\frac{\begin{array}{c}[w \rhd o] \\ \vdots \\ o : A\end{array}}{w : \Box A}\ \Box\mathrm{I} \qquad\qquad \frac{w \rhd o \qquad w : \Box A}{o : A}\ \Box\mathrm{E}$$

$$\frac{w \rhd o \qquad o : A}{w : \Diamond A}\ \Diamond\mathrm{I} \qquad\qquad \frac{w : \Diamond A \qquad \begin{array}{c}[w \rhd o, o : A] \\ \vdots \\ k : C\end{array}}{k : C}\ \Diamond\mathrm{E}$$

[39] Read's natural deduction system is presented and defended in Read (2008) and Read (2015). The only obstacle for this connection between Read's and Negri's systems is Negri's deployment of multiple-succedent sequents. We will not discuss this issue in detail, but it clearly is a solvable issue: see note 25.

[40] See Negri and von Plato (2001).

Negri's requirement that o must be a new term becomes the requirement that this label must not appear in any other assumption on which the premise of $\Box$I and the minor premise of $\Diamond$E depend, nor in the minor premise of $\Diamond$E. As for the other modal logics, both Read and Negri propose a set of rules that govern relational atoms in order to extend this basic modal system to stronger systems. Nonetheless, before evaluating these extensions, let us discuss Read's and Negri's basic systems in themselves.

According to the intended interpretation, the labels occurring in these systems refer to possible worlds, so that labeled sentences state which sentences hold in which possible worlds and relational atoms state which world is accessible from which world. The precise meaning of these labels should not bother us, because we will devote the entire next section to the relation between formal modal systems and their intended interpretations. Nevertheless, there are some issues regarding the usage of labeled formulas that do not seem to depend upon their interpretation and that we need to address before discussing semantic contamination and harmony for Read's and Negri's systems.

If we conceive Read's labeled system as any other natural deduction, that is as entitling someone to assert the conclusion of a derivation when they endorse its assumptions, the system does not entitle to assert modal sentences but just to assert sentences of form $w : A$ (labeled modal sentences) that contain modal sentences. In other words, the entire labeled formula says something *about* the modal sentence. Even worse, in assertions of form $w \triangleright u$ (relational atoms), modal sentences do not occur at all. By moving from natural deduction to sequent calculus, another layer of detachment is added, according to its metalinguistic reading. Hence, Negri's sequent calculus refers to derivability in Read's natural deduction system which, in turn, refers to sentences that "hold" in some *labels*. As a consequence, by establishing harmony of the system and excluding semantic contamination, we could at most obtain the inferential acceptability of this metalinguistic speech, not of the modal speech which is only mentioned.

The previous considerations clearly hold if we consider the labels in Read's system as referring to possible worlds and the labeled sentences as asserting that the given sentence holds in the world denoted by the label. Indeed, when someone asserts a sentence, they become committed to its truth, but when someone asserts $w : A$, they are not committed to the truth of A.[41] Moreover, I claim that this use/mention problem holds regardless of our interpretation of these labels. Let me explain further my position.[42]

Read argues that the possible-worlds interpretation of the labels is not a necessary ingredient of the system, and that if inferentialism is right, then meaning can be given to labels without the need to find referents for them.[43] Surely I agree with him that

[41] Someone could argue that they are committed to the truth of A in w. However, in order to consider this as a commitment to truth *stricto sensu*—like with the assertion of "there are kangaroos in Australia"—we need *at least* to interpret possible worlds as existing alongside the actual world, severed only spatio-temporally from it. See Lewis (1986).

[42] I really thank an anonymous referee for stressing this issue.

[43] "if inferentialism is right.. meaning (semantics) does not consist solely or properly in a reference to a realm of objects, a model theory, but meaning is already given by the rules for the use of the expression." Read (2015), p. 656.

terms can have meaning without referring to anything. However, there are different ways this can happen, and I think none of them saves labeled sentences from mentioning modal sentences instead of being modal sentences themselves.

Let us check the details of Read's ideas about non-referring labels. To support his claim that reference is not needed, in Read (2015) the author reports Leibniz's position that *calculus* makes sense even if infinitesimal quantities do not exist. With this example, Read seems to mean that it is not important whether possible worlds exist or not, they just work to characterize modal logic, just like infinitesimals can be used in *calculus*. What Read seems to imply is that labels can be used to characterize modal logic without taking any position about the metaphysical status of their referents: as an example, they can be fictional objects and the system still works fine. Even granting this assumption, it changes nothing about the fact that A is not asserted but just mentioned in $w : A$. It changes only the metaphysical status of the object referred to by w, so that now $w : A$ has the same meaning of "the sentence 'A' is true in the fictional possible world w".

Interestingly, the same analysis applies to the interpretation of tree-hypersequents in terms of *circumstances* described in Poggiolesi and Restall. The authors propose the following reading of a tree-hypersequents derivation of $\Diamond A \vee \Diamond B$ from $\Diamond(A \vee B)$[44]

> Suppose $\Diamond(A \vee B)$. So, in *some circumstance*, $A \vee B$. There are two cases: Case (i) A, and Case (ii) B. Take case (i) first. Then in *this circumstance*, A and so, back where we started, $\Diamond A$, and hence $\Diamond A \vee \Diamond B$. On the other hand, we might have case (ii). There we have B, so back in the *original circumstance*, $\Diamond B$, and hence, $\Diamond A \vee \Diamond B$. So, in either case, we have $\Diamond A \vee \Diamond B$, which is what we wanted.

