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An epistemic-logical model for analysis of students' argumentation in mathematics education research

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In this theoretical paper we present an epistemic-logical model for analysis of students' argumentation. The model is based on Vergnaud's concepts- and theorems-in-action, on Duval's distinction between epistemic and logical value of verbalised propositions, and on elements of Oostra's intuitionistic existential graphs: a kind of graphical topological logic rooted in the Peircean thought, adapted to Mathematics Education research by considering also shifts to classical existential graphs. We use the model to discuss two examples taken from literature.

Keywords: Analytic tools and methods of inquiry; epistemic logic, argumentation and proof, continuity and discontinuity between argumentation and proof.

Introduction

While in Mathematics Education (ME) research it is taken for granted that logic taught in the mathematics classroom should consist of elements of classical propositional or first order predicate logic (e.g., Durand Guerrier, 2005; Durand Guerrier et al., 2012), the situation may change when logic is intended as a means for analysis of students' discursive productions. While classical logic is a bivalent truth-value logic that perfectly grasps the epistemological constraints of mathematical knowledge, it struggles in capturing purely epistemic aspects related to knowledge acquirement, such as knowing appropriate properties or procedures. Furthermore, research in ME shows that non-classical logical tools can produce different results in analysis as classical ones (Asenova, 2022; D'Amore, 2005). On the other hand, while some scholars (e.g., Duval, 2007) state that there is a discontinuity between argumentation and mathematical proof, due to their cognitive distance and the concomitant linguistic proximity, other researchers are more oriented towards possible conditions of continuity, identifying in some cases a cognitive unity between arguments used to support a conjecture made by the students and the proof they provide after (e.g., Garuti et al. 1998; Pedemonte, 2007). For these researchers argumentation is a reasoning that supports the acquisition of knowledge during a conjecture-phase, while proof is a special, logically stricter kind of argumentation (Durand-Guerrier et al., 2012). A model able to frame both the epistemic and logical aspects, in a way specific to ME research (Asenova, 2022), is still missed. We first introduce such an epistemic-logical model and then show how it works in analysing and discussing two examples taken from literature.

The epistemic-logical model

The model we expose here is composed by three parts: the first two are taken from ME research and are used to frame the epistemic-logical aspect from a didactical viewpoint; the third part of the model is taken from logic and provides a system of signs with transformation and interpretation rules.

Concepts-in-action and theorems-in-action. The first ingredient of the model consists of two elements of the theory of conceptual fields (Vergnaud, 2009): concepts-in-action (cs-in-a) and theorems-in-

action (ts-in-a). Cs-in-a and ts-in-a differ from ‘usual’ concepts and theorems because they are not explicit and have to be inferred from the students’ behaviour. An important distinction to be made is that while ts-in-a are propositional sentences, and thus can be true or false, cs-in-a cannot be classified as true or false, but only as relevant or not relevant (Vergnaud, 2009). Another important point, according to Vergnaud, is that even if one may think a sentence is true that in fact is false, it is still a t-in-a: “There is little difference, from the point of view of activity, between a true proposition and a false one considered as true” (Vergnaud, 2009, p. 88). In this sense, Vergnaud’s viewpoint is truly epistemic because it concerns the knowledge of the individual independently from its correspondence to normative aspects.

The meaning space of a verbalised proposition. The second ingredient of the model is taken from Duval’s distinction between *content*, *value* and *status* of a verbalised proposition (Duval, 2007). The content-dimension refers to the objects of discourse and can be informative (provides information about objects or situations) or theoretical (provides information about the relations between the involved objects). The value-dimension refers to the relation of the proposition to what it enunciates. It can be epistemic (obvious, absurd, possible, probable etc.), logical (true, false, undecidable) or communicative (order, promise, question, assertion etc.). The status-dimension of a verbalised proposition refers to its relation to other verbalised propositions in the discourse as a whole. In our model we refer only to the first two components of the value-dimension: The epistemic value fits the idea of t-in-a, based on cs-in-a, because it is related to the implicit, proposition-like, knowledge, while the logical value fits the idea of theorem, based on explicit concepts.

