Field Theories, Stable Homotopy Theory, and Khovanov Homology

by

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ABSTRACT. In this paper, we discuss two topics: First, we show how to convert 1+1-topological quantum field theories valued in symmetric bimonoidal categories into stable homotopical data, using a machinery by A. D. Elmendorf and M. A. Mandell. Then we discuss, in this framework, two recent results (independent of each other) on refinements of Khovanov homology: our refinement into a module over the connective k-theory spectrum and a stronger result by Robert Lipshitz and Sucharit Sarkar refining Khovanov homology into a stable homotopy type.

1. INTRODUCTION

The present paper has a somewhat peculiar history. Essentially, all the work took place in Fall 2011 and Winter 2012. It was a conglomerate of several mathematical projects. We put the outcome on the arXiv, but no author had strong feelings about publication. Recently, however, interest in these topics was rekindled (see for example [19]). We, therefore, decided to revise the manuscript, and publish it in the present volume.

As for the ingredients of the project, Po Hu and Igor Kriz were long interested in topological modular functors, i.e., $1+1$-topological quantum field theories (TQFTs) valued in finite-dimensional $\mathbb{C}$-vector spaces, and developing a "realization" construction which would convert such a structure into a $1+1$-topological quantum field theory valued in k-modules,

where k denotes the connective K-theory spectrum. Daniel Kriz, on the other hand, studied Khovanov homology as a part of another project [16]. From joint discussion, there arose a project of writing down a realization construction and applying it to constructing a k-module refinement of Khovanov homology.

In this, we ultimately succeeded, and we learned quite a bit. The realization into k-modules is an intuitively compelling idea, but technical details are tricky due to the difficulty of multiplicative infinite loop space theory. We decided to use the machinery of A. D. Elmendorf and M. A. Mandell [7] which uses multicategories enriched in groupoids. We then discovered that the multicategory language is quite a convenient tool for axiomatizing modular functors as well. A multicategory has objects and n-tuple morphisms

$$(1.1) \qquad\qquad X_1, \ldots, X_n \to Y,$$

which compose in the same way as the elements of an operad. Multicategories are also called colored operads or multi-sorted operads. In this paper, by a $\star$-*category*, we mean a multicategory enriched in groupoids where for every n-tuple $X_1, \ldots, X_m$, there is a universal morphism (1.1) (in the 2-category sense). We denote $Y = X_1 \star \cdots \star X_n$. For detail, see Definition 2.1. By a $\star$-*functor*, we mean a multifunctor which preserves this structure. (Although we focused on the 2-category context, there are, of course, similar concepts in ordinary multicategories and multicategories enriched in topological spaces.) As examples of $\star$-categories, we have the 1+1-oriented cobordism multicategory $\mathcal{A}$ (and its many variants) and also a certain $\star$-category $\mathcal{C}_2$ associated with any symmetric bimonoidal category $\mathcal{C}$ (at least when its 2-morphisms form a groupoid). By a $\mathcal{C}$-*valued modular functor* on a $\star$-category $\mathcal{Q}$, we then mean a $\star$-functor

$$\mathcal{Q} \to \mathcal{C}_2.$$

Our realization theorem is then the following result (see §2 for precise definitions).

Theorem 1.1. *A $\mathcal{C}$-valued modular functor gives rise, in a canonical way, to a multifunctor*

$$(1.2) \qquad\qquad M : B_2\mathcal{Q} \to k_{\mathcal{C}}\text{-}modules$$

where $k_{\mathcal{C}}$ is the E_∞-ring spectrum associated with $\mathcal{C}$ and $B_2\mathcal{Q}$ denotes the topological multicategory obtained by taking classifying spaces of the 1-morphism groupoids.

Furthermore, a universal multimorphism

$$X_1, \ldots, X_n \to X_1 \star \cdots \star X_n,$$

maps, under (1.2), *into an equivalence*

$$(1.3) \qquad\qquad M_{X_1} \wedge_{k_\mathcal{C}} \cdots \wedge_{k_\mathcal{C}} M_{X_n} \to M_{X_1 \amalg \cdots \amalg X_n}.$$

Comment 1.2. 1. By "in a canonical way," we mean that we have a specific construction in mind. It is given by the Elmendorf-Mandell construction.

2. The second statement requires some explanation. What is relevant here is that we work in the category of symmetric spectra where we have a symmetric monoidal structure under which E_∞-ring spectra are, by definition, precisely commutative monoids. For an E_∞-ring spectrum E in this category, the multicategory of E-modules is then a $\star$-category where $\star = \wedge_E$. The morphism (1.3) is then the morphism whose existence is the defining property of the $\star$-structure.

In view of this, it would be interesting to know if one could devise a construction where the map (1.3) would be an isomorphism instead of just an equivalence, i.e., such that our construction would be a $\star$-functor. Our construction does not give this, because the Elmendorf-Mandell machine does not give a $\star$-functor. We suspect that such a $\star$-functor might not exist.

The main application we had in mind was refining Khovanov (sl_2)-homology of links in S^3 to a k-module invariant where k is connective k-theory. We hoped to achieve this by refining Khovanov's $1 + 1$-TQFT $\Lambda[x]$ into a $1 + 1$-modular functor valued in finite-dimensional $\mathbb{C}$-vector spaces on the oriented $1 + 1$-cobordism category $\mathcal{A}$. This turns out to be impossible, but we succeeded in constructing a modular functor on the $\star$-category $\mathcal{A}_s^A$ of spin $1+1$-cobordisms where the objects are antiperiodic 1-manifolds. (For a detailed definition of $\mathcal{A}_s^A$, see Example 2.4. For an explanation why $\mathcal{A}_s^A$ is needed instead of $\mathcal{A}$, see §3.2.) It, therefore, came as a surprise when the spin structure dropped out in the end and we were able to use this construction to define a k-theory lift of Khovanov homology on links without spin structure. We then thought that there must be a geometric guiding principle which explains this simplification.

Soon afterwards, the paper [20] by Robert Lipshitz and Sucharit Sarkar appeared on the arXiv. This paper contains a construction of a stable homotopy refinement of Khovanov homology. The paper [20] uses a different technique, namely Cohen-Jones-Segal flow categories arising from Morse theory, but after some initial skepticism, we realized that Lipshitz and Sarkar discovered the geometric principle we were looking for, while at the same time rendering our k-theory refinement obsolete: In our language, they realized that Khovanov's construction takes place in the category enriched in groupoids $\mathcal{A}_K$ of embedded cobordisms (in $S^2 \times [0, 1]$—see §2 for

precise definitions). They additionally observed what amounts to saying that the Khovanov TQFT refines into a lax 2-functor into $\mathcal{S}_2$ where $\mathcal{S}$ is the symmetric bimonoidal category of finite sets. The $\star$-functor structure here is lost as $\mathcal{A}_K$ is not a $\star$-category, but a $\star$-functor structure turns out to be unnecessary because the target of the construction is the category of symmetric spectra (instead of modules over another rigid ring spectrum), so the module structure does not have to be discussed (although an analogue of (1.3) is relevant and an equivalence follows from more special arguments). We, therefore, end up with an alternate proof of the following result, without requiring the language of Morse theory and flow categories.

Theorem 1.3 (Lipshitz-Sarkar [20]). *There exists an explicitly defined k-module symmetric spectrum $M(L)$ assigned to an oriented link L such that, for isotopic oriented links $L \cong L'$, there exists an equivalence*

$$M(L) \simeq M(L')$$

and such that

$$M(L) \wedge H\mathbb{Z}$$

corresponds to the Khovanov chain complex under the equivalence of derived categories of strict $H\mathbb{Z}$-modules and chain complexes [6], where $S \to H\mathbb{Z}$ is the strictly commutative strict symmetric ring spectrum unit. In other words, the homology of $M(L)$ is the Khovanov homology of L. In [20], $M(L)$ is denoted by $\chi_{Kh}(L)$.

The convention in [14] is that the Khovanov complex is written as a cochain complex. However, in our treatment, we reverse this by reversing the conventions for the 0-resolution and 1-resolution of link crossings (see Figure 1, page 19). This has the effect of changing cohomology into homology, which is more natural from our point of view.

As already remarked, strictly speaking, the full strength of Theorem 1.1 is unnecessary for our proof of Theorem 1.3. However, our investigation of stable homotopy realization of modular functors, including the construction of the Khovanov topological modular functor on $\mathcal{A}_s^A$, provides an excellent motivation for understanding our proof of Theorem 1.3, and thereby makes the argument easier to understand. Because of this, we decided to report on both investigations in the same paper and also to include a discussion of the spin-dependent modular functor.

The present paper is organized as follows: In §2, we review the main point of the Elmendorf-Mandell formalism, introduce the notion of a $\star$-category and $\star$-functor, and also prove Theorem 1.1. In §3, we construct our main example of the spin modular functor refinement of the Khovanov $1 + 1$-dimensional TQFT $\Lambda(x)$ and also the reinterpretation of Lipshitz-Sarkar's construction in terms of 2-functors. In §4, we construct the refinements of the Khovanov functor. In §5, we construct the refined invariant and state a more specific version of Theorem 1.3 (Theorem 5.3). Section 6 is dedicated to proving link invariance (theorems 1.3 and 5.3), refining, essentially, the proof of link invariance of Khovanov homology [14] (see also [3]).