Note that, when dealing with sentences that do not hold in the actual world, the only solution remains to claim that they hold *elsewhere* (using the term of the authors) and this cannot be done by asserting the sentences, but only by asserting something *about* the sentences. Apparently, the only advantage in speaking of circumstances instead of possible worlds is that the first term is less 'philosophically loaded'.[45]

Admittedly, the proposal that labels refer to non-existing or fictional entities is not the only option to dismiss the problem of the referent of the labels. In Read (2008), Read makes a more interesting comparison, claiming that labels for possible worlds should be intended just like the symbol ∞ in the expression $\lim_{n\to\infty} a_n$. Indeed, the term ∞ does not denote any value for n, but it is nonetheless meaningful. This observation is much finer than the previous one regarding infinitesimals in general. However, as it is well known after 19th-century arithmetization of analysis, the behavior of a sequence (convergence, divergence, etc) can be described without making any use of ∞, and it is precisely for this reason that we can regard this symbol as a special non-denoting term. In other words, the term ∞ does not need to have a refer-

[44] Poggiolesi and Restall (2012), p. 54 (*emphasis*: mine).

[45] For an interesting analysis of the philosophical similarities between labeled calculi and tree-hypersequents, see De Martin Polo (2024).

ent, because the expressions in which it occurs are abbreviations for expressions in which only terms for numbers occur. That something similar can be done for labels in labeled calculi seems to be at least doubtful, and the burden of proof is clearly on proponents of this idea. Having clarified this point, we can now address harmony and contamination for Read's and Negri's systems.

Taking for granted the metalinguistic reading of sequent calculi considered in the previous section, R-rules can be conceived as meaning-conferring and L-rules as justified by them. With this assumption, let us now check the status of Negri's systems in respect to dependence of meaning, keeping in mind that any conclusion will automatically apply to Read's systems as well. It is quite clear that the modal R-rules that define the meaning of $\Box$ and $\Diamond$ rely on the relational atoms $w \rhd u$. The dependence of meaning of the modalities on the relational atoms will be the key element in the discussion of semantic contamination.

On the one hand, since the R-rules for necessity and possibility rely only on relational atoms, they define these notions in a quasi-pure way, if compared to the rules of Troelstra and Schwichtenberg (2000) for S4 seen in Sect. 2.2, in which $\Box$ was characterized using both itself and $\Diamond$. Moreover, the L-rules are clearly not semantically contaminated, since the only logical term occurring in them apart from the main one is the term $\rhd$ for the accessibility relation. Also regarding separability, it clearly follows from the Cut-elimination theorem, which can be proved for the system.[46] On the other hand, the dependence of meaning between modal terms and accessibility relation raises the problem of whether accessibility can be harmonically characterized, that is of whether the rules that govern the relational atoms are harmonious. Indeed, it is a natural requirement that if the meaning of a logical term is defined relying on some other terms, the harmony of the *dependent* term should be evaluated together with that of the terms from which it depends.

Before applying this requirement to labeled modal systems, let us consider a simpler example taken from natural deduction for intuitionist logic. Since in natural deduction negation is usually defined using the constant for absurdity, in order to have a proper characterization of negation we have to address the problem of whether the rules for $\perp$ are harmonious or not. Indeed, much of the discussions regarding negation are focused primarily on the rules for absurdity and not on those that characterize the negation using absurdity, since harmony of the rules for absurdity (and its adequacy to describe the intended meaning of this term) is much more controversial. Moreover, the need to properly characterize $\perp$ in order to have a harmonious treatment of $\neg$ seems a point of agreement between researchers in this field.[47] A controversial but quite common solution for absurdity is that *ex falso quodlibet* works as an E-rule for $\perp$ and that this rule is harmonious with the *absence* of I-rules for the same constant. This solution is not without problems, but we will stay with it, since many logicians, including Read, endorse it.[48]

[46] Negri and von Plato (2011), p. 199.

[47] See Kürbis (2015) for a recent exposition of the problems regarding absurdity in proof-theoretic semantics. These problems have led some logicians to try a direct characterization of negation, in order to bypass the problem with absurdity. As an example, see Dummett (2000), p. 89.

[48] Jacinto and Read (2017).

Having clarified why we need harmony for relational atoms, let us now evaluate whether the rules that determine their behavior in Negri's and Read's labeled systems are harmonious. As for the basic system G3K, this issue can be easily solved, since we have no rules for the relational atoms, except from the Axiom. The situation is the same with Read's rules for modal logic K, where no rules are endorsed for relational atoms. However, both Negri's sequent calculus and Read's natural deduction get problematic when we move from the basic modal system K to more interesting systems like T, S4 and S5. This extension indeed asks for the adoption of extra rules that govern relational atoms. Read introduces them acknowledging that they constrain the accessibility relation but stressing that they do so without giving it any meaning at all.[49] However, this seems both implausible and problematic. If these rules do not confer any meaning to $\triangleright$, its meaning should be given in some other way, and these rules should be *justified* by it, or at least this is what should happen if we want to interpret Read's natural deduction system as the basis for a theory of meaning.

Read claims that the rules of the system do not confer any meaning to the relational atoms because they do not provide any ground from which these atoms could be asserted. If this were the case, the situation could at best be analogous with that of $\perp$, which we have just seen. Indeed, the idea behind the harmony for $\perp$ is that if the I-rules do not provide any possible ground to assert it, then everything follows from its assertion. Hence, following Read's idea that relational atoms have no ground because their I-rules do not give any situation in which it would be possible to assert it, relational atoms should behave like absurdity and this is clearly not the case. Moreover, an attempt to obtain harmony from the absence of grounds for assertibility is clearly desperate: we are not dealing with a set of E-rules for relational atoms, which could, in theory, be in harmony with an empty set of I-rules or an empty set of grounds, but with several sets of rules (one set for each modal logic), and at most one of them could be in harmony with the absence of I-rules or grounds.

Moreover, even though Read highlights that relational atoms have no ground for their assertion, this property is not obtained like for $\perp$, just by considering a void set of I-rules. Read shapes his rules for relational atoms in such a way to give no ground for them, but in doing so he just obtains something like Milne's general introduction rules, especially for Reflexivity.[50] However, even though general I-rules do not directly give ground for their main formula, harmony still applies as a requirement, which does not become vacuous just because of this property of the I-rules. In particular, Milne follows Read and argues that E-rules are harmonious if they are a function of the respective I-rules. Hence, harmony should not be considered vacuously satisfied for relational atoms either.[51]

To obtain a more in-depth analysis, the status of Read's rules for accessibility can be evaluated by looking at how they are obtained from their unrestricted version. Read's rules have an obvious strengthened formulation that gives ground for asserting relational atoms. As an example,

[49] Read (2015), p. 654.