The graphical topological logic of existential graphs. The third ingredient of the model is provided by elements of a graphical topological logic based on Oostra’s intuitionistic version of the Peircean Existential Graphs (EGs). Peirce (1960) elaborated a graphical logic correspondent to classical propositional logic (Alpha EGs), classical first order logic (Beta EGs) and modal logic (Gamma EGs). Starting from Peirce’s work, Oostra proposed an intuitionistic version of EGs. Intuitionistic logic is, according to Hintikka (2001), “truly epistemic” because the crucial notion in it: “is not knowing that, but knowing what (which, who, where, ...) (...) and this knowing-what-logic cannot be analysed in terms of knowing that plus the apparatus of received first order logic” (pp. 10-11). In this sense, intuitionistic logic refers to *what* and *how* a subject actually knows, while a *knowing-that-logic* is based on objective combinatorial aspects and concerns a truth value.

On one hand, this conception of epistemic logic fits the idea of epistemic value as well as the idea of ts-in-a and puts into the forefront aspects related to the way the student knows what she knows. As Oostra (2022) states, quoting Brouwer, in the intuitionistic approach, “a mathematical proposition becomes true when the subject experiences or intuits its truth, after having carried out a suitable mental construction” (p. 123). It is misleading to talk here about truth in classical sense, as to be true means that a suitable construction is known by the individual. In this sense, intuitionistic logic is not ‘truth preserving’ in the same way as classical logic, but it postulates that constructions ‘stand over time’, that is, they are truth-preserving. This concept of truth requires a different interpretation of logical connectives: e.g., the negation of a sentence means that something absurd may be constructed from it; the disjunction of two sentences is true if an effective construction of any of them is possible (or known); implication means that a construction of the consequent is possible (or

known), provided a construction of the antecedent is possible (or known), etc. On the other hand, it is important that mathematical knowledge is explicitly perceived as truth-functional and students must accomplish a shift from an epistemic to a logical value and from ts-in-a to theorems. To account for this shift, the model must make possible the passage from the epistemic to classical logic.

We introduce only some elements of the Alpha intuitionistic EGs (Alpha-IEGs); for more details, see Oostra (2022), where it is also shown how classical Alpha-EGs can be formalized. For our purposes, it is sufficient to treat Alpha-IEGs intuitively, as Peirce did when he devised this graphical logic. The Alpha-IEGs are here introduced with some additional elements respect to the system exposed by Oostra, due to the specific needs of ME research. According to Oostra (2022), Alpha-IEGs are build by the following elements: sheet of assertion, propositional letters, two types of curves. The sheet of assertion is the plane surface upon which the graphs are drawn, and it represents the universe of the possibilities of truth. Graphs can be composed of propositions, represented by capital letters, closed curves, called cuts (Figure 1a) or scrolls (Figure 1b): each scroll is composed by a cut with one loop (single scroll, Figure 1b) ore more loops (e.g., double scroll, Figure 1c) folded inside it. Peirce talks about a node, when he refers to the scroll: “The node merely serves to aid the mind in the interpretation and will be used only when it can have this effect” (Peirce, CP, 4.436).

In a scroll, the region limited by the cut and the loops is the outer area of the scroll and the interior part of the loop is the inner area. An area of the sheet of assertion is called *even* or *odd* if there is an even or odd number of curves around it, counting both cuts and loops (Figure 1d-f).

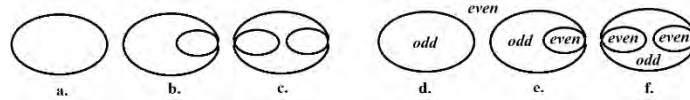


Figure 1: Elements and areas in Alpha-IEGs (Oostra, 2022, p. 136)

An Alpha-IEG is thus a diagram consisting of a finite combination of propositional letters, cuts, and scrolls, represented upon the sheet of assertion. The semantic dimension is introduced, according to Oostra (2022), by stating that drawing a graph on the sheet means asserting its interpretation and writing a letter means asserting the proposition it represents. The interpretation of the four basic and the two derived connectives (negation and implication of disjunction) for Alpha-IEG are represented in Figure 2, where ‘=’ stands for equivalence.