2. Multicategories and Topological Field Theories

Following [7], a *multicategory* $\mathcal{C}$ has a class of *objects* $Obj(\mathcal{C})$ and classes of *morphisms* $Mor_n(\mathcal{C})$ where $n = 0, 1, 2, \ldots$ written as

$$\phi : (x_1, \ldots, x_n) \to y, \quad x_1, \ldots, x_n, y \in Mor(\mathcal{C}).$$

We also write

$$\phi \in \mathcal{C}(x_1, \ldots, x_n; y).$$

There are composition, equivariance, and unit axioms analogous to the definition of an operad. Details may be found in [7]. In this paper, we will be dealing with multicategories enriched in groupoids. This means that while $Obj(\mathcal{C})$ is a class, $\mathcal{C}(x_1, \ldots, x_n; y)$ are groupoids, and compositions and units are functors. Associativity, unitality, and equivariance are satisfied up to natural isomorphisms, which, in turn, satisfy coherence axioms modeled on cocycle conditions. Details of this context are also fully discussed in [7].

Therefore, we are in a 2-categorical context. The objects of a morphism groupoid will sometimes be referred to as 1-morphisms and morphisms of a morphism groupoid as 2-morphisms. This is the standard language of 2-category theory. The reader should realize that a 2-category where the 2-morphisms are isomorphisms is the same thing as a category enriched in groupoids.

The most fundamental examples discussed in [7] are the multicategory $Perm$ (enriched in groupoids) of (small) permutative categories and the multicategory Sym (enriched in topological spaces) of symmetric spectra. In the multicategory Sym, morphisms $X_1, \ldots, X_n \to Y$ are the same thing as morphisms

$$X_1 \wedge \cdots \wedge X_n \to Y$$

where $\wedge$ is the commutative, associative, and unital smash product of symmetric spectra.

In some sense, the main result of [7] is constructing a continuous multifunctor

$$B_2 Perm \to Sym$$

where B_2 means taking the classifying spaces of the 1-morphism groupoids, thereby obtaining a topological multicategory. For a multicategory M enriched in groupoids, let Sym^M denote the category of multifunctors $M \to Sym$. The other main result of [7] is Theorem 1.4, stating that for M and M' multicategories enriched in groupoids, and $f : M \to M'$ a multifunctor that is a weak equivalence, the induced functor $Sym^{M'} \to Sym^M$ is a Quillen equivalence. In other words, the construction of [7] preserves weak equivalences of multicategories.

Definition 2.1. A $\star$-*category* is a multicategory enriched in groupoids such that for every $x_1, \ldots, x_n \in Obj(\mathcal{C})$ where $n \geq 0$, there exists an object $x_1 \star \cdots \star x_n$ and a 1-morphism

$$\phi : (x_1, \ldots, x_n) \to x_1 \star \cdots \star x_n$$

(in the case of $n = 0$, one denotes the right-hand side as 1), such that for every 1-morphism

$$\psi : (x_1, \ldots, x_n) \to y,$$

there exists a 1-morphism

$$h : x_1 \star \cdots \star x_n \to y$$

and a 2-isomorphism

$$\iota : \psi \xRightarrow{\cong} h \circ \phi$$

and furthermore, for other such data

$$h' : x_1 \star \cdots \star x_n \to y,$$

$$\iota' : \psi \xRightarrow{\cong} h \circ \phi,$$

there exists a unique 2-isomorphism

$$\lambda : h \xRightarrow{\cong} h'$$

such that

$$\lambda \circ Id_\phi = \iota' \circ \iota^{-1}.$$

Note that for two objects u and v satisfying the definition of $x_1 \star \cdots \star x_n$, there exist 1-morphisms $u \to v$ and $v \to u$ (unique up to 2-isomorphism) whose compositions are 2-isomorphic to the identity.

In the context of multicategories enriched in groupoids, one has a notion of lax multifunctors, analogous to lax functors of 2-categories, where the composition and identity axioms are satisfied up to 2-isomorphisms satisfying the standard coherence diagrams.

Definition 2.2. A $\star$-*functor* is a lax multifunctor $F : \mathcal{C} \to \mathcal{D}$ between multicategories enriched in groupoids such that $F(x_1 \star \cdots \star x_n)$ is a choice for $F(x_1) \star \cdots \star F(x_n)$.

Comment 2.3. There is a canonical $\star$-category which comes from a (lax) symmetric monoidal category: If the symmetric monoidal structure is $\otimes$, then morphisms

$$x_1, \ldots, x_n \to y$$

are, by definition, the morphisms

$$x_1 \otimes \cdots \otimes x_n \to y.$$

This is always a $\star$-category, with

$$x_1 \star \cdots \star x_n = x_1 \otimes \cdots \otimes x_n.$$

Not every $\star$-category, however, comes from a symmetric monoidal category. As an example, consider the operad A where $A(k)$ is the commutative monoid of non-negative integers $(\mathbb{N}_0, +)$ and composition

$$A(k) \times A(n_1) \times \ldots A(n_k) \to A(n_1 + \cdots + n_k)$$

is

$$(x, y_1, \ldots, y_k) \mapsto x + y_1 + \cdots + y_k - k + 1.$$

The only 2-isomorphisms are, by definition, identities. The reader should check that this operad (and hence multicategory) satisfies the $\star$-category axioms but does not come from a symmetric monoidal category.

However, most of the $\star$-categories discussed in this paper, in fact, come from (lax) symmetric monoidal categories. The reason we prefer the $\star$-category language is that the conditions on both $\star$-categories and $\star$-functors are much simpler to verify in comparison with symmetric monoidal 2-categories and 2-functors since there is only a universal property to check.

Example 2.4 (Cobordism Categories)**.** (1) The "basic" cobordism category $\mathcal{A}$: The objects of $\mathcal{A}$ are oriented compact smooth 1-manifolds. The 1-morphisms

$$(X_1, \ldots, X_n) \to Y$$

are oriented cobordisms between $X_1 \amalg \cdots \amalg X_n$ and Y. The 2-morphisms are orientation preserving diffeomorphisms which are the identity on the boundary. The $\star$-category structure is given by

$$X_1 \star \cdots \star X_n = X_1 \amalg \cdots \amalg X_n$$

with the universal 1-morphism $(X_1, \ldots, X_n) \to X_1 \star \cdots \star X_n$ being the identity. The unit object is $\emptyset$.

(2) There are a number of variants of $\mathcal{A}$. One example of interest is $\mathcal{A}_s$ where $Obj(\mathcal{A}_s)$ is the class of oriented 1-manifolds with spin structure and 1-morphisms are oriented spin cobordisms between $X_1 \amalg \cdots \amalg X_n$ and Y. Recall that a spin structure on a 1-manifold M with tangent bundle τ_M can be specified by giving a real bundle $\tau^{1/2}$ and an isomorphism of real bundles

$$\tau^{1/2} \otimes_{\mathbb{R}} \tau^{1/2} \cong \tau.$$

An oriented circle has two spin structures called *periodic* and *antiperiodic*, depending on whether $\tau^{1/2}$ is trivial or is a Möbius strip. The antiperiodic spin structure is cobordant to $\emptyset$, while the periodic one is not. The 2-morphisms are orientation preserving diffeomorphisms which are Id on the boundary including spin, which means also identity on $\tau^{1/2}$. One is also interested in the $\star$-category $\mathcal{A}_s^A$ which is defined in the same way, but one restricts to objects which are spin 1-manifolds with antiperiodic spin structure on each connected component.

(3) Another variant of $\mathcal{A}$ is $\mathcal{A}_K$, the embedded $1+1$-bordism category. Objects are smooth compact 1-dimensional submanifolds of S^2. The 1-morphisms $X_1 \to X_2$ are compact 2-submanifolds Y of $S^2 \times [0,1]$ whose boundary is in $S^2 \times \{0,1\}$ (which Y meets transversally), and such that $Y \cap S^2 \times \{0\} = X_1$ and $Y \cap S^2 \times \{1\} = X_2$. The 2-isomorphisms $Y \to Y'$ are isotopy classes of diffeomorphisms $\phi : S^2 \times [0,1] \to S^2 \times [0,1]$ which are the identity on the boundary and restrict to diffeomorphisms $\phi|_Y : Y \to Y'$ (the isotopies are required to restrict to isotopies of diffeomorphisms $Y \to Y'$). Note, however, that this 2-category has no canonical multicategory structure.

Example 2.5 (Target $\star$-Category)**.** Let $\mathcal{C}$ be a symmetric bimonoidal groupoid. The examples we are thinking of are

$$\mathcal{C} \;=\; R, \quad \text{a commutative semiring } R \text{ (considered as a discrete category, i.e., the only morphisms are identities), } +, \cdot\,;$$

$$\mathcal{C} \;=\; \mathcal{V}, \quad \text{the category of finite-dimensional } \mathbb{C}\text{-vector spaces and isomorphisms, } \oplus, \otimes;$$

$$\mathcal{C} \;=\; \mathcal{S}, \quad \text{the category of finite sets, } \amalg, \times.$$

The $\star$-category $\mathcal{C}_2$ has as objects the class of all finite sets. A 1-morphism $(S_1, \ldots, S_n) \to T$ is a $T \times (S_1 \times \ldots S_n)$-matrix (thinking of T as the set of rows and $(S_1 \times \ldots S_n)$ as the set of columns) of objects of $\mathcal{C}$. Composition

is "matrix multiplication" with respect to the additive and multiplicative operation of $\mathcal{C}$. The 2-isomorphisms are matrices of $\mathcal{C}$-isomorphisms.

