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Design and implementation of Digital Mathematical Discussion: focus on students' collaborative processes and the development of collective algebraic thinking

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Introduction

The integration of digital technologies into educational environments provides significant opportunities to transform the teaching and learning of mathematics, particularly by enabling collaboration and communication beyond the traditional classroom setting (Ball & Barzel, 2018; Ruthven, 2012). This research issue has been addressed in mathematics education over the past decades and has received increased attention during the Covid-19 pandemic period as institutions all over the word faced the challenge to implement distance learning (Padayachee & Campbell, 2022). However, during that period, digital tools were often used merely as substitutes for face-to-face teaching, replicating established practices rather than leveraging their features to actively engage students (Baccaglini-Frank, 2022). As a result, transmissive teaching methods prevailed, limiting the effectiveness of digital technologies in promoting deep, meaningful learning (Aldon et al., 2021). This thesis contributes to the ongoing investigation on the role of digital technologies in facilitating communication and collaboration between students and teachers for mathematics teaching and learning and, specifically, in supporting mathematical discussion (Bartolini Bussi, 1996). Mathematical discussion is a didactic methodology designed to actively involve students in constructing mathematical knowledge through interaction with peers and experts. Research in mathematics education has already started addressing the support digital technologies can give to the design and unfolding of mathematical discussions, but it has primarily focused on synchronous discussions taking place in classrooms or through conferencing platforms. In this research, instead, the focus is on asynchronous mathematical discussions. In fact, asynchrony enables participants to engage with content at their own pace and flexibility in time allows them to reflect, process information and contribute thoughtfully to the discussion (Wang, 2023). Moreover, written communication enhances clarity and allows for permanent records, which in turn support deeper understanding (Santos & Semana, 2015). Despite these features of asynchronous discussions, they remained underexplored in research in mathematics education, thus we aim to contribute to this research gap introducing and investigating Digital Mathematical Discussion.

Within this thesis, in chapter 1, we situate the research in a Vygotskyan paradigm (Vygotsky, 1962, 1978) to the teaching and learning of mathematics, constituting the basis of the theoretical frameworks followed in the study. Framed within a Vygotskyan perspective, we introduce the didactic methodology of mathematical discussion (Bartolini Bussi, 1996), highlighting the role of the teacher in designing and orchestrating mathematical discussions (Stein et al., 2008; Cusi & Malara, 2009, 2013) and the support provided by digital technologies for the implementation of synchronous mathematical discussions (e.g., Cusi et al., 2017; Albano et al., 2022; Giberti et al., 2022). We then introduce the didactic methodology of asynchronous discussion (Fehrman and Watson, 2021) and report the results of a systematic review on mathematical asynchronous online discussions. The review highlights that the theme of mathematical asynchronous online discussion is still emerging in mathematics education research, particularly with respect to high-school contexts. Thus, it suggests that more research is needed to further explore whether mathematical discussions can be conducted and enhanced by digital technologies, extending time in asynchronous modality and supporting communication in written form. In particular, we are interested in exploring both students' engagement and the collaborative processes realised, and the development of collective mathematical thinking. Concerning the collaborative processes realised by students, we introduce the dimension of social modes of co-construction within the framework by Weinberger and Fischer (2006) to analyse argumentative knowledge construction in computer-supported collaborative learning.

In order to address the research problem concerning how digital technologies can foster remote collaboration as an added value in mathematical discussions by extending classroom time and space beyond the physical structure and constraints of the school, we introduce the didactic methodology of Digital Mathematical Discussion in chapter 2. The introduction and the design of Digital Mathematical Discussion seek to complement traditional mathematical discussion, providing an added value rather than replacing it. Digital Mathematical Discussion is designed following a design-based research methodology (Cobb et al., 2003; Design-Based Research Collective, 2003), encompassing an initial design based on principles derived from the theoretical background of mathematical and asynchronous discussions, and successive redesign following an iterative structure of enactment, analysis and refinement of the initial design. Digital Mathematical Discussion consists of three phases: group problem-solving in chat environments, whole-class balance discussion on Padlet, and a final wrap-up classroom discussion. While all phases contribute to the overall didactic methodology, this thesis focuses primarily on the first two, which are conducted asynchronously and require the support of specific digital environments.

Building on research experiences in promoting the use of algebraic language to develop proofs in arithmetic within discussion contexts, we conduct Digital Mathematical Discussions focused on algebra. In chapter 3, we introduce the vision of algebra as a thinking tool we want to promote and the tasks developed and already experimented in traditional mathematical discussions (Cusi, 2008, 2009A, 2009B; Swidan et al., 2023) addressed in the experimentations carried out during this research. These tasks encourage students to investigate numerical scenarios, develop conjectures supported by examples, and construct proofs using algebraic language. In order to investigate the use of algebra as a thinking tool developed by students, we focus on the epistemic aspects of algebraic thinking, encompassing the representation registers (Duval, 2006), conceptual frames (Arzarello et al., 2001), and anticipating thoughts (Boero, 2001).

In chapter 4, we present the research problem and the research questions guiding this study as well as the context of the experimentations and the methods of data analysis. As our research is situated in educational settings where mathematical discussion promotes learning as a social and collaborative activity, we are interested in investigating the collaborative processes occurring between students and the co-construction of collective mathematical thinking processes within the asynchronous phases of Digital Mathematical Discussion. Thus, the research questions we address concern students' social modes of co-construction fostered within the digital environments facilitating Digital Mathematical Discussions and the ways in which students collectively develop algebraic thinking, focusing on the interplay between collaborative processes and epistemic aspects of algebraic thinking. Moreover, we aim to investigate students' perspectives on their experience with the didactic methodology of Digital Mathematical Discussion focusing particularly on asynchrony and written communication as its key features (Trouche, 2004). The research project reported in this thesis has been approved by the ethical committee from the University of Milan (reference number 60/23).

In chapter 5, we report the analyses of the data collected within the experimentations through the lenses of the social modes of co-construction and the epistemic aspects of algebraic thinking.

The research carried out within this project also provides valuable insights for the design and development of a platform to facilitate such discussions. As confirmed by the systematic review of research experiences in mathematics education on mathematical asynchronous online discussions, the issues emerged during the design and implementation of Digital Mathematical Discussions and the results our data analysis, an adequate digital environment for supporting Digital Mathematical Discussions in an optimal way does not exist at the moment. The design and the development of a prototype platform to support the implementation of Digital Mathematical Discussion, developed in collaboration with a multidisciplinary team, is reported in chapter 6.

Finally, in chapter 7, we present the results obtained within the research reported in this thesis. The conclusions are drawn from the data analysis and discussed with respect to the research literature. The chapter ends with possible future developments identified through the conducted research.

Chapter 1

Setting the study and literature review

In this chapter, we introduce the theoretical foundations of the research presented in this thesis. Section 1.1 outlines the Vygotskyan paradigm of teaching and learning, which serves as the basis for the theoretical frameworks employed in the study. In particular, we emphasise the relationship between the appropriation of language and the development of thought, as well as the essential role played by the teacher during collective activities in fostering advancement in students' thinking processes. In section 1.2, we present the concept of mathematical discussion, focusing on the role the teacher can play to orchestrate mathematical discussions (subsection 1.2.1) and on the support digital technologies can give for the implementation of mathematical discussions (subsection 1.2.2). In section 1.3, we present the concept of asynchronous discussion, accompanied by a theoretical framework for argumentative knowledge construction in computer-supported collaborative learning (subsection 1.3.1) and a systematic review of literature on mathematical asynchronous online discussion (subsection 1.3.2). Finally, section 1.4 we introduce the research problem addressed in this study, in light of the characterisation of mathematical discussion and asynchronous discussion provided in the preceding sections.

1.1 The relationship between the development of thought, language and social interaction: a Vygotskyan paradigm

The research presented in this thesis is situated within a Vygotskyan approach to the teaching and learning of mathematics. The main aspects of Vygotsky's thought relevant to this research are presented in the texts *"Thought and Language"* (Vygotsky, 1962) and *"Mind in Society: The Development of Higher Mental Processes"* (Vygotsky, 1978). The aim of this section is to highlight some key elements of Vygotsky's thought that underpin the theoretical background of this research.

Vygotsky (1962) in his research addresses the relationship between thought and language and argues that their unity lies in the meanings of words. He emphasises that the meaning of words represents a synthesis of the social function of speech, enabling communication, and the intellectual function of speech, enabling the transmission and genuine understanding of thoughts. The researcher schematically represents the relationship between thought and language with two intersecting circles. The intersection represents *verbal thought*, where thought and language coincide. However, this overlap does not encompass all forms of thought and language. There remains a broad area of thought unrelated to language. The fusion of thought and language is therefore limited to a specific area: non-verbal thought and non-intellectual language do not participate in this fusion and are only indirectly influenced by verbal thought processes. Vygotsky highlights the social nature of verbal thought, noting that it is not innate but shaped by historical and cultural processes. The relationship between thought and language is thus not a prerequisite for the historical development of human consciousness but rather its product. The researcher views the meaning of words as a phenomenon of verbal thought, existing as a phenomenon of thought only when embedded in language and simultaneously as a phenomenon of language only when connected to and illuminated by thought. In the historical evolution of language, the essential structure of word meanings changes, as does their psychological nature. This transformation involves not only the content of the words but also the way reality is generalised and reflected through them. As word meanings evolve in their fundamental nature, taking on different roles in thought processes, the relationship between thought and language also changes. In his work *"Thought and Language"*, Vygotsky (1962) describes the development of thought and language, highlighting how their relationship evolves over time. He notes that during a child's development, the progression of thought and language use appears independent until the child feels the need to associate new words with specific objects. Vygotsky (1962) identifies four stages of language development:

1. *The primitive or natural stage*: This corresponds to pre-intellectual speech and pre-verbal thought.
2. *The stage of naïve psychology*: Children correctly use grammatical forms and structures without fully understanding the associated logical operations. The child can control the syntax of language but not the syntax of thought.
3. *The external operational stage*: The child uses external operations as supports to solve internal problems (e.g., counting with fingers). Language development reaches the level of egocentric speech, that is the one the child uses to accompany their activity with words.
4. *The ingrowth stage*: The child internalises external operations, learning to work with internal signs (e.g., counting mentally or using logical memory). This stage corresponds to the development of inner speech.

Vygotsky (1978) considers the child's egocentric speech as a transitional form between external and internal speech, serving as a foundation for both inner speech and, in its external form, communicative speech. The researcher states that a stimulus for producing egocentric speech can be provided to children by presenting them with a complex activity associated with constructing a less automatic solution. The verbal search for a strategy to address the problem highlights the close connection between egocentric and socialised speech. When children realise they are unable to solve a problem independently, they turn to an adult, attempting to verbally describe the method they could not complete on their own. The key moment in the development of the ability to use language as a problem-solving tool coincides with the point where socialised speech, initially used to communicate with an adult, is repurposed for communicating with oneself. In this way, language assumes an intrapersonal function in addition to its interpersonal one, enabling the children to apply a social approach to problem-solving on themselves. The deepest level of verbal thought, beyond inner speech, is thought itself. The flow of thought does not always coincide with the development of speech. These two processes do not align perfectly, and there is no strict correspondence between the units of thought and those of speech.

Vygotsky (1962) underscores that direct communication between different minds is impossible, either physically or psychologically. Communication occurs in a circular manner: thought must first pass through meanings and then through words. The development of verbal thought moves from the motives behind thought to its full formation, first in inner speech, then in word meanings, and finally in spoken words. To truly understand individuals' speech, one must not only comprehend their words but also their thoughts and motivations.

In the chapter dedicated to the interaction between learning and development, Vygotsky (1978) observes how these two concepts are strongly interrelated from the very first day of a child's life and he highlights that associating developmental levels with specific ages is insufficient to describe the actual relationships between developmental processes and learning capabilities. To address these limitations, he proposes two distinct levels of development. On one hand, there is the traditional concept of the *actual developmental level*, associated with the child's mental functions developed through the completion of specific developmental cycles. This level reflects what the child can accomplish by themselves when solving problems. On the other hand, Vygotsky introduces a focus on abilities related to the child's capacity to complete reasoning initiated by others or solve problems after observing a teacher's demonstration - skills that had not previously been considered indicative of mental development. Vygotsky (1978) suggests that what an individual can achieve with assistance is more indicative of their mental development than what they can do alone. This challenges the notion that a child's mental development is solely determined by their age or independent work abilities. Children of the same age and with the same actual developmental level can achieve significantly different outcomes when supported by someone else, such as a teacher or a more experienced peer. According to Vygotsky, this means that children of the same chronological age and actual developmental level do not necessarily share the same mental development.

To better describe this statement, Vygotsky introduces the concept of the *zone of proximal development*, which represents the distance between the actual developmental level, demonstrated when the child works independently, and the *potential developmental level*, revealed when the child solves a problem with guidance by an adult or collaboration with more skilled peers. The zone of proximal development thus defines not the functions that have already fully matured but those that are still in the process of development. While the actual developmental level provides a retrospective view of mental development, the zone of proximal development offers a prospective perspective, characterising the development that has yet to occur. Studying an individual through their zone of proximal development reveals not only the maturation processes that have been completed but also those still in their early stages. Thus, while two individuals may have the same mental development concerning completed developmental cycles, their incomplete developmental dynamics may unfold differently.

The introduction of the zone of proximal development requires, according to Vygotsky (1978), a re-evaluation of the role of imitation in learning. Contrary to the idea that only independent activity, and not imitation, indicates real mental development, Vygotsky argues that imitation is not a mechanical process. An individual can imitate problem-solving demonstrated by an adult only if the problem falls within the type of activities aligned with their developmental level. Children are capable of imitating many actions beyond their independent abilities, demonstrating that they can achieve more when collaborating with others or guided by adults. This observation leads to an essential conclusion about the relationship between development and learning: learning that targets levels of development already reached by the child is ineffective for their overall development. Rather than advancing them to the next level, it leaves them behind. The only effective learning is that which anticipates development. A key characteristic of learning is that it creates zones of proximal development, that is learning activates a range of internal developmental processes that can only occur through interaction or collaboration with others. Once these processes are internalised, they become part of the child's independent developmental achievements. Thus learning does not equate to development; however, appropriately organised learning results in mental development. It activates a range of developmental processes in psychological functions that would not be possible in contexts other than learning. In this view, mastering an activity does not represent the culmination of a developmental process but rather forms the foundation for the subsequent development of complex internal thought processes.

1.2 Mathematical discussion

The term *mathematical discussion* was formally introduced into educational research by Pirie and Schwarzenberger (1988) as a:

"*purposeful talk*

i.e., there are well-defined goals even if not every participant is aware of them. These goals may have been set by the group or by the teacher but they are, implicitly or explicitly, accepted by the group as a whole.

on a mathematical subject

i.e., either the goals themselves, or a subsidiary goal which emerges during the course of the talking, are expressed in terms of mathematical content or process.

in which there are genuine pupil contributions

i.e., input from at least some of the pupils which assists the talk or thinking to move forwards. We are attempting here to distinguish between the introduction of new elements to the discussion and mere passive response, such as factual answers, to teachers' questions.

and interaction

i.e., indications that the movement within the talk has been picked up by other participants. This may be evidenced by changes of attitude within the group, by linguistic clues of mental acknowledgement, or by physical reactions which show that critical listening has taken place, but *not* by mere instrumental reaction to being told what to do by the teacher or by another pupil." (Pirie & Schwarzenberger, 1988, p. 461)

This definition introduces mathematical discussion as a discourse focused on a mathematical topic that involves both students' original contributions and interaction. The framework within which the discussion is developed is Vygotskyan in nature (Bartolini Bussi et al., 1995). The first key aspect of Vygotsky's thought relevant to research on discussion lies in the central and foundational importance of interpersonal relationships. However, in Pirie and Schwarzenberger's definition, no emphasis is placed on the role of the teacher, as they do not differentiate between discussions conducted with the teacher's involvement and peer interactions. Framing mathematical discussion within a Vygotskyan perspective, the role of the teacher emerges as crucial for the importance of experts' contribution in helping students bridging the gap between their potential and their actual developmental level (Vygotsky, 1978). For this reason, Bartolini Bussi (1996) proposes the following definition of mathematical discussion:

"*Mathematical Discussion is a polyphony of articulated voices on a mathematical object (e.g. a concept, a problem, a procedure, a structure, an idea or a belief about mathematics), that is one of the motives of the teaching-learning activity.*" (Bartolini Bussi, 1996, p.16)

which explicitly references to the teaching and learning activity and, in the polyphony of voices, includes the teacher's voice, which represents the cultural community outside the classroom and is institutionally designated for the transmission of knowledge. The term voice is used by Bartolini Bussi in the sense of Bakhtin (1982), referring to a form of discourse and thought that reflects a subject's perspective, conceptual horizon, intent, and worldview. A voice thus has both an internal component (thought) and an external component (speech), enabling communication and interpretation. The term motive, on the other hand, is used in the sense of Leont'ev (1978) to describe the object (concrete or mental) that meets the needs of the acting subjects. In the definition of mathematical discussion, it refers to the overarching object of the collective activity, which unfolds over the long term and does not necessarily coincide with the specific goals of the individual actions that compose it (Bartolini Bussi et al., 1995).

The polyphony of voices present in the definition naturally evokes the musical metaphor of orchestration. Such metaphor has been used in different ways within research in mathematics education, for instance related to internal cognitive functions (Ruthven, 2002) or to the use of instruments (Trouche, 2004), and it is generally related to the management of a learning environment blending different teaching strategies (McKenzie, 2001). With respect to mathematical discussion, the metaphor should be interpreted as the teacher guiding students' different voices according to the vision of the class as a jazz band, with both novice and advanced musicians, and the teacher as the band leader who has prepared an overall arrangement but encourages improvisation and interpretation by the students, accommodating input at various levels (Drijvers et al., 2010).

In a discussion, the different voices do not necessarily correspond to the verbal expressions of the physically present interlocutors (Bartolini Bussi et al., 1995). Two interlocutors may share the same voice when their discourse is aligned, or a single interlocutor may employ different voices when quoting or imitating another's discourse or introducing a new perspective on the same problem. Moreover, in the polyphony of voices, Bartolini Bussi and colleagues consider the presence of imitative voices as essential, while Pirie and Schwarzenberger stress the value of students' original contributions. Furthermore, Wood (1999) emphasises the importance of guiding students to become effective listeners as listening entails the ability to focus on what others are saying with the aim of assessing its coherence and proposing potential counterarguments.

For the various voices to be recognised by participants in a discussion, time is needed for their explicit articulation. This perspective contrasts with traditional classroom settings strongly controlled by the teacher, where any voice diverging from the teacher's is immediately evaluated as irrelevant or incorrect, leading to a monologic form of knowledge (Bartolini Bussi et al., 1995). In fact, classroom observations typically reveal the predominance of IRF discourse, that is initiation (I) by the teacher, response (R) by the student, and feedback (F) by the teacher. This pattern leads teachers to direct students' contributions into the predefined direction of the lesson through a back-and-forth management style, often unconsciously (Bartolini Bussi et al., 1995). Furthermore, when a teacher realises that a student is struggling to provide the expected response, they progressively lower their expectations and pose increasingly leading questions that directly guide the student toward the solution (Bartolini Bussi & Boni, 1995). This teaching strategy can clash with the student's desire to engage in a constructive effort. However, the power dynamics and the obligations to teach and learn embedded in the social structure of the classroom often dictate a fixed outcome: the articulation of the expected response. This is typically provided by the teacher but sometimes also by the student, who is left to merely complete an almost fully formulated sentence (Bartolini Bussi & Boni, 1995). Pirie and Schwarzenberger (1988) also report that operational statements - assertions that merely describe actions and are therefore closely linked to instrumental understanding (Skemp, 1978) - often dominate over reflective statements. In contrast, reflective statements aim to describe concepts and the relationships between them, serving as indicators of relational understanding (Skemp, 1978), and should be actively encouraged. Kyriacou and Issitt (2007) highlight the limitations of the IRF approach and identify eight characteristics of teacher-student dialogues that are most effective in promoting mathematical understanding, which align with mathematical discussion:

1. Moving beyond the IRF approach: using open-ended questions to encourage students to justify their assertions or comment on the responses of others;
2. Fostering genuine mathematical discourse: engaging students in meaningful conversations about mathematics rather than merely aiming for the "correct" answer;
3. Creating a collaborative learning environment: encouraging students and teachers to work together to explore mathematical problems;
4. Practicing transformative listening: demonstrating openness to revising the teacher's ideas based on students' contributions, making them feel valued;
5. Providing scaffolding: using dialogue to support students in tackling tasks they could not handle independently;
6. Promoting awareness of the value of classroom dialogue: helping students recognise that class discussions can be a rich learning experience and encouraging them to develop the skills necessary to make the most of it;
7. Encouraging high-quality discourse: supporting students in developing a type of discussion that is thoughtful and productive;
8. Inclusive teaching: conveying to students that every contribution is taken seriously and evaluated equally, fostering a sense of inclusion.

Another classroom discourse practice that should be abandoned to enable the class to move forward mathematically is show-and-tell (introduced by Ball, 2001, and referenced by Stein et al., 2008). In show-and-tell, students who have produced correct solutions take turns presenting their problem-solving strategies. In show-and-tells, one student presentation follows another with minimal commentary from either the teacher or other students. Additionally, there is little to no guidance in drawing connections between the methods presented or linking them to broadly accepted disciplinary methods and concepts.

Bartolini Bussi (1991) distinguishes different types of mathematical discussions that can be promoted, depending on the objectives related to conducting a discussion and the focus set by the teacher, encompassing:

- *Problem discussion* (or mathematization discussion) that arises from a text or an open problem situation that can be modelled by means of mathematical concepts or procedures. Problem discussion can be a *problem-solving discussion*, that is collective solution of a mathematical problem, or a *balance discussion*, that is socialization and collective evaluation of strategies set up by students in individual or small-group work. In Bartolini Bussi's description of balance discussions, the discussion takes place some days (or even some weeks) after the individual or small-group work. This delay fosters the distancing of students from their own work and product, allowing for a more detached consideration. Moreover, rather than comparing all strategies, the teacher selects different prototype strategies or different representations of the problem and invites students to identify the strategy most similar to their own. Then, each student participates in the discussion, defending their position if necessary. The goal is to reach consensus on one or more strategies, but if agreement on a mathematically sound strategy cannot be reached, the teacher institutionalizes the fact that a shared strategy is not possible at that time, and the problem will be revisited later since embracing periods of uncertainty is an important aspect of classroom custom.
- *Conceptualization discussion* (or weaving discussion) is raised by a provocative question to encourage students to construct or enrich the meaning of some experiences – whether in or outside school – in which mathematical concepts may have been used only as tools to solve problems. This type of discussion is accessible even for very young students due to the presence of a particular word (e.g., number, graph, infinity) and related terminology in both mathematical and everyday language. In these discussions, students cope with a problem that recurs throughout the history of mathematics: defining a concept so it can be integrated into scientific knowledge, rather than used solely as a problem-solving tool. Students thus encounter genuine epistemological obstacles, and their expressions often echo those of mathematicians from the past, allowing for meaningful comparisons. Conceptualization discussions can be introduced at the beginning of a teaching sequence to collect and share individual experiences, or after certain classroom activities (e.g., individual tasks or balance discussions) to support the collective weaving of a conceptual network surrounding a particular concept. This type of discussion allows the teacher to observe the development of initial conceptions and to adjust the planning of subsequent activities towards formulations that increasingly align with socially recognised definitions, highlighting both similarities and contrasts between mathematical and everyday language registers.

- *Meta-discussion* initiates by a proposal to negotiate interaction norms, evaluate a task or discussion as a teaching aid, or assess the whole didactical sequence. The term meta-discussion is justified by evidence of metacognitive activity. This type of discussion can be introduced directly by the teacher or indirectly by collectively reading transcripts from a previous discussion. In the latter case, it aims explicitly at the construction of the history of the group. Meta-discussion provides valuable opportunities to engage in discourse about mathematics, free from the teacher's authoritative or evaluative statements.

1.2.1 The role of the teacher in orchestrating mathematical discussions

Stein and colleagues (2008) argue that it is essential for teachers to guide classroom discussions effectively, using student-generated work to develop mathematical thinking. In line with this vision, Wood (1999) supports the idea that teachers must ensure that students develop the ability to construct meaning through their own thinking and reasoning:

"In a culture that demands student understanding, teaching is more than merely telling or showing pupils; teachers must enable pupils to create meaning through their own thinking and reasoning." (Wood, 1999, p. 189)

To support this goal, Stein and colleagues propose a model of five practices for facilitating mathematical discussions around cognitively demanding tasks. Teachers can prepare for these discussions by anticipating potential student contributions, identifying challenges or questions that might arise, and planning how to manage classroom time during the discussion. The five practices they outline are as follows:

1. *Anticipating likely student responses to cognitively demanding mathematical tasks.* Teachers should envision how students might approach the mathematical problem. This includes predicting how students might interpret the problem, the range of both correct and incorrect strategies they might use, and how these strategies could connect to the mathematical concepts, representations, procedures, and practices the teacher aims to highlight. Anticipation involves teachers solving the tasks themselves, exploring multiple solution strategies, and considering how students with varying levels of mathematical sophistication might engage with the problem, including potential misunderstandings or points of confusion.

2. *Monitoring students' responses to the tasks during the explore phase.* Teachers should closely observe and pay attention to the mathematical thinking that students demonstrate as they work on a problem during the exploration phase.

3. *Selecting particular students to present their mathematical responses during the discuss-and-summarize phase.* After monitoring student responses, teachers should select specific students to share their work with the class. This selection process allows teachers to ensure that the discussion focuses on significant mathematical ideas. Teachers retain control over which students present and, by extension, the content of the discussion, while avoiding sidelining challenging ideas or fragile students.

4. *Purposefully sequencing the student responses that will be displayed.* Teachers should determine the order in which selected students present their responses. Purposeful

sequencing can maximize the likelihood of achieving mathematical goals. For example, teachers might begin with the most commonly used strategy to validate students' work and ensure accessibility before addressing less common or more advanced approaches. Alternatively, they may start with a strategy reflecting a misconception to clarify misunderstandings before advancing to more effective problem-solving methods. Presenting related or contrasting strategies consecutively can also facilitate mathematical comparisons.

5. *Helping the class make mathematical connections between different students' responses and the key ideas.* Teachers should guide students in drawing connections between the mathematical ideas reflected in their strategies and representations. This approach transforms classroom discussions from isolated presentations of individual problem-solving methods into a cohesive dialogue that builds on collective contributions to deepen understanding of powerful mathematical concepts.

By implementing these five practices, teachers can foster richer and more meaningful classroom discussions that support mathematical learning and reasoning. At the heart of the challenge associated with student-centred practices is the need to strike a balance between granting students' authority over their mathematical work and ensuring that this work remains accountable to the discipline. The teacher must guide students collectively toward developing ideas and processes aligned with those widely regarded as valuable and essential in mathematics, as well as foundational for future mathematical learning in school. Efforts to promote accountability to the discipline, however, can unintentionally undermine students' authority and their sense of engaged meaning-making. For instance, as a result, students may shift from sharing their authentic thoughts about a problem to reporting what they believe will garner a favourable response from the teacher. Additionally, some teachers, when students reveal misunderstandings, may struggle to resist the urge to directly correct students' answers. This tendency can further erode students' mathematical authority by discouraging them from relying on their reasoning to assess the validity of their own or their peers' ideas. The model by Stein and colleagues addresses this issue by supporting accountability to the discipline while preserving students' mathematical authority. It does so through a set of teaching practices that use students' ideas as the foundation for discussions, shaping class dialogue in ways that gradually surface significant mathematical concepts, expose contradictions, and develop or consolidate understanding.

Thus, the role of the teacher in discussions is central in promoting students' mathematical understanding by choosing appropriate tasks and providing students with expert assistance while encouraging the explanation of their ideas and conjectures, identifying the potential of these ideas and charting the direction of their further development (Leikin & Dinur, 2007). Facilitating classroom discussions remains a challenge for many teachers (Stein et al., 2008). Compared to delivering a lecture, guiding a discussion around a task that can be solved in multiple ways significantly reduces the teacher's control over the course of the lesson. Moreover, the flexibility required to teachers in such situations cannot be improvised but must be developed over time. Leikin and Dinur (2007), in discussing teacher flexibility, compare lesson plans with the actual events and processes that occur in the classroom. They differentiate between the "planned learning trajectory", which represents the hypothetical learning pathway outlined for a particular lesson or discussion, and the "actual learning trajectory", which encompasses the events and procedures that unfold in reality. While the research literature often broadly defines teacher flexibility as the ability to act on students' contributions, Leikin and Dinur refine this definition by distinguishing between expected and unexpected students' responses. When a student's response is anticipated by the teacher and aligns with the lesson plan, no adjustments are necessary, and the teacher naturally exhibits flexibility. However, when a student's response is unforeseen or disrupts the teacher's agenda – meaning it does not align with the planned trajectory – the teacher faces a challenge. In such cases, the teacher must navigate the tension between the need to adapt the planned trajectory and the effort to complete what was initially planned.

Concerning the orchestration of balance discussions, Bartolini Bussi and colleagues (1995) highlight different roles the teacher can play. Indeed, experiments they conducted have identified a standard framework for balance discussions, divided into several phases: the *real balance*, aimed at comparing strategies; the *explicitation of solution processes*, aimed at reconstructing and socializing individual processes; the *explicitation of learning*, aimed at identifying the new elements introduced by the problem in the students' individual experiences and the collective history of the class; and the *institutionalization of learning*, aimed at formulating the concepts and procedures that should be remembered and linking them to prior knowledge.

The *real balance* is introduced by the teacher, usually by presenting a prototype for each category of individual work that has already been identified, and students are invited to identify themselves with one of the works. Next, the different strategies are collectively discussed, reviewed, and evaluated. In this phase, the teacher often highlights not only the points of agreement but also the conflicts between different solutions. The teacher shapes the communication, providing suitable expressions for students who cannot find the right words on their own, and introduces correct terminology, especially when students are inclined to agree on phrasing that is not consistent with mathematical conventions. By introducing this phase, the teacher aims to foster a positive attitude among students towards the collective construction of mathematical knowledge, counteracting the widespread stereotype that the learning process is exclusively individual and reserved for a select few gifted students.

The *explicitation of solution processes* is introduced by the teacher through questions, such as: "What difficulties did you encounter?". In fact, Bartolini Bussi and colleagues (1995) have experimentally confirmed that simply asking how a particular solution was developed is not enough, as students tend to overlook unsuccessful attempts, either because they no longer consider them important or because they are reluctant to discuss false paths they followed. Yet, these false starts are actually valuable for understanding problem-solving strategies. The teacher assists students in this reconstruction not only through direct questions, but by using various techniques, among which *mirroring* (repeating a student's intervention) plays a special role. By introducing this phase, the teacher aims to promote a positive attitude towards mistakes, viewing them not as deviations but as necessary steps in the construction of knowledge.

The *explicitation of learning* is introduced by the teacher with a question like: "What have we learned from this problem?". The focus in this phase is on what is new compared to already established knowledge. Students are encouraged to generalise the problem-solving strategy so that they can move away from the specific problem at hand and apply it to a broader set of situations. The teacher supports students in this process by paraphrasing their strategies and introducing universal quantifiers. These more general strategies should be reapplied to new problems. By introducing this phase, the teacher also aims to convey to students that the problem has had an impact on their learning, making it a significant milestone in the life of the class as a community. It is an opportunity to reflect on both individual and collective learning journeys.

The *institutionalization of learning* is managed by the teacher, recognised as the representative of mathematical culture in the class. This phase is introduced whenever the previous discussion process has led to agreement on one or more mathematically acceptable problem-solving strategies. When this is not possible (for example, if the majority of students seem to agree on a solution that is not mathematically valid), the teacher pauses the institutionalization, possibly suggesting counterexamples. Bartolini Bussi and colleagues (1995) affirm that a mathematically acceptable solution does not always need to be expressed in the terms of the current mathematical framework; solutions reflecting historically documented stages of knowledge development are acceptable. The researchers have found that accepting even long periods of uncertainty is not only necessary but also feasible, although it may initially cause some difficulties. In the long run, the teacher creates a classroom culture in which students recognise that the teacher knows the answer but refrains from revealing it because they must discover it on their own.

The effects of the outlined framework for balance discussions are observable both in the short and long term (Bartolini Bussi et al., 1995). In the short term, the framework structures the discussion, creating order in a large-group interaction that might otherwise be chaotic and unproductive. In the long term, students' participation in discussions with similar frameworks becomes internalised as a pattern for individually verbalising problem solutions. This includes clarifying solution processes, discussing unsuccessful attempts, and commenting on the generalization of strategies to broader classes of problems. The quality of participation in discussions also evolves. In the long term, students engage in a type of discourse where the teacher's requests are not interpreted literally, and they begin to spontaneously reconstruct their learning history. Some students even start to play roles usually assumed by the teacher.

Following the fifth practice by Stein and colleagues (2008) for designing and implementing the actual orchestration of discussions, and Bartolini Bussi and colleagues' (1995) focus on the orchestration of balance discussions, Cusi and Malara (2009, 2013) introduce the Model of Aware and Effective Attitudes and Behaviours (M-AEAB). The M-AEAB is a theoretical construct aimed at characterising the role played by the teacher during classroom discussions to foster students' cognitive and metacognitive processes. The construct has been developed and applies specifically to discussions aimed at fostering the use of algebraic language as a thinking tool. This aligns with the mathematical content addressed within the research reported in this thesis, as it will be presented in chapter 3.

The framework within which the M-AEAB construct has been conceived is based on two sets of theoretical components. The first set is constituted by those components that frame the investigation of teaching-learning processes and the role played by the teacher:

- Vygotsky's (1978) ideas about the importance of social interaction in the processes of developing thinking, and the central role of contributions from adults or more knowledgeable peers in helping students overcome the gap between potential and actual development;
- Leont'ev's (1978) reflections on the role of activities carried out in social contexts, not only in promoting the sharing of collectively constructed meanings but also in helping individuals develop their personal sense of these activities and the reasons that facilitate or hinder the development of awareness of learning processes;
- Collins and colleagues' (1989) cognitive apprenticeship model, which emphasises the importance of teaching aimed at "making thinking visible" by activating specific methods that enable students to observe, discover, and/or invent strategies typical of an expert approach.

The second set focuses on the development of thinking processes through algebraic language (Arzarello at al., 2001), emphasising the importance of activating conceptual frames and suitable changes from one frame to another, appropriate anticipating thoughts (Boero, 2001), and coordination between different representation registers (Duval, 2006). This set of theoretical components will be deepened in section 3.2 as part of the theoretical framework of the research reported in this thesis too. The M-AEAB construct outlines the approach of a teacher who consciously activates specific roles aimed at making thinking visible, allowing students to focus not only on syntactic aspects but also on effective strategies and meta-reflections on the activated processes. Moreover, as Stein and colleagues (2008) suggest, helping the class establish mathematical connections between students' responses and key ideas is the most complex strategy to implement effectively. The M-AEAB construct provides insights into the roles teachers can adopt to design and implement effective prompts aimed at fostering these processes during discussions.

The M-AEAB construct provides transparent indicators to describe teachers' effective and aware actions aimed at making thinking visible to their students and at fostering metacognitive reflections during classroom discussions. These indicators are introduced by means of two groups of roles that the teacher can enact during mathematical discussions:

1) the first four roles are those performed when the teacher poses as a learner who faces problems making the hidden thinking visible, highlighting the objectives, the choices of the strategies and the interpretation of results;
2) the last three roles are those mainly played with the aim of fostering students' reflections at a metacognitive level, focusing on thinking processes and on the effectiveness of the implemented strategies.

In Table 1.1 the different roles a teacher can play as a M-AEAB, their characterisations and the indicators to code the roles are schematically presented.

First group of roles		
Role as a M-$_{AE}$AB	*Characterisation of the role*	*Indicators to code the role*
Investigating subject	The teacher encourages students to take an inquiry approach to problems and to feel involved in the activity as a group	The teacher uses the first-person plural when when asking questions and accepts the different proposals without judging them
Practical-strategic guide	The teacher stimulates research into strategies for approaching a problem to be solved and acts in order to clarify how an expert might pose themselves when analysing such a problem	Typical questions: • What does this question mean? • What are we being asked to do? • What might be useful to find in order to answer this question? • Is this the only possible way to solve the problem?
Activator of anticipating thoughts	The teacher focuses students' attention on the objectives (or specific sub-objectives) of the activity the class is undertaking, or makes them explore hypotheses and identify the effects of a possible change undertaken or a possible strategy activated	Typical questions/interventions: • What is the objective? • Remember that the objective is… • Are we achieving the objective? • What result do we expect to achieve? • If we made this choice, what could we achieve?
Activator of interpretative processes	The teacher aims at clarifying the meanings of the representations that are constructed or transformed by the pupils and at stimulating a continuous interpretation of the representations and results that are obtained	Typical questions: • What does this symbol/expression/graph mean? • How can we represent this information? • How can we interpret this result? • Is this solution acceptable?

Second group of roles		
Role as a M-AEAB	*Characterisation of the role*	*Indicators to code the role*
Guide in fostering a harmonised balance between syntactic and semantic levels	The teacher makes students reflect on the syntactic correctness of certain transformations carried out and make connections between the processes that characterise the solution of a problem and the corresponding meanings	Typical questions: • Is this transformation correct? • Is it legitimate to simplify this expression? • Why did you make this transformation? • How did we get this result?
Reflective guide	The teacher helps students to make explicit the meaning of effective strategies/approaches used in class activities to encourage reflections on them, so that students can identify effective practical/strategic models from which to draw inspiration in tackling problems	Typical questions/interventions: • Could you explain your reasoning to your classmates? • Is there anyone who could explain their reasoning? • The teacher argues in this way: "Since I want get this kind of result, I could…" (The teacher speaks as if she/he were the student, repeating the same words as the student or reformulating her/his argument) • Is it clear what she/he said? She/he noticed that… (the teacher repeats what a student said referring to them with the singular third-person)
Activator of reflective attitudes and metacognitive acts	The teacher stimulates and provokes meta-level attitudes, by helping students exchange and compare different arguments/strategies to reflect on their strengths/weaknesses	Typical questions: • Do you agree with what she/he said /what is written? • Do you think this is an effective choice/strategy? Why? • What are the differences between theses answers? What do they have in common? • Was this task difficult for you? Why?

Table 1.1 Characterisation of the roles within the M-AEAB construct (Cusi & Morselli, 2024)

The M-AEAB construct is used in research in mathematics education in different ways (Cusi & Malara, 2015; Cusi & Morselli, 2018, 2024), particularly as a tool for teacher education, as it could be used:

- A-priori, to plan a mathematical discussion, as the teacher can design a mathematical discussion (for example, by choosing a problem or selecting specific excerpts from students' productions) in line with specific roles of the M-$_{AE}$AB construct to foster particular cognitive and metacognitive reflections by students;
- A-posteriori, to interpret and analyse teachers' roles during classroom discussions.

1.2.2 The support provided by digital technologies for the implementation of mathematical discussions: tracing its evolution

The research presented in this thesis aims to contribute to the ongoing exploration of the opportunities that digital technologies offer to enhance the teaching and learning of mathematics through collaboration and communication beyond the traditional classroom setting (Ball & Barzel, 2018). This research issue has become central in mathematics education over the past decades, particularly for fully online courses (see, for example, Geraniou & Crisan, 2019), and it received a significant boost during the Covid-19 pandemic as institutions were compelled to adopt distance learning (see, for instance, Padayachee & Campbell, 2022). The challenge of transitioning to distance learning was not merely about making minor adjustments or integrating digital tools into traditional teaching methods, but about creating a new type of school that can address new challenges (Aldon et al., 2021). The struggle to meet this challenge was real, as most teachers faced difficulties in establishing online classroom environments where students could actively engage in their learning, resulting in a prevalence towards transmissive approach (Aldon et al., 2021). Moreover, the challenge is not merely about making minor adjustments or integrating digital tools into traditional teaching methods, but about creating a "new type of school" that can address modern needs (Aldon et al., 2021), also taking into account the cultural background of modern students, grown within the digital culture (O'Reilly, 2005). Therefore, the need to harness the potential of digital technologies to stimulate collaboration and communication between students and with the teacher in online settings has emerged as a significant challenge. In education, this issue becomes even more delicate, as digital environments affect both interaction between participants and the subsequent development of mathematical thinking. In our research, we have chosen to confront this challenge by investigating the potential offered by digital technologies in the context of mathematical discussions. In fact, mathematical discussion, as presented in section 1.2, is a didactic methodology aimed at actively engaging students in the construction of mathematical knowledge through interaction with peers and experts, thus it represents an excellent testing ground to explore the role of digital technologies in providing new modes of communication and collaboration outside traditional classroom environments to promote the development of mathematical thinking. Moreover, mathematical discussion has been intertwined with digital technologies in various ways and for different purposes since its early stages. Indeed, mathematical discussion in the sense of Bartolini Bussi (1996) has been closely related to the theory of semiotic mediation (Bartolini Bussi & Mariotti, 2008), as mathematical discussion fosters the collective production of signs related to activities with artifacts, that has been studied also with respect to the use of digital artifacts within classroom situations (Mariotti, 2000). The further step of the relation between mathematical discussion and digital technologies may be represented by the instrumental orchestration in technology-rich environments in which digital artifacts are embedded in computerised learning environments (Drijvers et al., 2010). In technology-rich classrooms, instrumental orchestration (Trouche, 2004) concerns the teacher's intentional and systematic organisation and use of digital artifacts in a computerised learning environment in order to guide students' instrumental genesis. Some orchestration types emerging within such contexts concern whole-class discussions (e.g., discuss-the-screen, spot-and-show, sherpa-at-work) involving students in discussions that concern both the digital mathematical environment in which the activity is carried out and the digital artifact embedded in the environment (Drijvers et al., 2010). The presence of the digital mathematical environment and the related orchestration types can also support the teacher in conducting the discussion since they also provide practical support in showing and addressing students' production to be presented and discussed with the whole class.

Concerning the support that digital environments can give to teachers while organising and conducting mathematical discussions, the project "Improving progress for lower achievers through Formative Assessment in Science and Mathematics Education" (FaSMEd) represents another relevant example. The project has witnessed the integration of a connected classroom technology to support teachers in organising mathematical discussions with the aim of helping them activate formative assessment strategies (Cusi et al., 2017). Such connected classroom technology allows teachers to send through a computer digital worksheets or polls to students' tablets and to show their responses to the worksheets or statistics about the polls on an interactive whiteboard supporting the sharing of results, opinions and reflections within a classroom discussion. This connected classroom technology enables the sharing of students' work and ideas both during and at the end of discussions, allowing for timely feedback from both peers and the teacher, and both on the mathematical content (e.g., identifying correct solutions) and on metacognitive aspects (e.g., reflecting on the difficulty of understanding the problem or what was learned during the discussion). This promotes a dynamic and reflective learning environment that enhances student engagement and understanding. Moreover, it is stressed how crucial are the roles the teacher could activate to orchestrate mathematical discussions, with formative assessment purposed, through the support provided by specific digital technologies (Cusi & Morselli, 2024).

Another significant example of the use of digital technologies to carry out mathematical discussions is the project "Digital Interactive Storytelling in Mathematics: a competence-based social approach" (DIST-M). Over time, the project has developed through various expansive spiral cycles (Engeström & Sannino, 2010), leading to its evolution (Albano et al., 2016, 2020, 2022, 2024). We present in the following the most updated evolution of the model (Albano et al., 2024). The methodological framework of DIST-M revolves around a digital story, presented as a comic, that unfolds through episodes. The progression of the story corresponds to the evolution of the process of mathematical proof, as each episode focuses sequentially on exploring, conjecturing, formalizing, and proving a mathematical problem. The story centres on five animated characters who face mathematical problems arising from the storyline. The story is structured around a basic script and evolves through the improvisation of the characters' actions and dialogues. Thus, each student becomes a protagonist of the story taking on the role of one of the story's characters and, within this narrative, performs specific actions, interacting with other students to shape the story's development. The same applies to the expert mediator's role, who is also a character in the story, generally played by the teacher (Albano et al., 2021). The characters in the story represent different voices (in line with Bartolini Bussi, 1996) in mathematical problem-solving and are associated with specific cognitive functions. The educational objective of the model is for students to appropriate and internalise these roles through social practice. This requires not only reflecting on the appropriate actions for each role but also on the nature and characteristics of those actions. For this reason, the model is designed for students to engage with the story in two distinct ways. For each episode: one group acts as actors, that is each student in the group assumes the role of one of the characters, and the other groups act as observers, that is they are active and mindful of the actors in the story and each student is assigned to observe a specific character in the story and reflect on how that character behaves both in relation to the mathematical problem and the role they are playing. The collaboration and interaction between roles unfold in each episode through a two-phase structure:

1. *Peer-interaction phase*: During this phase, interaction occurs exclusively within the group of peers without explicit intervention from the expert. However, the expert remains in the background, maintaining a private communication channel with a specific character of the story, if needed. The peer-interaction phase concludes when the group reaches a shared product in response to the problem posed by the current episode.
2. *Discussion phase*: The group's shared product is discussed explicitly with the expert. This phase serves to consolidate what the group has produced, enabling the transition to the next episode.

The two phases unfold synchronously in specific digital environments hosted in the Moodle[1] platform, that is chats, for the first phase, and a forum, for the second phase. Furthermore, the teacher in the role of the expert within the story can provide students with specific technologies for mathematics (such as spreadsheets, CAS, and DGS), as well as purpose-built technologies designed to support the development of particular competencies, such as linguistic toolkits (Albano & Dello Iacono, 2019).

It was the Covid-19 pandemic that gave further impulse to research concerning the relationship between digital technologies and mathematical discussions, since the integration of digital technology to facilitate mathematical discussions became necessary and there was a prominent shift from classroom discussions to online discussions mediated by different digital environments. This is the case, for example, of Baccaglini-Frank (2022) and Giberti and colleagues (2022, 2024). Baccaglini-Frank (2022) reports about mathematical online synchronous discussions realised through the support of Google Meet[2] and Jamboard[3]. Her study focuses on how digital technology can mediate the communication about mathematics, rather than mediating mathematics itself, and it is consistent with one of the roles technology can play according to Ruthven (2012), that is updating and enhancing the basic infrastructure that supports classroom communication and assisting the use of content-related resources within and beyond the classroom. Giberti and colleagues (2022, 2024) share their research experience within the M@t.abel 2020 project. The

[1] https://moodle.org/
[2] https://meet.google.com/
[3] https://jamboard.google.com/

goal the M@t.abel 2020 project wants to pursue is exploring how mathematical discussions can be conducted in multi-level hybrid learning environments, including both in-person and online activities, with synchronous and asynchronous sessions, and fostering the interaction between different classes and teachers. Their hybrid learning environment is technologically assisted by the platform Padlet[4]. Within Padlet, each student can post their solution to a mathematical task without being able to see their classmates' posts. This feature gives students the opportunity to take their time to think and explain their ideas. Once all students have submitted their solutions, the teacher reviews and approves the posts, making them visible to everyone. In this way, students can read and comment on the posts by their classmates and the Padlet with students' solutions can also be shared with other classes so that they can comment on it (multiclass level). Then a classroom mathematical discussion is started by the teachers focusing on the content on the Padlet. Giberti and colleagues (2022) consider their technologically-mediated discussion able to promote participation and inclusion by: plural representations of knowledge, for example both via verbal and visual means; plural opportunities of expression, for example letting the students choose between visual or written presentation of their work; and extension of the time at disposal to read and contribute to the notice board. Moreover, the researchers report that the role of Padlet as a mediator enables a more dynamic discussion. Indeed, traditionally, in classroom discussions, the introduction of a new issue often overshadows the previous one. However, Padlet supports the teacher and students in managing discussions with more interludes, allowing the same issue to be revisited and explored multiple times throughout the discussion. However, their research does not focus deeply into the guiding role that the teacher can assume when considering the mediation of Padlet or the role this technological mediation plays in addressing the mathematical problem under discussion. Furthermore, to address the issue of inclusion in discussions, it is essential to examine also the type of participation students engage in, focusing on how they contribute to the mathematical discourse of the class beyond their participation at stake.

1.3 Asynchronous discussions

In recent years, research in distance education and educational technology has extensively explored discussions in online environments and, especially after the Covid-19 pandemic period, remote collaboration has taken on a new and central role. Studies such as those by Johnson (2006), Gao and colleagues (2013), Fehrman and Watson (2021), and Wang (2023) have focused on online discussions and reported a general panorama of it. Fehrman and Watson (2021) define online discussions occurring in educational contexts as "learning spaces within an online learning context and [that] primarily allow students to construct knowledge within a community of other learners" (Fehrman and Watson, 2021, p. 201).

When considering discussions in online environments, different characteristics should be taken into account. First, the modality in which a discourse can be developed through online environments can be synchronous or asynchronous, whether participation occurs simultaneously or is delayed in time and space (Johnson, 2006). Secondly, it can involve multiple ways of communicating (speaking, writing text, sending images,…) depending on the technical features of the digital environment supporting the discussion and on the aims to pursue with the discussion.

[4] https://padlet.com/

Also the educational context in which such discourses can be developed vary a lot, ranging from full-online courses (e.g., MacKinnon et al., 2022) to asynchronous activities linked to in-person classes (e.g., Kontorovich, 2018), and involving students of different grades, from primary school (e.g., Jazby & Symons, 2015) to university (e.g., Savic, 2022), but also teachers' professional development programs (e.g., Taranto et al., 2020).

Within the research documented in this thesis, we are interested in online discussions occurring in asynchronous modality as they eliminate the time and place constraints imposed by classroom setting, potentially enhancing participation by offering students 24/7 access from virtually anywhere (Wang, 2023). Some of the advantages of engaging in asynchronous online discussions are that they accommodate individuals requiring additional time to participate, facilitating their contribution to the discussion by allowing them to interact with content, with other participants and with the instructor at their own pace. Moreover, the text-based nature of asynchronous online discussion is often considered an advantage, as it facilitates the integration of thinking and writing. Text-based communication may even surpass oral communication in effectiveness when the goal is to achieve higher-order cognitive learning (Garrison et al., 2000). Writing in online discussions serves as a safeguard against the impulsive and fleeting nature of traditional face-to-face discussions. Students can view their thoughts as they compose their contributions, taking the time to refine, adjust, and shape them into the desired form before sharing them publicly (Wang, 2023). Moreover, it offers a written transcript available after the discussion concludes (Andresen, 2009). The role of the teacher represents a crucial component for asynchronous online discussions too, especially considering that the interaction between the teacher and students changes significantly due to the loss of face-to-face contact (Andresen, 2009). Teacher's interventions can span from being entirely absent from the discussion ("ghost in the wings") to being the major participant ("sage on the stage"), but generally the extremes should be avoided considering that research literature suggests that the teacher plays an active and visible part in the discussion, but avoiding overdoing since it may limit students' interactions ("guide on the side"): the quantity and type of intervention should be regulated by the purposes that the teacher sets for the discussion (Mazzolini & Maddison, 2003).

1.3.1 A framework for argumentative knowledge construction in computer-supported collaborative learning

Weinberger and Fischer (2006) have introduced a framework to analyse argumentative knowledge construction in computer-supported collaborative learning. Computer-supported collaborative learning often involves learners communicating through text-based, asynchronous discussion boards and, in this setting, learners are expected to engage in argumentative discourse with the aim of acquiring knowledge. The framework aims to analyse different process dimensions of knowledge construction that characterise students' interactions within computer-supported collaborative learning environments and consists of four dimensions: the participation dimension, the epistemic dimension, the argument dimension, and the dimension of social modes of co-construction.

The *participation dimension* provides two pieces of information: whether learners participate at all and whether they participate equally. The quantity of participation refers to the extent to which learners contribute to the discourse and indicates whether learners log in and engage with the computer-supported collaborative learning environment. A high quantity of participation suggests that learners are theoretically positioned to acquire knowledge within the environment. In text-based computer-supported collaborative learning environments, participation is often higher than in traditional classrooms due to the parallel nature of asynchronous discussion threads, which can reduce the presence of blocking effects related to in-person participation. Indeed, learners can elaborate on their contributions without interruptions from co-present peers, potentially leading to longer and more detailed messages. On the other hand, heterogeneity of participation concerns (un)equal participation of learners into the discourse. Collaborative learning in small groups may reduce participation heterogeneity because all group members are expected to contribute, unlike in whole-class discussions where some learners may participate frequently and contribute a lot, while others may participate minimally or not at all.

The *epistemic dimension*, which refers to how learners engage in the knowledge construction of the tasks they face (Fischer et al., 2002), analyses not only the quantity but also the content of learners' contributions. First, it should be determined whether learners are engaging in activities relevant to solving the task (on-task discourse) or if they are distracted by off-task aspects. Second, on-task discourse can be further examined by identifying specific epistemic activities that detail how learners solve the task within a given domain. Tasks involving argumentative knowledge construction involve at least three types of epistemic activities: constructing the problem space, the conceptual space, and the relationships between these spaces (Fischer et al., 2002). Constructing the problem space involves understanding the problem by selecting, evaluating, and linking elements of case information. Constructing the conceptual space involves summarising, rephrasing, and discussing theoretical concepts and principles which are essential for understanding the theoretical material being learned. The construction of relationships between conceptual and problem spaces indicates how learners approach a problem in detail and their ability to apply knowledge effectively. In complex problem settings, learners must construct multiple relationships between various concepts and aspects of the problem. Collaborative application of theoretical concepts to problem spaces can signify the internalization of these relationships and learners who collaboratively apply theoretical concepts to problems are more likely to transfer and apply this knowledge individually in future cases (Vygotsky, 1978).

The *argument dimension* refers to the analysis of discourse corpora with respect to the construction of individual arguments and the construction of sequences of arguments. The analysis of the construction of individual arguments is based on Toulmin's model (Toulmin, 1958), according to which arguments encompass the following elements: data, such as observations or experiences; claims, which are statements that present the learner's position; the warrants, which are logical connections between the data and the claim and provide the rationale for why a claim is valid; and qualifiers, which are statements that limit the validity of a claim to specific conditions. In discourse, individual arguments should be organised into coherent sequences of argumentation, including arguments for justifying a position, counterarguments for challenging such position, and replies for refining the positions. In addition to argumentative moves, non-argumentative moves can also be identified. Non-argumentative moves, which do not contain a claim, include questions, coordinating statements, and meta-statements about the argumentation process.

The way learners solve a task and construct arguments can vary in how these processes are distributed among the members of a learning group. *Social modes of co-construction* describe the extent to which learners engage with and refer to the contributions of their peers, a factor that has been linked to knowledge acquisition (Fischer et al., 2002). The different social modes are characterised according to the degree with which learners refer to the contributions of the other participants as presented in Table 1.2.