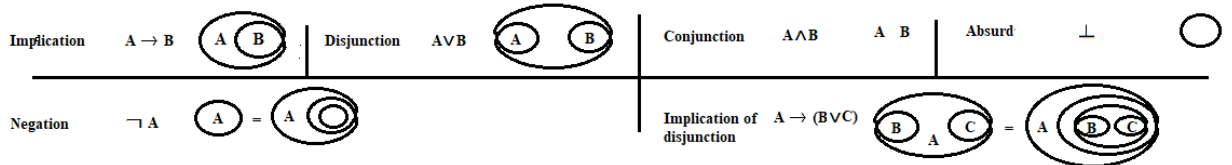


Figure 2: Basic and derived connectives of Alpha-IEGs (adapted from Oostra, 2022, pp. 135-136)

Finally, we introduce the transformation rules (TRs) for our system of EGs. Rules 1-5.1. are taken from Oostra’s (2022) Alpha-IEGs, where it is shown that they represent a sound system of graphical intuitionistic logic equivalent to the traditional one. These TRs contain both the TRs for classical Alpha EGs (the first part of TRs from 1 to 4, before the semicolon) and the intuitionist-additions: the second part of TRs from 1 to 4 and TR5.1. TRs 5.2-8 are added by the author as rules specifically needed in the context of ME research to express the increasing/decreasing epistemic value, the relations between different cs-in-a, the shifts from an epistemic to a logical value and from ts-in-a to

theorems. We denote this extended version of Alpha-IEGs, whose TRs are not in contradiction with the Peircean intuitive conception of EGs and with Oostra's system of Alpha-IEGs, with Alpha-IEGs*.

TR1: Erasure. In an even area, any graph may be deleted; any loop within an even area may be eliminated, together with its contents. **TR 2. Insertion.** In an odd area, any graph may be added; if the odd area is limited externally by a cut, a loop containing any graph may be added to the cut. **TR 3. Iteration.** A graph can be repeated in its own area or in any area contained in it, which is not part of the graph itself; any loop may be iterated, together with its contents, on its own cut. **TR 4. Deiteration.** Any graph may be deleted if a copy of it persists in the same area or in any area around it; a loop, together with its contents, may be erased if another loop with the same contents is present on its cut. **TR 5.1. Scrolling.** A scroll with empty outer area may be drawn around or removed from any graph on any area. **TR 5.2. Double cut.** A double cut may be drawn around or removed from any graph on any area. **TR 6. Increasing $\uparrow$ or decreasing $\downarrow$ loop.** In a scroll, the area enclosed by the loop can increase or decrease. **TR 7. Topological equivalence.** A graph can be transformed in other equivalent to it by continuously transforming the curves it is composed of. **TR 8. Detachment.** A loop of a scroll can be detached from the cut.

The idea of topological transformation anchored in TR 7 helps to understand what could be meant by logical continuity. It is well known that insofar as a given figure is continuously deformed (i.e., previously connected points were not separated and previously separated points were not connected), the obtained figures are topologically equivalent to it. Starting from this idea of continuous transformation, a topological space (in our case a bidimensional closed curve) associated to a connective (e.g., the implication $A \rightarrow B$) can be transformed in another (e.g., $A \rightarrow (B \vee C)$) by continuous transformations of the scroll that represents it. From the other hand, if we consider topological equivalence of two connectives as based on domain-equivalence between the cs-in-a they are based on, a relation of inclusion between domains corresponds to a continuous deformation of the corresponding curve. In this context, continuous transformations of the scroll-curve of implication can be used to express an increasing or decreasing epistemic value, related to the increasing or decreasing number of different classes of objects that fall under the same c-in-a (TR 6). As it is well known, in intuitionistic logic the law of the excluded middle does not hold in general and TR 6 accounts for this characteristic, considering an increasing or decreasing epistemic value. Classical Alpha-EGs and Alpha-IEGs differ from each other by representing implication and by the behaviour of negation. In classical Alpha-EGs the implication is represented in similar way as in Alpha-IEG, but the 'loop' and the cut do not touch each other, and the double negation corresponds to a double cut. To frame both, the classical and the intuitionistic perspective, the double cut (TR 5.2) and the detachment (TR 8) rules, that represents the shift from the epistemic to the logical value, and thus from ts-in-a to theorems, are introduced. This is in accordance with the Peircean intuitive conception of the scroll/double cut: "The scroll shall be a real curve of two closed branches (...) and these branches may or may not be joined at a node" (Peirce, CP, 4.437).