Definition 2.6. Let $\mathcal{Q}$ be a $\star$-category and let $\mathcal{C}$ be a symmetric bimonoidal category. Then a $\star$-functor $\mathcal{Q} \to \mathcal{C}_2$ is called a $\mathcal{C}$-*valued modular functor* on $\mathcal{Q}$. We are typically interested in examples such as $\mathcal{Q} = \mathcal{A}$, $\mathcal{Q} = \mathcal{A}_s^A$, etc.

Comment 2.7. The idea of modular functors originates with Graeme Segal [25], but the definition given in [25] was not rigorous (the coherence isomorphisms were treated as equalities, thereby neglecting the question of coherence diagrams). Thomas M. Fiore, Hu, and I. Kriz [8] developed a formalism defining modular functors rigorously, but the formalism is awkward from the point of view of infinite loop space theory, hence the variant introduced in the present paper.

Remark 2.8. A $\star$-functor

$$\mathcal{A}_A^s \to \mathcal{V}_2$$

is the flavor of 2-vector-space valued $1 + 1$-dimensional topological field theory with spin structure we will use in this paper. Generally speaking, one tends to call 2-vector-space valued $1 + 1$-dimensional topological field theories "topological modular functors." When using that term, however, one usually considers a larger source $\star$-category than $\mathcal{A}$. In one variant, one removes the spin structure; if there is spin structure, one usually removes the restriction on the spin structure on objects being antiperiodic. In the present paper, however, we are unable to work with these notions, as the relevant examples either don't exist or we are unable to construct them; when constructing the K-theory version of Khovanov homology, the source $\star$-category $\mathcal{A}_A^s$ is precisely what we need.

Lemma 2.9. *Let $\mathcal{C}$ be a symmetric bimonoidal category. Then there is a canonical lax multifunctor*

$$\Phi : \mathcal{C}_2 \to Perm$$

where $Perm$ is the lax multicategory of permutative categories (for $Perm$ see [7, Theorem 1.1]).

Proof. On objects, we set

$$X \mapsto \prod_X \mathcal{C}.$$

On 1-morphisms, a morphism $f : X_1 \times \cdots \times X_n \to Y$ in $\mathcal{C}_2$ is a $Y \times (X_1 \times \cdots \times X_n)$-matrix whose entries are objects in $\mathcal{C}$. For each $y \times (x_1, \ldots, x_n) \in Y \times X_1 \times \cdots \times X_n$, denote the corresponding entry by

$M_{(y,x_1,\ldots,x_n)} \in Obj(\mathcal{C})$. Then

$$\Phi(f): \prod_{X_1 \times \cdots \times X_n} \to \prod_Y \mathcal{C}$$

is given by matrix multiplication, using the $\otimes$ in $\mathcal{C}$ as the multiplication of entries.

On 2-morphisms, a 2-morphism $f \Rightarrow f'$ in $\mathcal{C}_2$ is a matrix of isomorphisms $M_{(y,x_1,\ldots,x_n)} \to M'_{(y,x_1,\ldots,x_n)}$ where $M_{(y,x_1,\ldots,x_n)}$ and $M'_{(y,x_1,\ldots,x_n)}$ are the $(y,x_1,\ldots,x_n)^{\text{th}}$ entries of f and f', respectively. The 2-isomorphism $\Phi(f) \Rightarrow \Phi(f')$ is induced by these isomorphisms. $\qquad\square$

Construction 2.10. Let $\mathcal{Q}$ be a category enriched in groupoids. Denote by $\mathcal{Q}_B$ the multicategory enriched in groupoids with objects $\{B\} \amalg Obj(\mathcal{Q})$, where

$$\mathcal{Q}_B(\underbrace{B,\ldots,B}_{n};B) = E\Sigma_n,$$

(recall from [7] that $E\Sigma_n$ means the torsor over Σ_n) and for $x,y \in Obj(\mathcal{Q})$,

$$\mathcal{Q}_B(\underbrace{B,\ldots,B}_{m}, x, \underbrace{B,\ldots,B}_{n}; y) = E\Sigma_{m+n} \times \mathcal{Q}(x,y).$$

Unspecified morphism sets are empty and composition rules are the obvious ones. Then the machine of [7] converts a lax multifunctor

$$F: \mathcal{Q}_B \to Perm$$

into an E_∞-symmetric spectrum R (obtained from $F(B)$) and an A_∞-functor (associative functor in [7])

$$B_2\mathcal{Q} \to E_\infty - R - \text{modules}.$$

Recall that $B_2\mathcal{Q}$ for a category $\mathcal{Q}$ enriched in groupoids is the topological category obtained by taking the classifying space on 2-morphisms. Theorem 1.4 in [7] further enables us to make this *strict*; i.e., we obtain a strictly commutative symmetric ring spectrum R and a strict functor

$$B_2\mathcal{Q} \to R - \text{modules}.$$

Specifically, by [7, Theorem 1.4], an E_∞-ring R in symmetric spectra is naturally equivalent to a strictly commutative ring, and an E_∞-module over R is naturally equivalent to a strict R-module.

Construction 2.11. Now let

$$F: \mathcal{Q} \to \mathcal{D}$$

be a lax functor of categories enriched in groupoids, and let $\mathcal{D}$ be a $\star$-category. Note that then we obtain a canonical lax multifunctor

$$F_B: \mathcal{Q}_B \to \mathcal{D}$$

given on objects by

$$F_B(x) = F(x) \text{ for } x \in Obj(\mathcal{Q}),$$

$$F_B(B) = 1.$$

The values of F_B on 1-morphisms and 2-morphisms are determined by universality. If we have, in addition, a multifunctor

$$\Phi : \mathcal{D} \to Perm,$$

then, by Construction 2.10, we obtain a strictly commutative symmetric ring spectrum R and a strict functor

$$B_2\mathcal{Q} \to R - \text{modules}.$$

Proof of Theorem 1.1. Our proof is similar to the situation of constructions 2.10 and 2.11 but with extra structure.

$\mathcal{Q}$ and $\mathcal{C}_2$ (which plays the role of $\mathcal{D}$) are $\star$-categories and F is a $\star$-functor. Accordingly, we replace $\mathcal{C}_B$ by a construction which takes into account the multiplication: Let us write, say,

$$\mathcal{Q}_B^{alg}(\underbrace{B, \ldots, B}_{n}; B) = E\Sigma_n,$$

$$\mathcal{Q}_B^{alg}(\underbrace{B, \ldots, B}_{n}, x_1, \ldots, x_n, \underbrace{B, \ldots, B}_{n}; y) = E\Sigma_{m+n} \times \mathcal{Q}(x_1, \ldots, x_n; y).$$

Reproducing Construction 2.11 verbatim in this context, we obtain a multifunctor

$$B_2\mathcal{Q} \to R - \text{modules},$$

as claimed (here $R = k_{\mathcal{C}}$). This is the first statement of Theorem 1.1.

To prove the second statement, recall that while the Elmendorf-Mandell machine does not preserve $\star$-structure, we may compose the multifunctor into R-modules with a functorial cofibrant resolution, in which case it turns universal multiplications into equivalences

$$M_{X_1} \wedge_R \cdots \wedge_R M_{X_n} \to M_{X_1 \star \cdots \star X_n},$$

as claimed. $\qquad \square$

3. A Special Example: Refinements of the Khovanov $\star$-functor $\mathcal{L}$

In this section, we construct our main example, which is the interpretation of Khovanov's construction [14], [3], as well as give an interpretation of Lipshitz-Sarkar's construction [20] in this language.

3.1. **Khovanov's original functor.**

Let us start with the "classical" example, i.e., with our interpretation of Khovanov's original construction [14], [3]. Assume that A is a commutative Frobenius algebra over a commutative ring R, i.e., that there is an augmentation R-module homomorphism

$$\epsilon : A \to R$$

such that the pairing

(3.1) $$A \otimes_R A \xrightarrow{\text{prod}} A \xrightarrow{\epsilon} R$$

is a non-degenerate bilinear pairing over R. It is well known that such an A gives rise to a $1 + 1$-dimensional TQFT where the field operators corresponding to pairs of pants with two inbound and one outbound (two outbound and one inbound, respectively) boundary components are given by the product and coproduct, respectively. Here the coproduct is the dual of the product with respect to the pairing (3.1). In our language, at least when A is a free R-module on a given basis Λ, this specifies a multifunctor

(3.2) $$\mathcal{L} : \mathcal{A} \to R_2.$$

The basis Λ becomes the value of the multifunctor on the object S^1. The example, interesting from the point of view of (SL_2) Khovanov homology, is $A = \Lambda_{\mathbb{Z}}[x]$. In this case, let $\Lambda = \{1, x\}$ (so $x^2 = 0$).

The reader should be reminded that in Khovanov's construction [14], [3], the special structure of $A = \Lambda_{\mathbb{Z}}[x]$ plays a crucial role. Essentially, one needs the sequence

$$0 \longrightarrow R \xrightarrow{1} A \xrightarrow{\epsilon} R \longrightarrow 0$$

to be exact. This is a property of $A = \Lambda_{\mathbb{Z}}[x]$ which does not happen often in Frobenius algebras. While analogues of Khovanov homology for other Frobenius algebras have since been discovered [15], [27], the construction is much more involved than a straightforward analogue of the original construction [14], [3].

