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**TOWARDS A
GENERALIZATION OF
THE NESTED SOFT-COLLINEAR
SUBTRACTION SCHEME FOR NNLO
INFRA-RED DIVERGENCES
IN QCD**

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*The task is not so much to see what no one has yet seen,
but to think what nobody has yet thought,
about that which everybody sees.*

Erwin Schrödinger

Contents

Abbreviations	1
Introduction	3
Part I: Infra-red divergences in QCD	11
1 Generalities on QCD	11
1.1 The Standard Model	11
1.2 Basics of QCD	13
1.2.1 QCD Lagrangian and the color group $SU(3)$	13
1.2.2 Local gauge invariance and Feynman rules	15
1.2.3 The running of the strong coupling constant	16
1.2.4 The anomalous dimensions	20
1.3 Definition of cross section	21
1.4 The factorization theorem in hadronic cross sections	23
1.4.1 The birth of the parton model	23
1.4.2 The factorization theorem	25
1.5 Introduction to infra-red divergences in QCD	28
2 Universal factorization formulas in QCD	31
2.1 Dimensional regularization, the color space, and the color algebra	32
2.2 IR singular behavior of QCD virtual amplitudes	35
2.2.1 Factorization formula for one-loop amplitudes	35
2.2.2 Factorization formula for two-loop amplitudes	38
2.3 Soft and collinear singularities of $\mathcal{O}(\alpha_s)$ in tree-level amplitudes	41
2.3.1 Soft singularities	42
2.3.2 Collinear singularities	46
2.4 IR singularities of $\mathcal{O}(\alpha_s^2)$ in tree-level amplitudes	54
2.4.1 Double-soft singularities	54
2.4.2 Triple-collinear singularities	57

2.5	IR singularities of $\mathcal{O}(\alpha_s^2)$ in real-virtual amplitudes	59
2.5.1	Soft limit in real-virtual amplitudes	59
2.5.2	Collinear limit in real-virtual amplitudes	60
2.6	The need for IR regularization schemes	63
2.6.1	IR slicing and subtraction methods for differential cross sections	65
Part II: The Nested Soft-Collinear Subtraction Scheme		71
3	Old vs New approach to NSC	71
3.1	Notation	71
3.2	NSC at NLO	75
3.2.1	Real, virtual, and PDFs renormalization corrections	76
3.2.2	Obstacles to generalizing the subtraction formalism at NLO	82
3.3	The new approach applied to processes $\mathcal{A}_0$ and $\mathcal{B}_0$ at NLO	83
3.3.1	NLO corrections to $ab \rightarrow \{g\}_N + X$ (process $\mathcal{A}_0$)	83
3.3.2	Poles cancellation of process $\mathcal{A}_0$ at NLO	87
3.3.3	NLO corrections to $gq \rightarrow \{g\}_N + X$ (process $\mathcal{B}_0$)	91
4	New approach to NSC at NNLO	99
4.1	Introduction to NSC at NNLO	100
4.1.1	An Overview of the NSC at NNLO	100
4.1.2	Collinear regularization of double-real cross section	105
4.2	The new approach to NSC	108
4.2.1	A motivation for generalization	108
4.2.2	Treatment of unresolved quarks and gluons	110
4.3	Analyzing double- and single-collinear contributions: $m_g n_g$ case	114
4.3.1	$\Sigma_{RR,1c}^{(a,c,dc)} _{\mathcal{H},gg}$ contribution	115
4.3.2	$\Sigma_{RR,1c}^{(b,d)} _{\mathcal{H},gg}$ contribution	120
4.3.3	Double-collinear contribution	128
4.4	Analyzing double- and single-collinear contributions: $m_q n_g$ case	128
4.4.1	$\Sigma_{RR,1c}^{(a,c,dc)} _{\mathcal{H},qg}$ contribution	129
4.4.2	$\Sigma_{RR,1c}^{(b,d)} _{\mathcal{H},qg}$ contribution	131
4.5	Rearranging single- and double-unresolved terms	133
4.5.1	The $m_g n_g$ case	134
4.5.2	The $m_q n_g$ case	137
5	Calculation of the NNLO double-unresolved contributions	139
5.1	Simplifying virtual and soft corrections	141
5.1.1	Double-virtual contribution	141
5.1.2	Double-soft contribution	142
5.1.3	Soft real-virtual contribution	143
5.1.4	Hard-collinear real-virtual contribution	145
5.1.5	Single-soft hard-collinear double-real contribution	148

5.2	Double-collinear contributions of $\mathcal{O}(\epsilon^{-2})$	150
5.2.1	Double-collinear sector	151
5.2.2	Triple-collinear sector	152
5.3	Computing all remaining double-unresolved contributions	156
5.3.1	Terms of $\mathcal{O}(\epsilon^{-2})$	157
5.3.2	Terms of $\mathcal{O}(\epsilon^{-1})$	159
5.3.3	Terms of $\mathcal{O}(\epsilon^0)$	164
6	Manifest cancellations of IR singularities at NNLO	167
6.1	Single-unresolved terms	167
6.2	Double-unresolved triple color-correlated terms	171
6.3	Other double-unresolved color-correlated terms	178
6.4	Analytic cancellation of poles in the double-unresolved terms	181
6.4.1	Double- and single-boosted contributions	183
6.4.2	Elastic pure collinear contribution	184
6.5	Final result	188
6.5.1	Final result for process $\mathcal{B}_0$	188
6.5.2	Final result for processes $\mathcal{A}_0, \mathcal{A}_0^{(\bar{q}q)}, \mathcal{A}_0^{(gg)}$	191
	Conclusions	195
	Appendices	199
A	Constants, angular integrals, splitting functions, anomalous dimensions and fundamental operators	201
A.1	Useful constants	201
A.2	Collection of simple angular integrals	203
A.3	Altarelli-Parisi splitting functions and PDFs renormalization	205
A.3.1	Altarelli-Parisi splitting functions	205
A.3.2	PDFs renormalization	206
A.4	Standard tree-level and one-loop splitting functions	208
A.4.1	Tree-level splitting functions	208
A.4.2	One-loop splitting functions	209
A.5	Generalized splittings and anomalous dimensions	211
A.5.1	Tree-level	211
A.5.2	Spin-correlations	212
A.5.3	One-loop	214
A.6	Definitions of I -operators, commutators, and expansions	215
B	Partitions at NLO and NNLO for an arbitrary number of final-state particles	221
C	Phase-space parametrization and collinear limits	227
C.1	Phase-space parametrizations for unresolved partons	227
C.2	Action of the collinear operators on the phase space	230

D Partition-dependent contributions	235
D.1 Derivation of $\mathcal{W}_i^{i n,\text{fin}}$	235
D.1.1 How $\mathcal{W}_i^{i n,\text{fin}}$ emerges in $\Sigma_{\text{DU}}^{\text{fin},(2)} _{\mathcal{H},gg}$	235
D.1.2 Explicit expression of $\mathcal{W}_i^{i n,\text{fin}}$	237
D.2 Derivation of $\mathcal{W}_i^{m n,\text{fin}}$	238
D.2.1 How $\mathcal{W}_i^{m n,\text{fin}}$ emerges in $\Sigma_{\text{RR},1c}^{(b,d),\text{sa},\text{II}} _{\mathcal{H},gg}$	238
D.2.2 Explicit expression of $\mathcal{W}_i^{m n,\text{fin}}$	241
E Spin correlations	243
E.1 Definition of the spin-correlated contribution	243
E.2 Integration of the spin-correlated contribution over the physical and un-physical regions	247
E.2.1 The physical region	247
E.2.2 The unphysical region	251
F Details on the single-soft hard-collinear double-real contribution	253
G Triple-collinear limit for the final-state quark leg	259
G.1 Derivation of the triple-collinear splitting function for $q^* \rightarrow ggq$	259
G.2 Singularities arising from interferences	265
H Triple color-correlated contributions to real-virtual corrections	267
I Collection of functions used in the final result	273
I.1 Splitting functions	273
I.2 Partition-dependent objects and relevant constants	279
I.3 Elastic operators	280
List of Publications	285
Bibliography	286

List of Figures

- 1.1 QCD Feynman rules in a covariant gauge for fermions (solid lines), gluons (curly lines), and ghosts (dotted lines). 17
- 1.2 Schematic representation of the *deep inelastic scattering* (DIS) process, where $\ell(k^\mu)$ and $\ell(k'^\mu)$ are the incoming and outgoing leptons, respectively, $H(P^\mu)$ is the initial-state hadron, and $\mathbb{H}$ stands for any hadronic final state. The diagram in red defines the *hard-scattering process*, which can be computed using QCD perturbation theory. 24
- 1.3 Schematic representation of the *factorization theorem* (1.55), where $H_1(P_1^\mu)$ and $H_2(P_2^\mu)$ are the two incoming hadrons, respectively, and the diagram in red defines the hard-scattering process, which can be computed using QCD perturbation theory. $\mathbb{H}_1$ and $\mathbb{H}_2$ are any hadronic final states that do not participate in the hard-scattering process. 26
- 1.4 Some examples of Feynman diagrams that contribute to higher-order corrections of the Drell-Yan process in Eq. (1.62), where X here corresponds to the $\ell^-\ell^+$ pair. Gluons in red are additional partons to the LO diagram in black. Specifically, the following corrections are listed: (a) (single-)virtual; (b) single-real; (c) double-virtual; (d) real-virtual; (e) double-real. Note that (a, b) and (c-e) are NLO and NNLO corrections to the process (1.62), respectively. 30
- 2.1 Graphical representation of the soft-gluon factorization formula (2.52) at $\mathcal{O}(\alpha_s)$. 42
- 2.2 Graphical representation of the collinear factorization formula (2.78) at $\mathcal{O}(\alpha_s)$, where we have drawn as an example the $q \rightarrow q(z) \times g(1-z)$ splitting process. The functions $P_{qq}(z, k_\perp)$ are the d -dimensional Altarelli-Parisi splitting functions (2.81). 47
- 2.3 Energy phase space in the (E_3, E_4) -plane (left) and after the change of integration variables, Eq. (2.93), in the $(E_{[34]}, z)$ -plane (right). The energy phase space after the substitution is split into two regions, $\Theta_{(I)}$ and $\Theta_{(II)}$, as defined in Eq. (2.95). 51

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- 2.4 Graphical representation of the collinear factorization formula (2.101) at $\mathcal{O}(\alpha_s)$, where we have drawn as an example the $q \rightarrow q(z) \times g(1-z)$ splitting process but crossed to initial state. Functions $P_{[hs]h,i}(z)$ are the d -dimensional initial-state Altarelli-Parisi splitting functions (2.102). 53
 - 2.5 Examples of Feynman diagrams for the double-soft emissions of two gluons, m and n . We have: (a) eikonal emission of two soft gluons from two different external partons; (b) eikonal emission of two gluons from the same external parton; (c) non-abelian contribution. 55
 - 2.6 Examples of triple-collinear splitting processes. We have the Feynman diagrams that contribute to the triple-collinear splitting processes: (a) $q_{[imn]} \rightarrow q_i g_m g_n$; (b) $g_{[imn]} \rightarrow g_i g_m g_n$. 58
 - 2.7 Example of a Feynman diagram representing a real-virtual amplitude with the emission of a real and a virtual gluon. 59

List of Tables

1.1	Quark masses extracted from Ref. [98]. The quark charges are written as fractions of the elementary charge e .	13
4.1	List of the terms that contribute to $\Sigma_{\text{RR},1c}^{(a,c,\text{dc})} _{\mathcal{H},gg}$.	120
4.2	List of the terms that contribute to $\Sigma_{\text{RR},1c}^{(b,d)} _{\mathcal{H},gg}$.	127
4.3	List of the terms that contribute to $\Sigma_{\text{RR},1c}^{(a,c,\text{dc})} _{\mathcal{H},qq}$.	132
4.4	List of terms contributing to the single-unresolved contributions in the $m_g n_g$ and $m_q n_g$ cases.	134
4.5	The list of terms contributing to the double-unresolved contributions in the $m_g n_g$ and $m_q n_g$ cases.	135

Abbreviations

CERN	Conseil Européen pour la Recherche Nucléaire
DGLAP	Dokshitzer-Gribov-Lipatov-Altarelli-Parisi
DIS	Deep-inelastic scattering
FKS	Frixione-Kunszt-Signer
FSR	Final-state radiation
IR	Infra-red
ISR	Initial-state radiation
KLN	Kinoshita-Lee-Nauenberg
LHC	Large Hadron Collider
LO	Leading order
$\overline{\text{MS}}$	Modified minimal subtraction scheme
NSC	Nested Soft-Collinear
NLO	Next-to-leading order
NNLO	Next-to-next-to-leading order
PDF	Parton distribution functions
QCD	Quantum Chromodynamics
QED	Quantum Electrodynamics
QFT	Quantum field theory
R	Real
RR	Double-real
RV	Real-virtual
RGE	Renormalization group equation
SLAC	Stanford Linear Accelerator
SM	Standard Model of particle physics
UV	Ultra-violet
V	Virtual
VV	Double-virtual

Motivation

Our understanding of nature at its most fundamental level is guided by experiments at particle colliders, the largest and most energetic being the Large Hadron Collider (LHC). Proton-proton collisions experiments running at the LHC discovered the Higgs boson in 2012, and are actively investigating its properties, as well as the properties of the other particles comprising the Standard Model (SM), with incredible precision, while also searching for signals of physics beyond the SM. The theoretical framework for describing hard scattering processes at the LHC is provided by the quantum field theories underlying the SM, most importantly perturbative Quantum Chromodynamics (QCD). Consequently, the development of theoretical techniques capable of providing predictions at progressively higher orders of perturbation theory has emerged as one of the most vibrant and engaging areas in theoretical particle physics over the last decade (see Refs [1–4] for recent reviews).

In the context of high-precision physics, numerous computational ingredients are essential for producing increasingly precise results that can be compared with experimental data. Among these are the determination of parton distribution functions (PDFs), the resummation of contributions that are enhanced in specific kinematic regions, the production of multiple final state partons through parton showers, modelling hadronization effects, and, the calculation of fixed-order results. In fixed-order calculations, two major challenges arise: the computation of multi-loop amplitudes and the construction of infra-red (IR) regularization schemes. This thesis focuses on the latter area.

It is well-known that IR singularities arise separately in virtual corrections, unresolved real-radiation contributions, and the collinear renormalization of PDFs, and cancel when these contributions are summed. The important question, therefore, is *how* to organize this cancellation in a *process-independent* manner and how to obtain finite remainders that are suitable for numerical evaluation.

This issue was fully resolved at next-to-leading order (NLO) in perturbative QCD many years ago [5–10] (see also Ref. [11] for more recent work), but extending this to next-to-next-to-leading order (NNLO) and beyond has proven to be very challenging. Numerous NNLO subtraction and slicing schemes [12–34] have been proposed and used for numerous impressive computations at this perturbative order (see Refs [35–59] for a

representative list of NNLO calculations by various collaborations). However, it is fair to state that the level of generality attained at NLO remains elusive at NNLO.

Recently, several groups have made important strides towards constructing a general NNLO subtraction procedure. Antenna subtractions have recently been recast in color space [60, 61], allowing an improved treatment of processes with multiple partons, and the construction of the antennas has also been refined [62–64]. For e^+e^- collisions, singularity cancellation for arbitrary final states has been demonstrated within the local analytic sector subtraction scheme [31, 32], but the treatment of initial-state radiation is still under development in this framework.

This thesis focuses on addressing this problem within the framework of the *Nested Soft-Collinear* (NSC) subtraction method [23]. This approach has proven effective in calculating the NNLO QCD corrections for several processes, including color singlet production [65] and decay [66], and deep inelastic scattering [67]. It has been used for phenomenological studies of Higgs production through associated VH production [68–70] and weak boson fusion (WBF) [71–73], mixed scalar-pseudoscalar Higgs production through gluon fusion [74] and non-factorizable corrections in t -channel single-top production [75]. It has also proven to be very flexible and has been adapted for mixed QCD-EW corrections for the production of electroweak gauge bosons and dilepton pairs [76–78]. On the one hand, all these processes have simple color configurations, in that only two partons are present in the leading-order (LO) process. On the other hand, the results indicate that the NSC offers the requisite flexibility and simplicity for the analysis of multi-particle final states. Furthermore, calculating double-unresolved soft and collinear contributions across arbitrary kinematics is often one of the most difficult tasks involved in creating a specific subtraction scheme. Notably, for the NSC scheme, these calculations were accomplished in full generality several years ago [79, 80]. This gives us further confidence that the NSC scheme can be generalized to handle arbitrary production processes at a hadron collider.

This thesis represents a significant step towards this goal. In particular, we will present the application of the NSC method to the computation of the NNLO QCD corrections for a hadron-hadron scattering process that produces an *arbitrary* number N of partons and a colorless object X in the final state, where we assume that *one* of these final-state partons is a quark, and *all* additional radiated partons are gluons. We define this process as follows:

$$\mathcal{B}_0 : \quad g_a + q_b \rightarrow \{g\}_{N-1} + q + X.$$

We emphasize that in our analysis we do not consider processes similar to $\mathcal{B}_0$ in which one or more gluons split into a quark-antiquark pair. Such processes would give rise to corrections analogous to those described in this work, but proportional to the number of light quark flavors n_f .¹ For the purposes of this thesis, we will assume $n_f = 0$ in the QCD β function, although we will present all relevant formulas with their n_f contributions in the β_0 factors for future extensions that may involve unresolved quark-antiquark pairs

¹To be precise, in this thesis, we will see that a specific process, namely $g_a + q_b \rightarrow \{g\}_{N-1} + \{q\}_2 + \bar{q} + X$, contributes to the NNLO corrections through *interference terms*, which yield a contribution proportional to $C_A - 2C_F$ instead of n_f .

in the final states.

Although our results apply to the process $\mathcal{B}_0$, they involve matrix elements with the most complex singularity structures that can arise, and we are confident that the insights we gain into IR cancellations at NNLO in QCD are broadly useful for generic final states. It is worth noting that the work in this thesis is based on the generalization of the NSC scheme presented for the first time in Ref. [81]. In that case, this approach was utilized to study the process $q\bar{q} \rightarrow \{g\}_N + X$, where both the initial and final states are symmetric. This generalization of the NSC scheme is presented here in an even more comprehensive form, revisited compared to Ref. [81], specifically to compute a more complicated process, $\mathcal{B}_0$, which, in contrast, is asymmetric in both the initial and final states. Furthermore, we will demonstrate how the results derived for $\mathcal{B}_0$ can be readily adapted to describe the process

$$\mathcal{A}_0 : \quad a + b \rightarrow \{g\}_N + X ,$$

which is the process addressed in [81], where the initial state can either be a pair $q\bar{q}$ or a pair gg . We emphasize that the primary challenge in generalizing these results from all-gluonic to arbitrary final states lies in the combinatorics of various collinear limits. This challenge does not arise prominently for all-gluonic final states due to the symmetry of the relevant matrix elements under permutations of the final-state gluons.

There is substantial evidence that IR subtraction terms can be arranged into iterative structures, extending from NLO to NNLO and potentially beyond. This is particularly evident in the case of leading collinear singularities, where the highest collinear poles at each perturbative order are governed by convolutions of leading-order splitting functions. Likewise, Catani's formula for the ϵ -poles of one- and two-loop amplitudes [82] suggests an iterative structure for soft emissions. However, this iterative structure may not be immediately visible in NNLO subtraction schemes, especially since such schemes could rely on partitioning the real-emission phase space and projecting matrix elements onto minimal singular kinematic configurations, similar to the FKS subtraction approach at NLO. In this thesis, we show how these iterative structures can be identified within the framework of the NSC method. We demonstrate that recognizing these structures simplifies the organization of NNLO QCD calculations, reducing their computational complexity and enabling the treatment of final states with *arbitrary* multiplicities.

The main result of this thesis is a formula that enables the calculation of NNLO QCD corrections for the process $\mathcal{B}_0$, using the NSC subtraction scheme. Importantly, we show that the complex cancellation of IR singularities for this process can be achieved for any multiplicity in a relatively straightforward way. In practice, this allows the differential cross section for this process to be expressed as a finite quantity, which can be directly implemented into numerical integration programs.

Structure of the thesis

This thesis is divided into two parts. *Part I* was written with a somewhat *pedagogical* intent, and deals with fundamental aspects of QCD and IR divergences. As such, everything presented in this part is readily available in the literature. However, we have

decided to collect all of this information into a single source, with the aim of providing a comprehensive review of material for a student embarking on a Ph.D. in theoretical particle physics. Part I consists of two chapters:

- Ch. 1 is a summary of various general aspects of QCD. After discussing the fundamental concepts of QCD in Secs 1.1 and 1.2, we dedicate Secs 1.3 and 1.4 to defining the cross section and discussing the QCD factorization theorem. Finally, in Sec. 1.5, we introduce the concept of IR divergences in QCD.
- Ch. 2 is a review of all possible sources of IR divergences in QCD necessary for the computation of cross-sections at NLO and NNLO. Sec. 2.1 is dedicated to introducing color space and color algebra. Sec. 2.2 summarizes the main formulas related to IR divergences in one- and two-loop amplitudes, while Sec. 2.3 focuses on the soft and collinear divergences in tree-level amplitudes. In Sec. 2.4, we discuss the IR divergences at order $\mathcal{O}(\alpha_s^2)$ in tree-level amplitudes, and in Sec. 2.5, we cover the soft and collinear limits in real-virtual amplitudes. Finally, in Sec. 2.6, we analyze why it is necessary to introduce IR regularization schemes in QCD.

Part II is entirely dedicated to the NSC scheme. We begin with the basics, starting from the NLO. The standard NSC approach is applied to specific examples, and based on these, we discuss why it is necessary to *generalize* this approach when studying more complex processes. We then introduce the key operators of this new approach, which we call the *I*-operators, and use them throughout the rest of the thesis to study the NLO and NNLO corrections for the $\mathcal{A}_0$ and $\mathcal{B}_0$ processes mentioned earlier. Part II consists of four chapters:

- Ch. 3 presents a comparison between the standard approach to NSC scheme and our generalization. Specifically, in Sec. 3.1, we introduce the notation that will be used throughout the rest of the thesis. Sec. 3.2 describes the NSC/FKS scheme at NLO and examines the challenges that motivate its generalization. Finally, in Sec. 3.3, we discuss the NLO corrections for the $\mathcal{A}_0$ and $\mathcal{B}_0$ processes using the new approach and introduce the *I*-operators.
- Ch. 4 describes the new approach to NSC scheme at NNLO. In Sec. 4.1, we introduce the NSC method at NNLO in full detail. In Sec. 4.2, we analyze the obstacles that arise when generalizing this method to arbitrary hadron production processes, and we introduce a new organizational principle that allows us to use the *I*-operators introduced in Ch. 3 to overcome these challenges. Sections 4.3 through 4.5 focus on reorganizing the single- and double-unresolved terms, laying the groundwork for the introduction of the *I*-operators.
- Ch. 5 incorporates the most complicated calculations of this thesis. Throughout the three sections 5.1, 5.2, and 5.3, we present in meticulous detail all the intermediate steps necessary for the calculation of the NNLO corrections to the $\mathcal{A}_0$ and $\mathcal{B}_0$ processes. The abundance of detail is justified by the pedagogical intent, and ensures that the calculations in this thesis are fully reproducible.

- Ch. 6 is dedicated to demonstrating the cancellation of the poles in the NNLO cross section for the process $\mathcal{B}_0$. First, in Sec. 6.1, we address the cancellation of the poles in the single-unresolved contributions. In Sections 6.2 and 6.3, we will deal with the cancellations of the poles arising from double-unresolved triple color-correlated terms and other double-unresolved terms containing color-correlations, respectively. Then, in Sec. 6.4, we will thoroughly discuss the *analytic* cancellation of the poles, up to $\mathcal{O}(\epsilon^{-2})$, for all double-unresolved terms, both boosted and elastic, and we briefly comment on the cancellation of the remaining $\mathcal{O}(\epsilon^{-1})$ poles. Finally, Sec. 6.5 is dedicated to the final results of this thesis.

Finally, additional information that complements the presentation of Chs 3-6 has been gathered into nine appendices.

Part I:

Infra-red divergences in QCD

Generalities on QCD

Contents

1.1	The Standard Model	11
1.2	Basics of QCD	13
1.2.1	QCD Lagrangian and the color group $SU(3)$	13
1.2.2	Local gauge invariance and Feynman rules	15
1.2.3	The running of the strong coupling constant	16
1.2.4	The anomalous dimensions	20
1.3	Definition of cross section	21
1.4	The factorization theorem in hadronic cross sections	23
1.4.1	The birth of the parton model	23
1.4.2	The factorization theorem	25
1.5	Introduction to infra-red divergences in QCD	28

In this introductory chapter, we provide essential information on the Standard Model (SM) of particle physics, and in particular on *quantum chromodynamics* (QCD), the well-known *quantum field theory* (QFT) governing the strong interactions of quarks and gluons. This discussion is essential for understanding the *infra-red* (IR), or low-energy, behavior of perturbative QCD, which will be explored in Ch. 2. The tools introduced here form the foundational background for understanding the main topic of this thesis, i.e., the *Nested Soft-Collinear* (NSC) subtraction scheme, which will be presented starting from Ch. 3.

The material discussed in this chapter can be found in standard textbooks, e.g. [83–89].

1.1 The Standard Model

During the 20th century, the development of the SM as a description of the behavior and interactions of the fundamental constituents of Nature marked a significant milestone in theoretical physics. The theory comprises two types of particles: matter constituents, called quarks and leptons, which obey Fermi-Dirac statistics, and force-carrying particles, which obey Bose-Einstein statistics. The latter describes three of the four fundamental interactions of nature: the electromagnetic, weak, and strong forces. The quantum

properties of gravity are currently the only unexplained interaction within this framework.

Within the SM, interactions can be divided into two sectors. The electroweak (EW) sector unifies the electromagnetic force, mediated by the photon, and the weak force, mediated by two charged W -bosons and a neutral Z -boson. The EW theory originated from the model developed by Sheldon Lee Glashow, Steven Weinberg, and Abdus Salam [90–92], now known as the GWS model, which they proposed in an attempt to provide a high-energy completion to Fermi’s theory of beta decay. The three authors were awarded the Nobel Prize in Physics in 1979 for their contributions. The other sector is that of the strong force, mediated by gluons. The theory of Quantum Chromodynamics (QCD), which describes the strong interactions, was developed in the mid-20th century by various physicists, starting from Murray Gell-Mann’s hypothesis on the existence of quarks [93], and later enlarged to include gluons that, like quarks, carry color charge. The experimental observation of the existence of a color charge was later provided by the Brookhaven National Laboratory and SLAC from the measurement of the decay of the J/ψ meson [94, 95]. The strong force is the central subject of this thesis and will be thoroughly reviewed in the following sections. Regarding matter, leptons only interact through the electroweak force, whereas quarks also interact through the strong force.

EW and QCD theories are based on gauge symmetries, and masses of gauge bosons are included through spontaneous symmetry breaking of electroweak interaction through the Higgs mechanism. This places the SM on a firm theoretical foundation. The Higgs mechanism leads to a scalar particle, the Higgs boson, which was the last particle of the SM to be discovered. Its discovery in 2012 by the Large Hadron Collider (LHC) collaboration [96, 97] placed the SM on a firm experimental foothold and marked a breakthrough in the history of theoretical and experimental fundamental physics. This discovery led to Peter Higgs and Franoise Englert being awarded the 2013 Nobel Prize in Physics. This initiated a thorough exploration of the properties of the Higgs boson and its interactions with the other particles in the SM, which will be one of the focuses of high-energy physics for the next decades.

In spite of its numerous successes, the SM does not explain all observed phenomena. For example, it does not provide a candidate for dark matter, lacks a framework to explain dark energy, fails to account for the matter-antimatter asymmetry, and does not offer a natural explanation for the smallness of the Higgs mass compared to the Planck mass, known as the hierarchy problem. In this regard, it is crucial that theorists are able to provide predictions with a precision that matches, or even surpasses, that obtained by the experiments. This endeavor is two-fold: on one hand, the agreement between increasingly precise measurements and theoretical predictions confirms the robustness of the theory; on the other hand, finding deviations between theoretical predictions and experimental results could open avenues to new physics.

Table 1.1: Quark masses extracted from Ref. [98]. The quark charges are written as fractions of the elementary charge e .

Flavour	Mass	Charge
u	$2.16^{+0.49}_{-0.26}$ MeV	$+\frac{2}{3}$
d	$4.67^{+0.48}_{-0.17}$ MeV	$-\frac{1}{3}$
s	$93.4^{+8.6}_{-3.4}$ MeV	$-\frac{1}{3}$
c	1.27 ± 0.02 GeV	$+\frac{2}{3}$
b	$4.18^{+0.03}_{-0.02}$ GeV	$-\frac{1}{3}$
t	172.69 ± 0.30 GeV	$+\frac{2}{3}$

1.2 Basics of QCD

1.2.1 QCD Lagrangian and the color group SU(3)

Quantum Chromodynamics (QCD) is the theory that describes the strong interactions between quarks and gluons, which are the fundamental constituents of protons, neutrons, and the other hadrons. As part of the *Standard Model* (SM) of particle physics, QCD is a quantum field theory (QFT), specifically a *Yang-Mills theory*, based on the non-abelian gauge *color* group SU(3). The *color charge* is a fundamental property in QCD, analogous to electric charge in electromagnetism but with three types, sometimes referred to as *blue*, *red*, and *green*.

The interactions between spin- $\frac{1}{2}$ Dirac (anti)quark fields and spin-1 gauge boson gluons are described through the QCD Lagrangian¹

$$\mathcal{L}_{\text{qcd}} = \sum_{\text{flavours}}^{n_f} \bar{q}_i (i\gamma^\mu (D_\mu)_{ij} - m_q \delta_{ij}) q_j - \frac{1}{4} G_a^{\mu\nu} G_{\mu\nu}^a. \quad (1.1)$$

In Eq. (1.1), $\bar{q}_i$ and q_j describe quark fields of mass m_q , where the sum runs over all the n_f flavours q of the theory, and the indices $i, j \in \{1, 2, 3\}$ are the color indices for the fundamental triplet representation q_i of SU(3) (spin indices are left understood). In QCD, there are $n_f = 6$ quark families. In Table 1.1, we list the name, mass, and electric charge of each.

In Eq. (1.1), gluons are described by the massless, electrically neutral gauge field $\mathcal{A}_a^\mu$, which belongs to the adjoint representation of SU(3). The index $a \in \{1, \dots, 8\}$ is the color index in the adjoint representation corresponding to the eight color degrees of freedom of the field $\mathcal{A}_a^\mu$. $G_a^{\mu\nu}$ describes the field strength tensor

$$G_a^{\mu\nu} \stackrel{\text{def}}{=} \partial^\mu \mathcal{A}_a^\nu - \partial^\nu \mathcal{A}_a^\mu + g_s f_{abc} \mathcal{A}_b^\mu \mathcal{A}_c^\nu, \quad (1.2)$$

¹In this thesis, we follow the mostly-minus signature for the metric. Moreover, we assume the natural units $\hbar = c = 1$.

where g_s is the gauge coupling representing the strength of the interaction between color states (it is a free parameter of the theory), and the symbol f_{abc} denotes the totally antisymmetric structure constants of $SU(3)$. Finally, the covariant derivative D^μ acting on triplet and octet fields is defined as

$$D_{ij}^\mu \stackrel{\text{def}}{=} \partial^\mu \delta_{ij} - ig_s (t_a \mathcal{A}_a^\mu)_{ij}, \quad D_{ab}^\mu \stackrel{\text{def}}{=} \partial^\mu \delta_{ab} - ig_s (T_c \mathcal{A}_c^\mu)_{ab}, \quad (1.3)$$

where t_a and T_a are the generators of $SU(3)$ in the fundamental and adjoint representations, respectively. They satisfy the properties

$$[t_a, t_b] = if_{abc} t_c, \quad [T_a, T_b] = if_{abc} T_c, \quad (T_a)_{bc} = -if_{abc}. \quad (1.4)$$

A representation for the generators t_a is provided by the eight 3×3 Hermitian traceless Gell-Mann matrices

$$t_a = \frac{\lambda_a}{2}, \quad (1.5)$$

whose explicit expressions are reported below:

$$\begin{aligned} \lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda_2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, & \lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, & \lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\ \lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, & \lambda_8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \end{aligned} \quad (1.6)$$

If we consider a theory built on a $SU(N_c)$ group, where N_c labels the number of colors of the theory, the convention often assumed in the literature for the normalization of the $SU(N_c)$ matrices is

$$\text{Tr}[t_a t_b] = T_R \delta_{ab}, \quad T_R = \frac{1}{2}. \quad (1.7)$$

Using this choice, one finds the following relations:

$$(t_a)_{ij} (t_a)_{jk} = C_F \delta_{ik}, \quad \text{Tr}[T_a T_b] = f_{cda} f_{cdb} = C_A \delta_{ab}, \quad (1.8)$$

where

$$C_F = \frac{N_c^2 - 1}{2N_c}, \quad C_A = N_c, \quad (1.9)$$

are the *quadratic Casimir operators* for the fundamental (F) and adjoint (A) representation of $SU(N_c)$, respectively. In the case of the gauge color group $SU(3)$ of QCD, i.e. $N_c = 3$,

Eq. (1.9) yields

$$C_F = \frac{4}{3}, \quad C_A = 3. \quad (1.10)$$

These two numbers are the colour factors that determine the intensity of gluon radiation off a quark and a gluon. It follows that radiation off a gluon is enhanced by the color factor with respect to radiation off a quark. Finally, we state the following anticommutation relations between t_a matrices in $SU(N_c)$,

$$\{t_a, t_b\} = \frac{\delta_{ab}}{N_c} \mathbb{1} + d_{abc} t_c, \quad d_{cda} d_{cdb} = \frac{N_c^2 - 4}{N_c} \delta_{ab}, \quad d_{cca} \equiv 0. \quad (1.11)$$

1.2.2 Local gauge invariance and Feynman rules

The QCD Lagrangian $\mathcal{L}_{\text{qcd}}$ in Eq. (1.1) is constructed to be invariant under local gauge transformations. Under such a transformation, the quark field q_i behaves as

$$q_i(x) \mapsto q'_i(x) = [\exp(i \theta_a(x) t_a)]_{ij} q_j(x) \equiv [\Omega(x)]_{ij} q_j(x). \quad (1.12)$$

The covariant derivative D_μ in Eq. (1.3) is defined such that, once applied to a field $q(x)$, the object $D_\mu q(x)$ behaves exactly like a field under a local gauge transformation, i.e.,

$$D_\mu q(x) \mapsto D'_\mu q'(x) \equiv \Omega(x) D_\mu q(x). \quad (1.13)$$

It follows that a gluon field $\mathcal{A}_a^\mu$ transforms under a local gauge transformation as

$$t_a \mathcal{A}_a^\mu \equiv (t \cdot \mathcal{A}^\mu) \mapsto (t \cdot \mathcal{A}'^\mu) = \Omega(x) \left[(t \cdot \mathcal{A}^\mu) + \frac{i}{g} \partial_\mu \right] \Omega^\dagger(x), \quad (1.14)$$

which implies that the field strength tensor $G_a^{\mu\nu}$ in Eq. (1.2) must transform as

$$t_a G_a^{\mu\nu} \equiv (t \cdot G^{\mu\nu}) \mapsto (t \cdot G'^{\mu\nu}) = \Omega(x) (t \cdot G^{\mu\nu}) \Omega^\dagger(x). \quad (1.15)$$

Unlike the electromagnetic field strength in *Quantum Electrodynamics* (QED), in QCD the field strength tensor is not gauge invariant due to the self-interactions of gluons. In fact, as we have seen above, gluons themselves carry color charge, unlike in QED, where photons are electrically neutral. In QCD this leads to the important property of *asymptotic freedom*, discussed in the next subsection. Additionally, we observe that $\mathcal{L}_{\text{qcd}}$ in Eq. (1.1) does not contain mass terms for gluons since

$$m^2 \mathcal{A}_a^\mu \mathcal{A}_\mu^a \quad (1.16)$$

is not gauge invariant under $SU(3)$, as one can see from Eq. (1.14). On the other hand, according to Eq. (1.12), the mass term for fermions

$$\sum_{\text{flavours}}^{n_f} \bar{q}_i (-m_q \delta_{ij}) q_j \quad (1.17)$$

is gauge invariant under $SU(3)$.²

It is well-known that the presence of gauge symmetry complicates the process of quantizing the theory. Indeed, one can see that, starting from Eq. (1.1), it is impossible to unequivocally define the two-point function of the gluon (i.e., the gluon propagator) without making a choice of the gauge.³ This phenomenon is solved by the introduction of the *Faddeev-Popov procedure*, which involves adding the following Lagrangian

$$\mathcal{L}_{\text{gauge-fixing}} = \frac{1}{2\xi} (\partial_\mu \mathcal{A}_a^\mu)^2 \quad (1.18)$$

to Eq. (1.1) in order to fix the class of *covariant gauges* with parameter ξ . Moreover, since QCD is built on the gauge group $SU(3)$ which is non-Abelian, the covariant gauge fixing term (1.18) must be supplemented with the so-called *ghost Lagrangian*

$$\mathcal{L}_{\text{ghost}} = \bar{c}_a (-\partial_\mu D_{ab}^\mu) c_b, \quad (1.19)$$

where D_{ab}^μ is defined in Eq. (1.3), and $c_a(x)$ are complex scalar fields that obey Fermi statistics, known as *Faddeev-Popov ghosts* [100, 101]. The ghost fields cancel unphysical degrees of freedom which would otherwise propagate in covariant gauges.⁴

The construction of perturbation theory for QCD has no further major subtleties. The final Lagrangian, including all terms relating to the Faddeev-Popov gauge-fixing, reads

$$\mathcal{L}_{\text{qcd}} = \sum_{\text{flavours}}^{n_f} \bar{q}_i (i\gamma^\mu D_\mu - m_q)_{ij} q_j - \frac{1}{4} G_a^{\mu\nu} G_{\mu\nu}^a + \frac{1}{2\xi} (\partial_\mu \mathcal{A}_a^\mu)^2 + \bar{c}_a (-\partial_\mu D_{ab}^\mu) c_b. \quad (1.20)$$

In Figure 1.1 we report Feynman rules for fermions, gluons, and ghosts, following from $\mathcal{L}_{\text{qcd}}$ in Eq. (1.20). The Lagrangian depends on $n_f + 1$ parameters: the masses m_q of the n_f different quark flavours, collected in Table 1.1, and the coupling constant g_s . The coupling g_s is usually written through the so-called *strong coupling constant* α_s , defined as

$$\alpha_s \stackrel{\text{def}}{=} \frac{g_s^2}{4\pi}. \quad (1.21)$$

1.2.3 The running of the strong coupling constant

The Lagrangian $\mathcal{L}_{\text{qcd}}$ in Eq. (1.20) depends on parameters that do not have a one-to-one correspondence with the physical quantities measured in experiments. Indeed, it is well known that in QCD, as in QFTs in general, the parameters of the Lagrangian are not constants across all energy scales but rather *run* with the variation of the energy scale. To connect the parameters of $\mathcal{L}_{\text{qcd}}$ with the corresponding physical quantities, it is

²In this work, we do not discuss how fermions acquire mass in the SM through the Higgs mechanism. This is treated in detail in standard textbooks, for instance Refs [86, 88, 99].

³This phenomenon is not unique to QCD but is present in all Yang-Mills theories built on a gauge group $SU(N)$. Indeed, even QED, which has an Abelian gauge group $U(1)$, suffers from the same issue.

⁴We do not discuss gauge fixing and ghosts further. More information can be found in standard textbooks, such as [83, 85, 86, 89].

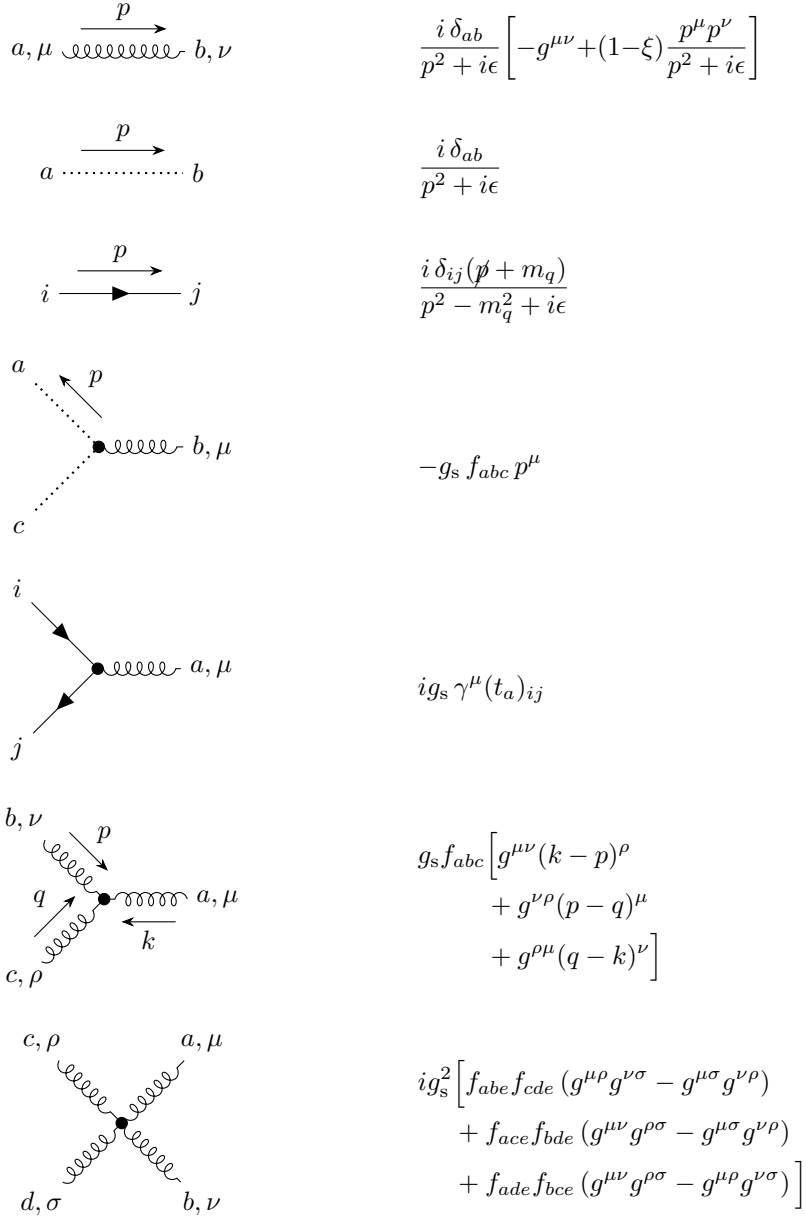


Figure 1.1: QCD Feynman rules in a covariant gauge for fermions (solid lines), gluons (curly lines), and ghosts (dotted lines).

necessary to subject the theory to a process known as *renormalization* [84–87, 89].

In order to illustrate how the coupling depends on the energy scale, let us consider a dimensionless physical observable $\hat{O}$ in QCD which depends on a single energy scale Q^2 . By assumption, the scale Q^2 is much bigger than all other dimensionful parameters, so we can set all masses in $\mathcal{L}_{\text{qcd}}$ to zero. Therefore, $\hat{O}$ is just a function of Q^2 and the strong coupling constant α_s in Eq. (1.21). We assume that α_s is small, and thus we write the operator the operator $\hat{O}$ as a perturbative series in terms of α_s ,

$$\hat{O} = \hat{O}_0 + \alpha_s \hat{O}_1 + \alpha_s^2 \hat{O}_2 + \dots \quad (1.22)$$

In computing this expansion, it is generally necessary to remove ultra-violet (UV) divergences which appear in loop diagrams, and this procedure introduces a second energy scale μ^2 . As $\hat{O}$ is dimensionless, it can depend on Q^2 and μ^2 only through the ratio Q^2/μ^2 . Additionally, as mentioned above, the strong coupling constant α_s must also be a function of the energy scale μ^2 , i.e., $\alpha_s = \alpha_s(\mu^2)$. However, the energy scale μ^2 is an arbitrary parameter introduced to regularize UV divergences, and does not appear in $\mathcal{L}_{\text{qcd}}$. Therefore, if we keep the bare coupling fixed, physical quantities such as $\hat{O}$ cannot depend on the choice of μ^2 .

All the above considerations lead to the conclusion that the physical observable $\hat{O}$ must satisfy the following differential equation:

$$\mu^2 \frac{d}{d\mu^2} \hat{O}\left(\frac{Q^2}{\mu^2}, \alpha_s\right) = \left[\mu^2 \frac{\partial}{\partial \mu^2} + \mu^2 \frac{\partial \alpha_s}{\partial \mu^2} \frac{\partial}{\partial \alpha_s} \right] \hat{O}\left(\frac{Q^2}{\mu^2}, \alpha_s\right) = 0. \quad (1.23)$$

It is useful to introduce the function

$$\beta(\alpha_s) \stackrel{\text{def}}{=} \mu^2 \frac{\partial \alpha_s}{\partial \mu^2}, \quad (1.24)$$

and rewrite Eq. (1.23) as

$$\left[\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_s) \frac{\partial}{\partial \alpha_s} \right] \hat{O}\left(\frac{Q^2}{\mu^2}, \alpha_s\right) = 0. \quad (1.25)$$

Eq. (1.25) is a particular case of a more general equation that plays a fundamental role in quantum field theory: the *Callan-Symanzik equation* [102, 103]. In order to solve Eq. (1.25), one can implicitly define the *running of the coupling* $\alpha_s(Q^2)$ as

$$\alpha_s \stackrel{\text{def}}{=} \alpha_s(\mu^2), \quad t \equiv \log \frac{Q^2}{\mu^2} \stackrel{\text{def}}{=} \int_{\alpha_s(\mu^2)}^{\alpha_s(Q^2)} \frac{d\bar{\alpha}_s}{\beta(\bar{\alpha}_s)}, \quad (1.26)$$

then differentiate (1.26) as

$$\frac{\partial \alpha_s(Q^2)}{\partial t} = \beta(\alpha_s(Q^2)), \quad \frac{\partial \alpha_s(Q^2)}{\partial \alpha_s} = \frac{\beta(\alpha_s(Q^2))}{\beta(\alpha_s)}, \quad (1.27)$$

and finally obtain the solution of Eq. (1.25), which corresponds to $\hat{O}(1, \alpha_s(Q^2))$. We can

see that all of the scale dependence in $\hat{O}$ enters through the running of the coupling constant $\alpha_s(Q^2)$. It follows that, if one knows the value of the observable at some energy scale by means of perturbation theory, then its value at a different energy scale can be derived by solving Eq. (1.25).

In Eq. (1.25), the β function defined in Eq. (1.24) plays a key role, where the latter is a *renormalization group equation* (RGE) that determines the running of the coupling constant α_s . At weak coupling, β can be expanded in perturbation theory as

$$\beta(\alpha_s) = - \sum_{n=0}^{\infty} \beta_n \alpha_s \left(\frac{\alpha_s}{2\pi} \right)^{n+1}. \quad (1.28)$$

The coefficients β_n are known up to $n = 4$ [104–109]. Here we report the expressions for β_0 and β_1 , which will be important throughout this work:

$$\beta_0 = \frac{11}{6} C_A - \frac{2}{3} T_R n_f, \quad \beta_1 = \frac{17}{6} C_A^2 - \frac{5}{3} C_A T_R n_f - C_F T_R n_f, \quad (1.29)$$

where n_f is the number of massless quark flavours in the theory. We note that $\beta_0 > 0$ as long as $n_f \leq 16$, and this property is intrinsically connected with an important phenomenon in QCD known as *asymptotic freedom* [110, 111]. Indeed, assuming that both $\alpha_s(Q^2)$ and $\alpha_s(\mu^2)$ are in perturbative region, we can replace the definition (1.28) of the β -function on the left-hand side of Eq. (1.24), truncated to the order $\mathcal{O}(\alpha_s^2)$, and solve Eq. (1.24) (neglecting the β_1 and higher coefficients). We obtain

$$\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + \beta_0 \frac{\alpha_s(\mu^2)}{2\pi} \log \frac{Q^2}{\mu^2}}, \quad (1.30)$$

which gives the relation between $\alpha_s(Q^2)$ and $\alpha_s(\mu^2)$ if both are in perturbative region. In the limit $Q^2/\mu^2 \rightarrow \infty$, we have $\alpha_s(Q^2) \rightarrow 0$, i.e., strong interactions between particles become asymptotically weaker as the energy scale increases and the corresponding length scale decreases. This phenomenon is known as asymptotic freedom. We point out that the sign of β_0 is crucial, because if this were the opposite, we would have found an increase in the coupling constant α_s as the scale Q^2 grows (like in QED).

On the other hand, $\alpha_s(Q^2)$ will become larger as the scale decreases, and at some point, the coupling will be non-perturbative. By convention, we define Λ_{qcd} as the energy scale that satisfies the condition (see Eq. (1.26))

$$\log \frac{Q^2}{\Lambda_{\text{qcd}}^2} \stackrel{\text{def}}{=} \int_{\infty}^{\alpha_s(Q^2)} \frac{d\bar{\alpha}_s}{\beta(\bar{\alpha}_s)}. \quad (1.31)$$

In terms of this scale, the coupling at the scale Q^2 becomes

$$\alpha_s(Q^2) = \frac{2\pi}{\beta_0 \log \frac{Q^2}{\Lambda_{\text{qcd}}^2}}, \quad (1.32)$$

namely $\alpha_s(Q^2)$ is singular when $Q^2 = \Lambda_{\text{qcd}}^2$. To estimate the magnitude of Λ_{qcd} , we

observe that at the scale $Q^2 = M_Z^2$, where M_Z is the mass of the Z boson, the strong coupling constant is $\alpha_s(M_Z^2) = 0.0179 \pm 0.0009$ [98], from which we obtain $\Lambda_{\text{qcd}} \sim 200$ MeV. The existence of Λ_{qcd} shows that QCD is strongly coupled at low energies, and this is the reason why isolated quarks or gluons cannot be observed in Nature. This phenomenon is known as *confinement*. Therefore, QCD can be treated as a perturbative theory only if the coupling constant α_s is small enough, which happens when $\Lambda_{\text{qcd}} \ll Q^2$. Throughout this work, we always assume to be in a high-energy regime, that is, at energy scales much larger than Λ_{qcd} .

To conclude, it is important to note that solving a renormalization group equation, such as the one mentioned in Eq. (1.24), requires defining a renormalization scheme. In the context of QCD, the *modified Minimal Subtraction* ($\overline{\text{MS}}$) scheme is frequently used in the literature to renormalize the strong coupling constant α_s . It is defined as

$$g_{s,b}^2 \mu_0^{2\epsilon} = g_s^2(\mu) \mu^{2\epsilon} S_\epsilon Z[\alpha_s(\mu)], \quad (1.33)$$

or, equivalently, as

$$\alpha_{s,b} \mu_0^{2\epsilon} = \alpha_s(\mu) \mu^{2\epsilon} S_\epsilon Z[\alpha_s(\mu)], \quad (1.34)$$

where $\alpha_{s,b}$ is the *bare* strong coupling constant, μ is the renormalization scale, $S_\epsilon = (4\pi)^{-\epsilon} e^{\epsilon\gamma_E}$ ($\gamma_E \approx 0.5772\dots$ is the Euler-Mascheroni constant), and $Z[\alpha(\mu)]$ is the renormalization factor. Up to two loops, $Z[\alpha(\mu)]$ reads

$$Z[\alpha_s] = 1 - \frac{\beta_0}{\epsilon} \left(\frac{\alpha_s}{2\pi} \right) + \left(\frac{\beta_0^2}{\epsilon^2} - \frac{\beta_1}{2\epsilon} \right) \left(\frac{\alpha_s}{2\pi} \right)^2 + \mathcal{O}(\alpha_s^3). \quad (1.35)$$

We stress that Eq. (1.24) provides information on the running of α_s , but not on its value at a given energy scale, which instead must be extracted from experiments.

1.2.4 The anomalous dimensions

In Eq. (1.25), we presented a simplified version of the Callan-Symanzik equation for the dependence of a dimensionless, mass-independent observable $\hat{O}$ on the renormalization scale μ . We now consider what happens when $\hat{O}$ remains dimensionless but depends on the quark masses. Recall that in the QCD Lagrangian (1.20), the quark masses are free parameters of the theory, just like the strong coupling constant α_s . Thus, the arguments that led us to Eq. (1.25) can be extended to the case where $\hat{O}$ also depends on the quark masses.

For simplicity, let us consider the case $n_f = 1$, meaning we assume the theory involves a single quark flavour with renormalized mass m . As in Eq. (1.25), $\hat{O}$ must be independent of μ , so it must satisfy a RGE, whose form depends on the choice of the renormalization procedure. This RGE, which also involves the derivative of the mass with respect to μ^2 , reads

$$\left[\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_s) \frac{\partial}{\partial \alpha_s} - \gamma(\alpha_s) m \frac{\partial}{\partial m} \right] \hat{O} \left(\frac{Q^2}{\mu^2}, \alpha_s, \frac{m}{Q} \right) = 0, \quad (1.36)$$

where we note that $\hat{O}$ depends on m through m/Q . In Eq. (1.36), we adopted Weinberg approach [112], where both β and γ are only functions of the renormalized coupling α_s .

The function γ is called the *mass anomalous dimension*, and it is defined as

$$\gamma(\alpha_s) \stackrel{\text{def}}{=} -\frac{\mu^2}{m(\mu^2)} \frac{\partial m}{\partial \mu^2} \equiv -\frac{d \log m}{d \log \mu^2}. \quad (1.37)$$

Similarly to the β function in Eq. (1.28), γ can be expressed as a power series in α_s as

$$\gamma(\alpha_s) = \sum_{n=0}^{\infty} \gamma_n \left(\frac{\alpha_s}{2\pi} \right)^{n+1}. \quad (1.38)$$

Throughout this thesis, the coefficient γ_0 will frequently reappear, so we report its value here, which in the $\overline{\text{MS}}$ scheme corresponds to

$$\gamma_0 = \frac{3}{2} C_F. \quad (1.39)$$

The solution of Eq. (1.37) is

$$\begin{aligned} m(Q^2) &= m(\mu^2) \exp \left[- \int_{\mu^2}^{Q^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2} \gamma(\alpha_s(\bar{\mu}^2)) \right] \\ &= m(\mu^2) \exp \left[- \int_{\alpha_s(\mu^2)}^{\alpha_s(Q^2)} d\bar{\alpha}_s \frac{\gamma(\bar{\alpha}_s)}{\beta(\bar{\alpha}_s)} \right], \end{aligned} \quad (1.40)$$

which gives the expression of the *running mass* $m(Q^2)$, i.e. the mass parameter evaluated at an energy scale Q^2 . In terms of the running mass and the running coupling, the solution of the RGE (1.36) is essentially the same as the massless case. We can include all the non-trivial scale dependence of $\hat{O}$ in the running coupling and mass, getting

$$\hat{O} \left(\frac{Q^2}{\mu^2}, \alpha_s, \frac{m}{Q} \right) \rightarrow \hat{O} \left(1, \alpha_s(Q^2), \frac{m(Q^2)}{Q} \right). \quad (1.41)$$

Using the perturbative expansions of the β and γ functions in Eq. (1.40) and, specifically, the asymptotic freedom in the high-energy limit $Q \rightarrow \infty$, from Eqs (1.24) and (1.37) one can also determine the running mass, which corresponds to

$$m(Q^2) = \hat{m} [\alpha_s(Q^2)]^{\gamma_0/\beta_0} \left[1 + \mathcal{O}(\alpha_s) \right], \quad (1.42)$$

where $\hat{m}$ is a renormalization-group invariant constant of integration. Since $\gamma_0/\beta_0 > 0$, the quark mass decreases as the energy scale increases.

1.3 Definition of cross section

In *Quantum Mechanics* (QM), quantities that can be measured experimentally are essentially differential *probabilities*, derived from the squared modulus $|\langle f, t_f | i, t_i \rangle|^2$ of the in-

ner products of states. As M.D. Schwartz writes in [89],

Quantum field theory is just quantum mechanics with lots of fields ,

thus the experimental quantities we are able to predict through QFT are also of the form $|\langle f, t_f | i, t_i \rangle|^2$. In the Heisenberg picture, which is the default picture for QFT, $|i\rangle$ and $\langle f|$ states are time-independent, and the time evolution is relegated to operators. When momentum eigenstates evolve from $t_i = -\infty$ to $t_f = +\infty$ (a standard scenario in collider physics), the time-evolution operator is known as the *scattering $\mathcal{S}$ -matrix*, defined as

$$\langle f | \mathcal{S} | i \rangle \stackrel{\text{def}}{=} \langle f, t_f = +\infty | i, t_i = -\infty \rangle, \quad \mathcal{S}^\dagger \mathcal{S} = \mathbb{1}, \quad (1.43)$$

where the states on the left- and right-hand sides are in the Heisenberg and Schrödinger pictures, respectively.