Example analysis and discussion

Example 1. In ME, the so called Lakatos-style-proof-approach is widely accounted in literature (e.g., Lada & Forbregd, 2022). According to this approach, there are some techniques that can be used to

improve ‘proofs’ when they are invalidated by a counterexample. Two techniques are exception barring and lemma incorporation, both concerning an exploration of the domain under consideration that is restricted or enlarged in order to exclude counterexamples or to include further properties. Lada and Forbregd (2022) discuss how prospective teachers (PTs) deal with these two techniques, while they were interviewed during an activity. The objective of the activity was to make PTs discover the theorem: A quadrilateral is cyclic if and only if its opposite angles are supplementary, but it was then judged to optimistic and thus all interviews ended once a necessary condition was conjectured by the PTs, namely ‘If a quadrilateral is cyclic, then its opposite angles are supplementary’. The assignments were: (a) Prove that the three perpendicular bisectors on the side of a triangle meet at one point and that this point is the centre of the circumscribed circle; (b) Investigate if a similar result holds for quadrilaterals. State a conjecture and try to prove it.

Marie is one of the PTs whose reasoning is exposed by Lada and Forbregd (2022). In the following, we analyse it using the tools provided by the presented epistemic-logical model. Marie’s conjecturing and proving activity can be subdivided into 9 steps. In the first part of the activity, that comprises steps from 1 to 5, she explores the problem using the c-in-a related to the intersection point of the bisectors. She implicitly uses the t-in-a $P(x)$: If x is a cyclic quadrilateral (antecedent A) then the bisectors of its sides intersect in a point that is the centre of the circumscribed circle (consequent B). In Figure 3 the 9 steps are represented by Alpha-IEGs* as they were introduced in this paper.

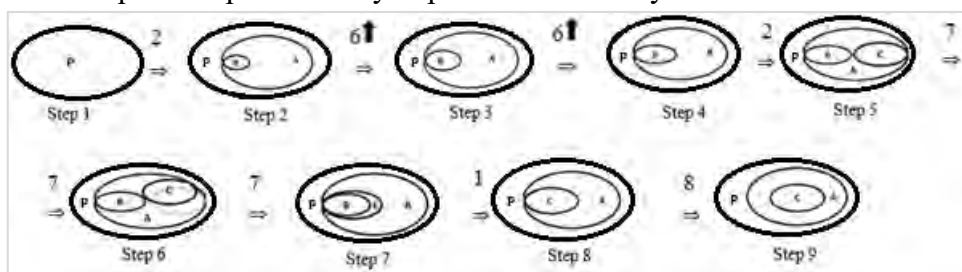


Figure 3: Marie’s reasoning represented by the system of IEGs* exposed in this paper