[50] Milne (2015).

[51] See Read (2010) for general-elimination rules and Milne (2015) for general-introduction rules.

$$\dfrac{\begin{matrix}[w \rhd w] \\ \vdots \\ u : A\end{matrix}}{u : A}\,\mathrm{T} \quad \text{is the weakened version of} \quad \dfrac{}{w \rhd w}\,\mathrm{T}^*$$

Here, rule T* not only allows the assertion of $w \rhd w$ in any circumstance, but it also has a quite clear I-rule structure, while Read's favored formulation is just a weakened version that cannot be used to introduce $w \rhd w$ but keeps all of its consequences. As for the other rules for the accessibility relation, they have a mixed structure, being both I and E-rules for $\rhd$. This is quite evident already when they are written in their general form, since they have relational atoms as major premises like general-elimination rules and relational atoms as discharged assumptions like general-introduction rules (even though their role as I-rules is impeded by the restriction on the required form $t : A$ of the conclusion):

$$\dfrac{w \rhd v \qquad v \rhd u \qquad \begin{matrix}[w \rhd u] \\ \vdots \\ t : A\end{matrix}}{t : A}\,4 \qquad \dfrac{w \rhd v \qquad \begin{matrix}[v \rhd w] \\ \vdots \\ t : A\end{matrix}}{t : A}\,\mathrm{B} \qquad \dfrac{w \rhd v \qquad w \rhd u \qquad \begin{matrix}[v \rhd u] \\ \vdots \\ t : A\end{matrix}}{t : A}\,5$$

Of course, this double I/E-nature of the rules for the accessibility relation is even more evident when they are formulated without any weakening of their introductory power and dropping the general formulation:

$$\dfrac{w \rhd v \qquad v \rhd u}{w \rhd u}\,4^* \qquad \dfrac{w \rhd v}{v \rhd w}\,\mathrm{B}^* \qquad \dfrac{w \rhd v \qquad w \rhd u}{v \rhd u}\,5^*$$

Read's strategy is just to reshape these rules so as to impede their usage as I-rules. In this way derivations containing alleged maximal-formulas such as $w \rhd u$ in

$$\dfrac{\dfrac{u \rhd w}{w \rhd u}\,\mathrm{B}^* \qquad u \rhd v}{w \rhd v}\,4^*$$

cannot be obtained.

In order to explain how this strategy works and why it disarms but does not resolve disharmony, let us focus on the following pair of *tonkish* rules for relational atoms:

$$\dfrac{w : A \qquad \begin{matrix}[w \rhd u] \\ \vdots \\ C\end{matrix}}{C}\,\mathcal{I} \qquad \dfrac{w \rhd u \qquad \begin{matrix}[u : B] \\ \vdots \\ C\end{matrix}}{C}\,\mathcal{E}$$

When added to any modal system, these rules lead to a trivial situation in which all formulas are satisfied in each world[52]:

[52] The only open assumption is that a world (w) forces at least a formula (A). This is clearly provable in any modal extension of each logic that has at least one theorem.

$$\dfrac{w : A \qquad \dfrac{[w \rhd u]^1 \qquad [u : B]^2}{u : B}\,\mathcal{E},_2}{u : B}\,\mathcal{I},_1$$

This behavior is clearly made possible by a disharmony between $\mathcal{I}$ and $\mathcal{E}$, which results in the irreducible derivation of $u : B$ from $w : A$.

If we impose a restriction, substituting $v \rhd z$ for C in $\mathcal{I}$ and $\mathcal{E}$, the system is not trivial anymore. However, by applying an analogous argument, it is still possible to prove that any world is accessible from any other world:

$$\dfrac{\dfrac{w : A \qquad [w \rhd u]^1}{w \rhd u}\,\mathcal{I},_1 \qquad \dfrac{[u : B]^3 \qquad [u \rhd o]^2}{u \rhd o}\,\mathcal{I},_2}{u \rhd o}\,\mathcal{E},_3$$

This result is of course much less problematic. What we obtain is a restriction to *total* accessibility relations, which correspond to (a special case of) Kripke's models for S5. This phenomenon is "innocuous" not because its consequences are justified by the grounds provided by $\mathcal{I}$, but just because deriving conclusions that are unjustified by these grounds (that is, violating the Inversion Principle) is less dangerous for relational atoms than for labeled atoms. Hence, disharmony is disarmed but still present. Even though our restriction is different from the one proposed by Read, the strategy is essentially the same: by imposing some restrictions on the applicability of the rules for $\rhd$, he prevents their disharmony from trivializing the system. So, just like for $\mathcal{I}$ and $\mathcal{E}$, we should not conclude that Read's rules for the relational atoms are harmonious.