In collider physics, where we have incoming beams in the high-energy regime, the primary quantities of interest are the $\mathcal{S}$ -matrix elements computed for $2 \rightarrow n$ scattering processes. For such a process, we want to relate the $\mathcal{S}$ -matrix with the *differential cross section*⁵

$$d\sigma = \frac{1}{T} \frac{1}{\Phi} d\mathcal{P}, \quad (1.44)$$

since the latter is a measure of the probability of a specific interaction or scattering event between particles. In Eq. (1.44), T is the time for the experiment, Φ is the incoming flux, and $d\mathcal{P}$ is the quantum mechanical differential probability of scattering. As said, in QFT, probabilities are determined by the squared modulus of amplitudes, hence the normalized differential probability $d\mathcal{P}$ can be expressed as

$$d\mathcal{P} = \frac{|\langle f | \mathcal{S} | i \rangle|^2}{\langle f | f \rangle \langle i | i \rangle} d\bar{\Pi}_n, \quad (1.45)$$

where $d\bar{\Pi}_n$ represents the n -particle final-state phase space, defined as

$$d\bar{\Pi}_n \stackrel{\text{def}}{=} \prod_{k=1}^n \frac{d^{(3)}\mathbf{p}_k}{(2\pi)^3} \frac{1}{2E_k}, \quad E_k = \sqrt{\mathbf{p}_k^2 + m_k^2}. \quad (1.46)$$

In Eq. (1.45), the $\mathcal{S}$ -matrix elements $\langle f | \mathcal{S} | i \rangle$ are computed by means of perturbation theory. In a free theory, the $\mathcal{S}$ -matrix is simply the identity matrix $\mathbb{1}$. Hence, to isolate its interesting part, that is, the part due to interactions, we rewrite it as

$$\mathcal{S} = \mathbb{1} + i\mathcal{T}. \quad (1.47)$$

The $\mathcal{T}$ -matrix is known as the *transfer matrix*, and accounts for deviations from the free theory. We note that $\mathcal{S}$ contains a factor $\delta^{(4)}(P_i^{\text{tot}} - P_f^{\text{tot}})$ due to momentum conservation, where P_i^{tot} and P_f^{tot} are the total initial- and final- state momenta, respectively, so does $\mathcal{T}$. Therefore, it is useful to factor out this δ -factor and introduce the *invariant matrix*

⁵Cross sections are measured in units of area, commonly *barns* (where $1 \text{ b} = 10^{-28} \text{ m}^2$).

element $\mathcal{M}$ such that

$$\langle f | i \mathcal{T} | i \rangle \stackrel{\text{def}}{=} (2\pi)^4 \delta^{(4)}(P_i^{\text{tot}} - P_f^{\text{tot}}) i \mathcal{M}(i \rightarrow f). \quad (1.48)$$

This approach allows us to focus on computing the non-trivial part of the $\mathcal{S}$ -matrix, that is, $\mathcal{M}$. In this thesis, we will typically use the words *matrix elements* or *amplitude* to refer to $\mathcal{M}$. Therefore, assuming $|f\rangle \neq |i\rangle$ (since $|f\rangle = |i\rangle$ means that nothing has happened), and, by noting that in center-of-mass frame the flux is given by the difference between the incoming particles' velocities, i.e., $\Phi = |\mathbf{v}_a - \mathbf{v}_b|/V$ (where V is the total volume, and the velocity is related to momentum by $\mathbf{v}_i = \mathbf{p}_i/p_0$), we conclude that⁶

$$d\sigma = \frac{1}{(2E_a)(2E_b)|\mathbf{v}_a - \mathbf{v}_b|} |\mathcal{M}(i \rightarrow f)|^2 d\Pi_n. \quad (1.49)$$

In Eq. (1.49),

$$d\Pi_n \stackrel{\text{def}}{=} \left(\prod_{k=1}^n \frac{d^{(3)}\mathbf{p}_k}{(2\pi)^3} \frac{1}{2E_k} \right) (2\pi)^4 \delta^{(4)}(P_i^{\text{tot}} - P_f^{\text{tot}}) \quad (1.50)$$

is the Lorentz-invariant n -particle final-state phase space. Eq. (1.49) is the definition of cross section that we will assume throughout this thesis.

1.4 The factorization theorem in hadronic cross sections

1.4.1 The birth of the parton model

As discussed in the previous section, QCD exhibits the property of asymptotic freedom, meaning that, if the energy scale Q^2 of the scattering process is sufficiently large ($\Lambda_{\text{qcd}} \ll Q^2$), then the strong coupling constant is small and it is possible to compute the amplitude for the scattering process using perturbation theory. However, due to the confinement phenomenon, in colliders we cannot accelerate free quarks or gluons, since in Nature they always cluster in hadrons. We therefore have the following problem: on one hand, our theory deals with elementary particles, namely quarks and gluons introduced in the Lagrangian (1.20); on the other hand, experimentally, we never deal with quarks and gluons, but rather with non-elementary particles, such as protons. In this section, we will describe a "bridge" that connects the theory describing interactions occurring between elementary particles with the fact that experiments involve hadrons.

A first solution to this problem was developed by Richard Feynman in the 1960s and 1970s, by means of a mechanism called *the parton model* [113]. This model was experimentally validated through deep-inelastic scattering (DIS)⁷ experiments at the Stanford Linear Accelerator (SLAC) during the same period [114]. Here we briefly discuss it using DIS as an example [86, 87].

⁶Intermediate steps can be found in standard textbooks.

⁷The *deep inelastic scattering* is a high-energy scattering process in which a lepton strikes a hadron, allowing the probing of its internal structure. *Deep* refers to the high energy of the incoming lepton. *Inelastic* means that the target hadron absorbs some kinetic energy, since it is shattered and emits many new particles.

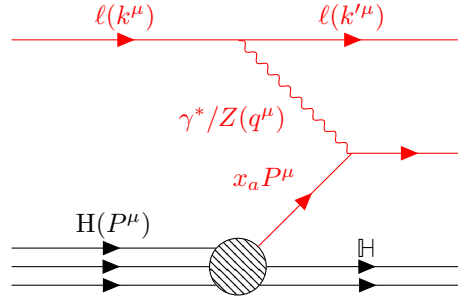


Figure 1.2: Schematic representation of the *deep inelastic scattering* (DIS) process, where $\ell(k^\mu)$ and $\ell(k'^\mu)$ are the incoming and outgoing leptons, respectively, $H(P^\mu)$ is the initial-state hadron, and $\mathbb{H}$ stands for any hadronic final state. The diagram in red defines the *hard-scattering process*, which can be computed using QCD perturbation theory.

In the parton model description of DIS, the hadron is considered as a composition of a number of simple particles, termed *partons*. The idea is that the interaction between the incoming lepton and hadron can be thought of as the interaction between a lepton and a single, pointlike parton, while the other partons inside the hadron do not interfere, as shown in Figure 1.2. This is a reasonable assumption because, in the reference frame where the hadron moves very fast and the energy of the process is large, the hadron will be Lorentz contracted in the direction of the collision and the lifetime of particles inside it increases. In its first version, this model was set in the infinite momentum frame where the hadron is ultra-relativistic, so its mass, and that of all its constituents, can be neglected. Therefore, not only does the hadron have an almost lightlike momentum, but its partons do as well, and these are all in the same direction, i.e., along the collision axis. If P^μ is the hadron momentum, the basic assumption is that each parton a takes a momentum $\hat{p}_a^\mu$ which is a fraction x_a of P^μ , namely

$$\hat{p}_a^\mu = x_a P^\mu, \quad 0 \leq x_a \leq 1. \quad (1.51)$$

The parameter x_a is called the *longitudinal fraction* of the constituent. Note that each constituent cannot acquire a large transverse momentum except through the exchange of a hard gluon, a process that is suppressed by the smallness of α_s at large momentum scales. Moreover, for the time being we consider *leading order* (LO) terms in α_s .

Regarding the parameter x_a , it cannot be derived using QCD perturbation theory, since it is related to the inner structure of the hadron as a bound state of quarks and gluons. Therefore, we define

$$dx_a f_a(x_a) \quad (1.52)$$

as the probability to find a parton a with a longitudinal fraction between x_a and $x_a + dx_a$ inside the hadron.

According to this notation, we can write the *hadronic cross section* of the DIS process

$$\ell(k^\mu) + H(P^\mu) \rightarrow \ell(k'^\mu) + \mathbb{H} \quad (1.53)$$

as

$$d\sigma_H = \sum_a \int_0^1 dx_a f_a(x_a) d\hat{\sigma}_a(x_a) \left[1 + \mathcal{O}\left(\frac{\Lambda_{\text{qcd}}^2}{Q^2}, \alpha_s\right) \right], \quad (1.54)$$

where $\ell(k^\mu)$ and $\ell(k'^\mu)$ are the incoming and outgoing leptons, respectively, $H(P^\mu)$ is the initial-state hadron, and $\mathbb{H}$ stands for any hadronic final state. Such a factorized expression is accurate up to corrections that are suppressed by powers of $\Lambda_{\text{qcd}}^2/Q^2$,⁸ as well as interactions between the partons, suppressed by $\mathcal{O}(\alpha_s)$. In Eq. (1.54), $d\hat{\sigma}_a(x_a)$ is the so-called *partonic cross section* between the incoming lepton and the parton a contained within the incoming hadron. Since $d\hat{\sigma}_a(x_a)$ involves only elementary particles, it can be computed directly through perturbation theory, and defines the so-called *hard scattering process*, as depicted in Figure 1.2. The functions f_a are called *parton distribution functions* (PDFs), and by definition, they are intrinsically non-perturbative objects. Their dependence on the longitudinal fraction x_a must be extracted from data.

Eq. (1.54) represents a fundamental result for QCD collider physics, since the hadronic cross section $d\sigma_H$ is a physical observable, i.e., its numerical value can be directly compared with experimental data. Early DIS experiments showed good agreement with the parton model, thereby providing strong support for QCD as a theory for the strong interaction, and Feynman's partons were soon associated with the quarks and gluons in the model based on the SU(3) gauge symmetry by Gell-Mann in 1964 [115], which led to QCD. However, Eq. (1.54) holds up to $\mathcal{O}(\alpha_s)$ corrections, since in our discussion so far we have neglected higher order QCD corrections to the partonic behavior. For this reason, the model presented here is sometimes called the *naïve* parton model. In the following pages, we will present a generalization of Eq. (1.54) and, additionally, all the implications of considering QCD corrections to this formula.

1.4.2 The factorization theorem

We have just discussed that the parton model led to a fundamental result in particle physics, namely Eq. (1.54). This early result needs to be generalized to include both corrections of $\mathcal{O}(\alpha_s^m)$ with $m \geq 1$, coming from the interaction of partons, and processes with two strongly interacting particles in the initial state. This led to the introduction of the *factorization theorem* [116–119], which constitutes the foundation of theoretical predictions for hard scattering processes at hadron colliders. It states that, up to power-suppressed terms, hadronic cross sections are described by the following formula

$$d\sigma_H = \sum_{ab} \int_0^1 dx_a dx_b f_a(x_a, \mu_F^2) f_b(x_b, \mu_F^2) \times d\hat{\sigma}_{ab}\left(x_a, x_b, \alpha_s(\mu_R^2), \frac{Q^2}{\mu_F^2}, \frac{Q^2}{\mu_R^2}\right) \left[1 + \mathcal{O}\left(\frac{\Lambda_{\text{qcd}}^n}{Q^n}\right) \right], \quad n \geq 1, \quad (1.55)$$

graphically represented in Figure 1.3. In Eq. (1.55), a and b are the flavours of the first and second incoming partonic legs, respectively, x_a and x_b are their longitudinal fractions, $\Lambda_{\text{qcd}} \sim 200$ MeV is the non-perturbative QCD scale, and $Q \gtrsim \mathcal{O}(10 \text{ GeV})$ is the

⁸Here $Q^2 = -q^2$, where $q^\mu = k^\mu - k'^\mu$ is the momentum transfer.

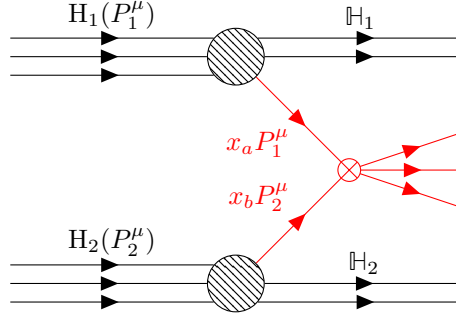


Figure 1.3: Schematic representation of the *factorization theorem* (1.55), where $H_1(P_1^\mu)$ and $H_2(P_2^\mu)$ are the two incoming hadrons, respectively, and the diagram in red defines the hard-scattering process, which can be computed using QCD perturbation theory. $\mathbb{H}_1$ and $\mathbb{H}_2$ are any hadronic final states that do not participate in the hard-scattering process.

typical scale of the hard process. The correction $\mathcal{O}(\Lambda_{\text{qcd}}^n/Q^n)$ accounts for the error introduced in Eq. (1.55) by neglecting non-perturbative effects that may spoil the validity of factorization. Notice that the power n is usually $n \geq 2$ in many processes, making this correction negligible. However, for some observables, $n = 1$ linear behavior has been found (see for instance Ref [120–122]), making this correction relevant for precise phenomenology.

Next, $f_{a,b}$ are the PDFs introduced in Eq. (1.52) and, as said, they are non-perturbative and process-independent objects. They can be extracted from a subset of experimental data and used to describe any process from a complementary dataset. PDFs satisfy the *Dokshitzer-Gribov-Lipatov-Altarelli-Parisi* (DGLAP) equation [123–125]

$$\mu^2 \frac{d}{d\mu^2} f_i(x, \mu^2) = \frac{\alpha_s(\mu^2)}{2\pi} \int_0^1 \frac{d\xi}{\xi} \hat{P}_{ij} \left(\frac{x}{\xi}, \alpha_s(\mu^2) \right) f_j(\xi, \mu^2), \quad (1.56)$$

where indices i, j can be a gluon g or any flavour of q or $\bar{q}$ (the sum over j is understood). Thus, Eq. (1.56) is a $(2n_f + 1)$ -dimensional matrix equation in the space of quarks, anti-quarks and gluons, describing how these partons evolve within the hadron through the emission of gluons. The functions $\hat{P}_{ij}$ are termed *splitting functions* and, again, they are calculable as a power series of the strong coupling constant α_s ,

$$\begin{aligned} \hat{P}_{q_i q_j}(z, \alpha_s) &= \delta_{ij} \hat{P}_{qq}^{(0)}(z) + \frac{\alpha_s}{2\pi} \hat{P}_{q_i q_j}^{(1)}(z) + \dots, \\ \hat{P}_{qg}(z, \alpha_s) &= \hat{P}_{qg}^{(0)}(z) + \frac{\alpha_s}{2\pi} \hat{P}_{qg}^{(1)}(z) + \dots, \\ \hat{P}_{gq}(z, \alpha_s) &= \hat{P}_{gq}^{(0)}(z) + \frac{\alpha_s}{2\pi} \hat{P}_{gq}^{(1)}(z) + \dots, \\ \hat{P}_{gg}(z, \alpha_s) &= \hat{P}_{gg}^{(0)}(z) + \frac{\alpha_s}{2\pi} \hat{P}_{gg}^{(1)}(z) + \dots. \end{aligned} \quad (1.57)$$

Here we report the expressions of the leading order $\hat{P}_{ij}^{(0)}$:

$$\begin{aligned}\hat{P}_{qq}^{(0)}(z) &\stackrel{\text{def}}{=} C_F \left[2\mathcal{D}_0(z) - (1+z) + \frac{3}{2}\delta(1-z) \right], \\ \hat{P}_{qg}^{(0)}(z) &\stackrel{\text{def}}{=} T_R \left[(1-z)^2 + z^2 \right], \\ \hat{P}_{gq}^{(0)}(z) &\stackrel{\text{def}}{=} C_F \left[\frac{1 + (1-z)^2}{z} \right], \\ \hat{P}_{gg}^{(0)}(z) &\stackrel{\text{def}}{=} 2C_A \left[\mathcal{D}_0(z) + z(1-z) + \frac{1}{z} - 2 \right] + \beta_0 \delta(1-z),\end{aligned}\tag{1.58}$$

where

$$\mathcal{D}_n(z) \stackrel{\text{def}}{=} \left[\frac{\log^n(1-z)}{1-z} \right]_+, \tag{1.59}$$

while the definitions of the next-to-leading order $\hat{P}_{ij}^{(1)}$ can be found in Eqs (A.27 - A.30). Note that, owing to the charge conjugation invariance and $\text{SU}(n_f)$ flavour symmetry, we have

$$\hat{P}_{q_i q_j} = \hat{P}_{\bar{q}_i \bar{q}_j}, \quad \hat{P}_{q_i \bar{q}_j} = \hat{P}_{\bar{q}_i q_j}, \quad \hat{P}_{q_i g} = \hat{P}_{\bar{q}_i g} \equiv \hat{P}_{qg}, \quad \hat{P}_{gq_i} = \hat{P}_{g\bar{q}_i} \equiv \hat{P}_{gq}. \tag{1.60}$$

We point out that the DGLAP evolution (1.56) introduces a dimensionful parameter μ^2 similarly to what the RGE does in the renormalization procedure discussed in Subsec. 1.2.3. This is why two different scales appear in Eq. (1.55): the *renormalization scale* μ_R^2 (see Eq. (1.24)), and the *factorization scale* on which the PDFs depend. A physical observable, like $d\sigma_H$, should in principle be independent on both μ_R^2 and μ_F^2 , but because we truncate the perturbative expansion at a finite order, a residual dependence on these scales is always present, and can be interpreted as a theoretical uncertainty due to missing higher-order corrections.

Finally, in Eq. (1.55), $d\hat{\sigma}_{ab}$ corresponds to the partonic cross section, which encodes the short-distance interactions for incoming quarks or gluons of flavours a and b that can be calculated in QCD perturbation theory. Expanding $d\hat{\sigma}_{ab}$ in powers of α_s , we write

$$d\hat{\sigma}_{ab} = d\hat{\sigma}_{ab}^{\text{LO}} + d\hat{\sigma}_{ab}^{\text{NLO}} + d\hat{\sigma}_{ab}^{\text{NNLO}} + \dots, \tag{1.61}$$

where $d\hat{\sigma}_{ab}^{\text{LO}}$ describes *leading order* (LO) contributions, and $d\hat{\sigma}_{ab}^{\text{NLO}}$ and $d\hat{\sigma}_{ab}^{\text{NNLO}}$ describe *next-to-leading order* (NLO) and *next-to-next-to-leading order* (NNLO) contributions, respectively, i.e., corrections of $\mathcal{O}(\alpha_s)$ and $\mathcal{O}(\alpha_s^2)$ with respect to $d\hat{\sigma}_{ab}^{\text{LO}}$. The theoretical description of $d\hat{\sigma}_{ab}^{\text{LO}}$ and $d\hat{\sigma}_{ab}^{\text{NLO}}$ is well-understood and largely automated [7, 126–136], while currently we still do not have a comprehensive description of NNLO phenomena. The goal of this thesis is to develop a new approach to compute $d\hat{\sigma}_{ab}^{\text{NNLO}}$ at a fully differential level.

1.5 Introduction to infra-red divergences in QCD

As anticipated at the end of the previous section, there are multiple aspects of the higher-order corrections to the partonic cross section $d\hat{\sigma}_{ab}$ in Eq. (1.61) that are not yet fully understood or automated. To understand this, we will consider the *Drell-Yan*⁹ process at LO precision, i.e.,

$$q + \bar{q} \rightarrow X, \quad (1.62)$$

as a prototype for the discussion, where X labels any colorless final state. As we will see shortly, the problem lies in the fact that $d\hat{\sigma}_{ab}^{\text{NLO}}$, $d\hat{\sigma}_{ab}^{\text{NNLO}}$, etc., can be written as sums of contributions that, when taken individually, are divergent. We examine in more detail what these divergences are.

To calculate higher-order perturbative contributions to the LO partonic cross section $d\hat{\sigma}_{ab}^{\text{LO}}$, we must include both *loop corrections* to this process and *inelastic processes* such as

$$q + \bar{q} \rightarrow X + g, \quad (1.63)$$

$$q + \bar{q} \rightarrow X + \{g\}_2, \quad (1.64)$$

where additional partons appear in the final state [138] ($\{g\}_2$ labels a set of two gluons). These contributions are known as *virtual* and *real corrections*, respectively. We point out that real emissions are *always* unresolved in those singular kinematic regions that are responsible for *infra-red* (IR) and collinear poles, no matter what process is under study. These kinematic configurations occur when a parton is emitted with vanishingly small energy (hence the label of IR singularity), or when the angle between partons approaches zero. These two cases are termed *soft limit* and *collinear limit*, respectively, and will play a central role in this thesis. We will dedicate the next chapter to analyze in detail how these divergences work.

One of the main issues with virtual and real corrections is that they are not separately IR-finite; on the contrary, in general they can contain multiple IR divergences of different kinds. However, combining both virtual and real corrections yields well-defined IR-safe observables, as guaranteed by the *Kinoshita–Lee–Nauenberg theorem* (KLN theorem) [138, 139]. Before proceeding, we make a brief explanatory digression. In QED, there is a theorem known as the *Bloch–Nordsieck theorem* [140], which states:¹⁰

IR singularities will always cancel when summing over final state radiation in QED with a massive electron as long as there is a finite energy resolution.

However, this statement does not hold in QCD. In general, IR singularities appearing at NNLO in QCD with massive quarks will not cancel summing over final-state radiation in $2 \rightarrow n$ processes only; one also needs to sum over $3 \rightarrow n$ processes [141]. On the other hand, the behavior of the singular leftover is proportional to a power of the quark mass,

⁹The Drell-Yan process [137] is a mechanism for producing lepton-antilepton pairs (typically e^-e^+ or $\mu^-\mu^+$ pairs) in hadronic collisions, such as those between protons. At LO, it occurs when a quark from one hadron interacts with an antiquark from another hadron, emitting a virtual photon γ^* or a Z boson, which then decays into the lepton-antilepton pair.

¹⁰A formal proof and discussion of both the Bloch–Nordsieck theorem and the Kinoshita–Lee–Nauenberg theorem is reported in Ref. [83].

so it vanishes as long as the high-energy limit $\frac{m_q}{Q} \rightarrow 0$ is assumed. It follows that at higher scales, where masses can be neglected, one can get an IR-finite result summing only cross sections with two particles in the initial state. This statement does not come from asymptotic freedom, and would hold even if there were enough flavors ($n_f \geq 17$) so that QCD were IR free, like QED. The *KNL theorem* states a broader result in QCD, i.e.,

IR divergences will cancel in any unitary theory when all possible final and initial states in a finite energy window are summed over.

To be fair, this theorem is more of formal interest, since one does not usually sum over initial states when computing cross sections. With this, we conclude the digression.

Following from the earlier discussion, we write the NLO and NNLO cross sections as

$$\begin{aligned} d\hat{\sigma}_{ab}^{\text{NLO}} &= d\hat{\sigma}_{ab}^{\text{V}} + d\hat{\sigma}_{ab}^{\text{R}} + d\hat{\sigma}_{ab}^{\text{pdf}}, \\ d\hat{\sigma}_{ab}^{\text{NNLO}} &= d\hat{\sigma}_{ab}^{\text{VV}} + d\hat{\sigma}_{ab}^{\text{RV}} + d\hat{\sigma}_{ab}^{\text{RR}} + d\hat{\sigma}_{ab}^{\text{pdf}}, \end{aligned} \quad (1.65)$$

where:

- $d\hat{\sigma}_{ab}^{\text{V}}$ and $d\hat{\sigma}_{ab}^{\text{VV}}$ represent *one-loop* and *two-loop virtual* corrections to the hard process (1.62), respectively;
- $d\hat{\sigma}_{ab}^{\text{R}}$ describes *single-real* corrections, i.e., a process with one additional parton in the final state (e.g. (1.63));
- $d\hat{\sigma}_{ab}^{\text{RV}}$ denotes *real-virtual* corrections, i.e., the one-loop correction to $d\hat{\sigma}_{ab}^{\text{R}}$;
- $d\hat{\sigma}_{ab}^{\text{RR}}$ accounts for *double-real* corrections, i.e., a process with two additional partons in the final state (e.g. (1.64));
- $d\hat{\sigma}_{ab}^{\text{pdf}}$ includes corrections due to the collinear renormalization of parton distribution functions (see Eq. (1.56)).

In Figure 1.4 we collect some examples of the corresponding Feynman diagrams for the aforementioned corrections.

In agreement with the above theorems, both $d\hat{\sigma}_{ab}^{\text{NLO}}$ and $d\hat{\sigma}_{ab}^{\text{NNLO}}$ in Eq. (1.65) are IR-safe quantities. However, as said, the individual contributions on the right-hand side of Eq. (1.65) are not. Virtual corrections in $d\hat{\sigma}_{ab}^{\text{V}}$ and $d\hat{\sigma}_{ab}^{\text{VV}}$ contain explicit IR and *collinear* (we will see shortly what this word means) poles in the dimensional regularization parameter $\epsilon = (d - 4)/2$ [142] (cf. Subsec. 2.1), which are proportional to the hard matrix elements [8, 82, 143–145]. For instance, if we consider the process (1.62), i.e., $q(p_a) + \bar{q}(p_b) \rightarrow X(p_X)$, the singular structure of the UV-renormalized one-loop contribution to the cross section is well-known and corresponds to¹¹ [8, 143, 144]

$$2 \operatorname{Re} [\mathcal{M}_0^* \mathcal{M}_1](p_a, p_b, p_X) = -2C_F \left[\frac{\alpha_s(\mu)}{2\pi} \frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)} \right] \left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon} \right] \cos(\pi\epsilon)$$

¹¹Integrating the left-hand side of the following formula over the phase space yields the virtual cross section $d\hat{\sigma}_{ab}^{\text{V}}$.

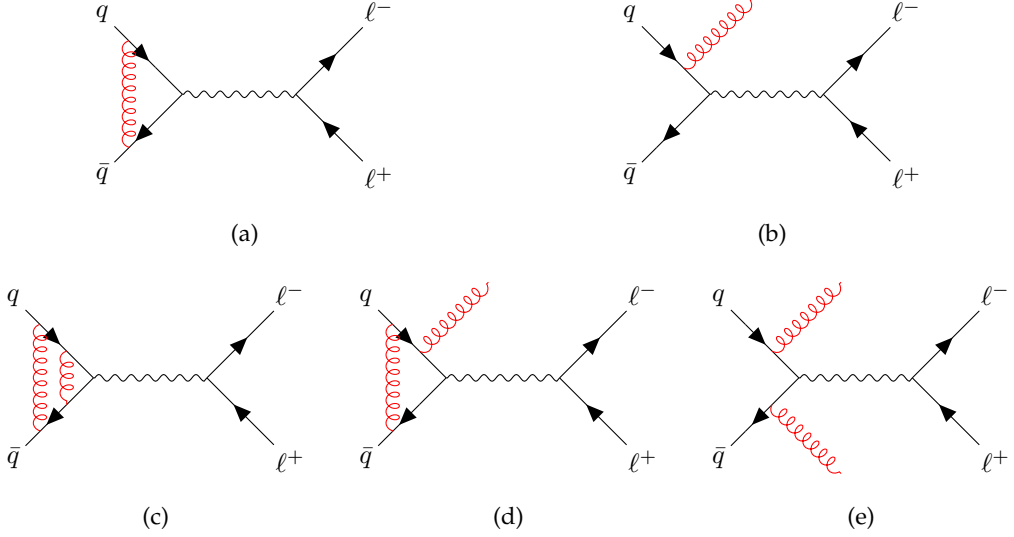


Figure 1.4: Some examples of Feynman diagrams that contribute to higher-order corrections of the Drell-Yan process in Eq. (1.62), where X here corresponds to the $\ell^-\ell^+$ pair. Gluons in red are additional partons to the LO diagram in black. Specifically, the following corrections are listed: (a) (single-)virtual; (b) single-real; (c) double-virtual; (d) real-virtual; (e) double-real. Note that (a, b) and (c-e) are NLO and NNLO corrections to the process (1.62), respectively.

$$\times \left(\frac{2p_a \cdot p_b}{\mu^2} \right)^{-\epsilon} |\mathcal{M}_0(p_a, p_b)|^2 + 2 \operatorname{Re} [\mathcal{M}_0^* \mathcal{M}_1^{\text{fin}}](p_a, p_b, p_X), \quad (1.66)$$

where $\mathcal{M}_0$ and $\mathcal{M}_1$ are the tree-level and 1-loop amplitudes, respectively, while $\mathcal{M}_1^{\text{fin}}$ is the IR-finite remainder of $\mathcal{M}_1$. Similar formulas can be obtained for any hadronic scattering process, and an analogous result is known for the two-loop case [82, 145]. We will dedicate Sec. 2.2 discussing IR virtual divergences.

As we can appreciate from Eq. (1.66), virtual contributions have explicit $1/\epsilon$ poles arising from IR divergences. In contrast, the real contributions, such as $d\hat{\sigma}_{ab}^{\text{R}}$, $d\hat{\sigma}_{ab}^{\text{RV}}$, and $d\hat{\sigma}_{ab}^{\text{RR}}$, contain kinematic singularities that become $1/\epsilon$ poles only upon integrating over the phase space of the additional final-state partons. Therefore, in order to keep partonic cross sections fully differential and to allow the numerical integration over the final-state phase space, one needs to develop specific techniques to extract and regularize implicit $1/\epsilon$ poles from real emission contributions before integrating over the resolved phase space.

Universal factorization formulas in QCD

Contents

2.1	Dimensional regularization, the color space, and the color algebra	32
2.2	IR singular behavior of QCD virtual amplitudes	35
2.2.1	Factorization formula for one-loop amplitudes	35
2.2.2	Factorization formula for two-loop amplitudes	38
2.3	Soft and collinear singularities of $\mathcal{O}(\alpha_s)$ in tree-level amplitudes	41
2.3.1	Soft singularities	42
2.3.2	Collinear singularities	46
2.4	IR singularities of $\mathcal{O}(\alpha_s^2)$ in tree-level amplitudes	54
2.4.1	Double-soft singularities	54
2.4.2	Triple-collinear singularities	57
2.5	IR singularities of $\mathcal{O}(\alpha_s^2)$ in real-virtual amplitudes	59
2.5.1	Soft limit in real-virtual amplitudes	59
2.5.2	Collinear limit in real-virtual amplitudes	60
2.6	The need for IR regularization schemes	63
2.6.1	IR slicing and subtraction methods for differential cross sections	65

In Subsec. 1.4.2 of the previous chapter, we introduced the factorization theorem (1.55) in its most general form. This important formula allows us to calculate the hadronic cross section $d\sigma_H$, which is a physical observable containing elements of non-perturbative physics, from the partonic cross section $d\hat{\sigma}_{ab}$, which is not a physical observable but can be calculated using perturbative QCD. In Sec. 1.5, we then saw that $d\hat{\sigma}_{ab}$ can be written as a power series in the strong coupling constant α_s (see Eq. (1.61)), and by virtue of the KLN theorem, each perturbative order of this series is IR-safe. However, in Eq. (1.65), we saw that at each perturbative order the cross section can be written as sum of contributions that are not individually IR-safe but contain multiple IR divergences of different kinds. Finally, we have shown that virtual contributions to the cross sections include explicit IR singularities, while real corrections contain IR kinematic singularities that become $1/\epsilon$ poles only upon integrating over the phase space of the additional final-state partons.

In this chapter, we present a summary of the main QCD IR divergences that can be encountered at $\mathcal{O}(\alpha_s)$ and $\mathcal{O}(\alpha_s^2)$ in both tree-level, and one- and two-loop amplitudes.

Specifically, in Sec. 2.1, we will introduce the so-called *color-space formalism* and *color-algebra*; in Sec. 2.2, we will analyze the IR poles of QCD one- and two-loop amplitudes; in Sec. 2.3, we will discuss *soft and collinear divergences* in tree-level amplitudes at order $\mathcal{O}(\alpha_s)$; in Sec. 2.4, we will summarize *double-soft* and *triple-collinear divergences*, which appear in tree-level amplitudes at order $\mathcal{O}(\alpha_s^2)$; in Sec. 2.5 we collect the main formulas for soft and collinear limits applied to real-virtual amplitudes.

We emphasize that this chapter does not contain any original content, as it covers phenomena and results that are well-established in the literature. Therefore, the purpose of this chapter is purely pedagogical. We hope that it could prove useful for researchers who are new to these topics and are looking for a (as much as possible) self-contained summary to gain a general overview.

2.1 Dimensional regularization, the color space, and the color algebra

As discussed in Sec. 1.5, scattering amplitudes are in general affected by IR and/or UV singularities, which require appropriate regularizations before being combined with contributions coming from real emissions. Despite different techniques being available, probably the most efficient way of simultaneously regularizing both types of singularities in gauge theories is *dimensional regularization* [142, 146–149]. This technique consists in analytically continuing the loop momenta to $d = 4 - 2\epsilon$ space-time dimensions, where $\epsilon > 0$ for UV singularities and $\epsilon < 0$ for IR ones. Such an analytic continuation allows some freedom in choosing both the dimensionality of the momenta of the external partons and the number of polarizations of internal and external partons, resulting in various regularization schemes within the dimensional regularization prescription. The *conventional dimensional regularization* (CDR) scheme is a convenient choice, as it treats partons in the loops and external partons uniformly. In CDR, all particle momenta are d -dimensional, gluons have $d - 2$ helicity states, and massless quarks have 2 helicity states. In what follows, we use the CDR scheme.

With this in mind, we proceed by defining the *color space* and the *color-charge algebra* [8, 82]. Let $\mathcal{M}(\{p\}_m)$ be a generic amplitude involving m external partons of momenta $\{p\}_m = \{p_1, \dots, p_m\}$, and let $c_1, \dots, c_m$ be the corresponding parton color indices, where $c_i = \{a\} = \{1, \dots, N_c^2 - 1\}$ for gluons, and $c_i = \{\alpha, \beta\} = \{1, \dots, N_c\}$ for (anti)quarks.¹ We make the following assumptions:

- i) We always consider incoming and outgoing parton momenta on-shell and in the physical region, i.e., every p_i is massless and with positive-definite energy.
- ii) Any dependence on non-strongly interacting particles is implicit in $\mathcal{M}$.
- iii) The unrenormalized amplitude $\mathcal{M}$ depends on powers of the bare strong coupling constant $\alpha_{s,b}$. The renormalized amplitude is obtained from the latter by writing

¹Differently from Sec. 1.2, here we label the color indices for the fundamental representation as α, β instead of i, j .

$\alpha_{s,b}$ in terms of the running coupling $\alpha_s(\mu^2)$. Here the standard $\overline{\text{MS}}$ scheme is assumed; since we evaluate $\alpha_{s,b}$ in the aforementioned CDR scheme, one can simply replace $\alpha_{s,b}$ according to Eq. (1.34).

- iv) $\mathcal{M}_0$ does not necessarily indicate a tree-level amplitude, but rather the leading-order (LO) of a given process under study. Therefore, $\mathcal{M}_1$ and $\mathcal{M}_2$ stand for the corresponding extra one-loop and two-loop corrections to $\mathcal{M}_0$, respectively.

In this m -parton color space, we introduce an orthogonal basis of unit vectors, termed $\{|c_1, \dots, c_m\rangle\}$, so as to write the color amplitude as

$$\mathcal{M}^{c_1 \dots c_m}(p_1, \dots, p_m) \stackrel{\text{def}}{=} \langle c_1, \dots, c_m | \mathcal{M}(\{p\}_m) \rangle, \quad (2.1)$$

where $|\mathcal{M}(\{p\}_m)\rangle$ defines an abstract vector in the color space. We write its square modulus, implicitly summed over color indices, as

$$|\mathcal{M}(\{p\}_m)|^2 = \langle \mathcal{M}(\{p\}_m) | \mathcal{M}(\{p\}_m) \rangle. \quad (2.2)$$

In Eq. (2.2), we represent color degrees of freedom at the QCD vertices by associating a color charge $T_i = \{T_i^a\}$ to the emission of a gluon from each parton i . The action of T_i on the color space is defined as

$$\langle c_1, \dots, c_i, \dots, c_m | T_i^a | b_1, \dots, b_i, \dots, b_m \rangle \stackrel{\text{def}}{=} \delta_{c_1 b_1} \dots T_{c_i b_i}^a \dots \delta_{c_m b_m}, \quad (2.3)$$

where (cf. Subsec. 1.2.1):

- $(T_i^a)_{cb} \equiv T_{cb}^a \stackrel{\text{def}}{=} if_{cab}$ if the emitting particle i is a gluon;
- $(T_i^a)_{\alpha\beta} \equiv T_{\alpha\beta}^a \stackrel{\text{def}}{=} t_{\alpha\beta}^a$ if the emitting particle i is a final-state quark or an initial-state antiquark;²
- $(T_i^a)_{\alpha\beta} \equiv T_{\alpha\beta}^a \stackrel{\text{def}}{=} \bar{t}_{\alpha\beta}^a = -t_{\beta\alpha}^a$ if the emitting particle i is a final-state antiquark or an initial-state quark.

Essentially, $T_i = \{T_i^a\}$ is a vector with respect to the color indices a , and an $\text{SU}(N_c)$ matrix with respect to the color indices of the parton i . According to this notation, each vector $|\mathcal{M}(\{p\}_m)\rangle$ is a color singlet, so the following *color conservation* holds:

$$\sum_{i \in \mathcal{H}} T_i |\mathcal{M}(\{p\}_m)\rangle = 0, \quad (2.4)$$

where $\mathcal{H}$ labels the list of the m external hard partons, that is,

$$\sum_{i \in \mathcal{H}} \stackrel{\text{def}}{=} \sum_{i=1}^m. \quad (2.5)$$

²The color-charge operator of an initial-state parton is defined by crossing symmetry.

The *color-charge algebra* for the product $\mathbf{T}_i \cdot \mathbf{T}_j \stackrel{\text{def}}{=} (\mathbf{T}_i)^a (\mathbf{T}_j)^a$ is

$$\mathbf{T}_i \cdot \mathbf{T}_j = \mathbf{T}_j \cdot \mathbf{T}_i \quad \text{if } i \neq j, \quad \mathbf{T}_i^2 = C_i, \quad (2.6)$$

where

$$\mathbf{T}_q^2 = \mathbf{T}_{\bar{q}}^2 = C_F, \quad \mathbf{T}_g^2 = C_A, \quad (2.7)$$

and C_i are the quadratic Casimir operators introduced in Eq. (1.9). We also define the square of *color-correlated tree-amplitudes*, $|\mathcal{M}^{(ij)}|^2$, as

$$\begin{aligned} |\mathcal{M}^{(ij)}|^2 &\stackrel{\text{def}}{=} \langle \mathcal{M}(\{p\}_m) | \mathbf{T}_i \cdot \mathbf{T}_j | \mathcal{M}(\{p\}_m) \rangle \\ &= [\mathcal{M}^{c_1 \dots b_i \dots b_j \dots c_m}(p_1, \dots, p_m)]^* T_{b_i c_i}^a T_{b_j c_j}^a \mathcal{M}^{c_1 \dots c_i \dots c_j \dots c_m}(p_1, \dots, p_m). \end{aligned} \quad (2.8)$$

Depending on the number m of partons that the amplitude $\mathcal{M}$ contains, the color algebra may or may not factorize in terms of linear combinations of the Casimir operators C_i . For instance, if $m = 2$, using color conservation (2.4), we find³

$$\mathbf{T}_1 + \mathbf{T}_2 = 0 \quad \implies \quad (\mathbf{T}_1 \cdot \mathbf{T}_2) = -\mathbf{T}_1^2 = -\mathbf{T}_2^2, \quad (2.9)$$

which states that all charge operators $[\mathbf{T}_1^2, \mathbf{T}_2^2, \mathbf{T}_1 \cdot \mathbf{T}_2]$ are factorizable in terms of the Casimir operators $C_1 = C_2$. Similarly, using (2.4) for the three-parton case, we find the relations

$$\mathbf{T}_1 \cdot (\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3) = 0 \quad \implies \quad (\mathbf{T}_1 \cdot \mathbf{T}_2) + (\mathbf{T}_1 \cdot \mathbf{T}_3) = -\mathbf{T}_1^2 \quad (2.10)$$

and

$$\begin{aligned} 0 &= (\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3)^2 \\ &= \mathbf{T}_1^2 + \mathbf{T}_2^2 + \mathbf{T}_3^2 + 2(\mathbf{T}_1 \cdot \mathbf{T}_2) + 2(\mathbf{T}_1 \cdot \mathbf{T}_3) + 2(\mathbf{T}_2 \cdot \mathbf{T}_3), \end{aligned} \quad (2.11)$$

which, once combined, yield

$$\begin{aligned} \mathbf{T}_1 \cdot \mathbf{T}_2 &= \frac{C_3 - C_1 - C_2}{2}, \\ \mathbf{T}_1 \cdot \mathbf{T}_3 &= \frac{C_2 - C_1 - C_3}{2}, \\ \mathbf{T}_2 \cdot \mathbf{T}_3 &= \frac{C_1 - C_2 - C_3}{2}. \end{aligned} \quad (2.12)$$

However, with $m \geq 4$, the color algebra is not closed anymore. Let us consider for example the case $m = 4$. The algebra is composed of 10 color operators, i.e., four $\mathbf{T}_i^2$ operators and six products $\mathbf{T}_i \cdot \mathbf{T}_j$, with $i \neq j$. We use the second identity in Eq. (2.6) to find the first four operators

$$\mathbf{T}_i^2 = C_i, \quad i = 1, \dots, 4. \quad (2.13)$$

In order to get the remaining six products $\mathbf{T}_i \cdot \mathbf{T}_j$ ($i \neq j$), we can only use Eq. (2.4) and

³In the following equations, we are using a slight abuse of notation. Indeed, any $\mathbf{T}_i$ operator should act on $|\mathcal{M}(\{p\}_m)\rangle$. However, for the sake of keeping the notation as light as possible, we leave $|\mathcal{M}(\{p\}_m)\rangle$ implicit.

multiply it by T_i , with $i = 1, \dots, 4$, as follows:

$$0 = T_i \cdot \left[\sum_{j=1}^4 T_j \right] = \sum_{i=1}^4 T_i \cdot T_j, \quad i = 1, \dots, 4. \quad (2.14)$$

In this way, we have identified a system of eight equations for ten operators that need to be determined. Therefore, we express all the charge operators in terms of Casimir invariants and two out of six products $T_i \cdot T_j$, say $T_1 \cdot T_2$ and $T_1 \cdot T_3$:

$$\begin{aligned} T_1 \cdot T_4 &= -C_1 - T_1 \cdot T_2 - T_1 \cdot T_3, \\ T_2 \cdot T_3 &= \frac{C_4 - C_1 - C_2 - C_3}{2} - T_1 \cdot T_2 - T_1 \cdot T_3, \\ T_2 \cdot T_4 &= \frac{C_1 + C_3 - C_2 - C_4}{2} + T_1 \cdot T_3, \\ T_3 \cdot T_4 &= \frac{C_1 + C_2 - C_3 - C_4}{2} + T_1 \cdot T_2. \end{aligned} \quad (2.15)$$

As for $T_1 \cdot T_2$ and $T_1 \cdot T_3$, they must be computed explicitly using Eq. (2.8).

In general, with $m \geq 4$, the color algebra is composed of $m(m+1)/2$ color operators $T_i \cdot T_j$. Of these, m operators can be found using the second identity in Eq. (2.6), which gives $T_i^2 = C_i$, with $i = 1, \dots, m$. Of the remaining $m(m-1)/2$ operators, another m can be determined via charge conservation (2.4), which yields

$$0 = T_i \cdot \left[\sum_{j \in \mathcal{H}} T_j \right] = \sum_{j \in \mathcal{H}} T_i \cdot T_j, \quad i \in \mathcal{H}. \quad (2.16)$$

Thus, $m(m-3)/2$ operators need to be explicitly calculated using Eq. (2.8).

2.2 IR singular behavior of QCD virtual amplitudes

2.2.1 Factorization formula for one-loop amplitudes

In Sec. 2.1, we introduced the formalism necessary to present here a first significant result: a formula that describes the IR-singular behavior of QCD amplitudes at one-loop [8, 82, 143, 144].

We start by considering a generic scattering process with m external partons, and we call $\mathcal{M} = \mathcal{M}(\alpha_s(\mu), \mu^2 \{p\}_m)$ its $\overline{\text{MS}}$ -renormalized amplitude, where μ^2 is the renormalization scale.⁴ We expand $\mathcal{M}$ to first order in powers of the strong coupling constant α_s ,

$$\mathcal{M} = \mathcal{M}_0 + \left[\frac{\alpha_s(\mu)}{2\pi} \right] \mathcal{M}_1 + \mathcal{O}(\alpha_s^2), \quad (2.17)$$

where $\mathcal{M}_0 = \mathcal{M}_0(\mu^2, \{p\}_m)$ and $\mathcal{M}_1 = \mathcal{M}_1(\mu^2, \{p\}_m)$ are the tree-level and the one-loop amplitudes, respectively. We saw in Eq. (1.49) that the cross section is proportional to

⁴Recall that we always consider the running coupling $\alpha_s(\mu)$ as that defined in the $\overline{\text{MS}}$ scheme, see Eq. (1.34).

$|\mathcal{M}|^2$, which reads

$$\begin{aligned} |\mathcal{M}|^2 &= \left| \mathcal{M}_0 + \left[\frac{\alpha_s(\mu)}{2\pi} \right] \mathcal{M}_1 + \mathcal{O}(\alpha_s^2) \right|^2 \\ &= \underbrace{|\mathcal{M}_0|^2}_{\text{LO}} + \underbrace{\left[\frac{\alpha_s(\mu)}{2\pi} \right] 2 \operatorname{Re}[\mathcal{M}_0^* \mathcal{M}_1]}_{\text{NLO}} + \underbrace{\mathcal{O}(\alpha_s^2)}_{\text{NNLO}}. \end{aligned} \quad (2.18)$$

In this section, we focus on the second term of (2.18), and write it using the color space formalism as

$$\left[\frac{\alpha_s(\mu)}{2\pi} \right] 2 \operatorname{Re}[\langle \mathcal{M}_0 | \mathcal{M}_1 \rangle]. \quad (2.19)$$

The main result is that the IR singularities of the one-loop amplitude $|\mathcal{M}_1\rangle$ can be expressed as a *Laurent series* in ϵ (the variable introduced through dimensional regularization), with non-zero coefficients up to $\mathcal{O}(\epsilon^{-2})$. More importantly, these coefficients are *universal*, to the extent that they depend only on colors, momenta, and flavours of the external partons. The IR singularities of $|\mathcal{M}_1\rangle$ can be encapsulated in the following formula [82]:

$$|\mathcal{M}_1(\mu^2, \{p\}_m)\rangle = I_1(\epsilon) |\mathcal{M}_0(\mu^2, \{p\}_m)\rangle + |\mathcal{M}_1^{\text{fin}}(\mu^2, \{p\}_m)\rangle. \quad (2.20)$$

In (2.20), $|\mathcal{M}_1^{\text{fin}}\rangle$ stands for the IR finite part of the amplitude in the limit $\epsilon \rightarrow 0$, while the singular dependence is embodied in *Catani's operator*

$$I_1(\epsilon) \stackrel{\text{def}}{=} \frac{1}{2} \frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)} \sum_{(ij)} \frac{\mathcal{V}_i^{\text{sing}}(\epsilon)}{\mathbf{T}_i^2} \mathbf{T}_i \cdot \mathbf{T}_j \left(\frac{\mu^2}{2p_i \cdot p_j} \right)^\epsilon e^{i\pi\lambda_{ij}\epsilon}, \quad (2.21)$$

where:

- (ij) means $i, j \in \mathcal{H}$ (the set of all external partons in the process), with $i \neq j$.
- The function $\mathcal{V}_i^{\text{sing}}(\epsilon)$ depends only on the parton flavour and is given by

$$\mathcal{V}_i^{\text{sing}}(\epsilon) = \frac{\mathbf{T}_i^2}{\epsilon^2} + \frac{\gamma_i}{\epsilon}, \quad (2.22)$$

where

$$\gamma_q = \gamma_{\bar{q}} = \frac{3}{2} C_F, \quad \gamma_g = \beta_0 \equiv \frac{11}{6} C_A - \frac{2}{3} T_R n_f. \quad (2.23)$$

Note that $\gamma_{q/\bar{q}}$ and γ_g are the quark and gluon anomalous dimension, respectively, at zeroth order (see Subsec. 1.2.4). In contrast with the notation adopted in Eq. (1.39), here we have dropped the subscript 0.

- The coefficient λ_{ij} is defined as

$$\lambda_{ij} \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } i \text{ and } j \text{ are both incoming or outgoing partons} \\ 0 & \text{otherwise.} \end{cases} \quad (2.24)$$

According to the discussion of the color algebra in Sec. 2.1, $I_1(\epsilon)$ is just a complex number if $m < 4$, and a *matrix operator* in the color space if $m \geq 4$.

Throughout this thesis we will often express our results in terms of the strong coupling constant

$$[\alpha_s] \stackrel{\text{def}}{=} \left[\frac{\alpha_s(\mu)}{2\pi} \right] \frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)} = \left[\frac{\alpha_s(\mu)}{2\pi} \right] [1 + \mathcal{O}(\epsilon^2)], \quad (2.25)$$

where $\alpha_s(\mu)$ is the renormalized strong coupling constant given in Eq. (1.34). Thus, we find it convenient to slightly modify the definition of Eq. (2.21) as

$$\bar{I}_1(\epsilon) \stackrel{\text{def}}{=} \frac{1}{2} \sum_{(ij)} \frac{\mathcal{V}_i^{\text{sing}}(\epsilon)}{\mathbf{T}_i^2} \mathbf{T}_i \cdot \mathbf{T}_j \left(\frac{\mu^2}{2p_i \cdot p_j} \right)^\epsilon e^{i\pi\lambda_{ij}\epsilon}, \quad (2.26)$$

such that

$$\left[\frac{\alpha_s(\mu)}{2\pi} \right] I_1(\epsilon) \equiv [\alpha_s] \bar{I}_1(\epsilon). \quad (2.27)$$

The NLO one-loop corrections in Eq. (2.19) involve the real part of $\langle \mathcal{M}_0 | \mathcal{M}_1 \rangle$. Hence, using Eq. (2.20) and the identity (2.27), we find

$$\left[\frac{\alpha_s(\mu)}{2\pi} \right] 2 \text{Re}[\langle \mathcal{M}_0 | \mathcal{M}_1 \rangle] = [\alpha_s] \langle \mathcal{M}_0 | \bar{I}_1(\epsilon) + \bar{I}_1^\dagger(\epsilon) | \mathcal{M}_0 \rangle + \mathcal{O}(\epsilon^0), \quad (2.28)$$

where $\mathcal{O}(\epsilon^0)$ stands for terms that are finite in the limit $\epsilon \rightarrow 0$. The operator on the right-hand side will play a crucial role in this thesis; we call it *virtual operator*, defined as

$$I_V(\epsilon) \stackrel{\text{def}}{=} \bar{I}_1(\epsilon) + \bar{I}_1^\dagger(\epsilon). \quad (2.29)$$

We could also write I_V as $2 \text{Re} [\bar{I}_1]$, but, as we will see later, it is more convenient to keep it in the form displayed in (2.29). It follows that Eq. (2.19) can be rewritten as

$$\left[\frac{\alpha_s(\mu)}{2\pi} \right] 2 \text{Re}[\langle \mathcal{M}_0 | \mathcal{M}_1 \rangle] = [\alpha_s] \langle \mathcal{M}_0 | I_V(\epsilon) | \mathcal{M}_0 \rangle + \left[\frac{\alpha_s(\mu)}{2\pi} \right] 2 \text{Re}[\langle \mathcal{M}_0 | \mathcal{M}_1^{\text{fin}} \rangle]. \quad (2.30)$$

The virtue of Eq. (2.30) is that it can describe the IR singular behavior of the single-virtual contribution to any NLO cross section in QCD. On the other hand, it does not provide any information on the second term on the right-hand side, which instead is process-dependent and depends also on the colorless particles of the amplitudes. This term must be computed explicitly.

2.2.2 Factorization formula for two-loop amplitudes

In the previous subsection, we discussed the generalized formula (2.20), which describes the IR-singular behavior of QCD amplitudes at one loop. In this subsection, we present an analogous result for QCD two-loop amplitudes.