Social mode of co-construction	*Description of the social mode of co-construction*
Externalization (EX)	Learners' contributions to the discourse are made without reference to previous contributions. Externalization can be typically encountered at the beginning of a discourse for presenting some new knowledge by a learner.
Elicitation (EL)	Information, feedback or specific actions from other participants are requested. Through elicitation, participants take advantage of their learning partners as resources. Elicitation facilitates knowledge acquisition only when learners receive and apply the help themselves.
Quick consensus building (QC)	Contributions are accepted by participants just to move on with the discourse and without taking charge of them. Quick consensus building may be counterproductive to individual knowledge acquisition when learners disregard other forms of consensus building in favour of it.
Integration-oriented consensus building (IC)	Learners' contributions are integrated and their perspectives are taken over. Integration happens when individual learners build upon the reasoning of their learning partners.
Conflict-oriented consensus building (CC)	Contributions are replaced, modified or supplemented. Facing critique may prompt learners to explore multiple perspectives or to find stronger arguments for their positions.

Table 1.2 Characterisation of the social modes of co-construction

1.3.2 A systematic review of research on mathematical asynchronous online discussions

While research on asynchronous online discussions has been well-established in various research contexts, it represents a relatively novel but emergent theme in mathematics education research. Therefore, I conducted a systematic review to present the current contributions of research in mathematics education on mathematical asynchronous online discussions. The systematic review of literature presented in the following is reported in a submitted article for a journal (Gagliani Caputo, under review). For the purpose of the systematic review, we consider mathematical asynchronous online discussions as purposeful (Pirie & Schwarzenberger, 1988) asynchronous online discussions (in the meaning of Fehrman and Watson, 2021) on mathematical topics. It is important to note that the meaning of "discussion" in online discussions, as proposed by Fehrman and Watson (2021), and subsequently in mathematical asynchronous online discussions within this context, differs from the meaning given to the term "discussion" in the definition provided by Bartolini Bussi (1996) as mentioned in section 1.2. The main difference concerns the subjects explicitly declared to be involved in a discussion and their interaction. In fact, Bartolini Bussi (1996) states that the teacher should guide a mathematical discussion, while for Fehrman and Watson (2021) that the learning space involves only a community of learners and no explicit reference to the presence and the role of the teacher is stated. We refer to the definition by Fehrman and Watson (2021) since it allows for the inclusion of a broader range of research concerning discussions on mathematical topics than that of Bartolini Bussi (1996), which is more restrictive and would not have provided a sufficiently comprehensive overview of the research conducted on mathematical asynchronous online discussions.

As part of this review, with the aim of collecting and reporting research in mathematics education dealing with mathematical asynchronous online discussions, we will specifically focus on two issues. The first issue concerns the characterisation of the various online environments used in the reviewed research for conducting mathematical asynchronous online discussions, with a focus on their relevant technical features for the unfolding of discussions. The second issue consists in characterising the different kinds of discussions that occur in the mathematical asynchronous online discussions present in the reviewed articles regarding the objectives and modality of discussions, teacher's role, and the contexts in which the discussions take place (types of course/program/activity in which the discussions are developed). In order to address the presented issues, the systematic review reported in this section was guided by the following questions:

(Q1) What kinds of digital environments are used to develop mathematical asynchronous online discussions? What features do these digital environments offer to support them?

(Q2) How are mathematical asynchronous online discussions characterised in terms of objectives of discussion and teacher's role?

Conducting a systematic review includes several aspects to be taken into account, among which the transparency and reproducibility of the search methods (Gusenbauer & Haddaway, 2020). The research method we followed was developed according to the seven stages identified by Petticrew and Roberts (2006) in carrying out a systematic review: (1) clearly articulate the question guiding the review or the hypothesis to be tested; (2) identify the types of studies essential for addressing the formulated question; (3) conduct an exhaustive literature search to pinpoint these studies; (4) evaluate the outcomes of the search by sifting through identified studies, determining which align with the inclusion criteria; (5) critically assess the studies deemed eligible for inclusion; (6) synthesize the findings from the included studies and evaluate heterogeneity among their results; (7) disseminate the review's findings.

Selection criteria

This systematic review includes research published between January 2015 and May 2024. The sources of information consist of high rank refereed journals and conference proceedings, both in the field of mathematics education. The journals to be included in the review were selected by combining three heterogeneous sources: the list of journals in mathematics education reported in the article “Journal Quality in Mathematics Education” (Williams & Leatham, 2017); the ranking of mathematics education journals done by the University of the Andes according to the JCR (Web of Science) impact factor and the SJR (Scopus) factor (Universidad de los Andes); the list of “class A” journals in mathematics education compiled by the Italian national agency for the evaluation of universities and research institutes (Anvur). The journals to be included in the review were identified as the intersection of the three lists excluding the journals that do not have articles in English. The selection has led to the identification of the journals reported in Table 1.3.

Name of the journal
Educational studies in mathematics (ESM)
For the learning of mathematics (FTLM)
International journal of mathematical education in science and technology (IJMEST)
International journal of science and mathematics education (IJSME)
Journal for research in mathematics education (JRME)
Journal of mathematical behavior (JMB)
Journal of mathematics teacher education (JMTE)
Mathematics education research journal (MERJ)
Research in mathematics education (RME)
ZDM The international journal of mathematics education (ZDM)

Table 1.3 List of the selected journals in mathematics education to be included in the review

The conference proceedings were included following Di Martino and colleagues (2023), since findings from international conferences often foreshadow trends in the evolution of research themes. The conference proceedings were selected on the basis of the popularity in terms of participation of the international conferences in mathematics education in order to encompass the widest range of research conducted on the topic within the international research community in mathematics education. The selected conferences are: the International Congress of Mathematical Education (ICME), the Conference of the International Group for the Psychology of Mathematics Education (PME) and the Congress of the European society for Research in Mathematics Education (CERME).

The search for articles was conducted by combining key terms and, when available in search bars, using logical connectors. For articles in journals, the search was performed directly using the search bar on the respective journal websites. For articles in conference proceedings, the search was carried out by directly searching the text within the documents of the proceedings. The key terms include: online AND (discussion OR communication OR discourse), computer mediated communication, asynchronous AND (communication OR discussion), discussion AND (board OR forum OR thread). Other terms related to discussions (e.g. orchestration) have not been inserted as key terms as they should refer to the topics of interest of this review and, indeed, be accompanied by the key terms already considered.

A first screening of the selected articles, following the methods of Yan and colleagues (2021), involves reviewing the title and abstract of each article (not considering duplicates) to identify those relevant to the topic of interest. If the title and abstract lack sufficient information to determine if the topic is addressed, the entire article is then read. Subsequently, a thorough reading of the remaining articles allows for the selection of those eligible for the review.

The initial search by key terms resulted in 2499 articles in journals and 83 articles in conference proceedings. It should be noted that the distribution over time of these articles is not uniform, since more than 75% of the articles have been published between 2020 and 2024. This discrepancy may be caused by the technological improvement and the more and more central role of distance education of the last years, also due to the Covid-19 pandemic remote teaching period.

An important aspect that should be considered in the initial search is that every time a search term including "discussion" is used, all articles containing a "discussion" section are included (if also the other key terms are encountered). This is a limitation of the search using these keywords and results in a significant decrease in the number of articles after the first screening based on title and abstract. Specifically, only 84 articles were selected from journals and 31 from conference proceedings after the first screening.

After reading the selected articles in their entirety following the first screening, 12 articles from journals and 13 from conference proceedings were found eligible for the review. Articles mentioning the theme of mathematical discussions or the use of digital technologies were excluded if these were not detailed due to being peripheral to the content of the article. This is the case, for example, with articles (n=15) that broadly outline some experiences of distance education during the Covid-19 period. Additionally, articles were excluded if they involved mathematical discussions supported by digital technologies for synchronous interaction (e.g., conferencing platforms) or if the interaction was not in an educational context (e.g., spontaneous discussions on the web - Parrish & Martin, 2022). In any case, the included studies had to offer enough information (at least, the objective of discussion and the digital environment supporting the discussion) to enable the author to determine the role that digital technology played in the learning process in a situation of discussion in an asynchronous modality.

Data extraction

The eligible papers were recorded using a data extraction form. We constructed the data extraction form on the basis of other forms used for systematic reviews (see, for instance, Di Martino et al., 2023; Yan et al., 2021) and included the following sections: study title, author name(s), year of publication, aim of study, reference to Covid-19 pandemic period, country or region, keywords, research questions, theoretical framework, target, educational context, type of research, data collection procedure, results of the study, objective of discussion, modality of discussion, role of the teacher, digital technology involved, use of the digital technology and remarks. This data extraction form assisted in organizing key information from each study relevant to the current review.

Even if the number of eligible papers seems scarce compared to more extended literature reviews, Petticrew and Roberts (2006) oppose the idea that "a literature review may be less valuable when a field is immature – that is when there are too few studies to yield data. In this situation however, a systematic review can highlight the absence of data, and point up the fact that any understanding is based on limited empirical underpinnings – in itself an important contribution. Further, even when a field is immature, it is important to cumulate prospectively rather than wait for some later date when "enough" evidence has accumulated, and consolidation can occur." (Petticrew & Roberts, 2006, p. 35). This systematic review is framed within this approach, aiming to identify gaps and guide future research efforts on an emergent theme.

Results of the systematic review

One of the aims of this systematic review was to organise what emerged in a chaotic and prolific manner during the Covid-19 period and previously in order to shed light for new research directions or deepening initial experiences on the basis of what reported in the considered articles. Such systematisation helped highlight different meanings given to the term 'discussion', which at times assumes specific connotations adhering to definitions, and in other cases, generic connotations derived from common sense, sometimes with a specific focus on mathematics and at other times more generally as a transversal didactical methodology.

The results of the systematic review are organised according to the different digital environments supporting mathematical asynchronous online discussions with relevant features and the characterisation of discussions in the mathematical asynchronous online discussions with respect to the objective of discussion and the teacher's role. Summary tables outlining the key aspects of each theme from the articles under review are reported in appendix A.

Before detailing mathematical asynchronous online discussions' characteristics, it should be noted that the considered articles refer to different targets and discussions are realised within different educational contexts, as expected. They range from primary school classes to university courses, and from initial training to professional development courses for teachers. In a general overview, it can be noted that the majority of targets in the reviewed papers concern the university level (n=13), considering that prospective teachers are university students as well. The educational contexts in which mathematical asynchronous online discussions are realised span between full-online programs to hybrid programs including in-person and online activities and both with the possibility of synchronous or asynchronous interactions. The detailed target and educational contexts of each article are reported in the tables in appendix A.

Digital environments and their features supporting mathematical asynchronous online discussions

Concerning the digital environments hosting mathematical asynchronous online discussions, a large number of articles refers to the use of discussion forums and, in particular, of Moodle forum, since Moodle is a quite spread learning management system, especially at university level. Besides forums, the platform Padlet is reported in some research. Less frequent, but still present, is the use of social networks (e.g., Facebook groups, Discord channels, Telegram chats) or other collaborative web platforms, such as Perusall or Tricider. It should be noted that in the reviewed articles no deep or technical presentation is done concerning the used digital environments and their features with respect to the objectives of the discussions.

Specific digital environments and their features supporting mathematical asynchronous online discussions presented in the analysed papers are:

- Forums: the forum activity allows students and teachers to exchange ideas by publishing posts and comments as part of a thread. Considering particular features of Moodle[5] forum, as it is the most used, files such as images and media may be included in forum posts and the teacher can choose to grade and/or rate forum posts or giving students permission to rate each other's posts, but no use of these features is mentioned in the analysed papers. In the reported articles, the forum (in general, not only the Moodle forum) is mainly used for sharing posts and commenting on them and, generally, the exchanges are by written text – there is no reference to the use of multimedia attachments to the posts in forum discussions. Among forums, VoiceThread[6] is mentioned as a particular type of forum in which it is possible to support both voice and text based discussion for peer collaboration.
- Social Networks: they are Internet services for managing relationships and social nets, typically accessible through a browser or mobile applications relying on the respective platform, which allows communication and sharing through textual and multimedia means. The used social networks in the analysed papers are Facebook[7], Discord[8], Google+[9], Telegram[10], Viaéduc[11] and WhatsApp[12]. In the reviewed articles, social networks are used to share written posts and comments, no more specific features are highlighted except for Viaéduc, which allows also creating, publishing, commenting and recommending documents.
- Collaborative web platforms: they are digital environments that facilitate teams' collaborative work and communication. The collaborative web platforms presented in the analysed papers are Padlet[13], Perusall[14], and Tricider[15]. Padlet is mainly used for collecting students' ideas or solutions to a problem through posts. Padlet allows posts to contain different multimedia (such as documents, images and videos) and to be anonymous. These features, reported in the use of such digital environments in the considered papers, allow students to use a multimodal approach to express themselves and to be more engaged through the use of nicknames. Posts on Padlet can also be commented through written comments. Perusall is a social annotation platform and allows written annotations on a shared document or video and replies to annotations by students and teachers. Tricider is for easy brainstorming, voting, decision making, crowdsourcing and idea generation. It helps facilitate decision making after any discussion by the request of a vote on statements.

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[5] https://moodle.org/
[6] https://voicethread.com/
[7] https://www.facebook.com/
[8] https://discord.com/
[9] https://www.google.com/
[10] https://web.telegram.org/
[11] https://www.viaeduc.fr/
[12] https://www.whatsapp.com/
[13] https://padlet.com/
[14] https://www.perusall.com/
[15] https://www.tricider.com/

Mathematical asynchronous online discussions' characteristics in terms of objectives of discussion and teacher's role

The concept of discussion in the reviewed articles encompasses a diverse array of activities aimed at fostering collaborative learning and deepening understanding of mathematical concepts.

Concerning the objectives of discussions outlined in the papers, we can distinguish between three main clusters in which discussions are associated with: (1) collaborative problem solving; (2) giving and receiving feedback or sharing reflections; and (3) social engagement. The discussions belonging to the second and third clusters can be considered general discussions, as they can be developed also with respect to other disciplines not relating specifically to mathematics, how it is instead for mathematical problem solving. Among these clusters we can highlight similarities, especially concerning the focus on active involvement and the promotion of deep understanding, usually associated to the construction of reasonings and explanations, and differences, since each type of discussion puts the emphasis on specific activities, highlighting different foci of mathematical discussion, such as problem solving, feedback, or communication, and focuses on different audiences, as some types focus on students' interactions with each other, while others emphasize interactions between students and teachers or engagement with course materials.

The discussions in the first cluster, i.e. those concerning related to collaborative problem solving, are characterised by the importance given to students' collaborative work and comparison between alternative solutions and strategies to the same mathematical problem. The comparison and evaluation of different solutions to a mathematical problem generally also offers feedback to the authors of such solutions, creating possible intersections between the first two clusters.

The discussions belonging to the second cluster, i.e those aimed at sharing participants' reflections or giving and receiving feedback, involve, in the collected papers, mainly in-service mathematics teachers sharing and confronting on their teaching experiences and reflecting on one's own and others' work to improve understanding and instructional practices, and university students, for whom the feedback provided by the teacher can also support individual study.

The papers focused on discussions belonging to the third cluster, i.e. those aimed at fostering social engagement, stress on the relevance of communication at stake in an educational environment. In particular, they refer to engaging participants in discussions to articulate their understanding, share insights, and deepen comprehension through dialogue on mathematical concepts or experiences not specifically related to problem solving but rather to course content in general.

Across these clusters, several commonalities emerge in terms of objectives related to the unfolding of discussions, such as: the development of critical thinking skills, achievable encouraging students to analyse, evaluate, and justify their reasoning; and the promotion of reflection, highlighted as essential for learning and growth in the majority of articles, whether through reflecting on teaching practices, evaluating solutions, or commenting on course content. In summary, while each perspective offers unique insights into specific content, they collectively emphasise collaborative learning, critical thinking, and reflection as essential components for effective teaching and learning in mathematics in a context of discussion.

Concerning the digital environments supporting the considered mathematical asynchronous online discussions, there is no evidence of particular relationships between the three clusters of discussions and specific digital environments chosen for their delivery. Particularly noteworthy regarding the various characterisations of the discussions presented and the corresponding use of digital technologies is that often the technologies do not have a specific purpose related to the mathematical needs under discussion but primarily serve as a means of supporting interaction. Indeed, this suggests that the last years have led to a flattening of discussions focusing on the general use of digital technology without necessarily linking it to the mathematical process (as also highlighted in Baccaglini-Frank, 2022).

Concerning the role of the teacher in the considered mathematical asynchronous online discussions, firstly, it should be noted that not all the mathematical asynchronous online discussions presented in the selected papers are guided by a teacher or teacher educator, as some researchers report that the teacher does not interrupt students' discussions or that the discussion forums are unfacilitated. On the other hand, when discussions are orchestrated by a teacher or teacher educator, their main roles concern the initiation of the asynchronous discussion, posting specific questions to be answered or opening discussion threads, and the unfolding of the discussion, relaunching students' posts asking them to justify their reasoning or taking the chance to publish additional questions on specific aspects emerged from peers' contributions.

From the analysed articles, it is not possible to clearly highlight the type of feedback provided to students in discussions pertaining to the second cluster (for example, what type of comments or corrections are proposed by the teachers within mathematical asynchronous online discussions), and these should be explicitly stated and analysed in more detail to see if the type of feedback concerns specific mathematical knowledge or not. Concerning the presence and the roles the teacher can play when orchestrating a discussion, the analysis reveals that almost 50% of the analysed articles declares that discussions are unfacilitated or gives no information on the presence of the teacher. The absence or the limited participation of the teacher can flatten the discussion, not giving students the opportunity to produce and deepen collective mathematical meanings with the appropriate guidance and knowledge, thus this suggests the need to further explore and investigate the teacher's interventions within mathematical asynchronous online discussions.

1.4 Introducing the research problem

The potentials and challenges that emerged during the Covid-19 pandemic have led research to continue exploring the interaction between digital technologies and mathematical discussion even in the post-pandemic period, with a particular attention towards remote collaboration, even in asynchronous modality. Moreover, the systematic review reported in subsection 1.3.2 reveals that the theme of mathematical asynchronous online discussion is present but still emerging in mathematics education research. At the current state, it still remains a new research topic to be explored, both in full-online courses but especially with its possible integrations with in-person classroom activities. Moreover, with respect to the targets addressed in the reviewed articles, it clearly opens a new path of exploration of mathematical online asynchronous discussions in school contexts, since the vast majority of research involved university students or in-service teachers as participants. In order to contribute to the presented research problem, the overarching goal of the research project reported in this thesis is to investigate how digital technologies can be used in educational contexts to create new possibilities for teaching and learning mathematics. The challenge is to further explore whether mathematical discussions can be conducted and enhanced by the mediation of digital technologies, engaging students with different modalities, including the extension of time in asynchronous modality and the written form of communication, while still fostering the development of mathematical thinking. In particular, we investigate how digital technologies can foster remote collaboration as an added value in mathematical discussions by extending classroom time and space beyond the physical structure and constraints of the school. Our proposal to address this research problem is to introduce and investigate the didactic methodology of Digital Mathematical Discussion which will be introduced in chapter 2.

Chapter 2

Design of Digital Mathematical Discussion

As presented in chapter 1, the research reported in this thesis aims to contribute to the research problem of supporting remote collaboration extending the didactic methodology of mathematical discussion beyond time and space constraints imposed by traditional classroom settings. This chapter is devoted to the introduction of the design of Digital Mathematical Discussion (DMD), which represents a first answer to address this research problem. In section 2.1, we recall the theoretical frameworks introduced in chapter 1 as background from which the design of DMD stems, including its roots in mathematical balance discussion (section 1.2) conducted with the didactic methodology developed within the FaSMEd project (subsection 1.2.2), and the M-$_{\text{AE}}$AB construct as reference for the roles a teacher can play when orchestrating a mathematical discussion (subsection 1.2.1). In section 2.2, the design of DMD is presented. A design-based research methodology is employed for its development. This methodology is introduced with its general characteristics at the beginning of section 2.2, followed by a detailed description of its implementation within the research conducted for this thesis project. In particular, in subsection 2.2.1 we present the principles guiding the design of DMD and, then, the design and redesign of the phases of DMD are reported through the successive cycles of design-based research in subsections 2.2.2 (design of the chats), 2.2.3 (design of the Padlet), and 2.2.4 (design of wrap-up discussion). In subsection 2.2.5, we present a summary of the redesign the DMD has undergone in the cycles of the research. Once the current design development of DMD is outlined, in section 2.3, we compare the design of DMD with the design of traditional mathematical discussion, offering our perspective on the added value that DMD can bring to traditional mathematical discussions when used in synergy.

2.1 Theoretical background for the design of DMD

In this section, we introduce the theoretical foundations for the design of Digital Mathematical Discussion (DMD). DMD stems from both traditional mathematical discussion and asynchronous discussion, as introduced in chapter 1. Specifically, within the research project reported in this thesis, we design DMD with reference to a particular type of mathematical discussion: the balance discussion, , i.e. socialization and collective evaluation of strategies set up by students in individual or small-group work (Bartolini Bussi, 1991). This choice builds on prior research experiences that utilised digital technologies to facilitate balance discussions (see, for instance, Cusi et al., 2017; Giberti et al., 2022) and aims to deepen the investigation into the support digital technologies can provide for the unfolding of mathematical discussions.

Concerning the design of balance discussions, we rely on the methodology developed and tested in the FaSMEd project (already introduced in subsection 1.2.2). Indeed, Cusi and colleagues (2017) report the support that digital technologies can give to teachers in monitoring students' work, in collecting their written solutions and in selecting and grouping excerpts from students' solutions to be displayed with the aim of fostering comparison and reflections. In particular, based on Stein and colleagues' (2008) pedagogical model for orchestrating mathematical discussions, the researchers highlight the practices of *selecting* particular students' responses to share during the discussion introducing, if needed, important strategies that no one in the class has used (e.g. by sharing the work done by students from other classes); and purposefully *sequencing* the students' responses to be shared and discussed to help compare related or contrasting strategies and make it more likely that meaningful mathematical ideas will be discussed by the class. The FaSMEd methodology builds on Bartolini Bussi's (1991) description of balance discussions, emphasising the importance of *helping the class make mathematical connections* among their peers' responses. This approach encourages students to reflect on and evaluate others' ideas while revising their own. Therefore, the teacher selects the responses with the aims of bringing to attention typical mistakes, so that students can receive feedback from their peers and teacher; of highlighting more or less efficient ways of processing problems, thus sharing criteria for success; and of contrasting different justifications or identifying similar ones. Concerning the role of the teacher for orchestrating mathematical discussions, we rely on the Model of Aware and Effective Attitudes and Behaviours (M-AEAB) by Cusi and Malara (2009, 2013), introduced in subsection 1.2.1. Since balance discussions focus on metacognitive reflections developed after the problem solving processes have been carried out, between the different roles of the M-AEAB construct, the second group of roles is the most suitable (Table 1.1). In particular, the role of *activator of reflective attitudes and metacognitive acts* could be played to prompt students to analyse and compare excerpts from the groups' solutions, the role of *reflective guide* to ask students to make the thinking processes subtended to the solution proposed by another student visible, and the role of *activator of interpretative processes* to encourage to further explore proposals made by other groups.

From the research experience of the M@t.abel 2020 project (Giberti et al., 2022, 2024), instead, we build on the opportunities given by digital technologies in balance discussions for allowing plural representations of knowledge, for example both via verbal and visual means; plural opportunities of expression, for example letting the students choose between visual or written presentation of their work; and extension of the time at disposal to read and contribute to the discussion.

2.2 Design-based research methodology for the design of DMD

The design of the didactic methodology of Digital Mathematical Discussion (DMD), presented in this section, has been developed following a Design-Based Research (DBR) methodology (Cobb et al., 2003; Design-Based Research Collective, 2003). DBR methodology stands on design experiments aimed at developing a greater understanding of learning ecologies. A learning ecology is a "complex, interacting system involving multiple elements of different types and levels" (Cobb et al., 2003, p.9), such as tasks or problems to be solved, kinds of discourses that are encouraged, norms of participation that are established, tools and material means provided, or practical means by which classroom teachers can orchestrate relations among these elements. Due to their inherent complexity, learning ecologies cannot be completely specified. Therefore, it is crucial to clearly distinguish between elements that are the focus of investigation and those that are secondary, accidental, or assumed as background conditions in the design specification. The metaphor of ecology is useful to highlight that designed contexts are viewed as interacting systems, rather than merely a collection of activities or a list of isolated factors influencing learning.

The creation of designs, which can be improved through incremental adjustments, grows into particular theoretical claims about teaching and learning and demonstrate a commitment to understanding the connections between theory, design, and practice. At the same time, a design theory explains why certain designs work and suggests how they can be adapted to new situations. Thus, design experiments serve as a testing ground for developing and validating new theories. Design experiments can vary for their type and scope but they are all characterised according to five features (Cobb et al., 2003):

1. *Purpose of design experimentation: develop theory* – The purpose of the design experimentation is to develop a class of theories about both the process of learning and the means that are designed to support that learning. As a practical matter, a design experiment is conducted in a limited number of settings. It is clear from the emphasis on theory that the intent is not merely to investigate the process of supporting new forms of learning in those specific settings. Instead, the purpose is to frame selected aspects of learning and the means of supporting it as paradigmatic cases of a broader class of phenomena.
2. *High interventionist methodology* – Design studies are typically test-beds for innovation, with the intent to explore possibilities for educational improvement by introducing and examining new forms of learning. As a result, there is often a significant discontinuity between conventional educational practices and those that are the focus of a design experiment.
3. *In context development of theory* – Considering the two previous features, design experiments provide an environment where new theories can be developed and tested. However, it is also important to ensure that these theories are placed within a relevant context to understand their applicability and relevance. Essentially, while design experiments help in formulating and refining theories, the theories must be appropriately contextualised to be meaningful and practical. Thus, design experiments always have two faces: prospective and reflective. The prospective side involves implementing designs with a hypothesised learning process to scrutinise and potentially uncover new learning pathways. The reflective side, instead, involves testing and refining these designs through conjectures, often resulting in the development and testing of more specific hypotheses during the study.
4. *Iterative design* – As conjectures are generated and tested, they may be refined or replaced, leading to an iterative design process characterised by cycles of invention and revision. This iterative approach involves systematic attention to evidence about learning and often requires the parallel development of measures that adapt to the evolving learning environment. The goal is to build an explanatory framework with specific expectations that guide subsequent cycles of design, enactment, analysis, and redesign.

5. *Humility of the developed theory* – Theories developed during the experiment process are humble not only because they focus on domain-specific learning processes, but also because they are closely tied to the design activity. The crucial question to address is whether the theory informs future design and, if so, how precisely it does so.

The research conducted within this doctoral research meets the mentioned five features. Indeed, the introduction of DMD methodology serves as fertile ground for the implementation of an innovative approach within educational and research contexts. Its theoretical foundations are rooted in mathematical discussion and asynchronous discussion, both established in research literature, as presented in chapter 1. However, there are also elements of novelty that allow for the introduction and exploration of new practical and theoretical aspects. Specifically, the design of DMD methodology itself represents an innovation since it stems from previous research experiences and results (detailed in subsection 2.2.1) but its design differs from those already introduced and tested and, especially, from the didactic methodologies commonly used in schools (feature 2). Additionally, the asynchronous nature of DMD for remote collaboration opens up opportunities to explore a relatively understudied area in mathematics education. This enables the generation of new theoretical knowledge about the modes of interaction and the development of mathematical thinking in students within asynchronous contexts (feature 1). As it will be introduced in the next subsections and detailed in chapter 4, the experimentations with DMD related to this research have been conducted in specific contexts (feature 3) in order to investigate the implementation of the DMD methodology in relevant environments. In our case, the relevance is related to the experience of students and teachers involved in the experimentations with mathematical discussions and to the mathematical content to address within discussions. As a consequence, the theoretical results we obtain through our investigation will come from specific environments with specific mathematical thinking processes (feature 5) and we will discuss their generalizability in the conclusion of this thesis. Finally, the design of DMD has been developed and tested through iterative design (feature 4), as detailed in the next subsections.

The iterative design of DMD is structured as shown in Figure 2.1. Over the three years of the PhD program, three experimental cycles can be identified in the DBR structure of the research: the 1st cycle encompasses the pilot study, the 2nd cycle involves the first experimentation, and the 3rd cycle includes the second experimentation. Figure 2.1 is organised chronologically to reflect the development of the research, with different colours representing the phases of the DBR cycles: orange for the design and redesign phases, blue for the enactment of the design, and green for the analysis based on the enactment, which informs the subsequent refinement.

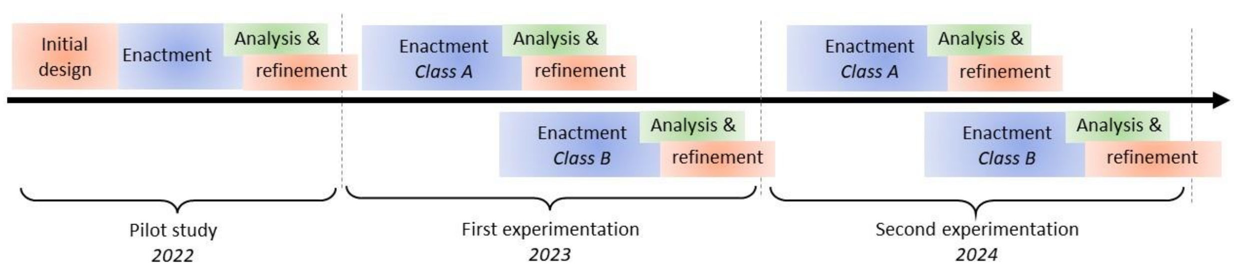


Figure 2.1 Iterative design of DMD

The first year was largely devoted to studying the research literature to map existing experiences related to mathematical discussions, asynchronous discussions and digital technologies and to identify the research gap in asynchronous modalities for conducting mathematical discussions. Based on this study, the initial design of the DMD didactical methodology was developed. The initial DMD design was tested in a pilot study in December 2022. Each enactment of the first and second experimentations, as shown in Figure 2.1, involved two classes (class A and class B), with partial overlap in both cycles. Both experimentations were conducted between February and April, in 2023 and 2024, respectively, and each consisted of the same four mathematical activities, presented in detail in chapter 3. Details about the classes involved in the experimentations and the structure of the DMDs conducted are presented in chapter 4.

In terms of analysis of the design of DMD, the pilot study aims to test the proper functioning of the digital environments (e.g., access to the platforms, or feasibility in sharing mathematical content in written form), of students' interaction with them (e.g., need of sharing technical instructions with students, or students' use of the affordances of the platforms), and between students (i.e., collaborative processes). Indeed, initially, the primary focus is on testing the internal coherence, design consistency, and practical implementation (Euler, 2017). Secondly, the evaluation focuses on identifying potential design optimizations. Here, both expected and unexpected processes are considered, along with contextual conditions that may impact the design. The refinements done between cycles have been informed by the analysis of students' perceptions and difficulties with the DMD methodology collected through the observation of the unfolding of the discussions, meta-discussions with students and their answers to final written questionnaires concerning the DMD methodology itself (the detailed analysis is presented chapter 5). For the first and second experimentations, the analysis aims not only at investigating relationships between the digital environments and students, and students' collaborative processes, but also at highlighting the mathematical thinking processes developed by students in the digital environments facilitating DMD and the guidance offered by the teacher in the different phases of DMD. The insights obtained from the analyses allow us to implement refinements between cycles and also within the same cycle. In fact, "one of the distinctive characteristics of the design experiment methodology is that the research team deepens its understanding of the phenomenon under investigation while the experiment is in progress" (Cobb et al., 2003, p.12). The refinements concerning the design of DMD are detailed in the next subsections.

2.2.1 Design principles for DMD

Within the framework of design-based research, design principles play a central role in connecting scientific knowledge production and innovative practice development (Euler, 2017). Acting as a bridge between theory and practice, these principles are both the product of empirical and theoretical research and the foundation for actionable design in real-world contexts. This dual role allows design principles to support both the advancement of knowledge and the application of this knowledge in practical settings. The design principles serve as foundational pillars for crafting interventions that empirically, theoretically, or plausibly support the achievement of the intended goals.

Design principles, as outlined by Euler (2017), are articulated at different levels of concretization. While research aims to develop general knowledge, designers seek also specific instructions tailored to the practical situations they need to address. The more abstract a principle's formulation, the broader its applicability, though this often results in less precise guidance for practical actions. Conversely, more specific principles offer clearer action steps but are limited in their applicability. Euler identifies three levels of principles: the first level involves abstract guiding principles aligned with general theories, while the third level addresses specific implementation for unique cases. Situated between these extremes, concrete design principles provide a medium degree of abstraction. Ranging from general to specific, design principles can be *guiding principles for design* (abstract guiding principles for concept design); *design principles* (principles for designing central components of the design object); *design characteristics* (concrete implementation of the design principles for a specific application context).

Design principles are initially established through a review of relevant literature and an exploration of the practice field to form preliminary design assumptions. Throughout the research process, these principles are refined as needed through targeted testing and evaluation. Thus, design principles are established at the beginning of the research to provide a foundation for the initial design. They are then revised and updated, representing an outcome of the experimental cycles. Within this subsection, we will present the initial design principles, articulated as guiding principles for design, design principles and design characteristics.

In our research, researchers and designers of DMD coincide. Thus, the different levels of design principles are developed by the researchers involved in this project, deriving the specific elements for practical implementation of DMD from the general theoretical principles. In our case, the initial design principles articulated over the three levels are as follows.

Guiding principles for design

The guiding principles for design of DMD stem both from mathematical and asynchronous discussion and, specifically, are derived from the theoretical background presented in chapter 1.

From the concept of mathematical discussion (section 1.2), we draw the importance of discussion in promoting the development of mathematical thinking. From balance discussions (section 1.2), we take the structure: group-work, aimed at collectively solving a mathematical problem, followed by a collective discussion comparing different proposed solutions. In the design of DMD, we follow the group-work modality (Bartolini Bussi considers both individual and group-work as options for solving the mathematical problem before balance discussions) because, at the research level, we aim to explore social dynamics among students and the co-construction of collective mathematical thinking processes, as it will be detailed in chapter 4. From the FaSMEd methodology for balance discussions (introduced in section 1.2.2), we adopt the teacher's practices of selecting and grouping excerpts from the groups' solutions to stimulate comparison and metacognitive processes. As a further consequence of DMD's roots in mathematical discussion, the teacher's presence and voice remain essential. Thus, the M-$_{AE}$AB construct (subsection 1.2.1) provides a reference both for designing collective discussion and for guiding teacher interventions during discussions. Teacher interventions are also closely connected to the goals of balance discussions within the FaSMEd methodology. Indeed, in the case of balance discussions, the teacher needs to adopt specific behaviours and attitudes to encourage effective comparison and evaluation of group solutions, or to prompt metacognitive reflection.

With respect to DMD's roots in asynchronous discussion (section 1.3), the guiding principles for design are related to asynchrony and written communication. In fact, asynchronous modality allows participants to engage with discussions at their own pace, enabling them to reflect, process information, and contribute thoughtfully. Written communication promotes clarity and permanence, helping participants to express and refine their ideas and to formally present and justify their problem-solving methods. In order to develop DMD according to these principles, an appropriate digital environment is needed. Within this digital environment students become *produsers* (neologism by Bruns' (2008) reflecting the intertwined roles of user and producer in digital environments) of mathematical thinking, sharing their proposals and reflections in a digital way.

Moreover, as we do not aim to promote DMD as a replacement for traditional mathematical discussion, but rather in synergy with it (as detailed in section 2.3), we support the presence of both asynchronous and synchronous phases in DMD, with the corresponding written and oral forms of communication.

Design principles

Design principles can be grouped into two main sets.

The first set includes those design principles based on the guiding principles introduced in the previous paragraph:

a) The work phases should include both synchronous and asynchronous timing;
b) Tasks should involve mathematical problems that encourage comparison and co-construction of knowledge among students;
c) Students should be encouraged to share all their thought processes, ideas, doubts, or challenges;
d) The teacher should decide to intervene or not depending on the activity requirements and goals;
e) When intervening, the teacher should take on different roles according to their objectives.

The second set includes those design principles based on preliminary assumptions emerged from the exploration of the practice field:

f) Different phases of work should be implemented, adjusting the activity requirements and the number of participants across phases;
g) The asynchronous phases of work should span some days;
h) The work phases should include various communication methods;
i) Students should have access to the content of each work phase whenever they wish, both during and after the work phases;

j) Students should be given a set of instructions to regulate their participation within the activity.

Design characteristics

DMD is structured in three different phases, differing for the activity requirements for students, the timing, the communication methods, and the presence and the role of the teacher as detailed in Table 2.1. Within the table, the design characteristics refer to the design principles presented in the previous paragraph (the design principles are reported in italics).

	Phase 1 of DMD	**Phase 2 of DMD**	**Phase 3 of DMD**
Activity requirements for students (*principle f*)	Group-work for collective solution of a mathematical problem (*principle b*)	Whole-class discussion aimed at comparison and evaluation of solutions produced by groups in phase 1	Whole-class discussion for further exploration of particularly interesting aspects that emerged in phase 2 and wrap-up of the discussions conducted in phases 1 and 2
Timing (*principle a*)	Asynchronous timing	Asynchronous timing	Synchronous timing
Communication methods (*principle h*)	Written communication through text or images	Written communication through text or images	Oral communication with written support provided by access to materials from phases 1 and 2 (*principle i*)
Presence and role of the teacher (*principles d and e*)	Presence of the teacher to supervise students' work but without intervening unless necessary (i.e., technical issues, explicit request from the group, or lack of group participation)	Presence of the teacher who acts as a guide by selecting, grouping, and presenting extracts from the group solutions, and when intervening to orchestrate the discussion	Presence of the teacher who acts as a guide by relaunching and connecting student contributions from phases 1 and 2, and orchestrating the discussion

Table 2.1 Structure of the phases of DMD

Concerning the phases of DMD reported in Table 2.1, phase 1 ends with the groups submitting their solutions to the mathematical problem. During phase 2, students can consult with their group from phase 1 and access materials produced by their group in that phase. Phase 2 ends when the teacher deems it appropriate, for example, when enough contributions have emerged in the collective discussion to meet the objectives the teacher had set.

In order to implement a DMD with such design characteristics, it is also necessary to identify suitable digital environments for mediating the first two phases of DMD. We identified chats from instant messaging platforms (e.g., WhatsApp[16], Telegram[17], Google Chat[18]) as suitable digital environments for conducting group-work in the first phase of DMD. Instant messaging platforms allow creating and joining groups' chats, support written communication, and allow for taking and sharing images, as required by the design principles. Additionally, many instant messaging platforms offer the option to enable or disable notifications for group members. We believe this feature is useful as students can receive alerts as soon as new contributions appear in the chat, while the teacher, who supervises multiple groups simultaneously, can choose to mute notifications. For the second phase of DMD, we selected Padlet[19] as digital environment for facilitating DMD because it allows the teacher to structure multiple discussion threads organised in a unique notice board. Using the section creation feature, the teacher can organise threads with different guiding questions. Within the Padlet, students and the teacher can add posts containing written text, images or attached documents, and can comment by written messages and react by upvoting or downvoting the posts. Furthermore, we considered that most students have access to a smartphone, though they may not have a tablet or computer available. Therefore, it was essential to select digital environments that are accessible via smartphone. The decision to use different digital environments for peer collaboration and teacher-led discussions has also been adopted in other research studies (Albano et al., 2020). In their case, two Moodle tools were chosen: the chat channel was used to manage all immediate and informal discussions among peers and for private exchanges with the teacher, while the forums were used to handle all formal discussions between students and the teacher. However, in our case, Moodle was not considered a suitable option because the chat environment is difficult for viewing the conversation history, and the forum environment does not provide a complete and well-organised spatial layout of threads like Padlet.

Thus, a typical DMD is structured as shown in Figure 2.2. The first phase consists of small-group work (4-5 students per group), aimed at fostering students' formulation of their strategies with respect to mathematical problems. It is realised in 3-4 days within chats of an instant messaging platform in which students collaboratively address the assigned mathematical problems. The teacher is present in the groups' chats but does not intervene unless explicitly requested by the students. At the end of the first phase, one student from each group sends the teacher the collective solution developed by the group to the tackled mathematical problems through an institutional platform (e.g., Google Classroom). The second phase is a whole-class discussion in which the teacher guides students in analysing, comparing and reflecting on the strategies proposed by the groups in the first phase. It is held on the collaborative web platform Padlet, it lasts 3-4 days, and it is structured by the teacher starting from selected excerpts from the groups' solutions. The Padlet is organised in columns. Each column starts from a guiding question from the teacher. The third phase of DMD, following the whole-class discussion on Padlet, is a 1 hour in-person discussion in the classroom to wrap up. Except for the 1 hour in-person discussion, the whole DMD activity is carried out in an asynchronous modality as homework and spans multiple days.

[16] https://www.whatsapp.com/
[17] https://web.telegram.org/
[18] https://mail.google.com/chat/
[19] https://padlet.com/

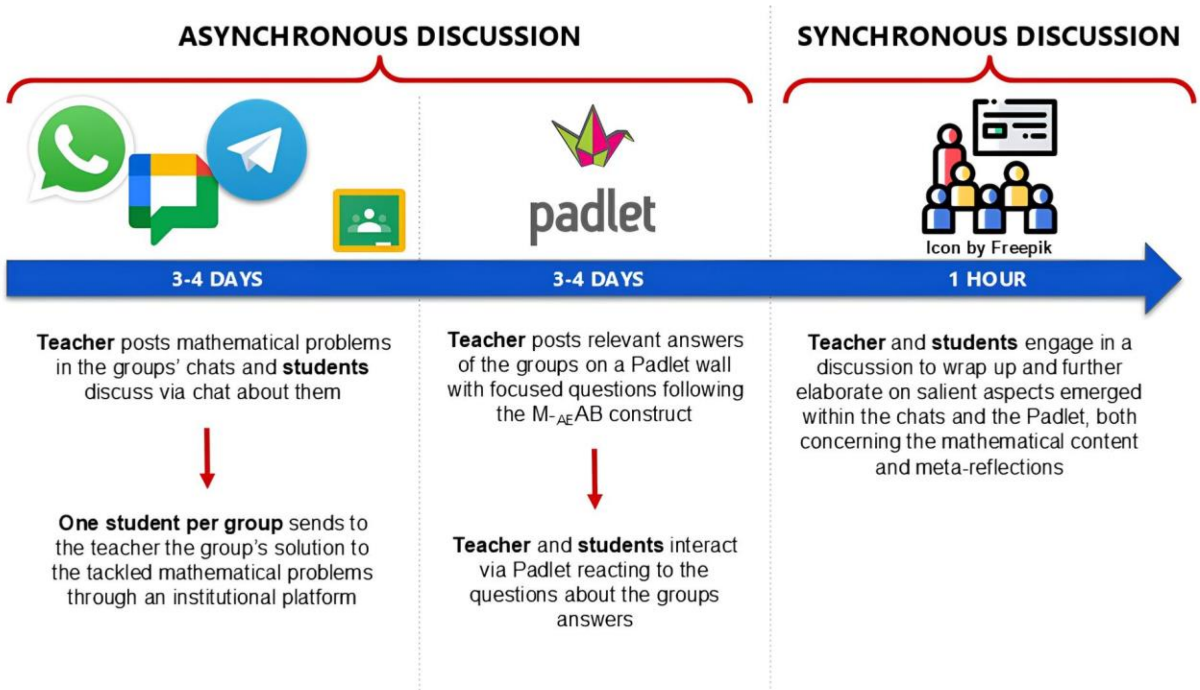


Figure 2.2 Schematic representation of the structure of a Digital Mathematical Discussion

Since research has documented pitfalls related to the distracting nature of social media, in particular when students work with mobile instant messaging platforms (Naidoo & Kopung, 2016), we decided to accompany the DMD activities with a set of indications to regulate students' use of the social media while discussing in DMDs (in line with design principle): (1) interact only on the chat and on the Padlet devoted to the activity, trying to avoid any other form of communication regarding the mathematical problem under discussion; (2) make all the thinking processes explicit, including the ones regarding the resolution of the problem, doubts, ideas, difficulties, individual explorations, etc. in order to let the thoughts become visible to all the participants; (3) share all the used materials, such as pictures of what is written (if paper and pencil are used to develop the reasoning); (4) do not use vocal messages in order to maintain the communication written.

2.2.2 Design and redesign of group-work within the chats of DMD

In the first phase of DMD, students engage in group-work within chats of an instant messaging platform. The choice of the instant messaging platform is left to the teachers of the classes involved in the experimentations as long as the platform allows for creation of group chats participants can join, supports written communication and the sharing of images and documents, and guarantees real-time transmission of messages. Moreover, the platform should be accessible via smartphone and, preferably, should include the possibility of activating or disabling notifications. Teachers may prefer different platforms based on the additional features they offer, such as the possibility to hide personal information (e.g., phone number) from other chat participants. Preferences may also depend on familiarity with certain platforms or the ability to access some platforms using institutional credentials (e.g., school email) instead of personal accounts. The instant messaging platforms chosen by the teachers involved in this research are Telegram, WhatsApp and Google Chat.

The initial design of the first phase of DMD for the pilot study involves group-work in chats where students collaborate to solve a mathematical problem. The maximum number of students per group depends on the size of the involved class and is determined by the teacher. This number is chosen to maximise students' participation (ensuring that the chats do not include too many students) while also allowing the teacher to effectively manage the chats (ensuring there are not too many of them). In the pilot study, each group consists of a maximum of 7 students. The teacher is present in the chat of each group, initiates the activity by sharing the mathematical problem, and provides guidelines on deadlines and submission procedures (see, for instance, Figure 2.3 left). In the pilot study, groups have 5 days to solve the mathematical problem, and one member from each group is responsible for uploading the group's solution in Moodle platform once the collective solution to the mathematical problem by the group is ready (Moodle is the institutional platform used by the students and teacher involved in the pilot study). The teacher supervises but does not intervene in the chats unless explicitly requested by the group or if technical issues arise. Students organise their work autonomously within the chats, both in terms of the timing for tackling the problem and the methods for constructing the collective solution. There are no restrictions on the format of the group's solution file set by the teacher, but all groups choose either to produce a solution on paper and submit a scanned version or to create a digital document. As an example of students' interventions in a group chat see Figure 2.3 right.

Figure 2.3 Screenshot of a chat from the first experimentation (personal data are redacted)

In the first experimentation, similar collaborative dynamics among students emerge as those identified in the pilot study. All groups worked, but some engaged in more effective collaborative processes than others. We hypothesise that teacher intervention in the chats could support the development of effective collaborative dynamics and of collective mathematical thinking processes in groups where they are not occurring. For this reason, in the second experimentation, the teacher intervenes in the chats where ineffective dynamics are observed, either to prompt student contributions, encourage the articulation of thinking processes, or foster deeper exploration of points raised in students' contributions.

Concerning some technicalities related to the implementation of the first phase of DMD, in the first and second experimentations, the maximum number of participants per group is reduced to 5, due to the smaller class sizes involved. The number of participants per group should be low enough to ensure that every student has a meaningful voice in the group-work, but it also needs to be balanced with the total number of chat groups created to avoid overburdening the teacher's supervision workload. The institutional platform to collect the groups' solution in the first and second experimentations is Google Classroom, as all the classes involved already used it for their curricular activities. The amount of days devoted to the first phase of DMD spans from 4 to 5 days.

2.2.3 Design and redesign of the whole-class discussion within the Padlet of DMD

The initial design of the second phase of the DMD during the pilot study includes students' participation in a whole-class balance discussion on Padlet, initiated by guiding questions from the teacher. These questions are based on selected and organised excerpts from the solutions produced by the groups in the first phase. Padlet is a collaborative board that the teacher can structure into sections, each containing different guiding questions for the discussion. The teacher's guiding questions can include both text and images. Students can interact on Padlet by creating new posts under the relevant section for their contribution, or by commenting and reacting to the specific posts they wish to respond to. Figure 2.4 reports a screenshot of the Padlet from the pilot study. In the pilot study, students are given 3 days to contribute to the Padlet. Teacher's guiding questions are posted at the beginning of each section (teacher's name is redacted in orange in Figure 2.4). Most of them (the white ones in Figure 2.4) are posted together at the beginning of the second phase of the DMD conducted in the pilot study, while the blue ones are posted by the teacher after 2 days from the beginning of the second phase of the DMD and concern more sophisticated proposals by the groups' solutions.

The Padlet is designed according to the FaSMEd methodology and the guiding questions by referring to the M-$_{AE}$AB construct in order to foster an effective balance discussion, as presented in section 2.1. Between the different roles of the M-$_{AE}$AB construct, the ones primarily referred to formulate the guiding questions pertain to the second group of roles, as detailed in section 2.1.

Figure 2.4 Screenshot of the Padlet from the pilot study (personal data are redacted)

As will be detailed in chapter 5, students' participation on Padlet has been problematic since the pilot study. In fact, compared to the total number of students involved in the study who actively participated in the first phase of the DMD in the chats, only a few students engaged on Padlet. Therefore, student participation on Padlet emerged as a sensitive issue from the very beginning, requiring close monitoring and encouragement. In order to address the absence of notifications, in redesign of the second phase of DMD before the first experimentation, we decided to use chats as a support tool to send reminders to students and encourage their participation on Padlet by notifying them when interesting contributions or new questions were posted. Additionally, starting with class B in the first experimentation, students were given more time to interact on Padlet by extending the number of available days (from 3 days to 6 days). Moreover, to address the third issue raised by students, we suggested them to discuss in the chats before writing a contribution on Padlet to address the difficulty they reported about exposing their individual ideas in front of the whole class, and that contributions on Padlet, although published individually, could also reflect the ideas of the group or of some members of the group.

Since low student participation in the discussion on Padlet persisted even in the first experimentation, before the second experimentation, we redesigned the second phase of DMD by posting the teacher's guiding questions gradually over time instead of all at once to help students focusing on one question for the discussion at a time. We also maintained longer time frames for interacting on Padlet and used the chats to send reminders to students about the activities on Padlet, as in the first experimentation. Despite the implemented changes, students remained reluctant to participate in the discussion on Padlet. Therefore, for class B's work in the second experimentation, we decided to further modify the design of the second activity of the DMD. To encourage greater participation, we first aimed to involve all students on Padlet and then conduct the discussion as in previous experimentations, posting guiding questions spread out over time as an initial stimulus for discussion. Involving all students was achieved by asking each of them to individually respond to guiding questions about the group's activities carried out in the first phase of the DMD while keeping the responses private. This was possible thanks to a mediation feature on Padlet that allows the teacher to see all the contributions written by the students while keeping them private to the class, so each student can only see their own contribution and not those of their peers. This initial form of involving all students on Padlet in a private manner seems promising, but it needs further experimentation to be consolidated.

Regarding the use of Padlet, another significant difference between the research cycles concerns the implementation of the third activity of the experimentations. In fact, the classes involved in the first and second experimentations followed the same didactic path (presented in section 3.3) consisting of various activities realised as DMDs. As will be presented in section 3.3, the third activity of the didactic path has a different nature from the other activities the students engaged in. The third activity requires evaluating proposals from fictional students, thus immediately demanding engagement on a metacognitive level rather than solving a mathematical problem. Due to the characteristics of the third activity, during the first experimentation, we decided to conduct it entirely on Padlet, initiating a full-class discussion from the start of the DMD instead of working in the groups' chats before. However, students still had access to the chats from previous activities if they needed to confer with one another. The idea of conducting the entire activity on Padlet was tied to the hope that it would help students become more familiar with the platform. In reality, student participation in the third activity during the first experimentation was very low. Therefore, to ensure greater student engagement in the third activity, in the second experimentation we decided to conduct it entirely via chat for class A, and using a combination of chat + private Padlet + public Padlet for class B, as previously described. This approach allowed students to explore the third activity in greater depth during the asynchronous phases of the DMD.

2.2.4 Design of wrap-up classroom discussion of DMD

The discussion in the third phase of DMD is conducted with the aim of deepening the reflections on interesting aspects or concepts that were not sufficiently explored during the whole-class discussion on Padlet and to wrap-up the activity carried out. This discussion takes place in the classroom and is structured as a meta-discussion based on the contributions published on Padlet. In fact, the Padlet is projected onto the interactive whiteboard (in Italy, every classroom has at least one) and the teacher guides the discussion by referring to elements from the Padlet that need further exploration. The teacher stimulates meta-level reflections, by helping students exchange and compare different arguments and strategies reported in the Padlet to further reflect on their strengths and weaknesses, in line with the role *of activator of reflective attitudes and metacognitive acts* (Table 1.1). The time allocated to the discussion varies between 1 and 2 hours, depending on the institutional time constraints that teachers have and the depth of analysis already achieved through the discussions on Padlet. This type of discussion aligns with those presented in Giberti and colleagues' paper (2024), but instead of being a balance discussion about a problem, it is a meta-discussion about a previously conducted balance discussion based on a problem.

During the cycles of the research, the third phase of the DMD did not undergo any redesign. Indeed, its structure is close to traditional mathematical discussions and the innovative aspects of

DMD rely on the asynchronous phases we focused the most. However, conducting in-person discussions as the final step of a DMD process makes these discussions different from traditional ones due to the preceding asynchronous phases. On the one hand, students have had the opportunity to delve deeply into the mathematical activity, having tackled the problem in chats and discussed it on Padlet before engaging in in-person class discussion. On the other hand, the support provided by the digital environments used in the first two phases of the DMD is also significant. In class, the final discussion is based on the Padlet where the second phase of the DMD took place, which students are already familiar with in terms of its organization and the content present. Moreover, students can also access the chats and Padlet from their own smartphones during class discussion to review and reconstruct what has already been discussed on a given topic.

2.2.5 Summary of DMD redesign throughout the cycles of the research

Within the DBR cycles of the research, the design of DMD has undergone some refinements. Table 2.2 includes the elements of the DMD design that were modified in at least one DBR cycle.