Arguably, it is not easy to find examples of rules that trivialize the system without any restrictions on the form of the conclusion, but become "innocuous" when Read's proper restriction is imposed. However, let us consider the following pair of rules:

$$\dfrac{\begin{matrix}[w \rhd u] \\ \vdots \\ C\end{matrix}}{C}\,\mathcal{I}' \qquad\qquad \dfrac{\vdash w \rhd u}{\vdash C}\,\mathcal{E}'$$

The first rule is a general I-rule for relational atoms and asserts that any relational atom used to derive a conclusion C can be discharged. It stands for a total accessibility relation, since it makes dispensable any assumptions about accessibility. The second rule is an E-rule that asserts that if a relational atom is provable, then any sentence is provable.[53] It means that no relational atom should be provable, since this fact would trivialize the system. These rules characterize the accessibility relation in opposite ways, and for this reason cannot be in harmony with each other, as evidenced by the following derivation, which contains an irreducible maximal-formula and shows that the system is trivialized by these rules:

[53] We use $\vdash A$ to stress that A is provable, i.e. that it is the conclusion of a derivation without open assumptions.

$$\dfrac{\dfrac{[w \rhd u]^1}{w \rhd u}\,\mathcal{I}', 1}{C}\,\mathcal{E}'$$

If we impose Read's restriction, substituting $v : D$ for C in $\mathcal{I}$ (and $\mathcal{E}$, even though this change is irrelevant), then $\mathcal{I}'$ cannot be used to prove relational atoms and so $\mathcal{E}'$ cannot be used anymore. Hence, even though from $\mathcal{I}'$ all consequences of the universal accessibility relation are still derivable, its disharmony with $\mathcal{E}'$ does not lead to maximal-formulas anymore:

$$\dfrac{\dfrac{[w \rhd u]^1 \qquad w : \Box A}{u : A}\,\Box\mathrm{E}}{u : A}\,\mathcal{I}', 1$$

Moreover, $\mathcal{E}'$ is clearly not in harmony with any of Read's rules, even though none of them lead to maximal-formulas in combination with this rule. Indeed, all Read's rules assume that some accessibility relations hold, while $\mathcal{E}'$ asserts that this is not the case. The only reason why we focused on $\mathcal{I}'$ is that in this case the disharmony is the strongest possible, since $\mathcal{E}'$ claims that no accessibility relation holds, while $\mathcal{I}'$ claims that each world is accessible from each world.

Disharmony is even clearer if considered from the point of view of sequent calculus. The rules for relational atoms that Sara Negri uses to deal with stronger modal logics are[54]:

$$\dfrac{w \rhd w, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,\mathrm{Ref} \qquad \dfrac{w \rhd r, w \rhd o, o \rhd r, \Gamma \Rightarrow \Delta}{w \rhd o, o \rhd r, \Gamma \Rightarrow \Delta}\,\mathrm{Trans}$$

$$\dfrac{o \rhd w, w \rhd o, \Gamma \Rightarrow \Delta}{w \rhd o, \Gamma \Rightarrow \Delta}\,\mathrm{Sym} \qquad \dfrac{w \rhd r, w \rhd o, o \rhd r, \Gamma \Rightarrow \Delta}{w \rhd o, w \rhd r, \Gamma \Rightarrow \Delta}\,\mathrm{Eucl}$$

Since all the rules for relational atoms handle them on the left-hand side of sequents, the Cut rule is clearly dispensable for them.[55] As an example, let us consider the following derivation:

$$\dfrac{\dfrac{}{u \rhd w \Rightarrow w \rhd u}\,\mathrm{Sym}^* \qquad \dfrac{}{w \rhd u, u \rhd v \Rightarrow w \rhd v}\,\mathrm{Trans}^*}{u \rhd w, u \rhd v \Rightarrow w \rhd v}\,\mathrm{Cut}$$

This is the most straightforward translation into sequent calculus of the natural deduction derivation of $w \rhd v$ from $u \rhd w$ and $u \rhd v$—which, as we have already shown, contains a seemingly maximal-formula. Even though it displays an application of Cut, it becomes Cut-free when reformulated using Negri's rules:

$$\dfrac{\dfrac{\dfrac{}{u \rhd w, w \rhd u, u \rhd v, w \rhd v \Rightarrow w \rhd v}\,\mathrm{Axiom}}{u \rhd w, w \rhd u, u \rhd v \Rightarrow w \rhd v}\,\mathrm{Trans}}{u \rhd w, u \rhd v \Rightarrow w \rhd v}\,\mathrm{Sym}$$

[54] Negri and von Plato (2011) p. 192. The repetition of the active formulas in the premise is needed to make Contraction admissible: Negri and von Plato (2011), p. 98.

[55] Negri and von Plato (2011), p. 198.

This could be considered as a good result, but by looking more carefully it is evident that this property is obtained by absorbing Cut itself in the rules for accessibility, in such a way that they can be used only to modify the left-hand side of sequents. As an example, Negri's rule for Symmetry of the accessibility relation is obtained from $w \rhd o \Rightarrow o \rhd w$ by *cutting away* the sentence on the right of the arrow:

$$\dfrac{\dfrac{}{w \rhd o \Rightarrow o \rhd w}\,\mathrm{Sym}^* \qquad w \rhd o, o \rhd w, \Gamma \Rightarrow \Delta}{w \rhd o, \Gamma \Rightarrow \Delta}\,\mathrm{Cut} \quad \rightsquigarrow \quad \dfrac{w \rhd o, o \rhd w, \Gamma \Rightarrow \Delta}{w \rhd o, \Gamma \Rightarrow \Delta}\,\mathrm{Sym}$$

This is the reason why Negri's rules delete formulas on the left, instead of introducing them.[56]

Admittedly, it is implausible that all the meaning of relational atoms is provided inferentially by these rules, but some parts of its meaning are, and specifically its *structural* properties surely are. Clearly, Negri's reformulation of the rules can be regarded as a formal solution to the problem of Cut-elimination, especially for proof-search and other formal properties. However, what both Read and Negri have done is essentially impeding the introductory content of the non-harmonious rules for relational atoms from being too problematic. This is an ingenious move, since neither maximal-formulas nor Cut can be obtained without proper I-rules and R-rules. However, harmony should require something more if it has to entail that the E-rules are justified by the respective I-rules. A more plausible conclusion of Read and Negri's proposal is that a fragment of the meaning of the accessibility relation that is so weakened that it cannot give rise to proper maximal-formulas, even though it is not harmonious, is sufficient to characterize inferentially $\Box$ and $\Diamond$. So, while the main problem of traditional non-labeled sequent calculi for modal logic is the circular characterization of $\Box$ and the semantic pollution of $\Diamond$L, here the problem is that all modal terms rely on relational atoms, which have an, at least, unclear proof-theoretic status.