We return to amplitude (2.17) of a generic scattering process with m external partons, and we expand it up to $\mathcal{O}(\alpha_s^3)$, i.e.,

$$\mathcal{M} = \mathcal{M}_0 + \left[\frac{\alpha_s(\mu)}{2\pi} \right] \mathcal{M}_1 + \left[\frac{\alpha_s(\mu)}{2\pi} \right]^2 \mathcal{M}_2 + \mathcal{O}(\alpha_s^3), \quad (2.31)$$

where $\mathcal{M}_2 = \mathcal{M}_2(\mu^2, \{p\}_m)$ labels the two-loop amplitude. Then we take the square modulus of (2.31),

$$\begin{aligned} |\mathcal{M}|^2 = & |\mathcal{M}_0|^2 + \left[\frac{\alpha_s(\mu)}{2\pi} \right] 2 \operatorname{Re}[\mathcal{M}_0^* \mathcal{M}_1] \\ & + \underbrace{\left[\frac{\alpha_s(\mu)}{2\pi} \right]^2 \left[2 \operatorname{Re}[\mathcal{M}_0^* \mathcal{M}_2] + |\mathcal{M}_1|^2 \right]}_{\text{NNLO}} + \mathcal{O}(\alpha_s^3), \end{aligned} \quad (2.32)$$

and we read the NNLO corrections (again using the color space formalism)

$$\left[\frac{\alpha_s(\mu)}{2\pi} \right]^2 \left[2 \operatorname{Re}[\langle \mathcal{M}_0 | \mathcal{M}_2 \rangle] + \langle \mathcal{M}_1 | \mathcal{M}_1 \rangle \right]. \quad (2.33)$$

The second term in Eq. (2.33) is the one-loop squared contribution to the NNLO correction, whose singular structure can be written using Eq. (2.20). The main result, presented in Ref. [82], is that a factorization formula for $|\mathcal{M}_2\rangle$ also exists, and can be written as

$$|\mathcal{M}_2\rangle = I_2(\epsilon) |\mathcal{M}_0\rangle + I_1(\epsilon) |\mathcal{M}_1\rangle + |\mathcal{M}_2^{\text{fin}}\rangle, \quad (2.34)$$

where $|\mathcal{M}_2^{\text{fin}}\rangle$ stands for the non-singular two-loop amplitude in the limit $\epsilon \rightarrow 0$, while the singular dependence is embodied in the operators $I_1(\epsilon)$ (see Eq. (2.21)), and $I_2(\epsilon)$, defined as

$$I_2(\epsilon) \stackrel{\text{def}}{=} -\frac{1}{2} I_1(\epsilon) \left(I_1(\epsilon) + \frac{2\beta_0}{\epsilon} \right) + c_\epsilon \left(\frac{\beta_0}{\epsilon} + K \right) I_1(2\epsilon) + \mathcal{H}_2, \quad (2.35)$$

with

$$c_\epsilon = e^{-\epsilon\gamma_E} \frac{\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)}, \quad K = \left(\frac{67}{18} - \frac{\pi^2}{6} \right) C_A - \frac{10}{9} T_R n_f. \quad (2.36)$$

In Eq. (2.35), the operator $\mathcal{H}_2$ contains $\mathcal{O}(\epsilon^{-1})$ poles only. We split it into a term containing *triple color-correlations* (which are proportional to $f_{abc} T_i^a T_j^b T_k^c$), and a *color-diagonal* term (which can be written as a combination of Casimir operators). Thus we write

$$\mathcal{H}_2(\epsilon) = \mathcal{H}_{2,\text{tc}}(\epsilon) + \mathcal{H}_{2,\text{cd}}(\epsilon). \quad (2.37)$$

The two quantities $\mathcal{H}_{2,\text{tc}}$ and $\mathcal{H}_{2,\text{cd}}$ were explicitly computed in Refs [150, 151]. According to Ref. [151], one can write $\mathcal{H}_{2,\text{tc}}$ as the commutator

$$\mathcal{H}_{2,\text{tc}} = \frac{1}{2\epsilon} [\Gamma, C], \quad (2.38)$$

where Γ and C are defined as

$$\Gamma \stackrel{\text{def}}{=} \frac{1}{2} \sum_{(ij)} \mathbf{T}_i \cdot \mathbf{T}_j (L_{ij} + i\pi \lambda_{ij}), \quad C \stackrel{\text{def}}{=} \frac{1}{2} \sum_{(ij)} \mathbf{T}_i \cdot \mathbf{T}_j (\delta_{ij}^+ + i\pi \delta_{ij}^-), \quad (2.39)$$

with λ_{ij} given in Eq. (2.24) and, in analogy with Eq. (2.21), we used the shorthand notation (ij) to indicate that the sum runs over all possible pairs of distinct partons. The quantities L_{ij} and $\delta_{ij}^\pm$ are defined as

$$\begin{aligned} L_{ij} &\stackrel{\text{def}}{=} \log \left(\frac{\mu^2}{s_{ij}} \right), \quad \text{with } s_{ij} = (p_i + p_j)^2, \\ \delta_{ij}^+ &\stackrel{\text{def}}{=} \frac{1}{2} L_{ij}^2 + \frac{\gamma_i}{\mathbf{T}_i^2} L_{ij} - \frac{\pi^2}{2} \lambda_{ij}^2, \\ \delta_{ij}^- &\stackrel{\text{def}}{=} L_{ij} \lambda_{ij} + \frac{\gamma_i}{\mathbf{T}_i^2} \lambda_{ij}. \end{aligned} \quad (2.40)$$

Eq. (2.38) will play an important role in Sec. 6.2. The expression for the color-diagonal piece can be taken from Ref. [151]. It reads

$$\mathcal{H}_{2,\text{cd}}(\epsilon) = \frac{1}{2\epsilon} \sum_{i \in \mathcal{H}} H_i, \quad (2.41)$$

where

$$\begin{aligned} H_g &\stackrel{\text{def}}{=} C_A^2 \left(\frac{5}{12} + \frac{11}{144} \pi^2 + \frac{\zeta_3}{2} \right) + C_A n_f \left(-\frac{29}{27} - \frac{\pi^2}{72} \right) + \frac{C_F n_f}{2} + \frac{5}{27} n_f^2, \\ H_q &\stackrel{\text{def}}{=} C_F^2 \left(\frac{\pi^2}{2} - 6\zeta_3 - \frac{3}{8} \right) + C_A C_F \left(\frac{245}{216} - \frac{23}{48} \pi^2 + \frac{13}{2} \zeta_3 \right) \\ &\quad + C_F n_f \left(\frac{\pi^2}{24} - \frac{25}{108} \right). \end{aligned} \quad (2.42)$$

Formula (2.35) states that $|\mathcal{M}_2\rangle$ can be written as a Laurent series, with non-zero coefficients up to $\mathcal{O}(\epsilon^{-4})$, since $I_1(\epsilon) \sim \mathcal{O}(\epsilon^{-2})$ and $I_2(\epsilon) \sim I_1^2(\epsilon)$. As in the one-loop case, we also want to rewrite Eq. (2.33) in a form more suited to the calculations we will present later on in this thesis. In fact, the main result (2.34) will play an essential role in our subtraction scheme, as well as Eq. (2.20).

Therefore, let us return to Eq. (2.33). We first consider the term

$$2 \operatorname{Re}[\langle \mathcal{M}_0 | \mathcal{M}_2 \rangle] = \langle \mathcal{M}_0 | \mathcal{M}_2 \rangle + \langle \mathcal{M}_2 | \mathcal{M}_0 \rangle^*. \quad (2.43)$$

Using both Eqs (2.20) and (2.34), we can write $\langle \mathcal{M}_0 | \mathcal{M}_2 \rangle$ as

$$\begin{aligned} \langle \mathcal{M}_0 | \mathcal{M}_2 \rangle &= \langle \mathcal{M}_0 | I_2(\epsilon) | \mathcal{M}_0 \rangle + \langle \mathcal{M}_0 | I_1(\epsilon) | \mathcal{M}_1 \rangle + \langle \mathcal{M}_0 | \mathcal{M}_2^{\text{fin}} \rangle \\ &= \left\langle \mathcal{M}_0 \left| \frac{1}{2} I_1^2(\epsilon) - \frac{\beta_0}{\epsilon} I_1(\epsilon) + c_\epsilon \left(\frac{\beta_0}{\epsilon} + K \right) I_1(2\epsilon) + \mathcal{H}_2(\epsilon) \right| \mathcal{M}_0 \right\rangle \\ &\quad + \langle \mathcal{M}_0 | I_1(\epsilon) | \mathcal{M}_1^{\text{fin}} \rangle + \langle \mathcal{M}_0 | \mathcal{M}_2^{\text{fin}} \rangle . \end{aligned} \quad (2.44)$$

To get the hermitian conjugate of this expression, we exploit the following standard properties

$$\begin{aligned} (\hat{O}^2)^\dagger &= (\hat{O}^\dagger)^2 , \\ \langle f | \hat{O} | i \rangle^* &= \langle i | \hat{O}^\dagger | f \rangle , \\ \langle f | \hat{O} | i \rangle + \langle i | \hat{O}^\dagger | f \rangle &= 2 \operatorname{Re} [\langle f | \hat{O} | i \rangle] , \end{aligned} \quad (2.45)$$

and we apply them to $\langle \mathcal{M}_2 | \mathcal{M}_0 \rangle^*$, which yields

$$\begin{aligned} 2 \operatorname{Re} [\langle \mathcal{M}_0 | \mathcal{M}_2 \rangle] &= \left\langle \mathcal{M}_0 \left| \frac{1}{2} I_1^2(\epsilon) + \frac{1}{2} [I_1^\dagger(\epsilon)]^2 \right| \mathcal{M}_0 \right\rangle \\ &\quad + \left\langle \mathcal{M}_0 \left| -\frac{\beta_0}{\epsilon} (I_1(\epsilon) + I_1^\dagger(\epsilon)) + c_\epsilon \left(\frac{\beta_0}{\epsilon} + K \right) [I_1(2\epsilon) + I_1^\dagger(2\epsilon)] + [\mathcal{H}_2(\epsilon) + \mathcal{H}_2^\dagger(\epsilon)] \right| \mathcal{M}_0 \right\rangle \\ &\quad + 2 \operatorname{Re} \left[\langle \mathcal{M}_0 | I_1(\epsilon) | \mathcal{M}_1^{\text{fin}} \rangle \right] + 2 \operatorname{Re} \left[\langle \mathcal{M}_0 | \mathcal{M}_2^{\text{fin}} \rangle \right] . \end{aligned} \quad (2.46)$$

Next, we move to the second term on the right-hand side of Eq. (2.33). Again using (2.20) and (2.45) we find

$$\begin{aligned} \langle \mathcal{M}_1 | \mathcal{M}_1 \rangle &= \left(\langle \mathcal{M}_0 | I_1^\dagger(\epsilon) + \langle \mathcal{M}_1^{\text{fin}} | \right) \left(I_1(\epsilon) | \mathcal{M}_0 \rangle + | \mathcal{M}_1^{\text{fin}} \rangle \right) \\ &= \langle \mathcal{M}_0 | I_1^\dagger(\epsilon) I_1(\epsilon) | \mathcal{M}_0 \rangle + 2 \operatorname{Re} \left[\langle \mathcal{M}_0 | I_1^\dagger(\epsilon) | \mathcal{M}_1^{\text{fin}} \rangle \right] + \langle \mathcal{M}_1^{\text{fin}} | \mathcal{M}_1^{\text{fin}} \rangle . \end{aligned}$$

Therefore, we substitute Eqs (2.46) and (2.47) in Eq. (2.33), and we obtain

$$\begin{aligned} &\left[\frac{\alpha_s(\mu)}{2\pi} \right]^2 \left[2 \operatorname{Re} [\langle \mathcal{M}_0 | \mathcal{M}_2 \rangle] + \langle \mathcal{M}_1 | \mathcal{M}_1 \rangle \right] \\ &= \left[\frac{\alpha_s(\mu)}{2\pi} \right]^2 \left[\left\langle \mathcal{M}_0 \left| \frac{1}{2} [I_1(\epsilon) + I_1^\dagger(\epsilon)]^2 - \frac{1}{2} [I_1(\epsilon), I_1^\dagger(\epsilon)] \right| \mathcal{M}_0 \right\rangle \right. \\ &\quad + \left\langle \mathcal{M}_0 \left| -\frac{\beta_0}{\epsilon} [I_1(\epsilon) + I_1^\dagger(\epsilon)] + c_\epsilon \left(\frac{\beta_0}{\epsilon} + K \right) [I_1(2\epsilon) + I_1^\dagger(2\epsilon)] + [\mathcal{H}_2(\epsilon) + \mathcal{H}_2^\dagger(\epsilon)] \right| \mathcal{M}_0 \right\rangle \\ &\quad \left. + 2 \operatorname{Re} \left[\langle \mathcal{M}_0 | I_1(\epsilon) + I_1^\dagger(\epsilon) | \mathcal{M}_1^{\text{fin}} \rangle \right] + \langle \mathcal{M}_1^{\text{fin}} | \mathcal{M}_1^{\text{fin}} \rangle + 2 \operatorname{Re} \left[\langle \mathcal{M}_0 | \mathcal{M}_2^{\text{fin}} \rangle \right] \right] , \end{aligned} \quad (2.47)$$

where we have used the identity

$$\frac{1}{2} I_1^2(\epsilon) + \frac{1}{2} [I_1^\dagger(\epsilon)]^2 + I_1^\dagger(\epsilon) I_1(\epsilon) = \frac{1}{2} [I_1(\epsilon) + I_1^\dagger(\epsilon)]^2 - \frac{1}{2} [I_1(\epsilon), I_1^\dagger(\epsilon)] . \quad (2.48)$$

At this point, we are almost satisfied. The final step consists of rewriting Eq. (2.47)

using the operators $\bar{I}_1(\epsilon)$ and $I_V(\epsilon)$ that we introduced in Eqs (2.26) and (2.29), respectively. Special attention should be paid to the operator $I_1(2\epsilon)$ in (2.47), which we can rewrite using Eqs (2.21), (2.25), and (2.26), as

$$\left[\frac{\alpha_s(\mu)}{2\pi} \right]^2 I_1(2\epsilon) = [\alpha_s]^2 \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \bar{I}_1(2\epsilon). \quad (2.49)$$

Lastly, as for $\mathcal{H}_2(\epsilon)$, we point out that

$$\left[\frac{\alpha_s(\mu)}{2\pi} \right]^2 \mathcal{H}_2(\epsilon) = [\alpha_s]^2 \mathcal{H}_2(\epsilon) + \mathcal{O}(\epsilon), \quad (2.50)$$

since $\mathcal{H}_2(\epsilon) \sim \mathcal{O}(\epsilon^{-1})$, and the expansion (2.25) holds. We thus conclude this subsection with the following formula:

$$\begin{aligned} & \left[\frac{\alpha_s(\mu)}{2\pi} \right]^2 \left[2 \operatorname{Re}[\langle \mathcal{M}_0 | \mathcal{M}_2 \rangle] + \langle \mathcal{M}_1 | \mathcal{M}_1 \rangle \right] \\ &= [\alpha_s]^2 \left[\left\langle \mathcal{M}_0 \left| \frac{1}{2} I_V^2(\epsilon) - \frac{1}{2} [\bar{I}_1(\epsilon), \bar{I}_1^\dagger(\epsilon)] \right| \mathcal{M}_0 \right\rangle \right. \\ &+ \left\langle \mathcal{M}_0 \left| -\frac{\beta_0}{\epsilon} \frac{\Gamma(1-\epsilon)}{e^{\epsilon\gamma_E}} I_V(\epsilon) + \frac{\Gamma(1-\epsilon)}{e^{\epsilon\gamma_E}} \left(\frac{\beta_0}{\epsilon} + K \right) I_V(2\epsilon) + [\mathcal{H}_2(\epsilon) + \mathcal{H}_2^\dagger(\epsilon)] \right| \mathcal{M}_0 \right\rangle \right] \\ &+ [\alpha_s] \left[\frac{\alpha_s(\mu)}{2\pi} \right] 2 \operatorname{Re} [\langle \mathcal{M}_0 | I_V(\epsilon) | \mathcal{M}_1^{\text{fin}} \rangle] + \left[\frac{\alpha_s(\mu)}{2\pi} \right]^2 \langle \mathcal{M}_1^{\text{fin}} | \mathcal{M}_1^{\text{fin}} \rangle \\ &+ \left[\frac{\alpha_s(\mu)}{2\pi} \right]^2 2 \operatorname{Re} [\langle \mathcal{M}_0 | \mathcal{M}_2^{\text{fin}} \rangle]. \end{aligned} \quad (2.51)$$

This result, which we will set aside for the moment, will prove useful in Subsec. 5.1.1. We note that the highest pole structure of the two-loop amplitude can be derived from operators present at one-loop; only the $\mathcal{O}(\epsilon^{-2})$ and $\mathcal{O}(\epsilon^{-1})$ require new terms (K , $\mathcal{H}_2$). Also, it is useful to point out the presence of term with $I_V(2\epsilon)$, since we will see analogous terms for I_S and I_C later.

2.3 Soft and collinear singularities of $\mathcal{O}(\alpha_s)$ in tree-level amplitudes

At the end of Sec. 1.5, we briefly mentioned the IR divergences arising from real emissions, with the promise that we would provide more details later. In this section, we fulfill that promise and delve into the origin of these divergences. In the case of QCD virtual amplitudes, the IR singularities are related to kinematic limits where virtual intermediate particles become on-shell. In contrast, in QCD amplitudes with real emissions, IR divergences of $\mathcal{O}(\alpha_s)$ can occur when the energy of an emitted gluon vanishes (*soft singularity*), or when a gluon or an (anti-)quark is emitted in the direction of another parton (*collinear singularity*).

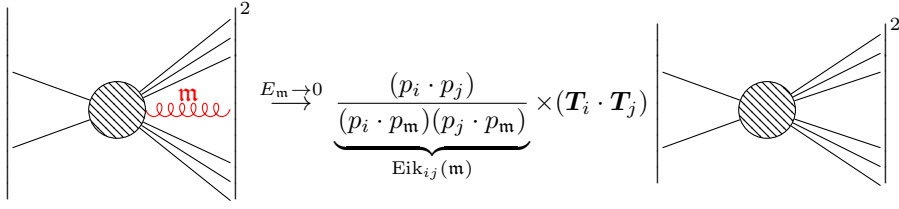


Figure 2.1: Graphical representation of the soft-gluon factorization formula (2.52) at $\mathcal{O}(\alpha_s)$.

2.3.1 Soft singularities

We begin with soft singularities. To illustrate this phenomenon, we consider a tree-level amplitude with $m+1$ external partons, which we denote as $\mathcal{M}_{f_1 \dots f_m g}(p_1, \dots, p_m, \mathbf{m})$, where f_i is the flavour of the parton with momentum p_i , while the parton with momentum p_m is by assumption a gluon. Indeed, in QCD, a soft particle produces a singularity *only if it is a gluon*. To simplify the notation, we leave the color and spin indices understood, hence $|\mathcal{M}_{f_1 \dots f_m g}(p_1, \dots, p_m, \mathbf{m})\rangle$ represents a vector in color-helicity space (cf. Sec. 2.1). In the soft limit, that is, in the limit where the energy of the gluon E_m drops to zero,⁵ the following *soft-gluon factorization formula* at $\mathcal{O}(\alpha_s)$ holds [152]:

$$S_m |\mathcal{M}_{f_1 \dots f_m g}(p_1, \dots, p_m, \mathbf{m})|^2 = -g_{s,b}^2 \mu_0^{2\epsilon} \sum_{(ij)} \text{Eik}_{ij}(\mathbf{m}) |\mathcal{M}_{f_1 \dots f_m}^{(ij)}(p_1, \dots, p_m)|^2, \quad (2.52)$$

with $g_{s,b}^2$ given in Eq. (1.33). Here,

$$S_m \stackrel{\text{def}}{=} \lim_{E_m \rightarrow 0} \quad (2.53)$$

is the *single-soft operator* that, by definition, acts on all quantities to its right and is to be understood in the sense of extracting the most singular soft contribution provided that limits in the conventional sense do not exist. The function $\text{Eik}_{ij}(\mathbf{m})$ is called the *eikonal function* and is defined as

$$\text{Eik}_{ij}(\mathbf{m}) \stackrel{\text{def}}{=} \frac{(p_i \cdot p_j)}{(p_i \cdot p_m)(p_j \cdot p_m)}, \quad (2.54)$$

while the square modulus of the color-correlated amplitude $\mathcal{M}^{(ij)}$ was introduced in Eq. (2.8). Graphically, Eq. (2.52) can be represented as in Figure 2.1. Similarly to the one-loop and two-loop cases described in the previous pages, the factorization formula (2.52) is also entirely general in QCD.

As we have said, the integral of the right-hand side of Eq. (2.52) over the phase space of the soft gluon is singular. To show it, we consider, as an explicit example, the tree-level

⁵Note that a gluon that is soft in a certain reference frame A may not be soft in another reference frame B, which is boosted with respect to A. For this reason, it is necessary first to define the reference frame in which the scattering process occurs.

amplitude of the process $q(p_1) + \bar{q}(p_2) \rightarrow X + g(p_m)$, where X represents any color-singlet final state. We take the soft limit of the relevant matrix element and integrate it over the phase space of the gluon, giving

$$\begin{aligned} & \int [dp_m] S_m |\mathcal{M}_{q\bar{q}g}(p_1, p_2, p_m)|^2 \Theta(E_{\max} - E_m) \delta_{12 \rightarrow m}^{(4)} \\ &= -g_{s,b}^2 \mu_0^{2\epsilon} 2C_F |\mathcal{M}_{q\bar{q}}(p_1, p_2)|^2 \int [dp_m] \text{Eik}_{12}(m) \Theta(E_{\max} - E_m) \delta_{12}^{(4)}, \end{aligned} \quad (2.55)$$

where we labeled the momentum conserving δ -functions (cf. Eq. (1.49)) as

$$\begin{aligned} \delta_{12}^{(4)} &\stackrel{\text{def}}{=} (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_X), \\ \delta_{12 \rightarrow m}^{(4)} &\stackrel{\text{def}}{=} (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_m - p_X), \end{aligned} \quad (2.56)$$

Additionally, $E_{\max}$ is an energy parameter that provides an upper bound on the energies of individual partons.⁶ As we will see shortly, it acts as a UV cut-off. Following the definition of S_m , we note that its action on the delta-function has removed the momentum p_m from its arguments. Although not strictly necessary, we will see in Ch. 3 that this approach is adopted within the framework of the Nested Soft-Collinear method, which is why it is introduced here. As a result, the only dependence on the kinematics of the unresolved gluon is through the eikonal function.

Next,

$$\begin{aligned} \int [dp_m] &\stackrel{\text{def}}{=} \int \frac{d^{(d-1)} \mathbf{p}_m}{(2\pi)^{d-1} (2E_m)} \\ &= \int dE_m E_m^{d-3} \int d\cos\theta_{im} (1 - \cos^2\theta_{im})^{d/2-2} \int \frac{d\Omega_m^{(d-2)}}{2(2\pi)^{d-1}} \end{aligned} \quad (2.57)$$

is the d -dimensional phase space measure of the emitted gluon. In Eq. (2.57), θ_{im} is the azimuthal angle between momenta $\mathbf{p}_i$ and $\mathbf{p}_m$, while $d\Omega_m^{(d-2)}$ is the $(d-2)$ -dimensional solid angle. Moreover, we point out that in Eq. (2.55) we could write

$$|\mathcal{M}_{q\bar{q}}^{(ij)}(p_1, p_2)|^2 = (\mathbf{T}_i \cdot \mathbf{T}_j) |\mathcal{M}_{q\bar{q}}(p_1, p_2)|^2, \quad \mathbf{T}_1 \cdot \mathbf{T}_2 = -C_F, \quad (2.58)$$

because, having two hard partons, the color algebra is closed (cf. Sec. 2.1).

In Eq. (2.55), IR singularities come from the integral⁷

$$\int [dp_m] \text{Eik}_{ij}(m) \Theta(E_{\max} - E_m), \quad (2.59)$$

which we need to compute explicitly. Before doing so, we introduce useful notation and some of the main angular integrals that will appear throughout the manuscript. First,

⁶More specifically, $E_{\max}$ should be greater than, or equal to, the maximal energy that the potentially unresolved final-state parton m can have according to the momentum conservation constraint.

⁷Since the following analysis holds for any i and j , we temporarily revert to a general notation.

we introduce the objects

$$\begin{aligned}\rho_{ij} &\stackrel{\text{def}}{=} 1 - \cos \theta_{ij}, & 0 \leq \rho_{ij} \leq 2, \\ \eta_{ij} &\stackrel{\text{def}}{=} \frac{\rho_{ij}}{2}, & 0 \leq \eta_{ij} \leq 1,\end{aligned}\tag{2.60}$$

and we use them to rewrite the integral over $\cos \theta_{ij}$ as

$$\int_{-1}^{+1} d\cos \theta_{ij} (1 - \cos^2 \theta_{ij})^{d/2-2} = \int_0^1 \frac{2d\eta_{ij}}{[4\eta_{ij}(1 - \eta_{ij})]^\epsilon} \stackrel{\text{def}}{=} \int [d\eta_{ij}]. \tag{2.61}$$

Since $p_i = (E_i, \mathbf{p}_i) = E_i(1, \mathbf{n}_i)$, where $\mathbf{n}_i$ is the unit vector that describes the direction of the momentum of parton i in $(d-1)$ -dimensional space, we can always write

$$p_i \cdot p_j = E_i E_j (1 - \mathbf{n}_i \cdot \mathbf{n}_j) = E_i E_j (1 - \cos \theta_{ij}) = E_i E_j \rho_{ij} = 2E_i E_j \eta_{ij}. \tag{2.62}$$

Next, we define the normalized element of the solid angle in $(d-1)$ - and $(d-2)$ -dimensions as

$$[d\Omega_{\mathbf{m}}^{(d-1)}] \stackrel{\text{def}}{=} \frac{d\Omega_{\mathbf{m}}^{(d-1)}}{2(2\pi)^{d-1}}, \quad [d\Omega_{\mathbf{m}}^{(d-2)}] \stackrel{\text{def}}{=} \frac{d\Omega_{\mathbf{m}}^{(d-2)}}{2(2\pi)^{d-1}}, \tag{2.63}$$

which implies⁸

$$\int [d\Omega_{\mathbf{m}}^{(d-1)}] \equiv \int [d\eta_{iq}] \int [d\Omega_{\mathbf{m}}^{(d-2)}]. \tag{2.64}$$

The integral over the $(d-2)$ -dimensional solid angle $[d\Omega_{\mathbf{m}}^{(d-2)}]$ reads

$$[\Omega^{(d-2)}] \stackrel{\text{def}}{=} \int [d\Omega^{(d-2)}] = \frac{1}{8\pi^2} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}. \tag{2.65}$$

Furthermore, we report an important general result,

$$\begin{aligned}\int \frac{[d\Omega_{\mathbf{m}}^{(d-1)}]}{[\Omega^{(d-2)}]} \frac{1}{\rho_{im}^\alpha \rho_{jm}^\beta} &= 2^{1-(\alpha+\beta)-2\epsilon} \frac{\Gamma(1-\epsilon-\alpha)\Gamma(1-\epsilon-\beta)}{\Gamma(2-(\alpha+\beta)-2\epsilon)} \\ &\times {}_2F_1(\alpha, \beta, 1-\epsilon, 1-\eta_{ij}),\end{aligned}\tag{2.66}$$

that can be used to compute the following list of angular integrals that will prove useful in Part II:

$$\begin{aligned}\int \frac{[d\Omega_{\mathbf{m}}^{(d-1)}]}{[\Omega^{(d-2)}]} \frac{\rho_{ij}}{\rho_{im} \rho_{jm}} &= -\frac{2^{1-2\epsilon}}{\epsilon} \eta_{ij}^{-\epsilon} K_{ij}, \\ \int \frac{[d\Omega_{\mathbf{m}}^{(d-1)}]}{[\Omega^{(d-2)}]} \frac{1}{\rho_{im}} &= -\frac{2^{-2\epsilon}}{\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)},\end{aligned}\tag{2.67}$$

⁸The axis with respect to which the azimuthal angle η is defined is entirely arbitrary.

$$\int \frac{[d\Omega_{\mathbf{m}}^{(d-1)}]}{[\Omega^{(d-2)}]} \left(\frac{\rho_{i\mathbf{m}}}{4}\right)^{-\epsilon} \frac{1}{\rho_{i\mathbf{m}}} = -\frac{2^{-2\epsilon}}{2\epsilon} \frac{2^\epsilon \Gamma(1-\epsilon) \Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)},$$

where K_{ij} is defined as

$$\begin{aligned} K_{ij} &\stackrel{\text{def}}{=} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \eta_{ij}^{1+\epsilon} {}_2F_1(1, 1, 1-\epsilon, 1-\eta_{ij}) \\ &= 1 + \left[\text{Li}_2(1-\eta_{ij}) - \frac{\pi^2}{6} \right] \epsilon^2 + \mathcal{O}(\epsilon^3). \end{aligned} \quad (2.68)$$

Therefore, substituting Eqs (2.60) and (2.64) in Eq. (2.57), we obtain

$$\int [dp_{\mathbf{m}}] = \int dE_{\mathbf{m}} E_{\mathbf{m}}^{d-3} \int [d\eta_{iq}] \int [d\Omega_{\mathbf{m}}^{(d-2)}], \quad d = 4 - 2\epsilon. \quad (2.69)$$

We return to the integral (2.59). According to the notation just introduced, we can compute (2.59) explicitly as follows:

$$\begin{aligned} &\int [dp_{\mathbf{m}}] \text{Eik}_{ij}(\mathbf{m}) \Theta(E_{\text{max}} - E_{\mathbf{m}}) \\ &= \int dE_{\mathbf{m}} E_{\mathbf{m}}^{d-3} \Theta(E_{\text{max}} - E_{\mathbf{m}}) \int [d\Omega_{\mathbf{m}}^{(d-1)}] \frac{1}{2E_{\mathbf{m}}^2} \frac{\rho_{ij}}{\rho_{i\mathbf{m}}\rho_{j\mathbf{m}}} \\ &= [\Omega^{(d-2)}] \int_0^{E_{\text{max}}} \frac{dE_{\mathbf{m}}}{E_{\mathbf{m}}^{1+2\epsilon}} \int \frac{[d\Omega_{\mathbf{m}}^{(d-1)}]}{[\Omega^{(d-2)}]} \frac{\rho_{ij}}{\rho_{i\mathbf{m}}\rho_{j\mathbf{m}}} \\ &= [\Omega^{(d-2)}] \left[-\frac{E_{\text{max}}^{-2\epsilon}}{2\epsilon} \right] \times \left[-\frac{2^{1-2\epsilon}}{\epsilon} \eta_{ij}^{-\epsilon} K_{ij} \right]. \end{aligned} \quad (2.70)$$

The terms in square brackets come from the integral over the energy and the solid angle, respectively. We call the former a *soft divergence* and the latter, which corresponds to an angular divergence within an eikonal function, a *soft-collinear divergence*. We note that, in order to obtain a meaningful result, it is crucial that the integration over $E_{\mathbf{m}}$ is bounded from above. This is why we introduced the parameter E_{max} , which, as mentioned, acts as a UV cut-off.

At this point, we have all the ingredients to write the final expression for Eq. (2.55), which reads

$$\begin{aligned} &\int [dp_{\mathbf{m}}] S_{\mathbf{m}} |\mathcal{M}_{q\bar{q}g}(p_1, p_2, p_{\mathbf{m}})|^2 \Theta(E_{\text{max}} - E_{\mathbf{m}}) \delta_{12 \rightarrow \mathbf{m}}^{(4)} \\ &= [\alpha_s] 2C_F \frac{(2E_{\text{max}}/\mu)^{-2\epsilon}}{\epsilon^2} \eta_{12}^{-\epsilon} K_{12} |\mathcal{M}_{q\bar{q}}(p_1, p_2)|^2 \delta_{12}^{(4)}. \\ &= [\alpha_s] \left[\frac{2C_F}{\epsilon^2} - \frac{2C_F (L_{\text{max}} + \log \eta_{12})}{\epsilon} \right] |\mathcal{M}_{q\bar{q}}(p_1, p_2)|^2 \delta_{12}^{(4)} + \mathcal{O}(\epsilon^0), \end{aligned} \quad (2.71)$$

where $L_{\text{max}} = \log(2E_{\text{max}}/\mu)$. As for the presence of the coupling constant $[\alpha_s]$, it comes

from the following steps (cf. Eqs (1.33), (1.34) and (2.65), (2.25)):

$$\begin{aligned} g_{s,b}^2 \mu_0^{2\epsilon} [\Omega^{(d-2)}] &= \frac{\alpha_{s,b}}{2\pi} \mu_0^{2\epsilon} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} = \left[\frac{\alpha_s(\mu)}{2\pi} \right] \mu^{2\epsilon} \frac{e^{\epsilon\gamma_E}}{(4\pi)^\epsilon \Gamma(1-\epsilon)} \\ &= \left[\frac{\alpha_s(\mu)}{2\pi} \right] \mu^{2\epsilon} \frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)} \equiv [\alpha_s] \mu^{2\epsilon}. \end{aligned} \quad (2.72)$$

In conclusion, in (2.71), we have derived the explicit formula for the Laurent series of the integral over the soft limit of the tree-level amplitude presented in (2.55). We emphasize that this approach is applicable to any tree-level amplitude, with any number of external partons.

2.3.2 Collinear singularities

Having analyzed the soft case, we now move on to explore the collinear case in more depth. The initial assumptions remain the same as Sec. 2.3.1, so we consider an amplitude $\mathcal{M}_{f_1 \dots f_{m+1}}(p_1, \dots, p_{m+1})$ with $m+1$ external partons. We aim to understand how to describe the case in which two final-state particles, say h and s , are emitted with an angle $\theta_{hs} \rightarrow 0$. Therefore, we consider the final-state splitting process $[hs] \rightarrow h(z) \times s(1-z)$, where $[hs]$ is the *mother (clustered) particle*, and h and s the two particles produced in the splitting process. To simplify the notation, we write the amplitude as $\mathcal{M}_{f_1 \dots f_{m-1} f_h f_s}(\{p\}_{m-1}, p_h, p_s)$, where as usual f_i is the flavour of parton with momentum p_i . Essentially, we are considering the following object:

$$\mathcal{M}_{f_1 \dots f_{m-1} qg}(\{p\}_{m-1}, p_h, p_s) \equiv \text{Diagram} \quad (2.73)$$

where we have drawn as an example the $q \rightarrow q(z) \times g(1-z)$ splitting process. We assume the limit $\theta_{hs} \rightarrow 0$ at $\mathcal{O}(\alpha_s)$, which occurs when the momenta of two partons p_h and p_s become parallel. This limit can be defined as follows:

$$\begin{aligned} p_h^\mu &= z p_{[hs]}^\mu + k_\perp^\mu + \beta_h n^\mu, \\ p_s^\mu &= (1-z) p_{[hs]}^\mu - k_\perp^\mu + \beta_s n^\mu, \end{aligned} \quad (2.74)$$

where $\beta_{h,s}^\mu$ are determined by on-shell conditions $p_h^2 = p_s^2 = 0$, and

$$\begin{aligned} p_{[hs]}^2 &= n^2 = 0, & p_{[hs]} \cdot k_\perp &= n \cdot k_\perp = 0, \\ k_\perp^2 &< 0, & s_{hs} &= 2p_h \cdot p_s = -\frac{k_\perp^2}{z(1-z)}. \end{aligned} \quad (2.75)$$

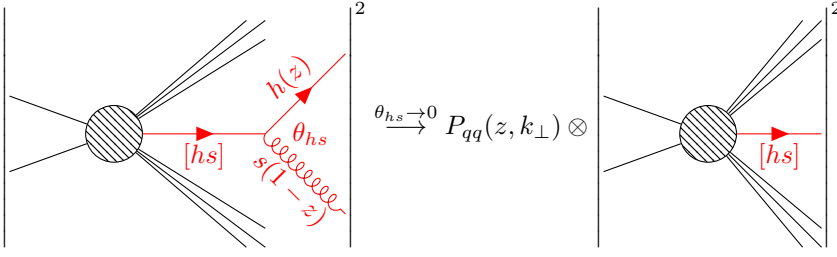


Figure 2.2: Graphical representation of the collinear factorization formula (2.78) at $\mathcal{O}(\alpha_s)$, where we have drawn as an example the $q \rightarrow q(z) \times g(1-z)$ splitting process. The functions $P_{qq}(z, k_\perp)$ are the d -dimensional Altarelli-Parisi splitting functions (2.81).

In Eq. (2.74), the light-like vector $p_{[hs]}^\mu$ denotes the collinear direction, and n^μ is an auxiliary light-like vector, which is necessary to specify the transverse component $k_\perp^\mu$ or, equivalently, how the collinear direction is approached. Note that the collinear limit $\theta_{hs} \rightarrow 0$ implies $k_\perp^\mu \rightarrow 0$. As for the variable z , it corresponds to the momentum fraction carried by the parton h in the collinear splitting, and reads

$$z = \frac{p_h \cdot n}{p_{[hs]} \cdot n}. \quad (2.76)$$

In the limit $k_\perp^\mu \rightarrow 0$, it can be expressed in terms of the energies of partons h , s and $[hs]$, as

$$E_h = z E_{[hs]}, \quad E_s = (1-z) E_{[hs]}. \quad (2.77)$$

In the literature, such a decomposition is known as *Sudakov decomposition* [153].

In the small- $k_\perp$ limit, i.e., neglecting corrections that are less singular than $1/k_\perp^2$, the following QCD *collinear factorization formula* at $\mathcal{O}(\alpha_s)$ holds [152]:

$$\begin{aligned} & C_{hs} \left| \mathcal{M}_{f_1 \dots f_{m-1} f_h f_s}(\{p\}_{m-1}, p_h, p_s) \right|^2 \\ &= \frac{g_{s,b}^2 \mu_0^{2\epsilon}}{p_h \cdot p_s} P_{[hs]h}^{rr'}(z, k_\perp) \left| \mathcal{M}_{f_1 \dots f_{m-1} f_{[hs]}}(\{p\}_{m-1}, p_{[hs]}) \right|_{rr'}^2, \end{aligned} \quad (2.78)$$

where r, r' are the spin indices of the parent parton $[hs]$ (flavour $f_{[hs]}$ is determined by flavour conservation). A graphical representation of (2.78) is shown in Figure 2.2. The operator

$$C_{hs} \stackrel{\text{def}}{=} \lim_{\theta_{hs} \rightarrow 0} \quad (2.79)$$

is the *collinear operator* that, by definition, acts on all quantities to its right and, similarly to the soft limit, is to be understood in the sense of extracting the most singular collinear contribution provided that limits in the conventional sense do not exist. As we will see shortly, by integrating the right-hand side of Eq. (2.78) over the phase space of particle s , we obtain poles starting at $\mathcal{O}(\epsilon^{-2})$. We will see that the pure-collinear and soft-collinear contributions will yield a pole of $\mathcal{O}(\epsilon^{-1})$.

In Eq. (2.78), the functions $P_{[hs]h}^{rr'}(z, k_\perp)$ are the d -dimensional Altarelli-Parisi *splitting functions* [125], and in general they can be thought as a matrix acting on the spin indices r, r' of $[hs]$ in the *spin-polarization tensor*

$$|\mathcal{M}_{f_1 \dots f_{m-1} f_M}(\{p\}_{m-1}, p_{[hs]})|_{ss'}^2. \quad (2.80)$$

This phenomenon is termed *spin correlations*. We list here the expressions of the functions $P_{[hs]h}^{rr'}(z, k_\perp)$:

$$\begin{aligned} P_{qq}^{rr'}(z, k_\perp) &\equiv \delta_{rr'} P_{qq}(z) \stackrel{\text{def}}{=} \delta_{rr'} C_F \left[\frac{1+z^2}{1-z} - \epsilon(1-z) \right], \\ P_{qg}^{rr'}(z, k_\perp) &\equiv \delta_{rr'} P_{qg}(z) \stackrel{\text{def}}{=} \delta_{rr'} C_F \left[\frac{1+(1-z)^2}{z} - \epsilon z \right], \\ P_{gq}^{\mu\nu}(z, k_\perp) &\stackrel{\text{def}}{=} T_R \left[-g^{\mu\nu} - 4z(1-z) \kappa_\perp^\mu \kappa_\perp^\nu \right], \\ P_{gg}^{\mu\nu}(z, k_\perp) &\stackrel{\text{def}}{=} 2C_A \left[-g^{\mu\nu} \left(\frac{1-z}{z} + \frac{z}{1-z} \right) + 2(1-\epsilon)z(1-z) \kappa_\perp^\mu \kappa_\perp^\nu \right], \end{aligned} \quad (2.81)$$

where the spin indices have been denoted by r, r' if $[hs]$ is a fermion, and μ, ν if $[hs]$ is a gluon. Additionally,

$$\kappa_\perp^\mu \stackrel{\text{def}}{=} \frac{k_\perp^\mu}{\sqrt{-k_\perp^2}}, \quad \kappa_\perp^2 = -1. \quad (2.82)$$

In case of gluon splitting, that is, when the mother parton $[hs]$ is a gluon, we also define the *spin-averaged splitting functions*

$$P_{gh}(z) \stackrel{\text{def}}{=} \frac{d_{\mu\nu}(p_{[hs]}) P_{gh}^{\mu\nu}(z, k_\perp)}{2(1-\epsilon)}, \quad d_{\mu\nu}(p) \stackrel{\text{def}}{=} -g_{\mu\nu} + \frac{p_\mu n_\nu + p_\nu n_\mu}{p \cdot n}, \quad (2.83)$$

with $p_{[hs]}^\mu$ and n^μ defined in (2.74). The spin-averaged splitting functions read

$$\begin{aligned} P_{qq}(z) &\stackrel{\text{def}}{=} C_F \left[\frac{1+z^2}{1-z} - \epsilon(1-z) \right], \\ P_{qg}(z) &\stackrel{\text{def}}{=} C_F \left[\frac{1+(1-z)^2}{z} - \epsilon z \right] \equiv P_{qq}(1-z), \\ P_{gq}(z) &\stackrel{\text{def}}{=} T_R \left[1 - \frac{2z(1-z)}{1-\epsilon} \right], \\ P_{gg}(z) &\stackrel{\text{def}}{=} 2C_A \left[\frac{z}{1-z} + \frac{1-z}{z} + z(1-z) \right]. \end{aligned} \quad (2.84)$$

Similarly to the soft case, the factorization formula (2.78) produces IR singularities once we integrate over the final phase space of parton s . To show this, we consider, as an explicit example, the tree-level amplitude of the process $g(p_1) + q(p_2) \rightarrow X + q(p_3) + g(p_4)$ (X is a color-singlet final state), where partons $q(p_3)$ and $g(p_4)$ become collinear to each other. For now, we choose $q(p_3)$ as the parton h and $g(p_4)$ as the parton s ; we will see later that making the opposite choice leads to the same result. It follows that this process involves the splitting $q \rightarrow q(z) \times g(1-z)$ and hence the splitting function $P_{qq}(z)$. We want to compute the expression

$$\begin{aligned} \mathcal{I} &= \int [dp_3][dp_4] C_{34} |\mathcal{M}_{g_1 q_2 q_3 g_4}(p_1, p_2, p_3, p_4)|^2 \Theta(E_{\max} - E_{3,4}) \delta_{12 \rightarrow 34}^{(4)} \\ &= \int [dp_3][dp_4] \frac{g_{s,b}^2 \mu_0^{2\epsilon}}{p_3 \cdot p_4} P_{qq}(z) |\mathcal{M}_{g_1 q_2 q_{[34]}}(p_1, p_2, p_{[34]})|^2 \Theta(E_{\max} - E_{3,4}) \delta_{12 \rightarrow [34]}^{(4)}, \end{aligned} \quad (2.85)$$

where we labeled the δ -factors as

$$\begin{aligned} \delta_{12 \rightarrow 34}^{(4)} &\stackrel{\text{def}}{=} (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4 - p_X), \\ \delta_{12 \rightarrow [34]}^{(4)} &\stackrel{\text{def}}{=} (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_{[34]} - p_X), \end{aligned} \quad (2.86)$$

and we bounded both E_3 and E_4 with the dimensionful parameter $E_{\max}$. Using the notation introduced in Subsec. 2.3.1, we have

$$p_3 \cdot p_4 = E_3 E_4 \rho_{34}, \quad [dp_4] = [\Omega_4^{(d-2)}] \int dE_4 E_4^{d-3} \int [d\eta_{34}] \int [d\Omega_4^{(d-2)}], \quad (2.87)$$

thus we can rewrite the right-hand side of (2.85) as

$$\begin{aligned} \mathcal{I} &= [\alpha_s] \mu^{2\epsilon} \left(\int [d\eta_{34}] \frac{1}{\rho_{34}} \right) \int [dp_3] \Theta(E_{\max} - E_3) \int_0^{E_{\max}} \frac{dE_4}{E_4^{-1+2\epsilon}} \frac{1}{E_3 E_4} P_{qq}(z) \\ &\quad \times |\mathcal{M}_{g_1 q_2 q_{[34]}}(p_1, p_2, p_{[34]})|^2 \delta_{12 \rightarrow [34]}^{(4)}. \end{aligned} \quad (2.88)$$

We point out that

$$|\mathcal{M}_{g_1 q_2 q_{[34]}}(p_1, p_2, p_{[34]})|^2 \delta_{12 \rightarrow [34]}^{(4)} \quad (2.89)$$

does not depend on the azimuthal angle $\rho_{34} = 2\eta_{34}$, nor on the solid angle $[d\Omega_4^{(d-2)}]$ (it has been integrated out in (2.88) and combined with various prefactors to form $[\alpha_s] \mu^{2\epsilon}$), which is why we could extract it from the angular integrals. At this point, we notice that the integral over the azimuthal angle on the right-hand side of Eq. (2.88) produces an IR pole (see Eq. (2.67)):

$$\int [d\eta_{34}] \frac{1}{\rho_{34}} = \int_0^1 \frac{2d\eta_{34}}{[4\eta_{34}(1-\eta_{34})]^\epsilon} \frac{1}{2\eta_{34}} = -\frac{2^{-2\epsilon}}{\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}. \quad (2.90)$$

Indeed, if we set $\epsilon = 0$ on the left-hand side of (2.90), the integral diverges in the kinematic region where $\eta_{34} = 1 - \cos \theta_{34} \rightarrow 0$, that is, when $\theta_{34} \rightarrow 0$. For this reason, this singularity is known as a *collinear divergence*. As anticipated above, this pure-collinear

singularity produces a single pole of $\mathcal{O}(\epsilon^{-1})$.

This pure-collinear divergence is not the only singularity in Eq. (2.88). Indeed, integrating over the energies reveals a soft singularity. To show this explicitly, we change the variables of integration as in Eq. (2.77), which yields

$$dE_3 dE_4 = E_{[34]} dE_{[34]} dz. \quad (2.91)$$

Then, we use Eq. (2.77) also to rewrite the integrals in Eq. (2.88) as

$$\begin{aligned} [dp_3] \frac{dE_4}{E_4^{-1+2\epsilon}} \frac{1}{E_3 E_4} &\mapsto \frac{dE_3 dE_4}{E_3^{2\epsilon} E_4^{2\epsilon}} [d\Omega_3^{(d-1)}] \\ &= \left(dE_{[34]} E_{[34]}^{1-2\epsilon} [d\Omega_3^{(d-1)}] \right) E_{[34]}^{-2\epsilon} dz [z(1-z)]^{-2\epsilon} \\ &= [dp_{[34]}] E_{[34]}^{-2\epsilon} dz [z(1-z)]^{-2\epsilon}, \end{aligned} \quad (2.92)$$

where we exploit the fact that the angular phase space of the clustered quark $q_{[34]}$, i.e. $[d\Omega_{[34]}^{(d-1)}]$, is the same as that of the hard quark q_3 , i.e. $[d\Omega_3^{(d-1)}]$, such that we can rename $[d\Omega_3^{(d-1)}]$ as $[d\Omega_{[34]}^{(d-1)}]$. Now, since

$$E_3 = z E_{[34]} \in [0, E_{\max}], \quad E_4 = (1-z) E_{[34]} \in [0, E_{\max}], \quad (2.93)$$

then

$$\begin{cases} 0 \leq E_{[34]} \leq 2E_{\max} \\ 0 \leq z E_{[34]} \leq E_{\max} \\ 0 \leq (1-z) E_{[34]} \leq E_{\max} \end{cases} \implies \begin{cases} 0 \leq E_{[34]} \leq 2E_{\max} \\ 0 \leq z \leq 1 \\ z \leq \frac{E_{\max}}{E_{[34]}} \\ z \geq 1 - \frac{E_{\max}}{E_{[34]}} \end{cases}, \quad (2.94)$$

which yields the following two regions of integration in the $(E_{[34]}, z)$ plane, represented in Figure 2.3:

$$\Theta_{(\text{I})} \stackrel{\text{def}}{=} \begin{cases} 0 \leq E_{[34]} \leq E_{\max} \\ 0 \leq z \leq 1, \end{cases} \quad \Theta_{(\text{II})} \stackrel{\text{def}}{=} \begin{cases} E_{\max} \leq E_{[34]} \leq 2E_{\max} \\ z \leq \frac{E_{\max}}{E_{[34]}} \\ z \geq 1 - \frac{E_{\max}}{E_{[34]}}. \end{cases} \quad (2.95)$$

Thus, Eq. (2.88) becomes (including the integral over the angular phase space in Eq. (2.90))

$$\begin{aligned} \mathcal{I} = & - \frac{[\alpha_s]}{\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \int [dp_{[34]}] dz [\Theta_{(\text{I})} + \cancel{\Theta_{(\text{II})}}] \left(\frac{2E_{[34]}}{\mu} \right)^{-2\epsilon} \\ & \times [z(1-z)]^{-2\epsilon} P_{qq}(z) |\mathcal{M}_{g_1 q_2 q_{[34]}}(p_1, p_2, p_{[34]})|^2 \delta_{12 \rightarrow [34]}^{(4)}. \end{aligned} \quad (2.96)$$

Thus, using the constraints on the energies E_3 and E_4 , we see that the energy of the clustered parton $E_{[34]}$ may exceed $E_{\max}$ and go all the way up to $2E_{\max}$. However, the two regions for the energy of the clustered particle, namely $\Theta_{(\text{I})}$ and $\Theta_{(\text{II})}$, are very different: the first one is physical whereas the second one is not. By this we mean that the

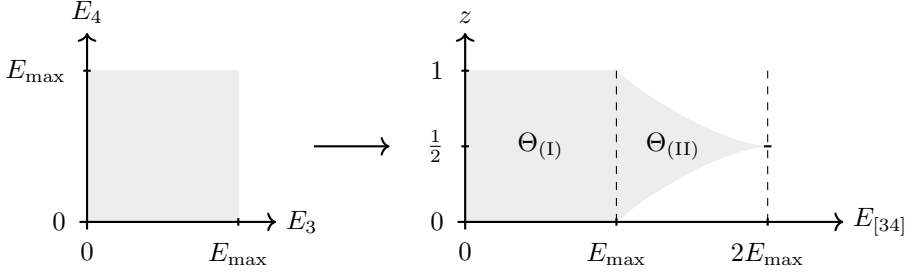


Figure 2.3: Energy phase space in the (E_3, E_4) -plane (left) and after the change of integration variables, Eq. (2.93), in the $(E_{[34]}, z)$ -plane (right). The energy phase space after the substitution is split into two regions, $\Theta_{(I)}$ and $\Theta_{(II)}$, as defined in Eq. (2.95).

integral must vanish for $E_{[34]} > E_{\max}$, since $E_{\max}$ is chosen to exceed the maximal energy that a parton can have in a physical process. It follows that the integral in Eq. (2.96) that contains $\Theta_{(II)}$ must be zero. In Ch. 4 we will see that, in the context of certain limits appearing in the NNLO calculation, the region $\Theta_{(II)}$ gives a non-zero contribution.

By virtue of this argument, Eq. (2.96) reduces to

$$\begin{aligned} \mathcal{I} = & -\frac{[\alpha_s]}{\epsilon} \int [dp_3] \left(\frac{2E_3}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} |\mathcal{M}_{g_1 q_2 q_3}(p_1, p_2, p_3)|^2 \Theta(E_{\max} - E_3) \delta_{12 \rightarrow 3}^{(4)} \\ & \times \int_0^1 dz [z(1-z)]^{-2\epsilon} P_{qq}(z), \end{aligned} \quad (2.97)$$

where we relabeled $[34] \mapsto 3$. After all this work, we succeeded in isolating the soft singularity mentioned above. In fact, if we write the integral over dz explicitly, i.e.,

$$\int_0^1 dz [z(1-z)]^{-2\epsilon} P_{qq}(z) = \int_0^1 dz [z(1-z)]^{-2\epsilon} C_F \left[\frac{1+z^2}{1-z} - \epsilon(1-z) \right], \quad (2.98)$$

we see that it diverges when $z \rightarrow 1$, that is, when $E_4 \rightarrow 0$. This is nothing but a soft singularity, producing a $1/\epsilon$ pole. We will see in Sec. 3.2 how to deal with such a singularity. Here we just note that if we had chosen $g(p_4)$ as the parton h , with a momentum fraction z , and $q(p_3)$ as the parton s , with a momentum fraction $1-z$, nothing would have changed. We would have obtained the integral

$$\int_0^1 dz [z(1-z)]^{-2\epsilon} P_{qg}(z) = \int_0^1 dz [z(1-z)]^{-2\epsilon} C_F \left[\frac{1+(1-z)^2}{z} - \epsilon z \right], \quad (2.99)$$

with $E_4 = z E_{[34]}$, which diverges when $z \rightarrow 0$, that is, when $E_4 \rightarrow 0$. This fact is not a coincidence. As stated at the beginning of Subsec. 2.3.1, in QCD, a soft particle produces a singularity only if it is a gluon. A soft (anti)quark does not produce IR singularities, and we showed it explicitly through this example. We also note that the results of these calculations are consistent with what was stated below Eq. (2.78), namely that the collinear factorization (2.78), once integrated over the phase space of parton s , contains

two singularities of $\mathcal{O}(\epsilon^{-1})$ each, one from Eq. (2.90) and the other discussed here.

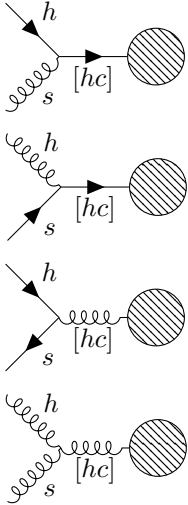
To conclude this subsection, we consider the collinear factorization formula (2.78) but for initial-state radiation. For definiteness, we take the initial-state splitting process $h \rightarrow [hs](z) \times s(1-z)$, where h is the incoming parton, and s and $[hs]$ radiate from it. The parton $[hs]$ enters the hard scattering process with momentum fraction z , while s is the final-state parton that takes momentum fraction $1-z$. It follows that here z corresponds to

$$z = 1 - \frac{E_s}{E_h}. \quad (2.100)$$

The initial-state version of Eq. (2.78) can be derived through crossing symmetry, and its expression up to $\mathcal{O}(\alpha_s)$ reads

$$\begin{aligned} & C_{hs} \left| \mathcal{M}_{f_h f_s f_1 \dots f_{m-1}}(p_h, p_s, \{p\}_{m-1}) \right|^2 \\ &= \frac{g_{s,b}^2 \mu_0^{2\epsilon}}{p_h \cdot p_s} P_{[hs]h,i}(z) \frac{\left| \mathcal{M}_{f_{[hs]} f_1 \dots f_{m-1}}(z \cdot p_h, \{p\}_{m-1}) \right|^2}{z}, \end{aligned} \quad (2.101)$$

where, as said, we used $p_{[hs]}^\mu = z \cdot p_h^\mu$. Functions $P_{[hs]h,i}(z)$ are the *initial-state Altarelli-Parisi splitting functions*, and read



$$\begin{aligned} P_{qq,i}(z) &\stackrel{\text{def}}{=} -z P_{qq}(1/z) = P_{qq}(z), \\ P_{qg,i}(z) &\stackrel{\text{def}}{=} \left[\frac{2N_c}{2(1-\epsilon)(N_c^2-1)} \right] z P_{qg}(1/z) = P_{gq}(z), \\ P_{gq,i}(z) &\stackrel{\text{def}}{=} \left[\frac{2(1-\epsilon)(N_c^2-1)}{2N_c} \right] z P_{gq}(1/z) = P_{qq}(z), \\ P_{gg,i}(z) &\stackrel{\text{def}}{=} -z P_{gg}(1/z) = P_{gg}(z). \end{aligned} \quad (2.102)$$

Notice that, since the functions $P_{[hs]h,i}$ are *averaged* splitting functions, their definitions contain a prefactor due to averaging over color and spin states

$$\frac{\text{ave}_h}{\text{ave}_{[hs]}}, \quad (2.103)$$

where

$$\text{ave}_q \stackrel{\text{def}}{=} \frac{1}{2N_c}, \quad \text{ave}_g \stackrel{\text{def}}{=} \frac{1}{2(1-\epsilon)(N_c^2-1)}. \quad (2.104)$$

The initial-state factorization formula (2.101) is conceptually similar to the final-

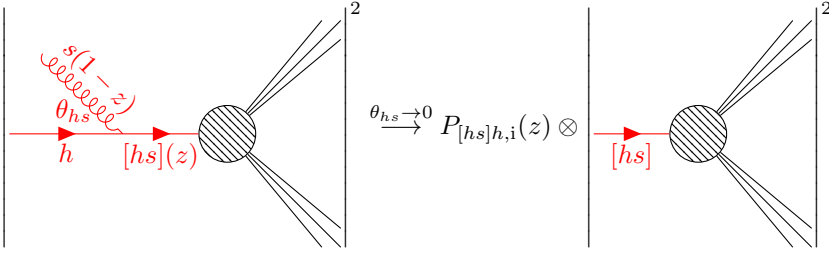


Figure 2.4: Graphical representation of the collinear factorization formula (2.101) at $\mathcal{O}(\alpha_s)$, where we have drawn as an example the $q \rightarrow q(z) \times g(1-z)$ splitting process but crossed to initial state. Functions $P_{[hs]h,i}(z)$ are the d -dimensional initial-state Altarelli-Parisi splitting functions (2.102).

state factorization formula (2.78). However, the integration over the phase space of the partons is somewhat different. To demonstrate this, we consider again the tree-level amplitude of the process $g(p_1) + q(p_2) \rightarrow X + q(p_3) + g(p_4)$, and we assume that the gluon $g(p_4)$ becomes collinear to the initial-state quark $q(p_2)$, that is, $\mathbf{p}_2 \parallel \mathbf{p}_4$. As usual, we aim to compute the expression

$$\begin{aligned} \mathcal{I} &= \int [dp_4] C_{24} |\mathcal{M}_{g_1 q_2 q_3 g_4}(p_1, p_2, p_3, p_4)|^2 \Theta(E_{\max} - E_4) \delta_{12 \rightarrow 34}^{(4)} \\ &= \int [dp_4] \frac{g_{s,b}^2 \mu_0^{2\epsilon}}{E_2 E_4 \rho_{24}} P_{qq,i}(z) \frac{|\mathcal{M}_{g_1 q_2 q_3}(p_1, z \cdot p_2, p_3)|^2}{z} \Theta(E_{\max} - E_4) \delta_{1(z \cdot 2) \rightarrow 3}^{(4)}, \end{aligned} \quad (2.105)$$

where $f_{[24]} = q$, and $z = 1 - E_4/E_2$. The integration over the angles proceeds in the same manner as for the final-state radiation discussed earlier, leading to

$$\begin{aligned} \mathcal{I} &= - \frac{[\alpha_s] \mu^{2\epsilon}}{\epsilon} 2^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \int_0^{E_{\max}} dE_4 \frac{E_4^{1-2\epsilon}}{E_2 E_4} P_{qq,i}(z) \\ &\quad \times \frac{|\mathcal{M}_{g_1 q_2 q_3}(p_1, z \cdot p_2, p_3)|^2}{z} \delta_{1(z \cdot 2) \rightarrow 3}^{(4)}, \end{aligned} \quad (2.106)$$

where

$$\delta_{1(z \cdot 2) \rightarrow 3}^{(4)} \stackrel{\text{def}}{=} (2\pi)^4 \delta^{(4)}(p_1 + z \cdot p_2 - p_3 - p_X). \quad (2.107)$$

For the integral over energy, we change the variable of integration as

$$\int_0^{E_{\max}} dE_4 \frac{E_4^{1-2\epsilon}}{E_2 E_4} = E_2^{-2\epsilon} \int_{z_{\min}}^1 dz (1-z)^{-2\epsilon}, \quad (2.108)$$

where $z_{\min} = 1 - E_{\max}/E_2 < 0$ since, by definition, $E_2 < E_{\max}$. Notice that the integral over the region $z \in [z_{\min}, 0]$ must vanish because the function $\delta_{1(z \cdot 2) \rightarrow 3}^{(4)}$ defines the physical region where an initial-state splitting process with a negative momentum fraction z

is forbidden. Consequently,⁹

$$\begin{aligned} \mathcal{I} = & -\frac{[\alpha_s]}{\epsilon} \left(\frac{2E_2}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \int_0^1 dz (1-z)^{-2\epsilon} P_{qq,i}(z) \\ & \times \frac{|\mathcal{M}_{g_1 q_2 q_3}(p_1, z \cdot p_2, p_3)|^2}{z} \delta_{1(z \cdot 2) \rightarrow 3}^{(4)}. \end{aligned} \quad (2.109)$$

Note that in the soft limit $z \rightarrow 1$ the amplitude becomes independent of z ,

$$|\mathcal{M}_{g_1 q_2 q_3}(p_1, z \cdot p_2, p_3)|^2 \delta_{1(z \cdot 2) \rightarrow 3}^{(4)} \rightarrow |\mathcal{M}_{g_1 q_2 q_3}(p_1, p_2, p_3)|^2 \delta_{12 \rightarrow 3}^{(4)}, \quad (2.110)$$

thus we are left with the integral over $(1-z)^{-2\epsilon} P_{qq,i}(z)$ that diverges in the kinematics region $z \rightarrow 1$. This singularity is another type of *soft singularity* and, as one may see, it produces a $1/\epsilon$ pole. We will see in Sec. 3.2 how to deal with it.