3.2. **Some remarks on refining the Khovanov functor to a $\mathcal{V}_2$-valued $\star$-functor: Why spin is needed.**

We originally tried to refine the Khovanov $\star$-functor (3.2) to a $\star$-functor from $\mathcal{A}$ to $\mathcal{V}_2$. We quickly realized, however, that this cannot work: We cannot construct a topological modular functor in the sense encountered, say, in the context of rational conformal field theory (RCFT) [2], [10], [8], [25]. One point is that in that setting, $\Lambda(x)$ would be the Verlinde ring of the modular functor $\mathcal{L}$. This is generally not allowed, as

the Verlinde conjecture [26] asserts that the Verlinde ring, when tensored with $\mathbb{C}$ (i.e., the Verlinde algebra), is semisimple, which is certainly not the case of $\Lambda(x)$. This is, however, not a definitive argument: While there are proofs of the Verlinde conjecture ([22], [11], [12]), these depend on concrete axiom systems for RCFT, which build in semisimplicity by requiring unitarity, so a generalization suitable for our purposes could still exist.

On the other hand, one can see more directly why a topological modular functor $\mathcal{L}$ in the naive sense cannot exist: The mapping class group of a genus 1 oriented surface is $SL_2(\mathbb{Z})$ and is generated by Dehn twists. However, Dehn twists are required to map to trivial 2-isomorphisms by the $\mathcal{L}$-functor because they can be realized on an annulus 1-morphism, on which the value of $\mathcal{L}$ is isomorphic to the value of $\mathcal{L}$ on a unit disk, which has a trivial mapping class group (since one can always attach a cap to one end of the annulus). On the other hand, consider the gluing in $\mathcal{L}$ corresponding to the coproduct in $\Lambda(x)$ followed by the product, which gives

(3.3)
$$1 \mapsto 1 \otimes x + x \otimes 1 \mapsto 2x.$$

Consider the non-trivial central element
$$z = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in SL_2(\mathbb{Z}),$$
which corresponds to switching the two components in the middle of the gluing. Hence, the value of $\mathcal{L}$ on z must switch the two summands corresponding to $1 \otimes x$ and $x \otimes 1$ in (3.3), and hence cannot be trivial, which is a contradiction. One clue was that it might actually help to replace $\mathcal{A}$ by $\mathcal{A}_s^A$. In the context of RCFT [22], modular functors are generally not topological, as they carry an invariant called *central charge*. Depending on the value of the central charge, however, the modular functors one encounters can sometimes be made topological by the following maneuver: One could tensor with the inverse of modular functors which are invertible with respect to the tensor product (i.e., 1-dimensional). What invertible modular functors one encounters depends on the exact axiomatization; a classification is given in [17]. Without adding any structure, the invertible modular functor of the smallest positive central charge is $Det^{\otimes 2}$ of central charge 4. Therefore, a modular functor of central charge divisible by 4 can be made topological by tensoring with a power of $Det^{\otimes 2}$. One has Det of central charge 2 if one allows *superstructure*, i.e., $\mathbb{Z}/2$-grading of the modular functor. Superstructure would not be fatal to our application, as the $\mathbb{Z}/2$ corresponding to the grading is known to twist K-theory (see [1]). In other words, one can replace the target category $\mathcal{V}$ by the category of super vector spaces (see also [18]).

However, even using *Det*, one can only rectify modular functors of even integral central charge into topological ones. One can do better if one introduces spin: There is an invertible supermodular functor of central charge 1 which corresponds to the 2-dimensional chiral fermion RCFT. There is *not* an invertible supermodular functor of central charge $1/2$ which would correspond to the 1-dimensional chiral fermion, but a part of the modular functor restricted to $\mathcal{A}_s^A$ (i.e., boundary components with antiperiodic spin structure) does exist (see [17]) and, moreover, on this restriction to $\mathcal{A}_s^A$, the superstructure trivializes.

Of course, since we have not constructed an RCFT in any generalized sense which would correspond to $\mathcal{L}$, we do not know what its central charge would be. However, we see that spin can help in making the functor topological, as long as the central charge is a multiple of $1/2$ and as long as we restrict to $\mathcal{A}$. We do not know if the restriction to $\mathcal{A}_s^A$ is necessary when defining a $\mathcal{V}_2$-refinement of $\mathcal{L}$, as constructing a modular functor with spin including periodic boundary components is much harder to do "by hand."

3.3. A $\mathcal{V}_2$-refinement of the Khovanov $\star$-functor.

We will now construct "by hand" a certain lax $\star$-functor

$$(3.4) \qquad\qquad \mathcal{L}_s : \mathcal{A}_A^s \to \mathcal{V}_2.$$

On objects, let C be a closed 1-manifold with spin structure such that every connected component is antiperiodic. Denote the set of connected components of C by $\pi_0(C)$. Then let

$$(3.5) \qquad\qquad \mathcal{L}_s(C) = \prod_{c \in \pi_0(C)} \{1, x\}.$$

Before specifying the effect of $\mathcal{L}$ on 1-morphisms and 2-morphisms, we introduce the following terminology for boundary components of a compact oriented surface Σ with spin structure, whose boundary components are labeled 1 or x: A *true inbound* boundary component is an inbound boundary component labeled 1 or an outbound boundary component labeled x. A *true outbound* boundary component is an outbound boundary component labeled 1 or an inbound boundary component labeled x.

Now, for a (2-dimensional) oriented spin cobordism Σ with antiperiodic boundary components, define $\mathcal{L}_s(\Sigma)$ as follows. Let $to(\Sigma)$ denote the number of true outbound boundary components of Σ, and let $g(\Sigma)$ denote the genus of Σ.

If Σ is connected, then

$$(3.6) \qquad \mathcal{L}_s(\Sigma) = \begin{cases} \mathbb{C} & \text{if } g(\Sigma) = 0 \text{ and } to(\Sigma) = 1 \\ \mathbb{C} \oplus \mathbb{C} & \text{if } g(\Sigma) = 1 \text{ and } to(\Sigma) = 0 \\ 0 & \text{else.} \end{cases}$$

By definition of a $\star$-functor, we must, of course, for a general cobordism Σ, have

$$\mathcal{L}_s(\Sigma) = \bigotimes_{\Sigma'} \mathcal{L}(\Sigma')$$

where Σ' runs through the connected components of Σ.

Lemma 3.1. *This defines a lax $\star$-functor:*

$$\mathcal{L}_s : \mathcal{A}_A^s \to \mathcal{V}_2.$$

Proof. Check the axioms. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Regarding 2-isomorphisms, any 2-isomorphism between spin cobordisms of genus 0 is sent to the identity. To go further, it is convenient to introduce some terminology. By a *reference curve* in a genus 1, Kervaire invariant 0 (2-dimensional) spin cobordism Σ with antiperiodic boundary components only, we mean an isotopy class of non-separating antiperiodic closed oriented curves in Σ. Let $\check{\Sigma}$ denote the surface obtained from Σ by gluing disks to all boundary components. Without loss of generality, a reference curve α_Σ is chosen in each Kervaire invariant 0 genus 1 spin cobordism Σ with antiperiodic boundary components.

Now let $f : \Sigma \to T$ be a 2-morphism where Σ and T are of genus 1, Kervaire invariant 0. Let $\alpha \in H_1(\check{T}, \mathbb{Z})$ be the homology class represented by α_T. Let (α, β) be any ordered basis of $H_1(\check{T}, \mathbb{Z})$ containing α. Let

$$f(\alpha_\Sigma) = k\alpha + \ell\beta \in H_1(\check{T}, \mathbb{Z}).$$

Then

$$(3.7) \quad \mathcal{L}_s(f) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } k \equiv 1 \mod 4, \ell \equiv 0 \mod 2 \\[2.5em] \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \text{if } k \equiv -1 \mod 4, \ell \equiv 0 \mod 2 \\[2.5em] \begin{pmatrix} \frac{1+i}{2} & \frac{1-i}{2} \\ \frac{1-i}{2} & \frac{1+i}{2} \end{pmatrix} & \text{if } k \equiv 0 \mod 2, \ell \equiv 1 \mod 4 \\[2.5em] \begin{pmatrix} \frac{1-i}{2} & \frac{1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{pmatrix} & \text{if } k \equiv 0 \mod 2, \ell \equiv -1 \mod 4. \end{cases}$$

It is easy to show that those are the only possibilities for k and ℓ, and that the definition does not depend on the choice of β.

Note that in all other (connected) cases of $f : \Sigma \to T$ and $\mathcal{L}_s(f) : 0 \to 0$, there is no choice.

This does not quite conclude the definition of $\mathcal{L}_s$. Since $\mathcal{L}_s$ is a lax multifunctor, we must specify a 2-morphism

$$\mathcal{L}_s(f) \circ (\mathcal{L}_s(g_1), \ldots, \mathcal{L}_s(g_n)) \to \mathcal{L}_s(f \circ (g_1, \ldots, g_n))$$

where applicable. As it turns out, the only non-trivial case occurs when we are gluing genus 0 connected cobordisms Σ and Σ' into a genus 1 connected cobordism. In this case, let

$$1 \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

if the true outbound boundary component c of Σ (or, equivalently, Σ') maps (with orientation) to $\alpha \in H_1(T, \mathbb{Z})$.