	1st DBR cycle	**2nd DBR cycle**	**3rd DBR cycle**
Teacher's involvement in the chats	Supervises but does not intervene	Supervises but does not intervene	Supervises and intervenes to encourage effective collaborative processes for the collective development of mathematical thinking processes
Organisation and management of the balance discussion on Padlet	The teacher posts multiple guiding questions at the beginning of the discussion on Padlet	During the second phase of the DMD, the teacher sends reminders to students to encourage their participation on Padlet Extension of the days available for discussion on Padlet Students are encouraged to discuss in chat before posting on Padlet	During the second phase of the DMD, the teacher sends reminders to students to encourage their participation on Padlet The teacher gradually publishes over time the guiding questions In class B, before conducting the discussion on Padlet as in the other experiments, students are engaged by responding privately to some guiding questions
Digital environment facilitating activity 3	Unscheduled activity	Only Padlet	Class A: only chats Class B: chat + private Padlet + public Padlet

Table 2.2 Summary of redesign elements throughout the DBR cycles of the research

Within the DBR cycles, the context of the experimentations also changed. The 1st DBR cycle, which includes the pilot study, involved university students who already had experience with traditional mathematical discussions to help us focus on the feasibility of discussions during the asynchronous phases of the DMD. These students participated in a single DMD activity. As for the context of the 2nd and 3rd DBR cycles, they involved grade 9 students with little to no experience in activities conducted as mathematical discussions. These students participated in a didactic path organised with a traditional mathematical discussion and three DMDs, as presented in section 3.3. Details about the context of the experimentations will be given in section 4.2.

Since in the first and second experimentations students were involved in multiple DMDs, this allowed us to observe more deeply their participation and engagement, the quality of their contributions, and the difficulties they encountered. This led us to implement analysis and refinement both between cycles and also within the same cycle.

We do not consider the DMD design created at the end of the 3rd DBR cycle to be final, but it will need to be further tested and refined in future experiments.

2.3 Synergies between digital and traditional mathematical discussion

In this section, we propose our analysis on the added value that the integration of Digital Mathematical Discussion (DMD) into the traditional approach to in-class mathematical discussion can give. When talking about mathematical discussion we consider the definition proposed by Bartolini Bussi (1996), presented in section 1.2, and we refer to this discussion as traditional mathematical discussion. The introduction of the didactic methodology of DMD (section 2.2) aims to complement, rather than replace, traditional in-person mathematical discussion, providing students with varied contexts for engaging in mathematical discussions, and allowing teachers to extend the time they can dedicate to this practice. We articulate the synergies between digital and traditional mathematical discussion by different perspectives. In subsection 2.3.1, we discuss how asynchrony and written communication can empower traditional mathematical discussion and, in subsection 2.3.2, we focus on how the integration between digital and traditional mathematical discussion can foster inclusion in the classroom. The content of subsection 2.3.1 is part of a study reported in a research report accepted for the 14th Congress of the European Society for Research in Mathematics Education (CERME14), while the content of subsection 2.3.2 is reported in Gagliani Caputo, Branchetti and Cusi (2024).

2.3.1 Empowering traditional mathematical discussion through asynchrony and written communication

Within this subsection we focus on asynchrony and written communication, which are key characteristics of asynchronous discussions (see section 1.3). In fact, asynchronous discussions, involving participants separated in space and time, allow each student and the teacher themselves to read and contribute to the discussion at their own pace (Andresen, 2009). Moreover, realising asynchronous discussions allows extending the time dedicated to this practice beyond the classroom, as students can work on it at home. The communication modality of asynchronous discussions, predominantly in written form, can encourage participants to make their reasoning and arguments more explicit, making their thinking visible (Collins et al., 1989).

We aim to explore how the features of asynchrony and written communication can be leveraged within a mathematical discussion context. When referring to features, we consider Trouche's

(2004, p. 290, footnote 6) distinction between tools' constraint (obliging or impeaching the user), enablement (making the user able to do something), potentiality (virtually opening possibilities), and affordance (favouring particular gestures). In the case of features for discussions, we interpret Trouche's definitions referring to *constraints* as the norms and rules that shape how the discussion unfolds, to *enablement* as the conditions or resources that allow participants to contribute effectively in the discussion, to *potentialities* as the range of possibilities that the discussion opens up for participants, and to *affordances* as the features that favour certain types of involvement in the discussion.

We are interested in investigating asynchrony and written communication as features for discussion for several reasons. First, the asynchronous modality allows participants to engage with discussions at their own pace, affording flexibility that enables students to reflect, process information, and contribute thoughtfully without the constraints of synchronous timing (Wang, 2023). This flexibility is particularly beneficial for accommodating diverse learning styles and schedules, as students can revisit discussion threads, formulate responses over time, and manage their commitments. Similarly, written communication supports clarity and permanence, and provides a transcript of the discussion that can be referenced later (Wang, 2023). Indeed, writing can make thinking visible (Collins et al., 1989), and what is written and shared in a discussion can become a collective resource for reflection. Moreover, as highlighted by Santos and Semana (2015), effective communication through writing is crucial for developing a deeper understanding of mathematics. Writing aids students in clarifying and refining their ideas, and in presenting and justifying their problem-solving methods formally.

Asynchrony and written communication can be characterised as features for discussion, according to our interpretation of Trouche's definitions as introduced at the beginning of this subsection, as presented in Table 2.3. The content of Table 2.3 provides a characterisation of asynchrony and written communication that reports our initial perspective as researchers-designers on the development of the design of DMD.

	Constraints	**Enablement**	**Potentialities**	**Affordances**
Asynchrony	- It is required not to interact with others in synchronous modality - Time to submit contributions and to refresh to see new contributions	- The duration of the discussion is set over multiple days - Engagement of participants who feel less comfortable in real-time interaction	- Exploration of concepts over time - Introduction of ideas that open up new directions for the discussion	- Flexibility in participation

	Constraints	Enablement	Potentialities	Affordances
Written communication	- Impossibility of exploring functionalities that do not involve written communication - Lack of body language for non-verbal communication	- Making thinking processes explicit for both the one writing a contribution and those who read it - Engagement of participants who feel writing as support for developing their reasoning or reflecting on others' contributions	- Control over the whole discussion thanks to the availability of its complete transcript and consequent potential activation of metacognitive reflections on the structure of the discussion and on the links between its parts	- Encourage structured contributions - Encourage the use of mathematical symbols

Table 2.3 Characterisation of asynchrony and written communication as features for discussion from the designers' perspective

2.3.2 Fostering inclusion in mathematics classroom

Although the notion of inclusion has received increasing attention in educational research over the past few decades, Roos (2019) highlights that there is no agreed definition for it in mathematics education research. This lack of agreement is due to varying perspectives and contexts in inclusion research, which spans ethnic, social, and special educational issues. In our research, the term 'inclusion' is used in line with Vislie (2003), who states that inclusion is a process by which a school strives to respond to *all* students. This perspective on inclusion expands the concept of integration, which often remains tied to the idea of inclusion, where the focus is on specific groups of students (e.g., students with disabilities or special educational needs) who are to be assimilated into existing forms of schooling, often with adapted curricula, different work, or support assistants (Vislie, 2003). Inclusive teaching methods, as intended in this research, should still accommodate students' individual differences, but with the perspective that these differences can serve as a starting point for teaching (Roos, 2019), leading to innovations that have the potential to benefit *all* students (Ainscow, 2016). The differences among students as individuals also highlight diverse needs in their learning process related, for example, to their previous school background or their interests (Sousa & Tomlinson, 2011). Differentiation represents one possible answer to embrace students' differences in the learning process and, as for inclusion, it is a somewhat fuzzy concept (Anthony et al., 2019). Differentiation is based on the idea that the teacher should proactively plan her/his activity having students' differences and similarities in mind in order to engage them more fully with learning (Tomlinson et al., 2003). Anthony and colleagues (2019) suggest a perspective on differentiation that shifts the focus from cognitive performance, by which students are classified and divided and that generally characterizes the perspective on differentiation, towards attention to students' well-being and productive mathematical disposition. Such focus in a differentiated perspective can promote inclusive teaching methods also taking into account students' moral, emotional and social development and not exclusively the cultivation of knowledge (Anthony et al., 2019).

Research on classroom experiences on traditional mathematical discussion allows us to highlight some strengths and challenges in terms of inclusion of this classroom practice. Traditional mathematical discussion can be considered an inclusive practice since it creates the opportunity to listen and to centre the discussion around diverse proposals developed by students. More specifically, traditional mathematical discussion fosters the expression of students' personal senses in meanings and the consciousness of their own intellectual processes (Bartolini Bussi, 1996). Moreover, it supports the development of interpersonal processes (Bartolini Bussi, 1996) in a "safe learning environment", in which all students are positioned as competent and capable and it promotes active listening and asking questions (Anthony et al., 2019). On the other hand, traditional mathematical discussion can involve more or fewer students. Students' participation generally represents a challenge while conducting traditional mathematical discussions since few students actively engage in discussions and the cognitive load related to discussions can be high, especially for problem solving discussions (Richland et al., 2017), resulting in an even more limited involvement. Therefore, since inclusion is a delicate aspect in relation to mathematical discussions, we have decided to complement traditional in-person mathematical discussions with DMD in order to amplify their potentialities in terms of inclusion. As for traditional mathematical discussions, DMDs have potential and limitations with respect to inclusion. For instance, a chat-based environment can provide a secure space for students to interact, share ideas, and seek help without the fear of judgment or exposure face-to-face. This setting not only encourages open communication, but also enables teachers to have a broader and more detailed view of their students. Teachers can interact with more students simultaneously and they can more easily identify issues and needs among students, allowing for timely interventions and support. This proactive approach ensures that all students feel seen, heard, and safe, contributing to a more effective and nurturing educational experience.

In line with this perspective, we aim to highlight the added value that the integration of DMD can give to traditional mathematical discussion in terms of inclusion. In accordance with the vision on inclusion (Vislie, 2003) and differentiation (Anthony et al., 2019) already presented, which promotes the recognition and appreciation of students' heterogeneity and its exploitation to support all students, we have chosen the framework of didactical differentiation (Sousa & Tomlinson, 2011) for analysing the added value of DMD with respect to traditional mathematical discussion. Differentiation arises from the research-based perspective that students are more likely to engage thoroughly in learning and experience more robust learning outcomes when teachers proactively plan with both their differences and similarities in mind (Tomlinson et al., 2003). The model of differentiation of Sousa and Tomlinson outlines essential principles that drive effective differentiation, integral to a classroom system wherein all components collaborate to foster peak learning. In our research, these principles are used in alignment with the core tenets of differentiation presented in the previous paragraphs with the aim of highlighting how the design for the integration of DMD and traditional mathematical discussion can support mathematical discussion as a more inclusive practice. According to the differentiation model (Sousa & Tomlinson, 2011), teachers can modify four variables within the classroom to address various student needs: (1) *content*, that is what students will learn or how they will gain access to what they are asked to learn; (2) *process*, that is activities guiding students' understanding or ownership of essential content; (3) *product*, that is how students showcase their knowledge, comprehension, and abilities after extended learning periods; and (4) *affect*, that is consideration of students' feelings and emotional requirements. By modifying these variables, teachers accommodate differences in students' *readiness* (closeness to learning objectives), *interests* (curiosity in specific ideas, topics, or skills), and *learning profile* (preferences for learning approaches or modes). In this section, the framework of differentiation is intended as an analytical tool that provides criteria for evaluating design in inclusive terms. It supports the analysis of the design of integrating DMD with traditional mathematical discussion, aiming to identify the design elements of such integration that can empower mathematical discussion as an inclusive practice, allowing as many students as possible to participate in mathematical discourse within a discussion context. Based on the characteristics that distinguish DMDs, we hypothesise that by varying the modes of delivering the discussion among traditional and digital, it is possible to expand the ways in which one can be part of the discussion, thereby enabling more students to participate in the mathematical discourse realised within a discussion situation. For traditional mathematical discussion, we focus on balance discussions as presented in section 1.2.

Analysis of the design for the integration between traditional and Digital Mathematical Discussions to foster inclusion

Traditional balance mathematical discussion and DMD share several elements regarding the variables within the framework of differentiation, as they both originate from common theoretical foundations. However, they also differ in certain aspects due to the influence of the digital environment in DMDs, making it interesting to analyse how their integration can promote inclusion.

Regarding the variable of *content*, considering "what students will learn", both discussions stem from mathematical problems and evolve from group solutions proposed by students to such problems, aiming to compare related or contrasting strategies and stimulate metacognitive reflections (Cusi et al., 2017). Regarding the dimension of "how students gain access to what they are asked to learn", the two types of discussion differ. For traditional balance discussions, access to content is primarily oral, both in communication with the teacher and with classmates, supported by written mode in group-work, when students submit their proposals, and in the whole-class discussion, when key aspects of the discussion are written on the board. Moreover, interaction is synchronous. The additional opportunities offered by DMD concern access to content in asynchronous mode and mainly in written form, still giving students the possibility to interact through images, audio messages, videos, emoticons and all the various forms of interaction that the digital environment supports.

Concerning the *product* variable, the two types of discussions share the possibility of assessing students' knowledge, understanding, and skills through their group productions of mathematical problem solutions. Furthermore, DMD enables the creation of a greater variety of products concerning the tackled mathematical problems. It is not limited solely to the final group production, but also facilitates an increase in shared materials encompassing written messages, images, audio messages, and all productions shared by students within both the chats and the Padlet during the whole discussion. Regarding the *product* variable, we can also consider students' communication abilities developed within discussions as a product. In traditional balance discussions, students acquire the ability to orally argue and respond quickly during a discussion. Additionally, through DMD, students also acquire the ability to argue in writing and to refer back to their or other peers' interventions and reasoning over extended periods given by the asynchronous modality.

For the *process* variable, we can identify similar activities for both traditional and digital discussions. In particular, we analyse the *process* in both cases by referencing the activities carried out according to the FaSMEd design (section 2.1). According to such design, the *process* of discussions unfolds in various activities in which students are involved, among which group-work and comparison with peers, comparison with the teacher, metacognitive reflections. The added value given by the integration of traditional balance mathematical discussions with DMDs concern the expansion of time at disposal to conduct the discussion and the availability of a complete transcript of the discussion. Concerning the timing of DMDs, it enables students to engage in the same activities as traditional balance mathematical discussion in an extended amount of time, also allowing the alternation of some of them (e.g., alternation between in-the-moment or more reflected comparison with peers or the teacher). Concerning the availability of the transcript, the added value lies in students having access anytime to all productions, which enables the activation of a comprehensive individual reflection process. With access to the entirety of written productions, students can engage in autonomous reconstruction of the entire discussion. Unlike in a classroom setting, where the teacher would need to reconstruct the discussion and might not have direct access to all students' contributions, having all written productions available allows for a more thorough reflection process and enables students to do it autonomously.

Finally, regarding the *affect* variable, both types of discussions stem from making all students comfortable in a discussion situation, giving the possibility to all of them to contribute, but they differ concerning students' sensitivities to their preferred modes of learning and interaction that can influence the *affect* variable. The modes of traditional balance discussions are favourable for students who prefer immediate feedback, can follow reasoning flows orally and at a quick pace, and enjoy expressing themselves in-person when participating in class. The integration of traditional mathematical discussion with DMD seems particularly rich in terms of inclusion considering that the modes of DMDs are favourable for students who need more time to assess teacher and classmates' statements and to formulate their proposals, who see written form as a support for following a reasoning process, and who do not enjoy exposing themselves in-person in face-to-face settings, so adding relevant differences to the modes the solely traditional mathematical discussion can offer. Additionally, the modes of DMDs may also be favourable for teachers who struggle with managing the immediacy of exchanges and keeping track of all students' work in class, allowing them to have more opportunities to observe students' progresses within the activities and their engagement, that may not always be visible in person, and to intervene accordingly, giving students a more central role.

In conclusion, the theoretical foundations on which the two types of discussion are based allowed us to hypothesise that the combined use of discussions could result in positive outcomes in terms of inclusion. In fact, in line with Vislie's perspective (2003), we do not believe that students should adapt to the given context, but rather the context should adapt to the diverse needs of the students. For this reason, we hypothesise that we can complement the practice of traditional mathematical discussion with a digital form modifying the timing and traditional interaction modes of discussions in order to make discussion a more inclusive practice. Promoting mathematical discussion as a more inclusive practice is achievable by basing the design of the practice on principles of differentiation, which take into account students' differences and learning preferences and results in a practice for all students. Indeed, traditional mathematical discussion already possesses characteristics that consider students as individuals and aims to be developed based on the personal meanings students attribute to mathematical concepts, giving voice to their diverse perspectives and involving as many students as possible in the mathematical discourse. However, classroom experiences show limited student participation in mathematical discourse within discussion contexts. The integration of traditional mathematical discussion with DMD can amplify the characteristics of mathematical discussion, allowing for new forms of participation in the discussion through the potentialities of the introduction of the digital modality and, then, fostering more participation in the mathematical discourse realised within the discussions. Traditional balance mathematical discussion suits students who are ready to engage in real-time oral exchanges and can quickly process information, while DMD caters to those who need more time to reflect and respond, thus accommodating different levels of readiness. Additionally, DMD engages students with a keen interest in technology and multimedia, offering a variety of interaction forms that make engaging in discussions more appealing. This integration also addresses diverse learning profiles by supporting students who prefer written communication and visual aids and those who thrive in oral, real-time discussions. The integration of technology in DMDs enhances inclusion by providing asynchronous access to content and a variety of interaction methods, such as written messages, images, audio messages, videos, and emoticons. This contrasts with the primarily oral and synchronous nature of traditional mathematical discussions. DMDs enable a broader range of student-produced materials and offer a complete transcript of discussions, facilitating individual reflection and autonomous reconstruction. Additionally, the asynchronous format allows students more time to engage with and reflect on content, benefiting those who need longer to formulate responses and follow reasoning processes. This technological integration also supports teachers in observing student progress and engagement more effectively, allowing for timely interventions and the creation of a more inclusive learning environment.

Chapter 3

Mathematical content implemented in Digital Mathematical Discussions: a focus on algebra as a thinking tool

Following research experiences on the development of algebra as a thinking tool within discussion contexts, we chose algebra as the mathematical content to be addressed in our research as well. In particular, we refer to tasks developed and already experimented in traditional mathematical discussions to promote the use of algebraic language for developing proofs in arithmetic (Cusi, 2009B). Within this chapter, we first present the vision of algebra that such an approach to its teaching wants to promote (section 3.1). Then, in section 3.2, we introduce a theoretical framework to investigate the use of algebra as a thinking tool by focusing on the development of the epistemic aspects of algebraic thinking, encompassing *representation registers* (Duval, 2006), *conceptual frames* (Arzarello et al., 2001) and *anticipating thoughts* (Boero, 2001).

In the second part of the chapter, we present the mathematical tasks addressed within the experimentations carried out during this research. The tasks belonging to a didactic path on proof in arithmetic through the use of algebraic language (Cusi, 2008, 2009A, 2009B) are presented in section 3.3, while a further task, experimented during the pilot study, is presented in section 3.4. The tasks have already been experimented for traditional mathematical discussions in previous research (e.g., Cusi, 2009A, 2009B; Swidan et al., 2023). The prior experiences with these tested tasks enable us to anticipate students' responses and difficulties and to rely on an already developed model of the teacher's role (M-$_{\text{AE}}$AB, see subsection 1.2.1). The presentation of the tasks is accompanied by their a-priori analysis based on the epistemic aspects of algebraic thinking, introduced in section 3.2.

3.1 Algebra as a thinking tool

The teaching and learning of what is commonly referred to as "school algebra" has traditionally been associated with secondary school students and it has largely focused on working with polynomial and rational expressions, representing problems through algebraic expressions and equations with variables and unknowns, and solving these equations using axiomatic principles and equivalence properties (Kieran, 2020). However, over the past several decades, perspectives on the nature of school algebra have evolved, leading to different conceptualizations as reported by Drijvers and colleagues (2011) and Kieran (2020). Freudenthal (1977) describes school algebra as not only the solving of linear and quadratic equations, but also as involving algebraic thinking, which encompasses the capability to describe relationships and solve problems in a general, abstract manner. Arcavi and colleagues (2017) similarly define the goals of school algebra as including "expressing generalizations, establishing relationships, solving problems, exploring properties, proving theorems, and calculating" (pp. 2–3). Stacey and Chick (2004) describe it as "a way of expressing generality; a study of symbol manipulation and equation solving; a study of functions; a method for solving certain classes of problems; and a tool for modelling real-world situations" (p. 16). The diversity of views on what constitutes school algebra is highlighted by Leung and colleagues (2014), who present evidence that algebra lessons vary significantly across countries, and even within the same country. This variation is not only in terms of content but also in whether the emphasis is placed on procedural fluency, conceptual understanding, or a combination of both. Educational research too, over the past fifty years, has highlighted an evolution in the vision of school algebra. The focus has shifted from viewing school algebra as a discipline for solving equations, where students are required to memorize rules for symbolic manipulation and practice equation solving, to a perspective that values algebraic reasoning, considering students' thinking processes that involve indeterminate objects and the use of algebraic symbols (Kieran, 2020).

When trying to answer to the question 'what is meant by algebraic thought?', Bell (1996A) proposes different mathematical activities:

- resolution of complex arithmetic problems, both step-by-step, working from given data to unknowns, and by global perceptions and the use of multiple arithmetic relations;
- codification and use of systematic general methods for different types of problems;
- finding and proving of number (and geometric) generalisations;
- recognition and use of general properties of the number system and its operations;
- recognition, naming and use of standard functions;
- the use of manipulable symbolic language.

According to the researcher (Bell, 1996A), all six are recognisable aspects of mathematical activity, but the term algebraic can refer to all of them only if the sixth is also present. Indeed, the use of the manipulable symbolic language clearly distinguishes algebra from other types of mathematical activity. Even if "using letters does not amount to doing algebra" (Radford, 2006, p.3) and symbols can be used without the need to check their meanings or understand their significance (Skemp, 1987), using and interpreting symbols to designate the objects of algebra is fundamental to the development of algebraic thinking (Warren et al., 2016) and algebraic thinking is recognised as being inseparable from the formalised language by means of which it expresses itself (Arzarello et al., 2001). In line with Vygotsky's (1962) theory regarding the connection between thought and language (section 1.1), algebraic thought and language are viewed as two intertwined and mutually dependent aspects of the same process. Language is seen as a tool for thinking that progresses over time, and in turn, the evolution of thought inspires the use of more complex and sophisticated linguistic forms (Arzarello et al., 2001). Mathematical language, and symbolism in algebra in particular, is not only for expression but also for manipulation, and syntactic transformations of symbolic expressions can be done to expose conceptual equivalences with a reduced degree of mental effort required to establish those equivalences by working with the concepts themselves (Bell, 1996B). Working with symbolic language and performing manipulations and transformations with it maintaining the semantic control is always a delicate matter in educational contexts. It is widely recognised that mastering algebra demands practice, though the nature of this practice remains a point of consideration oscillating between developing automated procedural skills and fostering flexible algebraic thinking. The two approaches do not necessarily complement each other, as tasks centred on procedural skills in a routine manner often fail to encourage insightful thinking. In fact, a tension can arise between the pursuit of automatization and the cultivation of deeper understanding. On one hand, students are expected to carry out basic procedures "without thinking"; on the other hand, they are expected to remain vigilant and recognise any deviations from standard pathways (Arcavi et al., 2017). The risk of manipulation without thinking is the loss of control over the significance of what is done, called evaporation by Arzarello and colleagues (2001). Evaporation refers to the loss of meaning associated with symbols, experienced by most students when they lose semantic control during algebraic problem-solving. This typically occurs when students are no longer able to express the meaning of mathematical objects and relationships in ordinary language, referring to actions, the processes of their construction and generation, or any other relevant information about them. In such cases, students can no longer use algebraic language as a record of their thinking processes, nor effectively mediate with natural language (Arzarello et al., 2001). Students' perception of algebraic expressions as mere strings of symbols to which routine procedures are applied is addressed also by Sfard and Linchevski (1994) as a pseudo-structural conception of algebra. In these cases, students equate the signifier (i.e., the representation) with the signified and they consider formal manipulations of algebraic expressions as the only form of meaning associated with them. The students may still be able to perform manipulations, but their understanding remains instrumental (Skemp, 1978). Thus, the challenge is to enable students to correctly manipulate algebraic expressions and to maintain the control over the significance of their manipulations.

Behind the idea that algebraic thinking is inseparable from the formalised language in which it is expressed and from its manipulations, it is limiting to think that it exists only at this level, otherwise, everything would be reduced to a series of mechanical manipulations. It is the functions of identification and transformation that allow mathematics to be not only a language but also a powerful tool for reasoning and prediction. Through the "shaping in formulas" of knowledge or hypotheses about phenomena, and the derivation of new insights through transformations allowed by algebraic language, mathematics becomes a means of discovering new knowledge about those phenomena (Arzarello et al., 2001). When algebraic language is encouraged as a tool for representing relationships and exploring their various aspects, while also developing the manipulative skills needed to transform symbolic expressions into different forms to reveal new features of these relationships, students can be guided through the essential algebraic cycle, described as an alternation of three types of algebraic activities: representing, manipulating, and interpreting (Bell, 1996B).

Algebraic expertise involves the interplay of various types of skills. First, it is essential for students to develop procedural skills, such as solving simple equations or simplifying expressions. This involves procedural algebraic calculations, which often have a local focus. It is important for students to perform these procedures fluently. Achieving this fluency requires both insight and consistent practice (Drijvers et al., 2011). Fluent mastery of procedural skills is important and requires practice, but simply repeating procedures until they become automatic is not motivating and may be counterproductive. Instead, tasks should offer insightful, varied, and challenging forms of practice that keep students alert and encourage creativity. The effectiveness of these tasks also depends on the context - whether they are individual, collaborative, tool-based, or involve discussion (Arcavi et al., 2017). However, algebra encompasses much more than just mastering basic skills. It also involves selecting appropriate strategies to tackle problems, maintaining an overview of the solution process, creating models, taking a global perspective on expressions, wisely choosing the next steps, distinguishing between relevant and irrelevant characteristics, and interpreting results meaningfully (Drijvers et al., 2011). In research literature, this type of meta-knowledge is referred to as symbol sense. Symbol sense is "a complex and multifaceted 'feel' for symbols", that is "a quick or accurate appreciation, understanding, or instinct regarding symbols" (Arcavi, 1994, p.31). Arcavi (1994) articulates symbol sense over eight behaviours which illustrate paradigmatic situations:

1) *Making friends with symbols* – It involves developing an instinct for when to use symbols in problem-solving and, equally, when to set them aside in favor of other representations. Symbols can be valuable allies, serving as tools for sense-making and aiding in the recognition of the meaning behind a symbolic solution. Ideally, students should also appreciate the "power" of symbols in validating or rejecting conjectures supported by argumentation. At the same time, symbol sense includes the capability to abandon symbolic methods when they are not suitable for solving a problem;

2) *Manipulations and beyond: reading through symbols* – Symbol sense involves interrupting a mechanical symbolic procedure to inspect and reconnect with the underlying meanings. This can manifest in various ways, whether through the manipulation or interpretation of symbols. Symbol sense can arise when one pauses an almost automatic routine to observe and recognise a symbolic relationship (*reading instead of manipulating*), when an initial inspection of the symbols provides insight into the problem and its meaning (*reading and manipulating*), or when the search for symbolic meaning is key to solving a problem or adds new insights (*reading as the goal of manipulations*). Furthermore, symbol sense may include developing the habit of re-reading and checking for the reasonableness of symbolic expressions (*reading for reasonableness*), while maintaining awareness that symbolic illusions may occur;

3) *Engineering symbolic expressions* – Symbol sense involves, first, recognising that a symbolic expression can be purposefully crafted to meet a specific goal and that we have the capability to engineer it. Second, it includes the awareness that a particular type of expression may be exactly what is required. Finally, it encompasses the capability to successfully engineer that expression;

4) *Equivalent expressions for "non-equivalent" meanings* – Another facet of symbol sense is the intuition and confidence in using symbols to explore and uncover new aspects of the original meanings through equivalent representations of expressions;
5) *The choice of symbols* – When translating a situation into symbols, the initial decision on what and how to represent can significantly impact both the solution process and the outcome. Symbol sense aids in making the right choice by aligning it with the objective of the problem. However, the selection of symbols can sometimes obscure aspects of the situation or even block the solution entirely;
6) *Flexible manipulation skills* – Even in situations where meaning can be set aside and manipulations are performed mechanically, a guiding technical or formal symbol sense should still govern the process. Proper manipulation of symbols involves much more than simply following rules; it encompasses various aspects of understanding the symbols, even when their referents are momentarily forgotten or ignored. This includes recognising circularity, having a "gestalt" perspective, and identifying formal targets. Circularity refers to symbolic manipulations that result in tautological identities that lack informative value. The capability to foresee such circularity demonstrates symbol sense, and when it goes unanticipated, symbol sense can prevent stagnation by prompting the search for alternative approaches. A "gestalt" perspective allows one to view symbols not merely as a sequence of letters but as arranged in a specific structure. As for formal targets, symbols sense emerges when the solver can first envision a symbolic target and its form that is manageable and interpretable, and then purposefully select the necessary formal manipulations to achieve that target;
7) *Symbols in retrospect* – When solving a problem by creating a mathematical model of a situation, the connection between symbols and meanings typically occurs at the beginning (when establishing the model) and at the conclusion (when interpreting the results in relation to the modelled situation). During the intermediate steps, progress is often made through formal derivations, where the meanings related to the situation are usually overlooked. In these instances, symbol sense allows for the recognition of meaning at intermediate steps, even when it is not typically necessary or common to do so;
8) *Symbols in context* – Symbol sense involves recognising the varied roles symbols play, requiring the capability to navigate their multiple meanings based on context and manage different mathematical objects and processes. While immediate recognition of these roles can be challenging, symbol sense can still manifest in ones' resourcefulness and common sense, enabling them to identify mistakes and resolve confusion, particularly in instances of symbolic illusions.

We refer to symbol sense to characterise the vision of algebra we aim to pursue in our research. Arcavi (1994) supports the idea that symbol sense can be nurtured through appropriate supportive instructional practices, including interventions in which mathematics is seen as a way for expressing ideas, students can express freely how they see and sense symbols versus other representations, and the classroom culture supports sense making. Thus, mathematical discussion seems suiting an appropriate instructional practice to support the development of symbol sense, and sense making in general. Moreover, the symbolic function of language is a long-term process that can be developed over time through a cognitive apprenticeship where the teacher encourages thinking as an ongoing stream rather than as a single act of language (Arzarello et al., 2001), as facilitated by discussions.

As the vision of school algebra expanded significantly over the decades, moving from a focus on letter-symbolic and symbol-manipulation views to one that included multiple representations, realistic problem settings, and the use of technological tools, so did the vision of how algebra is learned. The previously held notion that students learn algebra by memorizing rules for symbol manipulation and by practicing equation solving and expression simplification was replaced by perspectives that consider a multitude of factors and sources from which students derive meaning for algebraic objects and processes (Kieran, 2020). In school context, even students who are able to successfully apply algebraic techniques often struggle to view algebra as a means for understanding, expressing, and communicating generalizations, uncovering underlying structures, or constructing and presenting mathematical arguments or proofs. Instruction often lacks opportunities for students to not only learn and remember rules, but also to "forget" them - to see beyond the specific details in order to engage in abstract thinking, generalization, and strategic problem-solving (Arcavi, 1994). For this reason, the activities on which our experimentations are based are focused on mathematical tasks aimed at guiding students in appreciating algebra as a tool for reasoning. Specifically, in line with previous studies (Cusi, 2009A, 2009B), we chose to focus on activities that involve exploration in arithmetic, conjecturing, argumenting and proving through the support of algebraic language.

3.2 Epistemic aspects of algebraic thinking

A student facing proof problems in algebra must possess a range of different competencies in order to become able to succeed in this kind of activity and to acquire an awareness of the role played by algebraic language within these processes, including the capability to: (A) understand the meaning of mathematical terms in the problem text and interpret them accurately within context; (B) correctly translate from verbal to algebraic language; (C) interpret the results of transformations applied to algebraic expressions in relation to the problem context; and (D) evaluate the consequences of their assumptions (Cusi, 2009A).

The capability to interpret and to translate from verbal language to algebraic language (capabilities A and B) and to perform syntactic transformations on expressions during a proof can be seen as specific cases of transformations of representations (Duval, 2006). Duval argues that mathematical objects cannot be directly observed and accessed with instruments, as it happens in other branches of science, but access to mathematical objects is tied to the use of signs and semiotic representations. This specific epistemological situation in mathematics radically changes the cognitive use of signs, requiring learners to navigate two conflicting demands: first, the necessity of using semiotic representations to engage in any mathematical activity, and second, the requirement not to confuse these representations with the mathematical objects themselves. Semiotic systems of representation are essential not just for designating or communicating about mathematical objects, but for engaging with and manipulating them. Mathematical work relies on transforming one representation into another, as mathematical processing is fundamentally about substituting one set of signs for another, rather than substituting signs for objects. What truly matters according to Duval is not the representations themselves, but their transformations. Duval distinguishes between two different kinds of transformations: treatments and conversions. Treatments are transformations of representations that happen within the same system, while conversions consist in changing a system without changing the object being denoted. When transformations of representations are possible, the semiotic systems can be addressed as *representation registers*. Duval identifies four different registers of representation in mathematics and supports the idea that comprehension in mathematics assumes the coordination of at least two of them: natural language; symbolic systems (including algebraic); drawings and geometrical figures; and diagrams and graphs. These registers can be divided into multi-functional and mono-functional, depending on their use for only one mathematical function (mathematical processing) or for a large range of cognitive functions (e.g., communication, information processing, awareness, imagination). This functional difference is connected to the way mathematical processes run because within mono-functional registers most processes take the form of algorithms, while within multi-functional registers processes cannot be made into algorithms. Specifically, symbolic systems and diagrams and graphs are mono-functional registers, while natural language and drawings and geometrical figures are multi-functional. Another difference standing between natural language and symbolic systems is that natural language can be expressed through two equivalent modalities, that are orally and written, while the symbolic systems can be expressed only in written form (Duval, 2006).

The importance of interpretative processes (capability C) as key aspects in using algebra as a tool for reasoning has been highlighted in the theoretical model for analysing algebraic thinking processes proposed by Arzarello, Bazzini and Chiappini (2001). The model stems from Frege's semantics. The ideas of Frege on semantics seem well-suited for examining the interpretation of symbolic expressions in algebra, particularly concerning learning. In this context, it is essential to distinguish between sense and denotation of an expression. The denotation of an expression refers to the object that the expression denotes, while the sense represents the manner in which the object is perceived by the mind. In other words, sense is the thought that is conveyed by the expression. Mathematics, akin to a natural language, is stuffed with expressions that share the same denotation but encompass different senses. The most evident sense of an algebraic expression concisely represents the method by which the denoted object is obtained through the computational rules articulated in the formula itself; this is referred to as the algebraic sense. However, the same formula can embody additional senses beyond the algebraic one. It can be applied across various knowledge domains, whether mathematical or otherwise, each generating at least one new sense based on the characteristics of the domain. The contextualized sense of an expression relies on the knowledge domain in which it exists, differentiating it from the algebraic sense. The strength of algebra lies in the multiple senses inherent in a single formula and/or those that can be derived through syntactic manipulations. Conversely, the didactic challenge arises from the significant imbalance among senses, denotations, and expressions, which obscures the status of algebraic signs for students. The second ingredient of the model is the notion of conceptual frame. Arzarello and colleagues affirm that actions and decisions made during activities involving algebraic reasoning are governed by organised sets of knowledge that suggest how to reason, manipulate formulas, and anticipate outcomes. These sets of knowledge, which are called *conceptual frames,* also include the subject's expectations regarding what might happen as a consequence of the information they possess, and the set of actions with which to react if their expectations are not confirmed. The term "frame" is borrowed by the authors from studies in artificial intelligence, where it refers to a data structure capable of generating a stereotyped representation of a piece of knowledge. However, in their model, it is further enriched with specific conceptual aspects of knowledge, organised as a set of conceptual notions and operational skills related to particular aspects of mathematics (e.g., even-odd numbers or multiples). A conceptual frame is like a condensed story with a script, where the subject is expected to take action based on the task's objective. These frames are activated as virtual texts when interpreting, for example, a problem, depending on the context and circumstances. As such, they have both socio-cultural and individual characteristics. In summary, the model proposed for describing algebraic thinking by Arzarello and colleagues revolves around two key concepts: Frege's semiotic triangle (sense-denotation-expression) and the notion of conceptual frames. According to the model, the dynamics of the algebraic thinking develop among the signs, senses, and denotations of algebraic formulas, as well as the activation of conceptual frames, their mutual relationships and changes from one frame to another. Essentially, the dynamic aspects of algebraic thought processes can be effectively captured by examining how students adjust their semiotic triangles within a given conceptual frame or transition between different frames (coordination of conceptual frames). This approach highlights the fluidity of algebraic thinking and the importance of understanding how students navigate their understanding of mathematical concepts.

Arzarello and colleagues (2001) also emphasise that the interpretation of the various expressions obtained along the way (capability C) allow students to make accurate predictions about the effects and objectives of the syntactic transformations they perform (capability D). Changes in the conceptual frame used to interpret symbolic expressions enable students to make accurate predictions about the effects and goals of the syntactic transformations they perform (Arzarello et al., 2001). To carry out a transformation efficiently, one needs to anticipate certain aspects of the final form of the object being transformed in relation to the desired goal, as well as the range of possible transformations that can be applied. According to Boero (2001), *anticipating* is "the mental process through which the subject foresees the final (and/or some intermediate) shape of an algebraic expression useful for solving the problem, and the general direction of the transformations needed to get it" (Boero, 2001, p.14). Boero introduces the concept of anticipation as a key element in transformation processes, noting that anticipations can occur before, after, or without algebraic formalization. When anticipations happen prior to formalization, they often involve manipulations of variables through arithmetic or geometric strategies, referred to as pre-algebraic. After formalization, anticipations rely on the transformational function of algebraic language, requiring control of standard transformation models. To efficiently carry out a transformation, one must anticipate certain aspects of the final form of the object to be transformed concerning the goal to be achieved and the range of possible transformations that can be applied. It is this anticipation process that allows for the design of strategies and continuous feedback. To model transformation processes and analyse problem-solving processes in algebra, Boero introduces sem/form diagrams (Figure 3.1). "Form" encompasses all written expressions that utilize algebraic language, while "sem" consists of all own external non-algebraic representations of objects (e.g., a geometric figure or a sentence by natural language). In sem/form diagrams, upward arrows represent formalization, while downward arrows indicate interpretation. Formalization involves translating a sem representation into an algebraic expression, while interpretation refers to generating both a mental and a non-algebraic external representation consistent with the form. Continuous horizontal arrows represent transformations: between form 1 and form 2 (Figure 3.1 left), they denote transformations following algebraic rules; between sem 1 and sem 2 (Figure 3.1 right) reflect transformations of mental and corresponding external representations. Dotted horizontal arrows (Figure 3.1 left) indicate conjectures or guesses, such as hypotheses to be proved. Through the sem/form diagrams, Boero provides new insights into the cognitive pathway underlying algebraic transformations, highlighting that these transformations require students to integrate some of the following activities: transforming the nature of the problem to manage the new problem more simply; anticipating, or imagining the consequences of certain choices made regarding algebraic expressions and/or variables, and/or through formalization processes; suspending the initial reference meaning and working on the expressions by applying algebraic transformations; and using the reference meaning to plan subsequent transformations to be applied to the form or to interpret the consequences of those transformations. Finally, Boero suggests educational activities that can either hinder or foster the relationship between anticipations and transformations, advocating for exercises focused on simplifying expressions, discussing necessary transformations, and demonstrating conjectures using algebraic language.

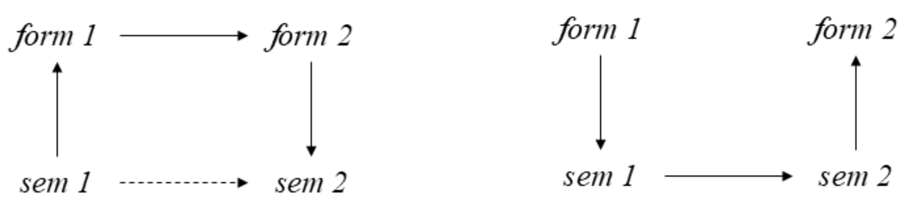


Figure 3.1 Sem/form diagrams from Boero (2001)

In this thesis, we refer to the transformations between representation registers, the activation and coordination of conceptual frames and to anticipating thoughts as the *epistemic aspects of algebraic thinking processes*. Through these epistemic aspects of algebraic thinking, the typical way in which a student navigates the solution process of an algebraic problem in which a conjecture needs to be proven can be described in detail following Arzarello and colleagues (2001) and referencing to the sem/form diagrams (Boero, 2001). First, the student interprets the text of the problem and, if needed, translates it from natural to algebraic language (formalization phase). The need to share interpretations among students is the determining factor that activates one or more conceptual frames. The manner in which this interpretation is conducted is crucial since subjective factors influence how concepts are understood and shape the trajectory of both thinking (i.e., anticipating thinking) and writing (i.e., naming). With respect to the sem/form diagram, sem 1 contains information about the data at disposal from the problem, and form 1 its representation in algebraic language derived from the interpretation done. The outcome of the interpretation (form 1) is another text - potentially written, spoken, or a combination of both - into which the original text has been interpreted, and this process may continue with further interpretations of the new text transitioning from form 1 to form 2. The transition from form 1 to form 2 may require several intermediate steps and is guided by proper anticipating thoughts. Achieving this transition allows the activation of the interpretation phase, during which a mental representation of sem 2 (containing the content of the conjecture to be proven) coherent with form 2 can be generated.

3.3 Didactic path on proof in arithmetic through the use of algebraic language

The didactic path on proof in arithmetic through the use of algebraic language presented in this section has been derived from the research conducted by Annalisa Cusi (Cusi, 2008, 2009A, 2009B, 2010; Cusi & Malara, 2009, 2013) and aligns with the promotion of school algebra as presented in this chapter. Cusi (2009A, 2009B) suggests that activities of proof in arithmetic would both help students to appreciate the value of algebraic language as a tool for the investigation and solving of situations that are difficult to manage through natural language only, and provide them with the opportunities they need to progress gradually from argumentation to proof. Furthermore, elementary number theory provides an appropriate domain for formulating conjectures and tackling proof-related problems because it offers easy access to the relevant content, a wide range of problems that can be assigned, and the capability to generate examples without much difficulty (Edwards & Zazkis, 2002). Concerning the latter aspect, working with integers helps students feel more comfortable with activities involving problem-solving, reasoning, generalization, abstraction, and proof (Selden & Selden, 2002). Moreover, problems in arithmetic and algebra can give students the freedom to explore and create something on their own and their different creations can lead to interesting classroom discussions (Arcavi et al., 2017).

The didactic path is aimed at high school students, particularly to students in grade 9 and 10. The content of the tasks in the didactic path aligns with the Italian high school curriculum. However, in schools, the focus on algebraic proofs is often overlooked in favour of proofs in (Euclidean) geometry, even though algebra offers many opportunities to apply deductive reasoning (Arcavi et al., 2017). The didactic path has been designed to complement traditional syntactic activities with the aim of fostering the development of symbol sense in students. Specifically, the didactic path is designed to guide students in developing the following skills:

- Understanding when to use symbols in the problem solving process and, conversely, when to abandon symbolic treatment in favour of better tools;
- Seeing symbols as tools that carry meaning, which includes: knowing how to invoke appropriate symbols and recognise the meaning of a symbolic solution; feeling the need to seek the meaning of symbols, whether it is essential for solving the current problem or simply for gaining new insights; and viewing equivalent symbolic expressions not merely as results but as potential sources of new meanings;
- Appreciating the communicative properties of symbols and their power to reveal and demonstrate relationships.

3.3.1 Structure of the didactic path

The didactic path is structured in six phases (Cusi 2008, 2009A): (1) translations of expressions from verbal to algebraic language and vice-versa; (2) study of the relationship between properties of a given formula and properties of the variables it contains; (3) analysis of the truthfulness/falseness of statements concerning natural numbers and justification of the given answers; (4) exploration of numerical situations, formulation of conjectures and related proofs; (5) analysis of proving strategies; and (6) construction of proofs of given theorems.

Phase 1 – Translation from natural to algebraic language and viceversa

The first phase is designed with the aim of helping students develop a relational understanding of algebra, encouraging them to recognise the meanings behind algebraic expressions. Students have the opportunity to experience the clarity and simplicity of algebraic language in contrast to the complex morphological and syntactic structure of natural language, and to learn how to "read" the meaning conveyed by algebraic expressions. For this reason, students are given tasks on the algebraic representation of propositions of varying complexity and tasks that require them to translate a series of statements into algebraic language in order to use the formalization produced to highlight the correctness or otherwise of these statements. Through the tasks of the first type, students are guided in developing representations of concepts such as odd numbers, two-digit numbers, multiples of a number, and divisibility and non-divisibility by a number. To consolidate these representations, they are also asked to formalize other, more complex statements, which require them to coordinate the various representations they have produced. Through the tasks of the second type, students can experience how an approach to statements' analysis involving their formalization in algebraic language allows them to more clearly and distinctly highlight their correctness or possible problematic elements contained within them.

Examples of tasks from phase 1 are*: Translate the following statements into algebraic language: an even number, a multiple of 3, the successive of an odd number*. *Translate the following sentences into algebraic language, expressing the equality between two expressions. Then verify whether the stated equality is correct. If it is not, identify the error: The sum of the triples of two numbers is equal to the triple of the sum of the numbers themselves*.

Phase 2 – Interpretation of expressions with variables

The tasks of the second phase of the didactic path are primarily aimed at encouraging students to develop interpretive skills, helping them to understand not only the existence but also the importance of the relationships between the values taken on by the variables within a given algebraic expression and the properties of the expression itself. The tasks in this phase require students to highlight the connections between the properties of given algebraic expressions and those of the variables contained within them.

An example of task from phase 2 is: *Find for which values of n (natural number)* n^2 *is a multiple of 4*.

Phase 3 – Analysis of the truthfulness/falseness of statements

The tasks proposed in the third phase are designed to foster the development of a critical approach towards the meaning and validity of given mathematical statements. Another key objective of this phase is to cultivate an awareness of the role played by algebraic language in proofs. In this phase, students are asked to analyse statements describing numerical properties, with the goal of determining their truth or falsity and providing a justification for their answers.

These tasks aim to guide students in comparing the process that establishes the truth of a statement with the one that leads to its refutation, making them aware of the inadequacy of relying solely on numerical verification for proof purposes, the effectiveness of using algebraic language as a tool for generalization to support their reasoning, and the significance of employing counterexamples as a means of refutation.

An example of task from phase 3 is: *Determine which of the following statements are true and which are false, providing justification for your answer. If the sum of two numbers is odd, then their product is also odd. The sum of a natural number and its square is always an even number.*

Phase 4 – Exploration of numerical situations, formulation of conjectures and related proofs

The main goal of the fourth phase is to enable students to use the skills developed in the previous phases to tackle simple proofs. The tasks require them to observe patterns, formulate conjectures based on numerical explorations, and attempt to prove the conjectures they generate. In this way, the various aspects explored in earlier phases, which characterize the use of algebraic language as a support for constructing arguments, become essential for developing an algebraic approach to proofs. The tasks provided offer students an opportunity to gain a deeper awareness of the use of algebraic language: by supporting the interpretation and highlighting of properties, it proves to be a powerful tool for constructing arguments that would be difficult, or sometimes impossible, to achieve using only natural language. These tasks also aim to enhance students' exploratory capabilities by demonstrating the importance of numerical examples as a means of investigation and to make them aware of the inadequacy of relying solely on numerical verification for proving statements.

An example of task from phase 4 is: *Consider a natural number. Determine the difference between its square and the square of the preceding number. Which regularities do you observe? Could you prove what you state?*

Phase 5 – Analysis of proving strategies

The key objectives of the fifth phase of the didactic path are to help students develop a critical understanding of the validity of deductive reasoning and, more broadly, the mathematical thinking that underpins proofs in the field of arithmetic, with particular emphasis on the role played by different types of language that can be used in these proofs. Students are asked to analyse four different fictitious proofs or attempts at proving the same statement in order to evaluate their correctness and completeness. The analysis of these proofs allows students to reflect on the importance of correctly translating the hypothesis, the necessity of constructing coherent deductive processes regardless of the language in which they are conducted, the fact that referencing numerical examples can support but not prove a statement, and the distinctions between argumentative and demonstrative activities.

An example of task from phase 5 is formulated as: *Consider the following statement: [a statement is given]. Analyse each of the following proofs of the statement proposed by four students, clarifying whether they are actual proofs of the statement and justifying your conclusions. Then, compare the four proofs, attempting to highlight any connections that may exist between some of them. [four proofs are given]* A complete example is provided in the following subsection.

Phase 6 – Construction of proofs of given theorems

The sixth and final phase, dedicated to proving theorems assigned to the students, aims to enable them to apply the skills progressively developed during the previous phases and to consolidate their understanding of the role algebraic language plays as a tool for thinking. During this phase, students are presented with statements that are more complex than those encountered in the fourth phase and are asked to prove them.

An example of task from phase 6 is: *To square a number ending in 5, simply multiply the number without the digit 5 by its successor, and then append the digits 2 and 5 to the result.*
Example 1: To calculate the square of 35, take 3 and multiply it by its successor: $3 \cdot 4 = 12$. *Then, append the digits 2 and 5 to this result. The number obtained, 1225, is precisely* 35^2*!*
Example 2: To calculate the square of 175, take 17 and multiply it by its successor: $17 \cdot 18 = 306$. *Then, append the digits 2 and 5 to this result. The number obtained, 30625, is exactly* 175^2*!*
Try to justify algebraically why this rule works.

The didactic path also includes two assessment sessions. Before the first phase, an individual assessment is planned to highlight the students' capability to translate and interpret algebraic expressions, perform simple transformations (substitutions) to solve problems, coordinate verbal and symbolic language, and recognise the relationships between the properties of an algebraic expression and the properties of a variable contained in it. At the end of the didactic path, an additional individual assessment is designed to evaluate the level of mastery of the skills and capabilities the path aims to promote. This assessment includes tasks similar to those featured in the core phases of the path: analysing the truthfulness of statements and justifying the provided answers, analysing proofs or pseudo-proofs, and proving assigned theorems. The text of the assessments is provided in appendix B.

Due to contextual demands by the classes involved in the experimentations, in our study the original didactic path on the use of algebraic language for the construction of proofs has been rearranged to condense the six phases in four activities as detailed in Figure 3.2. To condense the first three phases, the most representative tasks were identified in terms of the thinking processes students might engage in, to be included in activity 1. With respect to other modifications made to the original didactic path, in our study, activity 3 – which encompasses the task of phase 5 of the didactic path – has been presented to students in two parts: part 1 involving the analysis of proofs and pseudo-proofs, while part 2 focusing on their comparison.The didactic path arranged into the four activities and experimented within this research is provided in appendix B. In subsection 3.3.2, we provide the a-priori analysis of all the tasks from the didactic path experimented within this research.

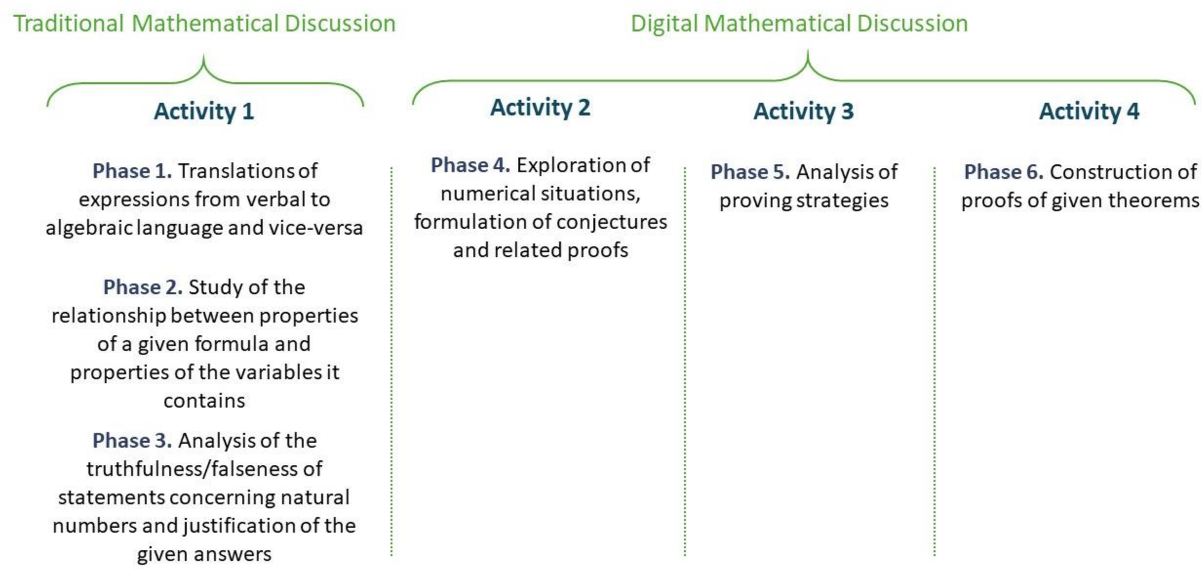


Figure 3.2 Rearrangement of the six phases of the didactic path on proof in arithmetic through the use of algebraic language into four activities

3.3.2 A-priori analysis of the tasks

In the a-priori analysis of the mathematical tasks of the didactic path experimented within this research, we report the expected solutions, possible difficulties or mistakes students can be caught and the potentialities of the tasks to activate a mathematical discussion. Moreover, in line with Stein and colleagues' (2008), the a-priori analysis of the mathematical tasks represents a powerful tool for the teacher orchestrating the discussion, since it provides a possible panorama of difficulties students can encounter and it suggests possible foci of attention for the discussion. In the following, all the tasks experimented in our research are presented and equipped with their a-priori analysis with respect to the epistemic aspects of the algebraic thinking presented in section 3.2.

Activity 1

In activity 1, we include two tasks from phase 1 of the didactic path, one task from phase 2 and one task from phase 3. The two tasks of phase 1 focus on the translation from natural to algebraic language, stimulating conversions from the natural language register to the symbolic one. The aim is to encourage students to begin reflecting on the choice of symbols (symbol sense #5) as informative of specific properties of numbers, and reinforce the formalization between information in natural language (sem 1) to their algebraic expression (form 1).