2.4 IR singularities of $\mathcal{O}(\alpha_s^2)$ in tree-level amplitudes

2.4.1 Double-soft singularities

In Subsec. 2.3.1, we saw that, when a soft gluon is emitted, the QCD amplitude factorizes as in Eq. (2.52). At NLO, we have to consider the emission of a single real parton, which may be soft. In the case of $\mathcal{O}(\alpha_s^2)$ corrections, we have to consider the emission of two real partons, and in this section we analyze what happens when both of these become soft simultaneously. This is the so-called *double-soft singularity*. Note that we referred to two *partons* and not just two gluons. In the single-soft case, only a gluon emission generates IR divergences, as showed with the explicit example at the end of the previous subsection. In contrast, a double-soft divergence can be caused either by a pair of soft gluons gg , or by a soft quark-antiquark pair of the same flavor, $q\bar{q}$. The two-gluon emission is represented in Figure 2.5.

To discuss the double-soft singularity, we consider a tree-level amplitude with $m+2$ external partons, which we denote by $\mathcal{M}(\{p\}_m, p_m, p_n) \equiv \mathcal{M}_{f_1 \dots f_m f_m f_n}(\{p\}_m, p_m, p_n)$ (cf. Sec. 2.1). Here, m and n are the two final-state soft partons, and f_m and f_n are their flavours, respectively. The double-soft limit implies $E_{m,n} \rightarrow 0$. We formally write this limit as

$$S_{mn} \stackrel{\text{def}}{=} \lim_{E_m, E_n \rightarrow 0} \quad \text{at fixed } \frac{E_m}{E_n}, \quad (2.111)$$

and similarly to the single-soft and collinear limits, by definition, it acts on all quantities to its right and is to be understood in the sense of extracting the most singular collinear contribution provided that limits in the conventional sense do not exist. Using this notation, the square modulus of the aforementioned matrix element in the double-soft limit

⁹Unlike in the final-state collinear limit, we cannot compute the integral over z , since the matrix element depends on it.

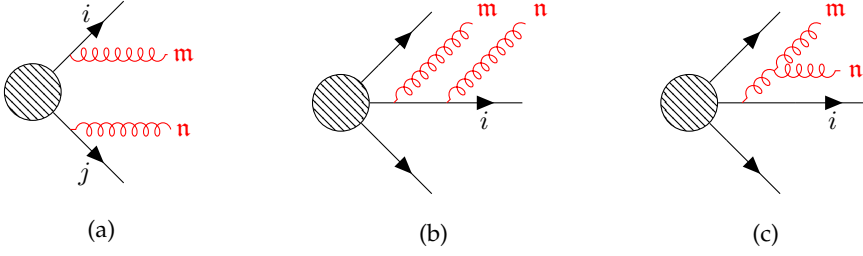


Figure 2.5: Examples of Feynman diagrams for the double-soft emissions of two gluons, m and n . We have: (a) eikonal emission of two soft gluons from two different external partons; (b) eikonal emission of two gluons from the same external parton; (c) non-abelian contribution.

$E_{m,n} \rightarrow 0$ factorizes, and can be written as [152]

$$S_{mn} |\mathcal{M}(\{p\}_m, p_m, p_n)|^2 = [g_{s,b}^2 \mu_0^{2\epsilon}]^2 \left[\frac{1}{2} \sum_{(ij),(kl)} S_{ij}(p_m) S_{kl}(p_n) |\mathcal{M}^{(ij)(kl)}(\{p\}_m)|^2 - C_A \sum_{\substack{i,j \in \mathcal{H} \\ i < j}} \widetilde{S}_{ij}(p_m, p_n) |\mathcal{M}^{(ij)}(\{p\}_m)|^2 \right], \quad (2.112)$$

if m, n are two gluons, and

$$S_{mn} |\mathcal{M}(\{p\}_m, p_m, p_n)|^2 = [g_{s,b}^2 \mu_0^{2\epsilon}]^2 T_R \sum_{\substack{ij \in \mathcal{H} \\ i < j}} \widetilde{\mathcal{I}}_{ij}(p_m, p_n) |\mathcal{M}^{(ij)}(\{p\}_m)|^2, \quad (2.113)$$

if m, n are a quark and an antiquark. In Eqs (2.112) and (2.113), we used the following *color-correlated amplitudes* (cf. Eq. (2.8)):

$$\begin{aligned} |\mathcal{M}^{(ij)}(\{p\}_m)|^2 &\stackrel{\text{def}}{=} \langle \mathcal{M}(\{p\}_m) | \mathbf{T}_i \cdot \mathbf{T}_j | \mathcal{M}(\{p\}_m) \rangle, \\ |\mathcal{M}^{(ij)(kl)}(\{p\}_m)|^2 &\stackrel{\text{def}}{=} \langle \mathcal{M}(\{p\}_m) | \{ \mathbf{T}_i \cdot \mathbf{T}_j, \mathbf{T}_k \cdot \mathbf{T}_l \} | \mathcal{M}(\{p\}_m) \rangle. \end{aligned} \quad (2.114)$$

In this thesis, we will refer to the object $|\mathcal{M}^{(ij)(kl)}(\{p\}_m)|^2$ as *double color-correlated* or *quartic color-correlated*. We also recall that $\mathcal{H}$ labels the list of the m external hard partons. All other eikonal functions are defined as follows [152]:¹⁰

$$S_{ij}(k) \stackrel{\text{def}}{=} \frac{(p_i \cdot p_j)}{(p_i \cdot k)(p_j \cdot k)}, \quad (2.115)$$

$$\begin{aligned} S_{ij}(p_m, p_n) &\stackrel{\text{def}}{=} S_{ij}^{\text{so}}(p_m, p_n) - \frac{2(p_i \cdot p_j)}{(p_m \cdot p_n)[p_i \cdot (p_m + p_n)][p_j \cdot (p_m + p_n)]} \\ &\quad + \frac{(p_i \cdot p_m)(p_j \cdot p_n) + (p_i \cdot p_n)(p_j \cdot p_m)}{[p_i \cdot (p_m + p_n)][p_j \cdot (p_m + p_n)]} \end{aligned}$$

¹⁰The function $S_{ij}(k)$ is precisely the eikonal function introduced in Eq. (2.54).

$$\times \left[\frac{(1 - \epsilon)}{(p_m \cdot p_n)^2} - \frac{1}{2} S_{ij}^{\text{so}}(p_m, p_n) \right], \quad (2.116)$$

$$S_{ij}^{\text{so}}(p_m, p_n) \stackrel{\text{def}}{=} \frac{p_i \cdot p_j}{p_m \cdot p_n} \left[\frac{1}{(p_i \cdot p_m)(p_j \cdot p_n)} + \frac{1}{(p_i \cdot p_n)(p_j \cdot p_m)} \right] - \frac{(p_i \cdot p_j)^2}{(p_i \cdot p_m)(p_j \cdot p_m)(p_i \cdot p_n)(p_j \cdot p_n)}, \quad (2.117)$$

$$\mathcal{I}_{ij} \stackrel{\text{def}}{=} \frac{(p_i \cdot p_m)(p_j \cdot p_n) + (p_i \cdot p_n)(p_j \cdot p_m) - (p_i \cdot p_j)(p_m \cdot p_n)}{(p_m \cdot p_n)^2 [p_i \cdot (p_m + p_n)] [p_j \cdot (p_m + p_n)]}, \quad (2.118)$$

and $\widetilde{S}_{ij}$ and $\widetilde{\mathcal{I}}_{ij}$ read

$$\begin{aligned} \widetilde{S}_{ij}(p_m, p_n) &\stackrel{\text{def}}{=} 2S_{ij}(p_m, p_n) - S_{ii}(p_m, p_n) - S_{jj}(p_m, p_n), \\ \widetilde{\mathcal{I}}_{ij}(p_m, p_n) &\stackrel{\text{def}}{=} 2\mathcal{I}_{ij}(p_m, p_n) - \mathcal{I}_{ii}(p_m, p_n) - \mathcal{I}_{jj}(p_m, p_n). \end{aligned} \quad (2.119)$$

The integration of Eqs (2.112) and (2.113) over the phase space of the unresolved partons m and n is highly non trivial to perform. In Ref. [80], this integration has been computed under the following constraints on the energies of partons m and n:

$$E_m < E_{\text{max}}, \quad E_n < E_m. \quad (2.120)$$

This result is of particular interest to us, as it will play a crucial role when discussing the NNLO corrections in the Nested Soft-Collinear subtraction scheme. According to Ref. [80], a double-soft contribution to any differential cross section with constraints (2.120) can be constructed if the following integrals are known:

$$\mathfrak{S}_{ij,kl} \stackrel{\text{def}}{=} \int [dp_m][dp_n] \Theta(E_{\text{max}} - E_m) \Theta(E_m - E_n) S_{ij}(p_m) S_{kl}(p_n), \quad (2.121)$$

$$\mathfrak{S}_{ij}^{(gg)} \stackrel{\text{def}}{=} \int [dp_m][dp_n] \Theta(E_{\text{max}} - E_m) \Theta(E_m - E_n) \widetilde{S}_{ij}(p_m, p_n), \quad (2.122)$$

$$\mathfrak{S}_{ij}^{(qq)} \stackrel{\text{def}}{=} -2 \int [dp_m][dp_n] \Theta(E_{\text{max}} - E_m) \Theta(E_m - E_n) \widetilde{\mathcal{I}}_{ij}(p_m, p_n). \quad (2.123)$$

The object $\mathfrak{S}_{ij,kl}$, once computed explicitly, reads

$$\mathfrak{S}_{ij,kl} = [\alpha_s]^2 \frac{(2E_{\text{max}}/\mu)^{-4\epsilon}}{2\epsilon^4} \eta_{ij}^{-\epsilon} \eta_{kl}^{-\epsilon} K_{ij} K_{kl}, \quad (2.124)$$

with K_{ij} given in Eq. (2.68). As for $\mathfrak{S}_{ij}^{(gg)}$ and $\mathfrak{S}_{ij}^{(qq)}$, we do not report them here as they do not have compact expressions. These can be found in Eqs (61) and (62) of Ref. [80], respectively. Note that, if we expand (2.124) in powers of ϵ , we obtain an expression with poles of $\mathcal{O}(\epsilon^{-4})$, exactly as for the double-virtual contributions in Subsec. 2.2.2. As we will see in Part II, these poles will cancel out among each other.

2.4.2 Triple-collinear singularities

In Subsec. 2.3.2, we saw that a QCD amplitude factorizes as in Eq. (2.78) under the action of the collinear limit (2.79). As mentioned there, the matrix element on the left-hand side of Eq. (2.78) represents a correction of $\mathcal{O}(\alpha_s)$ to the Born level. At $\mathcal{O}(\alpha_s^2)$, the so-called *triple-collinear singularity* can occur. This happens when two final-state partons, say m and n , both become collinear with a third parton i , which can be either in the initial or final state. In other words, the condition $p_m \parallel p_n \parallel p_i$ holds.

To discuss this phenomenon, as in the previous subsection we consider the tree-level amplitude $\mathcal{M}(\{p\}_{m-1}, p_i, p_m, p_n)$ with $m+2$ external partons. The triple-collinear kinematics $p_m \parallel p_n \parallel p_i$ is defined through [152]

$$p_k^\mu = x_k p_{[imn]}^\mu + k_{\perp,k}^\mu + \beta_k n^\mu, \quad k = i, m, n, \quad (2.125)$$

where β_k^μ are determined by on-shell conditions $p_i^2 = p_m^2 = p_n^2 = 0$, and

$$p_{[imn]}^2 = n^2 = 0, \quad p_{[imn]} \cdot k_\perp = n \cdot k_\perp = 0, \quad k_{\perp,k}^2 < 0. \quad (2.126)$$

As in Eq. (2.74), the light-like vector $p_{[imn]}^\mu$ denotes the collinear direction, and the auxiliary n^μ specifies how the collinear direction is approached. Next, we introduce the boost-invariant quantities

$$z_k \stackrel{\text{def}}{=} \frac{x_k}{x}, \quad \tilde{k}_k^\mu \stackrel{\text{def}}{=} k_{\perp,k}^\mu - z_k (k_{\perp,i}^\mu + k_{\perp,m}^\mu + k_{\perp,n}^\mu), \quad (2.127)$$

with

$$x \stackrel{\text{def}}{=} x_i + x_m + x_n, \quad (2.128)$$

which satisfy

$$z_i + z_m + z_n = 1, \quad \tilde{k}_i^\mu + \tilde{k}_m^\mu + \tilde{k}_n^\mu = 0. \quad (2.129)$$

In terms of these variables, we find

$$s_{kl} = (p_k + p_l)^2 = -x_k x_l \left(\frac{k_{\perp,l}}{x_l} - \frac{k_{\perp,k}}{x_k} \right)^2 = -z_k z_l \left(\frac{\tilde{k}_l}{x_l} - \frac{\tilde{k}_k}{x_k} \right)^2, \quad (2.130)$$

and we use it to define the object

$$t_{kl,j} \stackrel{\text{def}}{=} 2 \frac{z_k s_{lj} - z_l s_{kj}}{z_k + z_l} + \frac{z_k - z_l}{z_k + z_l} s_{kl}. \quad (2.131)$$

Finally, we introduce the *triple-collinear operator*

$$C_{mn,i} \stackrel{\text{def}}{=} \lim_{\rho_{im}, \rho_{in} \rightarrow 0}, \quad \text{with fixed } \frac{\rho_{im}}{\rho_{in}}, \frac{\rho_{mn}}{\rho_{im}}, \frac{\rho_{mn}}{\rho_{in}}, \quad (2.132)$$

which as usual must be understood in the sense of extracting the most singular contribution, provided that the limit in the conventional sense does not exist.

Using the notation just outlined, we can write the following QCD *triple-collinear*

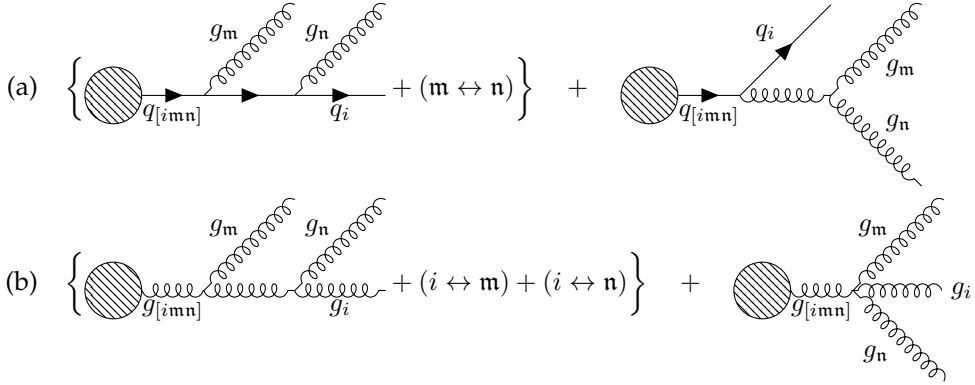


Figure 2.6: Examples of triple-collinear splitting processes. We have the Feynman diagrams that contribute to the triple-collinear splitting processes: (a) $q_{[imn]} \rightarrow q_i g_m g_n$; (b) $g_{[imn]} \rightarrow g_i g_m g_n$.

factorization formula for the splitting process $[imn] \rightarrow i(x_i) \times m(x_m) \times n(x_n)$ [152]:

$$C_{mn,i} \left| \mathcal{M}_{f_1 \dots f_{m-1} f_i f_m f_n}(\{p\}_{m-1}, p_i, p_m, p_n) \right|^2 = \frac{[2g_{s,b}^2 \mu_0^{2\epsilon}]^2}{s_{imn}^2} P_{mni}^{rr'}(\{z_k\}, \{s_{kl}\}, \{\tilde{k}_i\}) \left| \mathcal{M}_{f_1 \dots f_{m-1} f_{[imn]}}(\{p\}_{m-1}, x \cdot p_{[imn]}) \right|_{rr'}^2, \quad (2.133)$$

where $s_{imn} = (p_i + p_m + p_n)^2$. As in Eq. (2.78), the spin-polarization tensor

$$\left| \mathcal{M}_{f_1 \dots f_{m-1} f_{[imn]}}(\{p\}_{m-1}, x \cdot p_{[imn]}) \right|_{rr'}^2 \quad (2.134)$$

is obtained by replacing the partons $\{i, m, n\}$ with a single parent parton $[imn]$, whose flavour is determined by flavour conservation in the splitting process. Specifically, $[imn]$ is an (anti)quark if the set $\{i, m, n\}$ contains an odd number of (anti)quarks, and $[imn]$ is a gluon otherwise. In Figure 2.6, two explicit examples are reported. The three-parton splitting functions $P_{mni}^{rr'}(\{z_k\}, \{s_{kl}\}, \{\tilde{k}_i\})$ generalize the Altarelli-Parisi splitting functions (2.81). Owing to their invariance under longitudinal boosts along the collinear direction $p_{[imn]}^\mu$, they depend in a non-trivial way on sub-energy ratios s_{kl}/s_{imn} and $t_{kl,j}$, and on longitudinal- and transverse-momentum variables z_k and $\tilde{k}_k$. As in Eq. (2.81), if the mother parton $[imn]$ is a quark, the spin-dependence is trivial, i.e.,

$$P_{mni}^{rr'}(\{z_k\}, \{s_{kl}\}, \{\tilde{k}_i\}) = \delta_{rr'} P_{mni}(\{z_k\}, \{s_{kl}\}, \{\tilde{k}_i\}). \quad (2.135)$$

In contrast, if $[imn]$ is a gluon, spin correlations are highly non-trivial. The definitions of the triple-collinear splitting $P_{mni}^{rr'}$ can be found in Ref. [152].

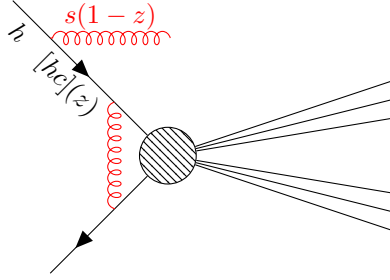


Figure 2.7: Example of a Feynman diagram representing a real-virtual amplitude with the emission of a real and a virtual gluon.

2.5 IR singularities of $\mathcal{O}(\alpha_s^2)$ in real-virtual amplitudes

The final case we need to discuss at $\mathcal{O}(\alpha_s^2)$ is that of real-virtual amplitudes, which involve the emissions of both a real and a virtual parton, as shown in Figure 2.7. In this case, the IR singularities arising from the loop amplitude can be described as discussed in Subsec. 2.2.1. The description of the soft and collinear limits of the real gluon follows along similar lines as for the tree-level amplitude (see Sec. 2.3), but the limits are more involved. We will discuss these in this section.

2.5.1 Soft limit in real-virtual amplitudes

We start from the soft case and demonstrate how the factorized structure of the soft limit of the QCD amplitudes at tree level, namely (2.52), can be generalized to one-loop. We recall the notation established in Subsec. 2.2.1, where we wrote the tree-level and one-loop amplitudes with m external partons as $|\mathcal{M}_0\rangle = |\mathcal{M}_0(\{p\}_m)\rangle$ and $|\mathcal{M}_1\rangle = |\mathcal{M}_1(\{p\}_m)\rangle$, and extend this notation to amplitudes with an additional emitted gluon, $|\mathcal{M}_0(p_m)\rangle = |\mathcal{M}_0(\{p\}_m, p_m)\rangle$ and $|\mathcal{M}_1(p_m)\rangle = |\mathcal{M}_1(\{p\}_m, p_m)\rangle$. The parton with momentum p_m^μ is by assumption a gluon. Under the action of the soft limit S_m (2.53) the following *soft-gluon real-virtual factorization formula* at $\mathcal{O}(\alpha_s^2)$ holds [154, 155]:

$$\begin{aligned}
 & S_m \left[2 \operatorname{Re} \left[\langle \mathcal{M}_0(p_m) | \mathcal{M}_1(p_m) \rangle \right] \right] \\
 &= -g_{s,b}^2 \mu_0^{2\epsilon} \sum_{(ij)} \left\{ \left[\frac{\alpha_s(\mu)}{2\pi} \right] 2S_{ij}(p_m) 2 \operatorname{Re} \left[\langle \mathcal{M}_0 | \mathbf{T}_i \cdot \mathbf{T}_j | \mathcal{M}_1 \rangle \right] - \left[\frac{\alpha_s(\mu)}{2\pi} \right] \frac{\beta_0}{\epsilon} 2S_{ij}(p_m) \right. \\
 &\quad \times \langle \mathcal{M}_0 | \mathbf{T}_i \cdot \mathbf{T}_j | \mathcal{M}_0 \rangle - 2 \frac{[\alpha_s]}{\epsilon^2} C_A A_K(\epsilon) [S_{ij}(p_m)]^{1+\epsilon} \langle \mathcal{M}_0 | \mathbf{T}_i \cdot \mathbf{T}_j | \mathcal{M}_0 \rangle \\
 &\quad \left. - [\alpha_s] \frac{4\pi \Gamma(1+\epsilon) \Gamma^3(1-\epsilon)}{\epsilon \Gamma(1-2\epsilon)} \sum_{\substack{k \in \mathcal{H} \\ k \neq i,j}} \kappa_{ij} S_{ki}(p_m) [S_{ij}(p_m)]^{1+\epsilon} f_{abc} \langle \mathcal{M}_0 | T_k^a T_i^b T_j^c | \mathcal{M}_0 \rangle \right\}. \tag{2.136}
 \end{aligned}$$

In Eq. (2.136), the coefficient κ_{ij} is defined as

$$\kappa_{ij} \stackrel{\text{def}}{=} \begin{cases} \lambda_{ij} - \lambda_{im} - \lambda_{jm} = +1 & \text{if } i \text{ and } j \text{ are both incoming partons} \\ -1 & \text{otherwise.} \end{cases} \quad (2.137)$$

with λ_{ij} given in Eq. (2.24), $S_{ij}(p_m)$ is the eikonal factor (2.54), $\mathcal{H}$ denotes the list of the m hard partons, and f_{abc} refers to the totally antisymmetric structure constants of SU(3). Moreover, we have introduced the constant

$$A_K(\epsilon) \stackrel{\text{def}}{=} \frac{\Gamma^3(1+\epsilon) \Gamma^5(1-\epsilon)}{\Gamma(1+2\epsilon) \Gamma^2(1-2\epsilon)} = 1 + \mathcal{O}(\epsilon^2). \quad (2.138)$$

Notice that the one-loop amplitude $|\mathcal{M}_1\rangle$ on the right-hand side of Eq. (2.136) contains IR divergences encapsulated in Eq. (2.20), and thus we expect the term proportional to $2 \text{Re} [\langle \mathcal{M}_0 | \mathbf{T}_i \cdot \mathbf{T}_j | \mathcal{M}_1 \rangle]$ to involve the virtual operator $I_V(\epsilon)$ introduced in Eq. (2.29). Upon integrating over the unresolved phase space of gluon m_g , we obtain further poles up to $\mathcal{O}(\epsilon^{-2})$, which combine with the poles up to the same order in $I_V(\epsilon)$ to give poles up to $\mathcal{O}(\epsilon^{-4})$. Moreover, the integral of the terms in the first line of Eq. (2.136) gives rise to color-correlated terms of the form

$$-\frac{(2E_{\text{max}}/\mu)^{-2\epsilon}}{\epsilon^2} \sum_{(ij)} \eta_{ij}^{-\epsilon} K_{ij} \langle \mathcal{M}_0 | \mathbf{T}_i \cdot \mathbf{T}_j | \mathcal{M}_0 \rangle. \quad (2.139)$$

This is not the first time that we have encountered this structure. We already saw it in Eq. (2.71), in the case of the single-soft limit of a tree-level amplitude with two hard partons. This observation, that singular structures present at NLO recur at NNLO, is central to the development of the Nested Soft-Collinear scheme presented in this thesis. Finally, the last term on the right-hand side of Eq. (2.136) contains explicit *triple color correlators*

$$f_{abc} \langle \mathcal{M}_0 | T_k^a T_i^b T_j^c | \mathcal{M}_0 \rangle. \quad (2.140)$$

Integrating this term over $[dp_m]$ is non-trivial and is discussed at length in Appendix H. In what follows, we will refer to it as the triple color-correlated real-virtual subtraction term, $I_{\text{tri}}^{\text{RV}}$.

2.5.2 Collinear limit in real-virtual amplitudes

In this subsection we analyze the collinear limit of a real-virtual amplitude. As in the soft case, we begin with a one-loop amplitude with $m+1$ external partons. We consider the final-state splitting process $[hs] \rightarrow h(z) \times s(1-z)$, where $[hs]$ is the mother (clustered) particle, h and s the two particles produced in the splitting process, and z is the momentum fraction (cf. Subsec. 2.3.2).

Under the action of the collinear limit C_{hs} (2.79), the following *averaged final-state collinear factorization formula for real-virtual amplitudes* at $\mathcal{O}(\alpha_s^2)$ holds:

$$C_{hs} \left[2 \text{Re} [\langle \mathcal{M}_0(\dots, p_h, p_s) | \mathcal{M}_1(\dots, p_h, p_s) \rangle] \right]$$

$$\begin{aligned}
&= \frac{g_{s,b}^2 \mu_0^{2\epsilon}}{p_h \cdot p_s} \left\{ P_{[hs]h}(z) \left[2 \operatorname{Re} \left[\langle \mathcal{M}_0(\dots, p_{[hs]}) | \mathcal{M}_1(\dots, p_{[hs]}) \rangle \right] \right. \right. \\
&\quad \left. \left. - [\alpha_s] \frac{\Gamma(1-\epsilon)}{\mu^{2\epsilon} e^{\epsilon\gamma_E}} \frac{\beta_0}{\epsilon} |\mathcal{M}_0(\dots, p_{[hs]})|^2 \right] \right. \\
&\quad \left. + [\alpha_s] \frac{\Gamma^3(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} \cos(\pi\epsilon) (2\rho_{hs})^{-\epsilon} E_{[hs]}^{-2\epsilon} \frac{P_{[hs]h}^{\text{1L}}(z)}{[z(1-z)]^\epsilon} |\mathcal{M}_0(\dots, p_{[hs]})|^2 \right\}. \tag{2.141}
\end{aligned}$$

In Eq. (2.141), $|\mathcal{M}_0(\dots, p_h, p_s)\rangle$ and $|\mathcal{M}_1(\dots, p_h, p_s)\rangle$ are the tree-level and one-loop amplitudes, respectively, with $m+1$ external partons, whose momenta are left implicit except for those of particles h and s . On the right-hand side, $|\mathcal{M}_0(\dots, p_{[hs]})\rangle$ and $|\mathcal{M}_1(\dots, p_{[hs]})\rangle$ are tree-level and one-loop amplitudes, respectively, with m external partons, where particles h and s have been replaced with the clustered particle $[hs]$, with momentum $p_{[hs]}^\mu = p_h^\mu + p_s^\mu$, energy $E_{[hs]}$, and flavour determined by the flavours of h and s . As at NLO, the momentum fraction z is related to the energies E_h , E_s , $E_{[hs]}$, through the relations

$$E_h = z E_{[hs]}, \quad E_s = (1-z) E_{[hs]}. \tag{2.142}$$

It follows that Eq. (2.141) is written in a form that is already averaged over spins and contains the NLO spin-averaged final-state splitting functions $P_{[hs]h}(z)$, shown in Eq. (2.84). Finally, $P_{[hs]h}^{\text{1L}}(z)$ are the spin-averaged final-state spitting functions at one-loop. They can be extracted from Ref. [156],¹¹ and correspond to

$$\begin{aligned}
P_{qq}^{\text{1L}}(z) &\stackrel{\text{def}}{=} P_{qq}(z) \left[C_A \frac{a(z)}{2} + (2C_F - C_A) \frac{\tilde{b}(z)}{2} \right] + \tilde{P}_{qq}^{\text{new}}(z) \left[\frac{C_A - C_F}{1-2\epsilon} \right], \\
P_{qg}^{\text{1L}}(z) &\stackrel{\text{def}}{=} P_{qg}(z) \left[C_A \frac{b(z)}{2} + (2C_F - C_A) \frac{\tilde{a}(z)}{2} \right] + \tilde{P}_{qg}^{\text{new}}(z) \left[\frac{C_A - C_F}{1-2\epsilon} \right] \\
&\equiv P_{qq}^{\text{1L}}(1-z), \\
P_{gg}^{\text{1L}}(z) &\stackrel{\text{def}}{=} P_{gg}(z) \left\{ C_A \frac{a(z) + \tilde{b}(z)}{2} + \frac{C_A - C_F}{\epsilon^2} \left[\frac{2}{1-2\epsilon} - \epsilon \right] - [n_f - C_A(1-\epsilon)] \right. \\
&\quad \left. \times \left[\frac{2(1-\epsilon)}{\epsilon(1-2\epsilon)(3-2\epsilon)} \right] \right\}, \\
P_{gg}^{\text{1L}}(z) &\stackrel{\text{def}}{=} P_{gg}(z) \left[C_A \frac{a(z) + \tilde{b}(z)}{2} \right] + \tilde{P}_{gg}^{\text{new}}(z) \left[\frac{n_f - C_A(1-\epsilon)}{(1-\epsilon)(1-2\epsilon)(3-2\epsilon)} \right], \tag{2.143}
\end{aligned}$$

with $P_{\alpha\beta}$ listed in Eq. (2.84), and where

$$\begin{aligned}
a(z) &= (1-z)F_1(1-z), & \tilde{a}(z) &= a(z) - \frac{2}{\epsilon^2}, \\
b(z) &= zF_1(z) \equiv a(1-z), & \tilde{b}(z) &= b(z) - \frac{2}{\epsilon^2} \equiv \tilde{a}(1-z), \\
F_1(z) &= \frac{2}{z\epsilon^2} \left[-\Gamma(1-\epsilon)\Gamma(1+\epsilon)z^{-\epsilon}(1-z)^\epsilon - 1 + (1-z)^\epsilon {}_2F_1(\epsilon, \epsilon, 1+\epsilon, z) \right], \tag{2.144}
\end{aligned}$$

¹¹Here we extract an overall normalization factor w.r.t. Ref. [156].

and

$$\begin{aligned}\tilde{P}_{qq}^{\text{new}}(z) &= C_F [1 - \epsilon(1 - z)] , \\ \tilde{P}_{qg}^{\text{new}}(z) &= C_F [1 - \epsilon z] , \\ \tilde{P}_{gg}^{\text{new}}(z) &= -C_A \left[\frac{1 - 2z(1 - z)\epsilon}{1 - \epsilon} \right] .\end{aligned}\tag{2.145}$$

Now we consider Eq. (2.141) but for an initial-state collinear limit. For definiteness, we take the initial-state splitting process $h \rightarrow [hs](z) \times s(1 - z)$ as described on page 52, and represented in Figure 2.7. Under the action of the collinear limit C_{hs} , the following *averaged initial-state collinear factorization formula for real-virtual amplitudes* at $\mathcal{O}(\alpha_s^2)$ holds:

$$\begin{aligned}& C_{hs} \left[2 \operatorname{Re} \left[\langle \mathcal{M}_0(p_h, \dots, p_s) | \mathcal{M}_1(p_h, \dots, p_s) \rangle \right] \right] \\ &= \frac{g_{s,b}^2 \mu_0^{2\epsilon}}{p_h \cdot p_s} \left\{ P_{[hs]h,i}(z) \left[\frac{2 \operatorname{Re} \left[\langle \mathcal{M}_0(z \cdot p_h, \dots) | \mathcal{M}_1(z \cdot p_h, \dots) \rangle \right]}{z} \right. \right. \\ &\quad \left. \left. - [\alpha_s] \frac{\Gamma(1 - \epsilon) \beta_0}{\mu^{2\epsilon} e^{\epsilon \gamma_E}} \frac{1}{\epsilon} \frac{|\mathcal{M}_0(z \cdot p_h, \dots)|^2}{z} \right] \right. \\ &\quad \left. + [\alpha_s] \frac{\Gamma^3(1 - \epsilon) \Gamma(1 + \epsilon)}{\Gamma(1 - 2\epsilon)} (2\rho_{hs})^{-\epsilon} E_h^{-2\epsilon} P_{[hs]h,i}^{\text{1L}}(z) \frac{|\mathcal{M}_0(z \cdot p_h, \dots)|^2}{z} \right\} ,\end{aligned}\tag{2.146}$$

where we just recall that $z = 1 - E_s/E_h$, and $p_{[hs]}^\mu = z \cdot p_h^\mu$. The functions $P_{[hs]h,i}^{\text{1L}}(z)$ are the *averaged initial-state one-loop Altarelli-Parisi splitting functions*, and read

$$\begin{aligned}P_{qq,i}^{\text{1L}}(z) &\stackrel{\text{def}}{=} \left\{ -\frac{C_A}{\epsilon^2} \left[\frac{\Gamma^2(1 - \epsilon) \Gamma^2(1 + \epsilon)}{\Gamma(1 - 2\epsilon) \Gamma(1 + 2\epsilon)} (1 - z)^{-\epsilon} + 2 \sum_{n=1}^{\infty} \epsilon^{2n} \operatorname{Li}_{2n}(1 - z) \right] \right. \\ &\quad \left. + \frac{2C_F}{\epsilon^2} \sum_{n=1}^{\infty} \epsilon^n \operatorname{Li}_n(1 - z) \right\} (1 - z)^{-\epsilon} P_{qq,i}(z) \\ &\quad + \left[\frac{C_A - C_F}{1 - 2\epsilon} \right] (1 - z)^{-\epsilon} \tilde{P}_{qq,i}^{\text{new}}(z) ,\end{aligned}\tag{2.147}$$

$$\begin{aligned}P_{qg,i}^{\text{1L}}(z) &\stackrel{\text{def}}{=} \left\{ \frac{C_A}{\epsilon^2} \left[\frac{\Gamma^2(1 - \epsilon) \Gamma^2(1 + \epsilon)}{\Gamma(1 - 2\epsilon) \Gamma(1 + 2\epsilon)} (1 - z)^{-\epsilon} - 2 + 2 \sum_{n=1}^{\infty} \epsilon^{2n} \operatorname{Li}_{2n}(1 - z) \right] \right. \\ &\quad \left. - \frac{2C_F}{\epsilon^2} \left[\frac{\Gamma^2(1 - \epsilon) \Gamma^2(1 + \epsilon)}{\Gamma(1 - 2\epsilon) \Gamma(1 + 2\epsilon)} (1 - z)^{-\epsilon} - 1 + \sum_{n=1}^{\infty} (-\epsilon)^n \operatorname{Li}_n(1 - z) \right] \right\} \\ &\quad \times (1 - z)^{-\epsilon} P_{qg,i}(z) + \left[\frac{C_A - C_F}{1 - 2\epsilon} \right] (1 - z)^{-\epsilon} \tilde{P}_{qg,i}^{\text{new}}(z) ,\end{aligned}\tag{2.148}$$

$$P_{gg,i}^{\text{1L}}(z) \stackrel{\text{def}}{=} \left\{ \frac{C_A}{\epsilon^2} \left[-\frac{\Gamma^2(1 - \epsilon) \Gamma^2(1 + \epsilon)}{\Gamma(1 - 2\epsilon) \Gamma(1 + 2\epsilon)} (1 - z)^{-\epsilon} + 2 \sum_{n=1}^{\infty} \epsilon^{2n-1} \operatorname{Li}_{2n-1}(1 - z) \right] \right\}$$

$$\begin{aligned}
& + \frac{C_A - C_F}{\epsilon^2} \left[\frac{2}{1 - 2\epsilon} - \epsilon \right] - [n_f - C_A(1 - \epsilon)] \left[\frac{2(1 - \epsilon)}{\epsilon(1 - 2\epsilon)(3 - 2\epsilon)} \right] \Big\} \\
& \times (1 - z)^{-\epsilon} P_{gq,i}(z), \tag{2.149}
\end{aligned}$$

$$\begin{aligned}
P_{gg,i}^{1L}(z) & \stackrel{\text{def}}{=} \frac{C_A}{\epsilon^2} \left[- \frac{\Gamma^2(1 - \epsilon)\Gamma^2(1 + \epsilon)}{\Gamma(1 - 2\epsilon)\Gamma(1 + 2\epsilon)} (1 - z)^{-\epsilon} + 2 \sum_{n=1}^{\infty} \epsilon^{2n-1} \text{Li}_{2n-1}(1 - z) \right] \\
& \times (1 - z)^{-\epsilon} P_{gg,i}(z) + \left[\frac{n_f - C_A(1 - \epsilon)}{(1 - \epsilon)(1 - 2\epsilon)(3 - 2\epsilon)} \right] (1 - z)^{-\epsilon} \tilde{P}_{gg,i}^{\text{new}}(z), \tag{2.150}
\end{aligned}$$

with $P_{\alpha\beta,i}$ given in Eq. (2.102), and

$$\begin{aligned}
\tilde{P}_{qq,i}^{\text{new}}(z) & = -C_F [z + \epsilon(1 - z)], \\
\tilde{P}_{qg,i}^{\text{new}}(z) & = T_R \left(\frac{z - \epsilon}{1 - \epsilon} \right), \\
\tilde{P}_{gg,i}^{\text{new}}(z) & = \frac{C_A}{1 - \epsilon} \left[z + 2\epsilon \left(\frac{1}{z} - 1 \right) \right]. \tag{2.151}
\end{aligned}$$

Looking back to Subsec. 2.3.2, we note that the tree-level splitting functions describing initial-state collinear limits could easily be related to those for final-state limits, cf. Eq. (2.102). This is not the case for the one-loop splitting functions in Eqs (2.141) and (2.146). The reason is that the final-state limit includes a factor of $(-s_{hs})^{-\epsilon}$ with $s_{hs} > 0$. When we take the real part in Eq. (2.141), we find

$$\begin{aligned}
\text{Re} [(-s_{hs})^{-\epsilon}] & = \text{Re} [(-2p_h \cdot p_s)^{-\epsilon}] = \text{Re} [(-1)^{-\epsilon}] (2\rho_{hs})^{-\epsilon} (E_h E_s)^{-\epsilon} \\
& = \cos(\epsilon\pi) (2\rho_{hs})^{-\epsilon} \frac{E_{[hs]}^{-2\epsilon}}{[z(1 - z)]^\epsilon}. \tag{2.152}
\end{aligned}$$

This is precisely what we wrote in (2.141). However, when we cross (2.141) to initial state, then $-s_{hs}$ turns positive, and we obtain

$$(-s_{hs})^{-\epsilon} = (2\rho_{hs})^{-\epsilon} (E_h E_s)^{-\epsilon} = (2\rho_{hs})^{-\epsilon} E_h^{-2\epsilon} (1 - z)^{-\epsilon}, \tag{2.153}$$

where $E_s = (1 - z)E_h$, and the factor $(1 - z)^{-\epsilon}$ has been absorbed into the definitions of $P_{[hs]h,i}^{1L}(z)$ in Eqs (2.147 - 2.150).

With that, we have presented the expressions for all the IR singularities that appear to NNLO in QCD. In what follows, we will employ these expressions to discuss ways to handle the IR singularities at this order, eventually demonstrating how they will cancel, leaving a finite result.

2.6 The need for IR regularization schemes

It is worth recapitulating what has been discussed throughout Ch. 1 and Ch. 2. We considered the behavior of QCD amplitudes that describe the radiation of a soft or collinear

parton(s), i.e., the singular limits of real emission corrections at NLO or NNLO. We observed the following at NLO:

- The leading singular behavior of QCD amplitudes in the soft and collinear limits gives rise to non-converging integrals in the phase space of the radiated partons.
- In the soft limit, QCD amplitudes factorize, as in Eq. (2.52), into:
 - a color-correlated amplitude describing the hard scattering,
 - an eikonal function (2.54), which depends on the momenta and color charges of the soft and hard partons.
- In the final-state collinear limit, the QCD amplitude factorizes, as in Eq. (2.78), into:
 - an amplitude describing the hard scattering with one of the final-state momenta rescaled to account for the collinear emission,
 - a splitting function (2.84), which depends on the types of the collinear partons and their energies, and gives the probability of the collinear emission.
- Similarly, in the initial-state collinear limit, the QCD amplitude factorizes as in Eq. (2.101).
- The divergent behavior can be regulated by performing the phase space integral in $d = 4 - 2\epsilon$ dimensions, which results in poles of $\mathcal{O}(\epsilon^{-1})$.
- These poles are guaranteed to cancel against the IR poles from the one-loop integration in the virtual corrections (2.30), leaving a finite result.

Analogous observation can be made at NNLO, although in this case, the factorization formulas and the cancellation is more complicated. One option is to integrate the real emission amplitude analytically over the entire phase space of the radiated partons, extract the $1/\epsilon$ poles, and cancel these against the virtual poles. However, this approach would result in the loss of all information about the kinematics of the radiated partons. While this may be acceptable in a few cases, it is generally unsuitable for the majority of interesting processes studied. One reason for this is that partons produce hadronic energy flows known as *jets*, and the properties of these jets often provide crucial information when analyzing experimental signatures. For this reason, it is common to apply cuts on jet observables, e.g. to enhance the number of signal events relative to the background. Since jets are typically defined through iterative algorithms, analytic integration over the phase space becomes unfeasible. Consequently, we cannot simply integrate over the entire phase space of the radiated partons, as these partons have observable effects that we want to preserve. However, soft or collinear radiated partons should not have an experimental signature because they are *unresolved*: soft partons have too low an energy to be detected, while two (or more) collinear partons cannot be distinguished from each other. We can define a broad class of observables, known as IR-safe observables, which remain unchanged in the presence of unresolved (i.e., soft or collinear) partons. Almost all observables routinely studied and reported at the LHC are

IR-safe. Therefore, one can integrate over the phase space of unresolved partons without losing information about the final state. The objective, then, is to *separate the resolved and unresolved phase spaces* of the radiated partons.

2.6.1 IR slicing and subtraction methods for differential cross sections

The two predominant methods for isolating and removing the divergences arising from singular phase space integrations are known as *slicing* and *subtraction*. Although this thesis focuses on subtraction, it is useful to first outline the key differences between these two techniques.

The slicing technique involves dividing phase space into regions where the IR singularities can be treated separately. Specifically, it works by introducing a *cutoff* parameter, typically denoted by δ_s , which slices the phase space into *resolved* and *unresolved* regions based on a property such as the energy or transverse momentum of partons, or the number of unresolved jets. The resolved region corresponds to the situation where the emitted particles have an energy and angular separation above a certain threshold, determined by the cutoff parameter δ_s . In this region, the matrix element does not contain IR divergences and can thus be integrated numerically. In contrast, in the unresolved region, the energy or angular separation of the emitted partons is below the threshold set by δ_s . In this case, the unresolved contributions can be approximated by their universal IR behavior. The terms omitted in this approximation vanish as a power of δ_s when it approaches zero, necessitating the choice of a small value for the parameter. In general, the computation of the resolved region for small values of δ_s leads to numerical instabilities (the smaller δ_s is, the closer we get to the singularity), hence there is a trade-off between smaller and larger values of δ_s . The resolved and unresolved regions depend logarithmically on the slicing parameter δ_s , and this dependence cancels only when the two contributions are combined, up to (logarithmic enhanced) power corrections. Indeed, the final result cannot depend on how the phase space is sliced; therefore, the cutoff parameter δ_s is chosen to be sufficiently small, while also ensuring that the final result is independent of it.

Unlike the slicing approach, subtraction schemes involve subtracting the divergent parts of the real emission contribution and adding back an analytically computed equivalent to the virtual corrections, so as to cancel the divergences. This results in a modified matrix element that is regularized and can be computed directly in four dimensions, while the integrated counterterm, which retains the explicit singularities, cancels against the corresponding virtual corrections. Ideally, the counterterms should be systematically constructed using process-independent components. We will discuss the philosophy of subtraction in more detail in the following subsection.

Comparing slicing and subtraction methods, we see that the cancellation of the IR singularities in slicing schemes occurs in a *non-local* manner. Indeed, in slicing, the divergence is not removed at the specific phase-space points where it occurs, but rather within a region of phase space defined by the cutoff. This is in contrast to subtraction schemes, which are (at least in principle) *local*, as they aim to construct a subtraction term that cancels the divergence exactly at every point in phase space where it arises.

Nonetheless, due to its approximate nature, the slicing approach can be more readily extended to higher perturbative orders.

At NLO, the regularization of IR divergences is a solved problem. There are two main approaches that are completely general and have been implemented in numerous public codes:

- *Catani-Seymour subtraction* [8],
- *Frixione-Kunszt-Signer subtraction* (FKS) [7, 126].

While NLO subtractions have been fully automated, this is not yet the case at NNLO, where the structure of IR singularities becomes significantly more complex, making a fully general subtraction procedure still out of reach. Nonetheless, tremendous progress has been made in differential NNLO calculations, effectively completing most of the relevant $2 \rightarrow 1$ and $2 \rightarrow 2$ processes [157]. Many different slicing and subtraction schemes for NNLO calculations are now available. Below is a list of some of the main approaches, presented in alphabetical order:

- *Antenna subtraction* [13, 14],
- *ColorFul subtraction* [43],
- *Geometric subtraction* [30],
- *Local analytic sector subtraction* [31, 158, 159],
- *Nested Soft-Collinear subtraction* (NSC) [23, 79, 80],
- *N-jettiness* [27, 28, 160],
- *Projection to Born* [40],
- *q_T -subtraction* [24],
- *Sector-improved residue subtraction* [19, 20, 26].

Before concluding this section, we focus on local subtraction schemes and aim to understand the main characteristics that they have in common. Very generically, we have a function F , which includes the matrix element squared for the emission of parton(s) $\mathbf{m}$ in perturbative QCD, and which we need to integrate over the d -dimensional phase space of the radiated parton(s), $[d^{(d)}p_{\mathbf{m}}]$. Thus we are interested in computing the integral

$$\mathcal{I} = \int [d^{(d)}p_{\mathbf{m}}] F(\mathbf{m}). \quad (2.154)$$

Now, suppose we have an operator $\mathcal{S}$, which has the following two properties when it acts on the function F :

- i) it reproduces the leading singular behavior of $F(\mathbf{m})$ in all soft and collinear limits;
- ii) it can be integrated analytically over the d -dimensional phase space $[dp_{\mathbf{m}}]$ (3.18) of the emitted parton.

Then, we can write

$$\int [d^{(d)}p_m] F(\mathbf{m}) = \int [d^{(d)}p_m] \mathcal{S} F(\mathbf{m}) + \int [d^{(d)}p_m] (\mathbb{1} - \mathcal{S}) F(\mathbf{m}). \quad (2.155)$$

Note that the operator $\mathcal{S}$ is *inside* the integral. Following property ii), the first term on the right-hand side of Eq. (2.155) can be analytically integrated, and will produce $1/\epsilon$ poles which describe the singular behavior of $\mathcal{S}$, and thus of the amplitude, as shown in Subsecs 2.3.1 and 2.3.2. In contrast, the second term on the right-hand side of Eq. (2.155) is regularized with respect to the divergence caused by $\mathcal{S}$, following from property i), and it can therefore be readily integrated in four dimensions. Typically, this is done numerically, e.g. by using Monte Carlo integration.

Following the philosophy in Eq. (2.155), we note that the cancellation of the IR singularities occurs in a *local* manner, because the divergence described by $\mathcal{S}$ is removed at the specific phase-space points where it occurs. The specific form of the subtraction operator $\mathcal{S}$ is ultimately irrelevant, provided it possesses the two properties mentioned earlier. Different subtraction schemes employ various forms of $\mathcal{S}$, but these are formally equivalent. For instance, in FKS subtraction,¹² $\mathcal{S}$ is constructed directly from the soft and collinear limits, S_m and C_{im} , defined in Eqs (2.53) and (2.79), respectively. The possibility of analytically integrating the subtraction term is due to the fact that the matrix element factorizes in these limits, as discussed in this chapter.

To cancel the poles generated by soft and real corrections (2.155), as discussed in Sec. 1.5, both virtual contributions and those arising from the renormalization of the PDFs must be introduced. The general structure of the poles in virtual contributions has been described in Subsec. 2.2.1. For the PDF part, the expression is derived from DGLAP equations (1.56), and explicit formulas are collected in Appendix A.3.2.

Here we close Part I of this thesis, dedicated to a general introduction to IR divergences in QCD. In Part II we will focus on a specific subtraction scheme, i.e., the Nested Soft-Collinear scheme.

¹²At NLO, the Nested Soft-Collinear subtraction scheme is identical to FKS.

Part II:

The Nested Soft-Collinear Subtraction Scheme

Old vs New approach to NSC

Contents

3.1	Notation	71
3.2	NSC at NLO	75
3.2.1	Real, virtual, and PDFs renormalization corrections	76
3.2.2	Obstacles to generalizing the subtraction formalism at NLO	82
3.3	The new approach applied to processes $\mathcal{A}_0$ and $\mathcal{B}_0$ at NLO	83
3.3.1	NLO corrections to $ab \rightarrow \{g\}_N + X$ (process $\mathcal{A}_0$)	83
3.3.2	Poles cancellation of process $\mathcal{A}_0$ at NLO	87
3.3.3	NLO corrections to $gq \rightarrow \{g\}_N + X$ (process $\mathcal{B}_0$)	91

In Part I, we discussed the universal factorization of QCD in the infrared (IR) limits, both at NLO and NNLO. We also examined the necessity of IR subtraction methods to obtain IR-safe, fully differential predictions, as well as the challenges that arise, particularly at NNLO.

In this Part, we discuss one such method, the so-called *Nested Soft-Collinear* (NSC) scheme. We begin at NLO, where the scheme coincides with the well-established FKS method. In this sense, the NSC can be viewed as a direct extension of FKS to NNLO. Specifically, this chapter focuses on the fundamentals of the NSC method, with particular emphasis on a detailed comparison between the historical approach and its generalization within this framework. Although subtraction is a solved problem at NLO, we revisit it to introduce the new notation, address the challenges of fully generalizing the scheme to NNLO, and outline the approach taken to overcome these difficulties, which constitutes the main objective of this thesis.

In this chapter, Sec. 3.1 is devoted to introducing the notation used in this and the following chapters. We provide a broad overview in Sec. 3.2, and demonstrate its application to the $X + \text{jet}$ process in Sec. 3.3. Sec. 3.3 is also dedicated to a generalization of the method applicable to arbitrary hadron production processes at NLO.

3.1 Notation

Before delving into the details of the Nested Soft-Collinear (NSC) subtraction scheme, in this section, we will provide a brief overview of the basic notation that will employ in

this chapter and the ones that follow.

We begin by considering the most general process at hadron colliders, namely the production of a color singlet X with N jets, i.e.,

$$p + p \rightarrow X + N \text{ jets} . \quad (3.1)$$

The cross section for this process is given by Eq. (1.55), reproduced here for convenience:

$$\begin{aligned} d\sigma_{pp} = & \sum_{ab} \int_0^1 dx_a dx_b f_a(x_a, \mu_F^2) f_b(x_b, \mu_F^2) \\ & \times d\hat{\sigma}_{ab} \left(x_a, x_b, \alpha_s(\mu_R^2), \frac{Q^2}{\mu_F^2}, \frac{Q^2}{\mu_R^2}, \hat{O} \right) \left[1 + \mathcal{O} \left(\frac{\Lambda_{\text{qcd}}^2}{Q^2} \right) \right] , \end{aligned} \quad (3.2)$$

Here,

$$a + b \rightarrow X + N \text{ jets} \quad (3.3)$$

corresponds to the partonic process, and the observable $\hat{O}$ ensures that the N jets are defined in an IR safe manner. We label the two initial-state partons of the Born as $\{a, b\}$, the N final-state partons of the Born as

$$\mathcal{H}_f \stackrel{\text{def}}{=} \{1, \dots, N\} , \quad (3.4)$$

and the list of all the partons at the Born level as

$$\mathcal{H} \stackrel{\text{def}}{=} \{a, b, 1, \dots, N\} \equiv \{a, b\} \cup \mathcal{H}_f . \quad (3.5)$$

In what follows, we will occasionally engage in a slight abuse of notation, by using the i -index of a parton to indicate its flavor f_i . This is done both to lighten the formulas and to align our notation with some well-established conventions in the literature. Additionally, in Eq. (3.2), we recall that $d\hat{\sigma}_{ab}$ is the partonic cross section in the ab partonic channel, $f_{a,b}$ are the parton distribution functions (PDFs), μ_R and μ_F are the renormalization and factorization scales, respectively, $\alpha_s(\mu)$ is the normalized strong coupling constant (1.34), and $\hat{O}$ is an observable, which provides (among other things) an IR-safe definition of the N -jet final state. The partonic cross section can be expanded in powers of $\alpha_s(\mu)$ as

$$d\hat{\sigma}_{ab} = d\hat{\sigma}_{ab}^{\text{LO}} + d\hat{\sigma}_{ab}^{\text{NLO}} + d\hat{\sigma}_{ab}^{\text{NNLO}} + \mathcal{O}(\alpha_s^{q+3}) . \quad (3.6)$$

The leading order (LO) term, also called *Born level*, is proportional to $\mathcal{O}(\alpha_s^q)$, and the next-to-leading order (NLO) and next-to-next-to-leading order (NNLO) terms are corrections of $\mathcal{O}(\alpha_s)$ and $\mathcal{O}(\alpha_s^2)$ to the LO term, respectively, that is, they are of $\mathcal{O}(\alpha_s^{q+1})$ and $\mathcal{O}(\alpha_s^{q+2})$, respectively.¹ See Secs 1.4 and 1.5 for more details.

The computation of partonic cross sections and kinematic distributions requires integrating matrix elements squared over phase spaces of relevant final states. For a generic process, we find it convenient to treat matrix elements as vectors in color space.

¹Note that we have suppressed the arguments of all the functions for brevity.