Step 1: $P(x)$ as general statement is negated because Marie believes there are no quadrilaterals with this property; Step 2: she discovers that a restricted version of $P(x)$: $A \rightarrow B$ is true for squares; Step 3: Marie discovers that $P(x)$ is true for rectangles; Step 4: $P(x)$ is believed true by Marie for quadrilaterals with two parallel sides (overgeneralization), that is then corrected later. Marie struggles to explore further the problem and is thus pushed by the researcher to use another c-in-a, related to the supplementary angles in a quadrilateral. This implies an implicit use of another t-in-a, $Q(y)$: If y is a cyclic quadrilateral (antecedent A) then its opposite angles are supplementary (consequent C). In step 5, the shift to the supplementary angles-property is expressed as the introduction of a second loop (C) that leads to the implication with disjunction $A \rightarrow (B \vee C)$. In steps 6 and 7 in Figure 3, the ‘move’ of loop C is represented, until it overlaps with loop B. The domain of the c-in-a related to loop B is included into the domain of the c-in-a related to loop C – quadrilaterals with bisectors that intersect at the same point have supplementary opposite angles – and thus loop C can be erased (TR 1). The shift from step 8 to step 9 represents the shift to truth-functional logic by detachment (TR 8).

The two conjectures (P and Q) are two ts-in-a driven by two different cs-in-a and the steps 2, 3 and 4 represent the increasing epistemic value of $P(x)$ experienced by Marie as she discovers new classes

of quadrilaterals that satisfies it. The steps from 5 to 9 are related to the (implicit) reasoning needed to finally shift to the theorem: ‘If a quadrilateral is cyclic, then its opposite angles are supplementary’.

Our analysis shows that the topological character of the employed logic allows to highlight the distinction between the explorations based on the two different cs-in-a. From purely extensional viewpoint, the distinction between loop B and loop C (step 5), and thus between the cs-in-a they are based on, does not make sense, but from an epistemic viewpoint it does, because the problem is explored by using two different properties. Indeed, Marie does not discover the usefulness of the second c-in-a (the one of the supplementary angles); she is pushed by the researcher to consider it. Furthermore, the topological logic allows to highlight in a diagrammatic manner the shift from an epistemic to a logical value (from step 8 to step 9). We can notice that steps from 2 to 8 are performed by continuously deforming the initial curve. In this sense, the graphs in steps 2 to 8 are topologically equivalent and topologically different from the graph in step 9. This can be considered as the representation of an epistemic continuity described by the topological epistemic logic, joined to a logical discontinuity when shifting to step 9.

Example 2. Boero and Turiano (2019) discuss examples of 10th grade-students’ reasoning on Euclidean Geometry-statements that make use of what the authors call a “continuity principle” (p. 139). A student, Ale, is working on a task aiming to make him discover the construction of a circle inscribed in a triangle. He just knows how to construct a circle tangent to the two sides of an angle having the centre on the bisector. Ale’s behaviour is described in Figure 4.

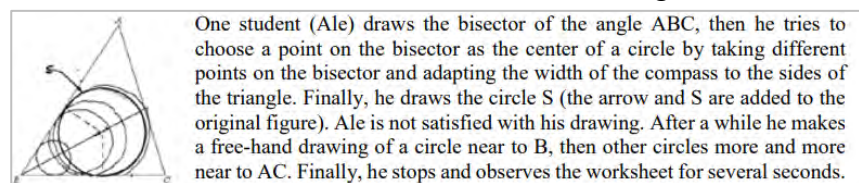


Figure 4: Example of student’s use of the “continuity principle” (Boero & Turiano, 2019, p. 137)

The student seems to perform a mental transformation on the geometrical figure, supported by drawing some snapshots of it – a circle that moves the centre on the bisector of an angle and is tangent on two sides – with the purpose to ‘prove’ the existence of a circle inscribed in the triangle, and considers this a proof of the existence of the required mathematical object: the circle S in Figure 4. How could Ale’s reasoning be framed within the model exposed in this paper? Ale first draws the circle near to the angle B of the triangle in Figure 4. As stated above, he just knows the Euclidean procedure to construct a circle with centre on the bisector and tangent to the sides of the angle. By drawing circle S he exhibits the required object: a circle inscribed in the triangle. Let us consider two propositions: Proposition A - In a triangle a circle with centre on the bisector of an angle and tangent to the sides of the angle can be constructed, and Proposition B - A circle tangent to the three sides of a triangle can be constructed. The implication $A \rightarrow B$ can be considered in intuitionistic sense: If I know how to construct the object of proposition A, then I am able to construct the object of proposition B. This knowledge is provided by the procedure that ‘transforms’ the object of the proposition A into the object of the proposition B (see the drawing in Figure 4). Globally, the student

seems to behave according to the modus ponens inferential rule (A and $A \rightarrow B$, thus B), a rule that holds also in intuitionistic logic (Figure 5a).