Task 1: *Translate the following statements and procedures into algebraic language: A) an even number, B) an odd number, C) add 5 to the successive of the square of a multiple of 5, D) subtract 2 from a three-digit number, E) multiply the successive of a number divisible by 7 by a two-digit number.*

In order to address requests A and B from task 1, the activation of the conceptual frame 'even-odd' is needed. Possible mistakes students can fall into are the listing of examples of even or odd numbers instead of a generalised representation of them or a verbal characterisation of them, such as "n such that n is even". Since in the statement of the task it is not specified the numerical set in which to work, during the discussion, after agreeing on $2n$ and $2n+1$ as suitable representations, it is interesting to explore whether and how these representations are valid in different number sets, or if other representations, such as $2n-1$ for odd numbers, are acceptable and in which number sets. Concerning request C, the 'multiple' and 'preceding-successive' frames should be activated and coordinated in order to correctly translate the expression. In this case, students may wrongly coordinate the different elements of the expression, resulting in expressions such as $(5n+1)^2 + +5$. During the discussion, sharing different students' proposals, such as $((5n)^2+1)+5$ or $(5n)^2+6$, may stimulate reflections on the opaqueness/transparency of representations with respect to the construction of the expression. Finally, requests D and E deal with the activation of the frame 'polynomial notation' to represent two-digit and three-digit numbers. In the case of E, it is also needed to activate and coordinate the 'preceding-successive' and 'multiple' frames. Dealing with multi-digit numbers, students might erroneously activate the 'positional notation' frame or characterise them by imposing limitations on the numbers themselves, such as "x such that $99 < x < 1000$" for three-digit numbers. During the discussion, it is possible to evaluate potentialities and limitations of different representations and to arrive at polynomial notation by starting with the need to explicitly express the role of the digits (which may already be perceived by students who have activated the 'positional notation' frame) and, then, using any proposals with constraints on the numbers to reflect on the constraints that the digits of multi-digit numbers must satisfy.

Task 2: *Translate the following sentences into algebraic language, which express the equality between two expressions. Then, verify whether the stated equality is correct. If it is not, identify the mistake. A) The difference between the square of a number and the number itself is equal to the product of the number and its preceding. B) By adding 3 to the double of the preceding of a number, you obtain the next even number of an even number.*

Concerning request A, a correct translation from natural to algebraic language is possible through the activation of the 'preceding-successive' frame and the coordination between the different elements of the expression. The equality $n^2 - n = n(n-1)$ is true and can be verified multiplying the factors in the second term or recognising the factoring out of n. For request B, the translation of the equality requires the activation of the frames 'multiple', 'even-odd' and 'preceding-successive', with the latter also applicable within the domain of even numbers. The equality $2(n-1)+3 = 2m+2$ is incorrect and it is expected that students activate the 'even-odd' frame to interpret the first term as an odd number $(2((n-1)+1)+1)$ and the second term as even $(2(m+1))$, thus concluding that they can never be equal. We expect students to test the equality through examples. Their approach in finding examples can be detailed during the whole-class discussion with respect to the role of counterexamples. We also expect that some students may mistake the initial expression using the same letter to express the numbers in the first and second term, writing $2(n-1)+3 = 2n+2$. In this case, they would obtain a particular case of the statement of task 2 that, through simple transformations, gives $1 = 2$. Such result requires students' interpretation to conclude that, even in the particular case $n = m$, the equality is not satisfied.

Task 3: *Determine for which values of b (natural) the following conditions are satisfied: A) 3b is odd. B) b^2 is a multiple of 4.*

Task 3 is taken from phase 2 of the didactic path. For both requests A and B students are supposed to study the cases in which b is even and odd, activating the 'even-odd' frame, in order to determine the required answers. In the case of request A, the 'even-odd' conceptual frame should be activated to anticipate the final form of the expression $3b$, thus guiding the syntactic transformations in order to conclude that $3b$ is odd if b is odd. In the case of request B, instead, the activation of the 'multiple' frame is required to translate the request and, then, to guide the transformations and interpret the results to conclude that b^2 is a multiple of 4 if b is even.

Task 4: *Determine which of the following statements are false and which are true, providing justification for your answers. A) The expression $n^2 + 1$ always represents an odd number. B) If the product of two numbers is odd, then their sum is even.*

Task 4 comes from phase 3 of the original didactic path. The statement in request A is false (e.g., $n = 1$, $n^2 + 1 = 2$), while the one in request B is true. In order to prove the truthfulness of the second statement, students should investigate the properties of evenness/oddness of the two considered numbers, given the hypothesis that their product is odd. Indeed, the activation of the 'even-odd' frame and its coordination with the 'divisibility' frame should guide students in concluding that if at least one between the two initial numbers is even, then its product is even as well. Thus the only case to be studied is the one in which both initial numbers are odd and can be represented as $x = 2a + 1$, $y = 2b + 1$ thanks to the correct activation of the frame 'even-odd'. Again, the 'even-odd' frame should be activated to guide students in the syntactical transformations and anticipating the final form of the expression and its interpretation ($x + y = = 2a + 1 + 2b + 1 = 2(a + b + 1)$).

Within the discussion, tasks 2, 3 and 4 aim at making students reflect on how to deal with justifications for correct statements and rejections of wrong ones, introducing the concept of counterexample. They can serve also as stimuli for students for recognising the power of symbols to validate or reject statements supported by arguments (symbol sense #1) and reinforce the algebraic transformations for guiding students from form 1 to form 2 in the sem/form diagrams.

Activity 2

Activity 2 includes tasks from phase 4 of the original didactic path. Tasks in activity 2 allow students to start addressing a complete sem/form diagram with guided inputs, such as using examples to create conjectures. After a correct formalization of the arithmetic expressions involved in the problems, with particular attention to the choice of symbols (symbol sense #5), and suitable algebraic transformations guided by anticipating thoughts, the interpretation of the final algebraic expressions obtained is related to the recognition of non-equivalent meanings in equivalent expressions (symbol sense #4).

Task 1: *What can you say about the difference between the cube of a natural number and the number itself? Give some numerical examples to support you in the formulation of the requested conjecture. Could you prove what you state?*

The regularity to be observed is that the difference between the cube of a natural number and the number itself is always a multiple of 6. Assuming that n is a natural number, the difference is $n^3 - n$, which, through a proper syntactic transformation, can be written as $n^3 - n = n(n^2 - 1) = n(n-1)(n+1)$. By activating the conceptual frame 'preceding-successive', such an expression can be recognised as the product of three consecutive natural numbers. In order to recognise that this expression represents a multiple of 6, the 'preceding-successive', 'even-odd' and 'multiple' frames should be coordinated to observe that at least one factor of this product is an even number and exactly one factor is a multiple of 3, so that the product is a multiple of 6.

Task 2: *Write down a two digit number and the number that you get when you invert the digits. Calculate the difference between the greater number and the smaller one. Repeat this procedure with other two digit numbers. What kind of regularity can you observe? Could you prove what you state?*

As presented by Cusi (2009A), the main regularity to be observed is that the difference between the two numbers is always a multiple of 9; precisely it is the product between 9 and the difference between the digits of the chosen number. In order to prove it, the initial conceptual frames to be activated and coordinated are the 'polynomial notation' and 'positional notation' frames. The required numbers can be written as $10m + n$ and $10n + m$ and, assuming $10m + n$ is the greater between the two of them, that is assuming $m > n$, the difference can be represented as $10m + n - (10n + m)$. Through simple syntactic transformations it is possible to turn the initial expression into a form that makes the required property explicit: $10m + n - (10n + m) = 9m - 9n = 9(m - n)$. The conceptual frame that should be activated in order to anticipate the final form of the expression, thus guiding the syntactic transformation, is 'multiple' as the desired final expression should be in the form 9·k, where k is a natural number. Moreover, this frame should be coordinated with the frame 'polynomial notation' to correctly interpret the expression $9(m - n)$ as the product between 9 and the difference between the two digits of the considered number.

The two proposed problems in activity 2 are interesting for conducting a balance discussion since they allow students to formulate various conjectures, albeit incomplete, and they require to explicitly interpret symbolic expressions in order to make sense of the proposed conjectures. Concerning task 1, students can just observe a partial conjecture, that is the difference is always even or it is always a multiple of 3. The lack of activation of the 'preceding-successive' frame to interpret the final expression $n(n-1)(n+1)$ as a product of three consecutive numbers may generate a block in the proving process. For task 2, different students' difficulties may arise (Cusi, 2009A), ranging from the wrong activation of the frame 'difference between polynomials' (e.g., leading to the expression $10m + n - 10n + m$ with a resulting block), to the representation of a two digit number as mn due to the predominant activation of the 'positional notation' frame, which causes blocks. Moreover, the wrong coordination between the 'polynomial notation' and the 'positional notation' frames can lead to wrong expressions such as $10m + n - (n + 10m)$. Other possible difficulties students can encounter concern the limitations to be given to the digits and their relationship in order to define the greater and the smaller number, although these obstacles can be not blocking in performing the needed syntactic transformations. An interesting situation to be discussed with students within task 2 concerns also the case in which $m = n$. Finally, we can observe that many students could end their numerical explorations after having observed that the difference between the two numbers is always a multiple of 9, without recognising the relationship that exists between the difference of the two digits of the numbers and the difference between the two numbers. Consequently, the analysis of the final expression $9(m-n)$ could stimulate further students' interpretative capabilities.

Activity 3

The task in activity 3 coincides with the task in phase 5 of the original didactic path.

Consider the following statement: "If n is a natural number not divisible by 3, then the preceding of its square is divisible by 3". Analyse each of the following proofs of the statement proposed by four students, clarifying whether they are actual proofs of the statement and justifying your conclusions. Then, compare the four proofs, attempting to highlight any connections that may exist between some of them.

<u>*Luca's proof:*</u>

If n is 2, then n^2*-1=4-1=3, that is divisible by 3.*
If n is 4, then n^2*-1=16-1=15, that is divisible by 3.*
If n is 5, then n^2*-1=25-1=24, that is divisible by 3.*
If n is 7, then n^2*-1=49-1=48, that is divisible by 3.*
Therefore, the statement is true because it always results a number divisible by 3.

<u>*Alice's proof:*</u>

The preceding of the square of n is equal to the product of its preceding and successive. If n is not divisible by 3, then one of its preceding or successive must be divisible by 3, because among any three consecutive numbers, one is always a multiple of 3, since multiples of 3 appear every three numbers. Therefore, n^2 *-1 is the product of two factors, one of which is definitely a multiple of 3, so* n^2 *-1 is also a multiple of 3. Thus, I have proven that the statement is true!*

<u>*Filippo's proof:*</u>

If n ≠ multiple of 3, then $\frac{n}{3} = x + y$ *(y is the reminder of the division of n by 3 and x is the quotient).*
Therefore $n = 3(x + y)$.
Then $n^2 - 1 = [3(x + y)]^2 - 1 = 9(x + y)^2 - 1 = 9(x^2 + y^2 + 2xy - 1)$.
Therefore, I have shown that the preceding of the square of n is a multiple of 9, and if it is a multiple of 9, it is also a multiple of 3!

<u>*Elena's proof:*</u>

n^2*-1=(n+1)(n-1)*
If n is not a multiple of 3, there are two cases: n=3x+1 or n=3x+2.
If n=3x+1 then (n+1)(n-1)=(3x+2)(3x)=3(x)(3x+2)
If n=3x+2 then (n+1)(n-1)=(3x+3)(3x+1)=3(x+1)(3x+1)
In both cases, I have proven that n^2*-1 is a multiple of 3, so I have proven the statement!*

The four proofs/pseudo-proofs include: a list of numerical examples that confirm the statement (Luca); a correct verbal argument supporting the validity of the statement (Alice); an algebraic approach to the proof containing a serious error in the formalization of the hypothesis and another error, easier to identify, in the syntactic transformations conducted (Filippo); a correct proof constructed using algebraic language (Elena).

This task is interesting also for developing a meta-discussion with students on their point of view about proofs in mathematics. Indeed, the four proofs/pseudo-proofs contain different elements related to proving, such as the presence of examples to make sense of the analysed situation, the verbal argumentations that can anticipate the development of a proof, the role of the mathematical language as synthetic and bearer of concepts, the identification of the hypothesis and their interpretation and use combined with other acquired mathematical knowledge in order to reach the thesis, and the logical connections in the reasoning. The importance of making explicit such elements related to proofs stands in allowing students to shape their idea of proofs, that generally remains implicit.

Activity 4

The task presented in activity 4 is included in phase 6 of the original didactic path.

A divisibility rule states that if the sum of the digits of a number is a multiple of 3, then the number itself is also a multiple of 3. Consider a two-digit number. Prove that if the sum of its digits is a multiple of 3, then the number is a multiple of 3.

In order to address activity 4, students are asked to apply the knowledge acquired during previous activities. The task involves dealing with two-digit numbers, so their representation from activity 1 should be recalled by activating the ‘polynomial notation’ frame, leading to $n = 10x + y$. To correctly interpret and formulate the hypothesis as $x + y = 3k$, the ‘multiple’ frame needs to be activated. The frame ‘multiple’ is also the one allowing students to anticipate the final form of the expression and, thus, guiding the syntactic transformation $10x + y = 9x + x + y = 9x + 3k =$ $= 3(3x + k)$ and the interpretation of the obtained result. Moreover, students are expected to reflect and include in the structure of their proof the elements discussed and emerged as relevant when proving within activity 4.

During the discussion, students can be asked to extend the reasoning to the case of three-digit numbers, evaluating any necessary modifications to the original reasoning. Afterwards, they can reflect on the generalizability of the proof to numbers with any number of digits. Moreover, students can be challenged in considering the statement with inverted hypothesis and thesis.

3.4 Task “exploration of tables” and its a-priori analysis

The task “exploration of tables”, addressed by students during the pilot study of this research, is described below. The task was developed and analysed within previous research experience (Swidan et al., 2023) of the teacher of the course where the pilot study was developed. The task is aligned with the objectives of the didactic path on proof in arithmetic through the use of algebraic language presented in the previous section and could be located in its fourth phase, as it aims at enabling students to explore numerical situations, formulate conjectures through the support of examples, and provide proofs for the formulated conjectures through the use of the algebraic language.

This task is particularly stimulating in a discussion context, as students can identify different regularities, which may vary in complexity and the level of justification provided. Most conjectures can be formulated by the majority of students since they relate to the structure of the table or to properties that are easily recognisable through examples, and their justifications require only the activation of the 'even-odd' frame. Thus, students can enrich their view on the table and deepen their understanding by discovering new regularities they had not previously noticed and by evaluating and refining the proofs presented by peers and guided by the teacher.

Task "exploration of tables"

Look closely at this table. Do you notice any regularities?

1	*3*	*3*
2	*4*	*8*
3	*5*	*15*
4	*6*	*24*
5	*7*	*35*

After listing all the observed regularities, try to justify them.

The regularities to be observed in the table are of two types: the principles on which the construction of the table is based (hereinafter referred to as "construction principles") and a list of conjectures that emerge from the exploration of the table. During the discussion, a first topic to address could be the distinction between unprovable construction principles and provable conjectures, which students might not be aware of.

The construction principles are:

- *The numbers in the first column are natural numbers starting from 1;*
- *The numbers in the second column are natural numbers starting from 3;*
- *The numbers in the second column are obtained by adding 2 to the numbers in the first column;*
- *The numbers in the third column are obtained by multiplying the numbers in the first and second columns.*

These principles concern the way the table has been built, while the exploration of the table can lead students to identify conjectures that can be proven. In the following, we report possible conjectures students can come up with and possible ways in which they can be proved.

- *The rows alternate between odd and even numbers;*

  For the first two columns, this is a direct consequence of the construction principles. For the third column, it can be observed that the product of two odd numbers is always odd, $(2n+1)\cdot(2m+1) = 2(2nm+m+n)+1$, and the product of two even numbers is always even, $2n \cdot 2m = 2(2nm)$, activating the frame 'even-odd' both for translating the numbers from verbal to algebraic language and for anticipating the desired final form and guiding the syntactic transformations.

The next conjectures can be proved by considering the representations of the numbers in three generic rows of the table:

a	$a+2$	$a\cdot(a+2)$
$a+1$	$a+3$	$(a+1)\cdot(a+3)$
$a+2$	$a+4$	$(a+2)\cdot(a+4)$

The generic rows of the table should be obtained by the construction principles considering symbols suitable for a generic representation of the terms of the table (symbol sense #1) and represent the formalization from sem 1, given by the numerical table, to form 1. We expect students to elaborate the following conjectures exploring the numerical table (the conjecture is a conditional statement of the if-then type, expressing a connection between sem 1 and sem 2) and to justify them through algebraic transformations on the generic table (from form 1 to form 2) in order to be able, in the end, to recognise the desired conjecture via the interpretation of form 2 into sem 2.

- *The difference between two numbers in consecutive rows in the third column is always odd;*

  The difference between two consecutive numbers in the third column can be expressed as: $(a+1)\cdot(a+3) - a\cdot(a+2) = a^2+4a+3-a^2-2a = 2\cdot(a+1)+1$.

  The activation of the 'even-odd' frame promotes the activation of a correct anticipating thought concerning the final form of the expression and guiding the syntactic transformations.

- *Considering the numbers in consecutive rows in the third column, their differences are increasing odd numbers;*

  The difference between two pairs of consecutive numbers in the third column can be expressed as:

  $$(a+1)\cdot(a+3) - a\cdot(a+2) = a^2+4a+3-a^2-2a = 2\cdot(a+1)+1$$

  $$(a+2)\cdot(a+4) - (a+1)\cdot(a+3) = a^2+6a+8-a^2-4a-3 = 2\cdot(a+2)+1$$

  From which it can be observed that they are odd numbers activating the frame 'even-odd', which is also guiding the transformations as anticipating thought, and that they are consecutive odd numbers since: $2\cdot(a+2)+1 - 2\cdot(a+1)-1 = 2$.

- *Each number in the third column is the antecedent of the square of the arithmetic mean of the numbers in the first two columns and in the same row;*

A generic element of the third column can be transformed in order to be written as the number that precedes the square of the arithmetic mean of the numbers in the first two columns: $a \cdot (a+2) = a^2 + 2a = a^2 + 2a + 1 - 1 = (a+1)^2 - 1$. This conjecture is the most sophisticated of those reported and requires a deep study of the table.

Chapter 4

Research questions, research methods and context of the experimentations

In this chapter, we present the focus of the research, the context and structure of the experimentations, and the research methods. First, the research problem is outlined and the research questions are presented (section 4.1). The research problem encompasses the interest of the research in investigating the potential of digital technologies in extending the teaching and learning of mathematics beyond the traditional classroom settings, specifically in the case of mathematical discussion. Section 4.2 focuses on the contextualization of the research, specifically outlining the context of the pilot study (subsection 4.2.1) as well as the first and second experimentations (subsection 4.2.2). In particular, for the first and second experimentations, the common context of the schools and classes involved in the study is presented together with some information about the class teachers and about the initial abilities of students in activating conceptual frames (from subsection 4.2.3 to subsection 4.2.6, one for each class involved in the experimentations). The experimental plan and the data collected are then outlined in section 4.3. The methods of data analysis with respect to the theoretical frameworks in order to address the research questions are presented in section 4.4. Specifically, the methods for the analysis are characterised for the analysis of students' social modes of co-construction (subsection 4.4.1), for the analysis of students' development of algebraic thinking processes (subsection 4.4.2), and for the analysis of students' perspective on asynchrony and written communication as key features of DMD in terms of constraints, potentialities, enablement and affordances (subsection 4.4.3).

4.1 Research problem and research questions

The research presented in this thesis aims to contribute to the ongoing exploration of the opportunities that digital technologies offer to enhance the teaching and learning of mathematics through collaboration and communication beyond the traditional classroom setting, as introduced in chapter 1. In particular, we focus on the investigation of the potential offered by digital technologies to support and enhance mathematical discussions. In order to do so, we introduced the didactic methodology of Digital Mathematical Discussion (chapter 2). As presented in section 2.2, DMD is a didactic methodology that involves the use of various digital environments to extend the possibility of conducting mathematical discussions over time and in a space outside the physical classroom, involving both students and teachers in an asynchronous modality, with communication primarily in written form. The introduction of the DMD design already partially addresses the research problem presented, offering a didactic methodology that leverages digital environments to support collaboration and communication between students and teachers beyond the traditional methods of conducting mathematical discussions. To thoroughly investigate the effects of the implementation of DMD, we explore both the students' engagement and the collaborative processes developed, as well as the development of mathematical thinking. Specifically, in this research, mathematical thinking is instantiated in the development of algebraic thinking within arithmetic proofs and is investigated relying on the epistemic aspects of algebraic thinking introduced in section 3.2. In particular, we focus on the asynchronous phases of DMD as they enable students and teachers to engage with mathematical discussions with different modalities compared to traditional ones, encompassing the extension of time out of the class for engaging with the discussion, and the communication modalities, moving beyond oral exchanges.

As our research is situated in educational settings where mathematical discussion promotes learning as a social and collaborative activity, we are interested in investigating the collaborative processes realised between students and the co-construction of collective mathematical thinking processes within the asynchronous phases of DMD. We believe collective mathematical thinking cannot be developed effectively if inadequate collaborative dynamics between participants are realised. We refer to *collective mathematical thinking* as the thinking processes developed by a group of students (either small groups or the entire classroom) as joint construction of ideas and understanding of concepts aiming at meaning making, in line with Eriksson and Sumpter (2021). This joint construction is generated by the interactions between students as they share, evaluate, and build upon each other's contributions, resulting in insights that extend beyond what any individual could achieve alone. This construction can also be stimulated by an expert, such as the teacher.

In the investigation of the collaborative processes between students, we refer to the dimension of the social modes of co-construction (subsection 1.3.1) from the framework for argumentative knowledge construction in computer-supported collaborative learning (Weinberger & Fischer, 2006), guided by the following research question:

(RQ1) What kind of students' social modes of co-construction are fostered in the digital environments (chat and Padlet) facilitating a DMD?

Answering this research question we aim to characterise the digital environments mediating DMDs in terms of collaborative processes that are developed within them. Our hypothesis related to the interaction aspects among DMD participants is that effective collaborative dynamics are linked to the activation of different types of social modes of co-construction and to the presence of social modes where a discussion is built upon the contributions of other participants. We are also interested in investigating which are the possible factors influencing the quantity and quality of collaborative processes in students' participation in the asynchronous phases of DMD.

Concerning the development of mathematical thinking processes, we are interested in investigating whether and how students develop algebraic thinking during the asynchronous phases of DMD with respect to the transformations between representation registers (Duval, 2006), the activation and coordination of conceptual frames (Arzarello et al., 2001) and to anticipating thoughts (Boero, 2001), i.e. epistemic aspects of algebraic thinking processes (section 3.2). Specifically, we aim to investigate how the epistemic aspects of algebraic thinking processes develop in relation to the social modes in order to highlight whether and how individual interventions by students contribute to co-construct a *collective algebraic thinking* (conceived as a specific case of collective mathematical thinking). This investigation is guided by the following research question:

(RQ2) Do students develop collective algebraic thinking within the asynchronous phases of DMDs? If so, how do the social modes of co-construction intertwine with the epistemic aspects of algebraic thinking in students' collective co-construction of algebraic thinking?

The investigation on research questions 1 and 2 can also provide us with insights that underpinned the redesign elements already presented in section 2.2 and with new insights that can be gathered for future redesigns.

Concerning the design of DMD, we are interested in investigating how the designers' expectations on design elements on the key features of DMD (presented in subsection 2.3.1) are perceived and experienced by students when engaging with DMD. In particular, we want to investigate students' reflections on asynchrony and written communication as key features of DMD with respect to potentialities, constraints, enablement and affordances (Trouche, 2004) and to highlight, if any, synergies and dissonances with the designers' perspective. This investigation is guided by the following research question:

(RQ3) How do students perceive asynchrony and written communication as key features of DMD in their articulation as constraints, enablement, potentialities and affordances after experiencing them?

In order to answer to the research questions, we collected and analysed data from different experimentations presented in the next sections. The answers to the research questions will be provided in chapter 5.

4.2 Context of the experimentations

The research involved different groups of teachers and students and, as presented in section 2.2, it has been articulated over three cycles of experimentations, namely an initial pilot study, the first and the second experimentations. In the following, the details of the context of the pilot study (subsection 4.2.1) and of the first and second experimentations (subsection 4.2.2) are outlined. Then, the specific background of the four classes involved in the experimentations is outlined from subsection 4.2.3 to 4.2.6. The background of the classes is outlined with respect to students' initial knowledge on symbolic representations and arithmetic explorations, assessed through the initial test of the didactic path on proof in arithmetic through the use of algebraic language, and with information on the mathematics teachers involved in the experimentations.

4.2.1 Context of the pilot study

The pilot study involved 31 university students enrolled at the first year of the master-degree course "Primary education sciences" at Sapienza University of Rome. The DMD activity was implemented within a 48 hours course aimed at making students develop reflections, in an integrated way, on specific mathematical contents, mathematical processes and on specific pedagogical aspects of mathematics teaching-learning. The teacher of the course is one of the researcher involved in the research reported in this thesis. This sample was chosen for conducting the pilot study since, during the course, students got used to a teaching methodology characterised by small group collaborative problem solving activities, followed by classroom discussions designed by the teacher starting from groups' written productions and orchestrated with the aim of fostering comparison between different approaches and reflections on the thinking processes activated to face the given problems (aligned with Cusi et al., 2017). Thus, the students and the teacher's experience with traditional mathematical discussions suggested to us that this particular context was ideal for investigating the emerging behaviours and engagement of participants in an online asynchronous discussion context. As students from the pilot study had already experienced, during the same course, some of the tasks of the didactic path on the use of algebraic language to construct proofs, a different task was used during the pilot study, i.e. the task "exploration of tables" (presented in section 3.4).

4.2.2 General context of the first and second experimentations

The first and second experimentations involved four grade 9 classes (2 classes for each experimentation). The same scientific-oriented high schools were involved in both the 2nd and 3rd cycles of the research. We refer to them as school A and school B, and to their respective classes as class A and class B, specifying the number of the DBR cycle they participated in, as each school was involved with one class per cycle (i.e., class A 2nd DBR cycle and class A 3rd DBR cycle for school A, class B 2nd DBR cycle and class B 3rd DBR cycle for school B).

The classes involved in the experimentations are part of the national project "Liceo Matematico" (namely mathematics high school) associated to the University of Milan (https://liceomatematicomilano.unimi.it/). The "Liceo Matematico" project offers Italian high schools to include additional mathematics hours beyond the regular curriculum, with at least one extra hour per week (equivalent to 33 hours per year), and the support of mathematics education experts in universities to co-design and collaborate on the activities realised within the additional hours. The project aims to develop students' collaborative skills and critical thinking, and the additional hours focus on deepening mathematical topics through a hands-on, laboratory, interdisciplinary approach, aiming to broaden the cultural education of students. The management of the additional hours is entrusted to teachers, with support from experts in mathematics education from the participating Italian universities (in our case, the University of Milan). Typically, the additional hours involve the joint presence of the class mathematics teacher and another teacher of other scientific or non-scientific subjects for co-design and assistance to the activities.

The schools involved in the experiments are public, with access that is inclusive of students from all social and economic backgrounds. The classes involved voluntarily in the experimentations and teachers and students' participation was confirmed by written consent. DMD activities within the 2nd and 3rd DBR cycles of the research were realised during the additional "Liceo Matematico" hours, guided by a researcher (the author) and with the involvement of the respective class teachers. Specifically, the researcher facilitated the discussions, maintaining continuous dialogue with the class teacher. This decision to have the researcher lead the experimentations, rather than the teachers, was made for several reasons. First, the teachers involved in the experimentations were not familiar with the didactic methodology of mathematical discussion, thus allowing the teachers to lead the discussions would have introduced another variable to the research: the teachers' capability and vision for conducting mathematical discussions. Moreover, teachers participating in the experimentations were enrolled in a professional development course based on research in mathematics education, delivered by the University of Milan as part of the "Liceo Matematico" project. The context of the experimentations allowed the teachers to benefit from the researcher's presence and her guidance during the discussions, providing them with real-world experience of the didactic methodology of mathematical discussion they have been introduced theoretically during the course. However, as the next steps in our research, we aim to provide teachers with all the necessary knowledge and experience to enable them to autonomously conduct (traditional and digital) mathematical discussions.

Concerning the students, we carried out our experimentations with grade 9 classes since the mathematical content of the proposed didactic path (presented in section 3.3) is aligned with the curriculum for grade 9 classes (first year of high school) of scientific-oriented high schools in Italy. We decided to experiment the DMD activities in the second half of the school year in order to let students and teachers know each other and, particularly, to allow students engaging in the DMD methodology after getting used the new high school system. Concerning the background of the four classes before the experimentations, all of them have few experiences with traditional in-person mathematical discussions. This is due to the fact that the teachers involved in the experimentations have few or none experience with classroom discussion, which is not part of their ordinary didactic. Concerning their mathematical knowledge, all classes have been introduced to algebra before the start of the experimentations. More detailed information about the mathematical background of the students is provided, class by class, in the following subsections.

Concerning the teachers, all the teachers involved in the experimentations attended the professional development courses aimed at "Liceo Matematico" teachers, offered by the University of Milan, for at least one year. The mathematics teachers involved in the first and second experimentations were 7, 3 from school A and 4 from school B. The three teachers from school A are: Gianmarco, Costanza, and Alice (the names are pseudonyms). Gianmarco is the teacher of class A 2nd DBR cycle, Costanza is the teacher of class A 3rd DBR cycle, while Alice is the local coordinator of the project "Liceo Matematico" and works in joint presence with the mathematics class teachers during the additional hours of the project. The four teachers from school B are: Sabrina, Lucia, Arianna, and Marta (the names are pseudonyms). Sabrina is the teacher of class B 2nd DBR cycle, Arianna is the teacher of class B 3rd DBR cycle, and Lucia and Marta participate in joint presence during the hours of "Liceo Matematico" in the two classes respectively. More detailed information about the background of the teachers is given in the following subsections.

As reported in subsection 3.3.1, due to contextual demands by the classes involved in the experimentations, in our study the original didactic path on the use of algebraic language for the construction of proofs has been rearranged to condense the six phases in four activities as detailed in Figure 3.2 in subsection 3.3.1. Even if the time at disposal to students to address the first three phases of the didactic path is reduced, the mathematical background of the classes (presented in the following subsections) allowed us to involve them in different tasks from the three phases only in one activity. Also activity 3, including the task from phase 5 of the original didactic path, has been rearranged and divided in two parts: part 1 involving the analysis of proofs and pseudo-proofs, and part 2 focusing on their comparison.

As shown in Figure 3.2 in subsection 3.3.1, activity 1 has been conducted as a traditional in-person balance mathematical discussions, while the following activities as DMDs, structured as presented in section 2.2. This choice to conduct the activities using both didactic methodologies (in-person and digital) was made for several reasons. First, as previously mentioned, the teachers participating in the experimentations had little or no experience with mathematical discussions in their teaching practice and had only been introduced to the concept theoretically during the professional development sessions offered within the context of the "Liceo Matematico" project. Consequently, also the students involved in the experimentations had not yet had the opportunity to experience mathematical discussions with their teachers. Thus, our experimentations represented a suitable situation for allowing both teachers and students to engage in mathematical discussions. Moreover, the in-person discussion led by a researcher (the author) served as an opportunity for teachers to observe and reflect on the different ways in which discussions and classroom interactions can be conducted by an expert. Secondly, as discussed in section 2.3, we believe that DMD should be developed in synergy with traditional in-person mathematical discussions. Exposing the participants to both modes of discussion allow them, despite the limited time of the experimentations, to explore different ways of engaging and interacting during discussions. Lastly, from a research standpoint, having all the discussions conducted by the same researcher in collaboration with the class teachers helps reducing the variability of teachers' guidance.

In the following subsections, we present the background of the four classes involved in the first and second experimentations. The background includes information on the class mathematics teachers involved in the experimentations and students' initial capabilities on the mathematical content of the experimented activities. Initial capabilities of students are assessed on the basis of the results of the initial test, which were analysed through the framework of epistemic aspects of algebraic thinking processes (section 3.2). In particular, we focus on the capability of students to activate conceptual frames for representing mathematical expressions in symbolic language. From the first exercise of the initial test it is possible to pinpoint students activation of the conceptual frames 'multiple' (request 1.A), 'even-odd' (request 1.C), and 'preceding-successive' (request 1.G), and the coordination between the 'multiple' and 'even-odd' frames (request 1.D) and between the 'preceding-successive' and 'even-odd' frames (request 1.F). It is important to note that in cases where the activation of frames is not performed correctly, it is not relevant to discuss coordination between frames that include those not activated correctly. The details of the analysis conducted to delineate the backgrounds of the classes are reported in appendix D.

4.2.3 Background of class A 2nd DBR cycle

Gianmarco, the mathematics teacher of the class involved in the experimentation, has been teaching for over ten years. He prefers a frontal lecture style and does not encourage much student interaction during his lessons. During the experimentation, and in general during the "Liceo Matematico" additional hours, he is in joint presence with Alice. Alice has been teaching for more than twenty years mathematics and physics in high school and has participated in many professional development courses by experts in mathematics education over the years.

Regarding students' background, the class demonstrates a wide range of capabilities in activating the conceptual frames under consideration. Indeed, all students correctly activate the 'multiple' frame, while 25% do not correctly activate the 'even-odd' frame, and 35% do not correctly activate the 'preceding-successive' frame. Only one student fails to correctly activate either of these frames. The incorrect activation of the 'even-odd' frame was testified by the construction of representation of an even number by means of mere numerical examples (such as 2), or by symbolic expressions that do not characterise an even number, such as "n", "$n+1$", "$x+2$". The incorrect activation of the 'preceding-successive' frame was testified by the students' construction of expressions such as "$x+1, x-1$", "a, b", "$n, 2n$", the use of numerical examples or blank answers (in the case of the translation of the sentence "two consecutive numbers" into algebraic language). We also identified five cases in which conceptual frames are activated correctly but are not coordinated properly afterward. This lack of coordination led to responses such as "$2n^2$" in the case of the translation of the sentence "the double of an even number" into algebraic language (testifying lack of coordination between the 'multiple' and 'even-odd' frames), or such as "$n, n+2$" or partial or blank answers in the case of the translation of the sentence "the even consecutive to an even number" into algebraic language (testifying lack of coordination between the 'even-odd' and 'preceding-successive' frames).

4.2.4 Background of class B 2nd DBR cycle

From school B, Sabrina is the teacher of the class of the 2nd DBR cycle. She has been teaching mathematics and physics for more than twenty years and she engages students with questions during her lessons. During the hours of "Liceo Matematico" project, she works in joint presence with Lucia who, within the experimentations, took part as an observer.

As for class A of the same DBR cycle, students display different levels of capabilities. In this case as well, all students correctly activate the 'multiple' frame, while the majority of them experience difficulties with the activation of the 'preceding-successive' frame (40% of students). Students who cannot correctly activate the frame provide numerical examples of consecutive numbers, consider particular situations (as consecutive numbers of even numbers), or leave the answer blank. Additionally, 33% of the students do not activate the 'even-odd' frame correctly, providing symbolic representations such as "$n \pm 1$ with n odd, $n \pm 2$ with n even", "$n{:}\,2, n \in N$", "a^n with n even", or providing numerical examples or leaving the request blank. Three students have difficulties with the activation of both conceptual frames. Two students, even if they can correctly activate both the 'even-odd' and the 'preceding-successive' frames, cannot effectively coordinate them.

4.2.5 Background of class A 3rd DBR cycle

The mathematics teacher of class A in the 3rd DBR cycle is Costanza, who works in joint presence with Alice during "Liceo Matematico" hours. She has been teaching for more than five years and in her ordinary teaching practice, Costanza tries to engage her students with questions and group-work.

Regarding the initial capabilities of the students in class A 3rd DBR cycle, all students except one correctly activate the 'multiple' frame, and all students except four correctly activate the 'preceding-successive' frame. The greatest difficulty for the students lies in activating the 'even-odd' frame, as more than half of the class (64%, precisely) do not activate this frame correctly, providing representations such as "$n \in P$" or "n even" without constructing specific symbolic representations. Furthermore, among the students who correctly activate all the frames, three are unable to coordinate them.

4.2.6 Background of class B 3rd DBR cycle

Arianna is the mathematics teacher of class B 3rd DBR cycle. She has been teaching for over five years and generally involves students with questions during her lessons. During the "Liceo Matematico" hours she works jointly with Marta, who has been teaching for more than twenty years and takes part as observer in the experimentations.

The global capabilities of this class are higher with respect to the other classes, since more than half of students can activate and coordinate correctly all the conceptual frames taken into account. More specifically, all students in class B 3rd DBR cycle correctly activate both the 'multiple' frame and the 'even-odd' frame and coordinate them properly. The most challenging frame to activate is the 'preceding-successive', with 33% of the students experiencing difficulties. Actually, even in the cases of students who wrongly activate the frame 'preceding-successive', the need to consider a number and its successive adding 1 has been considered. This influences also the phenomenon observed in this class that among the students who cannot correctly activate the 'preceding-successive' frame, most provide a correct representation for the request to translate "the even consecutive of an even number" into algebraic language, which would require coordination between the 'even-odd' and 'preceding-successive' frames.

4.3 Experimental plan and data collection

The research was structured into three Design-Based Research (DBR) cycles, as presented in section 2.2: the pilot study (1st DBR cycle), the first experimentation (2nd DBR cycle), and the second experimentation (3rd DBR cycle). This section provides a detailed overview of all the experiments conducted, encompassing the timeline, the environment (chat, Padlet, classroom) in which the activities have been carried out, and data collection. The timeline includes both the date and the day of the week to highlight the structure of the DMD phases in line with the design presented in section 2.2, such as the preference for group-work to unfold over weekends. Regarding the activities, for the group-works, the composition of the groups was arranged by the teachers to ensure that each group included students with a diverse range of interests and abilities. Groups remained the same along the whole experimentation. The choice of the specific instant messaging platform to unfold the chat phase of DMD was left to the teachers, as long as the platform allowed participants to write text messages, send and take pictures, and share documents.

Concerning the data collection, different types of data have been collected, including the transcripts from the chats and the Padlets, students' solutions created during the group-work, students' answers to two assessment tests and to a final questionnaire, and the audio-video recording of all the in-person meetings. The peculiarity of data collection in experimentations on DMD lies in the format of the data collected during the asynchronous phases. Indeed, the problem-solving process carried out by the groups and the collective discussion on Padlet are generated in the same format as the data itself, namely in written form. This limits the interpretive processes of the researcher, who transcribes audio recordings of discussions that occurred in-person and must represent the oral interactions among participants in written form. The transcripts of the chats and Padlets collected encompass both the written text and the pictures, documents, and other shared materials. The questionnaires have been administered through survey tools that are GDPR compliant and the methods for collecting and processing the data have been approved by the ethical committee from the University of Milan (reference number 60/23).

Regarding the pilot study, which was conducted as part of a course for university students (as described in subsection 4.2.1), Table 4.1 outlines the details of the experimentation. Students participated in only one activity with the DMD methodology, related to the task "exploration of tables" (section 3.4), under the guidance of one of the researchers involved in this study. Chats were facilitated through the instant messaging platform Telegram, and the groups' solutions were collected via the Moodle page of the course. The 31 students participating in the pilot study were divided into 5 groups, each consisting of 5 to 7 students, for the work conducted in the chats.

Timeline	**Activity – environment**	**Data collected**
Wed. 7/12/22 – Sun. 11/12/22 (national holidays in between)	Group-work – chat	Transcript of the chats Groups' solutions (digital)
Mon. 12/12/22 – Thu. 15/12/22	Collective discussion – Padlet	Transcript of the Padlet
Fri. 16/12/22	Wrap-up discussion – classroom (1h)	Audio-video recording
After	Questionnaire – online	Individual answers (digital)

Table 4.1 Experimental plan and data collected in the pilot study

As reported in the last row of Table 4.1, after their experience with DMD, students were asked to complete an online questionnaire regarding both their group-work on Telegram and the collective discussion on Padlet. The questions related to the small group collaborative activity within the Telegram chats aimed to investigate various aspects, such as students' difficulties in adhering to the provided guidelines, their opinions about the volume of messages exchanged in the chat, their feelings of being listened to by their peers during the activity, the usefulness of having access to their mates' written messages, the benefits of working asynchronously, and their reflections on comparing group-work in-person with that conducted in the Telegram chats. The questions asked in relation to the second phase of the DMD activity developed within the Padlet were focused on aspects such as students' opinions regarding the quantity and quality of messages written within the Padlet, the difficulties they encountered in participating in the second phase of DMD compared to the first, the usefulness of asynchronous work, and their reflections on comparing classroom discussions held in-person versus those conducted on Padlet. The complete questionnaire is available in appendix C.

Concerning the experimentations of the 2nd and 3rd DBR cycles, students were involved in the activities of the didactic path on the use of algebraic language to construct proofs as arranged in Figure 3.2 in subsection 3.3.1. In all the experimentations, activity 1 of the didactic path was conducted in-person in the classroom over two hours as a traditional balanced discussion (one hour for group-work and one hour for collective discussion), while the other activities were carried out as DMDs. Moreover, students were asked to complete two individual assessment sessions (presented in section 3.3) in the classroom under the supervision of the teachers: one before the beginning of the activities (initial test) and one at the end (final test). At the end of the DMD experimentations, all students were asked to respond to an online questionnaire (available in appendix C). The questions addressed their views of algebra and the possible influence of the activities conducted during the didactic path on these views, as well as their reflections on the DMD methodology. The questions related to their views of algebra were taken from a questionnaire already associated with the didactic path (Cusi, 2009B), while the questions about DMD are similar to those from the pilot study and aimed to investigate students' perspectives on the challenges they encountered during the asynchronous phases of the DMD, the usefulness of having access to their peers' written messages, the benefits and weaknesses of working asynchronously, and their reflections on the differences between working in-person and in the different digital environments involved in the DMD. In the following, details regarding each experimentation are provided.

During the experimentation of class A 2nd DBR cycle, chats were facilitated through the instant messaging platform WhatsApp, and the groups' solutions were collected via the Google Classroom page of the mathematics lessons. For the group-work, the 29 students of the class were divided into 6 groups, each consisting of 4 to 5 students. Table 4.2 reports the complete experimental plan and data collection of class A 2nd DBR cycle.

Concerning the experimentation in class B 2nd DBR cycle, chats were facilitated through the instant messaging platform Telegram, and the groups' solutions were collected via the Google Classroom page of the mathematics lessons. For the group-work, the 18 students of the class were divided into 4 groups, each consisting of 4 to 5 students. In Table 4.3 the complete experimental plan and data collection of class B 2nd DBR cycle is presented. In the table, it can be observed that the time allocated for activity 3 is disproportionate between part 1 and part 2 due to a request from the students to have more time for part 1, as it fell on holiday days.

For the second experimentation, class A 3rd DBR cycle's chats were facilitated through the platform Google chat, and the groups' solutions were collected via the Google Classroom page of the mathematics lessons. The class was composed by 23 students, who were divided into 5 groups of 4-5 participants for the group-work. Table 4.4 shows the complete experimental plan and data collection for class A 3rd DBR cycle.

Regarding the experimentation in class B 3rd DBR cycle, chats were facilitated through the platform Telegram, and the groups' solutions were collected via the Google Classroom page of the mathematics lessons. The 25 students of the class taking part in the experimentation were divided into 6 groups of 4-5 participants for the group-work. Table 4.5 reports the complete experimental plan and data collection for class B 3rd DBR cycle.

Timeline	Activity – environment	Data collected
Before	Initial test – classroom	Individual answers (paper)
Tue. 7/2/23	Activity 1 – group-work and collective discussion – classroom (2h)	Audio-video recording Groups' solutions (paper)
Fri. 10/2/23 – Mon. 13/2/23	Activity 2 – group-work – chat	Transcript of the chats Groups' solutions (digital)
Tue. 14/2/23 – Thu. 16/2/23	Activity 2 – collective discussion – Padlet	Transcript of the Padlet
Fri. 17/2/23	Activity 2 – wrap-up discussion – classroom (1h)	Audio-video recording
Mon. 20/2/23 – Tue. 28/2/23 (local holidays in between)	Activity 3 part 1 – collective discussion – Padlet	Transcript of the Padlet
Tue. 28/2/23 – Thu. 2/3/23	Activity 3 part 2 – collective discussion – Padlet	Transcript of the Padlet
Fri. 3/3/23	Activity 3 – wrap-up discussion – classroom (1h)	Audio-video recording
Fri. 3/3/23 – Mon. 6/3/23	Activity 4 – group-work – chat	Transcript of the chats Groups' solutions (digital)
Tue. 7/3/23 – Thu. 9/3/23	Activity 4 – collective discussion – Padlet	Transcript of the Padlet
Fri. 10/3/23	Activity 4 – wrap-up discussion – classroom (1h)	Audio-video recording
After	Final test – classroom	Individual answers (paper)
After	Questionnaire – online	Individual answers (digital)

Table 4.2 Experimental plan and data collected in class A 2^{nd} DBR cycle

Timeline	Activity – environment	Data collected
Before	Initial test – classroom	Individual answers (paper)
Tue. 14/3/23	Activity 1 – group-work and collective discussion – classroom (2h)	Audio-video recording Groups' solutions (paper)
Fri. 17/3/23 – Mon. 20/3/23	Activity 2 – group-work – chat	Transcript of the chats Groups' solutions (digital)
Tue. 21/3/23 – Mon. 27/3/23	Activity 2 – collective discussion – Padlet	Transcript of the Padlet
Tue. 28/3/23	Activity 2 – wrap-up discussion – classroom (2h)	Audio-video recording
Wed. 30/3/23 – Tue. 11/4/23 (national holidays in between)	Activity 3 part 1 – collective discussion – Padlet	Transcript of the Padlet
Tue. 11/4/23 – Wed. 12/4/23	Activity 3 part 2 – collective discussion – Padlet	Transcript of the Padlet
Wed. 12/4/23	Activity 3 – wrap-up discussion – classroom (1h)	Audio-video recording
Fri. 14/4/23 – Mon. 17/4/23	Activity 4 – group-work – chat	Transcript of the chats Groups' solutions (digital)
Wed. 19/4/23 – Fri. 21/4/23	Activity 4 – collective discussion – Padlet	Transcript of the Padlet
Wed. 26/4/23	Activity 4 – wrap-up discussion – classroom (2h)	Audio-video recording
After	Final test – classroom	Individual answers (paper)
After	Questionnaire – online	Individual answers (digital)

Table 4.3 Experimental plan and data collected in class B 2^{nd} DBR cycle

Timeline	**Activity – environment**	**Data collected**
Before	Initial test – classroom	Individual answers (paper)
Tue. 6/2/24	Activity 1 – group-work and collective discussion – classroom (2h)	Audio-video recording Groups' solutions (paper)
Fri. 9/2/24 – Tue. 13/2/24	Activity 2 – group-work – chat	Transcript of the chats Groups' solutions (digital)
Wed. 14/2/24 – Mon. 19/2/24	Activity 2 – collective discussion – Padlet	Transcript of the Padlet
Tue. 20/2/24	Activity 2 – wrap-up discussion – classroom (1h)	Audio-video recording
Wed. 21/2/24 – Sat. 24/2/24	Activity 3 part 1 – group-work – chat	Transcript of the chats
Sun. 25/2/24 – Wed. 28/2/24	Activity 3 part 2 – group-work – chat	Transcript of the chats
Thu. 29/2/24	Activity 3 – wrap-up discussion – classroom (2h)	Audio-video recording
Fri. 1/3/24 – Sun. 3/3/24	Activity 4 – group-work – chat	Transcript of the chats Groups' solutions (digital)
Mon. 4/3/24 – Wed. 6/3/24	Activity 4 – collective discussion – Padlet	Transcript of the Padlet
Thu. 7/3/24	Activity 4 – wrap-up discussion – classroom (1h)	Audio-video recording
After	Final test – classroom	Individual answers (paper)
After	Questionnaire – online	Individual answers (digital)

Table 4.4 Experimental plan and data collected in class A 3rd DBR cycle

Timeline	Activity – environment	Data collected
Before	Initial test – classroom	Individual answers (paper)
Fri. 8/3/24	Activity 1 – group-work and collective discussion – classroom (2h)	Audio-video recording Groups' solutions (paper)
Fri. 8/3/24 – Mon. 11/3/24	Activity 2 – group-work – chat	Transcript of the chats Groups' solutions (digital)
Tue. 12/3/24 – Thu. 14/3/24	Activity 2 – collective discussion – Padlet	Transcript of the Padlet
Fri. 15/3/24	Activity 2 – wrap-up discussion – classroom (1h)	Audio-video recording
Fri. 15/3/24 – Mon. 18/3/24	Activity 3 part 1 – group-work– chat	Transcript of the chats
Tue. 19/3/24 – Thu. 21/3/24	Activity 3 part 2 – collective discussion – Padlet	Transcript of the Padlet
Fri. 22/3/24	Activity 3 – wrap-up discussion – classroom (1h)	Audio-video recording
Fri. 22/3/24 – Tue. 26/3/24	Activity 4 – group-work – chat	Transcript of the chats Groups' solutions (digital)
Tue. 27/3/24 – Thu. 4/4/24 (national holidays in between)	Activity 4 – collective discussion – Padlet	Transcript of the Padlet
Fri. 5/4/24	Activity 4 – wrap-up discussion – classroom (1h)	Audio-video recording
After	Final test – classroom	Individual answers (paper)
After	Questionnaire – online	Individual answers (digital)

Table 4.5 Experimental plan and data collected in class B 3rd DBR cycle

4.4 Methods for the analysis

In this section, we present how the data analysis is conducted to address the research questions (section 4.1). The methods for the analysis are presented with respect to the three research questions: collaborative processes developed between students, analysed with respect to the social modes of co-construction (subsection 4.4.1); students' development of the algebraic thinking, analysed with respect to the epistemic aspects of algebraic thinking processes (subsection 4.4.2); students' perspectives of asynchrony and written communication as key features of DMD, analysed with respect to potentialities, enablement, constraints and affordances (subsection 4.4.3).

Among the collected data, in this research we analyse the transcripts of the chats and Padlets and students' answers to the final questionnaire. In fact, within the work presented in this thesis, we focus only on the asynchronous phases of DMD as they provide different modalities by which students and teachers can engage in mathematical discussions compared to traditional ones, including extended time outside the classroom written form of communication, in line with the research problem we are addressing (sections 1.4 and 4.1). A detailed report on how the analysis of this data contribute to answer to the research questions is presented in the following subsections. All the analyses are conducted in different phases. First, a researcher of the research group (the author) codes all the units of analysis with respect to the relevant theoretical frameworks. If a unit of analysis could be analysed with respect to multiple aspects of the theoretical frameworks used for the analysis, all of them are recorded in the coding. Then, the coding principles are shared with the other researchers of the research group for the triangulation of coding, which leads to the verification and confirmation of the coding. Non-converging elements of the analysis are discussed further in the research group so as to reach an agreement. For the data analysis of the transcripts of the chats and Padlets, the units of analysis are the individual contributions (messages, posts and comments) sent and published by students. When students' contributions involve fragmenting a single line of reasoning into multiple consecutive interventions, the consecutive interventions are considered as one single unit of analysis. Moreover, the qualitative analysis of the transcripts of the chats is also conducted by examining the groups' solutions and comparing them with individual contributions within the chat, in order to identify implicit choices in the development of the final group's solution. For the answers to the final questionnaire, the units of analysis are the answers given by students.

4.4.1 Methods for the analysis of students' social modes of co-construction

The analysis related to the social modes of co-construction aim to answer to the first (RQ1) and to part of the second (RQ2) research questions and involve the transcripts of chats and Padlets of both the pilot study and the first and second experimentations. Specifically, the analysis of the data from the pilot study contributes to answer to the first research question since we focus on how students develop different social modes of co-construction within the different digital environments supporting DMD. The results from the analysis on students' collaborative dynamics in the pilot study are then compared and confirmed with analogous analysis on the data from the first and second experimentations. The analysis of data from the first and second experimentations also contributes to RQ2, as the analysis with respect to the dimension of social modes of co-construction is intertwined with the analysis focused on the epistemic aspects of algebraic thinking in order to highlight how the development of collective algebraic thinking occurs in DMDs.

Based on Weinberger and Fischer's framework (subsection 1.3.1), we focus specifically on the dimension of social modes of co-construction. For the epistemic dimension, our research employs specific frameworks related to the epistemic aspects of algebraic thinking, thus we put aside the general dimension offered by Weinberger and Fischer in favour of a specific one to the mathematical content addressed in our experimentations. We do not take into consideration the argumentative dimension in the current research, though it is closely related and could be examined in future studies. As regards the participation dimension, we refer to it for analysing data with the aim of gaining insights into the initial design of the DMD in terms of feasibility and participants' engagement with its key features. In particular, we measure the quantity of students' participation by counting the number of interventions each student contributes within the chats and the Padlet of the pilot study, although, after analysing the pilot study, we have realised we lack sufficient data to derive meaningful and complete results on participation in the DMD. In fact, counting each student's interventions offers only a partial perspective on participation, as it does not account for students who engage with the mathematical discourse by reading rather than writing. Consequently, for subsequent experiments, we opt not to quantify participation by counting students' interventions and, for future studies, we plan to employ digital environments that provide more detailed information on students' platform activity, such as access times and time spent on each post/message, to more comprehensively address participation by including various forms of engagement beyond writing.

The qualitative analysis of students' interventions within the chats and Padlets is conducted by coding them according to the categories of social modes of co-construction. The coding process of students' interventions in the chats and Padlets with respect to the dimension of the social modes of co-construction is conducted as described in Table 4.6.

Code	**Description for coding**
Externalization (EX)	A new proposal is presented, either as an initial contribution or because it is not related to previous interventions. The author usually presents it as new knowledge for others (e.g., I tried this, I found that), and the intervention is not connected to the flow of the discussion.
Elicitation (EL)	There is an explicit reference to other participants, requesting their intervention either through a question (e.g., Why did you do this? How did you obtain that?) or a statement (e.g., If you have other ideas, let me know).
Quick consensus building (QC)	Nothing new is added compared to previous interventions, but it shows the author's presence in the discussion. These are usually very brief (e.g., Ok, I agree).
Integration-oriented consensus building (IC)	Previous statements are recalled or addressed, and the author agrees with what has been expressed. The author usually adds something new to co-construct a group discourse based on others' interventions. References to previous interventions can be explicit (e.g., I obtained your same solution *and*…) or implicit, as they fit into the flow of the discourse (e.g., responding to another participant's question or using the same words as another participant to say something).

Code	Description for coding
Conflict-oriented consensus building (CC)	Previous statements are recalled or addressed, and the author disagrees with what has been expressed. The author usually adds something new to modify others' interventions. References to others' interventions can be explicit (e.g., I read your reasoning *but*…) or implicit, as they fit into the flow of the discourse.