Let us consider a possible objection to our analysis of harmony for relational rules. Someone could argue that the rules for the accessibility relation do not need to be harmonious, but they should just preserve harmony of the system to which they are added. More formally, harmony should hold for the basic modal logic—and in particular for the rules for necessity and possibility—and it should be provable that if we extend the system with the rules for relational atoms, maximal-formulas for necessity and possibility remain reducible. As for basic modal logic K, the natural deduction system can indeed be described as harmonious, as we have already claimed, since relational atoms are not characterized by any rule and the rules for necessity and possibility satisfy the Inversion Principle. Moreover, the argument goes, when we extend the system with rules for the relational atoms, harmony still holds for the logical vocabulary *without considering relational atoms themselves* but only propositional sentences augmented with modal terms. In practice, it is required that every provable consequence that has only atoms as open assumptions and conclusion should be derivable using only atomic rules.[57]

[56] This deletion gives some problems for the subformula property as well. Negri and von Plato (2011), p. 201.

[57] Read himself seems to have in mind this kind of justification at the end of the fourth section of Read (2008).

To be fair, this kind of *criterion* has been proposed by Prawitz for the extension of logic with rules for atoms (atomic basis) already in the seventies and it is still regarded as convincing in recent contributions.[58] Moreover, if we take this route, there is no need to restrict the rules for relational atoms as Read does. Nevertheless, this *criterion* was designed for atomic *extensions* of logical systems that can be fully characterized without reference to any specific atomic basis. As an example, when we extend a logical system with a set of rules for Peano Arithmetic, since the rules of logic do not refer to the arithmetical terms, we can just apply this *criterion* and prove that purely arithmetical theorems (that is, atomic) are provable without any application of logical rules. However, as stressed at the beginning of this section, when the meaning of a logical term is *defined* using a specific kind of atoms (such as relational atoms are used to define modal rules), we cannot be satisfied with this weak requirement, as exemplified by the case of negation and absurdity.[59]

Before moving on and evaluating *realist* pollution of Negri and Read's modal systems in the next section, I should briefly defend my choice to address these systems instead of the well-known alternative that relies on hypersequents.[60] My main reason for doing so is that while neither Avron's hypersequent calculi, nor their tree-generalization investigated by Poggiolesi can easily be interpreted as if they were referring to natural deduction derivations, this interpretation is quite obvious for Negri's calculi, as we have seen. Hypersequents could of course be the basis for an inferentialist theory of meaning for modal terms, and to some extent have been developed for this reason. However, I claim that, contrary to what could seem to be the case at first glance, they require a much greater departure from the traditional proof-theoretic approach to meaning grounded on harmony. As an example, a traditional approach suggests that Cut-elimination should be required for a sequent calculus because it entails normalization for the natural deduction system that the metalinguistic interpretation of this calculus describes. Of course, there can be other reasons to require Cut-elimination in an inferential theory of meaning. Nevertheless, if we want to stick as closely as possible to the established proof-theoretic theory of meaning, which is based primarily on natural deduction, its philosophical justification relies on its connection with harmony and normalization of the corresponding derivation in natural deduction.[61] This paper addresses the problem of justifying modal logic in this traditional framework, choosing not to discuss other flavors of inferentialism.

Admittedly, both Avron and Poggiolesi give an interpretation of their hypersequents, but they only give an object-language interpretation for them, choosing not to provide a metalinguistic interpretation for their calculi. As an example, Poggiolesi focuses on tree-hypersequents, which have more or less the following shape:

[58] See Prawitz (1973) for an early exposition and Schroeder-Heister (2006) for a more recent application.

[59] Arguably, Prawitz himself uses this *criterion* for atomic basis to evaluate *ex falso quodlibet*. However, he does it only in some early papers, while this approach is nowadays quite uncommon. Moreover, Read himself applies harmony requirements to absurdity, as previously stressed; see for example (Jacinto and Read, 2017).

[60] See Avron (1996) and Lellmann and Poggiolesi (2023) for an overview.

[61] We have seen at the beginning of Sect. 2 some of the reasons for this preference.

$$\Gamma_1 \Rightarrow \Delta_1/(\Gamma_2 \Rightarrow \Delta_2/\Gamma_3 \Rightarrow \Delta_3); (\Gamma_4 \Rightarrow \Delta_4/(\Gamma_5 \Rightarrow \Delta_5/\Gamma_6 \Rightarrow \Delta_6); \Gamma_7 \Rightarrow \Delta_7)$$

Let us use G_i to denote tree-hypersequents. Poggiolesi proposes the following intended interpretation for them:

- $(\Gamma \Rightarrow \Delta)^\tau := \bigwedge \Gamma \supset \bigvee \Delta$
- $(\Gamma \Rightarrow \Delta/G_1; ...; G_n)^\tau := (\Gamma \Rightarrow \Delta)^\tau \vee \Box G_1^\tau \vee ... \vee \Box G_n^\tau$

Notwithstanding the interest and informativeness of this interpretation, it is not the kind of interpretation that can be used to give a proof-theoretic *criterion* of soundness. As we have seen in the previous section, while the metalinguistic reading of sequents works as a bridge between sequent calculus and natural deduction—suggesting the reading of R-rules as meaning-conferring, and Cut-elimination and the avoidance of semantic contamination as *criteria* of proof-theoretic acceptability—, the conditional reading of sequents cannot do the same. It seems plausible that a proof-theoretic soundness *criterion* for these systems should start from a completely different basis, since there seem to be few possibilities of adapting the traditional *criterion* of harmony to this framework. Moreover, while natural deduction is quite near to how natural language is actually used, hypersequents are not, and we will see in the next section why a connection with actual linguistic usage is needed for a formal system to properly treat modal logic.