In this regard, Sec. 2.1 was entirely devoted to the introduction and discussion of color space and color algebra. Here we recall the following two definition:

$$|\mathcal{M}(\{p\}_{\mathcal{H}})|^2 = \langle \mathcal{M}(\{p\}_{\mathcal{H}}) | \mathcal{M}(\{p\}_{\mathcal{H}}) \rangle, \quad (3.7)$$

$$|\mathcal{M}^{(ij)}(\{p\}_{\mathcal{H}})|^2 \stackrel{\text{def}}{=} \langle \mathcal{M}(\{p\}_{\mathcal{H}}) | \mathbf{T}_i \cdot \mathbf{T}_j | \mathcal{M}(\{p\}_{\mathcal{H}}) \rangle, \quad (3.8)$$

where $\mathbf{T}_{i,j}$ is the color charge operator of parton $i, j \in \mathcal{H}$. To address color-correlations (3.8), it is convenient to introduce a tensor product of leading-order matrix elements $|\mathcal{M}_0\rangle \equiv |\mathcal{M}_0(\{p\}_{\mathcal{H}})\rangle$ in color space. Therefore, we define the function

$$\begin{aligned} \tilde{F}_{\text{LM}}^{ab}[\{i\}_{\mathcal{H}_f}; X] &\stackrel{\text{def}}{=} |\mathcal{M}_0(p_a, p_b, p_1, \dots, p_N; p_X)\rangle \otimes \langle \mathcal{M}_0(p_a, p_b, p_1, \dots, p_N; p_X)| \\ &\quad \times \text{dLips}_X \hat{O}(p_1, \dots, p_N; p_X), \\ &\equiv |\mathcal{M}_0(\{p\}_{\mathcal{H}}; p_X)\rangle \otimes \langle \mathcal{M}_0(\{p\}_{\mathcal{H}}; p_X)| \text{dLips}_X \hat{O}(\{p\}_{\mathcal{H}_f}; p_X) \end{aligned} \quad (3.9)$$

to describe the partonic process (3.3) at LO. In Eq. (3.9), the symbol $\otimes$ indicates a tensor product in color space, and dLips_X the Lorentz-invariant phase space (cf. Eq. (1.50))

$$\text{dLips}_X \stackrel{\text{def}}{=} \left(\prod_{i \in X} \frac{d^{(3)}\mathbf{p}_i}{(2\pi)^3} \frac{1}{2E_i} \right) (2\pi)^d \delta^{(d)} \left(p_a + p_b - p_X - \sum_{k \in \mathcal{H}_f} p_k \right) \quad (3.10)$$

for the colorless system X , which also includes the momentum-conserving delta function. The matrix element squared is obtained by taking the trace in color space

$$\text{Tr} \left[\tilde{F}_{\text{LM}}^{ab}[\{i\}_{\mathcal{H}_f}; X] \right] = |\mathcal{M}_0(\{p\}_{\mathcal{H}}; p_X)|^2 \text{dLips}_X \hat{O}(\{p\}_{\mathcal{H}_f}; p_X). \quad (3.11)$$

This element will be the fundamental building block in our subtraction scheme, and we will refer to it as

$$\begin{aligned} F_{\text{LM}}^{ab}[\{i\}_{\mathcal{H}_f}] &\stackrel{\text{def}}{=} \text{Tr} \left[\tilde{F}_{\text{LM}}^{ab}[\{i\}_{\mathcal{H}_f}; X] \right] \\ &\equiv |\mathcal{M}_0(\{p\}_{\mathcal{H}}; p_X)|^2 \text{dLips}_X \hat{O}(\{p\}_{\mathcal{H}_f}; p_X). \end{aligned} \quad (3.12)$$

Note that in $F_{\text{LM}}^{ab}[\{i\}_{\mathcal{H}_f}]$ we leave the color singlet X understood to lighten the notation. This notation allows us to introduce, in a fairly straightforward way, functions of color-space operators that act on the matrix elements. Denoting such a function as $\mathcal{A}$, we introduce the notation

$$\begin{aligned} \mathcal{A} \cdot F_{\text{LM}}^{ab}[\{i\}_{\mathcal{H}_f}] &\stackrel{\text{def}}{=} \text{Tr} \left[\mathcal{A} \tilde{F}_{\text{LM}}^{ab}[\{i\}_{\mathcal{H}_f}; X] \right] \\ &\equiv \langle \mathcal{M}_0(\{p\}_{\mathcal{H}}; p_X) | \mathcal{A} | \mathcal{M}_0(\{p\}_{\mathcal{H}}; p_X) \rangle \text{dLips}_X \hat{O}(\{p\}_{\mathcal{H}_f}; p_X). \end{aligned} \quad (3.13)$$

In particular, in this thesis we will need to consider the following color-correlated functions (see Secs 2.2 and 2.4.1):

$$F_{\text{LM}}^{(ij)} \stackrel{\text{def}}{=} (\mathbf{T}_i \cdot \mathbf{T}_j) \cdot F_{\text{LM}} = \langle \mathcal{M}_0 | \mathbf{T}_i \cdot \mathbf{T}_j | \mathcal{M}_0 \rangle \text{dLips}_X \hat{O},$$

$$\begin{aligned}
F_{\text{LM}}^{(ijk)} &\stackrel{\text{def}}{=} f_{abc}(T_i^a T_j^b T_k^c) \cdot F_{\text{LM}} = \langle \mathcal{M}_0 | f_{abc}(T_i^a T_j^b T_k^c) | \mathcal{M}_0 \rangle \text{dLips}_X \hat{O}, \\
F_{\text{LM}}^{(ij)(kl)} &\stackrel{\text{def}}{=} \{\mathbf{T}_i \cdot \mathbf{T}_j, \mathbf{T}_k \cdot \mathbf{T}_l\} \cdot F_{\text{LM}} = \langle \mathcal{M}_0 | \{\mathbf{T}_i \cdot \mathbf{T}_j, \mathbf{T}_k \cdot \mathbf{T}_l\} | \mathcal{M}_0 \rangle \text{dLips}_X \hat{O},
\end{aligned} \tag{3.14}$$

where the arguments of $|\mathcal{M}_0\rangle$, and $\hat{O}$ are understood.

The LO partonic cross section can be obtained by integrating $F_{\text{LM}}(\{i\}_{\mathcal{H}})$ over the phase space of the final-state partons. We write

$$\langle F_{\text{LM}}^{ab}[\{i\}_{\mathcal{H}_f}] \rangle \stackrel{\text{def}}{=} 2s_{ab} \text{d}\hat{\sigma}_{ab}^{\text{LO}} = \mathcal{N} \int \left(\prod_{i \in \mathcal{H}_f} [\text{d}p_i] \right) F_{\text{LM}}^{ab}[\{i\}_{\mathcal{H}_f}], \tag{3.15}$$

where

$$s_{ab} = (p_a + p_b)^2 = 2p_a \cdot p_b \tag{3.16}$$

is the *partonic center-of-mass energy squared*, the angular brackets $\langle \dots \rangle$ indicate the integration over the final-state phase space, and $\mathcal{N}$ is a normalization factor that takes into account color and spin averages, but no symmetry factors. We emphasize that this is a different convention from that assumed in Ref. [81]. This is because, in Ref. [81], only the gluon final states were considered, while here we will also have quark(s) in the final state, and explicitly writing the symmetry factors makes it easier to keep track of all of them.

We saw in section Subsec. 2.3.2 that an initial-state collinear singularity results in one of the incoming momenta being rescaled by a factor z . If one of the initial-state partons, say a , is involved in a splitting process, and its momentum is scaled by a factor z under the action of the collinear operator C_{im} in Eq. (2.79), then we express this as

$$F_{\text{LM}}^{(z \cdot a)b}[\{i\}_{\mathcal{H}_f}] \equiv F_{\text{LM}}(z \cdot a, b, \{i\}_{\mathcal{H}_f}). \tag{3.17}$$

Finally, $[\text{d}p_i]$ in Eq. (3.15) is the phase-space element of a final-state parton i ,

$$[\text{d}p_i] \stackrel{\text{def}}{=} \frac{\text{d}^{(d-1)}\mathbf{p}_i}{(2\pi)^{d-1}(2E_i)} \Theta(E_{\text{max}} - E_i). \tag{3.18}$$

In contrast to the definition given in Eq. (2.57), here we include the function $\Theta(E_{\text{max}} - E_i)$ in the definition of $[\text{d}p_i]$. Specifically, we assume that the condition $E_i \leq E_{\text{max}}$ holds for every parton $i \in \mathcal{H}_f$, regardless of the process under consideration.²

As a final comment, we note that the definition (3.12), which generally applies to tree-level amplitudes squared, can also be extended to the one-loop, two-loop, and real-virtual cases. We will introduce these definitions in the following sections.

²We recall that the parameter E_{max} , introduced on page 43, is a dimensionful parameter that can be determined by energy conservation requirements and provides an upper bound on the energies of individual partons [23].

3.2 NSC at NLO

In Sec. 2.6, we discussed the necessity of introducing IR regularization schemes to handle the various types of soft and collinear divergences introduced in Ch. 2. We observed that there are two predominant methods for isolating and removing the divergences arising from loops and phase space integrations, known as slicing and subtraction. As stated earlier, throughout the remainder of this thesis, we focus on a particular subtraction method, the Nested Soft-Collinear scheme. In Sec. 2.6, we reviewed the general idea behind subtraction schemes. We now turn our attention to understanding how the NSC specifically works, beginning with its application at NLO.

It is well-known that at NLO, the NSC coincides in all respects with the FKS approach [7, 126]. Nevertheless, it is worthwhile to revisit the essential steps of the NSC at NLO, as this will provide insights into the motivation for generalizing the method. Following the philosophy of the FKS approach, the NSC is characterized as a *fully local* and *fully analytic* method. Recall that *fully local* means that the cancellation of IR singularities occurs in a *local* manner, that is, divergences described by soft and collinear operators are removed at the precise phase-space points where they arise. As for *fully analytic*, this indicates that poles are computed and cancelled analytically.

The FKS/NSC method is designed by exploiting two key properties of scattering amplitudes. The first property is their factorization in the soft and collinear limits into a product of universal kernels and lower-multiplicity on-shell amplitudes, extensively described in Ch. 2. The second property is QCD *color coherence* [161–163], which is most commonly considered in the context of parton showers. Consider, for instance, the emission of a soft gluon from a quark line, which then undergoes multiple splittings that may or may not be collinear. The wavelength of the soft gluon is too large to resolve the details of the splittings, including any potential collinear singularities: it only depends on the total color charge of all the radiated partons. Therefore, the factorization of the amplitude as a result of the soft radiation is insensitive to any other radiation. As a result, *soft and collinear emissions can be treated independently of one another*, and the soft and collinear limits are sufficient to describe and regulate all singularities in arbitrary real-emission NNLO QCD scattering amplitudes (in other words, they commute). In FKS/NSC, these features are employed to build up an iterative subtraction procedure that starts with the subtraction of soft divergences.

To see the NSC in action, we analyze the LO process in which a quark-antiquark pair annihilate to produce a color singlet system X and a gluon

$$q_a + \bar{q}_b \rightarrow X + g. \quad (3.19)$$

Our goal now is to write a fully IR-finite formula for this process at NLO. Before delving into the details of this process, we must fix the reference frame where we set calculation. The center-of-mass frame of the incoming partons ($E_a = E_b$) is a common choice. In this thesis, we prefer to be as general as possible, and we set the calculation in an arbitrary reference frame, that is, E_a and E_b can be any two energies (if desired, the condition $E_a = E_b$ can be restored a posteriori). We emphasize that once the reference frame

is fixed, it cannot be changed when calculating the counterterms for soft and collinear divergences, since our construction of the subtraction terms is frame-dependent and not Lorentz-invariant.

3.2.1 Real, virtual, and PDFs renormalization corrections

We start with the real emission corrections at NLO, which come from the process

$$q_a + \bar{q}_b \rightarrow X + \{g\}_2, \quad (3.20)$$

where $\{g\}_2$ labels a set of two gluons. We write these corrections as

$$2s_{ab} d\bar{\sigma}_{q\bar{q}}^R = \left\langle \frac{1}{2!} F_{\text{LM}}^{q\bar{q}}[\{g\}_2] \right\rangle. \quad (3.21)$$

where we keep the $1/2!$ symmetry factor explicit, and the reason why will become clear as we continue.

The observable $\mathcal{O}$ in the definition (3.12) of F_{LM} requires at least one resolved parton, hence one and only one parton of the final-state gluons in the above equation can become unresolved. To identify the unresolved parton, we introduce *damping factors* $\Delta^{(i)}$ such that they provide a partition of unity,

$$1 = \sum_{i \in \{g\}_2} \Delta^{(i)}. \quad (3.22)$$

The explicit form of the damping factors that we employ in this thesis can be found in Appendix B. They are constructed in such a way that a damping factor $\Delta^{(i)}$ vanishes when any parton, with the exception of parton i , becomes either soft or collinear to any other parton, including the incoming ones. This implies that in the combination $\Delta^{(i)} F_{\text{LM}}$, only soft and collinear limits of parton i can lead to non-integrable singularities and, eventually, to the appearance of $1/\epsilon^n$ poles. We then write

$$\left\langle \frac{1}{2!} F_{\text{LM}}^{q\bar{q}}[\{g\}_2] \right\rangle = \sum_{i \in \{g\}_2} \left\langle \frac{1}{2!} \Delta^{(i)} F_{\text{LM}}^{q\bar{q}}[\{g\}_2] \right\rangle. \quad (3.23)$$

Since we have only gluons in the final state, F_{LM} is unchanged under a permutation of the final-state partons. Then we obtain

$$\left\langle \frac{1}{2!} F_{\text{LM}}^{q\bar{q}}[\{g\}_2] \right\rangle = \langle \Delta^{(\text{m})} F_{\text{LM}}^{q\bar{q}}[g|\mathbf{m}_g] \rangle. \quad (3.24)$$

Note that, in the above result, we have relabelled the arguments of F_{LM} in such a way that the damping factors become identical for each term in the sum and we denote the potentially-unresolved gluon as $\mathbf{m}_g$. We have also introduced the following notation:

$$F_{\text{LM}}^{q\bar{q}}[\{g\}_{N_g}|\mathbf{m}_g] \stackrel{\text{def}}{=} \frac{1}{N_g!} F_{\text{LM}}^{q\bar{q}}[\{g\}_{N_g+1}]. \quad (3.25)$$

Essentially, $\{g\}_{N_g}$ denotes the set of *hard gluons*, and we incorporate the symmetry factor $1/N_g!$ within $F_{\text{LM}}^{ab}[\{g\}_{N_g}|\mathbf{m}_g]$. In our specific case, $N_g = 1$, so we leave it implicit. The remaining final-state gluon is resolved. Finally, we point out that the soft and collinear operators acting on the damping factors lead to the following results³

$$S_{\text{m}}\Delta^{(\text{m})} = 1, \quad C_{a\text{m}}\Delta^{(\text{m})} = 1, \quad C_{i\text{m}}\Delta^{(\text{m})} = \frac{E_{\text{m}}}{E_i + E_{\text{m}}} \stackrel{\text{def}}{=} z_{i,\text{m}}, \quad (3.26)$$

where a and i are an initial- and final-state partons, respectively.

At this point, we can proceed to regularize the real-emission term $\langle F_{\text{LM}}^{q\bar{q}}[g|\mathbf{m}_g] \rangle$ using the NSC framework outlined in Sec. 2.6. We start from the soft limit, which yields:

$$\langle \Delta^{(\text{m})} F_{\text{LM}}^{q\bar{q}}[g|\mathbf{m}_g] \rangle = \langle S_{\text{m}} F_{\text{LM}}^{q\bar{q}}[g|\mathbf{m}_g] \rangle + \langle (\mathbb{1} - S_{\text{m}})\Delta^{(\text{m})} F_{\text{LM}}^{q\bar{q}}[g|\mathbf{m}_g] \rangle. \quad (3.27)$$

The first term on the right-hand side of (3.27) is the *soft counterterm*, and it contains pure-soft divergences of $\langle F_{\text{LM}}^{q\bar{q}}[g|\mathbf{m}_g] \rangle$. The second term on the right-hand side of (3.27) is *soft-regulated* but, it still contains collinear singularities, whose regularization is generally more challenging due to their *overlapping* nature. To be precise, at NLO, collinear singularities do not overlap, since only one parton is unresolved. Thus, in principle, we do not need to partition the phase space. However, it is a convenient thing to do, for two reasons:

- i) We will have to partition the phase space at NNLO.
- ii) It is physically intuitive: each resolved parton defines a direction to which the unresolved one can be collinear. We then partition the phase space such that each direction is associated with a partition.

Therefore, we introduce a set of partition functions $\{\omega^{\text{mi}}, i \in \mathcal{H}\}$ that satisfy the conditions

$$1 = \sum_{i \in \mathcal{H}} \omega^{\text{mi}}, \quad C_{j\text{m}} \omega^{\text{mi}} = \delta_{ij}. \quad (3.28)$$

Generic expressions that satisfy these constraints are presented in Eq. (B.23). More generally, Appendix B contains a detailed discussion of partition functions for both NLO and NNLO corrections. Thanks to ω^{mi} functions, we can write

$$\mathbb{1} - \sum_{i \in \mathcal{H}} C_{i\text{m}} = \sum_{i \in \mathcal{H}} (\mathbb{1} - C_{i\text{m}}) \omega^{\text{mi}}, \quad (3.29)$$

and we use this formula to remove collinear singularities from the second term on the right-hand side of Eq. (3.27) as

$$\begin{aligned} \langle \Delta^{(\text{m})} F_{\text{LM}}^{q\bar{q}}[g|\mathbf{m}_g] \rangle &= \langle S_{\text{m}} F_{\text{LM}}^{q\bar{q}}[g|\mathbf{m}_g] \rangle + \sum_{i \in \mathcal{H}} \langle \bar{S}_{\text{m}} C_{i\text{m}} \Delta^{(\text{m})} F_{\text{LM}}^{q\bar{q}}[g|\mathbf{m}_g] \rangle \\ &\quad + \langle \mathcal{O}_{\text{NLO}}^{(\text{m})} \Delta^{(\text{m})} F_{\text{LM}}^{q\bar{q}}[g|\mathbf{m}_g] \rangle, \end{aligned} \quad (3.30)$$

³Derivation of these results can be found in Appendix B.

where we have introduced the following shorthand notation:

$$\begin{aligned}\bar{S}_m &\stackrel{\text{def}}{=} \mathbb{1} - S_m, & \mathcal{O}_{\text{NLO}}^{(i,m)} &\stackrel{\text{def}}{=} \bar{S}_m \bar{C}_{im}, \\ \bar{C}_{im} &\stackrel{\text{def}}{=} \mathbb{1} - C_{im}, & \mathcal{O}_{\text{NLO}}^{(m)} &\stackrel{\text{def}}{=} \sum_{i \in \mathcal{H}} \mathcal{O}_{\text{NLO}}^{(i,m)} \omega^{m_i}.\end{aligned}\quad (3.31)$$

In Eq. (3.30), as previously stated, the first term on the right-hand side represents the *soft counterterm*. The second term is soft-regulated but gives rise to collinear singularities. We will frequently refer to this as the *hard-collinear term*, where *hard* indicates that the unresolved parton m_g is prevented from becoming soft. The third term is the *fully-regulated term*, as it no longer contains any soft or collinear singularities. This term can be integrated in $d = 4$ dimensions, typically using Monte Carlo integration.

We now need to perform the integration over the unresolved phase space of gluon m_g in the soft and hard-collinear counterterms. The former can be computed as in Subsec. 2.3.1. From Eq. (2.52) we read

$$S_m F_{\text{LM}}^{q\bar{q}}[g|m_g] = -g_{s,b}^2 \mu_0^{2\epsilon} \sum_{(ij)} \text{Eik}_{ij}(m) (\mathbf{T}_i \cdot \mathbf{T}_j) \cdot F_{\text{LM}}^{q\bar{q}}[g], \quad (3.32)$$

with

$$\text{Eik}_{ij}(m) \stackrel{\text{def}}{=} \frac{(p_i \cdot p_j)}{(p_i \cdot p_m)(p_j \cdot p_m)}, \quad (3.33)$$

and $(\mathbf{T}_i \cdot \mathbf{T}_j) \cdot F_{\text{LM}}^{q\bar{q}}[g]$ defined in Eq. (3.14). Integrating over the final phase space $[dp_m]$, we obtain (cf. Eq. (2.71))

$$\langle S_m F_{\text{LM}}^{q\bar{q}}[g|m_g] \rangle = [\alpha_s] \left\langle -\frac{(2E_{\text{max}}/\mu)^{-2\epsilon}}{\epsilon^2} \sum_{(ij)} \eta_{ij}^{-\epsilon} K_{ij}(\mathbf{T}_i \cdot \mathbf{T}_j) \cdot F_{\text{LM}}^{q\bar{q}}[g] \right\rangle, \quad (3.34)$$

where the strong coupling constant $[\alpha_s]$ is defined in Eq. (2.25), and (ij) stands for $i, j \in \mathcal{H}$, with $i \neq j$. We note that $(\mathbf{T}_i \cdot \mathbf{T}_j) \cdot F_{\text{LM}}^{q\bar{q}}[g]$ is the Born cross section, but instead of the matrix element squared, it contains *color-correlated* matrix element squared. In principle, since we have 3 hard partons, we could write the product $\mathbf{T}_i \cdot \mathbf{T}_j$ in terms of Casimirs invariants, and make it explicit that Eq. (3.34) can be expressed in term of the LO cross section. However, for the time being, we prefer to keep Eq. (3.34) in this form.

We turn now to the hard-collinear term, we start with the initial-state collinear factorization formula (2.101), i.e.,

$$C_{am} \Delta^{(m)} F_{\text{LM}}^{q\bar{q}}[g|m_g] = \frac{g_{s,b}^2 \mu_0^{2\epsilon}}{p_i \cdot p_m} P_{qq,i}(z) \frac{F_{\text{LM}}^{(z,q)\bar{q}}[g]}{z}, \quad (3.35)$$

where $z = 1 - E_m/E_a$, $C_{am} \Delta^{(m)} = 1$, $F_{\text{LM}}^{(z,q)\bar{q}}[g]$ is given in Eq. (3.17), and $P_{qq,i}(z)$ is the initial-state splitting function defined in Eq. (2.102). From Eq. (2.109), we read the

integration over $[dp_m]$, which yields

$$\begin{aligned} \langle \bar{S}_m C_{am} \Delta^{(m)} F_{LM}^{q\bar{q}}[g|m_g] \rangle = & - \frac{[\alpha_s]}{\epsilon} \left\langle \left(\frac{2E_a}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \right. \\ & \times \int_{z_{\min}}^1 dz \bar{S}_z \left[(1-z)^{-2\epsilon} P_{qq,i}(z) \right] \frac{F_{LM}^{(z,q)\bar{q}}[g]}{z} \Bigg\rangle, \end{aligned} \quad (3.36)$$

where $z_{\min} = 1 - E_{\max}/E_a$. Notice that, in contrast with Eq. (2.109), the integral over dz here is soft-regulated through the operator $\bar{S}_z \equiv \bar{S}_m$ in the limit $z \rightarrow 1$. We can rewrite this integral as

$$\begin{aligned} & \int_{z_{\min}}^1 dz \bar{S}_z \left[(1-z)^{-2\epsilon} P_{qq,i}(z) \right] \frac{F_{LM}^{(z,q)\bar{q}}[g]}{z} \\ &= \int_0^1 dz \left[(1-z)^{-2\epsilon} P_{qq,i}(z) \right] \frac{F_{LM}^{(z,q)\bar{q}}[g]}{z} - \int_{z_{\min}}^1 dz \frac{2C_F}{(1-z)^{1+2\epsilon}} F_{LM}^{q\bar{q}}[g], \end{aligned} \quad (3.37)$$

where, in the first integral on the right-hand side, we set the lower bound to 0 because the δ -factor within $F_{LM}^{(z,q)\bar{q}}[g]$ defines the physical region where an initial-state splitting process with a negative momentum fraction z is forbidden (see discussion on page 53). This constraint, however, does not apply to the second integral, where we have taken the limit

$$\lim_{z \rightarrow 1} P_{qq}(z) = \frac{2C_F}{1-z}. \quad (3.38)$$

After the action of the soft limit S_z , $F_{LM}^{q\bar{q}}$ becomes independent of z , meaning there is no z -dependent δ -factor to exclude the integral over the kinematic region $[z_{\min}, 0]$. To describe this with convenient notation, we introduce the function (cf. Appendix A.4)

$$\begin{aligned} \int_0^1 dz \mathcal{P}_{qq}^{(k)}(z, E_a) g(z) \stackrel{\text{def}}{=} & \int_0^1 dz \left[(1 - S_z) \left[(1-z)^{-k\epsilon} P_{qq,i}(z) \right] \right. \\ & \left. - 2T_q^2 \frac{1 - e^{-k\epsilon L_a}}{k\epsilon} \delta(1-z) \right] g(z), \end{aligned} \quad (3.39)$$

where $k = 2$ at NLO and $k = 4$ at NNLO, $L_i = \log(E_{\max}/E_i)$, and $g(z)$ represents a generic function of the momentum fraction z . Furthermore, we also introduce the shorthand

$$\mathcal{P}_{qq}^{(k)} \otimes F_{LM}^{q\bar{q}}[g] \stackrel{\text{def}}{=} \int_0^1 dz \mathcal{P}_{qq}^{(k)}(z, E_a) \frac{F_{LM}^{(z,q)\bar{q}}[g]}{z} \quad (3.40)$$

if the z -boost is along leg a (in this case, the quark), and

$$F_{LM}^{q\bar{q}}[g] \otimes \mathcal{P}_{qq}^{(k)} \stackrel{\text{def}}{=} \int_0^1 dz \mathcal{P}_{qq}^{(k)}(z, E_b) \frac{F_{LM}^{a(z,b)}[g]}{z} \quad (3.41)$$

if along leg b (in this case, the antiquark). Therefore, using the above definitions, we can

express Eq. (3.36) in a compact form as

$$\langle \bar{S}_m C_{am} \Delta^{(m)} F_{LM}^{q\bar{q}}[g|\mathbf{m}_g] \rangle = -\frac{[\alpha_s]}{\epsilon} \left\langle \left(\frac{2E_a}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} (\mathcal{P}_{q\bar{q}}^{(2)} \otimes F_{LM}^{q\bar{q}}[g]) \right\rangle. \quad (3.42)$$

Similarly, leg b yields

$$\langle \bar{S}_m C_{bm} \Delta^{(m)} F_{LM}^{q\bar{q}}[g|\mathbf{m}_g] \rangle = -\frac{[\alpha_s]}{\epsilon} \left\langle \left(\frac{2E_b}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} (F_{LM}^{q\bar{q}}[g] \otimes \mathcal{P}_{q\bar{q}}^{(2)}) \right\rangle. \quad (3.43)$$

Finally, the configuration where the final state gluon $\mathbf{m}_g$ becomes collinear to the other final state gluon. The factorization formula is given in Eq. (2.78), and reads

$$C_{1m} \Delta^{(m)} F_{LM}^{q\bar{q}}[g|\mathbf{m}_g] = \frac{g_{s,b}^2 \mu_0^{2\epsilon}}{p_1 \cdot p_m} z P_{gg}(z) F_{LM}^{q\bar{q}}[g|_{1m}], \quad (3.44)$$

where $[1m]$ is the clustered gluon with energy $E_{[1m]} = E_1 + E_m$, $E_m = (1-z)E_{[1m]}$, and $P_{gg}(z)$ is the final-state splitting function defined in Eq. (2.84). Note that the additional factor of z in Eq. (3.44) originates from the collinear limit of the damping factor, $C_{im} \Delta^{(m)} = z$. From Eq. (2.97) we read the integration over $[dp_m]$, which yields

$$\begin{aligned} \langle \bar{S}_m C_{im} \Delta^{(m)} F_{LM}^{q\bar{q}}[g|\mathbf{m}_g] \rangle = & -\frac{[\alpha_s]}{\epsilon} \left\langle \left(\frac{2E_{[1m]}}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \right. \\ & \times \int_{z_{\min}}^1 dz \bar{S}_z [z(1-z)]^{-2\epsilon} z P_{gg}(z) F_{LM}^{q\bar{q}}[g|_{1m}] \Big\rangle, \end{aligned} \quad (3.45)$$

Since F_{LM} is independent of z , we can compute the integral over z (see the discussion in Subsec. 2.3.2). Similarly to the initial-state case, we define the object (cf. Appendix A.4)

$$\begin{aligned} \gamma_{g(z), f_i \rightarrow f_i g}^{nk}(L_i) \stackrel{\text{def}}{=} & -\int_0^1 dz (\mathbb{1} - S_z) \left[z^{-n\epsilon} (1-z)^{-k\epsilon} g(z) P_{f_i f_i}(z) \right] \\ & + 2\mathbf{T}_{f_i}^2 \frac{1 - e^{-k\epsilon L_i}}{k\epsilon} g(1), \end{aligned} \quad (3.46)$$

where $g(z)$ represents any regular function of z , and we use it to write (3.45) as

$$\langle \bar{S}_m C_{1m} F_{LM}^{q\bar{q}}[g|\mathbf{m}_g] \rangle = \frac{[\alpha_s]}{\epsilon} \left\langle \left(\frac{2E_1}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \gamma_{z,g \rightarrow gg}^{22}(L_1) F_{LM}^{q\bar{q}}[g] \right\rangle, \quad (3.47)$$

where we relabelled $[1m] \mapsto 1$.

We can now combine the soft corrections with the hard-collinear corrections for both initial- and final-state radiation in Eqs (3.42) and (3.47), respectively, to obtain the following compact expression for the real corrections at NLO:

$$2s_{ab} d\bar{\sigma}_{q\bar{q}}^R = \langle \Delta^{(m)} F_{LM}^{q\bar{q}}[g|\mathbf{m}_g] \rangle$$

$$\begin{aligned}
&= [\alpha_s] \left\langle - \frac{(2E_{\max}/\mu)^{-2\epsilon}}{\epsilon^2} \sum_{(ij)} \eta_{ij}^{-\epsilon} K_{ij} (\mathbf{T}_i \cdot \mathbf{T}_j) \cdot F_{\text{LM}}^{q\bar{q}}[g] \right\rangle \\
&\quad - \frac{[\alpha_s]}{\epsilon} \left\langle \left(\frac{2E_a}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} (\mathcal{P}_{qq}^{(2)} \otimes F_{\text{LM}}^{q\bar{q}}[g]) \right\rangle \\
&\quad - \frac{[\alpha_s]}{\epsilon} \left\langle \left(\frac{2E_b}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} (F_{\text{LM}}^{q\bar{q}}[g] \otimes \mathcal{P}_{qq}^{(2)}) \right\rangle \\
&\quad + \frac{[\alpha_s]}{\epsilon} \left\langle \left(\frac{2E_1}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \gamma_{z,g \rightarrow gg}^{22}(L_1) F_{\text{LM}}^{q\bar{q}}[g] \right\rangle \\
&\quad + \langle \mathcal{O}_{\text{NLO}}^{(\text{m})} \Delta^{(\text{m})} F_{\text{LM}}^{q\bar{q}}[\mathbf{m}_g] \rangle.
\end{aligned} \tag{3.48}$$

Eq. (3.48) contains all the IR divergences of the real emission contribution $q_a + \bar{q}_b \rightarrow \{g\}_2 + X$, expressed as poles in $1/\epsilon$. Obtaining the IR poles of the other contributions is straightforward. Indeed, we can extract the virtual corrections from Eq. (2.20) and find

$$\begin{aligned}
2s_{ab} \, d\bar{\sigma}_{q\bar{q}}^{\text{V}} &= \langle F_{\text{LV}}^{q\bar{q}}[g] \rangle \\
&= \left[\frac{\alpha_s(\mu)}{2\pi} \right] \frac{1}{2} \frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)} \left\langle \sum_{(ij)} \mathcal{V}_i^{\text{sing}}(\epsilon) \left(\frac{\mu^2}{2p_i \cdot p_j} \right)^\epsilon 2 \text{Re} \left[e^{i\pi\lambda_{ij}\epsilon} \right] \right. \\
&\quad \times \langle \mathbf{T}_i \cdot \mathbf{T}_j \rangle \cdot F_{\text{LM}}^{q\bar{q}}[g] \left. \right\rangle + \langle F_{\text{LV,fin}}^{q\bar{q}}[g] \rangle,
\end{aligned} \tag{3.49}$$

where $F_{\text{LV}}^{ab}[\{i\}_{\mathcal{H}_f}]$ and $F_{\text{LV,fin}}^{ab}[\{i\}_{\mathcal{H}_f}]$ are defined in analogy with (3.13) as

$$\begin{aligned}
F_{\text{LV}}^{ab}[\{i\}_{\mathcal{H}_f}] &\stackrel{\text{def}}{=} \left[\frac{\alpha_s(\mu)}{2\pi} \right] 2 \text{Re} \left[\langle \mathcal{M}_0(\{p\}_{\mathcal{H}}, p_X) | \mathcal{M}_1(\{p\}_{\mathcal{H}}, p_X) \rangle \right] \\
&\quad \times \text{dLips}_X \hat{O}(\{p\}_{\mathcal{H}_f}; p_X),
\end{aligned} \tag{3.50}$$

$$\begin{aligned}
F_{\text{LV,fin}}^{ab}[\{i\}_{\mathcal{H}_f}] &\stackrel{\text{def}}{=} \left[\frac{\alpha_s(\mu)}{2\pi} \right] 2 \text{Re} \left[\langle \mathcal{M}_0(\{p\}_{\mathcal{H}}, p_X) | \mathcal{M}_1^{\text{fin}}(\{p\}_{\mathcal{H}}, p_X) \rangle \right] \\
&\quad \times \text{dLips}_X \hat{O}(\{p\}_{\mathcal{H}_f}; p_X).
\end{aligned} \tag{3.51}$$

Here, $|\mathcal{M}_0\rangle$ and $|\mathcal{M}_1\rangle$ are the tree-level and the one-loop amplitudes, respectively, and $|\mathcal{M}_1^{\text{fin}}\rangle$ stands for the one-loop finite amplitude in the limit $\epsilon \rightarrow 0$. Additionally, the collinear renormalization of parton distribution functions (PDFs) is standard, and reads (see Appendix A.3.2 for details)

$$2s_{ab} \, d\bar{\sigma}_{q\bar{q}}^{\text{pdf}} = \left[\frac{\alpha_s(\mu)}{2\pi} \right] \frac{1}{\epsilon} \left[\langle \hat{P}_{qq}^{(0)} \otimes F_{\text{LM}}^{q\bar{q}}[g] \rangle + \langle F_{\text{LM}}^{q\bar{q}}[g] \otimes \hat{P}_{qq}^{(0)} \rangle \right], \tag{3.52}$$

where $\hat{P}_{qq}^{(0)}$ is the tree-level Altarelli-Parisi splitting function given in Eq. (1.57).

Finally, we combine Eqs (3.48), (3.49), and (3.52), and we obtain the NLO corrections

for process $q_a + \bar{q}_b \rightarrow 2 + X$, which read

$$d\hat{\sigma}_{q\bar{q}}^{\text{NLO}} = d\bar{\sigma}_{q\bar{q}}^{\text{R}} + d\bar{\sigma}_{q\bar{q}}^{\text{V}} + d\bar{\sigma}_{q\bar{q}}^{\text{pdf}}. \quad (3.53)$$

At this point, it remains to explicitly verify the cancellation of poles.

3.2.2 Obstacles to generalizing the subtraction formalism at NLO

Having obtained the NLO cross section in Eq. (3.53), the next steps are to verify that the poles cancel and then write an expression that is explicitly ϵ -finite. Typically, software such as `Mathematica` is employed to expand expressions like Eq. (3.53) in powers of ϵ and to check for pole cancellation. However, there are some observations worth noting, which we list below.

- i) Eq. (3.48), as written, is process-dependent, particularly in the collinear limits, but especially in the eikonal term due to color-correlated contributions of the form

$$(\mathbf{T}_i \cdot \mathbf{T}_j) \cdot F_{\text{LM}}^{q\bar{q}}[g], \quad (3.54)$$

which cannot be trivially simplified for Born-level processes involving $n \geq 4$ partons (i.e., non-trivial color configurations). Naively, one might think that this does not pose a problem. Color-correlated terms in the first term on the right-hand side of Eq. (3.48), and in the virtual contribution in (3.49), should cancel among themselves, as do the non-color-correlated terms in the second, third, and fourth terms on the right-hand side of Eq. (3.48), as well as the term (3.52) related to PDFs. However, this is not the case; in general, there is a non-trivial *cross-talk* between these terms. To disentangle this cross-talk and successfully cancel the poles, it is necessary to rewrite some of the color-correlated contributions in terms of Casimir operators, which is feasible but process-dependent.

- ii) The NSC approach has already been successfully applied to compute the NNLO QCD corrections for a variety of processes [65–67, 73, 75–78], which suggests that it possesses the flexibility and simplicity required for studies of multi-particle final states. However, it has never been applied to processes whose Born-level configuration contains $n \geq 4$ partons. This introduces the additional complexity of color-correlations, as highlighted in point i).
- iii) Eq. (3.48) mixes *boosted* and *elastic* contributions. This implies that divergences must be canceled by summing all the contributions together. Therefore, one has to deal with large expressions where it is necessary to separate the elastic pieces from the boosted ones and distinguish the contributions containing color-correlations from those that do not, as noted in point i). This procedure, in addition to being highly inefficient, is process-dependent.

For these reasons, we seek to establish a formalism where pole cancellation is manifest for an arbitrary proton-proton scattering process, and this approach must account

for color-correlations. Indeed, we note that the challenge of color-correlations is significantly amplified at NNLO, which is the primary focus of this thesis. As outlined in Ch. 2, at NNLO, the IR poles involve intricate structures, including triple and quartic color-correlations. A detailed discussion of NNLO will be presented in next chapter. For now, our attention is directed toward understanding the general mechanism for achieving complete pole cancellation at NLO.

We emphasize that the theoretical framework for NLO is already well understood, thus the NLO results we present in next pages are not original. What is of particular interest to us are the IR structures that emerge, and more importantly, our approach to expressing them in an (almost) process-independent manner. These structures, as we shall see, will play a crucial role in the analysis of NNLO.

3.3 The new approach applied to processes $\mathcal{A}_0$ and $\mathcal{B}_0$ at NLO

Following on the previous section, we here present a generalization of the FKS/NSC scheme at NLO. In principle, the formalism presented here can be applied to demonstrate pole cancellation and produce IR-finite results for arbitrary production. However, with an eye on the NNLO computation, here we will focus on two classes of processes:

$$\mathcal{A}_0 : \quad a + b \rightarrow \{g\}_N + X, \quad (3.55)$$

$$\mathcal{B}_0 : \quad g_a + q_b \rightarrow \{g\}_{N-1} + q + X, \quad (3.56)$$

and consider corrections due to *gluonic radiation only*. We use the former to introduce operators associated with soft and collinear emissions, and the latter to discuss how initial state flavors change in collinear limits, and how this can be accommodated within the formalism. Both of these aspects will be vital in Ch. 5.

3.3.1 NLO corrections to $ab \rightarrow \{g\}_N + X$ (process $\mathcal{A}_0$)

We begin by considering a class of processes $\mathcal{A}_0$, where two initial-state partons, denoted by a and b , produce N final-state gluons along with a color singlet X . We keep N as a free parameter and will demonstrate that our results depend straightforwardly thereupon. Additionally, we do not specify the flavors of the initial-state partons. Given that only gluons are produced in the final state, there are only two possible initial configurations: a $q\bar{q}$ pair or a gg pair. Since the structure of the calculation does not change depending on these two cases, we opt to leave the initial-state partons denoted implicitly as a and b .

Let us quickly review the steps discussed in Subsec. 3.2.1, and generalize these to the case where N gluons (rather than a single gluon) are present in the final state at LO. We begin with the real corrections to process (3.55) where an extra final-state gluon is produced, i.e.,

$$\mathcal{A}_1 : \quad a + b \rightarrow (N + 1)g + X, \quad (3.57)$$

The cross section of (3.57) corresponds to ⁴

$$2s_{ab} \, d\bar{\sigma}_{\mathcal{A}_1}^R = \left\langle \frac{1}{(N+1)!} F_{\text{LM}}^{ab}[\{g\}_{N+1}] \right\rangle. \quad (3.58)$$

We introduce a sum over the N damping factors $\Delta^{(i)}$,⁵ relabel the final state gluon momenta and write

$$\left\langle \frac{1}{(N+1)!} F_{\text{LM}}^{ab}[\{g\}_{N+1}] \right\rangle = \langle \Delta^{(\text{m})} F_{\text{LM}}^{gq}[\{g\}_N | \mathbf{m}_g] \rangle \equiv \langle \Delta^{(\text{m})} F_{\text{LM}}^{\mathcal{A}_1} \rangle, \quad (3.59)$$

where

$$F_{\text{LM}}^{\mathcal{A}_1}[\mathbf{m}_g] \equiv F_{\text{LM}}^{gq}[\{g\}_N | \mathbf{m}_g] \stackrel{\text{def}}{=} \frac{1}{N!} F_{\text{LM}}^{ab}[\{g\}_{N+1}]. \quad (3.60)$$

In the above, we obtain a factor of $(N+1)$ when relabeling the damping factors, which combines with the symmetry factor in Eq. (3.58) to create a factor of $1/N!$. This is absorbed into the definition (3.60). Finally, we regularize the soft divergences first, followed by the collinear ones. We then find

$$\begin{aligned} \langle \Delta^{(\text{m})} F_{\text{LM}}^{\mathcal{A}_1}[\mathbf{m}_g] \rangle &= \langle S_{\text{m}} F_{\text{LM}}^{\mathcal{A}_1}[\mathbf{m}_g] \rangle + \sum_{i \in \mathcal{H}} \langle \bar{S}_{\text{m}} C_{im} \Delta^{(\text{m})} F_{\text{LM}}^{\mathcal{A}_1}[\mathbf{m}_g] \rangle \\ &\quad + \langle \mathcal{O}_{\text{NLO}}^{(\text{m})} \Delta^{(\text{m})} F_{\text{LM}}^{\mathcal{A}_1}[\mathbf{m}_g] \rangle. \end{aligned} \quad (3.61)$$

We begin by considering the soft counterterm on the right-hand side of Eq. (3.61), which is a generalization of Eq. (3.30). Referring to Eq. (3.34), we introduce the Catani-like *soft operator*

$$I_S(\epsilon) \stackrel{\text{def}}{=} -\frac{(2E_{\text{max}}/\mu)^{-2\epsilon}}{\epsilon^2} \sum_{(ij)} \eta_{ij}^{-\epsilon} K_{ij}(\mathbf{T}_i \cdot \mathbf{T}_j), \quad (3.62)$$

where we recall that

$$\begin{aligned} K_{ij} &\stackrel{\text{def}}{=} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \eta_{ij}^{1+\epsilon} {}_2F_1(1, 1, 1-\epsilon, 1-\eta_{ij}) \\ &= 1 + \underbrace{\left[\text{Li}_2(1-\eta_{ij}) - \frac{\pi^2}{6} \right]}_{\stackrel{\text{def}}{=} K_{ij}^{(2)}} \epsilon^2 + \mathcal{O}(\epsilon^3). \end{aligned} \quad (3.63)$$

and use it to express the soft counterterm as

$$\langle S_{\text{m}} F_{\text{LM}}^{\mathcal{A}_1}[\mathbf{m}_g] \rangle = [\alpha_s] \langle I_S(\epsilon) \cdot F_{\text{LM}}^{\mathcal{A}_0} \rangle, \quad (3.64)$$

where

$$F_{\text{LM}}^{\mathcal{A}_0} \stackrel{\text{def}}{=} \frac{1}{N!} F_{\text{LM}}^{ab}[\{g\}_N] \quad (3.65)$$

⁴In contrast to the notation used in Ref. [81], we explicitly retain the symmetry factor $1/(N+1)!$ due to the $(N+1)$ final-state gluons.

⁵Recall that these damping factors sum to one, cf. Eq. (3.22)

is the LO cross section for $\mathcal{A}_0$. The soft operator $I_S(\epsilon)$ encapsulates all the singularities associated with soft gluon emission, and we will see that it is one of the three building blocks of our new approach to the NSC scheme.

Regarding the hard-collinear term, we can generalize Eq. (3.42) for the initial-state contributions as

$$\begin{aligned} & \langle \bar{S}_m C_{am} \Delta^{(m)} F_{LM}^{\mathcal{A}_1}[\mathbf{m}_g] \rangle \\ &= -\frac{[\alpha_s]}{\epsilon} \left\langle \left(\frac{2E_a}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} (\mathcal{P}_{aa}^{(2)} \otimes F_{LM}^{\mathcal{A}_0}) \right\rangle. \end{aligned} \quad (3.66)$$

As noted in the previous section, the splitting function $\mathcal{P}_{aa}^{(2)}$ combines boosted and elastic terms, which complicates the structure of the poles. We separate them by rewriting the splitting function $\mathcal{P}_{\alpha\beta}^{(k)}$, with $\alpha, \beta = q, g$, as

$$-\left[\left(\frac{2E_a}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \right]^{\frac{k}{2}} \mathcal{P}_{\alpha\beta}^{(k)}(z, E_a) \stackrel{\text{def}}{=} \Gamma_{a,\alpha}^{(k)} \delta(1-z) \delta_{\alpha\beta} + \mathcal{P}_{\alpha\beta}^{(k),\text{gen}}(z, E_a), \quad (3.67)$$

with

$$\Gamma_{a,\alpha}^{(k)} \stackrel{\text{def}}{=} \left[\left(\frac{2E_a}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \right]^{\frac{k}{2}} \left[\gamma_\alpha + 2T_\alpha^2 \frac{1 - e^{-k\epsilon L_a}}{k\epsilon} \right], \quad (3.68)$$

$$\mathcal{P}_{\alpha\beta}^{(k),\text{gen}}(z, E_a) \stackrel{\text{def}}{=} \left[\left(\frac{2E_a}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \right]^{\frac{k}{2}} \left[-\hat{P}_{\alpha\beta}^{(0)}(z) + \epsilon \mathcal{P}_{\alpha\beta}^{(k),\text{fin}}(z) \right]. \quad (3.69)$$

The functions $\hat{P}_{\alpha\beta}^{(0)}$ are the Altarelli-Parisi splitting functions, defined in Eq. (1.58), while $\mathcal{P}_{\alpha\beta}^{(k),\text{fin}}$ corresponds to ϵ -expansion of $-\mathcal{P}_{\alpha\beta}^{(k)}(z, E_a = 0)$ starting from $\mathcal{O}(\epsilon)$. The parameter k takes the value $k = 2$ at NLO and $k = 4$ at NNLO, while γ_α in (3.68) is the anomalous dimension introduced in Eq. (2.23). We point out that $\gamma_g = \beta_0$ should be understood with $n_f = 0$. Additionally, to lighten the notation, we leave the index k understood when $k = 2$. Using this new notation, we can rewrite Eq. (3.66) as

$$\langle \bar{S}_m C_{am} \Delta^{(m)} F_{LM}^{\mathcal{A}_1}[\mathbf{m}_g] \rangle = [\alpha_s] \left\langle \frac{\Gamma_{a,fa}}{\epsilon} F_{LM}^{\mathcal{A}_0} \right\rangle + \frac{[\alpha_s]}{\epsilon} \langle \mathcal{P}_{aa}^{\text{gen}} \otimes F_{LM}^{\mathcal{A}_0} \rangle. \quad (3.70)$$

This expression applies analogously for leg b .

Next, we consider the final-state hard-collinear contribution of leg i , with $i \in \mathcal{H}_f$, and we read it from Eq. (3.47), obtaining

$$\langle \bar{S}_m C_{im} F_{LM}^{\mathcal{A}_1}[\mathbf{m}_g] \rangle = \frac{[\alpha_s]}{\epsilon} \left\langle \left(\frac{2E_i}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \gamma_{z,g \rightarrow gg}^{22}(L_i) F_{LM}^{\mathcal{A}_0} \right\rangle. \quad (3.71)$$

Then, we introduce the *generalized final-state anomalous dimensions*

$$\Gamma_{i,\alpha \rightarrow \rho\sigma}^{(k)} \stackrel{\text{def}}{=} \left[\left(\frac{2E_i}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \right]^{\frac{k}{2}} \gamma_{z,\alpha \rightarrow \rho\sigma}^{2k}(L_i), \quad \alpha \rightarrow \rho\sigma = \begin{cases} q \rightarrow qg \\ q \rightarrow gq \\ g \rightarrow q\bar{q} \\ g \rightarrow gg, \end{cases} \quad (3.72)$$

where the standard anomalous dimensions $\gamma_{z,\alpha \rightarrow \rho\sigma}^{2k}(\epsilon, L_i)$ are given in Eq. (A.59), and we use this definition to rewrite Eq. (3.71) as

$$\langle \bar{S}_m C_{am} \Delta^{(m)} F_{LM}^{\mathcal{A}_1}[\mathbf{m}_g] \rangle = [\alpha_s] \left\langle \frac{\Gamma_{i,g \rightarrow gg}^{(k)}}{\epsilon} F_{LM}^{\mathcal{A}_0} \right\rangle. \quad (3.73)$$

Putting together initial- and final-state terms, we can express the hard-collinear contribution as⁶

$$\sum_{i \in \mathcal{H}} \langle \bar{S}_m C_{im} \Delta^{(m)} F_{LM}^{\mathcal{A}_1}[\mathbf{m}_g] \rangle = [\alpha_s] \langle I_C(\epsilon) \cdot F_{LM}^{\mathcal{A}_0} \rangle, \quad (3.74)$$

where we introduce the Catani-like *collinear operator* for process $\mathcal{A}_0$

$$I_C^{(k), \mathcal{A}_0}(\epsilon) \stackrel{\text{def}}{=} \sum_{i=a,b} \frac{\Gamma_{i,fi}^{(k)}}{\epsilon k/2} + \sum_{i \in \mathcal{H}_f} \frac{\Gamma_{i,g \rightarrow gg}^{(k)}}{\epsilon k/2}, \quad k = 2, 4. \quad (3.75)$$

As stated, we will leave the index k understood if $k = 2$. This completes the real corrections, which yield

$$\begin{aligned} 2s_{ab} d\bar{\sigma}_{\mathcal{A}_1}^R &= [\alpha_s] \langle [I_S(\epsilon) + I_C(\epsilon)] \cdot F_{LM}^{\mathcal{A}_0} \rangle \\ &+ \frac{[\alpha_s]}{\epsilon} \left[\langle \mathcal{P}_{aa}^{\text{gen}} \otimes F_{LM}^{\mathcal{A}_0} \rangle + \langle F_{LM}^{\mathcal{A}_0} \otimes \mathcal{P}_{bb}^{\text{gen}} \rangle \right] \\ &+ \langle \mathcal{O}_{\text{NLO}}^{(m)} \Delta^{(m)} F_{LM}^{\mathcal{A}_1}[\mathbf{m}_g] \rangle. \end{aligned} \quad (3.76)$$

For virtual corrections, we utilize the notation introduced in Eq. (2.30) to obtain

$$2s_{ab} d\bar{\sigma}_{\mathcal{A}_0}^V = F_{LV}^{ab}[\{g\}_N] = [\alpha_s] \langle I_V(\epsilon) \cdot F_{LM}^{\mathcal{A}_0} \rangle + \langle F_{LV,\text{fin}}^{\mathcal{A}_0} \rangle, \quad (3.77)$$

where

$$F_{LV,\text{fin}}^{\mathcal{A}_0} \stackrel{\text{def}}{=} \frac{1}{N!} F_{LV,\text{fin}}^{ab}[\{g\}_N]. \quad (3.78)$$

The details are discussed in Subsec. 2.2.1. Finally, from Eq. (3.52) we read the contribu-

⁶Since I_C does not contain color-correlations, we could write the following equation with as a simple product of I_C and F_{LM} , without introducing the product in color space. However, we prefer to keep the latter notation, to facilitate the combination of the collinear, soft, and virtual operators.

tion of the renormalization of the PDFs:⁷

$$2s_{ab} d\bar{\sigma}_{\mathcal{A}_1}^{\text{pdf}} = \left[\frac{\alpha_s(\mu)}{2\pi} \right] \frac{1}{\epsilon} \left[\langle \hat{P}_{aa}^{(0)} \otimes F_{\text{LM}}^{\mathcal{A}_0} \rangle + \langle F_{\text{LM}}^{\mathcal{A}_0} \otimes \hat{P}_{bb}^{(0)} \rangle \right]. \quad (3.79)$$

At this point, we can write the final NLO result by combining Eqs (3.76), (3.77), and (3.79), obtaining

$$\begin{aligned} 2s_{ab} d\hat{\sigma}_{\mathcal{A}_0}^{\text{NLO}} &= 2s_{ab} \left[d\bar{\sigma}_{\mathcal{A}_1}^{\text{R}} + d\bar{\sigma}_{\mathcal{A}_0}^{\text{V}} + d\bar{\sigma}_{\mathcal{A}_1}^{\text{pdf}} \right] \\ &= [\alpha_s] \langle I_{\text{T}}(\epsilon) \cdot F_{\text{LM}}^{\mathcal{A}_0} \rangle + \langle F_{\text{LV,fin}}^{\mathcal{A}_0} \rangle + \langle \mathcal{O}_{\text{NLO}}^{(\text{m})} \Delta^{(\text{m})} F_{\text{LM}}^{\mathcal{A}_1}[\mathbf{m}_g] \rangle \\ &\quad + \frac{[\alpha_s]}{\epsilon} \left[\langle \mathcal{P}_{aa}^{\text{gen}} \otimes F_{\text{LM}}^{\mathcal{A}_0} \rangle + \langle F_{\text{LM}}^{\mathcal{A}_0} \otimes \mathcal{P}_{bb}^{\text{gen}} \rangle \right] \\ &\quad + \left[\frac{\alpha_s(\mu)}{2\pi} \right] \frac{1}{\epsilon} \left[\langle \hat{P}_{aa}^{(0)} \otimes F_{\text{LM}}^{\mathcal{A}_0} \rangle + \langle F_{\text{LM}}^{\mathcal{A}_0} \otimes \hat{P}_{bb}^{(0)} \rangle \right] \end{aligned} \quad (3.80)$$

In Eq. (3.80), we introduced the operator $I_{\text{T}}(\epsilon)$ for process $\mathcal{A}_0$, defined as:

$$I_{\text{T}}^{\mathcal{A}_0}(\epsilon) \stackrel{\text{def}}{=} I_{\text{V}}(\epsilon) + I_{\text{S}}(\epsilon) + I_{\text{C}}^{\mathcal{A}_0}(\epsilon). \quad (3.81)$$

The operator $I_{\text{T}}(\epsilon)$, which could potentially contain poles of $\mathcal{O}(\epsilon^{-2})$, is in fact IR-finite, meaning it is of $\mathcal{O}(\epsilon^0)$. More generally, the entire expression on the right-hand side of Eq. (3.80) is IR-finite. In the following subsection, we will demonstrate this statement.

3.3.2 Poles cancellation of process $\mathcal{A}_0$ at NLO

In this subsection, we demonstrate the cancellation of the poles of the NLO result we obtained in Eq. (3.80). In particular, we would like to show that the operator $I_{\text{T}}(\epsilon)$ introduced in Eq. (3.81), once expanded in powers of ϵ , is of $\mathcal{O}(\epsilon^0)$.