Table 4.6 Coding principles used to conduct the data analysis with respect to the categories of the social modes of co-construction

The analysis of students' collaborative processes also provides information on possible redesign elements since our assumption is that mathematical discussions can occur effectively only if students are participating meaningfully. Therefore, the interpretation of the results obtained by the analysis of students' participation and social modes of co-construction in the DMDs represents the basis for the redesign elements implemented within the DBR cycles of this research and presented in section 2.2. Moreover, we want to compare and support our interpretations with students' perspectives on their experiences with DMD. In order to do so, the answers given by students to the final questionnaire are analysed for providing further evidence to support our interpretation of the results arising from the analysis through the dimensions of participation and of social modes of co-construction. The analysis of students' answers to the questionnaire is conducted through a thematic analysis (Boyatzis, 1998), reading all students' answers and grouping them according to common themes.

4.4.2 Methods for the analysis of students' development of algebraic thinking

In order to address the second research question (RQ2), we focus on the development of students' mathematical thinking during the asynchronous phases of DMD. Regarding the development of algebraic thinking, we consider that, even in a digital and asynchronous discussion environment, high-quality outputs in arithmetic proofs through the use of algebraic language require the integration of the same components considered for in-person activities and presented in section 3.2, that is: (1) effective application of conversions and treatments, along with good flexibility in coordinating between verbal and algebraic registers (Duval, 2006), for both translation and interpretation; (2) appropriate activation of conceptual frames and their coordination (Arzarello et al., 2001); and (3) correct activation of anticipating thoughts (Boero, 2001). These frameworks have already been used in the context of discussions on proofs in arithmetic using algebraic language (Cusi, 2009A) and for investigating students' views on formulas while working in group with peers (Schou & Bikner-Ahsbahs, 2022). In both cases it has been examined how the use of algebraic language within communication with peers and experts relates to the development of thinking processes. In fact, the interaction with others within mathematical discussions requires the explicit articulation of these thinking processes. The development of algebraic thinking as a collective process has also been explored by Eriksson and Sumpter (2021), who analyse algebraic thinking in collective argumentative processes carried out by the class under the teacher's guidance. Moreover, these theoretical frameworks have already been applied to analyse transcripts of traditional discussions, examining how algebraic language contributes to the development of thinking processes (Cusi, 2009A). In these cases, although the data originated from spoken discourse, it was ultimately analysed in written form from transcripts. In our case, it is possible to apply the theoretical frameworks directly to the students' written productions, collected from the chat and Padlet transcripts, as well as from the collective solutions produced by the groups.

We investigate the epistemic aspects of algebraic thinking only in students' productions from the first and second experimentations. In fact, the pilot study is aimed at gaining initial insights on the engagement of students with the didactic methodology of DMD, thus we do not focus on the development of mathematical thinking in that case. The transcripts of students' interventions within the chats and Padlets of the first and second experimentations are analysed with respect to the epistemic aspects of algebraic thinking processes and coded as detailed in Table 4.7.

Code	**Description for coding**
Activating conceptual frames	A symbolic expression is proposed, an interpretation of a symbolic expression is given, or reasoning that requires referencing a conceptual frame is formulated. The reference to the frame can be explicit (e.g., The expression 8n does not justify that the number is even because there should be a 2, activating the 'even number' frame) or implicit (e.g., writing 10x+y, activating the 'two-digit number' frame with reference to polynomial notation). The activation can be correct or incorrect.
Coordinating conceptual frames	A symbolic expression is proposed, an interpretation of a symbolic expression is given, or reasoning involves activating multiple conceptual frames that need to be coordinated to formulate the symbolic expression, interpretation, or reasoning. As with activation, the coordination of frames can be explicit or implicit and can be correct or incorrect.
Anticipating thoughts	Syntactic transformations of symbolic expressions or reasoning guided by a goal are carried out. Usually, the activated conceptual frames can also play the role of anticipating thoughts when they serve as a guide (e.g., $8n^3-2n = 2(4n^3-n)$, a transformation guided by the anticipating thought 'even number' to obtain an expression in the form 2k). Like conceptual frames, the activation of anticipating thoughts can be explicit (e.g., I wrote the expression in this form because I wanted to recognise an even number) or implicit (e.g., performing the syntactic transformation), and they can either correctly guide students or cause them to lose control over the syntactic transformations or reasoning they are doing.

Table 4.7 Coding principles used to conduct the data analysis with respect to the epistemic aspects of algebraic thinking processes

After the initial analysis, which leads to the coding of the whole dataset of the first and second experimentations with respect to the epistemic aspects of algebraic thinking processes and the social modes of co-construction, a second review of the data and their coding is done to identify connections between the analysis using the two lenses in order to answer the second research question (RQ2).

4.4.3 Methods for the analysis of students' perspectives on key features of DMD

In order to address the third research question (RQ3) and to collect students' perspectives on asynchrony and written communication as key features of DMD (articulated as presented in subsection 2.3.1 following Trouche, 2004), we consider the analysis of the answers to the final questionnaire given by students involved the first and second experimentations. Specifically, to gather students' opinions on the asynchronous modality of DMD, we analyse their responses to the question "Do you think having the opportunity to take your time to read and send messages (both on chat and on Padlet) was helpful? Why?". To highlight their reflections on the use of written communication, we examine the answers to the question "Was having the written contributions of the group members available helpful to you? Why?". Although the latter question focuses specifically on the availability of peers' written contributions during group-work, students' answers were general enough to allow us to analyse their broader opinions about the written communication modality in DMD. For the analysis, we consider only the 63 responses containing answers to both questions.

The analysis of students' answers to the questionnaire is conducted through a thematic analysis approach (Boyatzis, 1998). The analysis is conducted reading all students' answers and grouping them according to common themes emerging from the content of their answers with respect to asynchrony and written communication in DMD. The emerging themes are articulated as constraints, enablement, potentialities and affordances (Trouche, 2004) recognised by students. In particular, a theme concerns constraints if it highlights students' perceived difficulties or limitations, enablement if it refers to what actually allows them to participate in the DMD, potentialities if it concerns new possibilities DMD offers to students, and affordances if it relates to what students experience in the DMD in relation to their modes of involvement.

After characterising asynchrony and written communication as key features of DMD in terms of potentialities, constraints, enablement and affordances, students' perspectives can be compared with the designers' one (presented in subsection 2.3.1) in order to highlight synergies and dissonances between the a priori design elements and the a posteriori students' experience.

Chapter 5

Analyses and results

Chapter 5 is devoted to the presentation of the analyses conducted and the results obtained within the research presented in this thesis. The methods for conducting the analyses are presented in section 4.4. The presentation of the analyses and results in this chapter follows the three main themes related to the research questions: collaborative processes developed among students with respect to the social modes of co-construction in the digital environments supporting DMD (section 5.1); students' development of collective algebraic thinking with respect to the epistemic aspects of algebraic thinking processes intertwined with the collaborative processes in the asynchronous phases of DMD (section 5.2); students' perspective on asynchrony and written communication as key features of DMD with respect to potentialities, enablement, constraints and affordances (section 5.3). At the end of each section, in the discussion subsections, we provide the answers to the first, second and third research questions, respectively. Throughout this chapter, the analysis of the complete database is not presented, but rather significant excerpts that exemplify the analysis conducted on the entire dataset from which the results are drawn.

5.1 Analysis and results on students' collaborative processes

In the following we present the analysis and results concerning the collaborative processes realised by students within the digital environments of DMD with respect to the dimension of the social modes of co-construction (introduced in section 1.3.1). We focus on the data collected during the pilot study, as this study was intended to conduct an initial targeted analysis of the collaborative processes developed by students in the different digital environments supporting the DMD. Indeed, we consider the collaborative processes developed by students to be essential in carrying out mathematical discussions, so we initially focused only on this aspect in order to test the initial design of DMD with respect to the variable of engagement with students already experienced with traditional mathematical discussion. Within this section, we analyse the transcripts from chats (subsection 5.1.1) and Padlet (subsection 5.1.2) of the pilot study. Using the dimension of social modes of co-construction, we also analyse the collaborative processes developed by students in the first and second experimentations. These are presented in detail in section 5.2, where their connection to the development of collective algebraic thinking is explored.

The content of subsection 5.1.1 refers to the studies reported in Cusi and Gagliani Caputo (2024) and in a chapter of a volume by invitation currently under review (Gagliani Caputo et al., under review). In the first contribution, the focus is on group-work realised in the Telegram chats in the first phase of DMD of the pilot study, in which the focus is on the collaborative dynamics activated by students as resources for their peers during the group-work in the first phase of DMD. The aim is to characterise the collaborative processes developed by students in the groups in terms of social modes of co-construction and of the quantity of their interventions. The answers provided by students to the final questionnaire concerning their experience within the chats are presented to support our interpretation of the analysis in the discussion subsection (5.1.3).

The content of subsection 5.1.2 is related to the study reported in Gagliani Caputo and colleagues (2023) and centres on the collective discussion unfolding in Padlet – second phase of DMD. The presented analysis and results concern, as in the previous case, the characterisation of the interactions between students with respect to the social modes of co-construction and on their perception of the factors affecting their participation and interactions. Moreover, since the collective discussion conducted on Padlet also includes the teacher as facilitator and guide, the analysis of students' participation is related to the roles activated by the teacher on Padlet according to the M-$_{\text{AE}}$AB roles (subsection 1.2.1). The analyses presented in this section contribute to the first research question.

5.1.1 Students' collaborative processes within the chats

In this section, we report the analysis and results of the transcripts of the chats of the groups of students of the pilot study addressing the task "exploration of tables" (presented in section 3.4). The analysis of the interventions by students in the chats during the first phase of the DMD conducted in the pilot study allows us to identify two categories of collaborative processes developed by the groups:

(1) interactions characterised by a widespread lack of shared reflections on each other's proposals, with a predominance of quick consensus building interventions and mainly aimed at producing a written answer to be sent to the teacher rather than sharing and co-constructing thinking processes;

(2) interactions characterised by a constant comparison between each other's proposals with many integration-oriented consensus building interventions proposed by most of the participants and mainly aimed at developing a really shared answer to the given task.

The analysis leading to the identification of such categories consists of the combined qualitative analysis of the content of students' interventions according to the social modes of co-construction, following the coding principles presented in subsection 4.4.1 and exemplified in Table 5.1, and to the counting of the number of interventions pertaining to each social mode conducted on all the chats of the groups involved in the pilot study. For showcasing the qualitative analysis, we present the one of the excerpt from the chat of group C of the pilot study due to the variety of categories of social modes of co-constructions that can be referred to students' interventions within the chat. The timing of students' interventions in the table is the time at which they send the message.

Excerpt from the transcript of a chat	Coding of the interventions according to social modes of co-construction
11:08 I: Good morning! Have you noticed any other regularity beyond the ones already written? So, if you agree, in the meantime we can reformulate the regularities and write them in order, so that then we can focus on the missing generalisations	EL – Student I implicitly requests some actions from the group in order to organise the following work. The focus is on organisational aspects.
11:10 E: Good morning, I agree with the regularities you identified. I haven't found any different regularity. So, we can stop here and put them all together.	QC – Student E agrees with the issues proposed by student I and reformulates her message. The focus is on organisational aspects.
11:22 C: I agree, I would start from the more "trivial" observations, that is the relationship between the first and the second [columns], second and third [columns] and first and third [columns], then I would move to the more articulated ones, such as the difference between results, etc…	IC – Student C proposes her point of view on the ways in which the observed regularities should be listed in the collective document
11:28 I: Perfect!	QC – Student I explicitly agrees
11:29 I: How do we want to organise ourselves for the file we have to submit? Is one person writing it and then sharing it here in the chat?	EL – Student I asks her mates how to organise the writing of the collective document
11:31 A: It can be done this way	QC – Student A explicitly agrees
16:23 *I shares a document with the collection of all the regularities emerged in the group*	
16:23 I: Girls, I decided to write in a unique file all the regularities we found. Just generalisations are missing. Let me know what you think about it.	EL – Student I is leading the writing of the document, but she wants to involve her mates, by explicitly asking them feedback on the file she started creating
16:25 A: Excuse me, according to me the first two regularities are not essential, but I don't know	IC+EL – Student A is pointing out her perspective on the content of the document; the final "but I don't know" can represent an implicit request for feedback
16:26 I: Indeed, on the first [regularity] I had your same doubt because I wasn't sure to write it	IC – Student I is boosting the discussion with A expressing her accordance and her point of view

Table 5.1 Qualitative analysis of an excerpt of the Telegram chat of group C

A similar analysis to the one presented in Table 5.1 is conducted for all the group chats. Subsequently, the qualitative information from these analyses is combined into tables that, for each group, show the number of interventions for each category of social modes, distinguished between interventions focused on the mathematical content of the activity or on organisational aspects, and indicating the heterogeneity of participation by reporting the number of group members who activated specific social modes. The combination of information from the qualitative analysis of the chat transcripts and the tabulation of the number of interventions by type allow us to identify the two categories of collaborative processes within the groups: category 1, lack of shared reflection on each other's proposals; category 2, sharing and comparing on each other's proposals. Tables 5.2 and 5.3 show the combined analysis of two groups (group A and group C), belonging, respectively, to category 1 and category 2. Moreover, both tables show the lack of CC interventions by students.

The interactions observed within the chat of group A (Table 5.2) are characterised by a predominance of QC and EL interventions focused on organisational aspects. Table 5.2 also highlights that the work is guided by few leaders, since EX and EL interventions are proposed only by 3 students among 7.

Types of interventions	**Number of interventions**	**Number of interventions focused on content**	**Number of interventions focused on organisational aspects**	**Number of participants who propose this type of interventions**
EX	1	1 (EX+EL)	0	1/7
EL	23	2 (1 EL+EX)	21	3/7
QC	28	4	24	7/7
IC	9	6	3	5/7

Table 5.2 Summary of the combined analysis of the interventions within the chat of group A

What is not evident from the table, but can be observed through the qualitative analysis of the chat, is that, although students do not propose reflections on what their mates write in the chat and limit themselves to agree or to add further proposals, there is an effort to take all these proposals into account to be integrated into the written document to be uploaded within the e-learning platform Moodle as collective solution to the task "exploration of tables".

The fact that the interactions observed within the Telegram chat of group C belong to category 2 is evident both from Table 5.3 and from the results of the qualitative analysis of the excerpt presented in Table 5.1. In fact, even if the work is led by only 3 of the 7 students (lines 2 and 3 in Table 5.3), all the students (except one of them, who never participated) proposed IC interventions, mainly focused on the content under discussion.

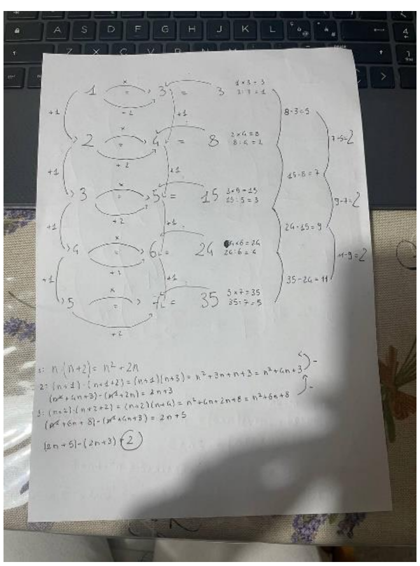
Types of interventions	Number of interventions	Number of interventions focused on content	Number of interventions focused on organisational aspects	Number of participants who propose this type of interventions
EX	8	8 (3 EX+EL)	0	3/7
EL	16	6 (3 EL+EX, 1 EL+IC)	10	3/7
QC	25	5	20	6/7
IC	20	19 (1 IC+EL)	1	6/7

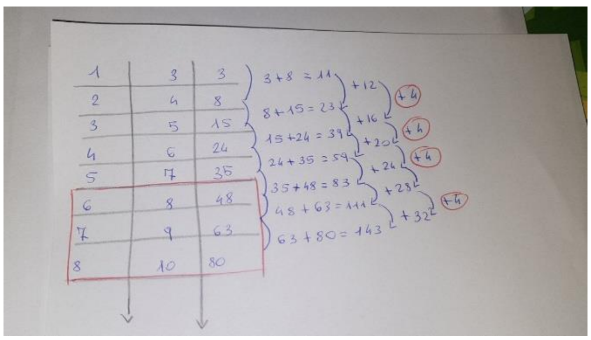
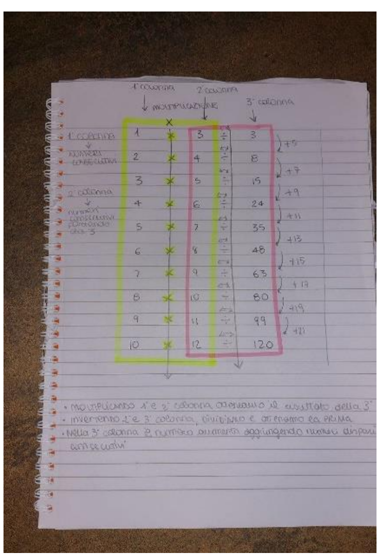
Table 5.3 Summary of the combined analysis of the interventions within the chat of group C

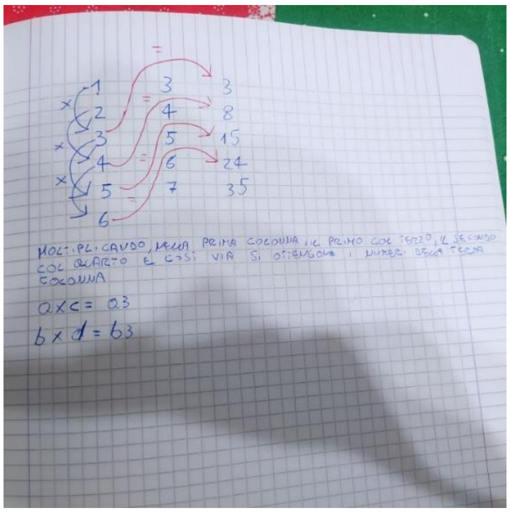
In this subsection, we also report the analysis conducted on the same transcripts already analysed with a focus on students' dynamics acting as resources for peers. The analysis reveals that in the groups, students develop different ways of supporting each other in the co-construction of a collective solution by connecting with what others have said. In the following, we report some excerpts from the groups' chats that exemplify the characterisation of the dynamics of students acting as resources for peers.

In Table 5.4, we present an excerpt from the chat of group B, where it is explicitly shown that many interventions aim to encourage all participants to contribute at their own pace and with their proposals. Additionally, students share their ideas while attempting to build upon what has already been said, creating a dynamic of integration while the group accommodates the timing and methods of its members. By *dynamic of integration* we refer to the interactive process through which participants acknowledge, analyse, and incorporate ideas, perspectives, or information shared by others into the ongoing conversation.

Excerpt from the transcript of a chat	Coding of the interventions according to social modes of co-construction
17:00 S: Good afternoon. Have you read the problem?	EL – S asks whether others have engaged with the task, inviting them to share their progress and initiating the collaborative process
23:24 C: Yes, I've read it, and I've found some regularities.	QC – C states that she has read the task and found regularities without sharing the content of her exploration of the table

Excerpt from the transcript of a chat	**Coding of the interventions according to social modes of co-construction**
07:55 E: Good morning, I’ve read it too and found some regularities. How should we organise ourselves?	EL – E confirms having found regularities and suggests organising the group’s approach, actively stimulating peers to engage in planning
08:57 C: Good morning, should we share what we’ve found and see if it’s the same?	EL – C proposes sharing findings to compare results and seeks for support or new proposals for organisation from peers
10:39 L: Good morning, I just joined this morning. Would it be an issue if I sent you my findings this evening or tomorrow morning?	EL – L informs the group about his late entry and requests flexibility to contribute findings later, demonstrating engagement
10:44 C: No worries, it’s not a problem. Here’s what I managed to find. Did you discover anything else?  *In the picture, C reports her exploration of the table in order to highlight the construction principles of the table and that the differences between consecutive numbers in the third column increase by 2*	EX+EL – C shares the regularities she observed within the table and their generalization. Then, she relaunches to her peers asking if they found something different

Excerpt from the transcript of a chat	**Coding of the interventions according to social modes of co-construction**
12:06 E: I’ll send mine later too, as I’m not home right now.	QC – E aligns with the group’s flexible timeline
12:43 S: I’m not home either, so I’ll send mine later.	QC – S aligns with the group’s flexible timeline
13:16 E: In addition to yours, I also found this.  *In the picture, E reports her exploration of the table in order to highlight that the differences between sums obtained by successively adding the numbers in the third column increase by a constant value, which is 4*	IC – E adds new findings while acknowledging and building upon previous contributions
14:18 F: I tried solving it like this. 	EX – F shares her solution, contributing new knowledge to the discourse and without referencing to previous contributions

Excerpt from the transcript of a chat	**Coding of the interventions according to social modes of co-construction**
In the picture, F repeats the construction principles of the table and observes that the differences between consecutive numbers in the third column are consecutive odd numbers	
16:00 L: Beyond what you all found, I came up with this.  *In the picture, L reports his exploration of the table in order to highlight that the product of two numbers in the first column belonging to alternate rows gives the numbers in the third column and in the row to which the first considered number belongs*	IC – L adds new findings while acknowledging and building upon previous contributions
17:29 S: This is what I discovered; it seems more or less in line with what you've also noted. Only the second part seems new to me.	IC – S adds new findings while acknowledging and relating her discovery to previous contributions, highlight the originality in her proposal

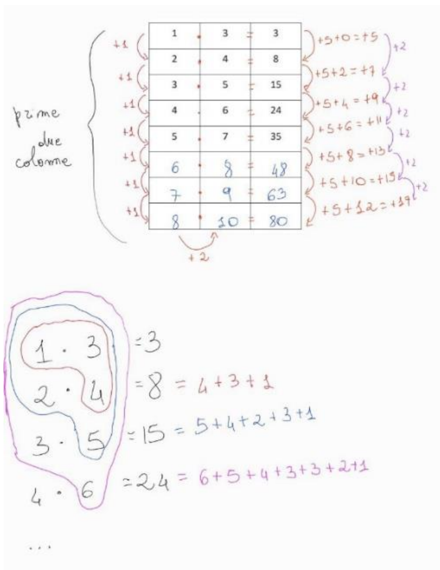
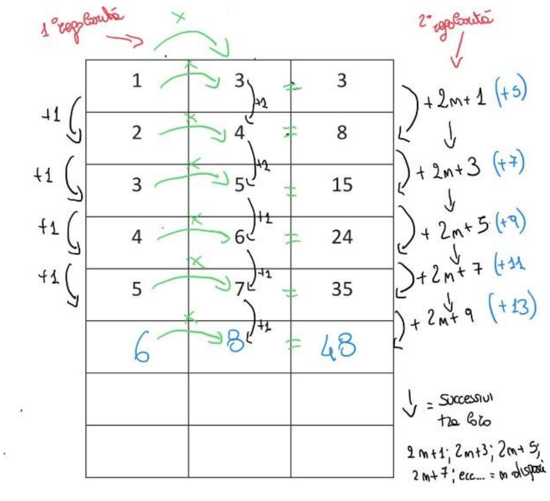
Excerpt from the transcript of a chat	Coding of the interventions according to social modes of co-construction
 *In the picture, S reports her exploration of the table first reporting the same observations made by C and, then, adding a new regularity regarding how the numbers in the third column are obtained: by summing the numbers in the first two columns for all the rows above the considered row, plus the number in the second column of the considered row*	
17:43 S: I'd suggest we wait for everyone to respond, then combine everything we've noticed. Regarding the reasoning, how do you want to proceed? Should we do what we just did—each of us tries to explain what they found, then we share everything here, compare, and combine it—or would you prefer dividing the observations, with each person explaining one?	EL – S proposes and asks to peers strategies for combining and organizing the group's findings, fostering a structured collaborative effort
17:44 C: I'm not sure; maybe we can try explaining all of them and send the ones we manage to complete.	QC – C proposes an answer to the request for organisation by S
17:45 S: Sure, that works for me too. Let's wait for the others and then decide.	QC – S agrees with C's proposal and implicitly asks the other peers to express their ideas about the organisation of their work

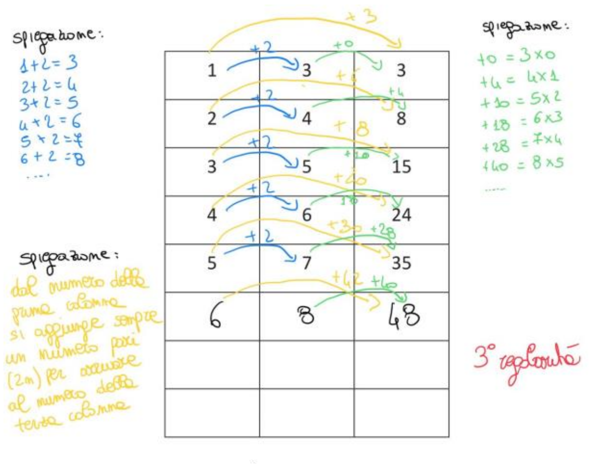
Table 5.4 Qualitative analysis of an excerpt from the chat of group B representative of dynamics of integration

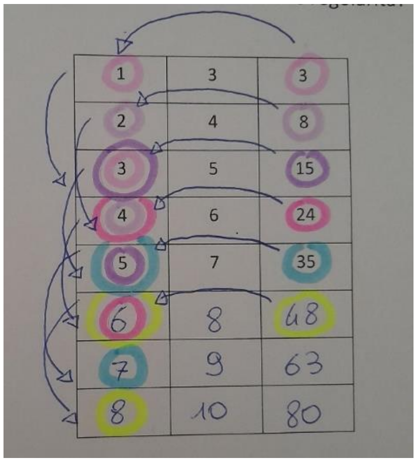
The excerpt reported in Table 5.4 exhibits multiple social modes of co-construction, reflecting varying levels of engagement and collaboration of students involved in the group problem-solving. Elicitation (EL) interventions play a pivotal role in stimulating participation and encouraging members to share their findings. This is particularly evident when students do not immediately share their personal findings at the beginning of the chat. Instead, they first ensure that others have also started working on the task and, then, they organize themselves on how to proceed with sharing their explorations and individual findings. Additionally, the group demonstrates that asynchronous communication allows more participants to share their ideas at their own pace (interventions from 10:39 to 12:43). With C's question (intervention at 10:44), the dynamic of integration begins, since the subsequent contributions from peers aim not to repeat what has already been noted in previous interventions but to propose new mathematical insights based on their observations of the table in the task. Even with the initiation of this dynamic of integration, the environment takes advantage of flexible timing to give everyone the opportunity to contribute. It is precisely the EL-type interventions made by students that encourage all participants to contribute, promoting the possibility of waiting for their interventions (interventions at 17:43 and 17:45). Thus, the group problem-solving environment promoted by students in group B aims to ensure all group members contribute actively, even if their contributions differ in timing or originality. This dynamic demonstrates how students intervening with EL contributions act as facilitators, fostering an environment where it is important to wait, listen, and consider everyone's input. Indeed, students who promote EL interventions also welcome the contributions of others with an attitude aimed at understanding their inputs. However, there is also student F, whose sole participation in the entire group-work involves sharing her observations without considering her peers' contributions.

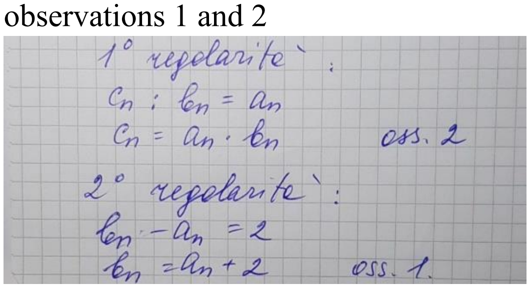
Table 5.5 includes an excerpt from the chat of group E which exemplifies different ways in which students could act as peer resources for each other by promoting dynamics aimed at connecting the discovered regularities that are shared within the chat.

Excerpt from the transcript of a chat	**Coding of the interventions according to social modes of co-construction**
23:39 A: *[We provide a summary of the first part of the content of the message. The message highlights two main regularities identified by A in the table: in each row, the product of the numbers in the first and second columns equals the number in the third column; in the third column, subtracting consecutive numbers results in odd numbers, forming a sequence of consecutive odd differences. A attempts to clarify the regularities with numerical examples. In the following we report the conclusion of A's message.]*	EX – A introduces detailed observations and regularities identified in the table, contributing new knowledge to the group

To make it clearer, I’m attaching a graphical demonstration. P.S.: For now, these are the only regularities I’ve noticed, but I’ll let you know if I find more.  *In the picture, A reports her exploration of the table in order to highlight the regularities she identified*	
00:14 A: *[We provide a summary of the first part of the content of the message. The message reports further relationships among the numbers in the table identified by A: from the first column to the second, the numbers increase by +2; from the second column to the third, the numbers follow a pattern where the value in the third column is derived by multiplying the number in the second column by a sequential multiplier starting from 0; for each row, the difference between a number in the third column and the corresponding number in the first column is an even number. A attempts to clarify the relationships with numerical examples. In the following we report the conclusion of A’s message.]* To make it clearer, I’ve attached a graphical representation (I hope it’s legible and not too messy; I used different colours to highlight the three distinct relationships I found, and I included a brief explanation on the side).	EX + EL – A expands on her findings by identifying additional relationships and inviting others to clarify or extend the explanations, positioning peers as resources for further analysis

 *In the picture, A reports her exploration of the table in order to highlight the regularities she identified*	
23:47 O: Good evening, I first read the problem and then A's observations (all excellent). For now, I don't have anything to add to the table's properties, so I tried to formally rewrite what A had already said. *[O attaches a document with the formalization of A's regularities]*	IC – O reviews A's contributions and formalises them contributing to shared co-construction of the final group solution to the task
02:33 O: My additional observations. *[O attaches a document with her exploration of the table in order to highlight regularities in the sums and differences between consecutive and alternate numbers in the third column]*	EX – O adds new observations to the discussion, expanding the group's knowledge on the task
11:40 L: Good morning, I've read the problem proposed by the professor and your observations. I confirm and agree with what has been said so far, but I'd like to add some additional observations about the activity: 1st regularity: If we divide the numbers in the third column by the numbers in the second column, we get the numbers in the first column. Example: 3÷3=1; 8÷4=2; 15÷5=3, and so on. 2nd regularity: If we subtract the numbers in the first column from the numbers in the second column, the result is always two. Example: 3-1=2; 4-2=2; 5-3=2, and so on. 3rd regularity: In the photo I'll send, we can observe that the result of dividing the numbers in the third column by the numbers in the first column appears (also) two rows later. Example: 3÷1=3; 8÷2=4; 15÷3=5, etc. To confirm this, I completed the table and continued verifying this relationship. The verification confirmed the regularity.	IC – L validates previous observations while introducing new regularities and providing evidence to support her claims. This integrates prior contributions and highlights additional perspectives.

<table>
<tr><td>

In the picture, L reports her exploration of the table in order to highlight the regularities she identified</td><td></td></tr>
<tr><td>11:43 A: Wonderful, thank you!
[In response to the first document shared by O]</td><td>QC – A acknowledges O's contribution</td></tr>
<tr><td>11:47 A: In the first two points, we could say these are opposite processes to what I observed. For instance, where you wrote 3:3=1, I had 3x1=3. … I'm not sure if I explained that clearly, but in any case, it's important to highlight this as well.
[In response to the last message by L]</td><td>IC – A reflects on the relationship between L's observations and her own, identifying complementary processes and emphasising connections between contributions</td></tr>
<tr><td>11:50 L: Exactly!</td><td>QC – L explicitly agrees with A</td></tr>
<tr><td>[...]</td><td>Discussion moves on with new EX interventions by S</td></tr>
<tr><td>00:41 O: Yes, L's first two regularities are the opposite of A's observations 1 and 2


In the picture, O showcases the relationship between L's and A's observations</td><td>IC – O recognises the inverse relationship between L's and A's observations and attaches a picture showing it explicitly foster a deeper exploration of such relationship</td></tr>
</table>

Table 5.5 Qualitative analysis of an excerpt from the chat of group E representative of dynamics of connection

The group discussion reported in Table 5.5 illustrates a dynamic interplay of externalization (EX) interventions, where participants introduced new ideas and findings, and integration-oriented consensus building (IC) interventions, as members synthesised, compared, and aligned observations. Peers acted as critical resources by connecting their observations and building upon each other's ideas. A's detailed initial externalization (interventions at 23:39 and 00:14) provides a foundation for others to refine and extend. O's synthesis and formalisation (intervention at 23:47) take up what was proposed by A and rework it in symbolic language, while L introduces fresh perspectives that complement existing findings (intervention at 11:40). The interplay of these social modes enables the group to collaboratively construct a nuanced understanding, demonstrating the value of peer engagement in connecting and expanding proposals. The collaborative focus shifts toward connecting individual contributions into a cohesive framework. For instance, A (intervention at 11:47) and O (intervention at 00:41) engage in reflective exchanges where they compare and contrast the findings of the group, comparing and contrasting their observations. Even in this case, as in the previous excerpt, the students attempt to create a dynamic of integration, where subsequent contributions do not repeat what has already been stated but instead add new insights. However, in this case, the chat begins with an extensive sharing of individual reflections that influences the subsequent dynamic of integration. In contrast to group B, not all participants are given space, but the space is taken spontaneously by the involved participants without any organizational agreements with the others. The key element that emerges from this excerpt is, however, the fact that the prevailing dynamic related to integration is one of explicitly linking close points from different proposals. This dynamic illustrates how students' interventions mainly aim at highlighting connections and at synthesising ideas. The emphasis is less on encouraging participation and more on highlighting meaningful relationships among the group's contributions. Moreover, it is interesting to observe A's contributions (time 23:39 and 00:14) as she explicitly outlines the methods for communicating with others in order to make her thoughts visible, thereby becoming a valuable resource for peers. What is notable is not only the use of these tools (graphic representations, in her case) for communication and sharing but also the explicit explanation of which tools were used and for what purpose.

The results of the analysis conducted on the transcripts of the chats from the pilot study enable us to highlight two main counterposed collaborative dynamics within the groups that interacted in the Telegram chats (i.e., categories 1 and 2). On one hand, two groups activated effective collaborative processes, characterised by many integration-oriented consensus building interventions proposed by most of the participants and aimed at developing a shared solution to the problem (category 2). On the other hand, the interactions that characterised the collaborative processes activated by three groups proved to be less effective, since we observed a widespread lack of shared students' reflections on each other's proposals and a predominance of quick consensus building interventions focused mainly on organisational aspects (category 1). Moreover, we highlighted how students can act as resources for peers in different ways. In group B, participation and engagement are prioritised, fostering a collaborative process that aims to include all participants. In group E, students focus on connecting ideas, fostering a deeper collective understanding. Moreover, students use specific tools (e.g., visual aids) in their communication with the explicit aim of better sharing their ideas with peers. Together, these examples showcase the multifaceted nature of peer collaboration, illustrating how different dynamics contribute to a rich and varied learning experience. Furthermore, the asynchronous mode proved effective in allowing flexibility of participation, and the ability to share pictures enabled students to use visual aids to more easily articulate their thought processes to their peers.

5.1.2 Students' collaborative processes within the Padlet

After completing the first phase of DMD and having collected the written answers uploaded by the groups, the teacher enacts the second phase of the DMD activity, by designing the Padlet for the collective discussion. The interventions written at the top of the six columns within the Padlet are designed by referring to the roles identified by the M-AEAB construct (as presented in subsection 1.2.1). At the top of the first, second and third column in the Padlet the teacher's interventions are in tune with the role of *activator of reflective attitudes and metacognitive acts*, since they are designed to stimulate reflections by asking to students to compare the different formulations of the conjectures, to reflect on them and to identify those that are the most complete, correct and clear from the ones selected from the groups' solutions. For example, the teacher's intervention at the top of the first column is: "All groups formulated the following conjectures. What are the formulations that are the most complete/clear/correct?". The intervention at the top of the fourth column in the Padlet refers both to the roles of *practical-strategic guide* and *activator of interpretative processes*, since the aim is to foster the sharing and comparison between strategies to generalise and to justify: "Some groups observed that the numbers within the even lines in the table are even, while the numbers within the odd lines in the table are odd. How could we justify this conjecture?". Finally, the roles to which the teacher's interventions refer in the fifth and sixth columns in the Padlet, added two days later to the beginning of the second phase of the DMD, are those of *activator of interpretative processes* and *reflective guide*, since students are asked to focus on some excerpts of the groups' answers to reflect on the role played by the use of algebraic language as a proving tool. This is, for example, one of the teacher's interventions within the sixth column in the Padlet: "Some groups introduced algebraic symbolism in their answers. Do you think that all the texts written in the following excerpts are proofs?".

As for the analysis of the chats, the transcript of the Padlet is analysed both qualitatively, through the framework of the social modes of co-construction, and by counting students' interventions. This combined analysis reveals difficulties for students in engaging within the Padlet.

The excerpt in Table 5.6, taken from the third column of the Padlet, exemplifies how the qualitative analysis of students' interactions within the Padlet is carried out by coding the interventions according to the categories of social modes of co-construction. Within the excerpt, a discussion is developed about a conjecture, submitted by one of the groups, on some relationships between consecutive numbers of the third column of the table of the task "exploration of tables" (section 3.4). At the top of the third column in the Padlet, the teacher writes the following guiding question, in tune with the role of activator of reflective attitudes and metacognitive acts: "Do you agree with the following conjectures proposed by two groups?". The two following posts in the third column, created by the teacher, consist in the screenshots of the two conjectures to be discussed. The excerpt in Table 5.6 concerns the discussion about the first conjecture posted in the Padlet, the conjecture is reported in the first line.

The first conjecture discussed in the third column of the Padlet: “Concerning the third column of the table we can notice that 3+5=8+7=15+9=24+11=35 and therefore each successive number is obtained by adding an odd number to the number you start with.”	
Excerpt from the transcript of the Padlet	**Coding of the interventions according to social modes of co-construction**
12/12/22, 23:24 O: I agree with the observation, but not with its representation. The sign “equal” is an equivalence sign: the expressions 3+5 and 8+7 are not equal.	CC – Student O agrees on the content of the observation but not on its representation. Indeed, she tries to point out the problematic nature of the representation proposed.
13/12/22, 22:15 T: I agree with what my colleague O said, it could have been written: 3+5=8; 8+7=15; 15+9=24; etc…	IC – Student T explicitly refers to O’s contribution and proposes an integration to O’s intervention by proposing a new correct representation.
15/12/22, 18:41 V: I agree with what has been observed by the group, but also with what has been specified by O and T, it would have been better to represent it in a different way.	QC – Student V explicitly agrees with O and T’s interventions and rephrases them without changing their meanings.

Table 5.6 Coding of the discussion on the first conjecture in the third column of the Padlet

We select this specific excerpt from the Padlet since it contains interventions that belong to different kinds of categories of social modes of co-construction occurring within the Padlet. Moreover, this excerpt highlights a trend observed in all the excerpts of discussion within the Padlet, that is the total absence of EL interventions.

Table 5.7 reports the analysis of the quantity of students’ interventions across the columns of the Padlet; the number of interventions is related to the roles associated with the teacher’s guiding questions that introduce the discussion within each column of the Padlet. Table 5.7 shows that only few comments have been written under each post within the Padlet and highlights the low number of students that intervened in the discussion.

	Teacher's roles in the guiding questions according to the M-$_{AE}$AB construct	**Number of students' comments**	**Students who participated in the Padlet (alphabetic order)**
Column 1 (3 posts by the teacher referring to different groups' conjectures)	Activator of reflective attitudes and metacognitive acts	4 (under the first post), 1 (under the second post), 5 (under the third post)	A, I, L, R, S, V (6/31)
Column 2	Activator of reflective attitudes and metacognitive acts	6	A, C, I, O, S, V (6/31)
Column 3 (2 posts by the teacher referring to different groups' conjectures)	Activator of reflective attitudes and metacognitive acts	3 (under the first post), 3 (under the second post)	A, I, L, O, T, V (6/31)
Column 4	Practical-strategic guide, Activator of interpretative processes	2	C, T (2/31)
Column 5	Activator of interpretative processes, Reflective guide	2	I, V (2/31)
Column 6	Activator of interpretative processes, Reflective guide	1	V (1/31)

Table 5.7 Summary of the analysis of the quantity and heterogeneity of students' interventions within the Padlet

5.1.3 Discussion

In the previous subsections, we presented the analysis and results obtained from data collected during the pilot study, specifically from the chat transcripts (subsection 5.1.1) and the Padlet transcripts (subsection 5.1.2). These results allow us to answer to the first research question (RQ1), which will be further elaborated in subsection 5.2.3 with findings from the analysis of data from the first and second experimentation. The first research question aims to investigate the types of social modes of co-construction that are fostered among students in the digital environments where DMD is conducted (chats and Padlet).

Regarding the collaborative processes developed by students within the chats, we identified two categories of collaborative processes among the groups: category 1, characterised by a lack of shared reflections or co-construction of thinking based on others' proposals, with the objective being to complete the assigned task to fulfil a teacher's request rather than to share and create thinking processes. In category 1, the prevalent social modes are EX (externalization), to introduce individual proposals, and QC (quick consensus building) to interact with other participants. Category 2, on the other hand, is characterised by interactions among participants to jointly construct a collective solution to the teacher-assigned problem, with knowledge shared by most participants and predominant IC (integration-oriented consensus building) interventions for engaging with others' ideas. Another emerging difference between the categories concerns the focus of each group's discourse: groups in category 1 (3 groups out of 5) primarily discussed organisational aspects related to producing, sharing, and submitting their work to the teacher, while groups in category 2 (2 groups out of 5) focused their discussion mainly on the mathematical content. Furthermore, focusing on students' dynamics acting as resources for peers, we can conclude that students utilise elicitation (EL) interventions to ensure that all members have the opportunity to contribute, and that asynchronous communication facilitates broader participation, allowing students to contribute at their own pace. Across the observed groups, both demonstrate a dynamic of integration, where participants build upon each other's ideas rather than repeating previous contributions. This process reflects a deep engagement with peers' inputs, aimed at collectively constructing a coherent understanding of the task. Moreover, students consciously utilise some visual aids to make their thinking processes explicit and clearer, enhancing transparency and their utility as resources for peers. Even if the students activated various dynamics to act as resources for their peers in the context of chat-facilitated group problem-solving, it is important to note that the students involved in the pilot study already had experience with group-work and classroom discussions, so they were accustomed to working in a similar manner. Nonetheless, certain distinctive features of the dynamics emerged in relation to the DMD context. Specifically, the chats highlighted the advantages of asynchronous timing, which allowed more students to participate, read others' contributions, and respond at a pace suited to their needs. This flexibility is significant, as extended time frames enable students to organize their inputs more effectively while the chat's digital support provides a transcript as a record of shared contributions. This facilitates the dynamic of integration of ideas. Additionally, the option to share images was utilised to complement written observations, examples, and justifications with visual aids, enhancing the clarity and depth of their explanations.

Regarding the collaborative dynamics on Padlet, we observed low student participation. In fact, among the 31 students who took part in the pilot study, only 9 commented on the Padlet, most of them with a significantly low number of interventions (as it is evident from the third and fourth columns of Table 5.7). Low students' participation also impacted the variety and depth of exchanges in terms of social modes of co-construction. Indeed, although various types of social modes of co-construction were noted (as shown in Table 5.6), student engagement was so limited that the discussion on Padlet cannot be considered satisfactory. This suggests that, although the roles activated by the teacher boost effective students' reflections and stimulate students' interventions that belong to different categories of social modes of co-construction, in tune with what usually happened during in-person mathematical discussions during the course, the quantity and heterogeneity of students' participation represents a critical variable. Moreover, the qualitative analysis of students' contributions reveals that no EL (elicitation) interventions are present in any of the Padlet columns. Our hypothesis is that this is due to the fact that students did not interpret the Padlet as a virtual place for comparison, open to new explorations to be developed with all the other peers. This interpretation is supported also by students' answers to the final questionnaire, as it is testified from this reflection: "Within the Padlet, the observations formulated by the groups were presented as they were and it was not possible a comparison as it happened within Telegram".

The findings reported in these subsections are further elaborated by their interpretation drawn from the pilot study students' responses to the final questionnaire.

Although the ineffective collaborative processes observed within some of the chats, students' answers to the final written questionnaire highlight a general positive perception of the work developed with their mates in the chats, due to the opportunity to use the flexible time at disposal to better reflect on each other's proposal and to the potentialities of focusing on a written form of communication as a way to make thinking more visible, as these two excerpts from the questionnaire testify:

"The fact that everything was written allowed me, first, to read all the proposed observations taking my own time and, then, to verify and interpret them calmly."

"If others express their thinking or even a thinking that does not have a clear form, when it is written, this thinking becomes more visible and it is possible to develop it and deepen the work."

We hypothesise that the ineffectiveness of the collaborative processes observed in some of the groups could be also related to the difficulties due to the lack of typical aspects of para-verbal communication during the interactions within the Telegram chats. Being not able to grasp the gaze of the group's mates when sharing their thinking within a chat prevents the participants from understanding the unwritten, making them possibly not realise the need of reformulating their thoughts to make them better understandable to others. This hypothesis has been confirmed by some students' reflections, who stress that within digital environments "there is less emotional involvement" and define the digital chats as "aseptic and detached situations", but this should be further investigated in light of the alternatives of body language for non-verbal communication in digital environments and social network settings (e.g., Al Tawil, 2019).

In spite of the difficulties related to these last aspects, most of the students stated that their participation and engagement in the DMD activity were higher than within in-person activities implemented during the course. This result is in tune with other research studies, which showed that, within text-based computer-supported collaborative learning environments, the quantity of participation is higher than in traditional classrooms (Weinberger & Fischer, 2006). The students who declared that their engagement has increased during the DMD activity identified the greater amount of time at disposal for developing reflections as a factor that influenced their way of participating, as the following reflection testifies:

"Having more time at disposal for reading and formulating observations and answers to my mates encouraged me to participate"

Another aspect highlighted by the qualitative analysis of the interactions within the Telegram chats is the complete lack of conflict-oriented consensus building (CC) interventions. This can be related to the explorative nature of the task, which makes almost all the students' statements on what they observe meaningful and useful for the development of the activity. However, we hypothesise that this result could also be due to the ways of working and interacting within the chats, which prompts most of the students to meditate on their interventions for a long time before writing them. This is testified by the following reflection written in the questionnaire:

“I noticed that, differently from what happens in class, I thought a lot more before sending my comments since I was afraid of making mistakes, so I made sure that what I was about to send was correct.”

Concerning the difficulties encountered by students within the Padlet, the analysis of students’ answers to the final questionnaire enable us to identify some factors that, in their opinion, prevented them from actively participating in the collective discussion.

Among the students who took part in the discussion within the Padlet, the majority declares that, although it was not difficult for them to work within the Padlet, the absence of notifications made the Padlet environment static and poor if compared with the environment that they experienced within the Telegram chats, where it was easier to receive feedback and to get “curious to check” what the other students wrote. Also affective factors play an important role, as two students state: “on Telegram we felt part of a group and we didn’t have fear to be wrong” whereas, when working on Padlet, “I reflected much more before sending my comment for fear of making mistakes”. Similar issues emerge within the answers provided by some students that didn’t participate in the collective discussion on Padlet but actively took part in the first phase of the DMD activity, within the Telegram chats. These students, in fact, identify the absence of notifications as a blocking factor for their participation in the activity, stressing that not receiving notifications made them forget about the discussion. Moreover, also these students stress on the role played by social and affective factors in limiting their participation, since they felt “more exposed and less comfortable in publishing observations”, due to the high number of people involved in the activity within the Padlet, and the “higher pression of feeling judged and fear of making mistakes”. Three additional blocking aspects are identified by the students who didn’t participate in the discussion on Padlet: the fact that Padlet was not a familiar platform for them; the role played by the Padlet’s graphic design and the modalities to intervene in the discussion, as these excerpts from the students’ answers to the questionnaire testify: “its graphic organisation is dispersive and it is an app I’m not used to use”, “my mates had already inserted everything I wanted to write” and “I considered useless rewriting what has already been said”.

In conclusion:

- Concerning the group-work within the chats, the collaborative processes observed reveal distinct approaches to interaction and task completion. Some groups primarily concentrated on fulfilling the assignment’s requirements, focusing more on organisational aspects and achieving consensus quickly without in-depth engagement with one another’s ideas. In contrast, other groups were more invested in building a shared understanding, collaboratively developing solutions through discussions centred on the mathematical content. Even if some groups pertain to the first category, students generally expressed positively with respect to the group-work within the chats due to the opportunity to use the flexible time at disposal to better reflect on each other’s proposal and to the potentialities of focusing on a written form of communication as a way to make thinking more visible. Moreover, students who aimed to co-construct a solution acted as resources for their peers, both fostering their engagement and integrating diverse proposals, also utilising purposefully some visual aids to make their thinking processes more transparent.

- Concerning the collective discussion within Padlet, although students' interventions belong to different categories of social modes of co-construction, the quantity and heterogeneity of their participation cannot be considered satisfactory. The low amount of students that actively participated in the discussion can be ascribed to different blocking factors that they mentioned in their answers, such as affective factors or factors related to the constraints of the digital environment where the DMD activity was implemented. The difficulties associated with low participation and engagement of students within the Padlet suggest this as a relevant variable to monitor across the first and second experimentations.

5.2 Analysis and results on the intertwining between students' collaborative processes and their development of algebraic thinking

This section is devoted to the analysis and results concerning the first and second experimentations (2nd and 3rd DBR cycles, respectively). The data analysis for the first and second experimentations is conducted and is presented together. Analysing a large dataset that includes multiple experiments and various classes allows us to derive general results that go beyond the specific contexts in which the experimentations took place. However, relevant aspects of the individual experiments are reported when necessary for the analysis or interpretation of the obtained results. The following subsections contain the analysis reported in the article by Gagliani Caputo and colleagues (2025) and concern the exploration of whether and how mathematical discussions on the use of algebra as a thinking tool can unfold in digital environments when in an asynchronous modality. The analysis aims at characterising the different collaborative processes activated by students and their thinking processes with respect to the development of algebraic thinking in the asynchronous phases of DMDs in chats (subsection 5.2.1) and in Padlet (subsection 5.2.2). The results are showcased by detailed analysis of excerpts from data on the second activity of the didactic path on arithmetic proof through the use of algebraic thinking (corresponding to the fourth phase of the original didactic path, section 3.3). We focus on the second activity of the didactic path as it is the first conducted as DMD within the experimentations (confront subsection 4.2.2) and as it comprehends relevant examples for all the presented results. The excerpts are organised in tables that exemplify the conducted qualitative analysis and exhibit the representative characteristics presented as results. In each table with excerpts, interventions within the transcript (column 1) are coded in terms of social modes of co-construction (column 2) and of epistemic aspects of algebraic thinking (column 3). Furthermore, in order to show that the results are consistent across the entire dataset and to compare how the identified categories are present and evolve throughout the experiments, we pair the detailed qualitative analysis of the second activity with the distribution of the categories in both the second and fourth activities (the last one of the didactic path in our experimentations). Since the classes completed the third activity using different modalities (entirely on Padlet, entirely in the chat, or a combination of both), a comparison in this case would not be meaningful. The presented analysis and results contribute to the second research question (RQ2), for which we provide an answer in subsection 5.2.3.

5.2.1 Intertwining students' social modes of co-construction and epistemic aspects of algebraic thinking within the chats

In this subsection, we present the results and analysis conducted on the transcripts of the chats from the first and second experimentations. The analysis of the transcripts of all the groups involved in the experimentations (6 groups class A and 4 groups class B in 2nd DBR cycle, 5 groups class A and 6 groups class B in 3rd DBR cycle) during the group-work within the asynchronous phases of DMDs lead us to the identification of three categories with respect to the collaborative processes developed within the groups and their collective development of algebraic thinking:

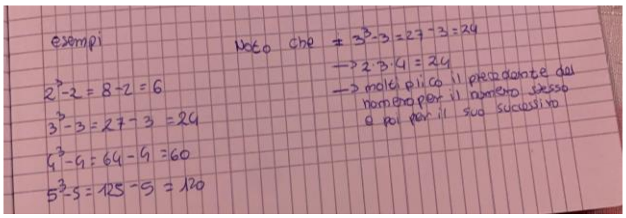
- Category A: *minimal interaction and individual algebraic thinking*. Chats are characterised by minimal interaction, with few messages exchanged (approximately 3 or less messages from each participant) and a lack of comparison around proposed ideas. The prevalent social modes of co-construction are externalization (EX) and quick consensus building (QC). Indeed, participants tend to engage in individual thinking without comparing their thoughts with others or accepting others' contributions without an explicit evaluation of them in terms of the mathematical content. This approach hinders the development of a collective solution, as the group often relies on individual proposals without building upon them collaboratively. The mathematical quality of the solutions is solely based on the participants' individual algebraic thinking.
- Category B: *leadership in articulating processes for collective development of algebraic thinking*. Within the chats there are one or more leaders in the group who articulate their thinking processes to their peers and who dominate (through the whole chat or partially) the discourse. Typically the leaders propose new solutions or ideas through EX interventions and recall them throughout the discourse with IC or CC interventions. The reasonings put forward are not necessarily already complete, and the leaders encourage others to participate and express their opinions on those reasonings through EL interventions. The other participants in the chat intervene also through EL interventions in order to ask clarifying questions related to mathematical content that prompt the explicit articulation and sharing of individual thinking processes with the group. There is an attempt to collectively develop a solution, but not all group participants engage in this collective construction. The explicitation of thinking processes inherently allows for highlighting the algebraic thinking that becomes shared in the group.
- Category C: *full participation and collective co-construction of algebraic thinking*. Chats are characterised by the participation of most group members and a full variety of social modes of co-construction. Contributions from all participants are acknowledged and further developed through questions or feedback (EL interventions) or subsequent elaborations, whether in agreement (IC) or in contrast (CC) with the proposals. The group's goal is to create a collective solution. Most group members articulate their thinking processes and contribute to the development of the group's algebraic thinking. Such development is not always effective, as students can encounter blocks or can wrongly

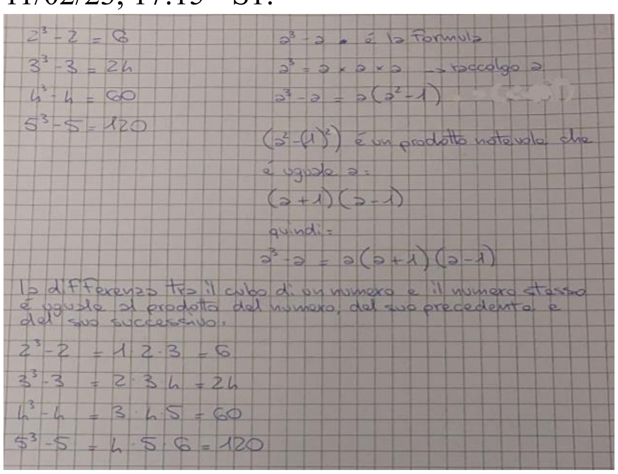
interpret or incorrectly manipulate algebraic expressions, but they work as resources for each other and, even when not succeeding in the development of algebraic thinking, they reach it collectively.