3 Realist pollution

Let us now address what we have labeled *realist pollution*, which is probably the main preoccupation for the logicians trying to give inferentially the meaning of modal terms. A realist pollution for a proof-system is not only the deployment of elements taken from model-theory in its syntax—like labels for possible worlds or truth-values. A proof-system is polluted with realism when some assumptions that are incompatible with an antirealist attitude toward meaning are needed to make sense of the proof-system itself. As an example, it cannot be just the occurrence of T and F that pollutes a proof-system (such as Smullyan's tableaux) with realist presuppositions. It is the need to interpret these elements as standing for *true* and *false* conceived as independent of any epistemic constraint that makes the calculus unacceptable for an antirealist.[62]

Labeled systems have been accused of being polluted with realist meaning because their labels refer to possible worlds, intended as metaphysical entities that are not accessible to the community of the speakers. This is of course the most obvious kind of realist pollution that can be ascribed to labeled systems. Since these labels stand for possible worlds, the argument goes, the system is adequate for the logic of meta-

[62] In the same way, it is not just the substitution of T and F with + and − that makes bilateral proof-systems (arguably) acceptable as a basis for an antirealistic theory of meaning. Some properties that make + and − more epistemically constrained must be added, in order for them to be interpretable as assertion and rejection. See Rumfitt (2015)

physical modality, as an example, if and only if the relations between the labels correspond to the relations between the metaphysically possible worlds that the labels denote.

A possible solution could be to just ignore the problem of which elements correspond to the labels in modal proof-systems. Indeed, we have seen that Read and Negri develop their systems without any reference to the intended meaning of the labels or the accessibility relation: their goal is only to find a purely inferential system for modal logics. But in this way, they manage to give only a pure formal system. In other words, if labels are not interpreted as possible worlds and $\triangleright$ as the accessibility relation between them, we lack any reason to interpret $\Box$ as necessity and $\Diamond$ as possibility. Hence, the question is whether it is possible to give the intended interpretation for the systems without presupposing a realist conception of meaning.

Another, more promising, solution could be to drop any controversial assumptions about the nature of possible worlds, and just interpret them as (fictional) entities that are epistemically accessible to the speakers. It is far from obvious that such an alternative can give the right meaning to modal terms, but we will concede this point. Nevertheless, we will show that, even granting the inferential acceptability of the possible-world vocabulary, antirealism about meaning rejects at least some proof-systems for modal logic, which can be endorsed only together with some realist presuppositions about meaning. Moreover, we will see that the realist pollution of these systems does not result from their inferential semantic contamination but is, on the contrary, a quite different problem. In what follows we will focus on metaphysical possibility, which arguably is both the most interesting and the most controversial modality, at least from a philosophical point of view.

3.1 Inferential practices

Inferential theories of meaning should use proof-systems to explain (and also in part to discipline) common usage of the language—of philosophical language, in this case. One should not underestimate this aspect of antirealism, especially in dealing with philosophical topics like metaphysical modality. The aim is not only to provide a formal system with a soundness *criterion* and an adequacy theorem regarding a set of formal models—arguably, this could be sufficient for an inferentialist theory of purely abstract terms, since they are almost completely detached from other areas of the discourse[63]—but to give a proper treatment of the *actual* modal terms that are used in metaphysical discourse. Hence, together with the formal *criteria* we also need to check the adherence of this system to the actual inferential practice. This is why in antirealism natural deduction is the preferred proof-system, and when sequent calculus is used in its place, a link with natural deduction is still required.[64]

Even if a harmonious system gives inferential meaning to the logical terms, it nonetheless cannot provide a connection with the linguistic practice. Dummett argued

[63] However, even in these cases the relation with other fragments of the language could be relevant. See for example Dummett's observations regarding quantum logic and complex numbers in Dummett (1976).

[64] As we have seen in Sect. 2.3, the availability of a translation into natural deduction is the main reason why we focused on Negri's labeled sequent calculi, as opposed to hypersequents.

that while a set of rules is enough to establish a practice, and so to give meaning to a logical term, in order to understand what is the purpose of this practice, we need to be able to connect it with the rest of our linguistic behavior. In this last part of the paper, I will argue that, even overlooking the formal problems with Negri and Read's modal systems, their main flaw is their incapacity to bridge a gap with the actual usage of the modal vocabulary.

In order to exemplify his position regarding formal systems and natural language, Dummett analyzes the following situation.[65] The rule

$$\frac{A \mapsto (B \vee C)}{(A \mapsto B) \vee (A \mapsto C)}$$

is not intuitively valid when $\mapsto$ is interpreted as a counterfactual conditional. Nonetheless, a speaker could adopt this rule and decide to have their linguistic behavior accord with it when using counterfactual conditionals, and we could respect their decision and try to entertain dialogues with them. Of course, we are assuming that the speaker can provide a harmonious set of rules for the counterfactual conditional that is consistent with this change. Now, Dummett's point is that, if the speaker does not propose any justification for this logical reform, we are not capable of understanding the reason for their speech. Hence, a real dialogue would be impossible, we would just speak past each other.

What this example shows is that, even if the new rules for the counterfactual conditional were in harmony, it would still be completely unclear to the listener what the *point* of this change of the linguistic practice is. Obviously, the meaning of the counterfactual conditional is changed, the problem is that the way in which it has changed is not clear. We know what the goal of this linguistic instrument in *our* everyday practice is, but we have no idea of the reasons why this speaker decided to adopt this new rule. If he manages to explain how this new kind of rule for the counterfactual conditional is connected with everyday speech or at least with other areas of language, we could understand their move, but without such an explanation their *reform* is just pointless to us.[66]

As Dummett stresses the listener is in the same situation as someone who, learning a new game, has understood everything about legitimate and illegitimate moves, but nothing about its goals.[67] In both cases the lack of understanding is real, it is not just a feeling. The only difference between the two situations is that the goal of a game is internal to the game itself, while logic has (also) purposes that are external both to the logical fragment of the language and to the language as a whole.[68] Hence, misunderstanding language is both more dramatic and easier. In conclusion, this example shows that we could decide to define arbitrarily the meaning of logical terms using sets of rules, but this could lead to logical systems that we do not know how to use. And even well-behaved sets of rules could define terms that are essentially unintelligible to us.