It then follows from the structure that if

$$c \mapsto k\alpha + \ell\beta \in H_1(T, \mathbb{Z}),$$

then

$$1 \mapsto \begin{cases} \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \text{if } k \equiv 1 \mod 4,\, \ell \equiv 0 \mod 2 \\[2ex] \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \text{if } k \equiv -1 \mod 4,\, \ell \equiv 0 \mod 2 \\[2ex] \begin{pmatrix} \frac{1+i}{2} \\ \frac{1-i}{2} \end{pmatrix} & \text{if } k \equiv 0 \mod 2,\, \ell \equiv 1 \mod 4 \\[2ex] \begin{pmatrix} \frac{1-i}{2} \\ \frac{1+i}{2} \end{pmatrix} & \text{if } k \equiv 0 \mod 2,\, \ell \equiv -1 \mod 4, \end{cases}$$

and no other possibility can arise.

Remark 3.2. It is possible to use the functor $\mathcal{L}_s$ to define a k-module refinement of Khovanov homology. When we did this in the original version of this paper, however, we eventually observed that spin completely drops out of the story (by a mechanism which we will briefly describe below). This is the effect of a geometric principle which we will now discuss.

3.4. The Lipshitz-Sarkar refinement of the Khovanov functor.

What is, in fact, happening is that it suffices to construct a "field theory" on $\mathcal{A}_K$, i.e., an "embedded field theory." Indeed, reinterpreting the

construction of Lipshitz and Sarkar [20], one can construct a lax functor

(3.8) $$\mathcal{L}_K : \mathcal{A}_K \to \mathcal{S}_2$$

(see §2.5 for the definition of $\mathcal{S}_2$). Note again that $\mathcal{A}_K$ is not a $\star$-category so we lose the possibility of a $\star$-structure, but, on the other hand, composing with the Elmendorf-Mandell machine (or, alternately, essentially any infinite loop space machine which lands in symmetric spectra), we obtain a functor

$$B_2 \mathcal{A}_K \to Sym,$$

which is sufficient, since symmetric spectra are the same thing as modules over the sphere spectrum in that category.

The construction of (3.8) is, in a way, similar to the construction of (3.4). On objects, use the same definition as for $\mathcal{L}_s$ (see (3.5)). On 1-morphisms, we also adapt definition (3.6): For a connected $1 + 1$-cobordism Σ embedded in $S^2 \times [0,1]$ whose boundary is in $S^2 \times \{0,1\}$ which Σ meets transversally, we set

(3.9) $$\mathcal{L}_K(\Sigma) = \begin{cases} \{1\} & \text{if } g(\Sigma) = 0 \text{ and } to(\Sigma) = 1 \\ \{1,2\} & \text{if } g(\Sigma) = 1 \text{ and } to(\Sigma) = 0 \\ \emptyset & \text{else.} \end{cases}$$

In general, we set

$$\mathcal{L}_K(\Sigma) = \prod_{\Sigma'} \mathcal{L}_K(\Sigma')$$

where Σ' runs through the connected components of Σ.

But how can we make consistent choices of $\mathcal{L}_K$ on 2-morphisms when the "square root" of the transposition map $c : \{1,2\} \to \{1,2\}$ cannot be a map of sets and only exists as a morphism of $\mathbb{C}$-vector spaces?

Remark 3.3. The answer is at the heart of the problem and was essentially discovered by Lipshitz and Sarkar [20] in their concept of ladybug matching. In the language of the present paper, the point is that embedding into $S^2 \times [0,1]$ severely restricts modular transformations. In fact, the embedded mapping class group of an unknotted torus T embedded in $S^2 \times [0,1]$ is $\mathbb{Z}/2$. For if we choose the reference curves α and β to be fundamental cycles representing the inside and outside of T, then α and β must be preserved up to orientation, and their orientations must be either both preserved or both reversed to preserve the orientation of T.

If σ is the generator of this $\mathbb{Z}/2$, we define

(3.10) $$\mathcal{L}_K(\sigma) = \tau.$$

Finally, we must define the composition isomorphism when gluing two genus 0 embedded connected cobordisms Σ and Σ' into a genus 1 connected cobordism. In this case, let

$$1 \mapsto 1$$

if the true outbound boundary component c of Σ (or, equivalently, of Σ') maps, with orientation, to α or β, and

$$1 \mapsto 2$$

if c maps to $-\alpha$ or $-\beta$. This definition depends on the choice of orientations of the generators α and β which indicates four possible choices, but we also have the possibility of simultaneously reversing the orientations of α and β (i.e., applying the modular transformation σ), which equates two and two of the choices. Therefore, there are two intrinsically different choices to make, which correspond to the left and right ladybug matchings of [20]. (Also see [19] for another description.)

4. The Khovanov Cube Functor

In this section, we construct our refinement of Khovanov's cube functor construction.

4.1. Lax categories.

Let $\mathcal{C}$ be a small category. We define a category $\mathcal{C}'$ enriched in groupoids where

$$Obj(\mathcal{C}') = Obj(\mathcal{C}) \text{ and}$$
$$Mor_1(\mathcal{C}') = \Gamma\mathcal{C}$$

where Γ denotes the free category on a directed graph (a *directed graph* is a pair of maps S and T from a set of *morphisms* to a set of *objects*). Here, we regard $\mathcal{C}$ as a graph by forgetting that compositions exist.

There is a canonical functor

$$\theta : \Gamma\mathcal{C} \to \mathcal{C}$$

(the monad structure). There is a single 2-isomorphism in $\mathcal{C}'$ between any two morphisms whose images under θ coincide.

4.2. **Links and link cobordisms.**

Let L be a link with spin structure, and let $\mathcal{D}$ be a non-degenerate projection of L, i.e., an immersion into S^2 with only at most transverse double points (i.e., where crossings occur at angles $\neq 0, \pi$). Label the crossings of $\mathcal{D}$ by $1, 2, \ldots, n$. For the ith crossing, select a disk D_i which is a neighborhood of the crossing, such that $D_1, \ldots, D_n$ are disjoint. Recall that for $\epsilon = 0, 1$, the ϵ-resolution is obtained by replacing a chosen crossing by a non-crossing according to Figure 1.
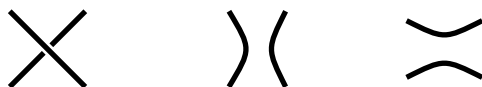
FIGURE 1. A crossing, the 0-resolution, the 1-resolution

Recall from [14] that the *link cobordism* $\Sigma_{\mathcal{D}}$ is obtained by taking

$$(4.1) \qquad \left(\mathcal{D} \smallsetminus \left(\bigcup_{i=1}^{n} D_i \cap \mathcal{D} \right) \right) \times I$$

and, for each crossing, gluing in an ϵ-resolution of the crossing at $(D_i \cap \mathcal{D}) \times \{\epsilon\}$ for $\epsilon = 0, 1$ and a saddle between the two crossings in $(D_i \cap \mathcal{D}) \times I$.

Observe that $\Sigma_{\mathcal{D}}$ can be obtained by taking a ribbon along L which takes a half-twist at each crossing (thus creating a horizontal square) and is vertical elsewhere, and identifying the two horizontal squares over each crossing. Note that the ribbon always has an even number of half-twists since there are two per crossing. Hence, the ribbon may be identified with $\tau_L \otimes_{\mathbb{R}} \mathbb{C}$.

Comment 4.1. To avoid confusion, note that in the present paper, by a link cobordism we mean the surface $(1 + 1$-embedded cobordism) associated with a link projection, *not* a cobordism of links.

Now let us observe that complete resolutions of a link projection are, by definition, objects of $\mathcal{A}_K$, and the cobordisms Σ_L are 1-morphisms on $\mathcal{A}_K$. Let us also make another observation: Let $\mathcal{D}$ be a non-degenerate link projection; label its crossings $D_1, \ldots, D_n$. Let $\mathcal{D}'$ be the projection obtained by taking 0-resolutions of $D_1, \ldots, D_k$ and let $\mathcal{D}''$ be the projection obtained by taking 1-resolutions of $D_{k+1}, \ldots, D_n$. Then there is a canonical 2-isomorphism in $\mathcal{A}_K$

$$(4.2) \qquad \Sigma_{\mathcal{D}''} \circ \Sigma_{\mathcal{D}'} \xrightarrow{\cong} \Sigma_{\mathcal{D}.}$$

We call these 2-isomorphisms *gluing isomorphisms*.

4.3. **Definition of the lax cube functor.**

Now let I be the category with two objects 0 and 1 and a single morphism $0 \to 1$ (and no morphism $1 \to 0$). Now let $\mathcal{D}$ be an admissible link projection with n crossings. Then $\mathcal{D}$ determines a lax functor

$$\mathcal{C} : (I^n)' \to \mathcal{A}_K$$

as follows. (The category $(I_n)'$ is defined in §4.1.)

To define $\mathcal{C}$ on objects,

$$(\epsilon_1, \ldots, \epsilon_n) \text{ where } \epsilon_i \in \{0, 1\},$$

maps to the complete resolution of $\mathcal{D}$ obtained by taking the ϵ_i-resolution at the ith crossing.