We begin with the ϵ -expansion of the operator I_{S} defined in Eq. (3.62). Although it is straightforward to perform such an expansion, arranging it in a particular way is helpful for an efficient demonstration of the cancellations of IR poles. We note that the I -operators include quantities raised to ϵ -dependent powers. For example, in the case of I_{S} , there are factors $(2E_{\text{max}}/\mu)^{-\epsilon}$ and $\eta_{ij}^{-\epsilon}$. The expansion of such quantities in ϵ starts with 1 and it is convenient to make this explicit. To this end, we introduce the function

$$f_k(x) \stackrel{\text{def}}{=} \frac{x^{-k\epsilon} - 1}{\epsilon}, \quad (3.82)$$

such that $f_k(x) \sim \mathcal{O}(\epsilon^0)$ as $\epsilon \rightarrow 0$. We then use this function to write the expansion of $I_{\text{S}}(\epsilon)$ as

$$\begin{aligned} I_{\text{S}}(\epsilon) &= -\frac{1}{\epsilon^2} \left[1 + \epsilon f_2(2E_{\text{max}}/\mu) \right] \\ &\quad \times \sum_{(ij)} \left[1 + \epsilon f_1(\eta_{ij}) + \epsilon^2 K_{ij}^{(2)} \right] (\mathbf{T}_i \cdot \mathbf{T}_j) + \mathcal{O}(\epsilon), \end{aligned} \quad (3.83)$$

⁷See Appendix A.3.2 for details.

where the coefficients $K_{ij}^{(2)}$ are reported in Eq. (3.63), and $\mathcal{O}(\epsilon)$ terms have been neglected. Since we only need terms up to $\mathcal{O}(\epsilon^0)$, we can simplify (3.83) and write it order-by-order in ϵ as follows:

$$\begin{aligned} I_S(\epsilon) = & -\frac{1}{\epsilon^2} \sum_{(ij)} \left[1 + \epsilon f_2(2E_{\max}/\mu) \right] (\mathbf{T}_i \cdot \mathbf{T}_j) \\ & - \sum_{(ij)} \frac{f_1(\eta_{ij})}{\epsilon} (\mathbf{T}_i \cdot \mathbf{T}_j) \\ & - \sum_{(ij)} \left[f_1(\eta_{ij}) f_2(2E_{\max}/\mu) + K_{ij}^{(2)} \right] (\mathbf{T}_i \cdot \mathbf{T}_j) + \mathcal{O}(\epsilon). \end{aligned} \quad (3.84)$$

Next, we note that in the first term on the right-hand side of Eq. (3.84), only the color charge operators depend on the summation indices i and j . For this reason, the summation over one of the indices can be performed using the color conservation condition (2.4), i.e.,

$$\sum_{k \in \mathcal{H}} \mathbf{T}_k = 0, \quad (3.85)$$

which yields

$$\sum_{\substack{j \in \mathcal{H} \\ j \neq i}} \mathbf{T}_i \cdot \mathbf{T}_j = -\mathbf{T}_i^2. \quad (3.86)$$

Thus, we obtain

$$\begin{aligned} I_S(\epsilon) = & \sum_{i \in \mathcal{H}} \left[1 + \epsilon f_2(2E_{\max}/\mu) \right] \frac{\mathbf{T}_i^2}{\epsilon^2} \\ & - \sum_{(ij)} \frac{f_1(\eta_{ij})}{\epsilon} (\mathbf{T}_i \cdot \mathbf{T}_j) \\ & - \sum_{(ij)} \left[f_1(\eta_{ij}) f_2(2E_{\max}/\mu) + K_{ij}^{(2)} \right] (\mathbf{T}_i \cdot \mathbf{T}_j) + \mathcal{O}(\epsilon). \end{aligned} \quad (3.87)$$

It is seen from the above equation that the residue of the $1/\epsilon^2$ pole is proportional to the sum of the Casimir factors $\mathbf{T}_i^2$. We recall that the IR poles of the one-loop amplitude described by Catani's function exhibit a similar feature. The $1/\epsilon$ pole in the second line of Eq. (3.87) contains color correlations, while the terms in the third line are ϵ -finite.

We now turn to the virtual corrections, and consider the operator $I_V(\epsilon)$ in Eq. (3.88), which we recall is defined as

$$I_V(\epsilon) \stackrel{\text{def}}{=} \bar{I}_1(\epsilon) + \bar{I}_1^\dagger(\epsilon), \quad (3.88)$$

where

$$\bar{I}_1(\epsilon) \stackrel{\text{def}}{=} \frac{1}{2} \sum_{(ij)} \frac{\mathcal{V}_i^{\text{sing}}(\epsilon)}{\mathbf{T}_i^2} \mathbf{T}_i \cdot \mathbf{T}_j \left(\frac{\mu^2}{2p_i \cdot p_j} \right)^\epsilon e^{i\pi\lambda_{ij}\epsilon}, \quad (3.89)$$

with

$$\begin{aligned} \mathcal{V}_i^{\text{sing}}(\epsilon) &= \frac{\mathbf{T}_i^2}{\epsilon^2} + \frac{\gamma_i}{\epsilon}, \\ \lambda_{ij} &\stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } i \text{ and } j \text{ are both incoming or outgoing partons} \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (3.90)$$

Expanding $I_V(\epsilon)$ in powers of ϵ , we find

$$I_V(\epsilon) = \sum_{(ij)} \frac{\mathcal{V}_i^{\text{sing}}(\epsilon)}{\mathbf{T}_i^2} \left[1 + \epsilon f_1(s_{ij}/\mu^2) - \frac{\pi^2}{2} \lambda_{ij} \epsilon^2 + \mathcal{O}(\epsilon^3) \right] (\mathbf{T}_i \cdot \mathbf{T}_j). \quad (3.91)$$

In the first term on the right-hand side of (3.91), we again use color conservation (3.85) to sum over the index j . Doing so allows us to write the virtual contributions as

$$\begin{aligned} I_V(\epsilon) &= - \sum_{i \in \mathcal{H}} \left[\frac{\mathbf{T}_i^2}{\epsilon^2} + \frac{\gamma_i}{\epsilon} \right] \\ &\quad + \sum_{(ij)} \frac{f_1(s_{ij}/\mu^2)}{\epsilon} (\mathbf{T}_i \cdot \mathbf{T}_j) \\ &\quad + \sum_{(ij)} \left[\frac{\gamma_i}{\mathbf{T}_i^2} f_1(s_{ij}/\mu^2) - \frac{\pi^2}{2} \lambda_{ij} \right] (\mathbf{T}_i \cdot \mathbf{T}_j) + \mathcal{O}(\epsilon). \end{aligned} \quad (3.92)$$

Since $f_k(x) \sim \mathcal{O}(\epsilon^0)$, poles in the color-correlated structures appear only at $\mathcal{O}(\epsilon^{-1})$, while all terms in the last line are finite.

Comparing Eqs (3.87) and (3.92), we observe that the $\mathcal{O}(\epsilon^{-2})$ poles cancel among these two contributions. Furthermore, we note that the function $f_1(s_{ij}/\mu^2)$ in Eq. (3.92) can be written as

$$f_1(s_{ij}/\mu^2) = f_1(\eta_{ij}) + f_1(2E_i/\mu) + f_1(2E_j/\mu) + \epsilon g_{ij}. \quad (3.93)$$

The first term on the right-hand side of the above equation appears in the soft contribution I_S , while the next two terms depend on one of the two indices i or j , and the last term,

$$g_{ij} \stackrel{\text{def}}{=} f_1(2E_i/\mu) f_1(2E_j/\mu) + f_1(4E_i E_j/\mu^2) f_1(\eta_{ij}), \quad (3.94)$$

is of $\mathcal{O}(\epsilon^0)$. Thus we can further simplify the expression for I_V by making use of color conservation. We find

$$\sum_{(ij)} \frac{f_1(2E_i/\mu) + f_1(2E_j/\mu)}{\epsilon} (\mathbf{T}_i \cdot \mathbf{T}_j) = -2 \sum_{i \in \mathcal{H}} \frac{\mathbf{T}_i^2}{\epsilon} f_1(2E_i/\mu). \quad (3.95)$$

Upon applying the above manipulations and combining the soft and virtual I -

operators, we obtain

$$\begin{aligned} I_V(\epsilon) + I_S(\epsilon) &= \sum_{i \in \mathcal{H}} \left[\frac{\mathbf{T}_i^2}{\epsilon} \left(f_2(2E_{\max}/\mu) - 2f_1(2E_i/\mu) \right) - \frac{\gamma_i}{\epsilon} \right] + \mathcal{O}(\epsilon^0) \\ &= - \sum_{i \in \mathcal{H}} \left(2L_i \frac{\mathbf{T}_i^2}{\epsilon} + \frac{\gamma_i}{\epsilon} \right) + \mathcal{O}(\epsilon^0), \end{aligned} \quad (3.96)$$

where, in the second line, we have substituted the expansions of $f_{1,2}(x)$ in ϵ .⁸ Since we aim to verify the cancellation of the poles, we will retain expansions only up to terms of $\mathcal{O}(\epsilon^0)$ from this point onward. The full expansions of the operators are provided in Appendix A.6. From Eq. (3.96), we see that the ϵ -divergences proportional to correlators of color charges cancel in the sum of the virtual and soft functions, I_V and I_S , leaving single poles which are proportional to either Casimir operators or anomalous dimensions. To cancel these poles, we need to combine the above result with the operator $I_C^{\mathcal{A}_0}(\epsilon)$ defined in Eq. (3.75). We expand in powers of ϵ the generalized collinear anomalous dimensions that appears in Eq. (3.75), and we obtain

$$\begin{aligned} \Gamma_{a,fa} &= \gamma_a + 2\mathbf{T}_a^2 L_a + \mathcal{O}(\epsilon), \\ \Gamma_{i,g \rightarrow gg} &= \gamma_i + 2\mathbf{T}_i^2 L_i + \mathcal{O}(\epsilon), \quad i \in \mathcal{H}_f, \end{aligned} \quad (3.97)$$

so that $I_C^{\mathcal{A}_0}(\epsilon)$ becomes

$$I_C^{\mathcal{A}_0}(\epsilon) = \sum_{i \in \mathcal{H}} \left(2L_i \frac{\mathbf{T}_i^2}{\epsilon} + \frac{\gamma_i}{\epsilon} \right) + \mathcal{O}(\epsilon^0). \quad (3.98)$$

Comparing this result with Eq. (3.96), we can easily see that the following combination of I -operators

$$I_T^{\mathcal{A}_0}(\epsilon) = I_V(\epsilon) + I_S(\epsilon) + I_C^{\mathcal{A}_0}(\epsilon) \quad (3.99)$$

is ϵ -finite, as stated at the end of the previous subsection.

Finally, we need to verify the cancellation of the *boosted* terms in Eq. (3.80), specifically those terms that contain explicit integrals with respect to the variable z . To accomplish this, we use the property $[\alpha_s] = \alpha_s(\mu^2)/(2\pi) + \mathcal{O}(\epsilon^2)$ (see Eq. (2.25)), and the following general expansion of the function $\mathcal{P}_{\alpha\beta}^{\text{gen}}$:

$$\mathcal{P}_{\alpha\beta}^{\text{gen}}(z, E_a) = -\hat{P}_{\alpha\beta}^{(0)}(z) + \epsilon \mathcal{P}_{\alpha\beta}^{\text{NLO}}(z, E_a) + \mathcal{O}(\epsilon^2), \quad (3.100)$$

where functions $\mathcal{P}_{\alpha\beta}^{\text{NLO}}$ are defined in Eq. (I.3). Therefore, we can rewrite the NLO cross section (3.80) for the process $\mathcal{A}_0$ in (3.55) as

$$\begin{aligned} 2s_{ab} d\hat{\sigma}_{\mathcal{A}_0}^{\text{NLO}} &= [\alpha_s] \langle I_T^{(0)} \cdot F_{\text{LM}}^{\mathcal{A}_0} \rangle + \langle F_{\text{LV,fin}}^{\mathcal{A}_0} \rangle + \langle \mathcal{O}_{\text{NLO}}^{(\text{m})} \Delta^{(\text{m})} F_{\text{LM}}^{\mathcal{A}_1}[\mathbf{m}_g] \rangle \\ &\quad + [\alpha_s] \left[\langle \mathcal{P}_{aa}^{\text{NLO}} \otimes F_{\text{LM}}^{\mathcal{A}_0} \rangle + [\alpha_s] \langle F_{\text{LM}}^{\mathcal{A}_0} \otimes \mathcal{P}_{bb}^{\text{NLO}} \rangle \right], \end{aligned} \quad (3.101)$$

⁸We recall that $L_i = \log(E_{\max}/E_i)$.

where $I_T^{(0)}$ is the $\mathcal{O}(\epsilon^0)$ coefficient in the expansion of $I_T^{A_0}(\epsilon)$, and its expression is reported in Eq. (A.103). Eq. (3.101) is now explicitly ϵ -finite. We observe that, thanks to the I -operators, the cancellation of color-correlated poles can be explicitly achieved through very simple expressions, often contained within just one or two lines. Additionally, this approach is entirely independent of the number of gluons in the final state. The parameter N , in fact, appears only as an upper bound for summations, as seen in Eqs (3.62), (3.75), (3.88). Lastly, we emphasize that the cancellation of poles in Eq. (3.101) should be understood with $n_f = 0$. To obtain the correct terms proportional to n_f , additional channels beyond that in Eq. (3.55) must be considered.

3.3.3 NLO corrections to $gq \rightarrow \{g\}_N + X$ (process $\mathcal{B}_0$)

In the previous pages of this section, we demonstrated that with the introduction of the I -operators, it is possible to express the NLO cross section for a scattering process with any number N of final-state gluons in a very compact and pole-free form, as shown in Eq. (3.101). In this context, we considered a process, $\mathcal{A}_0$, which is symmetric in both the initial and final states. This naturally raises the question of whether the above approach also applies when the symmetry is lifted. To address this, we consider process $\mathcal{B}_0$ in Eq. (3.56), and we define the corresponding LO F_{LM} function as

$$F_{\text{LM}}^{\mathcal{B}_0} \stackrel{\text{def}}{=} \frac{1}{(N-1)!} F_{\text{LM}}^{gq}[\{g\}_{N-1}, q]. \quad (3.102)$$

Here we stress that, contrary to Subsec. 3.3.1, we specify the nature of the initial state partons.

Compared to the process $\mathcal{A}_0$, here the symmetry of the final state is broken by the presence of a quark, which, in turn, also breaks the symmetry of the initial state. We therefore aim to study the NLO corrections for this process, considering the case where an additional gluon is included in the final state, i.e.,

$$\mathcal{B}_1 : \quad g_a + q_b \rightarrow \{g\}_N + q + X. \quad (3.103)$$

We follow the same steps presented for process $\mathcal{A}_0$ and begin with the real corrections,

$$2s_{ab} \, d\bar{\sigma}_{\mathcal{B}_1}^{\text{R}} = \left\langle \frac{1}{N!} F_{\text{LM}}^{gq}[\{g\}_N, q] \right\rangle = \sum_{i \in \mathcal{H}} \left\langle \frac{\Delta^{(i)}}{N!} F_{\text{LM}}^{gq}[\{g\}_N, q] \right\rangle. \quad (3.104)$$

In the second term, we introduced the partitions $\Delta^{(i)}$ discussed in Appendix B to identify the single final-state parton at NLO that can become unresolved. This can be either the quark or one of the N final-state gluons. In the latter case, we can use the symmetry of the final-state gluons, as we did in the previous subsection, and label the unresolved gluon as m_g . This gives an additional factor of N , which combines with the symmetry factor $1/N!$ to give $1/(N-1)!$. Similarly, if the quark is the parton that becomes unresolved, we label it as m_q . In this case, the symmetry factor remains as $1/N!$. We thus

have

$$2s_{ab} d\bar{\sigma}_{\mathcal{B}_1}^R = \langle \Delta^{(m)} (F_{LM}^{\mathcal{B}_1}[\mathbf{m}_g] + F_{LM}^{\mathcal{B}_1}[\mathbf{m}_q]) \rangle, \quad (3.105)$$

where the functions

$$\begin{aligned} F_{LM}^{\mathcal{B}_1}[\mathbf{m}_g] &\stackrel{\text{def}}{=} F_{LM}^{gq}[\{g\}_{N-1}, q|\mathbf{m}_g], \\ F_{LM}^{\mathcal{B}_1}[\mathbf{m}_q] &\stackrel{\text{def}}{=} F_{LM}^{gq}[\{g\}_N|\mathbf{m}_q], \end{aligned} \quad (3.106)$$

include factors of $1/(N-1)!$ and $1/N!$, respectively. In other words, the symmetry prefactor depends on the number of hard final-state gluons.

Next, we introduce the function

$$F_R^{\mathcal{B}_1} \stackrel{\text{def}}{=} F_{LM}^{\mathcal{B}_1}[\mathbf{m}_g] + F_{LM}^{\mathcal{B}_1}[\mathbf{m}_q], \quad (3.107)$$

which allows us to express the real corrections compactly as

$$2s_{ab} d\bar{\sigma}_{\mathcal{B}_1}^R = \langle \Delta^{(m)} F_R^{\mathcal{B}_1} \rangle, \quad (3.108)$$

and then we derive the standard soft- and collinear-regulated formula for real emission contributions,

$$2s_{ab} d\bar{\sigma}_{\mathcal{B}_1}^R = \langle S_m F_R^{\mathcal{B}_1} \rangle + \sum_{i \in \mathcal{H}} \langle \bar{S}_m C_{im} \Delta^{(m)} F_R^{\mathcal{B}_1} \rangle + \langle \mathcal{O}_{\text{NLO}}^{(m)} \Delta^{(m)} F_R^{\mathcal{B}_1} \rangle. \quad (3.109)$$

Here, an important phenomenon deserves discussion. In (3.109), let us focus on the hard-collinear term, specifically the initial-state collinear limit. If the potentially unresolved parton $\mathbf{m}$ is a gluon, as in the case of $F_{LM}^{\mathcal{B}_1}[\mathbf{m}_g]$, the poles are always proportional to the matrix elements squared of the process $\mathcal{B}_0$. However, when the unresolved parton is a quark and a collinear limit is taken in the initial state, this no longer holds. When a quark $\mathbf{m}_q$ becomes collinear with an initial-state parton, a flavor change always occurs: a quark collinear to a quark produces a gluon, while a quark collinear to a gluon results in a quark. Given this, two distinct collinear limits will lead to different initial states. First, the final-state quark $\mathbf{m}_q$ can become collinear with the initial-state gluon g_a , resulting in an antiquark entering the hard process:

$$\mathcal{A}_0^{(\bar{q}q)} : \quad \bar{q}_a + q_b \rightarrow \{g\}_N + X. \quad (3.110)$$

Second, the final-state quark $\mathbf{m}_q$ can become collinear with the initial-state quark q_b , changing the hard process to

$$\mathcal{A}_0^{(gg)} : \quad g_a + g_b \rightarrow \{g\}_N + X. \quad (3.111)$$

On the other hand, the virtual corrections to $\mathcal{B}_0$ produce poles proportional to the matrix elements squared of $\mathcal{B}_0$. Since the cancellation of poles must occur independently, it is convenient to separate the real-emission divergences based on the matrix elements

that they contain, i.e., the process that they project onto. To facilitate this, we introduce a *projection operator* $\mathcal{P}_{\mathcal{B}_0}$, which isolates contributions proportional to the matrix elements squared of the underlying Born process $\mathcal{B}_0$ plus an arbitrary number of gluons. We notice that soft limits remain unaffected by this projector, as in

$$\mathcal{P}_{\mathcal{B}_0} \langle S_{\mathbf{m}} F_{\text{LM}}^{\mathcal{B}_1}[\mathbf{m}_g] \rangle \equiv \langle S_{\mathbf{m}} F_{\text{LM}}^{\mathcal{B}_1}[\mathbf{m}_g] \rangle, \quad (3.112)$$

whereas specific collinear limits are removed, like the following:

$$\mathcal{P}_{\mathcal{B}_0} \langle C_{am} \Delta^{(\mathbf{m})} F_{\text{LM}}^{\mathcal{B}_1}[\mathbf{m}_q] \rangle = 0, \quad (3.113)$$

because, as discussed earlier, proportional to $F_{\text{LM}}^{\mathcal{A}_0^{(\bar{q}q)}}$. Similarly, we can introduce two additional projectors, $\mathcal{P}_{\mathcal{A}_0^{(\bar{q}q)}}$ and $\mathcal{P}_{\mathcal{A}_0^{(gg)}}$, which function in exactly the same way as described for $\mathcal{P}_{\mathcal{B}_0}$, but for the processes $\mathcal{A}_0^{(\bar{q}q)}$ and $\mathcal{A}_0^{(gg)}$, respectively. In the following, we will focus on those contributions that are non-zero once $\mathcal{P}_{\mathcal{B}_0}$ has acted, and we will return to $\mathcal{A}_0^{(\bar{q}q)}$ and $\mathcal{A}_0^{(gg)}$ later.

Therefore, let us begin from the following formula:

$$\mathcal{P}_{\mathcal{B}_0} [2s_{ab} d\bar{\sigma}_{\mathcal{B}_1}^{\text{R}}] = \langle S_{\mathbf{m}} F_{\text{R}}^{\mathcal{B}_1} \rangle + \sum_{i \in \mathcal{H}} \mathcal{P}_{\mathcal{B}_0} \langle \bar{S}_{\mathbf{m}} C_{im} \Delta^{(\mathbf{m})} F_{\text{R}}^{\mathcal{B}_1} \rangle + \langle \mathcal{O}_{\text{NLO}}^{(\mathbf{m})} \Delta^{(\mathbf{m})} F_{\text{R}}^{\mathcal{B}_1} \rangle. \quad (3.114)$$

Starting with the first term on the right-hand side, we notice that the soft quark does not develop singularities, thus

$$\langle S_{\mathbf{m}} F_{\text{R}}^{\mathcal{B}_1} \rangle = [\alpha_s] \langle S_{\mathbf{m}} F_{\text{LM}}^{\mathcal{B}_1}[\mathbf{m}_g] \rangle = [\alpha_s] \langle I_{\text{S}}(\epsilon) \cdot F_{\text{LM}}^{\mathcal{B}_0} \rangle, \quad (3.115)$$

where I_{S} is defined in Eq. (3.62).

The collinear limits require more attention owing to the asymmetry of the initial and final states. Starting with initial-state radiations, we find, with $i = a$,

$$\mathcal{P}_{\mathcal{B}_0} \langle \bar{S}_{\mathbf{m}} C_{am} \Delta^{(\mathbf{m})} F_{\text{R}}^{\mathcal{B}_1} \rangle = [\alpha_s] \left\langle \frac{\Gamma_{a,g}}{\epsilon} F_{\text{LM}}^{\mathcal{B}_0} \right\rangle + \frac{[\alpha_s]}{\epsilon} \langle \mathcal{P}_{gg}^{\text{gen}} \otimes F_{\text{LM}}^{\mathcal{B}_0} \rangle, \quad (3.116)$$

where we used the definitions (3.67 - 3.69). In a completely analogous way, for $i = b$, we get

$$\mathcal{P}_{\mathcal{B}_0} \langle \bar{S}_{\mathbf{m}} C_{bm} \Delta^{(\mathbf{m})} F_{\text{R}}^{\mathcal{B}_1} \rangle = [\alpha_s] \left\langle \frac{\Gamma_{b,q}}{\epsilon} F_{\text{LM}}^{\mathcal{B}_0} \right\rangle + \frac{[\alpha_s]}{\epsilon} \langle F_{\text{LM}}^{\mathcal{B}_0} \otimes \mathcal{P}_{qq}^{\text{gen}} \rangle. \quad (3.117)$$

Regarding final-state radiations, we now need to consider both $F_{\text{LM}}^{\mathcal{B}_1}[\mathbf{m}_g]$ and $F_{\text{LM}}^{\mathcal{B}_1}[\mathbf{m}_q]$ separately. Considering the case where the gluon $\mathbf{m}_g$ becomes unresolved, the hard-collinear limit reads

$$\sum_{i \in \mathcal{H}} \mathcal{P}_{\mathcal{B}_0} \langle \bar{S}_{\mathbf{m}} C_{im} \Delta^{(\mathbf{m})} F_{\text{LM}}^{\mathcal{B}_1}[\mathbf{m}_g] \rangle$$

$$\begin{aligned}
&= [\alpha_s] \sum_{i \in \mathcal{H}} \left\langle \frac{\Gamma_{i, f_i \rightarrow f_i g}}{\epsilon} F_{\text{LM}}^{\mathcal{B}_0} \right\rangle \\
&= [\alpha_s] \left\langle \left[\sum_{i \in \mathcal{H}_{f,g}} \frac{\Gamma_{i, g \rightarrow gg}}{\epsilon} + \frac{\Gamma_{q \rightarrow qq}}{\epsilon} \right] F_{\text{LM}}^{\mathcal{B}_0} \right\rangle,
\end{aligned} \tag{3.118}$$

where the definitions of $\Gamma_{i, f_i \rightarrow f_i g}$ are reported in Eq. (3.72). We introduced the set $\mathcal{H}_{f,g}$, which enumerates all the final-state gluons. In principle, we could define a similar set for quarks, denoted as $\mathcal{H}_{f,q}$, and sum over it. However, since there is only one quark in our case, we opt for a simpler notation and directly write $\Gamma_{q \rightarrow qq}$. Moving to the case where the final state quark becomes collinear to one of the final state gluons, the final-state collinear limit yields

$$\begin{aligned}
&\sum_{i \in \mathcal{H}_f} \mathcal{P}_{\mathcal{B}_0} \langle C_{im} \Delta^{(m)} F_{\text{LM}}^{\mathcal{B}_1}[\mathbf{m}_q] \rangle \\
&= [\alpha_s] \sum_{i \in \mathcal{H}_f} \left\langle \frac{\Gamma_{i, q \rightarrow gq}}{\epsilon} \frac{1}{N!} F_{\text{LM}}^{gq}[\dots, (i-1)_g, i_q, (i+1)_g, \dots] \right\rangle,
\end{aligned} \tag{3.119}$$

with $\Gamma_{i, q \rightarrow gq}$ given in Eq. (3.72). In the above equation, we have shown the symmetry factor $1/N!$ explicitly, and we point out that it originates from the N gluons originally present in $F_{\text{LM}}^{\mathcal{B}_1}[\mathbf{m}_q]$, while the F_{LM} function on the right-hand side contains $N-1$ gluons (as the i^{th} gluon has become a quark i_q after taking the collinear limit). However, all the terms in the sum over final state partons $\mathcal{H}$ are equivalent because the labels assigned to the hard partons are not physically meaningful. Given that we have N partons in the final state, we can consistently position the quark at index N , following the list $\{g\}_{N-1}$ of final-state gluons. This approach yields N identical copies of the same object, leading to

$$\begin{aligned}
&\sum_{i \in \mathcal{H}_f} \mathcal{P}_{\mathcal{B}_0} \langle C_{im} \Delta^{(m)} F_{\text{LM}}^{\mathcal{B}_1}[\mathbf{m}_g] \rangle \\
&= [\alpha_s] \underbrace{\sum_{i \in \mathcal{H}_f} \frac{1}{N}}_{=1} \left\langle \frac{\Gamma_{q \rightarrow gq}}{\epsilon} \underbrace{\frac{1}{(N-1)!} F_{\text{LM}}^{gq}[\{g\}_{N-1}, q]}_{\equiv F_{\text{LM}}^{\mathcal{B}_0}} \right\rangle \\
&= [\alpha_s] \left\langle \frac{\Gamma_{q \rightarrow gq}}{\epsilon} F_{\text{LM}}^{\mathcal{B}_0} \right\rangle,
\end{aligned} \tag{3.120}$$

where $\Gamma_{q \rightarrow gq}$ is given in (3.72). We combine Eqs (3.118) and (3.120) and obtain

$$\sum_{i \in \mathcal{H}} \mathcal{P}_{\mathcal{B}_0} \langle \bar{S}_m C_{im} \Delta^{(m)} F_{\text{R}}^{\mathcal{B}_1} \rangle = [\alpha_s] \left\langle \left[\sum_{i \in \mathcal{H}_{f,g}} \frac{\Gamma_{i, g \rightarrow gg}}{\epsilon} + \frac{\Gamma_q}{\epsilon} \right] F_{\text{LM}}^{\mathcal{B}_0} \right\rangle, \tag{3.121}$$

where we introduced the *generalized quark final-state anomalous dimension*

$$\Gamma_{i,q} \stackrel{\text{def}}{=} \Gamma_{i, q \rightarrow qq} + \Gamma_{i, q \rightarrow gq}. \tag{3.122}$$

Once more, since we have one final-state quark, we write Γ_q . Note that Eq. (3.122) satisfies the identity

$$\Gamma_{i,q} = 2L_i \mathbf{T}_q^2 + \gamma_q + \mathcal{O}(\epsilon), \quad (3.123)$$

with $\gamma_q = \frac{3}{2}C_F$ (see Eq. (2.23)). The expansion (3.123) coincides with that reported in Eq. (3.97).

It is worth spending a few words on the object $\Gamma_{i,q}$ in Eq. (3.122). To do so, we take a step back and consider the anomalous dimensions $\gamma_{z,q \rightarrow qg}^{22}(\epsilon, L_i)$ and $\gamma_{z,q \rightarrow gq}^{22}(\epsilon)$ in Eq. (A.59). Since the relation $P_{qg}(1-z) = P_{qq}(z)$ holds, we observe that

$$\begin{aligned} & \gamma_{z,q \rightarrow qg}^{22}(L_i) + \gamma_{z,q \rightarrow gq}^{22} \\ &= - \int_0^1 dz \left[[z(1-z)]^{-2\epsilon} \left(zP_{qq}(z) + zP_{qg}(z) \right) - \frac{2C_F}{(1-z)^{1+2\epsilon}} \right] + C_F \frac{1 - e^{-2\epsilon L_i}}{\epsilon} \\ &= - \int_0^1 dz \left[[z(1-z)]^{-2\epsilon} \left(zP_{qq}(z) + (1-z)P_{qg}(z) \right) - \frac{2C_F}{(1-z)^{1+2\epsilon}} \right] + C_F \frac{1 - e^{-2\epsilon L_i}}{\epsilon} \\ &= - \int_0^1 dz (\mathbb{1} - S_z) \left[[z(1-z)]^{-2\epsilon} P_{qq}(z) \right] + C_F \frac{1 - e^{-2\epsilon L_i}}{\epsilon} \\ &\equiv \gamma_{1,q \rightarrow qg}^{22}(L_i). \end{aligned} \quad (3.124)$$

In other words, the combination of the above two anomalous dimensions effectively removes the factor z within the integral and gives rise to $\gamma_{1,q \rightarrow qg}^{22}(L_i)$, that is the quark anomalous dimension. This is what we would expect because, in a certain sense, the factor z is not physical, but arises from the introduction of the damping factors $\Delta^{(i)}$ to disentangle collinear limits in the final state (recall that $C_{im}\Delta^{(m)} = z$). In other words, writing $z + (1-z) = 1$ reflects the unitarity property $\sum_i \Delta^{(i)} = 1$ used in Eq. (3.105). Physically, the fact that the z factor disappears indicates that we have correctly recombined the final-state divergences.

Having clarified these aspects, we return to our discussion of the collinear limits. We introduce the collinear operator $I_C^{\mathcal{B}_0}$, which is defined similarly to the definition given in (3.75) for process $\mathcal{A}_0$, but clearly adapted to process $\mathcal{B}_0$:

$$I_C^{(k),\mathcal{B}_0}(\epsilon) \stackrel{\text{def}}{=} \sum_{i=a,b} \frac{\Gamma_{i,f_i}^{(k)}}{\epsilon k/2} + \sum_{i \in \mathcal{H}_{f,g}} \frac{\Gamma_{i,g \rightarrow gg}^{(k)}}{\epsilon k/2} + \frac{\Gamma_q^{(k)}}{\epsilon k/2}, \quad (3.125)$$

with $f_a = g$ and $f_b = q$. Note that it still satisfies the property (we set $n_f = 0$)

$$I_C^{\mathcal{B}_0}(\epsilon) = \sum_{i \in \mathcal{H}} \left(2L_i \frac{\mathbf{T}_i^2}{\epsilon} + \frac{\gamma_i}{\epsilon} \right) + \mathcal{O}(\epsilon^0), \quad (3.126)$$

that we already saw in Eq. (3.98) for process $\mathcal{A}_0$. Therefore, putting together all the hard-collinear contributions, we obtain

$$\sum_{i \in \mathcal{H}} \mathcal{P}_{\mathcal{B}_0} \langle \bar{S}_m C_{im} \Delta^{(m)} F_R^{\mathcal{B}_1} \rangle = [\alpha_s] \langle I_C(\epsilon) \cdot F_{LM}^{\mathcal{B}_0} \rangle$$

$$+ \frac{[\alpha_s]}{\epsilon} \left[\langle \mathcal{P}_{gg}^{\text{gen}} \otimes F_{\text{LM}}^{\mathcal{B}_0} \rangle + \langle F_{\text{LM}}^{\mathcal{B}_0} \otimes \mathcal{P}_{qq}^{\text{gen}} \rangle \right]. \quad (3.127)$$

Having determined the IR singularities of the real emission contribution, we now consider virtual and PDFs renormalization contributions, which read, respectively,

$$2s_{ab} d\bar{\sigma}_{\mathcal{B}_0}^V = [\alpha_s] \langle I_V(\epsilon) \cdot F_{\text{LM}}^{\mathcal{B}_0} \rangle + \langle F_{\text{LV,fin}}^{\mathcal{B}_0} \rangle, \quad (3.128)$$

and⁹

$$\mathcal{P}_{\mathcal{B}_0} [2s_{ab} d\bar{\sigma}_{\mathcal{B}_1}^{\text{pdf}}] = \left[\frac{\alpha_s(\mu)}{2\pi} \right] \frac{1}{\epsilon} \left[\langle \hat{P}_{gg}^{(0)} \otimes F_{\text{LM}}^{\mathcal{B}_0} \rangle + \langle F_{\text{LM}}^{\mathcal{B}_0} \otimes \hat{P}_{qq}^{(0)} \rangle \right], \quad (3.129)$$

with $\hat{P}_{\alpha\beta}^{(0)}$ given in Eq. (1.58).

Finally, we can write the NLO contribution to the process $\mathcal{B}_0$ in (3.56) just by combining Eqs (3.114), (3.115), (3.127), (3.128) and (3.129). We find

$$\begin{aligned} \mathcal{P}_{\mathcal{B}_0} [2s_{ab} d\bar{\sigma}_{\mathcal{B}_1}^{\text{NLO}}] &= [\alpha_s] \langle I_{\text{T}}^{(0)} \cdot F_{\text{LM}}^{\mathcal{B}_0} \rangle + \langle F_{\text{LV,fin}}^{\mathcal{B}_0} \rangle + \langle \mathcal{O}_{\text{NLO}}^{(\text{m})} \Delta^{(\text{m})} F_{\text{R}}^{\mathcal{B}_1} \rangle \\ &+ [\alpha_s] \left[\langle \mathcal{P}_{gg}^{\text{NLO}} \otimes F_{\text{LM}}^{\mathcal{B}_0} \rangle + \langle F_{\text{LM}}^{\mathcal{B}_0} \otimes \mathcal{P}_{qq}^{\text{NLO}} \rangle \right], \end{aligned} \quad (3.130)$$

where for the functions $\mathcal{P}_{\alpha\beta}^{\text{NLO}}$ we used the expansion (3.100), and we introduced the operator

$$I_{\text{T}}^{\mathcal{B}_0}(\epsilon) \stackrel{\text{def}}{=} I_V(\epsilon) + I_S(\epsilon) + I_{\text{C}}^{\mathcal{B}_0}(\epsilon). \quad (3.131)$$

By virtue of the proof presented in the previous subsection, $I_{\text{T}}^{\mathcal{B}_0}(\epsilon)$ is ϵ -finite, and hence so is the cross section $d\bar{\sigma}_{\mathcal{B}_1}^{\text{NLO}}$ in Eq. (3.130). Additionally, we can appreciate that the NLO expression in Eq. (3.130) is analogous to the one in (3.101) for the $\mathcal{A}_0$ partonic process.

It remains to explore the fate of those terms that were nullified by the projector $\mathcal{P}_{\mathcal{B}_0}$. These terms correspond to purely initial-state collinear singularities, whose poles of $\mathcal{O}(\epsilon^{-1})$ cancel out when combined with the corresponding collinear singularities arising from PDFs renormalization. Thus, under the action of the operators $\mathcal{P}_{\mathcal{A}_0^{(\bar{q}q)}}$ and $\mathcal{P}_{\mathcal{A}_0^{(gg)}}$, we find

$$\begin{aligned} \mathcal{P}_{\mathcal{A}_0^{(\bar{q}q)}} [2s_{ab} d\bar{\sigma}_{\mathcal{B}_1}^{\text{R}}] &\equiv \langle C_{am} \Delta^{(\text{m})} F_{\text{LM}}^{\mathcal{B}_1}[\mathbf{m}_q] \rangle = \frac{[\alpha_s]}{\epsilon} \langle \mathcal{P}_{qq}^{\text{gen}} \otimes F_{\text{LM}}^{\mathcal{A}_0^{(\bar{q}q)}} \rangle, \\ \mathcal{P}_{\mathcal{A}_0^{(\bar{q}q)}} [2s_{ab} d\bar{\sigma}_{\mathcal{B}_1}^{\text{pdf}}] &\equiv \left[\frac{\alpha_s(\mu)}{2\pi} \right] \frac{1}{\epsilon} \langle \hat{P}_{qq}^{(0)} \otimes F_{\text{LM}}^{\mathcal{A}_0^{(\bar{q}q)}} \rangle, \end{aligned} \quad (3.132)$$

and

$$\begin{aligned} \mathcal{P}_{\mathcal{A}_0^{(gg)}} [2s_{ab} d\bar{\sigma}_{\mathcal{B}_1}^{\text{R}}] &\equiv \langle C_{bm} \Delta^{(\text{m})} F_{\text{LM}}^{\mathcal{B}_1}[\mathbf{m}_q] \rangle = \frac{[\alpha_s]}{\epsilon} \langle F_{\text{LM}}^{\mathcal{A}_0^{(gg)}} \otimes \mathcal{P}_{gg}^{\text{gen}} \rangle, \\ \mathcal{P}_{\mathcal{A}_0^{(gg)}} [2s_{ab} d\bar{\sigma}_{\mathcal{B}_1}^{\text{pdf}}] &\equiv \left[\frac{\alpha_s(\mu)}{2\pi} \right] \frac{1}{\epsilon} \langle F_{\text{LM}}^{\mathcal{A}_0^{(gg)}} \otimes \hat{P}_{gg}^{(0)} \rangle. \end{aligned} \quad (3.133)$$

⁹See Appendix A.3.2 for details.

It follows that¹⁰

$$\begin{aligned}\mathcal{P}_{\mathcal{A}_0^{(\bar{q}q)}}[2s_{ab} d\vec{\sigma}_{\mathcal{B}_1}^{\text{NLO}}] &= [\alpha_s] \langle \mathcal{P}_{qg}^{\text{NLO}} \otimes F_{\text{LM}}^{\mathcal{A}_0^{(\bar{q}q)}} \rangle, \\ \mathcal{P}_{\mathcal{A}_0^{(gg)}}[2s_{ab} d\vec{\sigma}_{\mathcal{B}_1}^{\text{NLO}}] &= [\alpha_s] \langle F_{\text{LM}}^{\mathcal{A}_0^{(gg)}} \otimes \mathcal{P}_{gq}^{\text{NLO}} \rangle.\end{aligned}\tag{3.134}$$

This result can be interpreted as a pure *off-diagonal* collinear contribution to be added to the NLO corrections in Eq. (3.101).

With this, we conclude our discussion of IR subtractions at NLO. To summarize, we have examined a method to extract the IR singularities from the real emission and cancel them with those arising from virtual corrections and PDF renormalization. We introduced the operators I_S and I_C , which encapsulate the soft and collinear poles, and used these to develop a more general framework. Finally, we showed that the elastic terms in NLO corrections can be described by an ϵ -finite operator, I_T , whose definition is (almost) process-independent. In the following chapters, we will extend this formulation to NNLO.

¹⁰Note that the above equations do not contain the real emission term that is regulated by the $\mathcal{O}_{\text{NLO}}$ operator. Such terms are fully contained in Eq. (3.130).

New approach to NSC at NNLO

Contents

4.1	Introduction to NSC at NNLO	100
4.1.1	An Overview of the NSC at NNLO	100
4.1.2	Collinear regularization of double-real cross section	105
4.2	The new approach to NSC	108
4.2.1	A motivation for generalization	108
4.2.2	Treatment of unresolved quarks and gluons	110
4.3	Analyzing double- and single-collinear contributions: $m_g n_g$ case	114
4.3.1	$\Sigma_{RR,1c}^{(a,c,dc)} _{\mathcal{H},gg}$ contribution	115
4.3.2	$\Sigma_{RR,1c}^{(b,d)} _{\mathcal{H},gg}$ contribution	120
4.3.3	Double-collinear contribution	128
4.4	Analyzing double- and single-collinear contributions: $m_q n_g$ case	128
4.4.1	$\Sigma_{RR,1c}^{(a,c,dc)} _{\mathcal{H},qg}$ contribution	129
4.4.2	$\Sigma_{RR,1c}^{(b,d)} _{\mathcal{H},qg}$ contribution	131
4.5	Rearranging single- and double-unresolved terms	133
4.5.1	The $m_g n_g$ case	134
4.5.2	The $m_q n_g$ case	137

In the previous chapter, we discussed the NSC/FKS subtraction method at NLO, demonstrating that it can be generalized to arbitrary production processes by introducing what we termed *I*-operators, namely I_V , I_S , I_C , and I_T . These operators allowed us to express NLO corrections in an explicitly IR-finite form for processes $\mathcal{A}_0$ and $\mathcal{B}_0$, as shown in Eqs (3.101) and (3.130), respectively.

However, formulating subtractions at NNLO is considerably more challenging than at NLO, primarily due to the issue of *overlapping singularities*. As we have previously noted in the context of NLO, the NSC method is a subtraction scheme that addresses this issue using a *fully local* and *fully analytic* approach. In the following, we will demonstrate how this method can be written using the *I*-operators and how it simplifies the cancellation of poles, providing a framework for arbitrary processes. The formulation described in this chapter and the subsequent two is *novel* in the literature, both in concept and in the results obtained, and represents a significant advance in the study of NNLO subtractions.

We consider the LO processes $\mathcal{A}_0$ and $\mathcal{B}_0$, which we rewrite below for convenience

$$\mathcal{A}_0 : a + b \rightarrow \{g\}_N + X, \quad (4.1)$$

$$\mathcal{B}_0 : g_a + q_b \rightarrow \{g\}_{N-1} + q + X, \quad (4.2)$$

where X is a final-state color singlet. Our goal is to calculate the NNLO corrections for these processes, with the important assumption that the parameter n_f is set to zero. Conceptually, the calculation of corrections proportional to n_f does not differ significantly from the corrections we are about to present. However, we prefer to exclude them to simplify the reading, which will already be quite dense due to the various technicalities that will be discussed. Moreover, as we will soon see, the NNLO corrections to process $\mathcal{A}_0$ can, at least conceptually, be derived from those of process $\mathcal{B}_0$. We will observe that there is no practical difference regarding the initial state, while for the final state, our approach will simply allow us to *turn off* the final-state quark, thereby reducing process $\mathcal{B}_0$ to $\mathcal{A}_0$.

We will structure this chapter as follows. In Sec. 4.1, we introduce the NSC at NNLO in full detail. In Sec. 4.2, we examine the obstacles that would arise in generalizing this method to arbitrary hadron production processes, and we introduce a new organizational principle that will allow us to use the I -operators introduced in Ch. 3 to overcome these obstacles. Secs 4.3 through 4.5 focus on reorganizing the single- and double-unresolved terms to lay the groundwork for the introduction of the I -operators. In this chapter, we do not present explicit calculations, but rather concentrate on restructuring the overarching framework of the NSC. The ultimate goal, in addition to introducing a new organizational approach to the NSC method, is to derive two fully general formulas, specifically Eqs (4.170) and (4.171) and Eqs (4.179) and (4.183) for the single- and double-unresolved contributions, respectively.

4.1 Introduction to NSC at NNLO

4.1.1 An Overview of the NSC at NNLO

In this section, we discuss some general aspects of NNLO calculations before providing an broad overview of the NSC scheme for the subtraction of IR divergences. Following the approach of Refs [23, 81], we will begin by considering the process $\mathcal{A}_0$ in (4.1) as a toy model, as it presents fewer peculiarities compared to $\mathcal{B}_0$. At this order of perturbation theory, we must combine the double-virtual, real-virtual, double-real, and PDF renormalization contributions to compute the differential cross section, as outlined in Sec. 1.5. Thus, we write

$$d\hat{\sigma}_{\mathcal{A}_0}^{\text{NNLO}} = d\hat{\sigma}_{\mathcal{A}_0}^{\text{VV}} + d\hat{\sigma}_{\mathcal{A}_1}^{\text{RV}} + d\hat{\sigma}_{\mathcal{A}_2}^{\text{RR}} + d\hat{\sigma}_{\mathcal{A}_2}^{\text{pdf}}, \quad (4.3)$$

where:

- i) $d\hat{\sigma}_{\mathcal{A}_0}^{\text{VV}}$ represents the *double-virtual* corrections to the hard process $\mathcal{A}_0$.
- ii) $d\hat{\sigma}_{\mathcal{A}_1}^{\text{RV}}$ denotes *real-virtual* corrections to process $\mathcal{A}_0$, i.e., is the virtual correction to process $\mathcal{A}_1$.

iii) $d\hat{\sigma}_{\mathcal{A}_2}^{\text{RR}}$ accounts for *double-real* corrections to process $\mathcal{A}_0$, i.e., a process

$$\mathcal{A}_2 : \quad a + b \rightarrow \{g\}_{N+2} + X, \quad (4.4)$$

with two additional gluon in the final state. For NNLO corrections, we only consider additional gluon radiation in the final state, as in Ch. 3.

iv) $d\hat{\sigma}_{\mathcal{A}_2}^{\text{pdf}}$ includes corrections due to the collinear renormalization of parton distribution functions.

We now write the various terms in Eq. (4.3) using the notation that we have introduced in the previous chapter. We start with $d\hat{\sigma}_{\mathcal{A}_0}^{\text{VV}}$, which we write as

$$2s_{ab} d\hat{\sigma}_{\mathcal{A}_0}^{\text{VV}} = \left\langle \frac{1}{N!} F_{\text{VV}}^{ab}[\{g\}_N] \right\rangle \equiv \langle F_{\text{VV}}^{\mathcal{A}_0} \rangle, \quad (4.5)$$

where

$$F_{\text{VV}}^{ab}[\{i\}_{\mathcal{H}_f}] \stackrel{\text{def}}{=} \left[\frac{\alpha_s(\mu)}{2\pi} \right]^2 \left[2 \text{Re}[\langle \mathcal{M}_0 | \mathcal{M}_2 \rangle] + \langle \mathcal{M}_1 | \mathcal{M}_1 \rangle \right] \text{dLips}_X \hat{O}(\{p\}_{\mathcal{H}_f}; p_X), \quad (4.6)$$

and $|\mathcal{M}_k\rangle \equiv |\mathcal{M}_k(\{p\}_{\mathcal{H}}, p_X)\rangle$.¹ Next, we consider $d\hat{\sigma}_{\mathcal{A}_1}^{\text{RV}}$. Similar to the NLO case, we distinguish between resolved and potentially unresolved partons using the partitions $\Delta^{(i)}$ defined in Appendix B, and write

$$\begin{aligned} 2s_{ab} d\hat{\sigma}_{\mathcal{A}_1}^{\text{RV}} &= \left\langle \frac{1}{(N+1)!} F_{\text{RV}}^{ab}[\{g\}_{N+1}] \right\rangle = \sum_{i \in \mathcal{H}_f} \left\langle \frac{\Delta^{(i)}}{(N+1)!} F_{\text{RV}}^{ab}[\{g\}_{N+1}] \right\rangle \\ &= \langle \Delta^{(\text{m})} F_{\text{RV}}^{ab}[\{g\}_N | \mathbf{m}_g] \rangle, \end{aligned} \quad (4.7)$$

where

$$\begin{aligned} F_{\text{RV}}^{ab}[\{i\}_{\mathcal{H}_f+1}] &\stackrel{\text{def}}{=} \left[\frac{\alpha_s(\mu)}{2\pi} \right] 2 \text{Re}[\langle \mathcal{M}_0(\{p\}_{\mathcal{H}+1}, p_X) | \mathcal{M}_1(\{p\}_{\mathcal{H}+1}, p_X) \rangle] \\ &\quad \times \text{dLips}_X \hat{O}(\{p\}_{\mathcal{H}_f+1}; p_X), \end{aligned} \quad (4.8)$$

and

$$F_{\text{RV}}^{ab}[\{g\}_N | \mathbf{m}_g] \stackrel{\text{def}}{=} \frac{1}{N!} F_{\text{RV}}^{ab}[\{g\}_{N+1}]. \quad (4.9)$$

In Eq. (4.8), symbol $\mathcal{H} + 1$ means that $\mathcal{M}_0$ and $\mathcal{M}_1$ amplitudes contain an extra parton with respect to the LO hard set $\mathcal{H}$, and the same holds for $\mathcal{H}_f + 1$ as well. Finally, we consider $d\hat{\sigma}_{\mathcal{A}_2}^{\text{RR}}$. Once more, we introduce damping factors $\Delta^{(ij)}$, defined Appendix B, to

¹We recall that $\mathcal{H}_f$ labels all the partons in the final state, and we note that Eq. (4.6) includes both two-loop and one-loop squared contributions.

distinguish the resolved partons from the two potentially unresolved ones. We obtain

$$\begin{aligned}
 2s_{ab} \, d\hat{\sigma}_{\mathcal{A}_2}^{\text{RR}} &= \left\langle \frac{1}{(N+2)!} F_{\text{LM}}^{ab}[\{g\}_{N+2}] \right\rangle = \sum_{\substack{i,j \in \{g\}_{N+2} \\ i < j}} \left\langle \frac{\Delta^{(ij)}}{(N+2)!} F_{\text{LM}}^{ab}[\{g\}_{N+2}] \right\rangle \\
 &= \left\langle \frac{1}{2!} \Delta^{(\text{mn})} F_{\text{LM}}^{ab}[\{g\}_N | \mathbf{m}_g, \mathbf{n}_g] \right\rangle. \tag{4.10}
 \end{aligned}$$

In order to get the second line in Eq. (4.10), starting from the last term in the first line, we write

$$\sum_{\substack{i,j \in \{g\}_{N+2} \\ i < j}} \Delta^{(ij)} = \sum_{\substack{i,j \in \mathcal{H}_f \\ i < j}} \Delta^{(ij)} + \sum_{i \in \mathcal{H}_f} \Delta^{(i\text{m})} + \sum_{i \in \mathcal{H}_f} \Delta^{(i\text{n})} + \Delta^{(\text{mn})}. \tag{4.11}$$

We then use the property that F_{LM} is invariant under exchange of any 2 gluons to relabel the unresolved ones as $\mathbf{m}_g$ and $\mathbf{n}_g$. The damping factors are then all denoted as $\Delta^{(\text{mn})}$, allowing us to write

$$\begin{aligned}
 \sum_{\substack{i,j \in \{g\}_{N+2} \\ i < j}} \Delta^{(ij)} &= \Delta^{(\text{mn})} \left[\sum_{\substack{i,j \in \mathcal{H}_f \\ i < j}} + \left(2 \sum_{i \in \mathcal{H}_f} \right) + 1 \right] = \Delta^{(\text{mn})} \left[\frac{N^2 - N}{2} + 2N + 1 \right] \\
 &= \Delta^{(\text{mn})} \frac{(N+2)(N+1)}{2}.
 \end{aligned}$$

Thus we find

$$\sum_{\substack{i,j \in \{g\}_{N+2} \\ i < j}} \frac{\Delta^{(ij)}}{(N+2)!} = \frac{\Delta^{(\text{mn})}}{2!} \times \frac{1}{N!}. \tag{4.12}$$

Note that the factor $1/N!$ is precisely the factor reabsorbed by $F_{\text{LM}}^{ab}[\{g\}_N | \mathbf{m}_g, \mathbf{n}_g]$ in Eq. (4.10), since it contains N hard gluons.

Combining the above results, we write the NNLO contribution to the cross section without the PDFs renormalization in the following way:²

$$2s_{ab} \, d\bar{\sigma}_{\mathcal{A}_0}^{\text{NNLO}} = \langle F_{\text{VV}}^{\mathcal{A}_0} \rangle + \langle \Delta^{(\text{m})} F_{\text{RV}}^{ab}[\{g\}_N | \mathbf{m}_g] \rangle + \left\langle \frac{1}{2!} \Delta^{(\text{mn})} F_{\text{LM}}^{ab}[\{g\}_N | \mathbf{m}_g, \mathbf{n}_g] \right\rangle. \tag{4.13}$$

We lighten the notation as

$$\begin{aligned}
 F_{\text{RV}}^{\mathcal{A}_1}[\mathbf{m}_g] &\stackrel{\text{def}}{=} F_{\text{RV}}^{ab}[\{g\}_N | \mathbf{m}_g], \\
 F_{\text{LM}}^{\mathcal{A}_2}[\mathbf{m}_g, \mathbf{n}_g] &\stackrel{\text{def}}{=} F_{\text{LM}}^{ab}[\{g\}_N | \mathbf{m}_g, \mathbf{n}_g], \tag{4.14}
 \end{aligned}$$

²The difference between $d\bar{\sigma}_{\mathcal{A}_0}^{\text{NNLO}}$ and $d\hat{\sigma}_{\mathcal{A}_0}^{\text{NNLO}}$ is that the former does not contain the contribution coming from the renormalization of the PDFs.

and we use it to write the compact formula

$$2s_{ab} \, d\bar{\sigma}_{\mathcal{A}_0}^{\text{NNLO}} = \langle F_{VV}^{\mathcal{A}_0} \rangle + \langle \Delta^{(m)} F_{RV}^{\mathcal{A}_1}[\mathbf{m}_g] \rangle + \left\langle \frac{1}{2!} \Delta^{(mn)} F_{LM}^{\mathcal{A}_2}[\mathbf{m}_g, \mathbf{n}_g] \right\rangle. \quad (4.15)$$

Eq. (4.15) provides the starting point for our calculation. It follows that the NNLO QCD corrections to the cross section contain contributions that exist in three distinct phase spaces. These phase spaces overlap in configurations where the gluons labelled as $\mathbf{m}_g$ and $\mathbf{n}_g$ become unresolved. When this happens, the corresponding amplitudes become singular, and integrating over unresolved phase spaces leads to the appearance of $1/\epsilon^n$ poles, similar to the NLO case, discussed in Ch. 2. Our goal is to isolate and remove these singularities locally in the phase space, demonstrate the cancellation of poles between the different contributions in Eq. (4.15), and determine the finite remainder.

One of the main challenges at NNLO is the issue of *overlapping singularities* in the double-real contribution. We recall from Ch. 2 that the double-real cross section can generally contain double-soft, single-soft, triple-collinear, and single-collinear contributions. Note that soft and collinear limits always commute when one considers on-shell, gauge-invariant amplitudes due to color coherence. It follows that the soft singularities can be regulated independently of the collinear ones. One typically starts with soft regularization and then proceeds with the collinear one. As we will see in the next subsection, overlapping collinear singularities are disentangled using a combination of *partitions* and *sectors*.