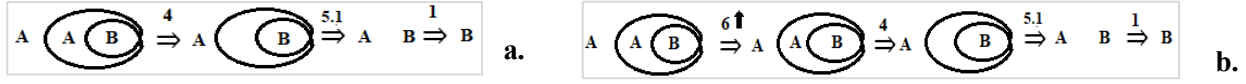


Figure 5: a. Modus ponens in Alpha-IEGs b. Modus ponens with increasing epistemic value

We can consider $A \rightarrow B$ as a t-in-a based on a c-in-a related to the property of tangency. The drawing of the intermediate circles represents the intermediate steps of this constructive ‘proof’, ending with the existence proof of the required object. In this sense, the epistemic value of the t-in-a increases by moving the centre of the circles on the bisector away from the edge B of the triangle. Thus, we could add an intermediate step in the modus ponens inference that makes no sense from classical logic-viewpoint but that makes sense from an epistemic one. We can highlight the increasing epistemic value of the t-in-a $A \rightarrow B$ by application of TR 6↑ to the loop of the first scroll (Figure 5b). Although the ‘proof’ provided by the student is completely convincing for him, it requires further reasoning and validation to be considered as such by the researchers. Indeed, the researcher pushes the student to think about what would happen if the radius of the circle S increases further and then decreases again. In this sense, the student is induced to validate his ‘proof’ by showing that S is *really* tangent to the side AC . To this purpose, the researcher introduces a new c-in-a, linked to the property of being secant. Conversely to the case of example 1, here the new c-in-a is related to the antecedent, not to the consequent and thus there is no transformation rule that allows to transform the $A \rightarrow B$ -graph to the $C \rightarrow B$ -graph. It could be conceived as an application of modus ponens to the implication $C \rightarrow B$, where C is the statement: The radius of the circle can be increased continuously keeping the centre on the bisector till it has two distinct intersection points with a side and then it can be decreased again continuously, till these two points coincide in a unique point. For the student, $C \rightarrow B$ becomes a t-in-a, in the same way as $A \rightarrow B$ was, although it seems to not have a higher epistemic value for him then $A \rightarrow B$. Indeed, he does not seem to feel the necessity to validate the ‘proof’, although he agrees with the researcher’s reasoning. To represent the shift to a logical value, as aimed by the researcher, we must add a step of detachment (TR 8) in the modus ponens-proof based on the secant-tangent-interplay (Figure 6).

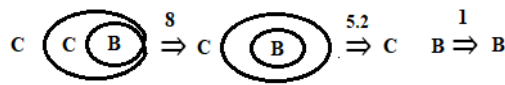


Figure 6: Modus ponens on $C \rightarrow B$ and shift to a logical value

Once validated by $C \rightarrow B$, the former t-in-a $A \rightarrow B$ becomes a theorem within a truth-functional context. In this example we can recognize two logical discontinuities: (1) in the shift from the first to the second modus ponens application (from Figure 5b to Figure 6); (2) in the shift to the validation, e.g. in analytical geometry (TR 8 in Figure 6 that signs the shift from argumentation to proof).

The two examples discussed above show that the model proposed in this paper allows, as aimed, to frame arguing within a ‘knowing-what and knowing-how’- epistemic context, as well as the shift to proof in a classical ‘knowing-that’- logical context. It also allows to frame the continuity-discontinuity issue between argumentation and proof from epistemic-logical viewpoint. Further research should be carried out to refine the model, considering the use of quantification (Beta-EGs).

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