On 1-morphisms, consider the I^n-morphism

$$(\epsilon_1, \ldots, \epsilon_n) \to (\epsilon'_1, \ldots, \epsilon'_n).$$

Let $J \subseteq \{1, \ldots, n\}$ be the subset such that $\epsilon'_i = \epsilon_i + 1$ for $i \in J$ and $\epsilon'_i = \epsilon_i$ for $i \notin J$. Then this I^n-morphism is sent to

$$\Sigma_{\mathcal{D}'}$$

where $\mathcal{D}'$ is the projection obtained from $\mathcal{D}$ by taking the ϵ_i-resolution at the ith crossing for all $i \notin J$.

The 2-morphisms of $(I^n)'$ are sent to gluing isomorphisms of the surfaces $\Sigma_{\mathcal{D}'}$ for different $\mathcal{D}'$, by their shared boundary component.

4.4. **The spin data.**

In this subsection, we will discuss directly spin structures on link projections. While this gives additional geometric insight into the k-theory refinement of Khovanov homology, this material is not strictly necessary to follow the progression of the paper, and the reader interested only in the proof of Theorem 1.3 may skip it.

By a *link with spin structure*, we mean a real bundle $\tau_L^{1/2}$ together with an isomorphism

$$\tau_L^{1/2} \otimes_{\mathbb{R}} \tau_L^{1/2} \to \tau_L$$

where τ_L is the tangent bundle of L. Note that this specifies an orientation on τ_L where we call a tangent vector *positive* if it has a square root in $\tau_L^{1/2}$.

By a *projection with spin*, we mean a non-degenerate projection $\mathcal{D}$ of L together with a *spin* of the self-identification of the ribbon $\tau_L \otimes_{\mathbb{R}} \mathbb{C}$ along each crossing square, namely an automorphism of the bundle $\tau_L^{1/2} \otimes_{\mathbb{R}} \mathbb{C}$ which covers the identity on $\tau_L \otimes_{\mathbb{R}} \mathbb{C}$. By gluing of bundles, this data, given in a projection with spin, specifies a spin structure on $\Sigma_{\mathcal{D}}$.

Recall that for a complex 1-manifold Σ with spin (i.e., a complex line bundle $\tau_\Sigma^{1/2}$ and an isomorphism $\tau_\Sigma^{1/2} \otimes_{\mathbb{C}} \tau_\Sigma^{1/2} \cong \tau_\Sigma$) and an oriented curve c in Σ, we have a determined spin structure on c where $(\tau_c^{1/2})_x$ is spanned by the $(\tau_\Sigma^{1/2})_x$-square roots of a positive tangent vector to c at x.

We call a projection $\mathcal{D}$ with spin of L *admissible* if the induced spin structure on every non-self-intersecting circuit in $\mathcal{D}$ is antiperiodic. (Recall Example 2.4(2)) It suffices to verify this condition for faces.

Now there is an obvious way (by sliding) to give spin structure to R2- and R3-moves. R1-moves require a more detailed discussion, as they do interfere with spin. When making an R1-move, we create a new face which borders the edge created by the R1-move only. Since we will be primarily interested in admissible projections with spin, we will be interested in only R1-moves where the new face has an antiperiodic spin structure. Given this condition, there are two possible ways of introducing a spin structure on the projection after the R1-move: One does not change the spin structure on the link L, but changes the spin structure on the two faces previously adjacent to the arc on which we performed the R1-move; we will call this an *R1L-move*. Taking an R1L-move and changing the spin structure of the resulting projection by reversing the gluing of the spin structure on the new crossing and also in the middle of the new arc created by the move, we obtain a move which does not change the spin structures of any of the faces of the old projection, but reverses the spin structure of the connected component of L on which we performed the move. We will call this an *R1A-move*.

Note that two R1L-moves on the same arc of a projection with spin is the same as a pair of R1A-moves on the same arc: The resulting move changes neither the spin structures of any of the faces of the old projection nor the spin structure of the link. We will call such a pair a *pair of adjacent R1A-moves*.

Lemma 4.2. *An admissible projection with spin of a link L with spin always exists.*

Proof. Start with any projection with spin. Making $\{A, P\}$ into a group by making A the neutral element, the spin structure of the infinite face is the product of the spin structures of the finite faces, and hence there are an even number of P-faces, including the infinite face. This specifies a $\mathbb{Z}/2$-valued 0-cycle ζ on the CW-decomposition of S^2 dual to $\mathcal{D}$, such that the augmentation of ζ is 0, and hence ζ is a boundary and $\zeta = dc$ for some $\mathbb{Z}/2$-valued 1-chain c. The 1-chains of $\mathcal{D}$ and its dual are the same; perform an R1L-move on each arc of $\mathcal{D}$ on which c has coefficient 1. $\square$

In fact, we have a stronger statement.

Lemma 4.3. *Consider a non-degenerate projection $\mathcal{D}$ (without spin) of a link L. Then an admissible spin structure on $\mathcal{D}$ always exists and any two admissible spin structures on $\mathcal{D}$ (for any spin structure on L) are isomorphic. In particular, the spin structure on L is determined.*

Proof. Consider the link cobordism $\Sigma_{\mathcal{D}}$ associated with $\mathcal{D}$. Then $\Sigma_{\mathcal{D}}$ is an oriented surface, so the embedding $\Sigma_{\mathcal{D}} \subset \mathbb{R}^3$ extends to an embedding $\Sigma_{\mathcal{D}} \times I \to \mathbb{R}^3$, and the spin structure extends, of course, uniquely to $\Sigma_{\mathcal{D}} \times I$. Smooth out $\Sigma_{\mathcal{D}} \times I$ into a manifold with boundary $\widetilde{\Sigma_{\mathcal{D}} \times I}$. Now since the spin structure on $\Sigma_{\mathcal{D}}$ is admissible, we may attach a disk D_f to each face f of $\mathcal{D}$ in $\Sigma_{\mathcal{D}}$ and extend the spin structure. Hence, we may attach a copy of $D_f \times I$ to $\widetilde{\Sigma_{\mathcal{D}} \times I}$ (and again smooth) for each face f of $\mathcal{D}$ and extend the spin structure to the resulting manifold Γ with boundary.

The manifold Γ, however, is diffeomorphic to D^3 and, hence, has a unique spin structure (up to isomorphism). This means that any two admissible spin structures on $\Sigma_{\mathcal{D}}$ are isomorphic.

Conversely, the same construction also implies that an admissible spin structure always exists. $\square$

Proposition 4.4. *Two admissible projections with spin $\mathcal{D}$ and $\mathcal{D}'$ represent isomorphic links with spin if and only if they are related by R2-moves, R3-moves, and pairs of adjacent R1-moves. Two admissible projections with spin $\mathcal{D}$ and $\mathcal{D}'$ represent isomorphic links without spin if and only if they are related by R2-moves, R3-moves, and R1A-moves.*

Proof. Consider first the second statement. Sufficiency is obvious as the Reidemeister moves do not change the isomorphism class of the link (without spin). To prove necessity, suppose $\mathcal{D}$ and $\mathcal{D}'$ are admissible projections which represent isomorphic links (without spin). As is well known, disregarding spin, $\mathcal{D}$ can be converted to $\mathcal{D}'$ by a sequence of R1-moves, R2-moves, and R3-moves. Now we may give spin to the moves (preserving admissibility) by interpreting the R1-moves as R1A-moves. By Lemma 4.3, the admissible spin structure on $\mathcal{D}'$ obtained by the moves is the same as the admissible spin structure originally given.

Now consider the first statement (on links with spin structure). Again, sufficiency is obvious as R2-moves, R3-moves, and pairs of adjacent R1-moves do not change the spin structure of the underlying link. To prove necessity, suppose $\mathcal{D}$ and $\mathcal{D}'$ represent the same link with spin structure. Proceed in the same way as in the part of the statement on links without spin. Note, in particular, that the argument there does not depend on the order of the Reidemeister moves chosen. By Alexander Coward [5, Theorem 1], we may choose the moves in such a way that all the R1 moves come first, followed by R2-moves, R3-moves, and reversed R2-moves. Now since

we interpret the R1-moves as R1A-moves when considering spin, there must be an even number of such moves on each connected component of the link in order for the spin structures on the links corresponding to $\mathcal{D}$ and $\mathcal{D}'$ to be the same. However, note that a pair of R1A-moves on the same connected component of a link L can always be obtained as a pair of adjacent R1-moves, followed by R3 and R2 (and possibly reversed R2) moves. $\qquad\square$

Analogously to §4.3, an admissible link projection with spin with n crossings now directly determines a lax functor

$$\mathcal{C}_s : (I^n)' \to \mathcal{A}_s^A.$$

Note that for $\mathcal{D}$ and $\mathcal{D}'$ as above, we have a unique, up to isotopy, inclusion

$$\Sigma_{\mathcal{D}'} \subseteq \Sigma_{\mathcal{D}}$$

commuting with the projection to $\mathbb{R}^2$, so there is a canonical spin structure on $\Sigma_{\mathcal{D}'}$ induced from the spin structure on $\Sigma_{\mathcal{D}}$.