Tables 5.8, 5.9 and 5.10 report the analysis of excerpts from the transcripts of the chats from the second activity of the didactic path that exemplify the different categories. Afterwards, it is reported a summary of the distribution of the groups' chats between the different categories in the second and fourth activities of the didactic path (Table 5.11).

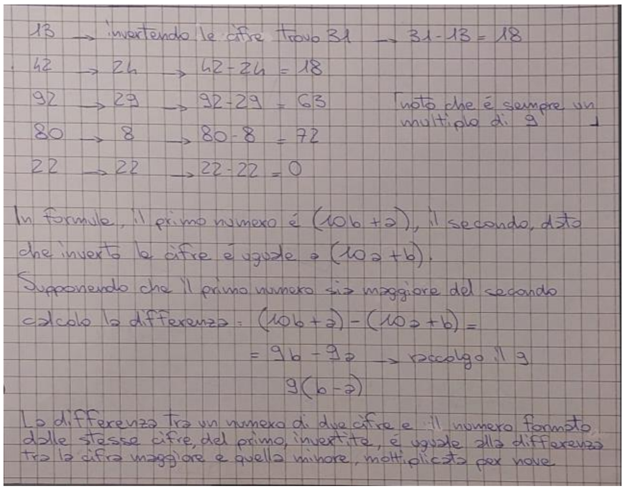
The excerpt reported in Table 5.8 exemplifies group discussions belonging to category A and it is taken from the chat of group 5 class A 2nd DBR cycle. This excerpt represents, actually, the whole discussion that occurred between the group concerning the proposed problems. Thus, it shows the limited exchange of messages between participants and the prevalence of EX and QC interactions that are characterised by the absence of considering others' contributions. The only EL message present in the chat concerns organisational aspects and no feedback or explicit intervention is asked to peers concerning the mathematical aspects of the proposed individual solutions to the problems. The group does not collaboratively create a solution but agrees on picking S1's. The scarce interaction and the exchanges through EX or QC social modes of co-construction result in the complete absence of development of collective algebraic thinking. Indeed, except for the individual productions of S1 and S2, there is no sharing at the mathematical content level.

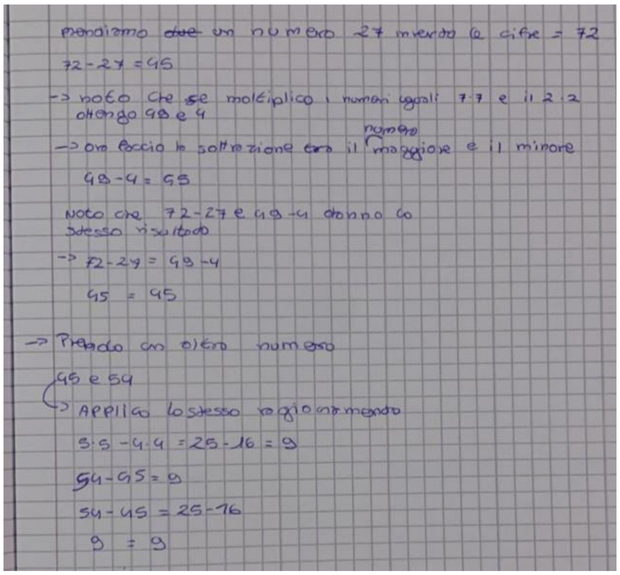
Excerpt from the transcript of a chat	**Coding of the interventions according to social modes of co-construction**	**Coding of interventions according to the epistemic aspects of algebraic thinking**
11/02/23, 11:44 - S1: Hello everyone. I would have found some solutions to the two questions that seem correct to me Have you already had a chance to think about it?	EL – Student 1 tries to involve the mates through a question	/
11/02/23, 12:25 - S2: I'll take a look at it this afternoon. I'll let you know	QC – Student 2 answers to S1's question in order to make explicit his presence. The focus is on organisational aspects	/

<table>
<tr>
<th>Excerpt from the transcript of a chat</th>
<th>Coding of the interventions according to social modes of co-construction</th>
<th>Coding of interventions according to the epistemic aspects of algebraic thinking</th>
</tr>
<tr>
<td>11/02/23, 16:59 - S2: I just completed the first problem. I’ll send you a photo of my reasoning.


In the picture, S2 proposes some numerical examples for task 1 and from them conjectures that the difference between the cube of a natural number and the number itself can be seen as the product between three consecutive natural numbers</td>
<td>EX – Student 2 presents a new proposal for task 1</td>
<td>Individual algebraic thinking – attempts of generalising through numerical explorations</td>
</tr>
<tr>
<td>11/02/23, 17:15 - S1:


In the picture, S1 proposes some numerical examples for task 1 and a symbolic transformation changing a^3-a into a(a+1)(a-1). He formulates the conjecture that the difference between the cube of a natural number and the number itself is the product between three consecutive natural numbers
I found this regarding the first exercise</td>
<td>EX – Student 1 presents a new proposal to task 1 without referencing to S2’s</td>
<td>Individual algebraic thinking – numerical explorations, syntactic transformations of the expression a^3-a into a(a+1)(a-1) and activation of the ‘preceding-successive’ frame to interpret the expression as the product of three consecutive numbers</td>
</tr>
</table>

Excerpt from the transcript of a chat	**Coding of the interventions according to social modes of co-construction**	**Coding of interventions according to the epistemic aspects of algebraic thinking**
11/02/23, 17:37 - S1:  *In the picture, S1 reports some numerical examples for task 2 and conjectures that the considered difference in task 2 is always a multiple of 9. Then he proposes the expected algebraic proof and completes the conjecture as "the difference between a two digit number and the number obtained inverting its digits is equal to the difference between the greater and the smaller digit multiplied by nine"* This instead is the photo of how I solved the second exercise	EX – Student 1 presents a new proposal for task 2	Individual algebraic thinking – activation and coordination of the frames 'difference between polynomials' and 'polynomial notation'. The proof is developed through the activation of the frame 'multiple', which guides both the reference to the correct anticipating thought and the interpretation of such expression

<table>
<tr><th>Excerpt from the transcript of a chat</th><th>Coding of the interventions according to social modes of co-construction</th><th>Coding of interventions according to the epistemic aspects of algebraic thinking</th></tr>
<tr><td>11/02/23, 17:37 - S2:


In the picture, S2 proposes two numerical examples for task 2 and tries to generalise what observed, that is the number obtained as the difference requested by task 2 is equal to the difference between the squares of the digits of the considered numbers

I did the second problem this way</td><td>EX – Student 2 presents new proposal for task 2 without referencing to S1’s</td><td>Individual algebraic thinking – attempts of generalising through numerical explorations</td></tr>
<tr><td>11/02/23, 18:51 - S3: I come up with the same solutions as you</td><td>QC – Student 3 states that his solutions are the same as his companions’ without making his reasoning explicit</td><td>/</td></tr>
<tr><td>12/02/23, 11:17 - S4: I am getting the same results as S2, but in my opinion S1’s seems more correct because she also wrote down the formula</td><td>IC – Student 4 agrees upon S1’s solutions explicitly recalling the presence of the formula as a criterion</td><td>/</td></tr>
</table>

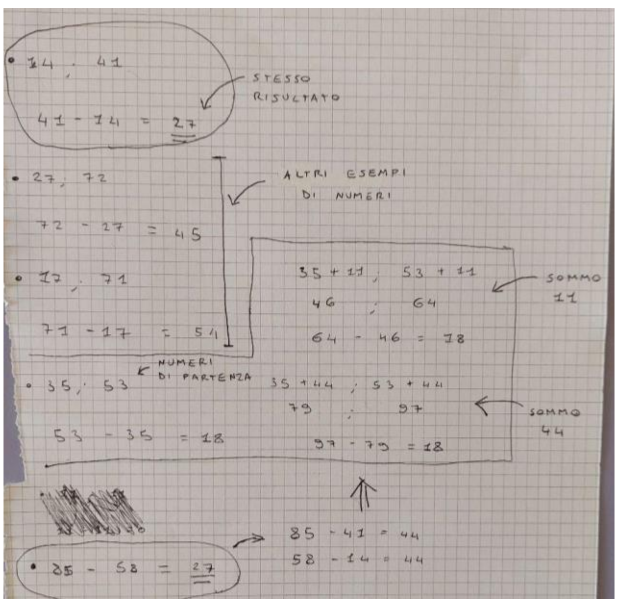
Excerpt from the transcript of a chat	Coding of the interventions according to social modes of co-construction	Coding of interventions according to the epistemic aspects of algebraic thinking
	for the correctness of that solution among others	
12/02/23, 11:41 - S5: In my opinion S4 is right, S1's result is more correct	QC – Student 5 explicitly agrees with S4	/

Table 5.8 Qualitative analysis of an excerpt of a chat exemplifying category A (group 5 class A 2nd DBR cycle)

Table 5.9 reports an excerpt referring to category B chats and taken from the chat of group 1 class B 2nd DBR cycle. In the transcript, S1 and S2 are the leaders. Indeed, from an initial EX by S1, the two students explicitly try to understand, through EL questions, and build upon, through IC and CC interventions, each other's messages and reasoning. This allows to highlight, at the epistemic level, the beginning of the development of collective algebraic thinking. From the participation dimension, it should also be noted that just two students are involved out of the 4 participants of the group (these dynamics characterise the whole transcript).

Excerpt from the transcript of a chat	Coding of the interventions according to social modes of co-construction	Coding of interventions according to the epistemic aspects of algebraic thinking
18/03/23, 10:24 - S1: So I tried to do the second one and I noticed something I had taken as an example a few numbers, including 14 and 58, which when subtracted from 41 and 85 (thus the numbers with reversed digits) both gave 27. So, I did 85-41=44, then I thought that the same could work for the other numbers and I tried: I had 35 and 53 (53-35=18) and I added 44 to both numbers (35+44 and	EX –Student 1 shares with his companions his numerical exploration concerning task 2	Individual algebraic thinking – Attempts of generalising from numerical explorations

Excerpt from the transcript of a chat	**Coding of the interventions according to social modes of co-construction**	**Coding of interventions according to the epistemic aspects of algebraic thinking**
53+44) and I noticed that the given numbers had reversed digits (79 and 97). Then I thought that 44 was a bit large because three-digit numbers could result, so I replaced it with 11		
18/03/23, 10:34 - S2: Okk, wait, I'll try to redo your reasoning on paper and I'll tell you if I understood it	IC – Student 2 takes in charge the numerical explorations proposed by S1	/
18/03/23, 10:34 - S1: Okk	QC – Student 1 explicitly agrees	/
18/03/23, 10:41 - S2: Just one question, why did you add 44 to both numbers?	EL – Student 2 asks for a clarification question concerning S1's exploration	/
18/03/23, 10:44 - S1: Because (taking as an example always 58; 85 and 41; 14) both 85-41 and 58-14 gave a result of 44 And so I thought that it should be added to both numbers in the other pairs as well Which then ultimately comes even if you only add it to one of the two and then reverse the digits	IC – Student 1 answers to the previous question making explicit his reasoning	Collective algebraic thinking – S1 makes his attempts of generalisation explicit thanks to the stimulus received by S2. In this way, the thinking processes are shared in a collective discourse
18/03/23, 11:03 - S2: Okay, so, looking back at what you did, I'm definitely not convinced because you take two numbers (85;41)	CC – Student 2 explicitly does not agree on S1's idea and proposes	/

Excerpt from the transcript of a chat	Coding of the interventions according to social modes of co-construction	Coding of interventions according to the epistemic aspects of algebraic thinking
completely randomly that have nothing to do with each other, then I wouldn't know So, I believe I have found a solution to the second problem I'll send a photo of what I wrote in the notebook because it's a bit difficult to write in a message	alternative examples and organises them trying to make sense of adding 44 to each number of the considered differences, as proposed by S1	
18/03/23, 11:03 - S1:  *In the picture, S1 reports some numerical examples in order to make explicit his reasoning* Here is the rough draft with the reasonings I hope it's understandable	IC – Student 1 answers to S2's intervention trying to make clearer his reasoning through examples	Collective algebraic thinking – S1 reorganises some numerical examples in order to make clear to S2 his attempt of generalisation

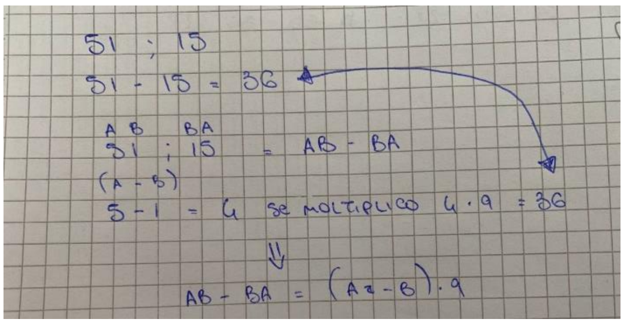
Excerpt from the transcript of a chat	Coding of the interventions according to social modes of co-construction	Coding of interventions according to the epistemic aspects of algebraic thinking
18/03/23, 11:17 - S2:  *In the picture, S2 proposes through an example and an attempt of algebraic representation a new conjecture for task 2, that is that the difference requested by task 2 is equal to the difference between the digits of the numbers multiplied by 9* I did the example with 51 I checked, and it works with any two digit number	EX – Student 2 presents new proposal for task 2	Individual algebraic thinking – S2 activates the frame 'positional notation to construct the representation of the example he is trying to generalise
18/03/23, 11:28 - S1: Well done	QC – Student 1 explicitly agrees	/

Table 5.9 Qualitative analysis of an excerpt of a chat exemplifying category B (group 1 class B 2nd DBR cycle)

The excerpt reported in Table 5.10 exemplifies chats pertinent to category C and comes from the transcript of group 5 class B 3rd DBR cycle. The excerpt, as the whole transcript of the group discussion, sees the active participation of all the group members interacting through the whole range of the social modes of co-construction. Concerning the epistemic aspects of algebraic thinking, it is the richest excerpt as students collaboratively engage in creating a group discourse around the use of algebraic language as a thinking tool. In the chat, although the epistemic aspects are more apparent compared to other categories, it is important to note that not all reasoning is made explicit. For example, there are instances where a transition occurs from numerical explorations to the construction of an algebraic proof without explicitly stating the underlying conjecture.

Excerpt from the transcript of a chat	**Coding of the interventions according to social modes of co-construction**	**Coding of interventions according to the epistemic aspects of algebraic thinking**
09/03/24, 18:33 - S1: Let's begin What can you say about the difference between the cube of a natural number and the number itself? Give some numerical examples to support you in the formulation of the requested conjecture. Could you prove what you state?	EL – Student 1 stimulates the group to start working and recalls the text of task 1	/
09/03/24, 18:35 - S2: Meanwhile the number belongs to N	EX – Student 2 explicitly notes the numerical set to work in as initial proposal for task 1	Individual algebraic thinking – Activation of the frame 'natural numbers'
09/03/24, 18:35 - S1: (n•n•n)-n	EX – Student 1 introduces a symbolic expression transforming task 1's text from verbal to symbolic representation	Individual algebraic thinking – Activation of the frame 'power' and its explicitation as product and activation of the frame 'difference between numbers' and their coordination
09/03/24, 18:35 - S3: (3•3•3)-3=27-3=24	IC – Student 3 takes in charge S1's symbolic expression and proposes some examples following the same structure	Collective algebraic thinking – Numerical exploration starting from the form the symbolic expression introduced by S1
09/03/24, 18:39 - S4: (4•4•4)-4=64-4=60 or (2•2•2)-2=8-2=6	IC – Student 4 takes in charge S1's symbolic expression and proposes some examples following the same structure	Collective algebraic thinking – Continuation of S3's numerical exploration

Excerpt from the transcript of a chat	**Coding of the interventions according to social modes of co-construction**	**Coding of interventions according to the epistemic aspects of algebraic thinking**
09/03/24, 18:39 - S2: (5•5•5)-5=125-5= 120	IC – Student 2 takes in charge S1's symbolic expression and proposes some examples following the same structure	Collective algebraic thinking – Continuation of S3 and S4's numerical exploration
09/03/24, 18:46 - S4: (2n•2n•2n)-2n=2n•(4n squared-1)=even	IC – Student 4, following the same structure of S1's proposal, presents the symbolic expression considering a particular natural number, an even one	Collective algebraic thinking – Activation of the 'even-odd' frame both in the initial step of considering, starting from S1's expression, the case of an even number and in the interpretation of the expression $2n(4n^2-1)$. The syntactic transformation is guided by the implicit anticipating thought that suggests that the final form should be $2 \cdot k$ (k natural number)
09/03/24, 18:48 - S1: It's wrong	CC – Student 1 explicitly disagrees with S4's proposal	/
09/03/24, 18:53 - S4: True, because it should be only the two outside	CC – Student 4 explicitly agrees with S1 in contrasting his own proposal	Collective algebraic thinking – Activation of the frame 'even-odd' to identify the structure of the final form of his expression

Excerpt from the transcript of a chat	Coding of the interventions according to social modes of co-construction	Coding of interventions according to the epistemic aspects of algebraic thinking
09/03/24, 18:54 - S3: (2n•2n•2n)-2n= $8n^3$ -2n = even	IC – Student 3 proposes a different syntactic transformation related to the considered case (the initial number is even)	Collective algebraic thinking – Implicit activation of the frame 'even-odd' and reference to the theorem "the difference of two even numbers is even" in the interpretation of $8n^3$-2n
09/03/24, 18:54 - S4: (2n•2n•2n)-2n=2•($4n^3$-n)=even written like this you do not justify the "even" *Direct answer to S3's message*	CC – Student 4 explicitly disagrees with S3's proposal and suggests a possible grouping to make explicit that the obtained expression represents an even number	Collective algebraic thinking – Activation of the frame 'even-odd' to interpret S3's expression $8n^3$-2n and consequent activation of the anticipating thought that guides the syntactic transformation of $8n^3$-2n into 2·($4n^3$-n)
09/03/24, 18:54 - S2: Right then grouped like this	QC – Student 2 agrees with the grouping proposed by S4 in addition to S3's proposal	/
09/03/24, 18:54 - S4: because it's not two times something How did you manage to place the power of three? *Direct answer to S3's message*	IC + EL – Student 4 strengthen the need of the grouping he proposed and asks for support to the peers for technical aspects	Collective algebraic thinking –Explicitation of the anticipating thought guiding the manipulation after the activation of 'even-odd' frame
09/03/24, 18:55 - S2: ³ Copy this	QC – Student 2 answers to technical needs of S4	/
09/03/24, 18:55 - S4: thanks	QC – Student 4 thanks S2	/

Table 5.10 Qualitative analysis of an excerpt of a chat exemplifying category C (group 5 class B 3rd DBR cycle)

Table 5.11 summarises the distribution of groups' chats across the three categories in the second activity (analysed in depth in the previous tables as representative) and the fourth activity (the final activity conducted in the experimentations). The table shows the number of groups' chats in each category out of the total number of groups' chats for each class involved in the experimentations (for example, column 2 shows that, in class A DBR 2, 3 groups out of 6 belong to category A and 3 groups to category B). We report the summary for the second and fourth activities to provide an overview of the results based on a more extensive portion of the dataset than what is presented in the excerpts.

	Second activity of the didactic path				**Fourth activity of the didactic path**			
	Class A DBR 2	Class B DBR 2	Class A DBR 3	Class B DBR 3	Class A DBR 2	Class B DBR 2	Class A DBR 3	Class B DBR 3
Cat. A	3/6	0/4	1/5	0/6	6/6	0/4	3/5	0/6
Cat. B	3/6	3/4	3/5	0/6	0/6	3/4	2/5	2/6
Cat. C	0/6	1/4	1/5	6/6	0/6	1/4	0/5	4/6

Table 5.11 Summary of the distribution of group chats across categories in the second and fourth activities of the didactic path

We can analyse the data reported in Table 5.11 by different perspectives: first, by viewing the information related to the second activity as snapshots of the initial state of the students' work in the different classes within the DMD; and second, by comparing the data from the second and fourth activities to characterise the evolution of students' group-work within the chats, class by class. As for the initial snapshots that the data provide on the classes, we can observe that class A from the 2nd DBR cycle in the second activity is distributed between categories A and B, reflecting a lack of interaction, collective knowledge construction, and idea sharing. In contrast, class B from the 2nd DBR cycle shows students' work distributed between categories B and C, indicating the presence of several leaders within the class. Class A from the 3rd DBR cycle is spread across all categories, suggesting a mix of student profiles in terms of group-work carried out in the chats. Finally, for class B in the 3rd DBR cycle, all groups in the second activity fall into category C, demonstrating strong social interaction skills and collective mathematical reasoning. Regarding the evolution of students' group-work throughout the DMD activities, we can observe an apparent lack of change or even a negative trend over time. In fact, aside from class B in the 2nd DBR cycle, where the groups remained in the same categories from the second to the fourth activity, the other classes experienced a convergence toward categories A and B. For example, in class A of the 2nd DBR cycle, by the end of the experimentation, all the groups fell into category A. In class A of the 3rd DBR cycle, the group that was in category C moved to B, and two groups shifted from category B to A. Similarly, in class B of the 3rd DBR cycle, two groups moved from category C to B.

5.2.2 Intertwining students' social modes of co-construction and epistemic aspects of algebraic thinking within the Padlet

In this subsection, we present the results and analysis conducted on the transcripts of the Padlets from the first and second experimentations. Each Padlet was designed according to the FaSMEd methodological approach presented in sections 1.2.2 and 2.1 and by referring to the M-$_{AE}$AB construct (subsection 1.2.1) in order to foster an effective balance discussion. Three types of teacher's guiding questions related to different roles of the M-$_{AE}$AB construct characterise the design of the Padlets:

- Type 1: request to analyse and compare more excerpts from the groups' solutions, in tune with the role of *activator of reflective attitudes and metacognitive acts*. The reflection can be guided by specific criteria such as the criteria of correctness, clearness, and completeness (Cusi et al., 2019) (e.g., "Which procedure among those proposed seems the most correct to you? Which one is the clearest? Which one is the most complete? Why?" from class B 2nd DBR cycle);
- Type 2: request to make students' thinking visible with respect to a single group's solution, in tune with the *role of reflective guide* (e.g., "A group has proposed the following conjecture. What do you think about it?" from class A 2nd DBR cycle);
- Type 3: request to further explore the proposals made by the groups regarding a conjecture, the interpretation of an expression, a reasoning or a proof, referring to the roles of *practical-strategic guide and activator of interpretative processes* (e.g., "Does the equivalent expression nx(n-1)x(n+1) allow us to observe characteristics of n^3-n that would otherwise not be evident? If so, which ones?" from class B 2nd DBR cycle).

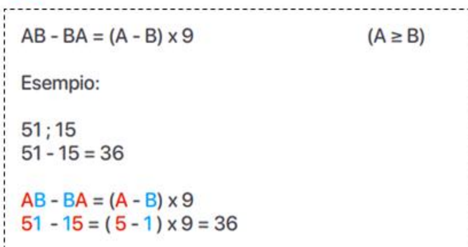
Concerning students' participation to the Padlets of the second activity, we can observe two different phenomena: the majority of students from classes B of both the 2nd (15 students out of 18) and 3rd (23 students out of 25) DBR cycles took part actively in the second phase of the DMDs, while both classes A from 2nd (13 students out of 29) and 3rd (10 students out of 23) DBR cycles have poor participation. Despite the difference in the number of participants, in all cases the Padlets turn out to be lacking in the variety of social modes of co-construction. They are all dominated by EX interventions from students. This is observed regardless of the type of guiding question posed by the teacher.

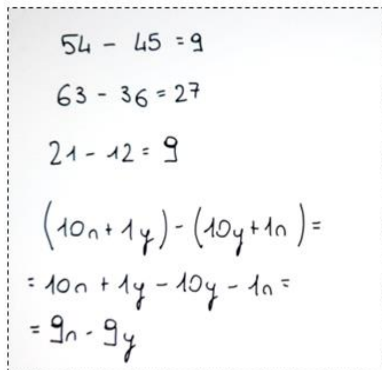
In the following, we exemplify the conducted analysis referencing to the data collected within the second activity of the didactic path on proof in arithmetic through the use of algebraic language (section 3.3). Tables 5.12 and 5.13 report the analysis of excerpts from the Padlets from the second activity for the different teacher's guiding questions and taken, respectively, from class B 2nd DBR cycle and class A 3rd DBR cycle. Afterwards, we report a summary of the distribution of the different types of the teacher's guiding questions in the Padlets of the second and fourth activities of the didactic path (Table 5.14).

In the excerpts shown in Tables 5.12 and 5.13, we can observe that multiple individual epistemic aspects of algebraic thinking are present in order to interpret and compare groups' proposals. The excerpts show that students activate the necessary processes to respond to the teacher and, in some cases, even if they do not explicitly refer to their classmates' statements, they build their reasoning in a way that completes what their classmates have proposed or adapts to it (see, for example, the first two interventions in Table 5.13). The excerpt in Table 5.13 also includes a second intervention by the teacher which can be categorised as a type 2 intervention, since the teacher plays the role of reflective guide by relaunching S2's proposal to elicit a response that triggers an IC intervention.

<table>
<tr><td colspan="3">T (guiding question of type 1): All groups observed that the difference between a two digit number and the number obtained by reversing its digits is a multiple of 9. Among the justifications provided here, proposed by some of the groups, which one seems the most correct to you? Which one is the most complete? Which one is the clearest? Why?

The question is accompanied by a screenshot of two excerpts (A and B) from groups' solutions to task 2

A

AB - BA = (A - B) x 9 (A ≥ B)
Esempio:
51 ; 15
51 - 15 = 36
AB - BA = (A - B) x 9
51 - 15 = (5 - 1) x 9 = 36

B

54 - 45 = 9
63 - 36 = 27
21 - 12 = 9
(10n + 1y) - (10y + 1n) =
= 10n + 1y - 10y - 1n =
= 9n - 9y</td></tr>
<tr><td>Excerpt from the Padlet</td><td>Coding of the interventions according to social modes of co-construction</td><td>Coding of interventions according to the epistemic aspects of algebraic thinking</td></tr>
<tr><td>S1: I find procedure A clearer because it is written more simply, immediately clarifying that AB and BA are the same numbers but with reversed digits. On the other hand, writing AB could be confusing as it might be read as A x B, so from this perspective, procedure B is more correct</td><td>EX – Student 1 inserts the first answer to the teacher's guiding question</td><td>Individual algebraic thinking – Activation of the frame 'monomials-polynomials' and coordination between the frames 'positional notation' and 'polynomial notation' for interpreting the expressions provided by his mates and detecting the mistake in the first one</td></tr>
</table>

Excerpt from the Padlet	Coding of the interventions according to social modes of co-construction	Coding of interventions according to the epistemic aspects of algebraic thinking
S2: I find procedure A clearer because the way it is written is easy to understand; it immediately becomes clear that the two variables A and B represent the two digits of the number. However, as mentioned by others, one might think that the variables do not represent a single number but rather a product of two numbers.	IC – Student 2 explicitly recalls others' contributions (in this case S1's, since it is the only one present before his intervention)	Collective algebraic thinking – Activation of the frame 'monomials-polynomials' to stress the mistake in the first representation, confirming S1' interpretation
S3: I find procedure B to be more correct and clear because the way the two digit number is written is more accurate, and procedure A does not include the steps that led to that equality.	EX – Student 3 expresses his opinion concerning the procedures without explicitly referencing other contributions. Moreover, in his motivations, he adds the lack of "steps" justifying the equality in procedure A	Individual algebraic thinking – Implicit activation of the frame 'polynomial notation' and identification of the fact that the equivalence between the two expressions in the first representation is not justified through syntactic transformations
S4: In my opinion, option B is written better as it represents the request more clearly and shows the equality clearly, but we can still gather the 9. Therefore, option A is slightly clearer only in the final result	EX – Student 4 expresses his opinion concerning the procedures without explicitly referencing other contributions	Individual algebraic thinking – Coordination between the frames 'multiple' and 'polynomial notation' for the interpretation of the two final expressions written in answers A and B in relation to the conjecture to be proven

Table 5.12 Qualitative analysis of an excerpt from a Padlet initiated by a teacher's guiding question of type 1 (class B 2^{nd} DBR cycle)

T (*guiding question of type 3*): Some groups have proposed the conjecture “the difference between the cube of a natural number and the number itself is a multiple of 6”, while other groups have proposed equivalent expressions for this difference. Can we prove the conjecture using the symbolic expressions proposed by some of you?

The question is accompanied by a screenshot of excerpts from groups' solutions to PB1 grouped by proposals of equivalent symbolic expressions, examples, and conjectures put forward by different groups

Alcune vostre proposte di scritture simboliche equivalenti per la differenza tra il cubo di un numero naturale e il numero stesso

1

Ho notato, anche attraverso esempi numerici, che il cubo di un numero naturale sottratto a se stesso, è equivalente al prodotto tra il suo precedente, il suo successivo e il numero stesso,ad esempio:
27-3=4x3x2=24, o 64-4=5x4x3=60
perciò possiamo dire che:
$(x)^3$- x = (x+1)(x)(x-1)

2

Della differenza tra il cubo di un numero naturale e il numero stesso è uguale al prodotto del numero stesso per il precedente del suo quadrato, perché A^3-A è uguale a AxA^2-A che si può scrivere anche come Ax(A^2-1)

Alcuni esempi proposti da voi

3

1) a^3 - a =
2^3 - 2 = 8 - 2 = 6
3^3 - 3 = 27 - 3 = 24
4^3 - 4 = 64 - 4 = 60
5^3 - 5 = 125 - 5 = 120
6^3 - 6 = 216 - 6 = 210
a ∈ N

4

1) n^3-n=6m (n,m appartenenti N)
Esempi: 2^3-2=6=multiplo di 6
3^3-3=24=multiplo di 6
4^3-4=60=multiplo di 6
5^3-5=120=multiplo di 6

Congetture proposte da uno dei gruppi

5

Osserviamo che facendo la differenza tra il cubo di un numero naturale e il numero stesso si ottiene sempre un numero divisibile per il numero dato e il numero è sempre divisibile per 6

Excerpt from the Padlet	**Coding of the interventions according to social modes of co-construction**	**Coding of interventions according to the epistemic aspects of algebraic thinking**
S1: In my opinion, to correctly express in symbols the difference between the cube of a natural number and the number itself, one should combine together the conjectures proposed by various groups: $(x)^3$-x=(x+1)(x)(x-1)=y=6n x,y,n ∈ N, y ∈ P	EX – Student 1 inserts the first answer to the teacher's guiding question	Individual algebraic thinking – Assembling different proposals by the groups with no explicit epistemic aspects of algebraic thinking guiding it
S2: we can also say that n^3 - n = n (n^2 - 1) = n (n - 1) (n + 1) = (n - 1) n (n + 1) with n in N	EX – Student 2 presents his proposal without	Collective algebraic thinking – Activation and coordination of the frames

Excerpt from the Padlet	Coding of the interventions according to social modes of co-construction	Coding of interventions according to the epistemic aspects of algebraic thinking
therefore since it is a product of 3 consecutive numbers in ascending order one could say that the result y is divisible by 3 an example can be 3^3 - 3 = 3 (3^2 - 1) = 3 (3 - 1) (3 + 1) = (3 - 1) 3 (3 + 1) = 2 x 3 x 4 = 24 (24 : 3 = 8)	explicitly referencing to the previous intervention	'preceding-successive' and 'multiple' to interpret the product (n-1)n(n+1) as divisible by 3, thus starting combining the different conjectures by the groups as proposed by S1
T: What do you think about S2's proposal to conclude that divisibility by 3 arises directly from writing a number as the product of three consecutive natural numbers?	EL – the teacher tries to relaunch to the whole class S2's proposal	The teacher wants to stimulate all students to interpret the expression as divisible by 3, posing herself as a reflective guide
S1: The statement is correct because by taking 3 consecutive natural numbers at least one of them will be a multiple of 3 and therefore their product will also be a multiple of 3. However it is also true that we necessarily take one which is also even, so the product will also be a multiple of 6 (6 = 3 x 2).	IC + EX – Student 1 recalls and evaluate S2'2 proposal as requested by the teacher and adds a new conjecture	Collective algebraic thinking (stimulated by the teacher) – Activation and coordination of the frames 'consecutive numbers' and 'multiple' to interpret the product of three consecutive natural numbers as both a multiple of 3 and an even number, therefore as a multiple of 6

Table 5.13 Qualitative analysis of an excerpt from a Padlet initiated by a teacher's guiding question of type 3 (class A 3rd DBR cycle)

Table 5.14 summarises the distribution of teacher's guiding questions between the three types in the second and in the fourth activities. The table shows the number of teacher's guiding questions of each type out of the total number of guiding questions starting a column for each Padlet of the classes involved in the experimentations. This table shows a decrease in the amount and types of guiding questions posed by the teacher in each Padlet as starting of new columns both from the fourth to the final activity, but also from the 2nd DBR cycle to the 3rd. The fact that the teacher starts less Padlet columns is due to the observed difficulty of students engaging within the Padlet and the reduction of columns was one possible answer to the problem we tried in our experimentations. What cannot be observed from the data reported in the table is that, despite a smaller number of questions and a lower variety of question types, students were asked to consider or compare mathematical aspects starting from more excerpts taken from the groups' solutions than in previous activities (as can also be seen in the screenshot in Table 5.12 compared to that in Table 5.13).

	Second activity				**Fourth activity**			
	Class A DBR 2	Class B DBR 2	Class A DBR 3	Class B DBR 3	Class A DBR 2	Class B DBR 2	Class A DBR 3	Class B DBR 3
Type 1	1/4	2/4	1/2	0/2	1/2	0/3	0/1	0/1
Type 2	2/4	1/4	0/2	1/2	0/2	0/3	1/1	1/1
Type 3	1/4	1/4	1/2	1/2	1/2	3/3	0/1	0/1

Table 5.14 Summary of the distribution of types of teacher's guiding questions as starting of columns in the Padlets of the second and fourth activities

5.2.3 Discussion

The analysis and results presented in the previous subsections allow us to address the second research question (RQ2), investigating if students develop collective algebraic thinking within the asynchronous phases of DMDs and how the social modes of co-construction intertwine with the epistemic aspects of algebraic thinking in students' collective co-construction of algebraic thinking within these phases. Moreover, the analysis with respect to the solely collaborative processes through the dimension of the social modes of co-construction allows us to integrate the answer to RQ1 provided in subsection 5.1.3 with the results from the analysis of data of the first and second experimentations.

Concerning RQ1, the analysis of a more extensive dataset from the first and second experimentation enables us to highlight clearer and more detailed collaborative processes compared to the limited cases in the pilot study, especially regarding interactions in the chats. We can expand the pilot study's categories 1 and 2 into the broader categories A, B, and C, detailed in subsection 5.2.1. These new categories align with the pilot study categories in differentiating the types of student group-work with respect to quantity of student participating and the social modes activated, but allow for a clearer identification of category B, where one or more leaders guide the group's work. Although within the pilot study we also identified some students acting as group leaders, the limited data available did not permit specific characterisation of this role. One notable difference between the two sets of categories is the focus on organisational aspects versus mathematical content. In the pilot study's category 1, group chats are primarily focused on organisational aspects, such as producing and submitting the solution, while in the first and second experimentation, all group chats are predominantly centred on mathematical content. Organisational concerns are still present across all groups but remain secondary and never dominant. Regarding the collaborative processes on Padlet, they retrace the same situation as the pilot study with social modes activated by few students. The main difference stands in the interventions done by the teacher during the collective discussion apart from the guiding questions that stimulate students to take into account each other's contributions.

Concerning RQ2, the analysis of group-work within chats reveals different levels of interaction and collaboration that impact the development of collective algebraic thinking. In category A, limited interaction and a focus on individual thinking inhibit the development of collective solutions, resulting in outcomes that rely solely on individual algebraic capabilities of few students in the group. Category B highlights the role of leadership in fostering a partially collective approach, with leaders guiding the discourse and encouraging participation, though not all members fully engage in constructing a shared solution. This leadership-driven structure allows for some algebraic thinking to be shared within the group, yet remains uneven in its reach. In category C, full participation and diverse collaborative interactions enable most group members to actively co-construct solutions, using each other's contributions as resources to address errors or deepen understanding, thus achieving a genuinely collective approach to algebraic thinking. These distinctions underscore the importance of engagement, shared contributions, and collaborative articulation for fostering collectively developed mathematical reasoning in group-work. Regarding the connection between social modes of co-construction and epistemic aspects of algebraic thinking processes, a link emerges across categories between the collective or individual creation of thinking processes and specific social modes. Individual thinking processes, primarily found in category A, involve EX interventions for introducing new contributions, with other participants responding mostly with QC interventions. Thinking processes become collective, instead, when a proposal is taken up through questions or requests by other participants (EL interventions) or when it is revisited, modified, or expanded with IC or CC interventions. This collective development occurs among a few students due to a leader's presence in category B chats and among most students in category C chats. The analysis conducted on the Padlet transcripts reveals that thinking processes remain mostly individual but can be encouraged to become collective through teacher interventions (see Table 5.13). This aspect warrants further investigation, as we believe the teacher's interventions were too limited in the current experiments. In future studies, more frequent teacher engagement may be beneficial, especially to promote effective processes like the one shown in Table 5.13, where the teacher encourages students to consider their peers' contributions and develop collective thinking processes. It should be noted that, in the analysis of the Padlet transcripts, the distinction between individual and collective thinking processes is subtler than in the chats. The students' contributions on Padlet refer to what the groups produced in the first phase of the DMD. Thus, following the teacher's guiding question, students review, reflect on, interpret, or add to what their peers have produced in the group-work. However, in this case, we do not consider all of the students' contributions on Padlet as collective thinking creation, since, despite referencing group outputs, the students' contributions are responses to the teacher's prompts rather than co-construction among students. On Padlet, we can speak of collective thinking development only if participants build on one another's contributions within the Padlet.

Moreover, concerning data reported in Table 5.11, we can deepen our understanding of classes' distribution across the different categories of group-work within chats. First, in subsection 5.2.1

we completed the initial background of the classes (redacted with respect to the initial students' capabilities in activating and coordinating conceptual frames in subsections from 4.2.3 to 4.2.6) with respect to the collaborative dynamics realised within the chats of the second activity of the didactic path. We can point out that classes with similar initial backgrounds on the activation and coordination of conceptual frames (see class A and B from 2nd DBR cycle) can differ in the collaborative dynamics (class A, poor interaction; class B, several leaders). Moreover, class B from 3rd DBR cycle results with stronger capabilities on the activation and coordination of conceptual frames and with strong social interaction skills and collective mathematical reasoning.

We can also relate the initial background of classes with respect to the collaborative processes with the teaching styles of their mathematics class teachers (presented in subsection from 4.2.3 to 4.2.6). Gianmarco's formal and traditional approach to teaching is reflected in the limited interaction and collaboration among his students (class A, 2nd DBR cycle), whereas Sabrina's encouragement of engagement of students (class B, 2nd DBR cycle) and Costanza's emphasis on group-work (class A, 3rd DBR cycle) resulted in students who were more inclined (though still limited) to interact and collaborate. The situation of class B from the 3rd DBR cycle, however, is somewhat anomalous compared to the teacher's perception of the class. Arianna described the class participating in the experimentation as one that typically faced some difficulties in mathematics and lacked significant engagement in activities prior to the experiment. What we observed in the DMD work carried out by this class, however, was deep collaboration among the students, both in terms of interaction and in the co-construction of mathematical reasoning. This discrepancy between the teacher's view and the findings from the students' work can be attributed to the novelty of the proposed activity, both in terms of teaching methodology (since, despite being encouraged to participate and work in groups during traditional lessons, they had never been involved in a discussion-based activity) and the mathematical content proposed (which was aligned with the school curriculum but focused on exploration and justification rather than merely performing calculations). Furthermore, it would be interesting to have an initial snapshot of the work carried out by students in a traditional classroom discussion context, to compare what was observed in the DMD with the activities conducted in class. Secondly, comparing the data in Table 5.11 from the second and fourth activities, at the end of subsection 5.2.1, we characterised the evolution of students' group-work within the chats, class by class. The discouraging data on the negative trend in the groups' dynamism within the categories actually suggest that there is a challenge to be acknowledged and addressed in classrooms: the promotion of effective long-term processes. The experimentations conducted within this research represent examples of medium-term activities (where the short term spans days, the medium term weeks/months, and the long term months/years) in which improvable collaborative dynamics do not improve, and good collaborative dynamics do not remain stable. From discussions with the teachers of the involved classes, we hypothesise that this may be influenced by today's cultural context, where students are accustomed to the consumption and stimulation of short, fast-paced content on social media. This observation points to the need for a deeper investigation into the dynamics of collaborative processes and co-construction of knowledge in DMD contexts, suggesting the importance of conducting long-term experiments in the future to explore this particular aspect further.

In conclusion:

- The chats analysis reveals a connection between the development of thinking processes and social modes of co-construction within group-work contexts in DMDs. In category A, limited interaction and reliance on individual contributions, characterised by EX and QC interventions, limit the group's algebraic reasoning to the abilities of individual members without promoting collective development of thinking. Category B illustrates the role of leadership in fostering collaborative thinking, with leaders using IC and CC interventions to engage peers, though not all members participate equally in co-constructing solutions. Finally, category C exemplifies a genuinely collective development of algebraic thinking, as high participation and diverse interaction modes (EL, IC, CC) allow members to build upon each other's contributions. One challenge emerging from the chat analysis concerns how to encourage groups to improve or sustain the development of collective thinking in chats over the long term.
- The Padlet analysis underscores the need for teacher intervention to encourage the development of collective thinking, as processes within Padlet largely remained individual regardless the type of guiding question asked by the teacher. Enhanced teacher engagement apart from the guiding questions in future experiments could stimulate effective collective processes, where teacher guidance leads students to build on their peers' ideas.

5.3 Analysis and results on students' perspective on key features of DMD

The analysis and results reported in this section are part of a research report accepted for the 14th Congress of the European Society for Research in Mathematics Education (CERME14) and deals with students' reflections on asynchrony and written communication as key features of DMD. In particular, we analyse students' answers to the final questionnaire from the first and second experimentations in order to highlight their vision on key features of DMD with respect to their articulation as constraints, enablement, potentialities and affordances by Trouche (2004), as presented in subsection 2.3.1. The analysis of the answers given by students to the final questionnaire allow us to identify themes of students' perceptions concerning the asynchronous modality and the written communication in DMD. In the following subsections we present the results of the analysis that consist in the identification of themes based on students' reflections and the percentage of answers pertaining to each theme (the total percentage does not add up to 100% because each response may fall into multiple themes) with respect to asynchrony (subsection 5.3.1) and written communication (subsection 5.3.2). Finally, in subsection 5.3.3, we discuss the obtained results providing an answer for the third research question (RQ3) and we compare the students' perspective emerged from the analysis with the designers' one presented in subsection 2.3.1.

5.3.1 Students' perspective on asynchrony

Concerning the asynchronous modality of DMD, four perceived themes emerge in students' answers:

(1) flexibility in work management (22% of answers);

(2) availability of time to reflect on one's contribution (60% of answers);

(3) availability of time to reflect on others' contributions (18% of answers);

(4) lack of pressure to respond immediately (3% of answers).

Students' answers in theme 1 concern how the asynchronous modality supports them at the organisational level both as enablement and affordance. Indeed, they noted that the flexibility in time allows them to effectively taking part in the discussion (enablement) but also that it shapes their time of interaction within the discussion (affordance), as they could choose when to participate in discussions without restrictions on their peers' timing ("I didn't have any problems coordinating with others, and I could write in the group whenever I had time"), and manage their participation alongside other school's activities ("I was able to organise myself better with the rest of my homework") and personal commitments ("With my personal commitments, I wouldn't have always been able to write").

Themes 2 and 3 show students' exploitation of the potentiality identified by the designers of exploring concepts over time. Students reported that the possibility to take their time allowed them to reason more thoroughly ("[taking my own time] gave me the opportunity to have all the time to reason") both revisiting their own reasoning multiple times before submission ("one can review their ideas more calmly") and reading and understanding others' contributions more easily due to the ample time available ("I had the opportunity to read and understand others' contributions more easily, given that I had a lot of time available [for the activity]").

Theme 4 includes students' interventions who state explicitly that they "didn't have the pressure to respond immediately". Even though theme 4 was identified in only a few students' contributions, it is still relevant to be considered, as it highlights a potentiality of DMD recognized by the students but only partially addressed in the design. With respect to asynchrony, students were generally positive in their comments, nevertheless some have identified some constraints. For theme 1, students noted that the freedom to organise their timing sometimes led to procrastination and close proximity to deadlines ("You could work on the activity whenever you wanted, but sometimes, there was a tendency to always postpone the work, thus delaying it until almost the deadline"), or to forgetting to respond ("Sometimes they [peers' contributions] would arrive while I was working on other assignments, and I needed to focus, and then I would forget to respond"), or losing the line of reasoning ("If you take time to respond you risk losing the line of reasoning, which in turn results in wasting more time").

5.3.2 Students' perspective on written communication

Regarding the use of written communication, three themes are identified. According to students, written communication in DMD supports:

(A) read and reflect on others' reasoning while discussing (25% of answers);

(B) compare others' reasoning with one's own (34% of answers);

(C) re-read old contributions (42% of answers).

In theme A, students reflected on the enablement of written communication for accessing others' contributions more easily within the discussion. Examples include statements such as "[having access to others' written contributions] was helpful for comparing and seeing the different types of solutions" and "I was able to better understand what my classmates wanted to communicate".

In theme B, students noted that having access to written contributions enabled them to revise their own opinions ("I was able to have direct comparisons between my opinions and those of my classmates, and on many aspects that I thought were clear, I changed my mind") and to find new insights ("[others' written contributions] were tools for gaining insights, but it was also challenging to read them and sometimes not understand them").

Theme C concerns students' exploitation of the potentialities identified by the designers concerning the availability of the transcript of the discussion. Students highlighted the possibility to re-read contributions once the discussion has moved on or finished to gain a better understanding of others' interventions ("since they [others' contributions] were written, they could be reviewed multiple times and reflected upon more deeply"), revisit one's own reasoning ("[having access to written contributions was useful] to revisit my reasoning even after days"), avoid forgetting important points during the discussion ("I didn't risk forgetting them [others' contributions]"), recover from missed parts of the conversation ("It was useful because I sometimes lost track of the conversation when others were writing in my absence, but by re-reading the messages, I was still able to understand what had been discussed"), or reconstruct the entire work ("It was easier to trace the development of the work"). However, few students expressed a perceived constraint that was not completely considered in the design, that the challenge to deal with long written contributions ("Having so much written in a single message is really annoying").

Students' comments focus also on combined features concerning asynchrony and written communication, such as the possibility to use the written contributions at their own pace to develop individual reasoning, and the availability of the written transcript of the entire workflow and discussion to all participants at any time. Regarding the first aspect, some students highlighted a potentiality of written communication, stressing that taking their own time to reflect on their own contribution was possible because they were writing their reasoning and that it wouldn't have been possible otherwise, such as using vocal messages ("before sending any message, there is time to review and correct any possible errors (which is not the case when sending a voice message)") or calls ("it was useful because I could respond when I had time. Therefore, it was more convenient than a phone call and offered more flexibility."). Regarding the second aspect, students shared their views on the availability of transcripts in response to the question "Do you think having access to the messages (both on chats and Padlet) even after the activities ended was helpful? Why? Did you find yourself going back to re-read messages (both on chats and Padlet) after the activities ended?". Forty students said that having access to the transcripts was beneficial and that they used them for various reasons: to clarify initially unclear issues ("I found myself going back to re-read the messages at the end of the activity to try to reflect on and understand some questions that were initially unclear"), to support subsequent class work ("In my opinion, keeping the messages was useful because, for example, while we were discussing in class, we could review our written reasoning from the chat"), to review the entire activity ("The messages show the progression of the entire activity, allowing one to 'retrace' it"), or to review parts of the activity when transitioning between phases of the discussion ("I found myself re-reading the passages when we moved from the chat to Padlet to understand our reasoning" or "[it was helpful] to compile the summary of the discussion that I then submitted on Classroom”). On the other hand, 29 students reported that they did not take advantage of the opportunity to revisit the messages. Many of them (21) do not recognise the enablement and affordances of written communication considered in the design, as they declare that having the transcripts was not useful at all (“I did not revisit the messages after the end of the activity because, once the posts were read, the reasoning was understood and memorised, making it unnecessary to go back”). Other 4 students, instead, acknowledged the potential future utility of having all contributions saved (“I did not review the messages after the end of an activity, but it is useful to have them saved in case they might be needed in the future”), even if at the moment they experienced DMD they have not recognised the availability of the transcript of the discussion as a potentiality.

5.3.3 Discussion

The analysis of students’ answers to the online questionnaire following their experience with DMD allow us to characterise their reflections on the asynchronous modality and the written communication in DMD, thus answering the third research question (RQ3), and highlight alignment and divergence in how the key features of DMD are characterised by designers as presented in subsection 2.3.1. Students’ responses reveal a more complex interaction with these features, introducing new insights that were only partially anticipated by the designers.

Concerning asynchrony, for designers, flexibility (theme 1) is seen primarily as an affordance, allowing participants to engage in the discussion at their own pace. For students, flexibility functions as both an enablement and an affordance. It enables participation by allowing students to engage in the discussion according to their own schedules, making it possible for them to manage their time effectively and participate despite other commitments. This aligns with the designers’ intentions but goes further as students perceive it as actively shaping their interaction with the discussion. However, this flexibility also introduces a constraint that was not fully considered by designers: some students experience a loss of control over their work, leading to procrastination, forgotten responses, and difficulties maintaining the flow of reasoning. This reveals a dissonance between the designers’ idea of flexibility and the practical challenges students face when this flexibility is not well managed. Moreover, designers emphasised the potentiality of asynchrony for exploring concepts over time. This potentiality is exploited by students, as reflected in their recognition of the benefits of having time to reflect on both their own and others’ contributions (themes 2 and 3). Students describe how this extended time allows them to carefully reason, revisit, and refine their ideas before submission, as well as to understand others’ contributions more deeply. This alignment between designers’ intentions and students’ experiences suggests that the potential of asynchrony to enhance reflective thinking is being effectively realised in practice. A key difference between students’ and designers’ perspectives emerges around the emotional dimension of participation, particularly the lack of pressure to respond immediately (theme 4). While designers did not focus on the emotional aspects of asynchrony, students specifically identify this as a significant feature. However, for students, the reduced pressure is perceived as a new potentiality of asynchronous discussions. This emotional benefit, which allows students to engage without the stress of immediate responses, fosters a more relaxed and thoughtful approach to participation. This aspect adds a new dimension to the designers’ understanding of asynchrony and suggests that the emotional impact of flexible timing should be given more consideration in future design iterations.

Concerning written communication, designers identify it as a key affordance that facilitates reflective and structured engagement within discussions, but students' experiences introduce additional insights and challenges that were not fully anticipated. Designers characterise the ability to read and reflect on others' contributions (theme A) as an enablement, intending to make thinking processes more explicit and accessible for participants. This aligns with students' perspectives, as they view this feature as an enablement that allows them to easily access others' reasoning within the discussion. Also concerning theme B, students' and designers' perspective align since written communication serves as an enablement that supports participants in comparing others' reasoning with their own, fostering deeper understanding and self-reflection. Designers identify the potentiality of re-reading old contributions (theme C) as a significant feature of written communication, allowing participants to engage in metacognitive reflection. This potentiality is actively exploited by students, who highlight the advantages of being able to re-read contributions once the discussion has moved on or concluded. However, some students perceive a constraint in managing long written contributions, finding them challenging to navigate, an issue that was not fully considered in the design. This suggests a need for more user-friendly ways to manage extensive written content in future redesign.

Overall, the comparison reveals that while there are significant synergies between designers' intentions and students' experiences, some dissonances and new insights also arise. Students highlight new potentialities regarding the emotional benefits of reduced response pressure, but also new constraints, suggesting that further refinements could enhance the usability and effectiveness of asynchrony and written communication in DMD. Concerning the potentiality of having a transcript of the discussion identified by designers, just some students used it for further work even if most of them recognised the importance of having it for possible future uses. The ability to go back, review, and rethink is an attitude promoted through discussion and students should be taught how to exploit the potentiality of having at disposal the transcript of the discussion. In future experimentations this can be realised asking students to re-read and comment on some excerpts of their previous asynchronous discussion from the transcripts.

Chapter 6

Design and implementation of a platform for facilitating Digital Mathematical Discussion

The research project was carried out in collaboration with a company, as requested by the program PON "Research and Innovation" 2014-2020, Axis IV "Education and research for recovery" with reference to Action IV.4 "Doctorates and research fellowships on innovation" ; the company is Wonderful Education - Future Education Modena. This chapter presents part of the work carried out with the company and an intern from the Department of Computer Science at the University of Milan on the design and development of a prototype platform to support the implementation of Digital Mathematical Discussion (DMD). The design was informed by the research conducted and reported within this thesis.

During the design of the DMD didactic methodology (section 2.2), we observed that none of the existing digital technologies to our knowledge were adequately suited for implementing a DMD. This led, during the design phase of DMD, to the identification of various platforms to support its implementation. Specifically, we selected instant messaging platforms such as WhatsApp, Telegram, or Google Chat for the first phase of the DMD, while Padlet was chosen for the second phase.

Furthermore, the experimentations highlighted challenges faced by students in transitioning from the first to the second phase of the DMD. According to student responses in the final questionnaire, these difficulties were partly attributed to limited familiarity with the Padlet platform, the lack of notifications, and its overly dispersed graphical organisation. These observations suggested the need to develop an educational platform specifically designed to support the DMD.

The creation of this platform was divided into four phases:

1. Problem analysis (section 6.1), which involved examining existing educational and non-educational applications to map their functionalities for supporting DMD, as well as analysing the actors and system requirements;
2. System design (section 6.2), including the creation of wireframes for the application's screens and the design of the system's relational database;
3. Development of a prototype (section 6.3) for a minimum viable product, focused on the student side, to support the second phase of the DMD;
4. Prototype evaluation (section 6.4), consisting of functionality and usability testing.

Prior to these phases, a dossier was prepared outlining the objectives related to the platform's development, the target audience, an overview of institutional documents concerning the Italian school context to highlight how the platform for DMD aligns with expected competency development, and a benchmark study. Additionally, a user journey for the platform was created.

The author of this thesis collaborated with the company and the intern throughout all phases of the project, monitoring the process and ensuring that the platform's features aligned with educational needs. This interaction also served as a capacity-building opportunity.