Therefore, we begin by isolating the soft limits of the real-emission contributions. As already discussed in Ref. [23], two soft limits are needed: one to describe the double-soft limit $E_m \sim E_n \rightarrow 0$, i.e. S_{mn} , and one for the single-soft limit $E_n \rightarrow 0$ at fixed E_m , i.e. S_n . The overlap between these soft limits can be removed by introducing the energy ordering of the unresolved gluons $\mathbf{m}_g$ and $\mathbf{n}_g$ as

$$\left\langle \frac{1}{2!} \Delta^{(mn)} F_{LM}^{ab}[\{g\}_N | \mathbf{m}_g, \mathbf{n}_g] \right\rangle = \langle \Delta^{(mn)} \Theta_{mn} F_{LM}^{ab}[\{g\}_N | \mathbf{m}_g, \mathbf{n}_g] \rangle, \quad (4.16)$$

where $\Theta_{mn} = \Theta(E_m - E_n)$. Then, we introduce soft regularization, which yields

$$\begin{aligned} \langle \Delta^{(mn)} \Theta_{mn} F_{LM}^{\mathcal{A}_2}[\mathbf{m}_g, \mathbf{n}_g] \rangle &= \langle S_{mn} \Theta_{mn} F_{LM}^{\mathcal{A}_2}[\mathbf{m}_g, \mathbf{n}_g] \rangle + \langle \bar{S}_{mn} S_n \Delta^{(mn)} \Theta_{mn} F_{LM}^{\mathcal{A}_2}[\mathbf{m}_g, \mathbf{n}_g] \rangle \\ &\quad + \langle \bar{S}_{mn} \bar{S}_n \Delta^{(mn)} \Theta_{mn} F_{LM}^{\mathcal{A}_2}[\mathbf{m}_g, \mathbf{n}_g] \rangle, \end{aligned} \quad (4.17)$$

where we used $S_{mn} \Delta^{(mn)} = 1$, and we recall that $\bar{S}_x = \mathbb{1} - S_x$ has already been introduced in the context of the NLO QCD computation. We will return to the first term on the right-hand side in the following chapter, where we will write an explicit expression for it. For now, we focus on the second term on the right-hand side of the above formula,

$$\langle \bar{S}_{mn} S_n \Delta^{(mn)} \Theta_{mn} F_{LM}^{\mathcal{A}_2}[\mathbf{m}_g, \mathbf{n}_g] \rangle, \quad (4.18)$$

make use of the fact that $S_{mn} S_n = S_m S_n$ and $S_n \Delta^{(mn)} = \Delta^{(m)}$, to write

$$\langle \bar{S}_{mn} S_n \Delta^{(mn)} \Theta_{mn} F_{LM}^{\mathcal{A}_2}[\mathbf{m}_g, \mathbf{n}_g] \rangle \equiv \langle \bar{S}_m \Delta^{(m)} S_n \Theta_{mn} F_{LM}^{\mathcal{A}_2}[\mathbf{m}_g, \mathbf{n}_g] \rangle. \quad (4.19)$$

We observe that the soft limit S_m in the above expression effectively removes the gluon n_g from the argument of F_{LM} , while the operator $\bar{S}_m$ regulates against soft limits of the gluon m_g . Thus, the right-hand side has the singular structure of a soft-regulated NLO real-emission expression. We therefore need to include collinear subtractions for the gluon m_g . We find

$$\begin{aligned} \langle \bar{S}_{mn} S_n \Delta^{(mn)} \Theta_{mn} F_{LM}^{A_2} [m_g, n_g] \rangle &= \langle \mathcal{O}_{\text{NLO}}^{(m)} \Delta^{(m)} \langle S_n \Theta_{mn} F_{LM}^{A_2} [m_g, n_g] \rangle \rangle \\ &+ \sum_{i \in \mathcal{H}} \langle \bar{S}_m C_{im} \Delta^{(m)} \langle S_n \Theta_{mn} F_{LM}^{A_2} [m_g, n_g] \rangle \rangle, \end{aligned} \quad (4.20)$$

with $\mathcal{O}_{\text{NLO}}^{(m)}$ defined in Eq. (3.31). As we will see, the order of the collinear operator C_{im} and the soft operator S_n in (4.20) plays a central role. The presence of the double parentheses $\langle \langle \dots \rangle \rangle$ on the right-hand side of (4.20) serves to isolate the order of the operations. In the final term of Eq. (4.20), the soft limit S_n is applied first, and then we integrate over the phase space of the particle n_g , as indicated by the innermost parentheses $\langle \dots \rangle$. Only afterward is the hard-collinear limit $\bar{S}_m C_{im}$ applied to the result of this previous step. The final integral, over the phase space of m_g , is indicated by the outermost parentheses $\langle \dots \rangle$. We emphasize this because, as we will see, applying these operations in the correct order will play a central role in the introduction of I -operators to the NSC.

To obtain a similar structure to (4.20) for the real-virtual contribution, we rewrite F_{RV} as

$$\langle \Delta^{(m)} F_{RV}^{A_1} [m_g] \rangle = \langle S_m F_{RV}^{A_1} [m_g] \rangle + \langle \bar{S}_m \Delta^{(m)} F_{RV}^{A_1} [m_g] \rangle, \quad (4.21)$$

where again $S_m \Delta^{(m)} = 1$. Note that the right-hand side of this formula contains collinear singularities that can be regularized as in the NLO case, which yields

$$\begin{aligned} \langle \Delta^{(m)} F_{RV}^{A_1} [m_g] \rangle &= \langle S_m F_{RV}^{A_1} [m_g] \rangle + \sum_{i \in \mathcal{H}} \langle \bar{S}_m C_{im} \Delta^{(m)} F_{RV}^{A_1} [m_g] \rangle \\ &+ \langle \mathcal{O}_{\text{NLO}}^{(m)} \Delta^{(m)} F_{RV}^{A_1} [m_g] \rangle. \end{aligned} \quad (4.22)$$

Thus, we can collect the results from Eqs (4.5), (4.17), (4.20), (4.22), and write

$$2s_{ab} d\bar{\sigma}_{\mathcal{A}_0}^{\text{NNLO}} = \Sigma_{\text{DU}}^{(1)}|_{\mathcal{A}_0} + \Sigma_{\text{SU}}^{(1)}|_{\mathcal{A}_0} + \Sigma_{\text{RR}}|_{\mathcal{A}_0}, \quad (4.23)$$

where

$$\begin{aligned} \Sigma_{\text{DU}}^{(1)}|_{\mathcal{A}_0} &\stackrel{\text{def}}{=} \langle F_{VV}^{A_0} \rangle + \langle S_{mn} \Theta_{mn} F_{LM}^{A_2} [m_g, n_g] \rangle + \langle S_m F_{RV}^{A_1} [m_g] \rangle \\ &+ \sum_{i \in \mathcal{H}} \left\langle \bar{S}_m C_{im} \Delta^{(m)} \left[F_{RV}^{A_1} [m_g] + \langle S_n \Theta_{mn} F_{LM}^{A_2} [m_g, n_g] \rangle \right] \right\rangle, \end{aligned} \quad (4.24)$$

$$\Sigma_{\text{SU}}^{(1)}|_{\mathcal{A}_0} \stackrel{\text{def}}{=} \left\langle \mathcal{O}_{\text{NLO}}^{(m)} \Delta^{(m)} \left[F_{RV}^{A_1} [m_g] + \langle S_n \Theta_{mn} F_{LM}^{A_2} [m_g, n_g] \rangle \right] \right\rangle, \quad (4.25)$$

$$\Sigma_{\text{RR}}|_{\mathcal{A}_0} \stackrel{\text{def}}{=} \langle \bar{S}_{mn} \bar{S}_n \Delta^{(mn)} \Theta_{mn} F_{LM}^{A_2} [m_g, n_g] \rangle. \quad (4.26)$$

The quantity $\Sigma_{\text{DU}}^{(1)}|_{\mathcal{A}_0}$ is *double-unresolved*, in the sense that both gluons $\mathbf{m}_g$ and $\mathbf{n}_g$ are either soft or collinear. The superscript indicates that this is the *first* of several contributions to $\Sigma_{\text{DU}}|_{\mathcal{A}_0}$ that has been identified. Similarly, the quantity $\Sigma_{\text{SU}}^{(1)}|_{\mathcal{A}_0}$ is the first *single-unresolved* term contributing to Σ_{SU} that we identify. On the contrary, $\Sigma_{\text{RR}}|_{\mathcal{A}_0}$ is a mix of fully-regulated, single-unresolved, and double-unresolved contributions. We devote the next subsection to extracting the singularities contained within it. As we will see, once we have done so, some parts of $\Sigma_{\text{RR}}|_{\mathcal{A}_0}$ will contribute to $\Sigma_{\text{DU}}|_{\mathcal{A}_0}$ and $\Sigma_{\text{SU}}|_{\mathcal{A}_0}$ and will play an important role in the cancellation of IR poles.

4.1.2 Collinear regularization of double-real cross section

We now focus on $\Sigma_{\text{RR}}|_{\mathcal{A}_0}$ in (4.26), which, as described in the previous subsection, is regulated against soft singularities but still contains collinear singularities, which overlap. In the NSC, we disentangle these in two stages. First, we partition the angular phase space [19, 20, 23, 65], in a way that is analogous to what was done at NLO. We begin with the identity

$$1 = \sum_{i,j \in \mathcal{H}} \omega^{\mathbf{m}i, \mathbf{n}j} = \sum_{(ij)} \omega^{\mathbf{m}i, \mathbf{n}j} + \sum_{i \in \mathcal{H}} \omega^{\mathbf{m}i, \mathbf{n}i}, \quad (4.27)$$

where (ij) is equivalent to $i, j \in \mathcal{H}$ with $i \neq j$. Here, $\omega^{\mathbf{m}i, \mathbf{n}j}$ are NNLO partition functions, defined in Appendix B. We note that the partitions $\omega^{\mathbf{m}i, \mathbf{n}j}$ with $i \neq j$ only involve collinear limits where partons $\mathbf{m}$ and $\mathbf{n}$ become collinear to different partons i and j . For this reason, we call these partitions *double-collinear*. On the other hand, the partitions $\omega^{\mathbf{m}i, \mathbf{n}i}$ involve collinear limits where partons $\mathbf{m}$ and $\mathbf{n}$ become collinear to the same parton i , and thus we refer to them as *triple-collinear partitions*. We note that there is no overlap of collinear limits in the double-collinear partition, but this is not the case in the triple-collinear one. To further disentangle the limits, we introduce the following *sectoring* of the phase space:

$$\begin{aligned} \theta^{(a)} &\stackrel{\text{def}}{=} \Theta\left(\eta_{\text{in}} < \frac{\eta_{\text{im}}}{2}\right), & \theta^{(c)} &\stackrel{\text{def}}{=} \Theta\left(\eta_{\text{im}} < \frac{\eta_{\text{in}}}{2}\right), \\ \theta^{(b)} &\stackrel{\text{def}}{=} \Theta\left(\frac{\eta_{\text{im}}}{2} < \eta_{\text{in}} < \eta_{\text{im}}\right), & \theta^{(d)} &\stackrel{\text{def}}{=} \Theta\left(\frac{\eta_{\text{in}}}{2} < \eta_{\text{im}} < \eta_{\text{in}}\right), \end{aligned} \quad (4.28)$$

which yields

$$\theta^{(a)} + \theta^{(b)} + \theta^{(c)} + \theta^{(d)} \equiv 1. \quad (4.29)$$

Applying this sectoring to the triple-collinear partitions, Eq. (4.27) becomes

$$1 = \sum_{(ij)} \omega^{\mathbf{m}i, \mathbf{n}j} + \sum_{i \in \mathcal{H}} \omega^{\mathbf{m}i, \mathbf{n}i} [\theta^{(a)} + \theta^{(b)} + \theta^{(c)} + \theta^{(d)}]. \quad (4.30)$$

The partitions constructed in Appendix B satisfy all the necessary properties for NNLO QCD computations. In particular, the combination of partitions and sectors isolates a minimal set of collinear singularities, and each partition adheres to the following

relations:

$$\begin{aligned}
C_{im} \omega^{mi,nj} &= \omega_{i||m}^{mi,nj} \stackrel{\text{def}}{=} \lim_{\rho_{im} \rightarrow 0} \omega^{mi,nj}, & C_{jn} \omega^{mi,nj} &= \omega_{j||n}^{mi,nj} \stackrel{\text{def}}{=} \lim_{\rho_{jn} \rightarrow 0} \omega^{mi,nj}, \\
C_{mn} \omega^{mi,nj} &= \delta_{ij} \omega_{m||n}^{mi,nj} \stackrel{\text{def}}{=} \lim_{\rho_{mn} \rightarrow 0} \omega^{mi,nj}, \\
C_{im} C_{mn} \omega^{mi,nj} &= \delta_{ij}, & C_{jn} C_{mn} \omega^{mi,nj} &= \delta_{ij}, \\
C_{im} C_{jn} \omega^{mk,nl} &= \delta_{ik} \delta_{jl}, & C_{im} C_{in} \omega^{mj,nj} &= \delta_{ij}, \\
C_{mn,i} \omega^{mi,nj} &= \delta_{ij}.
\end{aligned} \tag{4.31}$$

These relations are crucial for simplifying the required subtraction terms. For example, in sector $\theta^{(a)}$, only the triple-collinear limit $C_{mn,i}$ (see Eq. (2.132)) and the single-collinear limit C_{in} give non-zero results. Thus, we can write

$$\begin{aligned}
&\sum_{i \in \mathcal{H}} \omega^{mi,ni} \theta^{(a)} \\
&= \sum_{i \in \mathcal{H}} (\mathbb{1} - C_{mn,i} + C_{mn,i}) (\mathbb{1} - C_{in} + C_{in}) \omega^{mi,ni} \theta^{(a)} \\
&= \sum_{i \in \mathcal{H}} C_{in} \omega^{mi,ni} \theta^{(a)} + \sum_{i \in \mathcal{H}} C_{mn,i} \bar{C}_{in} \omega^{mi,ni} \theta^{(a)} + \sum_{i \in \mathcal{H}} \bar{C}_{mn,i} \bar{C}_{in} \omega^{mi,ni} \theta^{(a)}.
\end{aligned} \tag{4.32}$$

The same identity holds for sectors $\theta^{(b,c,d)}$, with the replacements $C_{in} \rightarrow C_{mn}$ in sectors $\theta^{(b,d)}$, and $C_{in} \rightarrow C_{im}$ in sector $\theta^{(c)}$. Performing analogous manipulations in the collinear partitions $\omega^{mi,nj}$, with $i \neq j$, we find

$$\begin{aligned}
&\sum_{(ij)} \omega^{mi,nj} \\
&= \sum_{(ij)} (\mathbb{1} - C_{im} + C_{im}) (\mathbb{1} - C_{jn} + C_{jn}) \omega^{mi,nj} \\
&= \sum_{(ij)} (C_{im} + C_{jn}) \omega^{mi,ni} - \sum_{(ij)} C_{jn} C_{im} \omega^{mi,nj} + \sum_{(ij)} \bar{C}_{im} \bar{C}_{jn} \omega^{mi,nj}.
\end{aligned} \tag{4.33}$$

Accounting for all the above terms, we find that $\Sigma_{\text{RR}}|_{\mathcal{A}_0}$ can be expressed as the sum of four distinct contributions:

$$\Sigma_{\text{RR}}|_{\mathcal{A}_0} = \sum_{i=1}^4 \langle \bar{S}_{mn} \bar{S}_n \Omega_i \Delta^{(mn)} \Theta_{mn} F_{\text{LM}}^{\mathcal{A}_2}[\mathbf{m}_g, \mathbf{n}_g] \rangle, \tag{4.34}$$

where the four quantities Ω_i provide the partition of unity

$$\sum_{i=1}^4 \Omega_i \equiv 1. \tag{4.35}$$

The four contributions read [23, 65, 66]

$$\Omega_1 = \sum_{(ij)} \bar{C}_{im} \bar{C}_{jn} [dp_m][dp_n] \omega^{mi,nj} + \sum_{i \in \mathcal{H}} \left[\bar{C}_{in} \theta^{(a)} + \bar{C}_{mn} \theta^{(b)} + \bar{C}_{im} \theta^{(c)} + \bar{C}_{mn} \theta^{(d)} \right] [dp_m][dp_n] \bar{C}_{mn,i} \omega^{mi,ni}, \quad (4.36)$$

$$\Omega_2 = \sum_{i \in \mathcal{H}} \left[\bar{C}_{in} \theta^{(a)} + \bar{C}_{mn} \theta^{(b)} + \bar{C}_{im} \theta^{(c)} + \bar{C}_{mn} \theta^{(d)} \right] [dp_m][dp_n] C_{mn,i} \omega^{mi,ni}, \quad (4.37)$$

$$\Omega_3 = - \sum_{(ij)} C_{jn} C_{im} [dp_m][dp_n] \omega^{mi,nj}, \quad (4.38)$$

$$\Omega_4 = \sum_{(ij)} \left[C_{im} [dp_m] + C_{jn} [dp_n] \right] \omega^{mi,nj} + \sum_{i \in \mathcal{H}} \left[C_{in} \theta^{(a)} + C_{mn} \theta^{(b)} + C_{im} \theta^{(c)} + C_{mn} \theta^{(d)} \right] [dp_m][dp_n] \omega^{mi,ni}. \quad (4.39)$$

We note that in the above definitions of Ω_i , $[dp_m]$ and $[dp_n]$ are phase-space elements for partons m and n, respectively, and that they appear to the *right* of the single collinear operators (C_{im} , C_{mn} , etc.) but to the *left* of the triple-collinear operators $C_{mn,i}$. Therefore, the single-collinear operators act on the phase-space elements, while the triple-collinear operators do not [65]. This choice was made in order to make Ω_2 more amenable to analytic integration using *reverse unitarity*, which was done in Ref. [79].

Using these results, we have succeeded in splitting $\Sigma_{\text{RR}}|_{\mathcal{A}_0}$ into four distinct terms. We name them as

$$\Sigma_{\text{RR}}|_{\mathcal{A}_0} = \Sigma_{\text{FR}}|_{\mathcal{A}_0} + \Sigma_{\text{DU}}^{(2)}|_{\mathcal{A}_0} + \Sigma_{\text{RR},2c}|_{\mathcal{A}_0} + \Sigma_{\text{RR},1c}|_{\mathcal{A}_0}. \quad (4.40)$$

In brief, the first term on the right-hand side in Eq. (4.40) is fully resolved, the second is the triple-collinear subtraction term, the third is the double-collinear term, and the last term is the single-collinear contribution. To elaborate further, the first term $\Sigma_{\text{FR}}|_{\mathcal{A}_0}$ is the *fully-regulated* contribution given by³

$$\Sigma_{\text{FR}}|_{\mathcal{A}_0} = \langle \bar{S}_{mn} \bar{S}_n \Omega_1 \Delta^{(mn)} \Theta_{mn} F_{\text{LM}}^{\mathcal{A}_2}[\mathbf{m}_g, \mathbf{n}_g] \rangle. \quad (4.41)$$

The quantity $\Sigma_{\text{FR}}|_{\mathcal{A}_0}$ represents the sole contribution to the NNLO cross section involving $N + 2$ resolved final-state partons, and it can be directly implemented into a numerical code without further ado. It is important to note that we are using a slight abuse of notation here. Specifically, the soft operators are placed to the left of the phase space terms $[dp_m]$ and $[dp_n]$, implying that they should, in principle, act on them. However, we clarify that none of the soft limits actually acts on the phase space. The operators are positioned to the left purely for graphical convenience. This statement holds also for the following terms as well.⁴

³We note that this contribution will contribute to final states with $N + 2$, $N + 1$ and N gluons, as a result of the subtracted limits contained in the definition of Ω_1 .

⁴To be precise, the phase space of a particle x is proportional to $[dp_x] \sim dE_x E_x^{d-3}$. Therefore, the action

The second term $\Sigma_{\text{DU}}^{(2)}|_{\mathcal{A}_0}$ in (4.40) is the triple-collinear contribution. It reads

$$\Sigma_{\text{DU}}^{(2)}|_{\mathcal{A}_0} = \langle \bar{S}_{\text{mn}} \bar{S}_{\text{n}} \Omega_2 \Delta^{(\text{mn})} \Theta_{\text{mn}} F_{\text{LM}}^{\mathcal{A}_2}[\mathbf{m}_g, \mathbf{n}_g] \rangle, \quad (4.42)$$

where Ω_2 is a triple-collinear projection operator (4.37). We note that this term was computed in Ref. [79] and can be immediately borrowed from there. It represents the second contribution to the fully-unresolved term Σ_{DU} that we have identified.

The third term $\Sigma_{\text{RR},2\text{c}}|_{\mathcal{A}_0}$ in (4.40) is the double-collinear contribution where the gluons $\mathbf{m}_g$ and $\mathbf{n}_g$ are emitted from different legs, i.e.,

$$\begin{aligned} \Sigma_{\text{RR},2\text{c}}|_{\mathcal{A}_0} &= \langle \bar{S}_{\text{mn}} \bar{S}_{\text{n}} \Omega_3 \Delta^{(\text{mn})} \Theta_{\text{mn}} F_{\text{LM}}^{\mathcal{A}_2}[\mathbf{m}_g, \mathbf{n}_g] \rangle \\ &= - \sum_{(ij)} \langle \bar{S}_{\text{mn}} \bar{S}_{\text{n}} C_{jn} C_{im} [dp_m] [dp_n] \omega^{mi,nj} \Delta^{(\text{mn})} \Theta_{\text{mn}} F_{\text{LM}}^{\mathcal{A}_2}[\mathbf{m}_g, \mathbf{n}_g] \rangle. \end{aligned} \quad (4.43)$$

Also this term is double-unresolved, and we will come back to it later.

We are left with $\Sigma_{\text{RR},1\text{c}}|_{\mathcal{A}_0}$, which is the double-real single-collinear contribution. It reads

$$\begin{aligned} \Sigma_{\text{RR},1\text{c}}|_{\mathcal{A}_0} &= \langle \bar{S}_{\text{mn}} \bar{S}_{\text{n}} \Omega_4 \Delta^{(\text{mn})} \Theta_{\text{mn}} F_{\text{LM}}^{\mathcal{A}_2}[\mathbf{m}_g, \mathbf{n}_g] \rangle \\ &= \sum_{(ij)} \langle \bar{S}_{\text{mn}} \bar{S}_{\text{n}} [C_{im} [dp_m] + C_{jn} [dp_n]] \omega^{mi,nj} \Delta^{(\text{mn})} \Theta_{\text{mn}} F_{\text{LM}}^{\mathcal{A}_2}[\mathbf{m}_g, \mathbf{n}_g] \rangle \\ &\quad + \sum_{i \in \mathcal{H}} \langle \bar{S}_{\text{mn}} \bar{S}_{\text{n}} [C_{in} \theta^{(a)} + C_{mn} \theta^{(b)} + C_{im} \theta^{(c)} + C_{mn} \theta^{(d)}] \\ &\quad \times [dp_m] [dp_n] \omega^{mi,ni} \Delta^{(\text{mn})} \Theta_{\text{mn}} F_{\text{LM}}^{\mathcal{A}_2}[\mathbf{m}_g, \mathbf{n}_g] \rangle. \end{aligned} \quad (4.44)$$

The challenge therefore is to write $\Sigma_{\text{RR},1\text{c}}|_{\mathcal{A}_0}$ as a sum of terms with a well-defined number of resolved partons. To do this, we need to extract the remaining collinear singularities from $\Sigma_{\text{RR},1\text{c}}|_{\mathcal{A}_0}$.⁵ We will do so in the next section.

4.2 The new approach to NSC

4.2.1 A motivation for generalization

To summarize what has been discussed so far, and neglecting the contribution from the renormalization of the PDFs, the NNLO cross section can be written as

$$2s_{ab} d\bar{\sigma}_{\mathcal{A}_0}^{\text{NNLO}} = \Sigma_{\text{DU}}^{(1)}|_{\mathcal{A}_0} + \Sigma_{\text{SU}}^{(1)}|_{\mathcal{A}_0} + \Sigma_{\text{RR}}|_{\mathcal{A}_0}, \quad (4.45)$$

of the soft limit S_x has no effect on it, since S_x keeps the leading singularity. The situation would be different if the phase space contained terms proportional to a power of $(1 - E_x)$, since $S_x(1 - E_x) = 1$ (note that this occurs in the collinear limit, where $[d\eta_{ij}] \sim (1 - \eta_{ij})^{-\epsilon}$, cf. Eq. (2.61)). However, since in the NSC approach we never assumed soft limits acting on the phase space, we emphasize this point.

⁵The label *single-collinear* for this contribution refers to the one collinear limit appearing in Eq. (4.44). Such collinear limit is relevant for only one potentially unresolved parton, but the remaining one is still unregulated. For this reason we will need to further extract the singularities associated to the second extra emission. This procedure will lead to double-unresolved terms.

where $\Sigma_{\text{DU}}^{(1)}|_{\mathcal{A}_0}$, $\Sigma_{\text{SU}}^{(1)}|_{\mathcal{A}_0}$ are given in Eqs (4.24) and (4.25), and $\Sigma_{\text{RR}}|_{\mathcal{A}_0}$ in Eq. (4.40), i.e.,

$$\Sigma_{\text{RR}}|_{\mathcal{A}_0} = \Sigma_{\text{FR}}|_{\mathcal{A}_0} + \Sigma_{\text{DU}}^{(2)}|_{\mathcal{A}_0} + \Sigma_{\text{RR},2c}|_{\mathcal{A}_0} + \Sigma_{\text{RR},1c}|_{\mathcal{A}_0}. \quad (4.46)$$

We point out that Eq. (4.46) is general, in the following sense:

- all the genuine NNLO-like singularities that appear have already been included in the triple-collinear or double-soft terms;
- the terms that still need to be treated are $\Sigma_{\text{RR},2c}|_{\mathcal{A}_0}$ and $\Sigma_{\text{RR},1c}|_{\mathcal{A}_0}$, where just contain iterations of the NLO-like collinear singularities.

This means that determining the IR poles of $\Sigma_{\text{RR}}|_{\mathcal{A}_0}$ for an arbitrary process is conceptually straightforward: they have to be extracted from $\Sigma_{\text{RR},2c}|_{\mathcal{A}_0}$ and $\Sigma_{\text{RR},1c}|_{\mathcal{A}_0}$.

In light of what we have listed above, the following question naturally arises:

Why is a generalization of the NSC necessary?

The first reason concerns the efficiency of computing processes with a varying number N of hard jets. Suppose we use Eq. (4.45) to explicitly calculate the cross section for process $\mathcal{A}_0$ with $N = 1$ hard gluons in the final state. We could then wonder: how much additional effort would be required to calculate the process with $N = 3$, knowing the result for $N = 1$? Unfortunately, the answer is: quite a lot. The difficulty arises from the terms $\Sigma_{\text{RR},2c}|_{\mathcal{A}_0}$ and $\Sigma_{\text{RR},1c}|_{\mathcal{A}_0}$ in Eq. (4.46). Indeed, by adding just two more gluons to the final state, numerous combinations of collinear limits must be added to $\Sigma_{\text{RR},2c}|_{\mathcal{A}_0}$ and $\Sigma_{\text{RR},1c}|_{\mathcal{A}_0}$, which were not present in the $N = 1$ case. The only way to address this would be to tediously identify all the missing combinations of collinear limits and incorporate them into the $N = 1$ result, while ensuring that all singularities are extracted. What is needed, therefore, is a procedure that allows for a streamlined calculation of the result, independent of the specific value of N .

The second reason refers to the issue of *color-correlations*, that we already encountered at NLO. As discussed in Ch. 2, once explicitly computed, the term $\Sigma_{\text{DU}}^{(1)}|_{\mathcal{A}_0}$ will contain $1/\epsilon^n$ poles involving both double and quartic color-correlations, i.e., terms proportional to the product of color matrices $T_i \cdot T_j$ and $\{T_i \cdot T_j, T_k \cdot T_l\}$, respectively. On the other hand, purely collinear contributions, such as $\Sigma_{\text{RR},2c}|_{\mathcal{A}_0}$, can be expressed as explicit combinations of Casimir operators and anomalous dimensions. The mixing of implicit and explicit color terms implies that, if each object is computed individually, expanded in powers of ϵ (e.g., using software such as `Mathematica`), and subsequently summed with the other terms in the calculation, the cancellation of poles can only be observed after using color algebra to rewrite some of the color-correlations in terms of Casimirs. Not only is this time-consuming, it is also process-dependent, since including an additional parton changes the color algebra. It is important to highlight that this issue has not arisen in previous NSC studies because the processes analyzed there involved no more than three hard particles. In such cases, the color algebra is closed, which means that any product $T_i \cdot T_j$ can be expressed in terms of Casimir operators.

We propose, therefore, to take a different *organizational approach* to the one taken in Refs [65–67, 73, 75–78]. In all these references, contributions are first explicitly computed

individually and *then* combined to verify the cancellation of poles and obtain an IR-finite result. This methodology leads to the two issues discussed above. Instead, we will aim to combine terms with similar structures *before* calculating them explicitly, with the aim of identifying the I -operators, as far as possible. This allows for the cancellation of divergent contributions using these macrostructures. Only after this has been done will the remaining terms be explicitly calculated. This new approach offers two significant advantages:

- i) the poles containing color-correlations will cancel explicitly;
- ii) the result will be an explicit formula where N is simply an arbitrary input parameter.

Although the NNLO computation is significantly more involved than the NLO one, we aim to replicate the latter as much as possible. In doing so, we face the following dilemma. On the one hand, the double-real contributions need to be split into partitions and sectors to define the approach to collinear singular limits in a unique way. On the other hand, this sectoring destroys the emergence of structures that can be combined naturally with the double-virtual and real-virtual corrections. Hence, finding an optimal balance between splitting the real-emission contributions into many well-defined pieces and identifying proper structures early in the calculation is the central challenge to efficiently organizing the NNLO computation.

In a sense, we are refining the NSC approach described in Ref. [23], and reorganizing it in a more efficient manner. However, as is often the case in physics, a well-written formula can open up new avenues and perspectives that might remain inaccessible with a poorly formulated one. We will demonstrate how a more organized formulation of the NSC enables us to achieve results that were previously unattainable with this subtraction scheme.

4.2.2 Treatment of unresolved quarks and gluons

As stated above, our intention is to write the NNLO singularities to processes $\mathcal{A}_0$ and $\mathcal{B}_0$ in terms of the I -operators (or variants thereof) introduced in Ch. 3. Before doing so, we need to rewrite the terms $\Sigma_{\text{RR},2\text{c}}|_{\mathcal{A}_0}$ and, more importantly, $\Sigma_{\text{RR},1\text{c}}|_{\mathcal{A}_0}$, as defined in Eqs (4.43) and (4.44), respectively, in a form that will allow us to do so.

However, before proceeding, an additional detail requires attention. Eq. (4.45) is a general formulation derived for the specific case of process $\mathcal{A}_0$. But does it have broader applicability? More importantly for our purposes, does it also extend to the NNLO corrections for process $\mathcal{B}_0$? The answer is yes, though certain aspects warrant further discussion. As said, the NNLO corrections for process $\mathcal{A}_0$ can be considered a special case of the NNLO corrections for process $\mathcal{B}_0$ in Eq. (4.2). Therefore, before efficiently rewriting the contributions of $\Sigma_{\text{RR},2\text{c}}|_{\mathcal{A}_0}$ and $\Sigma_{\text{RR},1\text{c}}|_{\mathcal{A}_0}$ (that will be the main topic of the next section), it is worthwhile to first examine how Eq. (4.45) can be adapted to process $\mathcal{B}_0$. This will be addressed in this section.

We start from process $\mathcal{B}_0$. In order to get the NNLO corrections to this process, we could follow all the steps described in Subsec. 4.1.1, considering that here we are always

considering NNLO corrections where $n_f = 0$. Specifically, we find

$$d\hat{\sigma}_{\mathcal{B}_0}^{\text{NNLO}} = d\hat{\sigma}_{\mathcal{B}_0}^{\text{VV}} + d\hat{\sigma}_{\mathcal{B}_1}^{\text{RV}} + d\hat{\sigma}_{\mathcal{B}_2}^{\text{RR}} + d\hat{\sigma}_{\mathcal{B}_2}^{\text{pdf}} + d\hat{\sigma}_{\tilde{\mathcal{B}}_2}^{\text{int}}, \quad (4.47)$$

where $\mathcal{B}_2$ and $\tilde{\mathcal{B}}_2$ are the following two NNLO processes:

$$\mathcal{B}_2: \quad g_a + q_b \rightarrow \{g\}_{N+1} + q + X, \quad (4.48)$$

$$\tilde{\mathcal{B}}_2: \quad g_a + q_b \rightarrow \{g\}_{N-1} + \{q\}_2 + \bar{q} + X. \quad (4.49)$$

Notice that the interference term $d\hat{\sigma}_{\tilde{\mathcal{B}}_2}^{\text{int}}$ was not present in (4.45). This contribution will be discussed in greater detail in Sec. 5.3.2. For now, we anticipate that it corresponds to singularities that arise in interference terms in $\tilde{\mathcal{B}}_2$ from the triple-collinear splitting process $q^* \rightarrow qq\bar{q}$. These interference terms occur when all the quarks in $\tilde{\mathcal{B}}_2$ have the same flavour, the three final-state quarks become collinear to each other. This limit projects $\tilde{\mathcal{B}}_2$ onto the Born process $\mathcal{B}_0$, and the resulting expression is not proportional to n_f , thus contributing directly to the cancellation of poles for the process $\mathcal{B}_0$.

Regarding the double-virtual contribution,

$$2s_{ab} d\hat{\sigma}_{\mathcal{B}_0}^{\text{VV}} = \langle F_{\text{VV}}^{\mathcal{B}_0} \rangle, \quad (4.50)$$

there is nothing further to add. As for the real-virtual contribution, we handle it similarly to the NLO case, as in Eq. (3.105). Thus, we apply soft and collinear regularization, yielding

$$\begin{aligned} 2s_{ab} d\hat{\sigma}_{\mathcal{B}_1}^{\text{RV}} &= \langle \mathcal{O}_{\text{NLO}}^{(\text{m})} \Delta^{(\text{m})} F_{\text{RV}}^{\mathcal{B}_1}[\mathbf{m}_g] \rangle + \langle S_{\text{m}} F_{\text{RV}}^{\mathcal{B}_1}[\mathbf{m}_g] \rangle + \sum_{i \in \mathcal{H}} \langle \bar{S}_{\text{m}} C_{i\text{m}} \Delta^{(\text{m})} F_{\text{RV}}^{\mathcal{B}_1}[\mathbf{m}_g] \rangle \\ &+ \sum_{i \in \mathcal{H}} \langle C_{i\text{m}} \Delta^{(\text{m})} F_{\text{RV}}^{\mathcal{B}_1}[\mathbf{m}_q] \rangle + \langle \mathcal{O}_{\text{NLO}}^{(\text{m})} \Delta^{(\text{m})} F_{\text{RV}}^{\mathcal{B}_1}[\mathbf{m}_q] \rangle, \end{aligned} \quad (4.51)$$

where we have introduced the shorthand notation

$$\begin{aligned} F_{\text{RV}}^{\mathcal{B}_1}[\mathbf{m}_g] &\stackrel{\text{def}}{=} F_{\text{RV}}^{gq}[\{g\}_{N-1}, q | \mathbf{m}_g], \\ F_{\text{RV}}^{\mathcal{B}_1}[\mathbf{m}_q] &\stackrel{\text{def}}{=} F_{\text{RV}}^{gq}[\{g\}_N | \mathbf{m}_q]. \end{aligned} \quad (4.52)$$

In Eq. (4.51), we are using a slight abuse of notation. According to the definition in (3.31), the operator $\mathcal{O}_{\text{NLO}}^{(\text{m})}$ includes the soft regularization limit $\bar{S}_{\text{m}}$, which has no effect when applied to $F_{\text{RV}}^{\mathcal{B}_1}[\mathbf{m}_q]$, as parton $\mathbf{m}_q$ is a quark ($\bar{S}_{\text{m}_q} = \mathbb{1}$). However, we deliberately use the notation $\mathcal{O}_{\text{NLO}}^{(\text{m})}$ because it serves as a visual reminder that $\langle \mathcal{O}_{\text{NLO}}^{(\text{m})} \Delta^{(\text{m})} F_{\text{RV}}^{\mathcal{B}_1}[\mathbf{m}_q] \rangle$ is a fully regulated term.

For the double-real contribution, we perform similar manipulations to those that led us to Eq. (4.10). We start with the formula

$$2s_{ab} d\hat{\sigma}_{\mathcal{B}_2}^{\text{RR}} = \left\langle \frac{1}{(N+1)!} F_{\text{LM}}^{gq}[\{g\}_{N+1}, q] \right\rangle, \quad (4.53)$$

where F_{LM} contains a list $\{g\}_N$ of N final-state gluons, and an additional gluon and quark that we take to be in positions m and n , respectively. We then implement the identity (note that $\Delta^{(mn)} = \Delta^{(nm)}$)

$$1 = \sum_{\substack{i,j \in \{g\}_N \\ i < j}} \Delta^{(ij)} + \sum_{i \in \{g\}_N} \Delta^{(im)} + \sum_{i \in \{g\}_N} \Delta^{(in)} + \Delta^{(mn)}. \quad (4.54)$$

Since we require the unresolved partons to be m and n , we need to map all partitions $\Delta^{(ij)}$ onto $\Delta^{(mn)}$. Let us now explain how this mapping is done.

i) $\Delta^{(ij)} \mapsto \Delta^{(mn)}$: we rename $i \leftrightarrow m$ and $j \leftrightarrow n$, such that

$$\begin{aligned} & \sum_{\substack{i,j \in \{g\}_N \\ i < j}} \frac{\Delta^{(ij)}}{(N+1)!} F_{\text{LM}}^{gq}[\{g\}_{N+1}, q] \\ &= \sum_{\substack{i,j \in \{g\}_N \\ i < j}} \frac{\Delta^{(mn)}}{(N+1)N} F_{\text{LM}}^{gq}[\{g\}_{N-1}, q | m_g, n_g] \\ &= \frac{N-1}{N+1} \times \frac{1}{2!} \Delta^{(mn)} F_{\text{LM}}^{gq}[\{g\}_{N-1}, q | m_g, n_g], \end{aligned} \quad (4.55)$$

where $F_{\text{LM}}^{gq}[\{g\}_{N-1}, q | m_g, n_g]$, by definition, has reabsorbed a factor $1/(N-1)!$, and we have used the sum

$$\sum_{\substack{i,j \in \{g\}_N \\ i < j}} = \frac{N(N-1)}{2} \quad (4.56)$$

to move from the second to the third line.

ii) $\Delta^{(im)} \mapsto \Delta^{(mn)}$: we rename $i \leftrightarrow n$, such that

$$\sum_{i \in \{g\}_N} \frac{\Delta^{(ij)}}{(N+1)!} F_{\text{LM}}^{gq}[\{g\}_{N+1}, q] = \frac{1}{N+1} \times \Delta^{(mn)} F_{\text{LM}}^{gq}[\{g\}_{N-1}, q | m_g, n_g]. \quad (4.57)$$

iii) $\Delta^{(in)} \mapsto \Delta^{(mn)}$: we rename $i \leftrightarrow m$, such that

$$\sum_{i \in \{g\}_N} \frac{\Delta^{(ij)}}{(N+1)!} F_{\text{LM}}^{gq}[\{g\}_{N+1}, q] = \frac{N}{N+1} \times \Delta^{(mn)} F_{\text{LM}}^{gq}[\{g\}_N | m_g, n_g], \quad (4.58)$$

where $F_{\text{LM}}^{gq}[\{g\}_N | m_g, n_g]$, having N hard gluons, has reabsorbed a factor $1/N!$.

iv) As for the last term on the right-hand side of Eq. (4.54), we have

$$\frac{\Delta^{(mn)}}{(N+1)!} F_{\text{LM}}^{gq}[\{g\}_{N+1}, q] = \frac{1}{N+1} \times \Delta^{(mn)} F_{\text{LM}}^{gq}[\{g\}_N | m_g, n_g], \quad (4.59)$$

as, once again, a factor $1/N!$ has been absorbed into F_{LM} .

Therefore, combining the results in points i) - iv), we obtain⁶

$$2s_{ab} d\hat{\sigma}_{\mathcal{B}_2}^{\text{RR}} = \left\langle \Delta^{(\text{mn})} \left[\frac{1}{2!} F_{\text{LM}}^{gq}[\{g\}_{N-1}, q | \mathbf{m}_g, \mathbf{n}_g] + F_{\text{LM}}^{gq}[\{g\}_N | \mathbf{m}_q, \mathbf{n}_g] \right] \right\rangle. \quad (4.60)$$

Finally, we remove the factor $1/2!$ from $F_{\text{LM}}^{gq}[\{g\}_{N-1}, q | \mathbf{m}_g, \mathbf{n}_g]$ by introducing the energy ordering of the unresolved gluons $\mathbf{m}_g$ and $\mathbf{n}_g$, which yields

$$2s d\hat{\sigma}_{\mathcal{B}_2}^{\text{RR}} = \left\langle \Delta^{(\text{mn})} \left[\Theta_{\text{mn}} F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g] + F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_q, \mathbf{n}_g] \right] \right\rangle, \quad (4.61)$$

where again we introduced the shorthand notation

$$\begin{aligned} F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g] &\stackrel{\text{def}}{=} F_{\text{LM}}^{gq}[\{g\}_{N-1}, q | \mathbf{m}_g, \mathbf{n}_g], \\ F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_q, \mathbf{n}_g] &\stackrel{\text{def}}{=} F_{\text{LM}}^{gq}[\{g\}_N | \mathbf{m}_q, \mathbf{n}_g]. \end{aligned} \quad (4.62)$$

An important point is worth noting here. Eq. (4.61) consists of two F_{LM} terms: the first, $F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g]$, with two unresolved gluons, and the second, $F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_q, \mathbf{n}_g]$ with unresolved quark and gluon. The soft and collinear regularization of $F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g]$ is analogous to the process $\mathcal{A}_2$ described in the previous section. Indeed, regardless of the hard particles involved in the F_{LM} function, if the unresolved particles are two gluons, the general formula remains unchanged. As for the case of $F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_q, \mathbf{n}_g]$, the regularization proceeds similarly to $F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g]$, but without any double-soft limit (cf. Subsec. 2.4.1), since S_{mn} produces no singularities with a qg pair. Thus, we can combine the double-virtual, real-virtual, and double-real contributions, as follows:

$$2s_{ab} d\hat{\sigma}_{\mathcal{B}_0}^{\text{NNLO}} = \Sigma_{\text{DU}}^{(1)}|_{\mathcal{B}_0} + \Sigma_{\text{SU}}^{(1)}|_{\mathcal{B}_0} + \Sigma_{\text{RR}}|_{\mathcal{B}_0} + 2s_{ab} d\hat{\sigma}_{\mathcal{B}_2}^{\text{int}}, \quad (4.63)$$

where

$$\begin{aligned} \Sigma_{\text{DU}}^{(1)}|_{\mathcal{B}_0} &= \Sigma_{\text{DU}}^{(1)}|_{\mathcal{B}_0,gg} + \Sigma_{\text{DU}}^{(1)}|_{\mathcal{B}_0,qg}, \\ \Sigma_{\text{SU}}^{(1)}|_{\mathcal{B}_0} &= \Sigma_{\text{SU}}^{(1)}|_{\mathcal{B}_0,gg} + \Sigma_{\text{SU}}^{(1)}|_{\mathcal{B}_0,qg}, \\ \Sigma_{\text{RR}}|_{\mathcal{B}_0} &= \Sigma_{\text{RR}}|_{\mathcal{B}_0,gg} + \Sigma_{\text{RR}}|_{\mathcal{B}_0,qg}. \end{aligned} \quad (4.64)$$

$\Sigma_{\text{DU}}^{(1)}|_{\mathcal{B}_0,gg}$, $\Sigma_{\text{SU}}^{(1)}|_{\mathcal{B}_0,gg}$, and $\Sigma_{\text{RR}}|_{\mathcal{B}_0,gg}$ are defined as in Eqs (4.24 - 4.25) and (4.40 - 4.44), with the replacements $\mathcal{A}_k \mapsto \mathcal{B}_k$, $k = 0, 1, 2$, while $\Sigma_{\text{DU}}^{(1)}|_{\mathcal{B}_0,qg}$, $\Sigma_{\text{SU}}^{(1)}|_{\mathcal{B}_0,qg}$, and $\Sigma_{\text{RR}}|_{\mathcal{B}_0,qg}$ read

$$\Sigma_{\text{DU}}^{(1)}|_{\mathcal{B}_0,qg} = \sum_{i \in \mathcal{H}} \left\langle C_{i\text{m}} \Delta^{(\text{m})} \left[F_{\text{RV}}^{\mathcal{B}_1}[\mathbf{m}_q] + \langle S_{\text{n}} F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_q, \mathbf{n}_g] \rangle \right] \right\rangle, \quad (4.65)$$

$$\Sigma_{\text{SU}}^{(1)}|_{\mathcal{B}_0,qg} = \left\langle \mathcal{O}_{\text{NLO}}^{(\text{m})} \Delta^{(\text{m})} \left[F_{\text{RV}}^{\mathcal{B}_1}[\mathbf{m}_q] + \langle S_{\text{n}} F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_q, \mathbf{n}_g] \rangle \right] \right\rangle, \quad (4.66)$$

$$\Sigma_{\text{RR}}|_{\mathcal{B}_0,qg} = \Sigma_{\text{FR}}|_{\mathcal{B}_0,qg} + \Sigma_{\text{DU}}^{(2)}|_{\mathcal{B}_0,qg} + \Sigma_{\text{RR},2c}|_{\mathcal{B}_0,qg} + \Sigma_{\text{RR},1c}|_{\mathcal{B}_0,qg}. \quad (4.67)$$

⁶In defining the following functions, we are free to decide which unresolved parton is m and which is n . For the discussion in the following pages, in F_{LM} , it is convenient to always choose m as the quark and n as the gluon.

All the terms on the right-hand side of $\Sigma_{\text{RR}}|_{\mathcal{B}_0,gg}$ are defined as in $\Sigma_{\text{RR}}|_{\mathcal{B}_0,gg}$ but without the double-soft limit S_{mn} .

From this analysis, we understand that the process $\mathcal{A}_0$, in the context of double-real corrections, consists of a single function $F_{\text{LM}}^{\mathcal{A}_2}[m_g, n_g]$, while the process $\mathcal{B}_0$ involves two F_{LMS} functions $F_{\text{LM}}^{\mathcal{B}_2}[m_g, n_g]$ and $F_{\text{LM}}^{\mathcal{B}_2}[m_g, n_g]$, but the regularization of $F_{\text{LM}}^{\mathcal{B}_2}[m_g, n_g]$ is entirely analogous to that of $F_{\text{LM}}^{\mathcal{A}_2}[m_g, n_g]$. Furthermore, as we observed at NLO, the I -operator approach allows for the independent treatment of each leg of the process under study. Specifically, considering the I_C operator, it sums the collinear contributions from each leg *individually*. It follows that the NNLO regularization of the processes $\mathcal{A}_0$ and $\mathcal{B}_0$ can be discussed simultaneously, since, as we will see, the result for $\mathcal{A}_0$ can be derived from that of $\mathcal{B}_0$ by *omitting* the final-state quark leg.

4.3 Analyzing double- and single-collinear contributions: $m_g n_g$ case

At this point, we can truly delve into our new approach to the NSC and begin a preliminary reorganization of the terms. Specifically, we address the double- and single-collinear contributions $\Sigma_{\text{RR},2c}$ and $\Sigma_{\text{RR},1c}$ in Eqs (4.43) and (4.44), respectively. As mentioned earlier, it is necessary to distinguish between the $m_g n_g$ and $m_q n_g$ cases, where the former applies to the NNLO double-real corrections for both $\mathcal{A}_0$ and $\mathcal{B}_0$, while the latter applies only to $\mathcal{B}_0$. We will discuss the $m_g n_g$ case here, and the $m_q n_g$ case in Sec. 4.4.

We consider Eqs (4.43) and (4.44) applied to an F_{LM} function with two unresolved gluons. We denote this function as $F_{\text{LM}}^{\mathcal{H}}[m_g, n_g]$, where, as usual, $\mathcal{H}$ represents the list of hard partons, and here is used as a placeholder for $\mathcal{A}_2$ and $\mathcal{B}_2$. Regarding $\Sigma_{\text{RR},1c}|_{\mathcal{H},gg}$, the $1/\epsilon^n$ singularities simplify if the contributions of different partitions and sectors are *combined*. To appreciate why doing so is both important and non-trivial, we need to remind ourselves why partitions and sectors were introduced in the first place. The reason was to disentangle overlapping singular limits, making them uniquely defined. However, it also complicates the identification of physical quantities such as, e.g., collinear anomalous dimensions and splitting functions. We emphasize that the ability to recognize these universal structures in the early stages of the calculation is very useful for canceling the IR divergences in an efficient and transparent manner. Hence, our strategy will be to remove sectors in a *controlled* way, eventually getting to the point where various contributions can be rearranged into recognizable universal structures. As a result of this analysis, we are able to represent $\Sigma_{\text{RR},1c}|_{\mathcal{H},gg}$ by a sum of:⁷

- five divergent double-unresolved quantities, i.e., $\Sigma_{\text{DU}}^{(4,\dots,8)}|_{\mathcal{H},gg}$,
- four finite double-unresolved quantities, i.e., $\Sigma_{\text{DU}}^{\text{fin},(2,\dots,5)}|_{\mathcal{H},gg}$,
- two divergent single-unresolved quantities, i.e., $\Sigma_{\text{SU}}^{(2,3)}|_{\mathcal{H},gg}$,
- three finite single-unresolved quantities, i.e., $\Sigma_{\text{SU}}^{\text{fin},(1,2,3)}|_{\mathcal{H},gg}$.

⁷The numbering of the following objects is arbitrary and does not reflect any physical property.

The remainder of this section describes manipulations of $\Sigma_{\text{RR},1\text{c}}|_{\mathcal{H},gg}$ that lead to such a representation. Finally, in Subsec. 4.3.3 we will quickly discuss the rearrangement of $\Sigma_{\text{RR},2\text{c}}|_{\mathcal{H},gg}$.

We begin with $\Sigma_{\text{RR},1\text{c}}|_{\mathcal{H},gg}$, and separate sectors $\theta^{(b)}$ and $\theta^{(d)}$ from the remaining contributions to $\Sigma_{\text{RR},1\text{c}}|_{\mathcal{H},gg}$,

$$\Sigma_{\text{RR},1\text{c}}|_{\mathcal{H},gg} = \Sigma_{\text{RR},1\text{c}}^{(a,c,\text{dc})}|_{\mathcal{H},gg} + \Sigma_{\text{RR},1\text{c}}^{(b,d)}|_{\mathcal{H},gg}, \quad (4.68)$$

where⁸

$$\begin{aligned} \Sigma_{\text{RR},1\text{c}}^{(a,c,\text{dc})}|_{\mathcal{H},gg} \stackrel{\text{def}}{=} & \left\langle \bar{S}_{\text{mn}} \bar{S}_{\text{n}} \left[\sum_{(ij)} (C_{i\text{m}} + C_{j\text{n}}) \omega^{\text{mi},\text{nj}} + \sum_{i \in \mathcal{H}} (C_{i\text{n}} \theta^{(a)} + C_{i\text{m}} \theta^{(c)}) \omega^{\text{mi},\text{ni}} \right] \right. \\ & \left. \times [\text{dp}_{\text{m}}][\text{dp}_{\text{n}}] \Delta^{(\text{mn})} \Theta_{\text{mn}} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \right\rangle, \end{aligned} \quad (4.69)$$

and

$$\begin{aligned} \Sigma_{\text{RR},1\text{c}}^{(b,d)}|_{\mathcal{H},gg} \stackrel{\text{def}}{=} & \sum_{i \in \mathcal{H}} \left\langle \bar{S}_{\text{mn}} \bar{S}_{\text{n}} C_{\text{mn}} (\theta^{(b)} + \theta^{(d)}) \omega^{\text{mi},\text{ni}} [\text{dp}_{\text{m}}][\text{dp}_{\text{n}}] \right. \\ & \left. \times \Delta^{(\text{mn})} \Theta_{\text{mn}} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \right\rangle. \end{aligned} \quad (4.70)$$

Since $\Sigma_{\text{RR},1\text{c}}^{(a,c,\text{dc})}|_{\mathcal{H},gg}$ and $\Sigma_{\text{RR},1\text{c}}^{(b,d)}|_{\mathcal{H},gg}$ require non-trivial manipulations, we dedicate a subsection to each.

4.3.1 $\Sigma_{\text{RR},1\text{c}}^{(a,c,\text{dc})}|_{\mathcal{H},gg}$ contribution

We first consider $\Sigma_{\text{RR},1\text{c}}^{(a,c,\text{dc})}|_{\mathcal{H},gg}$ in Eq. (4.69). To further simplify this contribution, note that the collinear limit $C_{i\text{m}}$ (or equivalently $C_{j\text{n}}$) decouples the two unresolved partons m and n . Thus, the double-soft limit S_{mn} behaves as the product of the two single-soft limits S_{m} and S_{n} . Consequently

$$\begin{aligned} & S_{\text{mn}} \bar{S}_{\text{n}} C_{i\text{m}} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \\ & \sim S_{\text{m}} S_{\text{n}} \bar{S}_{\text{n}} C_{i\text{m}} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \\ & = (S_{\text{m}} S_{\text{n}} - S_{\text{m}} S_{\text{n}}) C_{i\text{m}} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \\ & = 0, \end{aligned} \quad (4.71)$$

allowing us to write

$$S_{\text{mn}} \bar{S}_{\text{n}} [\dots] = 0, \quad \Rightarrow \quad \bar{S}_{\text{mn}} \bar{S}_{\text{n}} \mapsto \bar{S}_{\text{n}}. \quad (4.72)$$

⁸The superscript (a, c, dc) reminds us that $\Sigma_{\text{RR},1\text{c}}^{(a,c,\text{dc})}|_{\mathcal{H},gg}$ includes contributions of sectors a and c and of the double-collinear partitions.

It follows that

$$\begin{aligned} \Sigma_{\text{RR},1c}^{(a,c,\text{dc})}|_{\mathcal{H},gg} = & \left\langle \bar{S}_n \left[\sum_{(ij)} (C_{im} + C_{jn}) \omega^{\mathbf{m}i, \mathbf{n}j} + \sum_{i \in \mathcal{H}} (C_{in} \theta^{(a)} + C_{im} \theta^{(c)}) \omega^{\mathbf{m}i, \mathbf{n}i} \right] \right. \\ & \left. \times [dp_m][dp_n] \Delta^{(mn)} \Theta_{mn} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \right\rangle. \end{aligned} \quad (4.73)$$

We can simplify this expression by renaming the gluons $\mathbf{m}_g$ and $\mathbf{n}_g$ such that the collinear operators always refer to the gluon $\mathbf{m}_g$. We also exploit the fact that, under this relabeling, $\Delta^{(mn)}$, $[dp_m][dp_n]$, and $F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g]$ are invariant, while $S_n \Theta_{mn}$ becomes $S_m \Theta_{nm}$, with $\Theta_{nm} = \Theta(E_n - E_m)$, and sector $\theta^{(a)}$ becomes sector $\theta^{(c)}$, as one can see from Eq. (4.28). Hence, we obtain

$$\begin{aligned} \Sigma_{\text{RR},1c}^{(a,c,\text{dc})}|_{\mathcal{H},gg} = & \left\langle \mathcal{S}(\mathbf{m}, \mathbf{n}) \left[\sum_{(ij)} C_{im} \omega^{\mathbf{m}i, \mathbf{n}j} + \sum_{i \in \mathcal{H}} C_{im} \theta^{(c)} \omega^{\mathbf{m}i, \mathbf{n}i} \right] \right. \\ & \left. \times [dp_m][dp_n] \Delta^{(mn)} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \right\rangle, \end{aligned} \quad (4.74)$$

where the soft-regulating operator $\mathcal{S}(\mathbf{m}, \mathbf{n})$ reads

$$\mathcal{S}(\mathbf{m}, \mathbf{n}) \stackrel{\text{def}}{=} \bar{S}_n \Theta_{mn} + \bar{S}_m \Theta_{nm}. \quad (4.75)$$

Note that the operator $\mathcal{S}(\mathbf{m}, \mathbf{n})$ can be rewritten in several equivalent ways, for instance

$$\begin{aligned} \mathcal{S}(\mathbf{m}, \mathbf{n}) &= \bar{S}_n \Theta_{mn} + \bar{S}_m \Theta_{nm} \\ &= \mathbb{1} - S_n \Theta_{mn} - S_m \Theta_{nm} \\ &= \bar{S}_m \bar{S}_n + S_m \bar{S}_n \Theta_{mn} + S_n \bar{S}_m \Theta_{nm} \\ &= \bar{S}_n (\mathbb{1} - S_m \Theta_{nm}) + S_n \bar{S}_m \Theta_{nm}. \end{aligned} \quad (4.76)$$

Throughout our calculations, we will use all these different representations.

To simplify $\Sigma_{\text{RR},1c}^{(a,c,\text{dc})}|_{\mathcal{H},gg}$ further, we need to extract the remaining collinear singularities. Since we relabeled gluons so that the collinear operators refer to the gluon $\mathbf{m}_g$, the unregulated singularities affect gluon $\mathbf{n}_g$ only. However, there is an additional technical detail that should be highlighted before proceeding. As we already mentioned, the many single-collinear contributions will have to be combined with collinear limits from single-soft, real-virtual and other terms where the collinear operators do not act on the phase space. However, in $\Sigma_{\text{RR},1c}^{(a,c,\text{dc})}|_{\mathcal{H},gg}$, having the collinear limits which act on the phase space is necessary to be able to use the results for $\Sigma_{\text{DU}}^{(2)}|_{\mathcal{H},gg}$ (see Eq. (4.42)) from Ref. [79]. Indeed, in Ref. [79], the terms involving triple-collinear limits were computed using an approach integrated with the NSC subtraction. These terms were calculated under the assumption that single-collinear limits act on the phase space. The computations discussed in Ref. [79] are intricate, and it is advantageous for us to use them as they are. To do so, however, we are compelled to adopt the assumption of collinear limits acting on the phase space. Although this assumption is now disadvantageous for the

reorganization of terms, it explains why this also applies in $\Sigma_{\text{RR},1c}^{(a,c,\text{dc})}|_{\mathcal{H},gg}$. Therefore, it is useful to rewrite $\Sigma_{\text{RR},1c}^{(a,c,\text{dc})}|_{\mathcal{H},gg}$ in such a way that:

- i) C_{im} does not act on the phase space,
- ii) restrictions imposed by the presence of sector $\theta^{(c)}$ are lifted.