Note that Lemma 4.3 also implies that there is a canonical lax functor

$$(4.3) \qquad\qquad\qquad \mathcal{A}_K \to \mathcal{A}_s^A,$$

so we could simply obtain $\mathcal{L}_s$ as a composition of (4.3) with $\mathcal{L}_K$. This way, however, we lose the $\star$-structure since $\mathcal{A}_K$ is not a $\star$-category and (4.3) is not a $\star$-functor.

5. Stable Homotopy Realization and Link Invariance

Now let $\mathcal{D}$ be an admissible projection of a link L. (Note: Spin structure is not used in this section.) In §4.3, we constructed a lax functor

$$\mathcal{C} : (I^n)' \to \mathcal{A}_K.$$

In §3, we constructed a lax functor

$$\mathcal{L}_K : \mathcal{A}_{\mathcal{K}} \to \mathcal{S}_2.$$

In Lemma 2.9, we further constructed a lax multifunctor

$$\Phi : \mathcal{S}_2 \to Perm.$$

Then by the remark at the end of §2.10, the composition $\Phi\mathcal{L}_K\mathcal{C}$ is canonically converted into a *strict* functor

$$(5.1) \qquad\qquad\qquad \Delta_{\mathcal{D}} : I^n \to Sym.$$

Remark 5.1. We may, of course, smash the functor (5.1) with k in the category of symmetric spectra. Alternately, we may directly consider the composition

$$(5.2) \qquad\qquad\qquad (I^n)' \to \mathcal{A}_K \to \mathcal{A}_s^A.$$

By Construction 2.11, the $\star$-functor

$$\mathcal{L}_s : \mathcal{A}_s^A \to \mathcal{V}_2$$

determines a lax functor

$$(I^n)'_B \longrightarrow \mathcal{V}_2 \xrightarrow{\Phi} Perm$$

which, by Construction 2.10, gives a strict functor

$$I^n \to R - \text{modules}$$

where R is the strictly commutative symmetric ring spectrum arising by the Elmendorf-Mandell machine [7] from the bipermutative category $\Phi(1)$. However, $\Phi(1)$ is the category $\mathcal{V}$ of finite-dimensional complex vector spaces and isomorphisms with its usual bipermutative category structure, so R is k, the connective k-theory strictly commutative symmetric ring spectrum. We have, therefore, constructed a strict functor

$$(5.3) \qquad\qquad \Delta_{\mathcal{D},s} : I^n \to k - modules.$$

While this direct construction contributes nothing to Theorem 1.3 as stated, it is interesting to note that it shows that the k-theory realization "remembers less data" about the structure of the link, since it depends on only the composition (5.2) and not the embedded link cobordism.

5.1. **The higher cofiber.**

The higher cofiber is a functor C_n from the category of diagrams

$$\Gamma : I^n \to R - modules$$

to the category of R-modules where R is a strictly commutative symmetric ring spectrum. Functors of such form are used extensively, for example, in Goodwillie calculus. (See, for example, [13] for an overview of such functors.)

One description of the higher fiber proceeds as follows. Consider the category $\mathcal{I}$ whose objects are functions $\phi : J \to \{0,1\}$ where $J \subseteq \{1,\ldots,n\}$ and there is a unique morphism $\phi \to \psi$ if and only if ϕ is a restriction of ψ. In other words, $\mathcal{I}$ can be thought of as

$$(0 \leftarrow \cdot \to 1)^n.$$

Then Γ specifies a functor

$$\widetilde{\Gamma} : \mathcal{I} \to R - modules$$

where

$$\widetilde{\Gamma}(\phi) = \begin{cases} * & \text{if } 0 \in Im(\phi) \\ \Gamma(1 - \chi_J) & \text{else} \end{cases}$$

where $\chi_J(x) = 1$ if $x \in J$ and $\chi_J(x) = 0$ if $x \notin J$. The value of $\widetilde{\Gamma}$ on morphisms is given by the corresponding morphism values of Γ when the target is not $*$ and by the trivial map else.

One defines

$$C_n\Gamma = hocolim\widetilde{\Gamma}.$$

(The right-hand side is well defined using the simplicial realization in R-modules.)

The advantage of the above description is that it is obviously symmetrical in the coordinates. There is an alternate elementary inductive description which is not symmetrical in coordinates, but symmetry is readily proved by equivalence with the above description.

We define $C_0\Gamma = \Gamma$. Assuming we have already defined C_{n-1}, define

$$\Gamma_\epsilon : I^{n-1} \to R - modules \text{ where } \epsilon = 0, 1$$

by

$$\Gamma_\epsilon = \Gamma(?, \dots, ?, \epsilon).$$

Then Γ gives a natural transformation

$$\iota : \Gamma_0 \to \Gamma_1.$$

Inductively, we get a natural transformation

$$C_{n-1}\iota : C_{n-1}\Gamma_0 \to C_{n-1}\Gamma_1.$$

Let $C_n\Gamma$ be the homotopy cofiber of $C_{n-1}\iota$.

In fact, the symmetric description of the higher cofiber immediately gives the following fact, which will be useful to us.

Lemma 5.2. *Let $\Gamma : I^n \to R - modules$ be a functor, and let $f : \{1, \dots, k\} \to \{1, \dots, n\}$ and $g : \{1, \dots, m\} \to \{1, \dots, n\}$ be maps such that $f \amalg g$ is a bijection (so, in particular, $n = k+m$). For $\phi : \{1, \dots, k\} \to \{0, 1\}$, define a functor $\Gamma_\phi : I^m \to R - modules$ by $\Gamma_\phi(\psi) = \Gamma_{(\phi \amalg \psi)(f \amalg g)^{-1}}$ for $\psi : \{1, \dots, m\} \to \{0, 1\}$. Then $C_m\Gamma_? : I^k \to R - modules$ is a functor in the obvious way. We have*

$$C_k(C_m\Gamma_?) = C_n\Gamma.$$

From now on, we shall work only with the Lipshitz-Sarkar realization, i.e., in the category Sym of symmetric spectra. Analogous results in k-modules follow by applying $? \wedge k$ or, alternately, using analogous reasoning directly for the k-module realization.

Now, recalling (5.3), we can assign a k-module $C_n\Delta_{\mathcal{D}}$ to an admissible projection $\mathcal{D}$ of a link L.

Theorem 5.3. *If $\mathcal{D}$ and $\mathcal{D}'$ are nondegenerate projections of a link L, then there exists an equivalence of symmetric spectra*

$$\Sigma^{-n_-(\mathcal{D})} C_n \Delta_{\mathcal{D}} \simeq \Sigma^{-n_-(\mathcal{D}')} C_n \Delta_{\mathcal{D}'}$$

where $n_-(\mathcal{D})$ denotes the number of negative crossings of the projection $\mathcal{D}$ (a number which does not depend on spin).

6. Proof of the Main Theorem

The proof of Theorem 5.3 basically mimics Khovanov's proof of the invariance of Khovanov homology (see [14], [3]). Of course, we cannot refer to elements and take chain differentials; we must phrase everything in the language of categories. We begin with two lemmas on higher cofibers.

Lemma 6.1. *Consider a diagram M of symmetric spectra*

(6.1)
$$
\begin{array}{ccc}
M_{10} & \xrightarrow{\ \gamma\ } & M_{11} \\
\beta \uparrow & & \uparrow \delta \\
M_{00} & \xrightarrow{\ \alpha\ } & M_{01}
\end{array}
$$

and suppose there exists a map of R-modules $s : M_{11} \to M_{10}$ such that

$$\gamma s = Id,$$

$$\beta \vee s : M_{00} \vee M_{11} \to M_{10} \text{ is an equivalence.}$$

Then

$$C_2 M \simeq \Sigma M_{01}.$$

Proof. The commutative diagram

(6.2)
$$
\begin{array}{ccc}
M_{00} \vee M_{11} & \xrightarrow{\gamma \circ (\beta \vee s)} & M_{11} \\
\iota_0 \uparrow & & \uparrow \delta \\
M_{00} & \xrightarrow{\ \alpha\ } & M_{01}
\end{array}
$$

maps into (6.1) by the map $\beta \vee s$ in the upper left corner and identity elsewhere and, hence, has an equivalent 2-cofiber since $\beta \vee s$ is an equivalence. Now since $\gamma s = Id$, diagram (6.2) maps into

(6.3)
$$
\begin{array}{ccc}
M_{00} & \longrightarrow & 0 \\
Id \uparrow & & \uparrow \\
M_{00} & \xrightarrow{\ \alpha\ } & M_{01}
\end{array}
$$

with cofiber

$$
\begin{array}{ccc}
M_{11} & \xrightarrow{\ Id\ } & M_{11} \\
\uparrow & & \uparrow \\
0 & \longrightarrow & 0
\end{array}
$$

so the 2-cofiber of (6.3) is equivalent to the 2-cofiber of (6.2). But (6.3), in turn, maps into

$$
\begin{array}{ccc}
M_{00} & \longrightarrow & 0 \\
\scriptstyle Id\ \uparrow & & \uparrow \\
M_{00} & \longrightarrow & 0
\end{array}
$$

with fiber

(6.4)
$$
\begin{array}{ccc}
0 & \longrightarrow & 0 \\
\uparrow & & \uparrow \\
0 & \longrightarrow & M_{01}.
\end{array}
$$

So the 2-cofiber of (6.4) is equivalent to the 2-cofiber of (6.1).   □

Lemma 6.2. *Consider a diagram N of the form*

(6.5)