6.1 Problem analysis

The problem analysis includes an examination of the system's users, a review of existing state-of-the-art solutions, and a requirements analysis.

The user research was based on the analysis of the responses provided by students in the final questionnaire (appendix C) of the DMD activity conducted during the first experimentation. The responses from students involved in the pilot study were excluded, as the platform's primary target is secondary school students and in the pilot study university students have been involved. The answers from students from the second experimentation were not considered because the user research was conducted in February 2024, prior to the conclusion of the experimentation. From the user research, the following characteristics of the student profile were identified:

- Preference of devices among students: Students favour smartphones over devices like computers and tablets. They cite reasons such as greater convenience and practicality for sending messages and photos, as well as the ease of accessing activities anytime, even when away from home;
- Digital environments most frequently used by students: Social media, video games, and communication platforms are commonly used for leisure and entertainment. For school tasks and educational purposes, students prefer collaborative and work-oriented platforms;
- Varied use of social media applications: Social media platforms like Instagram are utilised differently by students. Some use them for information and research, while others focus on interacting with peers or creating their own content, resulting in diverse usage experiences;
- Most frequently used applications: Instagram, YouTube, messaging apps (mainly WhatsApp), and Google services such as Gmail and Google Classroom are highlighted as the most commonly used applications by students.

From interviews with some of the teachers involved in the first experimentation, focusing on their needs and goals for teaching mathematics, the following insights were obtained:

- Challenges in balancing experimental activities with the pace of classroom work;
- Importance of giving every student the opportunity to contribute, share opinions, and express doubts about topics discussed in class to better understand individual challenges;
- Desire to stimulate students to question why phenomena occur and guide them towards solving problems through their own reasoning.

Following the user research, hypothetical user archetypes were created using Personas. Based on the findings of user research, four Personas were developed: two for students and two for teachers. The Personas were created following criteria provided by the company (Designers Italia). Each Persona was assigned an age, a name, and an adjective to describe their attitude. The profiles included the following elements:

- Background: Outlining the user's personality and lifestyle;
- Digital experience: Describing how the user interacts with and integrates technology into daily life;
- Primary digital devices: Highlighting the devices most frequently used;
- Interaction with the product: Explaining how the user might engage with the service and their motivations for doing so;
- Needs and challenges in mathematical discussions: Detailing the user's requirements and potential obstacles in achieving their objectives.

The Personas created were as follows:

- An outgoing and extroverted student: Eager to participate in group activities;
- A shy and reserved student: Technologically proficient, preferring to use a computer as their primary device;
- A tech-savvy teacher: Confident and experienced in leveraging technology for teaching and managing activities;
- A rather traditional teacher: Less experienced with technology and favouring a traditional frontal approach to lessons.

On the technological side, various identifying features of existing tools were examined, divided into two categories: applications designed for educational purposes and those not explicitly developed for education but offering features aligned with the required functionalities. These functionalities included enabling teachers to organise and manage multiple activities, conducting the first phase of DMD through group chats and submitting solutions, and implementing the second phase via thread creation. The analysis focused on identifying the strengths and utility of each application for supporting DMD. Between March and May 2024, a comprehensive review was conducted, assessing the functionalities available in these applications up to May 2024. Educational applications, such as Google Classroom[20], Padlet[21], and Moodle[22], were evaluated alongside non-educational tools like Instagram[23], Reddit[24], Tinder[25], Microsoft Community[26], and

[20] https://edu.google.com/intl/it/workspace-for-education/classroom
[21] https://padlet.com
[22] https://moodle.org
[23] https://www.instagram.com
[24] https://www.reddit.com
[25] https://tinder.com/
[26] https://answers.microsoft.com/

WhatsApp[27]. This process highlighted the primary features of each application, explored their potential for supporting both phases of DMD, and examined deeper into those that showed the greatest alignment with the required functionalities (the educational applications). A detailed account of this analysis for educational applications is provided in appendix E.

The requirements analysis identified the following features a system must satisfy to support the implementation of DMD activities as articulated in section 2.2:

- Creation of a virtual classroom by the teacher, allowing students to join and enabling the teacher to assign DMD activities;
- Creation of new DMD activities by the teacher;
- Creation of group chats by the teacher, where students can participate and share text, images, documents, and mathematical formulas;
- Ability for students to request teacher intervention within the group chat;
- Submission of the group-generated solution during the first phase of the activity within the application;
- Sending messages to one or more students by the teacher during the first phase;
- Teacher access to group chats with the ability to save specific messages;
- Student evaluation of group-submitted solutions at the end of the first phase, required before proceeding to the second phase;
- Creation of questions by the teacher for students to answer publicly during the second phase, allowing responses in text, images, or mathematical formulas. Both students and teachers must be able to reply to responses or provide quick feedback through reactions;
- Consultation of group-submitted solutions from the first phase during the second phase of the discussion;
- Access to all messages and materials shared during all phases of the activity, both during and after the DMD activity.

6.2 System design

The system design process began with design synthesis, in which user stories for students and teachers were created based on the requirements outlined earlier. Then, the application pages were designed, including their hierarchy and the user flow during navigation. Finally, the relational database was designed.

The design synthesis process is defined as "an abductive sensemaking process, where data collected during research is manipulated, organised, selected, and filtered to produce information and knowledge" (Kolko, 2009). The design synthesis was implemented using the Insight Combination method (Kolko, 2012), according to which an insight is described as a clear, deep, and meaningful perception of human behaviour in a specific design context, helping to identify

[27] https://www.whatsapp.com

needs or motivations and guide the design process. User data analysis led to clustering instances by shared themes and affinities. Considering the information from the problem analysis, insights were developed for each cluster via abductive inference, explaining the observed phenomena and the information within each cluster (Kolko). The following insights were derived:

- Students prefer smartphones for discussions rather than PCs, due to device availability and ease of interaction. However, some teachers prefer using PCs for a broader overview of discussions;
- Mathematics serves as a tool for reasoning, beyond mere calculations, and mathematical discussions promote relational understanding;
- While students can mechanically solve problems, they often lack comprehension of underlying theories. Mathematical discussions encourage them to articulate the theoretical rationale behind their solutions;
- Students often struggle with mathematical reasoning; reviewing past discussions helps them recall and understand better the content of the discussion;
- Teachers frequently lack time to integrate practical activities into lessons. Automated functionalities can assist in managing mathematics discussions effectively;
- Students' discussions are influenced by the teacher's presence, which can discourage participation due to fear of making mistakes. Highlighting when teacher intervention is actively needed and emphasising the activity's formative, non-evaluative aspect could help.

The generated insights were combined with existing design patterns to develop design ideas addressing the issues identified. These ideas suggested features and supplementary aspects that, while not integral to the core design, represented potential improvements and solutions to user needs. The design patterns considered focused on navigation, feedback, interaction, and engagement functionalities. A subsequent brainstorming session generated and refined design ideas, with selections made through a voting process involving the company. The chosen design ideas were:

1. Teachers receive an overview of students participating or not in the discussion and notifications when students request help;
2. A dedicated section for reviewing and exploring specific topics, with access to educational materials and/or chatbot interaction;
3. A chatbot trained in DMD discussions assists students in formulating thoughts and fostering understanding the reasoning behind the result;
4. At the end of each activity, students receive feedback on their engagement and content of the discussion.

To explore the possible actions users can perform within the application and to design the screens that allow seamless navigation, user navigation flows were developed using the user story technique (Di Pascale, 2019). User stories consist of brief descriptions that outline a user's needs concerning a specific feature, following these structures:

As a <role>, I want <action> so that <result>.

As a <role>, I want <action>.

The user stories created for students include:

- As a student, I want to register;
- As a student, I want to participate in the current activity;
- As a student, I want to communicate with my peers to collaborate on tasks and share opinions;
- As a student, I want to work simultaneously with my peers on the same project;
- As a student, I want to write mathematical formulas effectively to express my ideas more clearly and quickly;
- As a student, I want to share my work to showcase what I have done and make my reasoning immediately clear;
- As a student, I want to receive notifications to stay updated on ongoing events;
- As a student, I want to view, evaluate, and comment on the solutions submitted by other groups;
- As a student, I want to respond to questions posed by the teacher;
- As a student, I want to react to another student's response to quickly provide feedback on a shared message;
- As a student, I want access to all messages and materials shared during all activity phases, even after the activity ends, so I can review the shared work at any time;
- As a student representative, I want to submit the work completed by my group.

For teachers, the user stories are as follows:

- As a teacher, I want to register;
- As a teacher, I want to create an activity from scratch or using a template;
- As a teacher, I want to write mathematical formulas effectively;
- As a teacher, I want to upload an existing file to simplify activity creation;
- As a teacher, I want to create groups to divide students;
- As a teacher, I want a dashboard to have an overview of the groups and activity progress;
- As a teacher, I want to define keywords to simplify activity monitoring;
- As a teacher, I want to manage and access various groups to moderate activities and assist when needed;
- As a teacher, I want to save group messages to revisit and share them with the entire class;
- As a teacher, I want to set and adjust the timelines for the phases of the DMD activity;

- As a teacher, I want to create threads to initiate and organise collective discussions during the second phase of the DMD activity;
- As a teacher, I want to post a question in each thread for students to respond to, fostering discussion on the posed topic;
- As a teacher, I want to review the list of solutions created by students;
- As a teacher, I want to assign the creation of the final post in each thread to one or more students;
- As a teacher, I want to create a poll;
- As a teacher, I want to write a concluding description for the completed activity.

The objectives described in the user stories were subsequently translated into navigation flows, considering the DMD activity structure and the identified requirements.

The system was designed with students as the primary users and teachers as secondary users. Due to this focus and the limited resources available, the application's wireframes were created with high fidelity for students and medium fidelity for teachers. In the following the main screens of the application are reported (the screens are in Italian since the platform is currently available only in Italian). For students, the system includes a home page that features:

- A list of ongoing activities and access to the history of completed activities for the currently selected class;
- Access to the classes the student is enrolled in;
- Access to a section for managing profile-related actions.

Within the section dedicated to student classes, users can select a class to view its activities, join a new class, or view details about the classes they are already part of (Figure 6.1).

For each ongoing activity, students can access its respective phases. The section dedicated to the first phase of the DMD is designed as a group chat (Figure 6.2), where students can collaborate to solve the mathematical problem assigned by the teacher. In addition to standard chat functionality, this section includes features such as the ability to communicate with and request assistance from the teacher, use a collaborative whiteboard where group members can draw freely, use post-it notes to jot down ideas, and upload images from their devices. Students can also view the problem posed by the teacher, the list of group members, and the group representative at any time.

The group representative has access to the submission area, where they can deliver the group's solution by uploading a file from their device, sharing a photo taken in real time, or submitting the work completed on the collaborative whiteboard. Based on the developed design ideas, an area for discussion with a chatbot has also been incorporated.

Figure 6.1 Student's home page (left) and list of classes the student is enrolled in (right)

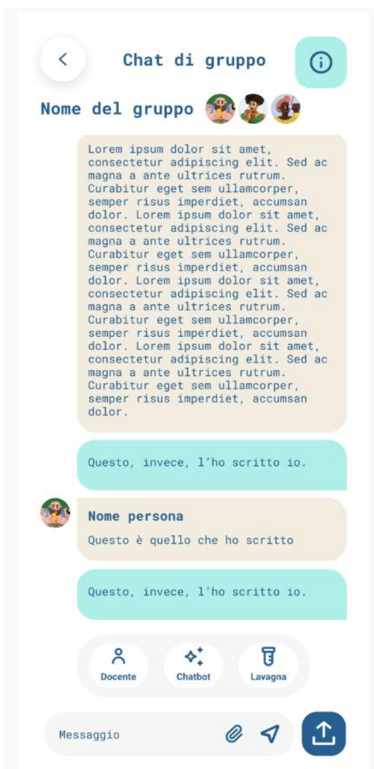


Figure 6.2 Group chat for the student user

Once the teacher completes the first phase of the DMD, students gain access to the section for the second phase, the collective discussion. Before entering this section, each student is required to evaluate the solutions submitted by other groups through a reaction and a comment. The section for the second phase of the DMD consists of several threads. The threads may vary in type: “runway of solutions”, “chat balance discussion”, and “solution balance discussion”. The first thread displays the solutions submitted by the groups at the end of the first phase, along with the reactions and comments left by the students during the evaluation process. This thread is view-only and cannot be modified (Figure 6.3).

Next, the threads created by the teacher are listed, where the teacher can ask a question regarding the work done in the previous phase. These threads are grouped under two main categories: “chat balance discussion” and “solution balance discussion”, based on the content of the question (Figure 6.4). Within the threads related to “chat balance discussion”, the teacher’s question focuses on the messages shared in the group chats during the previous phase. Threads related to “solution balance discussion” focus on questions about the solutions submitted by the groups at the end of the first phase of the DMD.

Students can respond individually to the teacher’s question by creating a post to share text and/or images. Users can also reply to an existing post or interact with it through reactions.

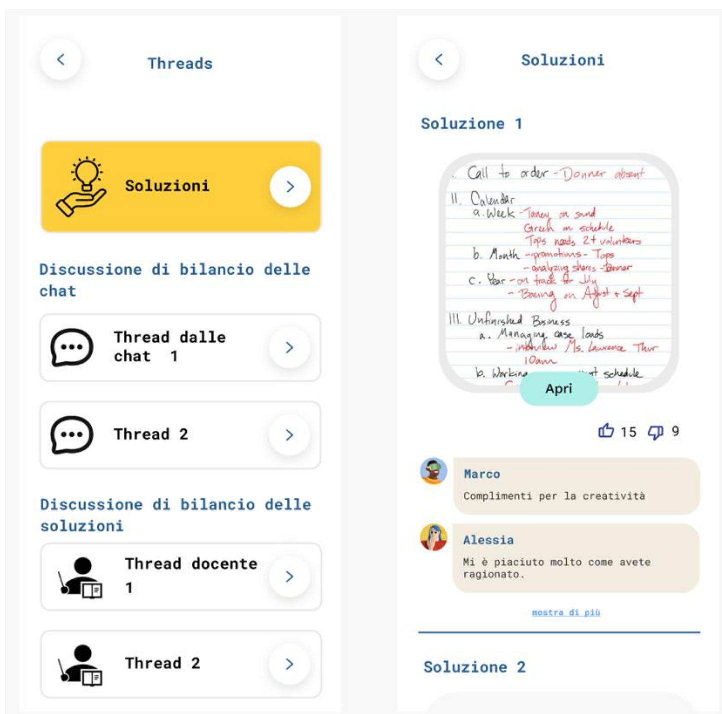


Figure 6.3 Menu related to the threads section for the second phase of the DMD (left) and the solution threads (right) for the student user

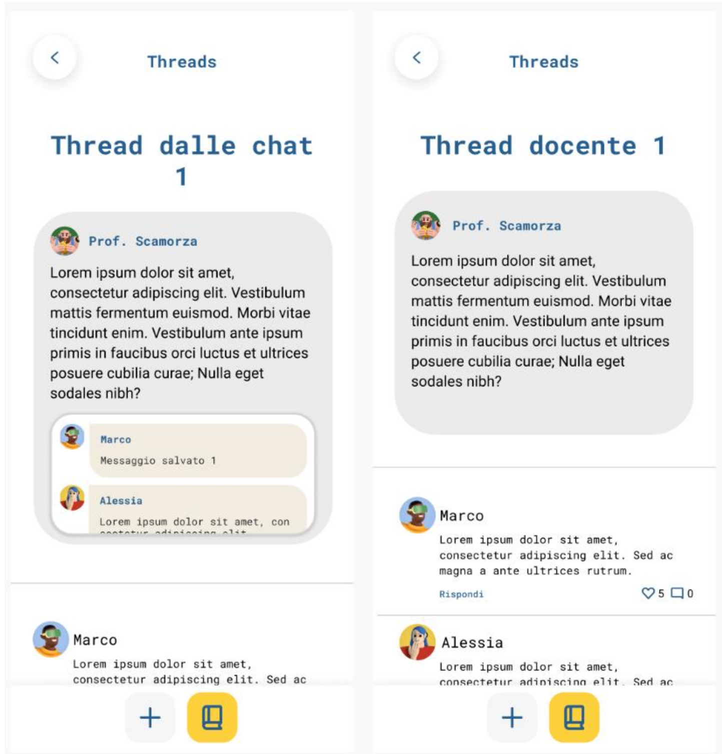


Figure 6.4 “Chat balance discussion” thread (left) and “solution balance discussion” thread (right) for the student user

The display of past activities is similar to what has been described so far, with the difference that each of them is only accessible in read-only mode and cannot be modified.

Regarding the teacher’s role, the system has been designed for desktop use. This decision was made based on information about the devices most commonly used by teachers and due to the greater volume of information that needs to be displayed and the number of actions available to the teacher. The home page includes a view of the classes the teacher is part of and the option to create new ones (Figure 6.5).

For each class, it is possible to view its details, the list of current activities, and the list of completed activities. Within the details page, the teachers and students of the class can be viewed, along with the code that allows others to join. The teacher who leads the class can also add and remove other teachers from the class and designate another teacher as the class owner (Figure 6.6).

Within the current activities page (Figure 6.7), it is possible to view the listed activities as well as create new ones (Figure 6.8). Creating an activity involves: entering the title and description; adding any attachments; defining the groups for the first phase of the DMD; setting the timelines for both phases of the activity; optionally creating work steps for the students, where the assigned work can be divided into various parts with deadlines for each; and defining keywords to be used as parameters for monitoring the activity. The activity can be created from scratch, starting with a completely empty template, or selected from preconfigured templates in the catalogue; in the latter case, the title and description in the template are already defined and can be modified if needed.

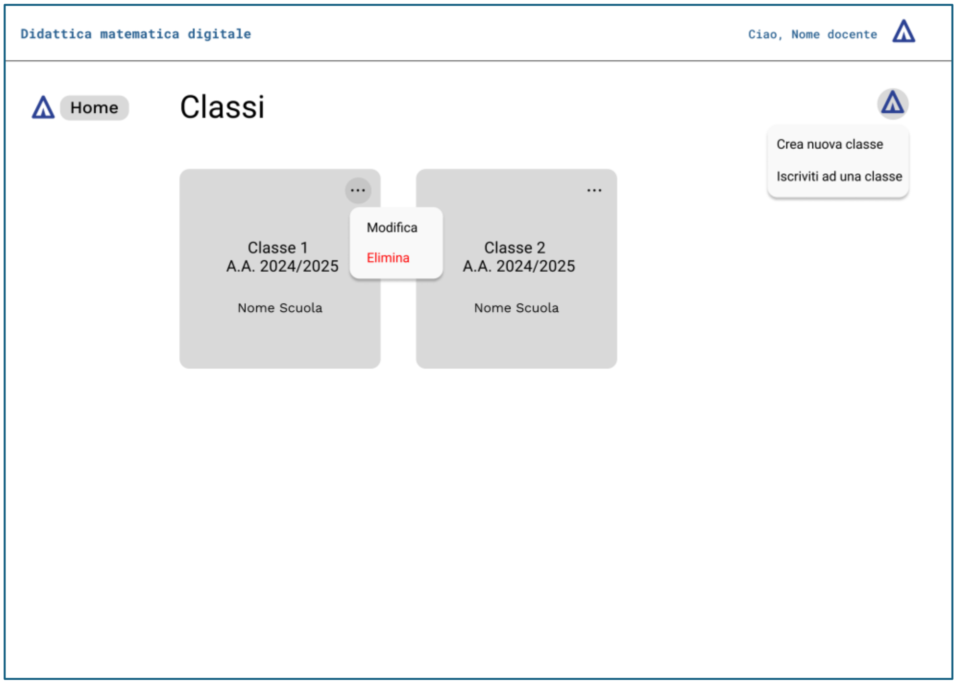


Figure 6.5 Home page for teacher user

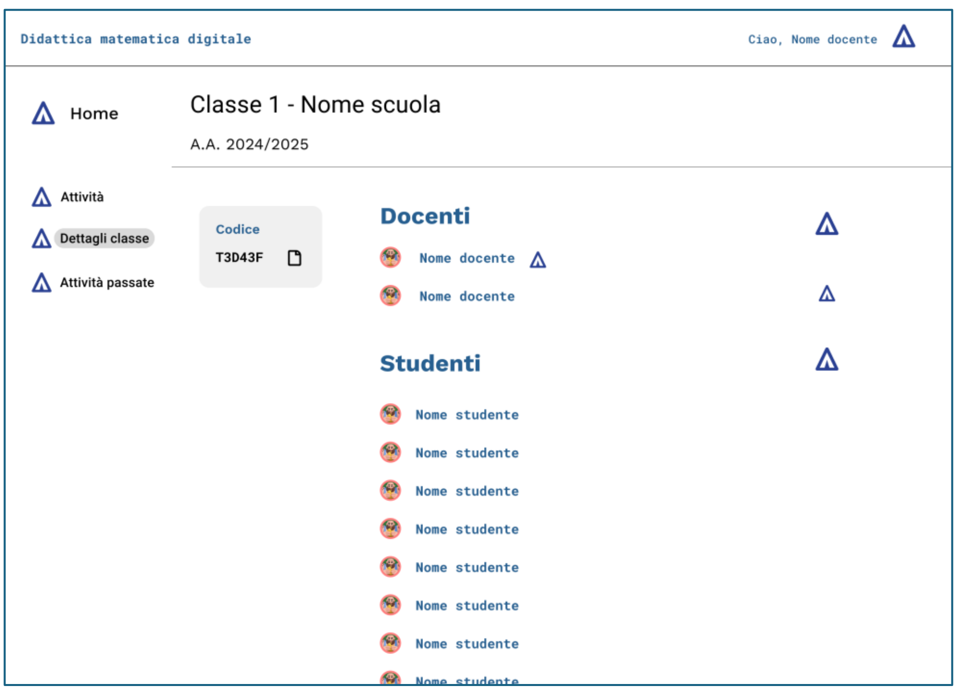


Figure 6.6 Class details for the teacher user

Figure 6.7 Current activities page for the teacher user

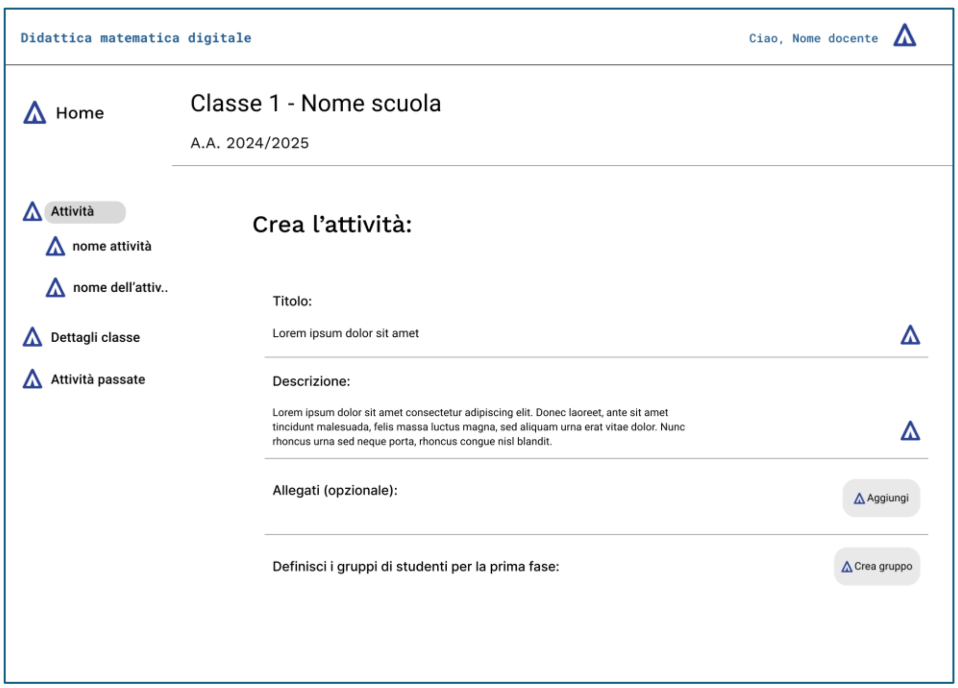


Figure 6.8 Create a new activity page for the teacher user

For each ongoing activity, it is possible to access the section that allows its control and management. During the first phase of the DMD, the management section enables the teacher to: view and access the group chats; view the solutions submitted by the groups; view and receive notifications related to the management of the activity; send a message to one or more students and/or one or more groups; modify the activity and interrupt the current phase (Figure 6.9).

Figure 6.9 Management section of the first phase of the activity for the teacher user

During the second phase of the DMD, similarly to the student role, the teacher can access and interact within the various threads. Additionally, the teacher can: access the group chats from the first phase; create specific guiding questions to assign to one or more students within each thread; create a new thread; modify the activity and finish it, provided they enter a concluding description of the activity (Figure 6.10).

Figure 6.10 Section for managing the second phase of the activity for the teacher user

The viewing of past activities follows the same approach as for the student role, allowing access and viewing of these activities in read-only mode.

Finally, the system's database was designed and developed. The company chose Supabase[28] as the application for its creation. Supabase is a platform for creating and managing relational databases based on PostgreSQL. The database design did not include managing data related to group chats. This decision was made for two main reasons: first, the primary focus of the development of the platform was on implementing and testing the section for the second phase of the DMD process; second, the company considered that more efficient and specific solutions for managing this type of data, such as NoSQL databases, could better optimize performance and scalability. The database design was both conceptual and logical.

The conceptual database design included the identification of the following entities and relationships for representing a database capable of storing and managing the necessary information for conducting a DMD activity:

- Activity: contains information about the DMD activity. An activity is designed by a teacher and assigned to a class, and it can include multiple work steps for students. During the second phase, the activity also contains the various threads created by the teacher. The activity records the following: title, description, start and end dates, end date of the first phase, any attachments, concluding description, keywords used for monitoring, status (active or completed), and the phase in which the activity currently resides (group phase for the first phase or collective phase for the second phase);
- Scenario: represents a preset activity template; each scenario has a title and description that are inherited as default values when creating an activity;
- Activity step: represents the work steps for students within an activity. Each step is characterised by an identifier, the activity it belongs to, title, goal to be achieved, and any deadline for completion;
- User: represents the system's users through generalization, dividing them into students and teachers. For each user, the system stores the email, password, username, and any profile image. Each user can create or react to a post within a thread;
- Student: a specialization of the user entity, represents users with the student role. A student can belong to none or several classes and none or several groups within a class. A student may be assigned guiding questions by the teacher. Additionally, every student is required to comment on the solutions of each group at the end of the first phase of the DMD;
- Teacher: a specialization of the user entity, represents users with the teacher role. A teacher can create a class, design an activity, create threads for the second phase of the activity, and assign guiding questions within those threads. Each teacher is assigned to one or more classes;

[28] https://supabase.com

- Class: represents a group of users consisting of students and teachers, to which various activities can be assigned. For each class, the following is stored: section, school, academic year, and the invitation code that allows users to join;
- Group: represents the student groups for the first phase of the activity. Each group is characterised by a title, the activity it participates in, and the student representative of the group. Each group belongs to a specific class and can only be composed of students from that class;
- Solution: includes the solutions submitted by groups at the end of the first phase of a specific activity. Each group must submit two solutions: the problem's solution and the conceptual map representing the group's problem-solving process. For each occurrence, the type of solution delivered, the group that submitted it, and the file containing the solution (which may be an image or a PDF) are stored. Additionally, for the problem solution, the reactions and comments added by students during evaluation are also stored;
- Solution comment: represents the comments of students on the problem solutions;
- Thread: represents, through generalization, the threads that make up the second phase of the activity. They are divided into two main categories, the "chat balance discussion" and the "solution balance discussion". Each thread contains one or more posts. For each thread, the title and the activity it belongs to are recorded;
- Post: represents the posts and responses to posts within the threads. Users can react to posts with likes or reply to an existing post by creating another post. For each post, the system stores any text and/or images it contains.

The translation of the conceptual design into the relational model involved restructuring the design, eliminating generalizations and multivalued attributes present within it. The generalization of the User entity was removed by replacing the generalization with identifying associations. This decision was made due to the numerous relationships the child entities participate in, the common attributes they share with the parent entity, and the significant differences between them. The generalization of the Thread entity was removed by eliminating the child entities, as there are no associations directly involving these entities. The multivalued attributes keywords and attachments in Activity were transformed into the Keyword entity, identified by the Word attribute, and the ActivityAttachment entity, with file and id attributes, identified by the latter.

6.3 Development of a prototype

The implementation of the system involved the development of a minimum viable product focused solely on the section of the application that manages the second phase of a DMD for the student role. The implementation was carried out by the intern Luca Molinari from the Department of Computer Science at the University of Milan.

The developed minimum viable product partially differs from the wireframes presented in the previous section, particularly regarding the functionalities included. It is important to emphasise that the wireframes, being high-fidelity representations of the final application, include additional functionalities not present in the minimum viable product, as the latter focuses on a smaller set of essential features. The main differences between the minimum viable product and the wireframes are also due to the following reasons: the limited time available for system development and the changes made following the results of functional testing, as described in the next section. The functionalities shown in the wireframes that are currently not present in the minimum viable product include: the ability to hide and show responses to a post or solution; the display of messages selected by the teacher and used to create their post in the "chat balance discussion" threads; the ability to assign specific guiding questions to one or more students within the threads; the presence of notifications; and the ability to easily write mathematical formulas within a post.

Figure 6.11 shows the main page of the section dedicated to the second phase of the DMD, while Figure 6.12 illustrates the thread "runway of solutions", containing the solutions submitted by the groups at the end of the first phase of the activity. The thread view pages for "chat balance discussion" and "solution balance discussion" are shown in Figures 6.13 and 6.14, respectively. Finally, Figure 6.15 shows the page for creating a new post. The data displayed within the screenshots relates to the test carried out with users, as described in the next section.

Figure 6.11 Main page of the section dedicated to the second phase of DMD

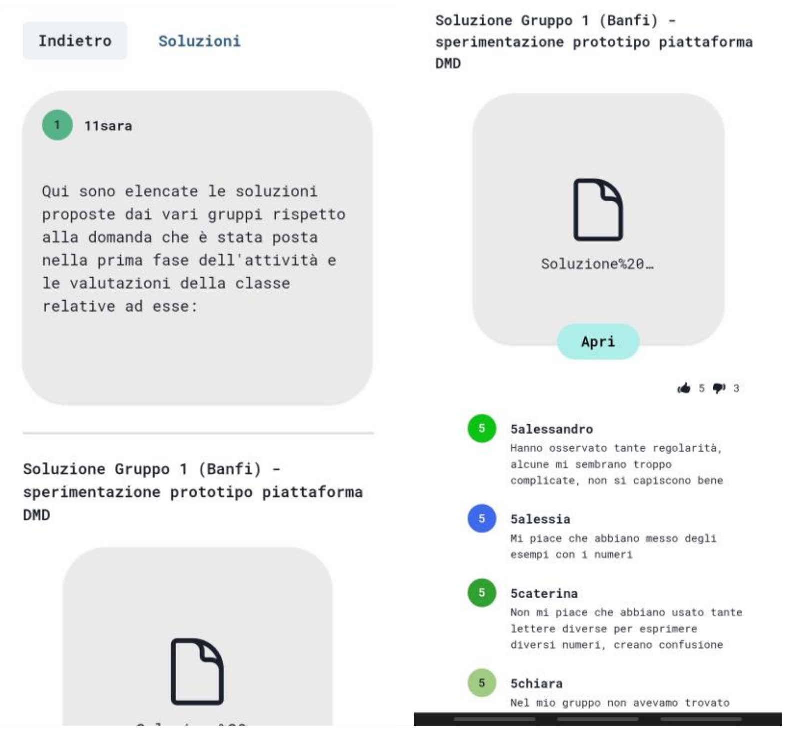


Figure 6.12 Thread "runway of solutions" with an opening post from the teacher (left) and a solution with its respective reactions and comments (right)

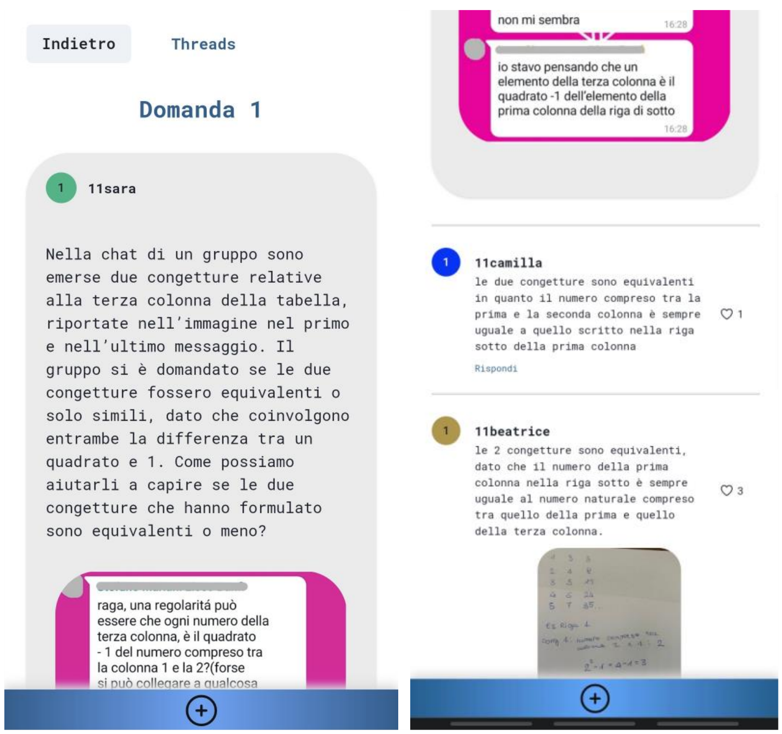


Figure 6.13 Thread "chat balance discussion", with the corresponding post from the teacher (left) and the student posts in response (right)

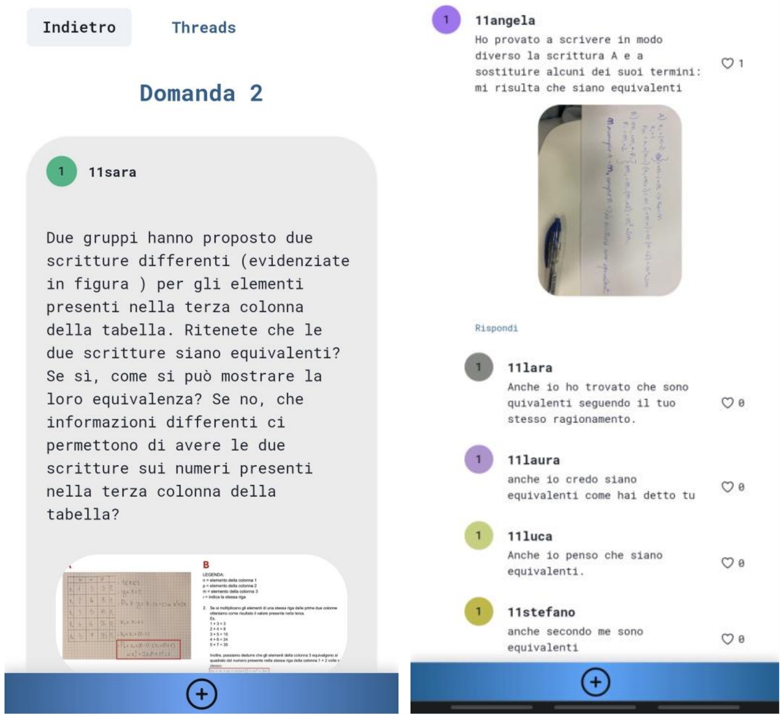


Figure 6.14 Thread "solution balance discussion", with the corresponding post from the teacher (left) and a student's response post (right), including the related comments by peers

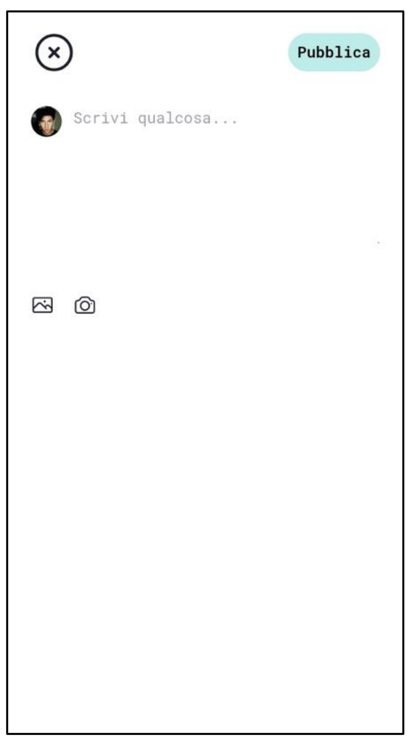


Figure 6.15 Page for creating a new post

6.4 Prototype evaluation

At the conclusion of the work carried out by the intern, an experimental evaluation of the developed minimum viable product was conducted, consisting of two tests: a functional test, aimed at verifying the correct operation of the system, and a usability test, conducted with end users to assess their satisfaction and the overall experience.

The experimentation of the developed minimum viable product first involved a functional test to verify whether it was working correctly, identify technical errors, and detect potential improvements. The test was carried out by two members of the research team using their personal smartphones, on which the web application was run through a web browser. This allowed for simulating a usage context similar to the one for which the application was designed.

As already mentioned, the developed minimum viable product focused solely on the section of the application managing the second phase of a DMD, namely the collective discussion phase. Therefore, for the functional test, it was necessary to manually insert fictional data concerning the first phase of a DMD into the database.

The test involved checking the following aspects:

- Login, using credentials for fictional users created before the experiment;
- Viewing the solutions submitted by the groups at the end of the first phase and the information shared within each thread;
- Viewing attachments and the information of a post within a thread;
- Creating and publishing a new post within a thread, sharing text and/or images (which could either be taken at the moment using the camera functionality or selected from the device gallery);
- Responding to an existing post within a thread, using text and/or images;
- Reacting to an existing post within a thread.

During the test, any technical errors were monitored, and participants' comments were noted regarding the issues encountered and potential improvements. The research team members who took part in the experiment also provided suggestions and opinions on usability and performance aspects, which they considered particularly important. Data was collected regarding the outcomes of the tasks, the technical errors identified, and the participants' comments, to pinpoint the system's limitations and ensure that it was ready to be tested by end-users in a real usage context. The reported errors and comments were prioritized based on the severity of the issues for the system's functionality and their impact on user experience.

The results of the functional test were positive for all assigned tasks, with no errors that would prevent their resolution. Below the identified issues are reported, listed in order of priority:

- The automatic updating of content within a thread is not implemented, meaning the user must manually refresh the page to see new posts;

- When creating a post, multiple photos can only be attached if they are selected from the device's gallery. Uploading a photo taken at the moment via the camera functionality removes any previously selected photo;
- The system is not designed to remember the user's position within a thread's page: after creating a post, the user is automatically brought back to the top of the thread page, not to the point where they were before posting.

The following comments on usability and performance aspects were made for possible improvements:

- The footer on the thread page partially covers the reply button on the last post, making it difficult for the user to complete the action;
- The button to create new posts within a thread is not very visible;
- The position of the icon indicating the number of likes on a post is not aligned with the position of the like icon on a solution. Also, its size, as well as that of the reply button, are too small. The icon for indicating the number of replies to a post is redundant and confusing, especially with the presence of the reply button;
- It is suggested to display the comment being replied to within the post creation screen, so it could be consulted at any time;
- A slowdown in loading and publishing posts containing several photos is observed.

However, due to the time constraints between the two tests and technical difficulties in addressing some issues, only some of these problems have been already addressed: the footer in the thread page was modified and highlighted so that it no longer covered the reply button on the last post, and the button to create new posts became more visible; the reaction icons for solutions were aligned with the like icon for a post; additionally, the size of the latter and the reply button was increased; the icon indicating the number of replies to a post was removed.

Following the functional test, a usability test was conducted with end-users to assess user satisfaction and experience, and to gather qualitative feedback in order to understand the limitations of the developed application and how it could be improved in the future. The participants in the test were nine high school students who had volunteered and who had previously participated in earlier DMD experimentations carried out by the research team. The students were informed that the aim of the experimentation was to test the minimum viable product developed for the second phase of the DMD. For each student, profiling data was collected through a questionnaire, covering various aspects related to the student's profile, such as: their level of participation in classroom and group activities; how they use social media applications; the devices most commonly used for leisure and academic activities; and the devices used in previous DMD experiments. Each participant was identified by a unique code, from P1 to P9, in sequential order. Seven participants stated they actively engage in class and group activities, while P4 and P6 mentioned they only intervene if they are confident in their response. All participants indicated they use social media apps at least to communicate and write with friends and family. Specifically, P1, P4, and P6 use WhatsApp for this purpose, while P1 and P6 also stated they do not use any other social media apps. Additionally, P2, P4, and P7 use social media apps for leisure activities, P8 for posting content, and P3 and P5 for research and information gathering. Regarding device usage, all participants reported using a computer for academic activities (except P2, who uses a tablet) and using a smartphone for leisure and communication. As for the devices used in previous DMD experiments, all participants except P9 used their smartphones. P1, P2, P3, and P5 also used computers and tablets. P9 used only a computer.

To test the minimum viable product, which includes only the functionalities related to the second phase of the DMD, the students first completed the first phase of the DMD in an online and asynchronous manner, similar to the previous DMD experimentations conducted by the research team. The author of this thesis acted as the teacher during the test. The first phase of the DMD lasted three days. The students, divided into three groups, tackled a mathematical problem in group chats. The groups were given the option to choose between WhatsApp and Telegram. Two groups chose WhatsApp, while the third group preferred Telegram, as it allowed them to participate without sharing their phone numbers. The solutions created by each group were sent via email to the teacher in the experimentation. Data from the first phase of the DMD, such as the composition of the groups and the solutions submitted, were manually entered into the database, and a fictitious profile was created for each student to allow login to the application. The second phase of the activity was carried out using the developed minimum viable product and took place in-person at the Department of Mathematics at the University of Milan. Although the second phase of the DMD is typically asynchronous, it was decided to conduct it in person in order to monitor the students during their interaction with the minimum viable product and provide technical support if needed. The students participated in the collective discussion via the minimum viable product using their personal smartphones, on which the web application was accessed through a browser. The collective discussion was structured by teacher. Three questions related to the work done during the first phase of the DMD and the solutions produced were posed to the students. The questions were organised into three threads according to their content: two in the "solution balance discussion" category and one in the "chat balance discussion" category. The students had one hour to respond to the questions posed in the threads and to reply and react to the posts created by other participants. To best simulate the second phase of the DMD, students were asked not to interact verbally during the minimum viable product test. During the test, students did not encounter any technical difficulties that required intervention. At the end of the second phase of the DMD, students were asked to complete a System Usability Scale questionnaire (Brooke, 1996) and respond to some open-ended questions about the tested minimum viable product, reflecting on any difficulties encountered, doubts, and suggestions for potential improvements. The System Usability Scale questionnaire was used to gather data regarding user satisfaction and the overall usability of the system. The responses were analysed according to the methodology of the questionnaire, and the results were compared with standard benchmarks. Qualitative feedback from the participants was used to gain a more in-depth understanding of their experience with the developed system.

The experiment showed an overall positive evaluation from the users, but also revealed some flaws and missing features, as well as various aspects that could be improved. The responses to the System Usability Scale questionnaire indicated good general usability of the system, with an average score of 83, compared to the acceptable threshold of 68. Regarding the qualitative feedback, all participants generally stated that the developed system was simple, clear, and intuitive, and that they did not encounter difficulties in participating in the proposed activity or interacting with other students. P4 mentioned that they appreciated the ability to upload images taken in real-time using the camera feature, as they find it easier to write mathematical formulas on paper. P2 and P8 found the platform useful for comparing their opinions with those of other students. Additionally, P2 believed that the application could be helpful for students who find it more difficult to interact with others.

Regarding the comparison with the Padlet platform for carrying out the second phase of the DMD, all participants, except for P9, stated that they preferred the prototype platform. The main reasons cited by the students for this choice were greater simplicity and intuitiveness in use, and a clearer, less confusing organization of content. P5 and P7 also mentioned that the ability to reply to a comment was a key reason for their preference, stating that this was not possible on Padlet. Additionally, P5 highlighted the ease of adding and using reactions to posts, while P7 found it useful to be able to view the solutions submitted within the same workspace. P9 expressed a preference for Padlet over the minimum viable product developed, citing greater convenience and intuitiveness in its use, the positioning of elements (such as buttons), and a better overall view of all the content.

The current solution developed has shown promising results during the tests conducted and could provide a solid foundation for the future development of a system dedicated to managing and implementing DMD. However, as of now, the project completed as part of this thesis does not yet constitute a definitive solution to the problem of creating a system that allows for the full execution of DMD. Although the system design is largely defined, its implementation and development have only been partially completed, leading to some limitations. Specifically, the following issues remain unresolved: automatic content updating within a thread to view new posts without needing to manually refresh the page; inability to attach multiple photos to a post during its creation unless all are selected from the device gallery; saving the user's position within a thread so they are brought back to the exact point they were at before posting; ability to view the comment being replied to within the post creation screen; and the excessive slowness when loading and publishing a post containing multiple photos.

Furthermore, the developed minimum viable product does not include all the features initially planned during the design phase. The features that were designed but not developed include: the ability to hide and show responses to a post or solution; displaying the messages selected by the instructor and used for creating the instructor's post in the "chat balance discussion" threads; the ability to assign specific guiding questions to one or more students within the threads; notifications to alert users about the creation of a new post within a thread and to notify them of likes and comments received on their posts; and the ability to easily write mathematical formulas within a post. Finally, for future development, consideration has been given to allowing additional file types (beyond PDF and images) to be submitted as part of the solutions generated by the groups at the end of the first phase of the activity, such as Word documents. In future developments, we aim to develop the designed platform and to extend its features.

Chapter 7

Conclusions and future developments

In this chapter, we present the results obtained from the research conducted in this thesis (section 7.1). The conclusions are drawn from the data analysis presented in chapter 5 and discussed with respect to the research literature. They are articulated following the research questions that guided the study (section 4.1). In the second part of this chapter (section 7.2), we outline the future developments stemming from the research conducted.

7.1 Discussion and conclusions

As presented in section 1.4, the research reported in this thesis aims to explore the opportunities that digital technologies offer to enhance the teaching and learning of mathematics through collaboration and communication beyond the traditional classroom setting (Ball & Barzel, 2018; Ruthven, 2012). With respect to this theme, there are several challenges to address, which became particularly evident through the experiences of remote teaching and learning during the Covid-19 pandemic. Indeed, teachers often used digital technologies somewhat "naively", merely as tools to replicate their usual classroom practices online (Baccaglini-Frank, 2022). This approach failed to fully leverage the potential of these technologies to foster active student engagement, leading to a predominance of a transmissive teaching approach (Aldon et al., 2021). Thus, we decided to address these challenges by exploring the opportunities for conducting mathematics teaching and learning activities through digital technologies, with a focus on the didactic methodology of mathematical discussion. As presented in section 1.2, mathematical discussion is a didactic methodology that actively involves students in building mathematical knowledge through interaction with peers and experts. This approach provides an ideal context for examining how digital technologies can enable new forms of communication and collaboration beyond traditional classroom environment. While recent research studies have focused on the use of digital technologies to support synchronous mathematical discussions (e.g., Cusi et al., 2017; Baccaglini-Frank, 2022; Giberti et al., 2022, 2024), we chose to focus on the potential of digital technologies in mediating asynchronous mathematical discussions. This choice is tied to the opportunity to conduct mathematical discussions across extended time and space through the support provided by asynchronous modality and written communication. The asynchronous modality allows participants to engage with the discussion at their own pace, providing the flexibility to reflect, process information, and contribute thoughtfully (Wang, 2023). Written communication, in turn, enhances clarity and permanence, and effective writing can foster a deeper understanding of mathematics (Santos & Semana, 2015). Additionally, this type of discussion generates a transcript that participants can revisit after the discussion has progressed or concluded (Wang, 2023). Research on asynchronous mathematical discussion remains an understudied emerging area within the field of mathematics education, as highlighted in the systematic review presented in subsection 1.3.2, in particular considering its development in high school contexts.

In order to address the challenges and potentialities of asynchronous interactions and of their integration with synchronous activities, we introduced and investigated the didactic methodology of Digital Mathematical Discussion (DMD). In chapter 2, we introduced the design of DMD, which draws on principles from both mathematical and asynchronous discussions and was developed considering the modern needs of students immersed in digital culture. Although we have introduced a new didactic methodology for discussion, we do not intend for it to replace traditional mathematical discussion. Instead, we aim for it to complement and integrate with the traditional approach, adding value to it. DMD is structured in three phases: group-work within chats to address a mathematical problem, whole-class balance discussion within Padlet, and wrap-up classroom discussion. The first two phases are developed in asynchronous modality through the support of digital environments, while the third phase is realised synchronously and in-person. For the scopes of the research reported in this thesis, we focused mainly on the first two phases of DMD, as we intended to deepen the investigation of the role of asynchronous interactions within mathematical discussions, as previously mentioned. In particular, we focused on the collaborative processes of co-construction developed by students and their intertwining with the development of collective mathematical thinking within the asynchronous phases of DMD. As regards the analysis of collaborative processes, we referred to the dimension of the social modes of co-construction within the Weinberger and Fischer's (2006) framework (subsection 1.3.1). As regards the mathematical content of the discussions, we relied on previously tested tasks (sections 3.3 and 3.4) that highlight the use of algebra as a thinking tool and enable students to explore numerical situations, formulate conjectures through the support of examples, and construct proofs for the formulated conjectures through the use of algebraic language. Concerning the analysis of the development of collective mathematical thinking, we focused on epistemic aspects of algebraic thinking processes (section 3.2), encompassing transformations between representation registers (Duval, 2006), the activation and coordination of conceptual frames (Arzarello et al., 2001) and anticipating thoughts (Boero, 2001). The investigation of these aspects was guided by two research questions:

(RQ1) What kind of students' social modes of co-construction are fostered in the digital environments (chat and Padlet) facilitating a DMD?

(RQ2) Do students develop collective algebraic thinking within the asynchronous phases of DMDs? If so, how do the social modes of co-construction intertwine with the epistemic aspects of algebraic thinking in students' collective co-construction of algebraic thinking?

Moreover, we also aimed at investigating students' perspectives on their experience with the didactic methodology of DMD focusing particularly on asynchrony and written communication as key features (Trouche, 2004) of DMD. This investigation was guided by the following research question:

(RQ3) How do students perceive asynchrony and written communication as key features of DMD in their articulation as constraints, enablement, potentialities and affordances after experiencing them?

In order to answer the research questions, we conducted a research using a Design-Based Research methodology (Cobb et al., 2003; Design-Based Research Collective, 2003). After a pilot study with university students familiar with traditional mathematical discussion, we designed and implemented two cycles of experimentations, involving four grade 9 classes. In this chapter, we present the main results of this research and the conclusions of the thesis, based on the analyses of the data collected during the pilot study and the two experiments described in chapter 5.

In the following subsections, we present the results, discussions and conclusions related to the themes addressed by the research questions: subsection 7.1.1 focuses on students' social modes of co-construction; subsection 7.1.2 centres on students' development of collective co-construction of algebraic thinking; and subsection 7.1.3 reports on students' experience on asynchrony and written communication as key features of DMD. Finally, in subsection 7.1.4, we report some concluding remarks.

7.1.1 Results related to students' social modes of co-construction within a DMD

In order to address the first research question and investigate the kinds of students' social modes of co-construction fostered within the digital environments facilitating a DMD, we analysed, through the dimension of the social modes of co-construction (Weinberger & Fischer, 2006), the transcripts from the chats and the Padlets collected during the pilot study and the first and second experimentations.

In the first phase of DMD, the group-work within the chats revealed varying approaches to interaction and task completion among students. Some groups focused primarily on fulfilling the assignment's requirements, emphasising organisational aspects and progressing without deeply engaging with one another's ideas. In contrast, other groups prioritised developing a shared understanding, considering each other's proposals and building on one another's ideas. Specifically, the analysis of the collected data within this research revealed three categories with respect to the collaborative processes activated in the groups:

- Category A: chats are characterised by low participation by students, with a prevalence of interventions belonging to the social modes of co-construction of externalization (EX), when introducing new proposals concerning the solution of the problem under discussion without referencing others' ideas, and quick consensus building (QC), to move on with the discussion reaching an agreement on how to proceed on organizational aspects but without deepening participants' reflections upon the mathematical content under discussion.
- Category B: chats are characterised by the presence of one or more leaders, who stimulate the discourse within the group. The leaders typically propose new solutions or ideas related to the problem under discussion through EX interventions and encourage the other participants to intervene through elicitation (EL) interventions. The other participants primarily engage with the group by using EL interventions to ask clarifying questions about the leaders' proposals. These EL interventions stimulate, in turn, integrated-oriented (IC) and conflict-oriented (CC) consensus building.
- Category C: chats are characterised by the active participation of most group members, engaging with a full variety of social modes. The discourse within the group unfolds

through the whole range of social modes and it reveals a genuine co-construction of the solution of the problem under discussion.

The results regarding the dynamics of students acting as resources for one another in chat-facilitated collaborative problem-solving align with the research conducted by Hurme & Järvelä (2005). In that study, students were encouraged to use networked technology for collaborative problem-solving and engaged in similar practices: they commented on each other's notes, provided alternative solutions, and supported one another's inquiry projects.

Concerning the second phase of DMD, the quantity and heterogeneity of students' participation were not satisfactory. The limited number of students actively participating in the discussion can be attributed to various blocking factors mentioned by the students themselves, such as affective factors or constraints imposed by the digital environment in which the DMD activity was conducted. The limited contributions observed in our study align with findings by Hew and Cheung (2012), who emphasised the impact of specific constraints in online discussions, such as technical issues within digital environments or difficulties arising from a lack of understanding about how to contribute effectively to the discussion. Moreover, a balance discussion facilitated by Padlet is not "linear", as a traditional verbal discussion in the classroom or a chat-facilitated discussion, since it allows the interweaving of "simultaneous" voices (Giberti et al., 2024). This condition is challenging for students and can hinder participation if they are not used to it. In future research, we will investigate whether and how the challenge of simultaneous voices in discussions can be addressed through education on "multi-voice" discourse to both students and teachers. With respect to the variety of social modes of co-construction, we identified the predominance of EX interventions, aimed at presenting one's proposal without relating to the other peers' contributions. Moreover, the differences detected in students' quantity of participation and the kinds of social modes activated can be attributed to the background of the classes. Some were more accustomed to active practices, such as discussions or group-work, while others were more familiar with transmissive teaching methods, which led to difficulties in participating effectively in discussions.