[65] Dummett (1991), p. 206.

[66] See also note 63 about this issue.

[67] Dummett (1991), p. 208.

[68] Dummett (1991), pp. 204-205.

3.2 Realism and antirealism on inferential practice

In this last part of the paper, I will argue that realist and antirealist theories of meaning lead to a different selection of the system that properly describes the usage of metaphysical modal terms in natural language. To do so, I will refer to Dummett's argument against the modal axiom B, which corresponds to the symmetry of the accessibility relation.[69] This case study will be useful to explain in which cases a reliance on purely realist aspects of the meaning is needed in order to give to a formal system its intended meaning in the inferential practice. In other words, the example will explain what realist pollution for harmonious systems looks like.

Dummett starts by considering a small passage of Kripke (1980) in which the author argues against the metaphysical possibility of unicorns. Kripke's argument is presented by Dummett in the following way. Assuming that unicorns never existed, the term "unicorn" has been introduced in the language without any reference. More precisely, let us assume that it has been introduced as a term for fictional objects, and ignore the possibility of an erroneous misidentification of the referent. Kripke argues that, lacking an initial baptism, the term "unicorn" does not refer to anything and, for this reason, the sentence "unicorns could have existed" cannot be true.[70]

Kripke's point is based on a theory of meaning for terms of natural kinds centered on initial baptism and samples. According to this theory, the truth of a sentence regarding a natural kind requires that the samples that are individuated by the initial baptism related to this natural kind satisfy this sentence. In particular, in the case of unicorns initial baptism would regard a single animal and reference would extend to all other animals of the same species. Moreover, according to Kripke natural kinds are rigid designators, so they detect the same objects in each possible world. Hence, initial baptism is needed also for modal sentences to be true, and since "unicorns" has never been used to baptize any kind of animal, the sentence asserting their possibility cannot be true.[71]

The thesis that initial baptism is needed in order to have denotation is of course controversial even for philosophers who endorse realist conceptions of the meaning.[72] However, Dummett criticizes it specifically because it is incompatible with antirealist theories of meaning. Indeed, even though realism does not entail Kripke's theory of initial baptism, antirealism entails the rejection of Kripke's theory of initial baptism. Dummett expresses his methodological *criterion* in the following way:

> Any thesis about the meaning or reference of a word must draw its substance from how we use it or should use it in hypothetical circumstances. There is no possibility of a statement's failing to be true, in virtue of some fact, if all speak-

[69] See Dummett (1996).

[70] See argument (1) in section 3 of Dummett (1996).

[71] The details about how reference extends from the sample to other elements of the same natural kind are not relevant here.

[72] In fact, Kripke's theory is raised in opposition to other *realist* (or at least non-antirealist) theories of reference, such as Frege (1952) and Russell (1905).

> ers of the language would agree that that fact did not invalidate its claim to be true.[73]

In other words, a fact that is inaccessible to the linguistic community cannot be a determinant for the linguistic correctness of the usage of a term. This is one of the clearest examples of an antirealist principle about meaning.

Moreover, he observes that if we were to encounter animals resembling unicorns we *would* obviously call them "unicorns", so linguistic practice speaks in favor of the possibility of unicorns:

> We should certainly still call the newly discovered creatures "unicorns"; the only question is whether, by so doing, we should be conferring a new sense on the word.[74]

Dummett argues that this discovery would not change the sense in which we use the term "unicorns". Indeed, even if unicorns never existed and the term has been introduced as a fictional term, this fact is unknown by the linguistic community. As an example, a speaker could be ignorant of the fictional origin of this term and hypothesize that it has been originally used to erroneously denote some kind of rhinoceros. When the memory of this fictional origin gets lost, the sense of the term "unicorns" changes and it becomes possible for unicorns to exist. According to Dummett, the loss of the memory of the fictional origin of the term is what causes a change in the sense of the term and makes it possible for unicorns to exist, not the discovery of unicorn-looking animals. Hence, by deciding to apply the term to real creatures (that is, by baptizing them with this term) we would not change the meaning of the term. From Dummett's analysis, it follows that the assumption that unicorns never existed is not enough to conclude that they could not have existed. If we do not want to cross the boundaries of an antirealist theory of meaning, in order to infer the impossibility of unicorns the extra (quite implausible) assumption that this fact is well-known by all competent speakers of the community is needed:

> The fictional origin either of a proper name or of a general term undoubtedly affects the use that we make of it, and thus its sense; but only when it is known that its origin was fictional. [...] Even if the word "unicorn" originated as deliberately fictional, the fact is not generally known, and perhaps will forever be unknown to anyone. There is therefore no reason why it may not have been subject to an unwitting change of sense, from being regarded as intrinsically fictional in view of its known fictional origin to being treated as having an application determined by its descriptive content.[75]

Dummett evaluates and rejects two other arguments against the possibility of unicorns. The details of his confutation are not relevant to our discussion. What is rel-

[73] Dummett (1996), p. 334.

[74] Dummett (1996), p. 335.