Assume there exists a map $s : N_{111} \to N_{101}$ such that

$$
\gamma s = Id \quad and
$$

$$
\beta \vee s : N_{001} \vee N_{111} \to N_{101} \text{ is an equivalence,}
$$

and assume further that there exists a map

$$
t : N_{100} \to N_{001}
$$

such that

$$
\beta t = \epsilon \text{ and } \zeta = \eta t.
$$

Then

$$C_3 N \simeq \Sigma C_2 M$$

where M is the diagram

(6.6)

$$
\begin{array}{ccc}
N_{100} & \xrightarrow{\;\mu\Pi\alpha t\;} & N_{110}\Pi N_{011} \\
\eta \uparrow & & \uparrow \lambda\Pi\kappa \\
N_{000} & \xrightarrow{\;\nu\;} & N_{010}.
\end{array}
$$

Proof. Into (6.5), there are maps

(6.7)

where the map on the 101-corner is the equivalence

$$\beta \vee s : N_{001} \vee N_{111} \to N_{101}.$$

As in Lemma 6.1, (6.7) maps into

(6.8)

with fiber

$$
\begin{array}{ccc}
N_{111} & \xrightarrow{\;Id\;} & N_{111}
\end{array}
$$

By a standard construction in homotopy theory, a diagram of the form
(6.8) can be "folded" into the suspension of the diagram

$$
(6.9) \qquad
\begin{array}{ccc}
(N_{100}\Pi N_{001})' & \xrightarrow{\;\phi\;} & N_{110}\Pi N_{011}\Pi\widetilde{N_{001}} \\
{\scriptstyle (\eta\Pi s)'}\uparrow & & \uparrow{\scriptstyle \lambda\Pi\kappa\Pi 0} \\
(N_{000})' & \xrightarrow{\;\nu\;} & N_{010}
\end{array}
$$

where $(?)'$ denotes cofibrant replacement and $\widetilde{(?)}$ denotes fibrant replace-
ment, and ϕ is the product of

$$
N_{100}\Pi N_{001} \xrightarrow{\;\mu p_1\;} N_{110}
$$

$$
N_{100}\Pi N_{001} \xrightarrow{\;\alpha p_2\;} N_{011}
$$

$$
(N_{100}\Pi N_{001})' \xrightarrow{\;t-Id\;} \widetilde{N_{001}}.
$$

Diagram (6.9) commutes up to homotopy, but can be converted into a
strict diagram by standard techniques (for example, by [7, Theorem 1.4]).
(Note: These complications are, of course, caused by the fact that the
canonical map $(Id \vee 0)\Pi(0 \vee Id) : A \vee B \to A\Pi B$ is an equivalence but
not an isomorphism in the category of R-modules.)

Now into (6.9), there are maps

$$
\begin{array}{ccc}
(N_{001})' & \xrightarrow{\;Id\;} & (N_{001})' \\
\uparrow & & \uparrow \\
0 & \longrightarrow & 0
\end{array}
$$

where the upper left corner maps by $0\Pi Id$ and the upper right corner by
$0\Pi\alpha\Pi Id$ (omitting fibrant and cofibrant replacements from the notation).
The cofiber is equivalent to (6.6). $\qquad\square$

Proof of Theorem 5.3. As usual, it suffices to prove invariance under R1-moves, R2-moves, and R3-moves.

INVARIANCE UNDER THE R1 MOVE: After performing an R1-move, consider the restrictions of the lax functor $\mathcal{L}_K\mathcal{C}$ to the subcategory (enriched over groupoids) where there is a 0-resolution (1-resolution, respectively) of the new crossing created by the move. The cobordism from the 0-resolution to the 1-resolution will give a lax natural transformation η between these functors. Denote these restrictions by $\mathcal{L}_K\mathcal{C}_\epsilon$ where $\epsilon = 0, 1$. Depending on the sign of the move (which is, by definition, the sign of the new crossing), one of the resolutions will have an extra boundary component (the 0-resolution in case of a negative move and the 1-resolution in case of a positive move). The new boundary component can be labeled 1 or x, and this makes this functor $\mathcal{L}_K\mathcal{C}_\epsilon$ laxly isomorphic to two copies of the functor $\mathcal{L}_K\mathcal{C}_{1-\epsilon}$. Further, we can laxly split η by choosing this label to be 1 (in case of a negative move) and by forgetting the label (in case of a positive move). In either case, after applying the Elmendorf-Mandell machine, the cofiber of the realization of η becomes isomorphic to the realization of the other factor of $\mathcal{L}_K\mathcal{C}_\epsilon$, its suspension, respectively, i.e., the invariant before the move, its suspension, respectively, depending on whether the move was negative or positive.

INVARIANCE UNDER THE R2 MOVE: We use the "Khovanov bracket" notation of Dror Bar-Natan [3, Figure 2], omitting the suspensions.
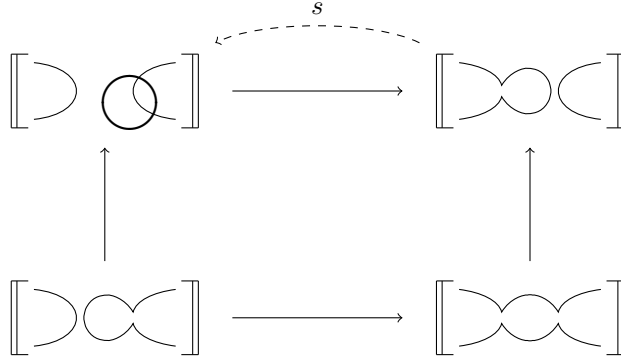

FIGURE 2

We give this picture, however, a modified interpretation: Each bracket denotes a lax functor $(I^n)' \to \mathcal{S}_2$ corresponding to the indicated partial resolution of the projection after the R2-move. The arrows in Figure 2 are lax natural transformations. With the notation of Lemma 6.1, the functor

s multiplies objects by the label 1 on the additional connected boundary component. On 1-morphisms, the functor s tensors a morphism with $\mathbb{C}$ and 2-isomorphisms with Id. Upon applying the Elmendorf-Mandell machine, including [7, Theorem 1.4], we can obtain a strict functor

$$\bullet \times I^n \to S - modules$$

which, up to equivalence, has the form

$$
\begin{array}{ccc}
M\Pi M & \xrightarrow{\ p_2\ } & M \\
\end{array}
$$

with $Id\Pi 0$ on top, $0\Pi Id$ on left, and $M \longrightarrow ?$, on bottom.

which implies the assumptions of Lemma 6.1. Here, M is as in Lemma 6.1, and the ? in the lower right corner is the argument of the functor.

INVARIANCE UNDER THE R3 MOVE: We again follow [3], adapting the proof to categories enriched in groupoids. In Figure 3, with the notation of Lemma 6.2, the construction of the s-map in the ??1-square in (6.5) is precisely the same as in the above treatment of the R2-move. Regarding the map t, note that the lax functors $(I^n)' \to \mathcal{A}_K$ at the 001 and 100 corners are canonically isomorphic (as are the partial resolutions drawn); let t be the canonical lax isomorphism. From this point on, apply the Elmendorf-Mandell machine and use Lemma 6.2.   □

Proof of Theorem 1.3. All that remains to show is that applying $? \wedge_k H\mathbb{Z}$ to our construction produces an $H\mathbb{Z}$-module which, using the equivalence [6, Chapter IV, section 2] produces a chain complex whose homology is Khovanov homology.

To prove this, we note that the strict symmetric ring spectrum unit

$$S \to H\mathbb{Z}$$

is realized, on the level of bipermutative categories, by the functor

$$\mathcal{S}_2 \to \mathbb{Z}$$

which assigns its cardinality to a finite set. We conclude that applying $? \wedge H\mathbb{Z}$ to our invariant is realized by taking the Khovanov cube functor
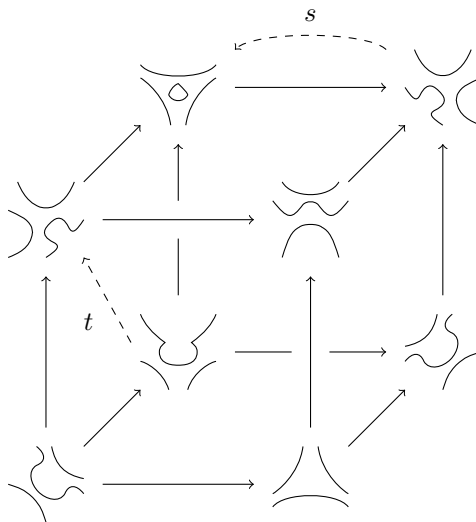
FIGURE 3

as mentioned in [3], and then applying to it the Elmendorf-Mandell machinery instead of the totalization described in [3]. (Smashing with $H\mathbb{Z}$ commutes with the Elmendorf-Mandell machine and with the iterated homotopy cofiber, since it is a left adjoint.)

One must, therefore, show that Elmendorf-Mandell machinery [7] to a diagram D of abelian groups (=$\mathbb{Z}$-modules) produces an $H\mathbb{Z}$-module corresponding, under the machinery of [6, Chapter IV, section 2], to the chain complex obtained as the homotopy colimit of the diagram D in the category of chain complexes. This follows from the fact that the equivalence [6, Chapter IV, section 2] commutes, up to equivalence, with simplicial realization. $\square$

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