7.1.2 Results related to students' collective co-construction of algebraic thinking processes within a DMD

Concerning the second research question, in order to investigate students' development of collective algebraic thinking within the asynchronous phases of DMD, we analysed the transcripts from the chats and Padlets of the first and second experimentations with respect to the activation and coordination of conceptual frames (Arzarello et al., 2001) and anticipating thoughts (Boero, 2001). We use the term "collective" to refer to the thinking processes developed by a group of students through the joint construction of ideas and understanding of concepts, based on each other's proposals.

The development of collective algebraic thinking differed in the two asynchronous phases of DMD. In fact, the categories identified for the group-work in the chats with respect to the social modes of co-construction intertwine also with the collective algebraic thinking processes developed in the groups. For instance, chats in category A are characterised by individual thinking processes by students who propose the EX interventions. Thus, the activation and coordination of conceptual frames and anticipating thoughts are developed individually and the proposals are discussed little, if at all, by the group, so they do not become a shared and collective resource. In chats from categories B and C, individual thinking becomes shared for collective development. Collective thinking is evident when a student's contribution refers to another student's input, either by continuing a line of reasoning or building a new argument in response to their peer's ideas. In category B chats, the epistemic aspects of algebraic thinking are first developed individually by the leaders and then explicitly shared within the group, typically in response to EL interventions. In category C chats, the creation and articulation of thinking processes are collective, and the epistemic aspects of algebraic thinking are shared and co-constructed by the students. In conclusion, regarding the development of collective algebraic thinking within the group-work in the chats, we can deduce that it relates to the social modes of co-construction that occur within the groups, given the clear characterisation between the development (or lack) of collective thinking processes and the categories of the chats in relation to the collaborative dynamics.

Concerning the development of algebraic thinking within the Padlet in the second phase of DMD, we can conclude that students' interventions highlighted the sharing of individual thinking processes, which remained as such since there is no evidence of co-construction of algebraic thinking. As in the case of the chats, also for the Padlet the development of collective algebraic thinking is related to the collaborative dynamics developed between students, that flatten to EX interventions in the case of Padlet. Specifically, it can be noted both from category A of the chats and the unfolding of the Padlets that poor interaction among students characterised by the prevalence of EX and QC interventions is often associated with a lack of collective development of algebraic thinking. On the other hand, the presence of a varied interaction with different social modes of co-construction also stimulates the sharing of individual thinking processes (through EL interventions) and the collective development of algebraic thinking (through IC and CC interventions). Moreover, a significant difference between the categories of the chats concerning the epistemic aspects and the interaction is that they all begin with EX interventions where epistemic aspects are personal, and the difference lies in how these contributions are received by the rest of the group (taken up and developed in categories B and C). As already discussed in subsection 5.2.3, the background of the classes involved can influence the processes activated by students, this aspect will be further explored in future research.

Regarding the identified categories for group chats, we do not consider them exhaustive. We plan to verify and refine this categorization in future studies, adapting them as needed in light of future experimentations.

7.1.3 Results related to students' experience on asynchrony and written communication as key features of DMD

The analysis of students' responses to the online questionnaire, conducted after their experience with DMD, enabled us to address the third research question. We examined students' reflections on the asynchronous modality and written communication in DMD, identifying both alignment and divergence in how the key features of DMD were characterised by students and designers.

Regarding the asynchronous modality, the analysis of students' responses allowed us to identify four relevant themes.

Theme 1 (flexibility in work management) emphasises how asynchrony supports students at an organisational level by offering flexibility. It enables them to participate in discussions at their own convenience and manage their contributions alongside school activities and personal commitments. However, some students noted challenges, such as procrastination, forgetting to respond, or losing the line of reasoning when delays occurred. The ideas emerging in theme 1 aligns with the designers' view as an affordance, allowing students to engage at their own pace. However, students add layers to this perspective, describing flexibility as both an enablement (supporting participation despite other commitments) and an affordance (shaping their interaction with the discussion). At the same time, this flexibility introduces challenges such as procrastination, forgotten responses, and difficulty maintaining the flow of reasoning which are constraints not fully anticipated by designers.

Themes 2 (availability of time to reflect on one's own contribution) and 3 (availability of time to reflect on others' contributions) focus on the ability to explore concepts over time. Students appreciated having ample time to reflect on their own reasoning, revisit their contributions, and better understand others' ideas due to the lack of time pressure. Themes 2 and 3 align with designers' perspective. Students value the extended time to reflect, revise their ideas, and better understand others' contributions, confirming the potential of asynchrony to enhance deeper reasoning and learning.

Theme 4 (lack of pressure to respond immediately) highlights the reduced pressure to respond immediately, a potential benefit recognised by students but less emphasised in the design. Students' responses reveal a more nuanced interaction with the feature of asynchrony, providing insights that extend beyond the designers' initial expectations. The emotional dimension related to this theme highlights a divergence between students' and designers' perspectives. While designers only partially considered emotional aspects, students emphasised the reduced pressure to respond immediately as a significant potentiality. This emotional relief fosters a more relaxed and thoughtful participation, adding a new dimension to the understanding of asynchrony.

Concerning written communication in DMD, designers highlight its role as a key feature for fostering reflective engagement, a perspective confirmed by students' experiences.

Specifically, students declare that written communication enables them to access and reflect on others' reasoning (theme A), making thinking processes more explicit and facilitating understanding. It also supports the comparison between one's own and others' reasoning (theme B), encouraging deeper self-reflection and insights. Moreover, it gives students the opportunity to exploit the possibility to re-read contributions (theme C) for metacognitive reflection, allowing them to revisit ideas and track the development of the discussion even after it has progressed or concluded. However, an unanticipated constraint emerged, as some students found long written contributions difficult to manage, suggesting a need for user-friendly strategies to handle extensive content in future designs.

Furthermore, students reflected on the combined features of asynchrony and written communication, emphasising two key aspects: the possibility to use written contributions at their own pace to develop individual reasoning and the availability of a written transcript of the entire workflow for all participants at any time. Regarding the first aspect, students noted that writing their reasoning allowed them to reflect and refine their contributions, a process they felt would not be possible with vocal messages or calls. For example, they appreciated the ability to review and correct errors before sending a message, which added flexibility and convenience compared to real-time communication. Concerning the second aspect, many students found the availability of transcripts beneficial for clarifying unclear issues, supporting subsequent classwork, reviewing the entire activity, or transitioning between phases of DMD.

7.1.4 Concluding remarks

The research process and the results obtained within the research reported in this thesis touch upon the five features characterising the Design-Based Research methodology (section 2.2) that guided the design of the DMD didactic methodology. The introduction and implementation of DMD in upper secondary schools represented an interventionist methodology aimed at introducing an innovative practice compared to traditional teaching methods (feature 2). Throughout the various cycles of iterative design (feature 4), students from diverse contexts and backgrounds were involved, encompassing both classes from grade 9 and university students. As a consequence, the findings reported in the previous subsections already stem from varied contexts, suggesting the potential generalizability of the results to new cases (features 1 and 5). Moreover, the research had both practical and theoretical impacts (feature 3).

On the practical side, the outcomes can be grouped into three areas. First, the teachers of the classes directly involved in the first and second experimentations gained direct experience with both traditional and digital methodologies for mathematical discussion. Reflections of these teachers after the experiments revealed that participating in activities conducted using innovative methodologies helped them uncover new possibilities and directions for classroom work. Second, the experiences and research results highlighted some difficulties teachers faced in adopting the methodologies for mathematical discussion, both traditional and digital. This realization prompted an initial study on creating content and methodologies for pre-service and in-service teachers' professional development. The aim is to involve teachers directly in mathematical discussion scenarios, as suggested approach within the MathTASK project[29] (Biza, 2023). Third, the research informed the design of a platform to support Digital Mathematical Discussions (chapter 6).

On the theoretical side, the analysis of data collected during the experimentations enabled the development of new theoretical knowledge about the intertwining between the social modes of co-construction and the development of collective algebraic thinking processes by students within asynchronous contexts of mathematical discussion. In fact, regarding the first phase of DMD, we identified varying interaction dynamics between students. In some cases, the interactions were more effective, with students striving to develop a shared understanding and actively considering and building on each other's proposals.

The limited participation, in both quantity and heterogeneity, observed in both the first and second phase of DMD, and the general flattening of social modes of co-construction observed in the Padlet (second phase of DMD) can be re-read in relation to the typical learning practices (Bini, 2023) of Web-based online discussions (Jenkins, 2006, 2009), as introduced in Gagliani Caputo, Bini and

[29] https://www.uea.ac.uk/groups-and-centres/a-z/mathtask

colleagues (2024). In fact, even if Web-based online discussions are not conceived as educational settings, they generally become informal learning environments and share structural similarities with the asynchronous phases of Digital Mathematical Discussions, allowing participants to share knowledge in online environments in written form, interacting with peers and experts, and contributing at their own pace and convenience. Despite their structural similarities, the two types of discussions are, in reality, quite distinct. Web-based online discussions, although they can function as informal learning environments, typically arise spontaneously. As such, participant engagement in these discussions cannot be equated with that of DMDs, which are conducted within a school setting with institutional educational objectives. For this reason, it is reasonable to expect that the epistemic needs (Kidron et al., 2011) of participants in DMD are stimulated and activated through the teacher's prompts. The epistemic needs in Web-based online discussions are the users' desire to develop a thorough understanding of the topic addressed within the discussion and are activated by shared digital content (Bini, 2024). These epistemic needs may or may not be internalised by the students within DMDs. The lack of internalisation is evident in the contributions of students who fail to co-construct collective knowledge, likely due to an incomplete devolution of the task. Therefore, simply providing students with an online environment for mathematical discussion is not enough to foster typical learning practices of Web-based online discussions. This highlights the importance of the teacher's role in guiding mathematical discussions, even in digital formats, in educational settings to achieve specific learning objectives and effectively stimulate students' epistemic needs in order to develop mathematical thinking.

Finally, it should be noted that the discussion of the results in this thesis does not refer to an extensive literature. On the one hand, this is because research in mathematics education concerning asynchronous discussions rarely characterises discussion in the same way as this thesis, for instance most studies report a lack of teacher guidance or set the objective of the discussion at a general level, such as developing critical thinking skills or stimulating reflection (see the findings of the systematic review presented in subsection 1.3.2). This implies that the activities realised as asynchronous discussions within other contexts in most cases cannot be compared to the ones reported in this thesis, as the objectives and implementation of the discussions differ a lot.

On the other hand, studies that do characterise mathematical discussion as a didactic methodology aimed at developing mathematical thinking through the interplay of students' and teachers' voices often do not deepen the development of students' mathematical thinking processes. Some of these research studies focus, for example, on students' interactions and inclusion in discussions (Giberti et al., 2022, 2024), without focusing on the mathematical thinking processes developed within the discussions. In our perspective, it is not sufficient to investigate a mathematical discussion solely based on modes of participation, as we consider participation in terms of students' engagement in mathematical discourse. Therefore, the analysis of a mathematical discussion should include its mathematical content and the educational objective linked to its implementation.

7.2 Future developments

The research conducted and reported within this thesis suggests new roads for investigation, laying the groundwork for future studies.

The first area of future investigation might concern the observation and analysis of long-term processes involved in the development of both in-person and asynchronous mathematical discussions. From the data analysis reported in chapter 5 and, in particular, from the distribution of groups' across the identified categories reported in Table 5.11 (subsection 5.2.1), it seems emerging a negative trend in the groups' dynamism within the categories. As already reported in subsection 5.2.3, this trend suggests the need to acknowledge and address the challenge faced in classrooms to promote effective long-term processes related to the development of mathematical thinking. From its origins, mathematical discussion has been tied to the achievement of long-term goals, referred to the overarching object of the collective activity (Bartolini Bussi et al., 1995). Thus, further research on DMD should consider class observation and experimentations across several months in order to investigate how the characteristics of the discussion evolves and how, if so, students get accustomed to the practice of discussion, both traditional and digital. Furthermore, comparing in-person and asynchronous discussions over the long-term would be relevant to observe if and how patterns of interaction or knowledge co-construction present in traditional settings shift when moving to a digital environment. From the data already available within this research and from new experimentations, the investigation of the relationship between the dynamics and the processes developed within in-person classroom discussions and within asynchronous ones should be deepened. We hypothesised that traditional and Digital Mathematical Discussions should be realised in synergy, in order to promote more inclusive class practices. At this stage, we have conducted an a-priori analysis of inclusion through the integrated use of traditional and digital discussions (subsection 2.3.2). However, this hypothesis requires further investigation by a-posteriori analysing students' progress. This would help determine whether the different formats in which discussions are designed effectively accommodate diverse learning styles and preferences in discussion modalities. Moreover, the integrated use of traditional and Digital Mathematical Discussion can accommodate also teachers' needs and challenges in designing and orchestrating discussions.

The second area of future studies is the exploration of the scaffolding provided by the teacher to promote DMDs. The need to further explore this aspect has emerged at various points throughout this research. First, teacher's interventions within the chats and Padlets remained limited within the experimentations already conducted. Only within the 3rd DBR cycle of the research the teacher started intervening in the chats to encourage effective collaborative processes for the development of collective mathematical thinking in groups whose spontaneous collaborative dynamics were not effective. Concerning the Padlet, instead, the teacher posted the guiding questions from the 1st DBR cycle. From the 2nd DBR cycle, the teacher started sending reminders to students to encourage their participation on Padlet and, in the 3rd DBR cycle, started orchestrating the "rhythm" of the discussion posting the guiding questions over time. However, their interventions within Padlet apart from the guiding questions remained limited. In recent experiments, we observed greater and more effective student participation in collective discussions on Padlet when the teacher appropriately stimulated engagement. This includes encouraging students to check the Padlet at suitable times without leaving too much time before engaging with the discussion and relaunching on previous interventions, particularly from a mathematical perspective. These findings suggest that teachers need to pay more attention to the "rhythm" of discussion in asynchronous contexts compared to face-to-face settings. This is also linked to the "simultaneous" nature of discussions enabled by Padlet (Giberti et al., 2024), as opposed to the linear progression in time and space typical of in-person discussions or those facilitated by chats. Thus, we aim to further explore the presence and the role of the teacher within the different phases of DMD with their increased participation. Moreover, the transition from chat to Padlet has proven to be a delicate passage. We aim to explore the types of tailored scaffolding that a teacher can implement in a DMD to facilitate the shift from group-work to collective discussion. This tailored scaffolding should also take into account the fact that in in-person settings, routines and social habits have been established and consolidated over time. In contrast, digital environments lack these established routines. Therefore, participation needs to be more actively encouraged by the teacher in the initial involvement of students with DMD to help it evolve into a new interaction routine for the class. Focusing more on the teacher's role in DMDs also suggests the need to further explore the relationship and balance between the spontaneous dynamics developed by students and the scaffolding provided by the teacher.

The third area of future investigation concerns the teachers' involvement in our studies to explore the role they play in DMDs. Another aspect related to the presence and role of the teacher in the research conducted is that the experiments were carried out by researchers experienced in mathematical discussion, rather than by classroom teachers. In the case of the pilot study, these two roles coincide: the teacher of the course in which DMD was tested is also a researcher of the team and an expert in mathematical discussions. However, for the first and second experimentations, the role of the teacher in leading the discussions was carried out by the author of this thesis, as detailed in subsection 4.2.2. The choice to give researchers the role of the teacher in discussions was preferable for their knowledge of mathematical discussion, thus reducing the variabilities in teachers' orchestration of discussions. However, the teachers were present both in traditional mathematical discussions and in DMDs carried out during the experimentations. They collaborated with the researcher in order to design the balance discussions, highlighting relevant aspects to be discussed for their classes, and were free to intervene in addition to the researcher conduction of discussions. In future experimentations, we will involve more actively the teachers in the design and orchestration of their own traditional and Digital Mathematical Discussions in order to investigate more also their role, considering its evolution within long-term experimentations. Moreover, some peculiar features of DMD, as the availability of a transcript of the whole asynchronous discussion, can represent a new resource for the teacher to be used beyond the discussion with new didactic aims (Albano et al., 2021). The presence and the role of the classroom teacher in orchestrating Digital Mathematical Discussions also emerge as relevant aspects to consider within research in mathematics education since the systematic review on mathematical asynchronous online discussions highlighted that in online and asynchronous contexts discussions are generally unfacilitated or with limited participation of the teacher. Also other relevant research projects have identified the role of the teacher as significant. It is the case, for instance, of the DIST-M project, introduced in subsection 1.2.2, in which the online discussions unfolding following a storytelling started as unfacilitated by the teacher but evolved including the teacher in the digital story (Albano et al., 2024) and with specific roles of guidance within the activities (Albano et al., 2021).

The fourth are of future studies concerns the design of professional development programs aimed at fostering teachers' development of competencies for the implementation of the methodology of mathematical discussion. The meaningful involvement of teachers in future experimentations, where they participate actively, can only be achieved if they are adequately trained in the methodology of mathematical discussion. Our interest lies not in analysing how different teachers (both novices and experts), provided with the same formation, internalise and handle their discussions. This raises the question of how to effectively train teachers in mathematical discussions, which often teachers struggle to manage. Stein and colleagues (2008) identified, for example, difficulties in teachers' degree of control over class situations in which students can propose numerous different strategies for solving a task or opinions. Similarly, our experience during the research conducted for this thesis highlighted that traditional, lecture-style teachers' professional development sessions did not effectively enable teachers to grasp the key concepts of the didactic methodology of mathematical discussion and to effectively face the challenges that may arise. This was evident from the reflections shared by the teachers involved in the research during the experiments. Thus, first, it is necessary to deepen our understanding of the perspectives that teachers develop regarding the methodology of mathematical discussion following traditional, lecture-based courses. Since our experience suggests that teachers' views on mathematical discussion can differ significantly from the concept of mathematical discussion defined by Bartolini Bussi (1996) and considered in this research, we believe it is essential to design appropriate teachers' professional development pathways. We initiated preliminary teachers' professional development sessions for pre-service and in-service teachers, where mathematical discussion was not only introduced in a lecture format but also included practical activities. These activities immersed teachers in situations specifically designed by trainers to bring them closer to the methodology of mathematical discussion. The idea of creating more practical professional development courses stems from the MathTASK project (Biza, 2023), where teachers are engaged in contextualised classroom situations through tasks, and relates to previous research experience linking theory and practice related to teachers' professional development, involving teachers in designing mathematical classroom discussions (Cusi & Malara, 2015) and creating fictional mathematical classroom discussions (Cusi & Morselli, 2018). Inspired by these approaches, we have attempted to recreate contextualised classroom situations in the form of discussions. The design and implementation of these teachers' professional development pathways, which involve the active participation of teachers in traditional and Digital Mathematical Discussion contexts, will be further explored in future studies.

The research conducted suggests that the fifth area of future studies should concern an investigation aimed at advancing in the design and development of a platform to support DMD. Many elements still need to be designed, beyond those identified in the already completed user journey, and developed, beyond the prototype already produced. In particular, we believe that with the increasing spread of artificial intelligence, AI tools can be incorporated to facilitate teachers' management of the content shared by students during Digital Mathematical Discussions. Specifically, textual outputs could be processed to generate insights that assist teachers in ease managing discussions.

Finally, the sixth area for future studies concerns the theoretical implications of the investigation of the relationship between the individual and the collective dimensions of mathematical thinking processes. Over the course of this research, in fact, the need for a theoretical shift to support this exploration emerged. We have conceptualised the development of collective thinking as the co-construction of understandings, based on the consideration and integration of participants' contributions to the discussion. The need of focusing on the relationship between the individual dimension and the collective dimension is evident in the progression of data analysis: initially, the pilot study data were analysed with a focus on individuals. The analysis then shifted to identifying the dynamics students employed to position themselves as resources for their peers (subsection 5.1.1) in group-work via chat. In the analysis of the first and second experiments, we examined how individual contributions supported the development of collective thinking (section 5.2). This shift is also visible in the tables we reported in chapter 5, where we moved from counting individual interventions within groups (Tables 5.2 and 5.3) to representing the distribution of groups based on social mode categories at the class level (Table 5.11). Specifically, our perspective shifted from examining how an individual fits into and contributes to the community to understanding how the community itself is shaped by individuals. This change evolved naturally and deliberately and it highlights the necessity to investigate further the interplay between individual and collective dimensions within the development of mathematical discussions.

As these future developments suggest, the design of DMD, as developed and presented in this research, is not in its final form. It can undergo further redesign and modifications to refine and update DMD in response to future experiences.

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Appendix

Appendix A: summary tables of the articles analysed within the systematic review on mathematical asynchronous online discussion

In the following we report the summary tables outlining the key information in articles analysed within the systematic review on mathematical asynchronous online discussion reported in subsection 1.3.2. In the tables, the columns are articulated by authors (in alphabetic order), target of the research, educational context, objectives of the discussion, modality of discussion, role of the teacher and digital technology and its use in supporting the discussion.

Table are divided by publication type: journal articles are covered in Table A.1, and conference proceedings articles are covered in Table A.2.

Table A.1 *Journal articles: the complete list of selected publications with relevant information for the themes of the review*

Author(s)	*Target*	*Educational context*	*Objective of discussion*	*Modality of discussion*	*Role of the teacher*	*Digital technology and its use*
Fernandez et al. (2020)	Prospective secondary mathematics teachers	Online distance mathematics teacher education program	Sharing narratives on their teaching and giving/receiving feedback on them	Online asynchronous discussion	Gives feedback to written narratives	Moodle forum – sharing and giving feedback to written narratives
Galleguillos & Borba (2018)	In-service mathematics teachers	Online continuing education course	Posing a modeling problem and resolving contradictions generated by opposing views	Online asynchronous discussion combined with online synchronous work	Scarcely intervenes within the discussion	Closed group on the social network Facebook – publishing posts
Giberti et al. (2022)	Primary school students (grade 6)	Hybrid classes combining in-person and online synchronous and asynchronous activities	Comparing alternative solutions to the same mathematical problem solved individually by students	Online asynchronous discussion and in-person discussion	Orchestrates the in-person discussion	Padlet – collecting students' ideas and solutions in posts (text, pictures, audio) and comments (text)
Hollebrands & Lee (2020)	In-service mathematics and statistics teachers	Online professional development MOOC	Communicating about what they are learning in the course and sharing additional information and resources	Online asynchronous discussion	Discussion forums are mostly unfacilitated	Moodle forum or forum from a Google Coursebuilder platform – publishing and answering posts

Table A.1 *Journal articles: the complete list of selected publications with relevant information for the themes of the review*

Author(s)	*Target*	*Educational context*	*Objective of discussion*	*Modality of discussion*	*Role of the teacher*	*Digital technology and its use*
Kurianski et al. (2022)	University mathematics students	In-person classes and/or online classes	Collaborating on assignments, explaining and commenting exercises, asking and responding about course content	Online asynchronous discussion combined with online or in-person synchronous work	Academic discussions are mostly unfacilitated	Discord – text channels for community building VoiceThread – voice and text based discussion forum for peer collaboration
Livy et al. (2022)	Prospective primary mathematics teachers	Hybrid classes combining online synchronous and asynchronous activities	Accessing and comparing alternative solutions to the same mathematical problem solved individually by students	Online discussion combined with online synchronous work (follow-up webinar)	Uses students' posts as a stimulus to encourage participants to post additional questions and discuss a range of possible answers	Padlet – recording multiple responses in different formats (files, photos,…) in an anonymous way
Matranga & Silverman (2022)	In-service secondary mathematics teachers	Online teacher education course in a mathematics education master's program	Supporting the development of mathematical content knowledge, sharing aspects of mathematical reasoning, revising others' mathematical discourse	Online asynchronous discussion	Supports the emergence of more sophisticated sociomathematical norms (e.g., asking to explain and justify mathematical reasoning)	Threaded discussion forum (no platform specified) – text based communication and archival of text in a public space

Table A.1 *Journal articles: the complete list of selected publications with relevant information for the themes of the review*

Author(s)	***Target***	***Educational context***	***Objective of discussion***	***Modality of discussion***	***Role of the teacher***	***Digital technology and its use***
Muir (2022)	Prospective primary mathematics teachers	Online classes in a teaching degree master's program	Making students' thinking visible, encouraging explanation, reasoning and justification	Online asynchronous discussion	Provides different stimulus questions to respond to and to interact with students	Online forum (no platform specified) – publishing and answering posts
Padayachee & Campbell (2022)	University engineering students	Online classes	Developing skills in asking questions and assessing responses in order to promote deeper learning	Online asynchronous discussion	Facilitates the discussion forums and supports students by regularly scanning the discussion forums for posts, raising issues, commenting on and directing the discussion	Online discussion forum (no platform specified) – posting questions and videos challenging students to engage with the course content
Santos-Trigo & Reyes-Martinez (2019)	Prospective secondary mathematics teachers	Hybrid classes combining in-person and online asynchronous activities	Sharing ideas, contrasting others' approaches, asking clarification questions and looking for different ways to solve a problem	Online asynchronous discussion combined with previous synchronous discussion	Not specified	Padlet – posting individual ideas and approaches and receiving feedback from other participants

Table A.1 *Journal articles: the complete list of selected publications with relevant information for the themes of the review*

Author(s)	***Target***	***Educational context***	***Objective of discussion***	***Modality of discussion***	***Role of the teacher***	***Digital technology and its use***
Shaker et al. (2023)	University engineering students	Hybrid classes combining online synchronous and asynchronous activities	Annotating a shared document and replying to other peers and teachers' annotations to increase students engagement with pre-class reading	Online asynchronous discussion	Not specified	Persusall – making annotations on a shared document and replying to annotations posted by peers and teachers
Taranto et al. (2020)	In-service mathematics teachers	Online asynchronous MOOC (Massive Online Open Course) for professional learning	Sharing and commenting on ideas and reflections on the course content, but also on participants' materials or didactic experiences	Online asynchronous discussion	Includes specific questions to be answered or a title that serves as discussion thread	Moodle forum – posting reflections or materials guided by a question or a theme and commenting on others' posts

Table A.2 *Conference proceedings articles: the complete list of selected publications with relevant information for the themes of the review*

Author(s)	***Target***	***Educational context***	***Objective of discussion***	***Modality of discussion***	***Role of the teacher***	***Digital technology and its use***
Biton et al. (2015)	Prospective university students	Online asynchronous activity	Supporting students' self-study posing and answering questions. Both teachers and peers answer questions	Online asynchronous discussion	Monitors the discussion forum and answers to students' questions	Groups on the social network Facebook – posting and answering questions both uploaded as images or text
Biton et al. (2016)	Prospective primary mathematics teachers	Hybrid classes combining in-person and online asynchronous activities	Answering pedagogical questions related to the solution of a problem	Online asynchronous discussion	Not specified	Paths forum
Gagliani Caputo et al. (2023)	Prospective primary mathematics teachers	Hybrid classes combining in-person and online asynchronous activities	Collaboratively solving a problem and confronting and evaluating other groups' solutions	Online asynchronous discussion combined with in-person work (wrap-up discussion)	Designs the collective discussion with initial guiding questions and boosts it with subsequent interventions	Closed groups of the social network Telegram – sending written messages and images Padlet – answering to guiding questions and to other participants' contributions
Geraniou & Crisan (2019)	University mathematics education students	Online asynchronous course	Collecting ideas from course readings, reflecting on points of agreement or disagreement, and exploring their application in learning contexts	Online asynchronous discussion	Contributes to online discussion	Moodle forum – posting ideas and questions

Table A.2 *Conference proceedings articles: the complete list of selected publications with relevant information for the themes of the review*

Author(s)	***Target***	***Educational context***	***Objective of discussion***	***Modality of discussion***	***Role of the teacher***	***Digital technology and its use***
Grønbæk & Winsløw (2015)	University students with a minor in mathematics	Online asynchronous course	Supporting students' self-study posing and answering questions on the course content	Online asynchronous discussion	Monitors the discussion forum and answers to students' questions	Discussion forum (no platform specified) – posting and answering questions or comments
Jazby & Symons (2015)	Primary school students (grade 5)	Hybrid classes combining in-person and online asynchronous activities	Collaboratively solving problems	Online asynchronous discussion	Introduces the problem and reviews the produced solutions	Edmodo – sending message board posts including text and uploading files
Koichu & Keller (2017)	High school students (grade 10 and 11)	Hybrid classes combining in-person and online asynchronous activities	Exploratory discourse for long-term problem solving	Online asynchronous discussion combined with in-person activities (preparatory tasks and closing discussion)	Discussion is unfacilitated	Closed groups on the social networks Google+ or WhatsApp

Table A.2 *Conference proceedings articles: the complete list of selected publications with relevant information for the themes of the review*

Author(s)	***Target***	***Educational context***	***Objective of discussion***	***Modality of discussion***	***Role of the teacher***	***Digital technology and its use***
Kontorovich (2018)	High school students (grade 9)	Hybrid classes combining in-person and online asynchronous activities	Collaboration on homework assignments, sharing and addressing questions and difficulties on course content	Online asynchronous discussion	Discussion is unfacilitated	Social network (no platform specified)
Koppitz & Schreiber (2015)	Prospective primary mathematics teachers	Hybrid classes combining in-person and online asynchronous activities	Supporting students' self-study posing and answering questions on the course content	Online asynchronous discussion	Monitors the discussion forum and answers to students' questions	Discussion forum (Ilias platform) – posting and answering questions in anonymous way
Panero et al. (2017)	In-service mathematics teachers	Online asynchronous MOOC (Massive Online Open Course) for professional learning	Sharing the design and analysis of a mathematical situation and giving and receiving feedback on it	Online asynchronous discussion	Discussion is unfacilitated	Social network Viaéduc – posting comments, creating and publishing, commenting, recommending, sharing documents Padlet – writing on a collaborative board
Postelnicu et al. (2023)	University mathematics students	Hybrid classes combining in-person and online asynchronous activities	Feedback on the individual solutions of each participant	Online asynchronous discussion	Discussion is unfacilitated	Blackboard forum – writing and commenting other participants' solutions

Table A.2 *Conference proceedings articles: the complete list of selected publications with relevant information for the themes of the review*

Author(s)	***Target***	***Educational context***	***Objective of discussion***	***Modality of discussion***	***Role of the teacher***	***Digital technology and its use***
Symons & Pierce (2017)	Primary school students (grade 5)	Hybrid classes combining in-person and online asynchronous activities	Collaboratively problem solving	Online asynchronous discussion combined with in-person work (teacher's comments)	Discussion is unfacilitated (in in-person work reviews the solutions and discuss online collaboration)	Computer supported collaborative learning environment (no platform specified)
Taranto et al. (2017)	In-service mathematics teachers	Online asynchronous MOOC (Massive Online Open Course) for professional learning	Sharing materials, crowdsourcing and idea generation	Online asynchronous discussion	Discussion is unfacilitated	Forum – reading and answering to messages, using nested replies Padlet – sharing materials (images, videos, documents, texts) on common tasks Tricider – brainstorming and voting

Appendix B: didactic path on proof in arithmetic through the use of algebraic language

In the following we report the didactic path on proof in arithmetic through the use of algebraic language introduced in section 3.3 articulated as experimented within this research.

Initial test of the path to introduce algebraic language for proving

Name and surname:..

1) Translate the following statements into algebraic language:

A) The triple of a number ....

B) The square of a number ....

C) An even number ....

D) The double of an even number ....

E) The opposite of an even number ....

F) Two consecutive even numbers ....

G) The product of two consecutive numbers ....

H) The successive of the square of a number ....

I) The square of the successive of a number ....

2) Translate the following propositions expressed in algebraic language into natural language:

$(k+2)\cdot 3$ ....

k^2-1 ....

$(k+1)^2$ ....

3) Identify, among the following expressions, those equivalent to 2k+4. Then express the observed equivalences in verbal language.

(2k+2)+2	2(k+4)
2(k+1)+1	2(k+2)

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4) What do you think the numbers represented by these expressions have in common?

$3 \cdot 5a$ $\quad$ $(h + k) \cdot 3$ $\quad$ $3h + 3b$ $\quad$ $9d$ $\quad$ $\frac{6k}{2}$

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5) Complete the blanks, explaining the reasoning you used to arrive at the result:

If $u = v + 3$ and $v = 1$ then $u =$………..

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If $e + f = 8$ then $e + f + g =$………..

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If $n - 246 = 762$ then $n - 247 =$………..

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6) Consider the expression $2n + 4$. Write down its meaning.

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Write for which values of n (n naturale) the integers of the form $2n + 4$ become:

- Multiples of 2
  ………………………………………………………………………………………
- Multiples of 4
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Activity 1: Translation, analysis of variable expressions, and analysis of statements

1) Translate the following statements and procedures into algebraic language:

A. *An even number*
B. *An odd number*
C. *Add 5 to the successive of the square of a multiple of 5*
D. *Subtract 2 from a three-digit number*
E. *Multiply the successive of a number divisible by 7 by a two-digit number*

2) Translate the following sentences into algebraic language, which express the equality between two expressions. Then, verify whether the stated equality is correct. If it is not, identify the mistake.

A. *The difference between the square of a number and the number itself is equal to the product of the number and its preceding.*
B. *By adding 3 to the double of the preceding of a number, you obtain the next even number of an even number.*

3) Determine for which values of b (natural) the following conditions are satisfied:

A. $3b$ *is odd.*
B. b^2 *is a multiple of 4.*

4) Determine which of the following statements are false and which are true, providing justification for your answers.

A. *The expression* $n^2 + 1$ *always represents an odd number.*
B. *If the product of two numbers is odd, then their sum is even.*

Activity 2: Formulation of conjectures and construction of proofs

1) What can you say about the difference between the cube of a natural number and the number itself? Give some numerical examples to support you in the formulation of the requested conjecture. Could you prove what you state?

2) Write down a two digit number and the number that you get when you invert the digits. Calculate the difference between the greater number and the smaller one. Repeat this procedure with other two digit numbers. What kind of regularity can you observe? Could you prove what you state?

Activity 3: Analysis of proving strategies

Consider the following statement*: "If n is a natural number not divisible by 3, then the preceding of its square is divisible by 3".*
Analyse each of the following proofs of the statement proposed by four students, clarifying whether they are actual proofs of the statement and justifying your conclusions.
Then, compare the four proofs, attempting to highlight any connections that may exist between some of them.

<u>Luca's proof:</u>

If n is 2, then n^2*-1=4-1=3, that is divisible by 3.*
If n is 4, then n^2*-1=16-1=15, that is divisible by 3.*
If n is 5, then n^2*-1=25-1=24, that is divisible by 3.*
If n is 7, then n^2*-1=49-1=48, that is divisible by 3.*
Therefore, the statement is true because it always results a number divisible by 3.

<u>Alice's proof:</u>

The preceding of the square of n is equal to the product of its preceding and successive. If n is not divisible by 3, then one of its preceding or successive must be divisible by 3, because among any three consecutive numbers, one is always a multiple of 3, since multiples of 3 appear every three numbers. Therefore, n^2 *-1 is the product of two factors, one of which is definitely a multiple of 3, so* n^2 *-1 is also a multiple of 3. Thus, I have proven that the statement is true!*

<u>Filippo's proof:</u>

If n ≠ multiple of 3, then $\frac{n}{3} = x + y$ *(y is the reminder of the division of n by 3 and x is the quotient).*
Therefore $n = 3(x + y)$.
Then $n^2 - 1 = [3(x + y)]^2 - 1 = 9(x + y)^2 - 1 = 9(x^2 + y^2 + 2xy - 1)$.
Therefore, I have shown that the preceding of the square of n is a multiple of 9, and if it is a multiple of 9, it is also a multiple of 3!

<u>Elena's proof:</u>

n^2*-1=(n+1)(n-1)*
If n is not a multiple of 3, there are two cases: n=3x+1 or n=3x+2.
If n=3x+1 then (n+1)(n-1)=(3x+2)(3x)=3(x)(3x+2)
If n=3x+2 then (n+1)(n-1)=(3x+3)(3x+1)=3(x+1)(3x+1)
In both cases, I have proven that n^2*-1 is a multiple of 3, so I have proven the statement!*

Activity 4 : Construction of proofs

A divisibility rule states that if the sum of the digits of a number is a multiple of 3, then the number itself is also a multiple of 3. Consider a two-digit number. Prove that if the sum of its digits is a multiple of 3, then the number is a multiple of 3.

Final test of the path to introduce algebraic language for proving

Name and surname:…...………………………………………………………………………..

1) Determine which of the following statements are false and which are true, providing justification for your answers.

a) If uv is even, then u+v is even.

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b) The sum of three consecutive odd numbers is divisible by 3.

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2) Consider the following statement: "*If n is a natural even number, then* $n^2 + 16$ *is a number divisible by 4*".
Analyse each of the following proofs of the statement proposed by four students, clarifying whether they are actual proofs of the statement and justifying your conclusions.

Luca's proof:
If n is even then it is divisible by 2, then it has a factor 2, which squared becomes 4, thus its square is divisible by 4.
Also 16 is divisible by 4, then I can factor out 4 from n^2 *and from 16, thus their sum is divisible by 4!*

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Giorgia's proof:
If n is 2, then $n^2+16=4+16=20$.
If n is 4, then $n^2+16=16+16=32$.
If n is 6, then $n^2+16=36+16=52$.
If n is 8, then $n^2+16=64+16=80$.
Therefore, the statement is true!

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Federica's proof:

$n=2k$ *then* $n^2=(2k)^2=4k^2$

then $n^2+16=4k^2+16=4(k^2+4)$

Thus, the statement is true!

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Edoardo's proof:

$n^2+16=(n+4)(n-4)$

$n=even$

$n+4=even$

$n-4=even$

Then $n^2+16=(n+4)(n-4)$

Thus it is divisible by 4 since it is the product between two even numbers!

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Compare the four proofs, attempting to highlight any connections that may exist between some of them.

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3) Prove that any three-digit number formed by three identical digits is divisible by 3.

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4) To create a table of squares of integers, the following rule can be followed. In the first column of a table, write the sequence of integers starting from 1; in the second column, write the sequence of odd numbers starting from 3 (see table). To find the square of a number *n*, simply add the square of the previous number of n to the odd number to its right.
For example: $8^2 = 49 + 15 = 64$; $9^2 = 64 + 17 = 81$. Provide a justification for this rule.

Number n	*Odd number d*	*Square of n*
1	3	1
2	5	4
3	7	9
4	9	16
5	11	25
6	13	36
7	15	49
8	17	64
9	19	81
10	21	100
11	23	121
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5) Consider a three-digit number. Prove that if the number formed by its last two digits is a multiple of 4, then the original number is also a multiple of 4.

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Appendix C: final questionnaire administered to students involved in the experimentations

As reported in chapter 4, at the end of each experimentations students were asked to answer to a final questionnaire. The questionnaire is divided into two parts. The questions in the first part address students' view of algebra and the possible influence of the activities conducted during the didactic path on these views. The questions related to their views of algebra are taken from a questionnaire already associated with the didactic path (Cusi, 2009B). The questions in the second part of the questionnaire concern students' reflections on the Digital Mathematical Discussion, aiming to investigate students' perspectives on the challenges they encountered during the asynchronous phases of the Digital Mathematical Discussion, the usefulness of having access to their peers' written messages, the benefits and weaknesses of working asynchronously, and their reflections on the differences between working in-person and in the different digital environments involved in the Digital Mathematical Discussion.

Students from the first and second experimentations completed the entire questionnaire, while students involved in the pilot study were only given the questions from the second part of the questionnaire. In the following we report the entire questionnaire.

Part 1 – view of algebra

To answer the next two questions, you may (but it is not compulsory!) use the terms in the following list:

- it is a language
- it is a set of rules
- it is a branch of mathematics
- it is abstract
- it requires great precision
- it is a tool for proving
- it is a tool for communication
- it is difficult
- it is easy
- it requires a lot of memory
- it is concrete
- it is useless
- it is useful in everyday life
- it is used for reasoning
- it requires calculation skills

  - it is used to generalise
  - it simplifies problems
  - it complicates problems
  - it is concise
  - it is a tool for understanding
- What was your idea of algebra before participating in the project? Try to describe it using 5 key words/phrases, justifying your choices.
- Has participating in the project changed your idea of algebra? If so, what has changed? Try to describe your current idea of algebra using 5 key words/phrases, explaining your choices.

Part 2 – reflections on Digital Mathematical Discussion

Reflections on the work done within the chats

- Which application did you mainly use for the work done within the chat?
  - mobile app
  - tablet app
  - Desktop application for pc
  - Web application for pc
  - Web application for tablet
  - other
- If you used more than one application for the work done within the chat (for example, mobile app and Web application for pc), specify which applications you used. Moreover, explain why you used different applications and if you used them in different ways (for example, you used the mobile and tablet versions to send pictures rather than the ones for pc).
- Did the presence of the teacher educator in the chat influenced your interaction with your group mates? If so, how?
- According to you, was the number of messages written within the chat of your group too high? Rate from 1 to 5 your answer (1 strongly disagree; 5 strongly agree)
- How much did you feel yourself being listened to by your group mates when you wrote in the chat? Rate from 1 to 5 your answer (1 at all, 5 a lot)
- Was it difficult for you to write messages in the chat to discuss with your groupmates? Why?
- Was it useful for you to have the interventions of your group mates written? Why?

Reflections on the work done within Padlet

- Which application did you mainly use for the work done within Padlet?
  - mobile app
  - tablet app
  - Windows app for pc
  - mobile website
  - tablet website
  - pc website
  - other
- If you used more than one application for the work done within Padlet (for example, app for pc and mobile website), specify which applications you used. Moreover, explain why you used different applications and if you used them in different ways (for example, you used the mobile and tablet versions to send pictures rather than the ones for pc).
- According to you, was the quantity of comments and reactions added by you and your classmates within Padlet adequate for a collective mathematical discussion? Rate from 1 to 5 your answer (1 strongly disagree; 5 strongly agree)
- How much did you feel yourself being listened to by your classmates when you wrote in the Padlet? Rate from 1 to 5 your answer (1 at all, 5 a lot)
- Was it difficult for you to insert comments and reactions and to interact with your classmates within the Padlet?

General reflections

- Was it useful for you to have the possibility of taking your own time for reading and sending messages (both on chat and Padlet)? Why?
- Do you think having access to the messages (both on chat and Padlet) after the activities ended was helpful? Why? Did you ever go back to reread any messages (either on chat or Padlet) after the activities had finished?
- According to you, why was the number of messages written in the Telegram chats significantly higher than the number of messages written on the Padlet?
- What differences did you notice between the work done in-person, the work done on the chat, and the work done on Padlet?
- Was it difficult for you to follow the instructions in the “guide” of the activity? If so, which ones?

- Did you feel the need to use additional communication tools (other chats with your colleagues, phone calls, video calls, in person meetings,...) than the chat and the Padlet devoted to the activity? Why?
- Would you recommend the activity to other peers? Why?
- Are there other reflections on the activity you want to share? If so, which ones?

Appendix D: analysis of the background on activation and coordination of conceptual frames of the classes from the first and second experimentations

In the following tables we report a snapshot of the initial state of the students in the classes involved in the first and second experimentations with respect to their ability to activate and coordinate conceptual frameworks. Specifically, from the first exercise of the initial test (appendix B) it is possible to pinpoint students activation of the conceptual frames ‘multiple’ (request 1.A), ‘even-odd’ (request 1.C), and ‘preceding-successive’ (request 1.G), and the coordination between the ‘multiple’ and ‘even-odd’ frames (request 1.D) and between the ‘preceding-successive’ and ‘even-odd’ frames (request 1.F).

For each class, a table is presented with the rows reporting the correct (✓) or incorrect (×) activation and coordination of these frames for each student who take the initial test. The activation of the frame ‘multiple’ is considered correct if an expression in the form $3n$ is provided as translation of “the triple of a number”. The activation of the frame ‘even-odd’ is correct if the statement ‘an even number’ is translated into the form $2n$. The correct activation of the frame ‘preceding-successive’ is recognised if consecutive numbers are expressed as n and $n+1$, or $n+1$ and $n+2$. The frames ‘multiple’ and ‘even-odd’ are correctly coordinated if the statement “the double of an even number” is translated into algebraic language in the form $2 \cdot (2n)$ or $4n$. The correct coordination between the frames ‘even-odd’ and ‘preceding-successive’ is recognised if $2n$, $2n+2$ or $2n+2$, $2n+4$ are presented as symbolic representations of the statement “two consecutive even numbers”. The activation and coordination of frames are incorrect if left blank, with numerical examples or other symbolic expressions not in the desired form.

Frame 'multiple'		*Frame 'even-odd'*		*Frame 'preceding-successive'*	**TOT**
Activation	**Coordination**	**Activation**	**Coordination**	**Activation**	
✓	✓	✓	✓	✓	5 ✓
✓	✓	✓	✓	✓	5 ✓
✓	✓	✓	✓	✓	5 ✓
✓	✓	✓	✓	✓	5 ✓
✓	✓	✓	✓	✓	5 ✓
✓	✓	✓	✓	✓	5 ✓
✓	✓	✓	✓	✓	5 ✓
✓	✓	✓	✓	×	4 ✓ 1 ×
✓	×	✓	✓	✓	4 ✓ 1 ×
✓	✓	✓	×	✓	4 ✓ 1 ×
✓	✓	✓	×	✓	4 ✓ 1 ×
✓	✓	✓	×	✓	4 ✓ 1 ×
✓	✓	✓	×	×	3 ✓ 2 ×
✓	✓	×	×	✓	3 ✓ 2 ×
✓	✓	✓	×	×	3 ✓ 2 ×
✓	✓	✓	×	×	3 ✓ 2 ×
✓	✓	✓	×	×	3 ✓ 2 ×
✓	✓	✓	×	×	3 ✓ 2 ×
✓	✓	✓	×	×	3 ✓ 2 ×
✓	✓	✓	×	×	3 ✓ 2 ×
✓	×	✓	×	✓	3 ✓ 2 ×
✓	×	×	×	✓	2 ✓ 3 ×
✓	×	×	×	✓	2 ✓ 3 ×
✓	×	×	×	✓	2 ✓ 3 ×
✓	×	×	×	✓	2 ✓ 3 ×
✓	×	×	×	×	1 ✓ 4 ×

Table D.1 Overview of students' activation and coordination of conceptual frames in class A 2nd DBR cycle

Frame 'multiple'		*Frame 'even-odd'*		*Frame 'preceding-successive'*	**TOT**
Activation	**Coordination**	**Activation**	**Coordination**	**Activation**	
✓	✓	✓	✓	✓	5 ✓
✓	✓	✓	✓	✓	5 ✓
✓	✓	✓	✓	✓	5 ✓
✓	✓	✓	✓	✓	5 ✓
✓	✓	✓	✓	✓	5 ✓
✓	✓	✓	×	✓	4 ✓ 1 ×
✓	✓	✓	×	✓	4 ✓ 1 ×
✓	✓	✓	✓	×	4 ✓ 1 ×
✓	✓	✓	✓	×	4 ✓ 1 ×
✓	✓	✓	×	×	3 ✓ 2 ×
✓	×	×	×	✓	2 ✓ 3 ×
✓	×	×	×	✓	2 ✓ 3 ×
✓	×	×	×	×	1 ✓ 4 ×
✓	×	×	×	×	1 ✓ 4 ×
✓	×	×	×	×	1 ✓ 4 ×

Table D.2 Overview of students' activation and coordination of conceptual frames in class B 2nd DBR cycle

Frame 'multiple'		*Frame 'even-odd'*		*Frame 'preceding-successive'*	**TOT**
Activation	**Coordination**	**Activation**	**Coordination**	**Activation**	
✓	✓	✓	✓	✓	5 ✓
✓	✓	✓	✓	✓	5 ✓
✓	✓	✓	✓	✓	5 ✓
✓	✓	✓	✓	✓	5 ✓
✓	✓	✓	×	✓	4 ✓ 1 ×
✓	✓	✓	×	✓	4 ✓ 1 ×
✓	✓	✓	✓	×	4 ✓ 1 ×
✓	×	✓	×	✓	3 ✓ 2 ×
✓	×	×	×	✓	2 ✓ 3 ×
✓	×	×	×	✓	2 ✓ 3 ×
✓	×	×	×	✓	2 ✓ 3 ×
✓	×	×	×	✓	2 ✓ 3 ×
✓	×	×	×	✓	2 ✓ 3 ×
✓	×	×	×	✓	2 ✓ 3 ×
✓	×	×	×	✓	2 ✓ 3 ×
✓	×	×	×	✓	2 ✓ 3 ×
✓	×	×	×	✓	2 ✓ 3 ×
✓	×	×	×	✓	2 ✓ 3 ×
✓	×	×	×	×	1 ✓ 4 ×
✓	×	×	×	×	1 ✓ 4 ×
✓	×	×	×	×	1 ✓ 4 ×
×	×	×	×	✓	1 ✓ 4 ×

Table D.3 Overview of students' activation and coordination of conceptual frames in class A 3rd DBR cycle

Frame 'multiple'		*Frame 'even-odd'*		*Frame 'preceding-successive'*	**TOT**
Activation	**Coordination**	**Activation**	**Coordination**	**Activation**	
✓	✓	✓	✓	✓	5 ✓
✓	✓	✓	✓	✓	5 ✓
✓	✓	✓	✓	✓	5 ✓
✓	✓	✓	✓	✓	5 ✓
✓	✓	✓	✓	✓	5 ✓
✓	✓	✓	✓	✓	5 ✓
✓	✓	✓	✓	✓	5 ✓
✓	✓	✓	✓	✓	5 ✓
✓	✓	✓	✓	✓	5 ✓
✓	✓	✓	✓	✓	5 ✓
✓	✓	✓	✓	✓	5 ✓
✓	✓	✓	✓	✓	5 ✓
✓	✓	✓	✓	✓	5 ✓
✓	✓	✓	✓	✓	5 ✓
✓	✓	✓	×	✓	4 ✓ 1 ×
✓	✓	✓	×	✓	4 ✓ 1 ×
✓	✓	✓	✓	×	4 ✓ 1 ×
✓	✓	✓	✓	×	4 ✓ 1 ×
✓	✓	✓	✓	×	4 ✓ 1 ×
✓	✓	✓	✓	×	4 ✓ 1 ×
✓	✓	✓	✓	×	4 ✓ 1 ×
✓	✓	✓	✓	×	4 ✓ 1 ×
✓	✓	✓	×	×	3 ✓ 2 ×
✓	✓	✓	×	×	3 ✓ 2 ×

Table D.4 Overview of students' activation and coordination of conceptual frames in class B 3rd DBR cycle

Appendix E: analysis of existing educational platforms with respect to relevant characteristics for supporting Digital Mathematical Discussions

As reported in chapter 6, the design of a platform that can effectively support Digital Mathematical Discussions took into account existing solutions. In particular, in the following we report the detailed analysis conducted on the educational applications considered as a reference for the design of the prototype platform. The educational applications taken into account are Padlet, Google Classroom and Moodle. For Moodle platform we focus specifically on the forum. The solutions examined were compared based on four main categories: general, social, messaging, and management. The first category includes information such as available app variants (mobile, desktop), the requirement for user registration to access the service, and kinds of shareable files within the application. The second category addresses the layout and organisation of posts, comments, and responses, if present. The third category covers the presence of a chat function within the solution, the possibility to send voice messages and to write mathematical functions. The fourth category lists the features available to administrators for managing and moderating content within the platform, as well as the possibility to organise and assign tasks.

The detailed analysis of the educational solutions is presented in Table E.1 for the general category, Table E.2 for the social category, Table E.3 for the messaging category, and Table E.4 for the management category. Moreover, we analysed the artificial intelligence functionalities supported by the platforms (Table E.5).

	General characteristics		
	Padlet	**Google Classroom**	**Moodle**
Mobile App	Yes	Yes	Yes
Desktop App	Yes	Yes	Yes
User registration	Yes	Yes	Yes
Sharable files	<u>For posts:</u> uploads from the device; links; images form camera; videos from camera; audio recorded from device; screen recorded from device; drawings; polls; uploads from Google Drive; images generated by AI; images from the Web; GIF; contents from YouTube, Spotify or the Web	<u>For posts:</u> uploads from Google Drive; uploads from the device; links; contents from YouTube <u>For delivering homework:</u> uploads from Google Drive; uploads from the device; PDF files; images; videos; audios	<u>For posts in the forum:</u> images; videos; documents; audios; links

Table E.1 General characteristics of Padlet, Google Classroom and Moodle

	Social characteristics		
	Padlet	**Google Classroom**	**Moodle**
Layout for posts	Available layouts: list, grid, wall, canva, timeline, map The administrator can make posts anonymous or nominal	Available layouts: list	Available layouts in the forum: standard forum, Q&A forum
Layout for comments and answers	Comments are shown as list under the post they refer The administrator can enable comments (written) and reactions (likes, grade, stars, points)	Comments are shown as list under the post they refer	Available layouts in the forum: list in chronological order; list in inverse chronological order; threaded; nested

Table E.2 Social characteristics of Padlet, Google Classroom and Moodle

	Messaging characteristics		
	Padlet	**Google Classroom**	**Moodle**
Chat function	No	No	Yes
Vocal messages	Yes (only posts)	No	Yes
Mathematical keyboard	No	No	Yes (support of LaTeX and calculator)

Table E.3 Messaging characteristics of Padlet, Google Classroom and Moodle

	Management characteristics		
	Padlet	**Google Classroom**	**Moodle**
Assignments	No (availability of breakout links to specific sections)	Available assignments: task, quiz, question Assignments can be directed to the whole class or to specific students Assignments can have a due date Assignments can be graded Students can send messages to the teacher when delivering an assignment	Assignments can be directed either individually or to members of a group Assignments can be graded and commented
Moderation	The administrator can approve posts before publication 'Filter Profanity' function to avoid the use of specific words Option to grant access to both registered and non-registered users Option to make contributions either anonymous or nominal	Ability to: - Move a post to the top - Delete a comment or a post - 'Deactivate a student' (who can still submit their work, but it will not be visible to other students; they cannot respond to peers' work and cannot send comments or posts)	Ability to: - Pin a post to the top of the forum - Limit the number of posts each user can make in a forum within a specified time interval - Make subscription to a forum either optional or mandatory (subscribers will receive a notification for every new message posted to the forum; non-subscribers will not receive such notifications) - Mark a discussion as special within a forum (this will move it to the top); or pin a discussion at the top

Table E.4 Management characteristics of Padlet, Google Classroom and Moodle

Supported AI functionalities		
Padlet	**Google Classroom**	**Moodle**
Layout creation	No	OpenAI ChatGPT plugin to generate questions based on a given text OpenAI Chat Block plugin to provide automatic assistance to students, answer questions, or generate interactive content in real-time AI Text-to-Image plugin, to generate images from text input

Table E.5 AI functionalities supported by Padlet, Google Classroom and Moodle