[75] Dummett (1996), pp. 336-7.

evant is that these two extra arguments, when neutralized as arguments against the possibility of unicorns, are used by Dummett to disprove symmetry of the accessibility relations between possible worlds, and so against axiom B, which corresponds to this restriction on the accessibility relation between possible worlds. The argument against symmetry goes as follows:

1. Unicorns do not exist but could exist;
2. The meaning of the term "unicorns" contains only information about their external look;
3. In particular, the meaning of the term "unicorns" does not specify their order, family or genus (from the previous point);
4. The term "unicorns" is nonetheless a term for a species, like dog, cat, etc;
5. Belonging to a species is an essential and so a necessary property (more or less a shared assumption in the metaphysical debate)[76];
6. Unicorns could be of the order Artiodactyla or of the order Perissodactyla (from 1–3 and the compatibility of these orders with the external characters of unicorns);
7. There is a possible world *u*, accessible from our world *w*, in which unicorns belong to a species of the order Artiodactyla (from 1 and 3);
8. There is a possible world *v*, accessible from our world *w*, in which unicorns belong to a species of the order Perissodactyla (from 1 and 3);
9. In *u*, unicorns are necessarily of the order Artiodactyla and cannot be of the order Perissodactyla (from 5 and 7);
10. In *v*, unicorns are necessarily of the order Perissodactyla and cannot be of the order Artiodactyla (from 5 and 8);
11. *u* is not possible relative to *v* and vice-versa (from 9 and 10);
12. *w* is possible relative to neither *u* nor *v* (from 6–8);
13. Hence, the accessibility relation between possible worlds is not symmetric (from 7, 8 and 12).

As we have seen, the first assumption of the argument is licit if we assume an antirealist theory of meaning. The only problem with it could be the actual existence of unicorns, which would lead to an identification of their order. Anyway, this would be just a contingent problem: another counterexample with a different term could easily be provided. On the contrary, the status of this assumption if a realist theory of meaning is endorsed is at least doubtful, and the possibility of rejecting it by endorsing a realist interpretation of the consequences of initial baptism is provided by Kripke himself. Hence, a realist could reject the argument and endorse axiom B.[77]

[76] According to Kripke's theory of denotation, it is necessary because it is reducible to an identity statement regarding natural kinds, which are rigidly designated by their terms.

[77] This aspect is to some extent overlooked in Williamson (2017), which highlights the abductive methodology applied by Dummett in pointing out his counterexample to principle B, but does not discuss the antirealist assumption used to defend the availability of this counterexample. Moreover, this coexistence of meaning-theoretic and abductive methodologies corresponds to some extent to Read's position regarding anti-exceptionalism. See Read (2019).

A consequence of this situation is that, even if harmonious systems that comprise axiom B exist and are available to logicians, in order to give to these systems their intended metaphysical meaning we need to endorse a realist conception of the meaning. Hence, if antirealism is right, these systems cannot properly describe the way in which the modal terms are actually used in the language, at least when they are used to speak of metaphysical necessity. For contraposition, for these systems to properly describe the actual inferential practice, their meaning must be *polluted* with realist assumptions about the correct usage of words. So, while the system in itself is neither semantically contaminated in the inferential sense of this term nor polluted with realist meaning, we need a *realist pollution* in order to attach to its modal terms their intended metaphysical meaning. Even allowing possible-world vocabulary, if we are antirealist about meaning there are counterexamples to some modal systems that cannot be avoided, at least if we want to use these systems to reason about genuine metaphysical modalities.[78]

The situation here is akin to that of Dummett's counterintuitive rule for the counterfactual conditional that we saw in the previous subsection. As observed, if we were to encounter unicorn-looking animals, we would clearly call them "unicorns". Hence, rejection of axiom B follows, as we have argued. This means that endorsing this axiom constitutes a reform of our linguistic practice, justifiable, as far as we have seen, only using the realist assumption that initial baptism is relevant for the correct application of terms even when it is not epistemically accessible by the community of the speakers. In other words, for this reform to be understandable, the meaning of modal terms needs to be polluted with realist assumptions regarding their meaning.

In conclusion, let me address a possible objection. While Dummett's argument seems sound, someone could reject it, maybe finding a flaw in it or contesting one of its presuppositions regarding meaning. If these criticisms do not regard antirealism, but other elements of the proof, we could conclude that there are no reasons for an antirealist to reject axiom B. Hence, we would be left without an example of a harmonious system that suffers realist pollution. Even conceding this point, the main lesson about realist pollution remains. Realist pollution is avoided for a proof-system if:

- It satisfies all the proof-theoretic formal *criteria* of acceptability (it is harmonious, its E-rules are not semantically contaminated, it is separable, etc.);
- It can be used to describe inferential practice in natural language without applying realist presuppositions on meaning.

If Dummett's argument is sound, then we cannot reject the possibility of unicorns without applying the realist idea that an initial baptism of unicorns with this term is needed for them to be possible. And since the metaphysical possibility of unicorns can be used to provide a counterexample to the axiom B, antirealism entails the rejection of this axiom. If Dummett's argument has some flaws and there are no counter-

[78] Interestingly enough, it seems natural to think that restricting possible worlds only to epistemically accessible ones excludes many of them. Dummett's example shows that sometimes antirealistically acceptable possible worlds can do things that antirealistically unacceptable worlds cannot, like having unicorns in them.

examples to axiom B, then maybe the intended meaning can be conferred to even S5 without requiring realist presuppositions on how the meaning is attached to the elements of the language. However, the availability of an inferentially harmonious system and its adequacy to characterize the intended meaning remain conceptually distinct *criteria* of antirealist acceptability. *Contra* Read, we still need to distinguish these two moments of inferential evaluation of a proof-system.

4 Conclusions

We have seen that in discussing semantic pollution we should distinguish between inferential semantic contamination, which concerns inferential meaning, and realist pollution, which concerns the pollution of inferential systems with realist meaning. As for inferential semantic contamination, we have distinguished between impure I-rules (which are not problematic in themselves), contaminated E-rules and contaminated derivations. After this methodological clarification, we have evaluated different proof-systems for modal logic. We have seen that traditional non-labeled systems for modal logic fail (also) because they are inferentially semantically contaminated. In particular, we have seen both L-rules for possibility that are polluted by the occurrence of necessity, and derivations that are polluted by the occurrence of non-atomic applications of Weakening. Then we have moved toward labeled systems, and argued that their impure I-rules require a harmonious treatment of relational atoms and shown that the usual sets of rules for them are not really harmonious. Lastly, we have addressed realist pollution, explaining how a well-behaved formal system could still be polluted with realist meaning when it receives its intended meaning.

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