We dedicate Appendix C.2 to show in details how to do this. Here, we just report the final result, which is obtained once we insert $\mathbb{1} = \bar{C}_{im} + C_{im}$ in Eq. (4.74) for $\Sigma_{\text{RR},1c}^{(a,c,\text{dc})}|_{\mathcal{H},gg}$. We find

$$\Sigma_{\text{RR},1c}^{(a,c,\text{dc})} = \Sigma_{\text{RR},1c}^{(a,c,\text{dc}),1} + \Sigma_{\text{RR},1c}^{(a,c,\text{dc}),2}, \quad (4.77)$$

where

$$\begin{aligned} \Sigma_{\text{RR},1c}^{(a,c,\text{dc}),1}|_{\mathcal{H},gg} \stackrel{\text{def}}{=} & \left\langle \mathcal{S}(\mathbf{m}, \mathbf{n}) \left[\sum_{(ij)} \bar{C}_{jn} C_{im} \omega^{m_i, n_j} + \sum_{i \in \mathcal{H}} (\eta_{in}/2)^{-\epsilon} \bar{C}_{in} C_{im} \omega^{m_i, n_i} \right] \right. \\ & \left. \times \Delta^{(\mathbf{mn})} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \right\rangle, \end{aligned} \quad (4.78)$$

and

$$\begin{aligned} \Sigma_{\text{RR},1c}^{(a,c,\text{dc}),2}|_{\mathcal{H},gg} \stackrel{\text{def}}{=} & \frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)} \left\langle \mathcal{S}(\mathbf{m}, \mathbf{n}) \left[\sum_{(ij)} C_{jn} C_{im} + \sum_{i \in \mathcal{H}} (\eta_{in}/2)^{-\epsilon} C_{in} C_{im} \right] \right. \\ & \left. \times \Delta^{(\mathbf{mn})} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \right\rangle. \end{aligned} \quad (4.79)$$

We point out that the phase space $[dp_m][dp_n]$ does not appear in these formulas anymore, indicating that collinear operators there do not act on it. We also note that the sector function $\theta^{(c)}$ disappeared from Eqs (4.78) and (4.79), leaving as a remnant the factor of $(\eta_{in}/2)^{-\epsilon}$. Furthermore, this two formulas become *potentially ambiguous* because the collinear operators C_{in} and C_{im} do not commute in general. Therefore, the order in which they appear in the above formula (and in similar formulas) is important.⁹ On the other hand, since the operator $\mathcal{S}(\mathbf{m}, \mathbf{n})$ represents a soft subtraction, it commutes with the collinear operators. Finally, we have omitted an overall factor

$$\frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)} \quad (4.80)$$

in $\Sigma_{\text{RR},1c}^{(a,c,\text{dc}),1}|_{\mathcal{H},gg}$ because it would only generate $\mathcal{O}(\epsilon)$ terms in the result.

We proceed with the discussion of the contribution $\Sigma_{\text{RR},1c}^{(a,c,\text{dc}),1}|_{\mathcal{H},gg}$. It is convenient to rewrite the factor $(\eta_{in}/2)^{-\epsilon}$ in Eq. (4.78) as

$$(\eta_{in}/2)^{-\epsilon} = [(\eta_{in}/2)^{-\epsilon} - 1] + 1, \quad (4.81)$$

⁹For unresolved $\mathbf{m}_g \mathbf{n}_g$, these limits will always commute. However, this statement does not hold in the $\mathbf{m}_q \mathbf{n}_g$ case.

and combine the $+1$ term with the sum over (ij) in (4.78). This yields

$$\begin{aligned} \Sigma_{\text{RR},1c}^{(a,c,\text{dc}),1}|_{\mathcal{H},gg} = & \left\langle \mathcal{S}(\mathbf{m}, \mathbf{n}) \left[\sum_{i,j \in \mathcal{H}} \bar{C}_{jn} C_{im} \omega^{\mathbf{m}i, \mathbf{n}j} + \sum_{i \in \mathcal{H}} [(\eta_{in}/2)^{-\epsilon} - 1] \bar{C}_{in} C_{im} \omega^{\mathbf{m}i, \mathbf{n}i} \right] \right. \\ & \left. \times \Delta^{(\text{mn})} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \right\rangle, \end{aligned} \quad (4.82)$$

where, for the sake of clarity, we emphasize that the first sum includes terms with $i = j$. Another important point is that the second term in Eq. (4.82) is *finite* in the limit $\epsilon \rightarrow 0$. The reason for this is that the only singularity present in this term comes from the collinear limit $i \parallel \mathbf{m}$, which gives an $\mathcal{O}(\epsilon^{-1})$ contribution once integrated over the phase space of gluon $\mathbf{m}_g$. On the other hand, since the collinear divergence of parton $\mathbf{n}_g$ is regulated through the operator $\bar{C}_{in}$, we can expand the object $[(\eta_{in}/2)^{-\epsilon} - 1]$ in power of ϵ before integrating over $[dp_n]$, which gives an $\mathcal{O}(\epsilon)$ quantity. Hence, the divergence of $\mathcal{O}(\epsilon^{-1})$ caused by parton $\mathbf{m}_g$ is compensated by the collinear regularization of parton $\mathbf{n}_g$, which is of $\mathcal{O}(\epsilon)$, and the combination of both is of $\mathcal{O}(\epsilon^0)$. Furthermore, in the first term on the right-hand side of Eq. (4.82), the partitioning can be replaced with another, more suitable one. Indeed, since by construction

$$\sum_{j \in \mathcal{H}} C_{im} \omega^{\mathbf{m}i, \mathbf{n}j} \equiv \sum_{j \in \mathcal{H}} \omega_{i \parallel \mathbf{m}}^{\mathbf{m}i, \mathbf{n}j} C_{im} = C_{im}, \quad C_{jn} C_{im} \omega^{\mathbf{m}i, \mathbf{n}j} = C_{jn} C_{im}, \quad (4.83)$$

one finds

$$\sum_{i,j \in \mathcal{H}} \bar{C}_{jn} C_{im} \omega^{\mathbf{m}i, \mathbf{n}j} = \sum_{i \in \mathcal{H}} C_{im} - \sum_{i,j \in \mathcal{H}} C_{jn} C_{im} \equiv \sum_{i,j \in \mathcal{H}} \bar{C}_{jn} \omega^{\mathbf{n}j} C_{im}, \quad (4.84)$$

where $\omega^{\mathbf{n}j}$ is, e.g., a NLO partition where the unresolved gluon is $\mathbf{n}_g$.

Finally, it is convenient to split the soft subtraction operator $\mathcal{S}(\mathbf{m}, \mathbf{n})$ acting on the first term in Eq. (4.82) in a particular way. Using the following representation (cf. Eq. (4.76)):

$$\mathcal{S}(\mathbf{m}, \mathbf{n}) = \bar{S}_n \Theta_{\text{nn}} + \bar{S}_m \Theta_{\text{nm}} = \bar{S}_n (\mathbb{1} - S_m \Theta_{\text{nm}}) + S_n \bar{S}_m \Theta_{\text{nm}}, \quad (4.85)$$

we rewrite the expression for $\Sigma_{\text{RR},1c}^{(a,c,\text{dc}),1}|_{\mathcal{H},gg}$ in such a way that partonic multiplicities are clearly separated:

$$\Sigma_{\text{RR},1c}^{(a,c,\text{dc}),1}|_{\mathcal{H},gg} = \left[\Sigma_{\text{SU}}^{(2)} + \Sigma_{\text{DU}}^{(4)} + \Sigma_{\text{SU}}^{\text{fin},(1)} + \Sigma_{\text{DU}}^{\text{fin},(2)} \right]_{\mathcal{H},gg}. \quad (4.86)$$

The individual contributions read

$$\begin{aligned} \Sigma_{\text{SU}}^{(2)}|_{\mathcal{H},gg} & \stackrel{\text{def}}{=} \sum_{i,j \in \mathcal{H}} \langle \bar{S}_n (\mathbb{1} - S_m \Theta_{\text{nm}}) \bar{C}_{jn} \omega^{\mathbf{n}j} C_{im} \Delta^{(\text{mn})} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \rangle \\ & \equiv \sum_{i \in \mathcal{H}} \left\langle \mathcal{O}_{\text{NLO}}^{(\text{m})} (\mathbb{1} - S_n \Theta_{\text{nn}}) C_{in} \Delta^{(\text{mn})} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \right\rangle, \end{aligned} \quad (4.87)$$

$$\Sigma_{\text{DU}}^{(4)}|_{\mathcal{H},gg} \stackrel{\text{def}}{=} \sum_{i,j \in \mathcal{H}} \langle S_n \bar{S}_m \bar{C}_{jn} \omega^{nj} C_{im} \Delta^{(mn)} \Theta_{nm} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \rangle, \quad (4.88)$$

$$\begin{aligned} \Sigma_{\text{SU}}^{\text{fin},(1)}|_{\mathcal{H},gg} &\stackrel{\text{def}}{=} \sum_{i \in \mathcal{H}} \left\langle [(\eta_{in}/2)^{-\epsilon} - 1] \bar{S}_n (1 - S_m \Theta_{nm}) \bar{C}_{in} C_{im} \omega^{mi,ni} \Delta^{(mn)} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \right\rangle \\ &\equiv \sum_{i \in \mathcal{H}} \left\langle \mathcal{O}_{\text{NLO}}^{(i,m)} \omega_{i\|n}^{mi,ni} [(\eta_{im}/2)^{-\epsilon} - 1] (1 - S_n \Theta_{mn}) \right. \\ &\quad \left. \times C_{in} \Delta^{(mn)} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \right\rangle, \end{aligned} \quad (4.89)$$

$$\begin{aligned} \Sigma_{\text{DU}}^{\text{fin},(2)}|_{\mathcal{H},gg} &\stackrel{\text{def}}{=} \sum_{i \in \mathcal{H}} \left\langle [(\eta_{in}/2)^{-\epsilon} - 1] S_n \bar{S}_m \bar{C}_{in} C_{im} \omega^{mi,ni} \Delta^{(mn)} \Theta_{nm} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \right\rangle \\ &\equiv \sum_{i \in \mathcal{H}} \left\langle [(\eta_{im}/2)^{-\epsilon} - 1] \bar{C}_{im} S_m \langle \bar{S}_n C_{in} \omega^{mi,ni} \Delta^{(mn)} \Theta_{mn} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \rangle \right\rangle, \end{aligned} \quad (4.90)$$

with $\mathcal{O}_{\text{NLO}}^{(i,m)}$ and $\mathcal{O}_{\text{NLO}}^{(m)}$ given in Eq. (3.31). Note that, when moving from the first to the second line in $\Sigma_{\text{SU}}^{(2)}|_{\mathcal{H},gg}$, $\Sigma_{\text{SU}}^{\text{fin},(1)}|_{\mathcal{H},gg}$, and $\Sigma_{\text{DU}}^{\text{fin},(2)}|_{\mathcal{H},gg}$, we have relabelled m to n and *vice versa*. Additionally, in Eq. (4.86), the first ϵ -finite contribution to the single-unresolved cross section is denoted as $\Sigma_{\text{SU}}^{\text{fin},(1)}|_{\mathcal{H},gg}$. We emphasize again that this does not correspond to the finite part of $\Sigma_{\text{SU}}^{(1)}|_{\mathcal{H},gg}$. Finally, in writing the term $\Sigma_{\text{DU}}^{\text{fin},(2)}|_{\mathcal{H},gg}$, after relabeling $m \leftrightarrow n$, we rearranged the operators as

$$\langle S_m \bar{S}_n \bar{C}_{im} C_{in} [\dots] \rangle = \langle \bar{C}_{im} S_m \langle \bar{S}_n C_{in} [\dots] \rangle \rangle, \quad (4.91)$$

where, once again, the double parentheses $\langle \langle \dots \rangle \rangle$ emphasize that there is an integration operation between the C_{in} and S_m operators. This rearrangement is justified by the properties of $\bar{S}_n$ and $\bar{C}_{im}$, which ensure that the soft divergence of $\mathbf{n}_g$ and the collinear divergence of $\mathbf{m}_g$ are avoided. We observe that, although the soft limit S_m is acting in $\Sigma_{\text{DU}}^{\text{fin},(2)}|_{\mathcal{H},gg}$, the expression remains ϵ -finite because this soft divergence is regulated by the energy-ordering Θ_{mn} .

We now return to $\Sigma_{\text{RR},1c}^{(a,c,\text{dc}),2}|_{\mathcal{H},gg}$ in Eq. (4.79). We rewrite it as

$$\Sigma_{\text{RR},1c}^{(a,c,\text{dc}),2}|_{\mathcal{H},gg} = \Sigma_{\text{DU}}^{(5)}|_{\mathcal{H},gg} + \Sigma_{\text{DU}}^{\text{fin},(3)}|_{\mathcal{H},gg} \quad (4.92)$$

where

$$\Sigma_{\text{DU}}^{(5)}|_{\mathcal{H},gg} \stackrel{\text{def}}{=} \left\langle \mathcal{S}(\mathbf{m}, \mathbf{n}) \left[\sum_{(ij)} C_{jn} C_{im} + \sum_{i \in \mathcal{H}} (\eta_{in}/2)^{-\epsilon} C_{in} C_{im} \right] \Delta^{(mn)} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \right\rangle, \quad (4.93)$$

$$\Sigma_{\text{DU}}^{\text{fin},(3)}|_{\mathcal{H},gg} \stackrel{\text{def}}{=} \left[\frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)} - 1 \right] \times \Sigma_{\text{DU}}^{(5)}|_{\mathcal{H},gg}. \quad (4.94)$$

Once again, we emphasize that $\Sigma_{\text{DU}}^{\text{fin},(3)}|_{\mathcal{H},gg}$ is finite because the soft-regulated collinear limits $C_{jn} C_{im}$ produce an $\mathcal{O}(\epsilon^{-2})$ pole when integrated over the angles of $\mathbf{m}_g$ and $\mathbf{n}_g$,

Table 4.1: List of the terms that contribute to $\Sigma_{\text{RR},1\text{c}}^{(a,c,\text{dc})}|_{\mathcal{H},gg}$.

Terms that contribute to $\Sigma_{\text{RR},1\text{c}}^{(a,c,\text{dc})} _{\mathcal{H},gg}$	
Name	Reference
$\Sigma_{\text{DU}}^{(4)} _{\mathcal{H},gg}$	Eq. (4.88)
$\Sigma_{\text{DU}}^{(5)} _{\mathcal{H},gg}$	Eq. (4.93)
$\Sigma_{\text{DU}}^{\text{fin},(2)} _{\mathcal{H},gg}$	Eq. (4.90)
$\Sigma_{\text{DU}}^{\text{fin},(3)} _{\mathcal{H},gg}$	Eq. (4.94)
$\Sigma_{\text{SU}}^{(2)} _{\mathcal{H},gg}$	Eq. (4.87)
$\Sigma_{\text{SU}}^{\text{fin},(1)} _{\mathcal{H},gg}$	Eq. (4.89)

while the expansion of the prefactor reads

$$\frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)} - 1 = \mathcal{O}(\epsilon^2). \quad (4.95)$$

This concludes our discussion of all single-collinear limits, except for those in the triple-collinear sectors $\theta^{(b,d)}$, which are treated in the following subsection. After all the manipulations, we have rewritten $\Sigma_{\text{RR},1\text{c}}^{(a,c,\text{dc})}|_{\mathcal{H},gg}$ as

$$\Sigma_{\text{RR},1\text{c}}^{(a,c,\text{dc})}|_{\mathcal{H},gg} = \sum_{i=4,5} \Sigma_{\text{DU}}^{(i)}|_{\mathcal{H},gg} + \sum_{i=2,3} \Sigma_{\text{DU}}^{\text{fin},(i)}|_{\mathcal{H},gg} + \Sigma_{\text{SU}}^{(2)}|_{\mathcal{H},gg} + \Sigma_{\text{SU}}^{\text{fin},(1)}|_{\mathcal{H},gg}. \quad (4.96)$$

For the reader's convenience, Table 4.1 summarizes the reference of each contribution.

4.3.2 $\Sigma_{\text{RR},1\text{c}}^{(b,d)}|_{\mathcal{H},gg}$ contribution

We now turn to $\Sigma_{\text{RR},1\text{c}}^{(b,d)}|_{\mathcal{H},gg}$, defined in Eq. (4.70). We begin by mapping sector $\theta^{(d)}$ onto sector $\theta^{(b)}$ by renaming gluons $\mathbf{m}_g$ to $\mathbf{n}_g$ and *vice versa*, where appropriate. It is important to emphasize that this exchange between sectors $\theta^{(b)}$ and $\theta^{(d)}$ is only valid at the level of *integrated* subtraction terms, and is therefore prohibited for the fully-regulated term $\Sigma_{\text{FR}}|_{\mathcal{H},gg}$ in Eq. (4.41). Under this renaming, $\Sigma_{\text{RR},1\text{c}}^{(b,d)}|_{\mathcal{H},gg}$ becomes

$$\begin{aligned}
\Sigma_{\text{RR},1\text{c}}^{(b,d)}|_{\mathcal{H},gg} &= \sum_{i \in \mathcal{H}} \left\langle \bar{S}_{\text{nn}} (\bar{S}_{\text{n}} \Theta_{\text{nn}} + \bar{S}_{\text{m}} \Theta_{\text{nm}}) C_{\text{nn}} \theta^{(b)} [dp_{\text{m}}] [dp_{\text{n}}] \omega^{\text{mi},\text{ni}} \right. \\
&\quad \times \Delta^{(\text{mn})} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \Big\rangle \\
&= \sum_{i \in \mathcal{H}} \left\langle \bar{S}_{\text{nn}} (\mathbb{1} - S_{\text{n}} \Theta_{\text{nn}} - S_{\text{m}} \Theta_{\text{nm}}) C_{\text{nn}} \theta^{(b)} [dp_{\text{m}}] [dp_{\text{n}}] \omega^{\text{mi},\text{ni}} \right. \\
&\quad \times \Delta^{(\text{mn})} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \Big\rangle.
\end{aligned} \quad (4.97)$$

Making use of the fact that the action of the collinear operator C_{mn} on the function $F_{LM}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g]$ is symmetric in $\mathbf{m}$ and $\mathbf{n}$, we can exchange $\mathbf{m} \leftrightarrow \mathbf{n}$ in the term with Θ_{nm} in Eq. (4.97), which yields

$$\begin{aligned} \Sigma_{RR,1c}^{(b,d)}|_{\mathcal{H},gg} &= \sum_{i \in \mathcal{H}} \left\langle \bar{S}_{mn}(\mathbb{1} - 2S_n \Theta_{mn}) C_{mn} \theta^{(b)}[dp_m][dp_n] \omega^{mi,ni} \right. \\ &\quad \left. \times \Delta^{(mn)} F_{LM}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \right\rangle. \end{aligned} \quad (4.98)$$

The action of the collinear operator C_{mn} on the phase space of two unresolved partons leads to a non-trivial result. To derive it, we consider the specific phase-space parametrization [19, 20] described in Appendix C, where all the required definitions are collected and a detailed calculation is presented. The final result is in Eq. (C.32), which we report here for convenience:¹⁰

$$\begin{aligned} &C_{mn}[d\Omega_m^{(d-1)}][d\Omega_n^{(d-1)}] \theta^{(b)} \omega^{mi,ni} F_{LM}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \\ &= \left[\frac{1}{8\pi^2} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \right] N_\epsilon^{(b,d)} \omega_{m||n}^{mi,ni} \eta_{i[mn]}^{-\epsilon} (1 - \eta_{i[mn]})^\epsilon [d\Omega_{[mn]}^{(d-1)}] \\ &\quad \times \left[\bar{\rho}_{mn} \frac{dx_4}{x_4^{1+2\epsilon}} \frac{[d\Omega_a^{(d-3)}]}{[\Omega^{(d-3)}]} d\Lambda \right] C_{mn} F_{LM}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g], \end{aligned} \quad (4.99)$$

where $[d\Omega_x^{(d-1)}]$ is defined in Eqs (2.61 - 2.65). In this formula, $[mn]_g$ denotes a clustered gluon with momentum $p_{[mn]}^\mu = p_m^\mu + p_n^\mu$, calculated in the strict collinear limit, and the expression for $C_{mn} F_{LM}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g]$ is given in Eq. (E.2). From Eq. (E.2), it follows that $C_{mn} F_{LM}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \sim F_{LM}^{\mathcal{H}}[[mn]_g]$. Since it depends solely on the kinematics of the clustered parton $[mn]_g$, we can integrate over dx_4 , $d\Omega_a^{(d-3)}$, and $d\Lambda$. After doing so, we find

$$\begin{aligned} \Sigma_{RR,1c}^{(b,d)}|_{\mathcal{H},gg} &= - \sum_{i \in \mathcal{H}} [\alpha_s] \mu^{2\epsilon} \frac{N_\epsilon^{(b,d)}}{2\epsilon} \left\langle \int_0^{E_{\max}} \frac{dE_m}{E_m^{2\epsilon-1}} \frac{dE_n}{E_n^{2\epsilon-1}} \int [d\Omega_{[mn]}^{(d-1)}] \sigma_{i[mn]}^{-\epsilon} \right. \\ &\quad \times \omega_{m||n}^{mi,ni} \bar{S}_{mn}(\mathbb{1} - 2S_n \Theta_{mn}) \Delta^{([mn])} \frac{1}{E_m E_n} \left[P_{gg}(z) F_{LM}^{\mathcal{H}}[[mn]_g] \right. \\ &\quad \left. \left. + \epsilon \left[P_{gg}^\perp(z) (r_{i,(b)}^\mu r_{i,(b)}^\nu + g^{\mu\nu}) - P_{gg}^{\perp,r}(z) g^{\mu\nu} \right] F_{LM,\mu\nu}^{\mathcal{H}}[[mn]_g] \right] \right\rangle. \end{aligned} \quad (4.100)$$

In Eq. (4.100), P_{gg} , $P_{gg}^\perp$ and $P_{gg}^{\perp,r}$ are splitting functions defined in Eqs (2.84) and (A.67) respectively, $z = E_m/(E_m + E_n)$, is the longitudinal fraction, and

$$\sigma_{ij} \stackrel{\text{def}}{=} \frac{\eta_{ij}}{1 - \eta_{ij}}. \quad (4.101)$$

Additionally, the four-vector $r_{i,(b)}^\mu$, defined in Eq. (C.39), describes *spin correlations* that arise in the collinear limit, as detailed in Appendix C.2. Notably, $r_{i,(b)}^\mu$ is *partition-dependent*, as indicated by the subscript i .

¹⁰We point out that the following formula holds for sector $\theta^{(d)}$ as well.

Following the discussion in Ref. [23], it is convenient to decompose Eq. (4.100) into two terms,

$$\Sigma_{\text{RR},1\text{c}}^{(b,d)}|_{\mathcal{H},gg} = \Sigma_{\text{RR},1\text{c}}^{(b,d),\text{sa}}|_{\mathcal{H},gg} + \Sigma_{\text{RR},1\text{c}}^{(b,d),\text{sc}}|_{\mathcal{H},gg}, \quad (4.102)$$

where the first term on the right-hand side is *spin-averaged*, and the second is *spin-correlated*. The spin-averaged contribution depends on the spin-averaged splitting function P_{gg} and yields the leading divergent part of $\Sigma_{\text{RR},1\text{c}}^{(b,d)}|_{\mathcal{H},gg}$, with its Laurent series beginning at $\mathcal{O}(\epsilon^{-2})$. The spin-correlated contribution, $\Sigma_{\text{RR},1\text{c}}^{(b,d),\text{sc}}|_{\mathcal{H},gg}$, encompasses all terms in Eq. (4.100) proportional to $F_{\text{LM},\mu\nu}^{\mathcal{H}}[[\text{mn}]_g]$. Since these terms are suppressed by a factor of ϵ , the spin-correlated part is less divergent than the spin-averaged one; its Laurent series starts at $\mathcal{O}(\epsilon^{-1})$. Thus, in this section, we focus on the spin-averaged contribution $\Sigma_{\text{RR},1\text{c}}^{(b,d),\text{sa}}|_{\mathcal{H},gg}$ and defer the detailed discussion of $\Sigma_{\text{RR},1\text{c}}^{(b,d),\text{sc}}|_{\mathcal{H},gg}$ to Appendix E.

Our starting point is the following expression for the spin-averaged contribution extracted from Eq. (4.100):

$$\begin{aligned} \Sigma_{\text{RR},1\text{c}}^{(b,d),\text{sa}}|_{\mathcal{H},gg} = & - \sum_{i \in \mathcal{H}} [\alpha_s] \mu^{2\epsilon} \frac{N_\epsilon^{(b,d)}}{2\epsilon} \left\langle \int_0^{E_{\text{max}}} \frac{dE_{\text{m}}}{E_{\text{m}}^{2\epsilon-1}} \frac{dE_{\text{n}}}{E_{\text{n}}^{2\epsilon-1}} \int [\text{d}\Omega_{[\text{mn}]_g}^{(d-1)}] \sigma_{i[\text{mn}]}^{-\epsilon} \right. \\ & \times \omega_{\text{m}||\text{n}}^{i,\text{ni}} \bar{S}_{\text{mn}} (\mathbb{1} - 2S_{\text{n}} \Theta_{\text{mn}}) \Delta^{([\text{mn}]_g)} \frac{1}{E_{\text{m}} E_{\text{n}}} P_{gg}(z) F_{\text{LM}}^{\mathcal{H}}[[\text{mn}]_g] \Big\rangle. \end{aligned} \quad (4.103)$$

To rewrite it, it is convenient to *undo* the collinear limit, as follows:

$$\begin{aligned} & - [\alpha_s] \mu^{2\epsilon} \frac{N_\epsilon^{(b,d)}}{2\epsilon} \frac{1}{E_{\text{m}} E_{\text{n}}} P_{gg}(z) F_{\text{LM}}^{\mathcal{H}}[[\text{mn}]_g] \\ & \equiv \frac{N_{\text{m}||\text{n}}(\epsilon)}{2} \int [\text{d}\Omega_{\text{n}}^{(d-1)}] C_{\text{mn}} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g], \end{aligned} \quad (4.104)$$

where

$$N_{\text{m}||\text{n}}(\epsilon) \stackrel{\text{def}}{=} 2^{2\epsilon} \frac{\Gamma(1+2\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1+\epsilon)\Gamma(1-\epsilon)}. \quad (4.105)$$

Note that the integration on the right-hand side of Eq. (4.104) is performed solely over the angular phase space of the unresolved parton $\mathbf{n}_g$. As a result, $\Sigma_{\text{RR},1\text{c}}^{(b,d),\text{sa}}$ becomes

$$\begin{aligned} \Sigma_{\text{RR},1\text{c}}^{(b,d),\text{sa}}|_{\mathcal{H},gg} = & \frac{N_{\text{m}||\text{n}}(\epsilon)}{2} \sum_{i \in \mathcal{H}} \left\langle \int_0^{E_{\text{max}}} \frac{dE_{\text{m}}}{E_{\text{m}}^{2\epsilon-1}} \frac{dE_{\text{n}}}{E_{\text{n}}^{2\epsilon-1}} \int [\text{d}\Omega_{[\text{mn}]}^{(d-1)}] [\text{d}\Omega_{\text{n}}^{(d-1)}] \sigma_{i[\text{mn}]}^{-\epsilon} \right. \\ & \times \omega_{\text{m}||\text{n}}^{i,\text{ni}} \bar{S}_{\text{mn}} (\mathbb{1} - 2S_{\text{n}} \Theta_{\text{mn}}) \Delta^{([\text{mn}]_g)} C_{\text{mn}} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \Big\rangle. \end{aligned} \quad (4.106)$$

Next, we note that the action of S_{mn} on $C_{\text{mn}} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g]$ is equivalent to the action of a soft operator $S_{[\text{mn}]}$, which refers to the zero-energy limit of a clustered parton $[\text{mn}]$. We also note that the joint action of S_{n} and S_{mn} can also be described as $S_{\text{mn}} S_{\text{n}} \equiv S_{\text{m}} S_{\text{n}}$, and that the action of S_{n} on the clustered parton $[\text{mn}]$ gives m . Following these observations,

we find

$$\begin{aligned} \Sigma_{\text{RR},1c}^{(b,d),\text{sa}}|_{\mathcal{H},gg} = & \frac{N_{m||n}(\epsilon)}{2} \sum_{i \in \mathcal{H}} \left[\left\langle \int_0^{E_{\max}} \frac{dE_m}{E_m^{2\epsilon-1}} \frac{dE_n}{E_n^{2\epsilon-1}} \int [d\Omega_{[mn]}^{(d-1)}] [d\Omega_n^{(d-1)}] \sigma_{i[mn]}^{-\epsilon} \right. \right. \\ & \times \omega_{m||n}^{mi,ni} \bar{S}_{[mn]} \Delta^{([mn])} C_{mn} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \Big\rangle \\ & \left. \left. - \langle 2\Theta_{mn} \bar{S}_m S_n \sigma_{im}^{-\epsilon} \Delta^{(m)} \omega_{m||n}^{mi,ni} C_{mn} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \rangle \right] . \end{aligned} \quad (4.107)$$

We focus on the first term on the right-hand side of Eq. (4.107). Thanks to the constraints on the energies of $\mathbf{m}_g$ and $\mathbf{n}_g$, the energy of the clustered parton $E_{[mn]}$ may exceed $E_{\max}$ and go all the way up to $2E_{\max}$. We faced the same issue in Subsec. 2.3.2, where we defined the two regions of integration $\Theta_{\text{(I)}}$ and $\Theta_{\text{(II)}}$ (see Eq. (2.95)). We report here their definitions for convenience.

$$\Theta_{\text{(I)}} \stackrel{\text{def}}{=} \begin{cases} 0 \leq E_{[mn]} \leq E_{\max} \\ 0 \leq z \leq 1, \end{cases} \quad \Theta_{\text{(II)}} \stackrel{\text{def}}{=} \begin{cases} E_{\max} \leq E_{[mn]} \leq 2E_{\max} \\ z \leq \frac{E_{\max}}{E_{[mn]}} \\ z \geq 1 - \frac{E_{\max}}{E_{[mn]}} , \end{cases} \quad (4.108)$$

In Subsec. 2.3.2, we argued that the integral over the unphysical region $\Theta_{\text{(II)}}$ was zero, since $E_{[mn]}$ exceeds the maximal energy that a parton can have in a physical process, i.e., $E_{\max}$. In contrast, region $\Theta_{\text{(II)}}$ gives a non-zero contribution in Eq. (4.107) in the soft limit because the parton $[mn]$ does not appear in the matrix element or in the momentum-conserving δ -function. In virtue of this, we write $\Sigma_{\text{RR},1c}^{(b,d),\text{sa}}|_{\mathcal{H},gg}$ as the sum of two terms,

$$\Sigma_{\text{RR},1c}^{(b,d),\text{sa}}|_{\mathcal{H},gg} = \Sigma_{\text{RR},1c}^{(b,d),\text{sa,I}}|_{\mathcal{H},gg} + \Sigma_{\text{RR},1c}^{(b,d),\text{sa,II}}|_{\mathcal{H},gg} , \quad (4.109)$$

where:

- i) $\Sigma_{\text{RR},1c}^{(b,d),\text{sa,I}}|_{\mathcal{H},gg}$ includes both the first term on the right-hand side of Eq. (4.107), integrated over the region $\Theta_{\text{(I)}}$, and the second term on the right-hand side of Eq. (4.107);
- ii) $\Sigma_{\text{RR},1c}^{(b,d),\text{sa,II}}|_{\mathcal{H},gg}$ includes the first term on the right-hand side of Eq. (4.107), integrated over the region $\Theta_{\text{(II)}}$.

The term $\Sigma_{\text{RR},1c}^{(b,d),\text{sa,I}}|_{\mathcal{H},gg}$ can be written in the following way

$$\begin{aligned} \Sigma_{\text{RR},1c}^{(b,d),\text{sa,I}}|_{\mathcal{H},gg} = & \frac{N_{m||n}(\epsilon)}{2} \sum_{i \in \mathcal{H}} \left\langle \bar{S}_m \sigma_{im}^{-\epsilon} \Delta^{(m)} \omega_{m||n}^{mi,ni} \langle (\mathbb{1} - 2\Theta_{mn} S_n) \right. \\ & \left. \times C_{mn} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \rangle \right\rangle , \end{aligned} \quad (4.110)$$

where in the first (Θ_{mn} -independent) term we renamed $[mn]_g \rightarrow \mathbf{m}_g$. Essentially, Eq. (4.110)

must be interpreted as follows: one first computes the integral

$$\langle (\mathbb{1} - 2\Theta_{\mathbf{mn}} S_{\mathbf{n}}) C_{\mathbf{mn}} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \rangle, \quad (4.111)$$

then renames $[\mathbf{mn}]_g \rightarrow \mathbf{m}_g$, and finally considers the remaining terms and/or limits, which are thus expressed using the label $\mathbf{m}$ instead of $[\mathbf{mn}]$. As previously done, we prefer to use the double integral symbols $\langle \langle \dots \rangle \rangle$ to keep track of this detail.

The expression in Eq. (4.110) contains further divergences which arise when gluon $\mathbf{m}_g$ becomes collinear to parton i . We extract these divergences by introducing collinear operators, and write

$$\Sigma_{\text{RR},1\text{c}}^{(b,d),\text{sa,I}}|_{\mathcal{H},gg} = \left[\Sigma_{\text{SU}}^{\text{fin},(2)} + \Sigma_{\text{DU}}^{(6)} + \Sigma_{N+1}^{(3)} \right]_{\mathcal{H},gg}, \quad (4.112)$$

where

$$\begin{aligned} \Sigma_{\text{SU}}^{\text{fin},(2)}|_{\mathcal{H},gg} \stackrel{\text{def}}{=} \sum_{i \in \mathcal{H}} \frac{1}{2} \langle \mathcal{O}_{\text{NLO}}^{(i,\mathbf{m})} \omega_{\mathbf{m}||\mathbf{n}}^{\mathbf{m}i,\mathbf{n}i} \Delta^{(\mathbf{m})} [N_{\mathbf{m}||\mathbf{n}}(\epsilon) \sigma_{i\mathbf{m}}^{-\epsilon} - 1] \langle (\mathbb{1} - 2\Theta_{\mathbf{mn}} S_{\mathbf{n}}) \\ \times C_{\mathbf{mn}} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \rangle \rangle, \end{aligned} \quad (4.113)$$

$$\Sigma_{\text{DU}}^{(6)}|_{\mathcal{H},gg} \stackrel{\text{def}}{=} \sum_{i \in \mathcal{H}} \frac{N_{\mathbf{m}||\mathbf{n}}(\epsilon)}{2} \langle \bar{S}_{\mathbf{m}} C_{i\mathbf{m}} \Delta^{(\mathbf{m})} \sigma_{i\mathbf{m}}^{-\epsilon} \langle (\mathbb{1} - 2\Theta_{\mathbf{mn}} S_{\mathbf{n}}) C_{\mathbf{mn}} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \rangle \rangle, \quad (4.114)$$

$$\Sigma_{\text{SU}}^{(3)}|_{\mathcal{H},gg} \stackrel{\text{def}}{=} \frac{1}{2} \langle \mathcal{O}_{\text{NLO}}^{(\mathbf{m})} \Delta^{(\mathbf{m})} \langle (\mathbb{1} - 2\Theta_{\mathbf{mn}} S_{\mathbf{n}}) C_{\mathbf{mn}} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \rangle \rangle, \quad (4.115)$$

where we remind the reader that $\mathcal{O}_{\text{NLO}}^{(i,\mathbf{m})}$ and $\mathcal{O}_{\text{NLO}}^{(\mathbf{m})}$ are given in Eq. (3.31). The term in Eq. (4.113) is finite in the limit $\epsilon \rightarrow 0$, the term in Eq. (4.114) is double-unresolved, and the last one in Eq. (4.115) is single-unresolved. In Eq. (4.115), we replaced the set of NNLO partition functions $\omega_{\mathbf{m}||\mathbf{n}}^{\mathbf{m}i,\mathbf{n}i}$ with a set of NLO partition functions $\omega^{\mathbf{m}i}$, following the argument given in Eq. (4.84).

We continue with the discussion of $\Sigma_{\text{RR},1\text{c}}^{(b,d),\text{sa,II}}|_{\mathcal{H},gg}$, which reads

$$\begin{aligned} \Sigma_{\text{RR},1\text{c}}^{(b,d),\text{sa,II}}|_{\mathcal{H},gg} \stackrel{\text{def}}{=} \frac{N_{\mathbf{m}||\mathbf{n}}(\epsilon)}{2} \sum_{i \in \mathcal{H}} \left[\left\langle \int_0^{E_{\text{max}}} \frac{dE_{\mathbf{m}}}{E_{\mathbf{m}}^{2\epsilon-1}} \frac{dE_{\mathbf{n}}}{E_{\mathbf{n}}^{2\epsilon-1}} \Theta_{(\text{II})} \int [d\Omega_{[\mathbf{mn}]}^{(d-1)}] [d\Omega_{\mathbf{n}}^{(d-1)}] \right. \right. \\ \left. \left. \times \sigma_{i[\mathbf{mn}]}^{-\epsilon} \omega_{\mathbf{m}||\mathbf{n}}^{\mathbf{m}i,\mathbf{n}i} \bar{S}_{[\mathbf{mn}]} \Delta^{([\mathbf{mn}])} C_{\mathbf{mn}} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \right\rangle \right] \end{aligned} \quad (4.116)$$

In Eq. (4.116), we can replace $\bar{S}_{[\mathbf{mn}]}$ with $-S_{[\mathbf{mn}]}$, because, as said, $F_{\text{LM}}^{\mathcal{H}}[[\mathbf{mn}]_g]$ has zero support if the energy of the clustered parton exceeds E_{max} . Then, we compute the collinear limit, change the integration variables to $E_{[\mathbf{mn}]} = E_{\mathbf{m}} + E_{\mathbf{n}}$ and $z = E_{\mathbf{m}}/(E_{\mathbf{m}} +$

E_n), and integrate over the angular phase space of the gluon n_g . We obtain

$$\begin{aligned} \Sigma_{\text{RR},1c}^{(b,d),\text{sa},\text{II}}|_{\mathcal{H},gg} &= [\alpha_s] \mu^{2\epsilon} \frac{N_\epsilon^{(b,d)}}{2\epsilon} \sum_{i \in \mathcal{H}} \left\langle \int_{E_{\text{max}}}^{2E_{\text{max}}} \frac{dE_{[\text{mn}]}}{E_{[\text{mn}]^{4\epsilon-1}}} \int_{1-\xi}^{\xi} dz [z(1-z)]^{-2\epsilon} P_{gg}(z) \right. \\ &\quad \times \left. \int [d\Omega_{[\text{mn}]^{(d-1)}}] \sigma_{i[\text{mn}]}^{-\epsilon} \omega_{\text{m}||\text{n}}^{\text{mi},\text{ni}} S_{[\text{mn}]} F_{\text{LM}}^{\mathcal{H}}[[\text{mn}]_g] \right\rangle, \quad \xi = \frac{E_{\text{max}}}{E_{[\text{mn}]}}. \end{aligned} \quad (4.117)$$

At this point, for the remaining soft limit $S_{[\text{mn}]} F_{\text{LM}}^{\mathcal{H}}[[\text{mn}]_g]$ in Eq. (4.117), we use the standard result (2.52), and get

$$\begin{aligned} \Sigma_{\text{RR},1c}^{(b,d),\text{sa},\text{II}}|_{\mathcal{H},gg} &= -[\alpha_s]^2 \frac{\delta_g^{\text{sa}}(\epsilon) (E_{\text{max}}/\mu)^{-4\epsilon}}{\epsilon} \sum_{i \in \mathcal{H}} \sum_{(kl)} \int \frac{[d\Omega_{[\text{mn}]^{(d-1)}}]}{[\Omega^{(d-2)}]} \\ &\quad \times \left\langle \sigma_{i[\text{mn}]}^{-\epsilon} \omega_{\text{m}||\text{n}}^{\text{mi},\text{ni}} \frac{\rho_{kl}}{\rho_{k[\text{mn}]} \rho_{l[\text{mn}]}} (\mathbf{T}_k \cdot \mathbf{T}_l) \cdot F_{\text{LM}}^{\mathcal{H}} \right\rangle, \end{aligned} \quad (4.118)$$

where

$$\delta_g^{\text{sa}}(\epsilon) \stackrel{\text{def}}{=} \frac{N_\epsilon^{(b,d)} E_{\text{max}}^{4\epsilon}}{2} \int_{E_{\text{max}}}^{2E_{\text{max}}} \frac{dE_{[\text{mn}]}}{E_{[\text{mn}]^{1+4\epsilon}}} \int_{1-\xi}^{\xi} dz [z(1-z)]^{-2\epsilon} P_{gg}(z), \quad \xi = \frac{E_{\text{max}}}{E_{[\text{mn}]}}. \quad (4.119)$$

The integration over the angle of the clustered gluon $[\text{mn}]_g$ in Eq. (4.118) is described in Appendix D.2. The result reads (cf. Eq. (D.33))

$$\Sigma_{\text{RR},1c}^{(b,d),\text{sa},\text{II}}|_{\mathcal{H},gg} = \Sigma_{\text{DU}}^{\text{sa}}|_{\mathcal{H},gg} + \Sigma_{\text{DU}}^{\text{sa},\text{fin}}|_{\mathcal{H},gg}. \quad (4.120)$$

In the above formula, $\Sigma_{\text{DU}}^{\text{sa}}|_{\mathcal{H},gg}$ is given by

$$\begin{aligned} \Sigma_{\text{DU}}^{\text{sa}}|_{\mathcal{H},gg} &\stackrel{\text{def}}{=} [\alpha_s]^2 2^{1+2\epsilon} \delta_g^{\text{sa}}(\epsilon) \left(\frac{2E_{\text{max}}}{\mu} \right)^{-2\epsilon} \\ &\quad \times \left[-\langle I_S(\epsilon) \cdot F_{\text{LM}}^{\mathcal{H}} \rangle + \frac{(2E_{\text{max}}/\mu)^{-2\epsilon}}{2\epsilon^2} N_c(\epsilon) \sum_{i \in \mathcal{H}} \mathbf{T}_i^2 \langle F_{\text{LM}}^{\mathcal{H}} \rangle \right], \end{aligned} \quad (4.121)$$

where

$$N_c(\epsilon) \stackrel{\text{def}}{=} \frac{2\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} - \frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)} = 1 + \mathcal{O}(\epsilon^3), \quad (4.122)$$

while the quantity $\Sigma_{\text{DU}}^{\text{sa},\text{fin}}|_{\mathcal{H},gg}$ is ϵ -finite and reads

$$\Sigma_{\text{DU}}^{\text{sa},\text{fin}}|_{\mathcal{H},gg} \stackrel{\text{def}}{=} [\alpha_s]^2 2^{2\epsilon} \delta_g^{\text{sa}}(\epsilon) \left(\frac{2E_{\text{max}}}{\mu} \right)^{-4\epsilon} \sum_{i \in \mathcal{H}} \langle \mathcal{W}_i^{\text{m}||\text{n},\text{fin}} \cdot F_{\text{LM}}^{\mathcal{H}} \rangle, \quad (4.123)$$

The partition-dependent function $\mathcal{W}_i^{\text{m}||\text{n},\text{fin}}$ has been computed in Appendix D.2, and

reads

$$\mathcal{W}_i^{\mathbf{m} \parallel \mathbf{n}, \text{fin}} \stackrel{\text{def}}{=} \sum_{(kl)} \int \frac{d\Omega_{\mathbf{m}}^{(3)}}{2\pi} \bar{C}_{im} \left[\log \left(\frac{\eta_{im}}{1 - \eta_{im}} \right) \frac{\rho_{kl}}{\rho_{km} \rho_{lm}} \omega_{\mathbf{m} \parallel \mathbf{n}}^{\mathbf{m}i, \mathbf{n}i} \right] (\mathbf{T}_k \cdot \mathbf{T}_l). \quad (4.124)$$

In Eq. (4.121), for the first time at NNLO, we encounter one of the I -operators introduced at NLO, specifically I_S . As we will see in the next chapter, these operators will play a central role in our calculations.

The final contribution to consider is the spin-correlated term $\Sigma_{\text{RR}, 1c}^{(b,d), \text{sc}} |_{\mathcal{H}, gg}$ in Eq. (4.100), i.e.,

$$\begin{aligned} \Sigma_{\text{RR}, 1c}^{(b,d), \text{sc}} |_{\mathcal{H}, gg} \stackrel{\text{def}}{=} & - \sum_{i \in \mathcal{H}} [\alpha_s] \mu^{2\epsilon} \frac{N_\epsilon^{(b,d)}}{2} \left\langle \int_0^{E_{\text{max}}} \frac{dE_{\mathbf{m}}}{E_{\mathbf{m}}^{2\epsilon-1}} \frac{dE_{\mathbf{n}}}{E_{\mathbf{n}}^{2\epsilon-1}} \int [d\Omega_{[\mathbf{mn}]}^{(d-1)}] \sigma_{i[\mathbf{mn}]}^{-\epsilon} \omega_{\mathbf{m} \parallel \mathbf{n}}^{\mathbf{m}i, \mathbf{n}i} \right. \\ & \times \bar{S}_{\mathbf{mn}} (\mathbb{1} - 2S_{\mathbf{n}} \Theta_{\mathbf{mn}}) \Delta^{([\mathbf{mn}])} \frac{1}{E_{\mathbf{m}} E_{\mathbf{n}}} \left[P_{gg}^\perp(z) (r_{i,(b)}^\mu r_{i,(b)}^\nu + g^{\mu\nu}) \right. \\ & \left. \left. - P_{gg}^{\perp, r}(z) g^{\mu\nu} \right] F_{\text{LM}, \mu\nu}^{\mathcal{H}} [[\mathbf{mn}]_g] \right\rangle. \end{aligned} \quad (4.125)$$

In Appendix E, we show that $\Sigma_{\text{RR}, 1c}^{(b,d), \text{sc}} |_{\mathcal{H}, gg}$ can be written as

$$\Sigma_{\text{RR}, 1c}^{(b,d), \text{sc}} |_{\mathcal{H}, gg} = \Sigma_{\text{RR}, 1c}^{(b,d), \text{sc}, \text{I}, 1} |_{\mathcal{H}, gg} + \Sigma_{\text{RR}, 1c}^{(b,d), \text{sc}, \text{I}, 2} |_{\mathcal{H}, gg} + \Sigma_{\text{RR}, 1c}^{(b,d), \text{sc}, \text{II}} |_{\mathcal{H}, gg}, \quad (4.126)$$

where the terms on the right-hand side are given in Eqs (E.23), (E.24) and (E.53). Among these contributions, there are two that are *identical* to $\Sigma_{\text{DU}}^{\text{sa}} |_{\mathcal{H}, gg}$ and $\Sigma_{\text{DU}}^{\text{sa}, \text{fin}} |_{\mathcal{H}, gg}$, provided we substitute $\delta_g^{\text{sa}} \mapsto \delta_g^{\perp, r}$, where

$$\delta_g^{\perp, r}(\epsilon) \stackrel{\text{def}}{=} \frac{N_\epsilon^{(b,d)} E_{\text{max}}^{4\epsilon}}{2} \int_{E_{\text{max}}}^{2E_{\text{max}}} \frac{dE_{[\mathbf{mn}]}}{E_{[\mathbf{mn}]^{1+4\epsilon}}} \int_{1-\xi}^{\xi} dz [z(1-z)]^{-2\epsilon} P_{gg}^{\perp, r}(z), \quad (4.127)$$

with $\xi = E_{\text{max}}/E_{[\mathbf{mn}]}$. We call these contributions $\Sigma_{\text{DU}}^{\text{sc}} |_{\mathcal{H}, gg}$ and $\Sigma_{\text{DU}}^{\text{sc}, \text{fin}} |_{\mathcal{H}, gg}$, and they correspond to the first and second term on the right-hand side of Eq. (E.53), respectively. Combining them with $\Sigma_{\text{DU}}^{\text{sa}} |_{\mathcal{H}, gg}$ and $\Sigma_{\text{DU}}^{\text{sa}, \text{fin}} |_{\mathcal{H}, gg}$, we obtain the quantities

$$\begin{aligned} \Sigma_{\text{DU}}^{(7)} |_{\mathcal{H}, gg} \stackrel{\text{def}}{=} & \Sigma_{\text{DU}}^{\text{sa}} |_{\mathcal{H}, gg} + \Sigma_{\text{DU}}^{\text{sc}} |_{\mathcal{H}, gg} = [\alpha_s]^2 2^{1+2\epsilon} \delta_g(\epsilon) \left(\frac{2E_{\text{max}}}{\mu} \right)^{-2\epsilon} \\ & \times \left[-\langle I_S(\epsilon) \cdot F_{\text{LM}}^{\mathcal{H}} \rangle + \frac{(2E_{\text{max}}/\mu)^{-2\epsilon}}{2\epsilon^2} N_c(\epsilon) \sum_{i \in \mathcal{H}} \mathbf{T}_i^2 \langle F_{\text{LM}}^{\mathcal{H}} \rangle \right] \end{aligned} \quad (4.128)$$

and

$$\begin{aligned} \Sigma_{\text{DU}}^{\text{fin}, (4)} |_{\mathcal{H}, gg} \stackrel{\text{def}}{=} & \Sigma_{\text{DU}}^{\text{sa}, \text{fin}} |_{\mathcal{H}, gg} + \Sigma_{\text{DU}}^{\text{sc}, \text{fin}} |_{\mathcal{H}, gg} \\ & = [\alpha_s]^2 2^{2\epsilon} \delta_g(\epsilon) \left(\frac{2E_{\text{max}}}{\mu} \right)^{-4\epsilon} \sum_{i \in \mathcal{H}} \langle \mathcal{W}_i^{\mathbf{m} \parallel \mathbf{n}, \text{fin}} \cdot F_{\text{LM}}^{\mathcal{H}} \rangle, \end{aligned} \quad (4.129)$$

Table 4.2: List of the terms that contribute to $\Sigma_{\text{RR},1\text{c}}^{(b,d)}|_{\mathcal{H},gg}$.

Terms that contribute to $\Sigma_{\text{RR},1\text{c}}^{(b,d)} _{\mathcal{H},gg}$	
Name	Reference
$\Sigma_{\text{DU}}^{(6)} _{\mathcal{H},gg}$	Eq. (4.114)
$\Sigma_{\text{DU}}^{(7)} _{\mathcal{H},gg}$	Eq. (4.128)
$\Sigma_{\text{DU}}^{(8)} _{\mathcal{H},gg}$	Eq. (4.131)
$\Sigma_{\text{DU}}^{\text{fin},(4)} _{\mathcal{H},gg}$	Eq. (4.129)
$\Sigma_{\text{DU}}^{\text{fin},(5)} _{\mathcal{H},gg}$	Eq. (4.133)
$\Sigma_{\text{SU}}^{(3)} _{\mathcal{H},gg}$	Eq. (4.115)
$\Sigma_{\text{SU}}^{\text{fin},(2)} _{\mathcal{H},gg}$	Eq. (4.113)
$\Sigma_{\text{SU}}^{\text{fin},(3)} _{\mathcal{H},gg}$	Eq. (4.132)

where

$$\delta_g(\epsilon) \stackrel{\text{def}}{=} \delta_g^{\text{sa}}(\epsilon) + \delta_g^{\perp,r}(\epsilon). \quad (4.130)$$

Additionally, we denote the remaining spin-correlated terms as¹¹

$$\Sigma_{\text{DU}}^{(8)}|_{\mathcal{H},gg} \stackrel{\text{def}}{=} \Sigma_{\text{RR},1\text{c}}^{(b,d),\text{sc},\text{I},1}|_{\mathcal{H},gg}, \quad (4.131)$$

$$\Sigma_{\text{SU}}^{\text{fin},(3)}|_{\mathcal{H},gg} \stackrel{\text{def}}{=} \Sigma_{\text{RR},1\text{c}}^{(b,d),\text{sc},\text{I},2}|_{\mathcal{H},gg} \quad (4.132)$$

$$\Sigma_{\text{DU}}^{\text{fin},(5)}|_{\mathcal{H},gg} \stackrel{\text{def}}{=} [\alpha_s]^2 \delta_g^{\perp} \left(\frac{E_{\text{max}}}{\mu} \right)^{-4\epsilon} \sum_{i \in \mathcal{H}} \langle \mathcal{W}_r^{(i)} \cdot F_{\text{LM}}^{\mathcal{H}} \rangle, \quad (4.133)$$

where $\mathcal{W}_r^{(i)}$ is a partition-dependent function given in Eq. (E.50), and $\delta_g^{\perp}$ is defined as

$$\delta_g^{\perp} \stackrel{\text{def}}{=} \frac{N_{\epsilon}^{(b,d)} E_{\text{max}}^{4\epsilon}}{2} \int_{E_{\text{max}}}^{2E_{\text{max}}} \frac{dE_{[\text{mn}]}}{E_{[\text{mn}]^{1+4\epsilon}}} \int_{1-\xi}^{\xi} dz [z(1-z)]^{-2\epsilon} P_{gg}^{\perp}(z), \quad \xi = \frac{E_{\text{max}}}{E_{[\text{mn}]}}. \quad (4.134)$$

This concludes our discussion of all single-collinear limits in $\Sigma_{\text{RR},1\text{c}}^{(b,d)}|_{\mathcal{H},gg}$. After all the manipulations, we have rewritten $\Sigma_{\text{RR},1\text{c}}^{(b,d)}|_{\mathcal{H},gg}$ as

$$\Sigma_{\text{RR},1\text{c}}^{(b,d)}|_{\mathcal{H},gg} = \sum_{i=6,7,8} \Sigma_{\text{DU}}^{(i)}|_{\mathcal{H},gg} + \sum_{i=4,5} \Sigma_{\text{DU}}^{\text{fin},(i)}|_{\mathcal{H},gg} + \Sigma_{\text{SU}}^{(3)}|_{\mathcal{H},gg} + \Sigma_{\text{SU}}^{\text{fin},(2)}|_{\mathcal{H},gg}. \quad (4.135)$$

¹¹Note that the term $\Sigma_{\text{SU}}^{\text{fin},(3)}|_{\mathcal{H},gg}$ corresponds to Σ_{N+1}^{sp} in Ref. [81].

Table 4.2 summarizes the reference of each contribution.

4.3.3 Double-collinear contribution

To recapitulate, we have succeeded in expressing the double-real single-collinear contribution $\Sigma_{\text{RR},1\text{c}}|_{\mathcal{H},gg}$ as a sum of contributions to the single-unresolved terms $\Sigma_{\text{SU}}|_{\mathcal{H},gg}$ and the double-unresolved terms $\Sigma_{\text{DU}}|_{\mathcal{H},gg}$. We now do the same for the double-collinear contribution $\Sigma_{\text{RR},1\text{c}}|_{\mathcal{H},gg}$. This quantity is defined in Eq. (4.43), which we report here for convenience:

$$\Sigma_{\text{RR},2\text{c}}|_{\mathcal{H},gg} = - \sum_{(ij)} \langle \bar{S}_{\text{mn}} \bar{S}_{\text{n}} C_{j\text{n}} C_{i\text{m}} [dp_{\text{m}}] [dp_{\text{n}}] \omega^{\text{m}i,\text{n}j} \Delta^{(\text{mn})} \Theta_{\text{mn}} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \rangle. \quad (4.136)$$

We note that both collinear operators $C_{i\text{m}}$ and $C_{j\text{n}}$ act on the phase space of partons $\mathbf{m}_g$ and $\mathbf{n}_g$.¹² As done in Subsec. 4.3.1, we would like to rewrite Eq. (4.136) such that $C_{i\text{m}}$ and $C_{j\text{n}}$ do not act on the phase space of the unresolved partons. Doing this, we can rewrite $\Sigma_{\text{RR},2\text{c}}|_{\mathcal{H},gg}$ as follows (cf. Appendix C.2)

$$\Sigma_{\text{RR},2\text{c}}|_{\mathcal{H},gg} = \Sigma_{\text{DU}}^{(3)}|_{\mathcal{H},gg} + \Sigma_{\text{DU}}^{\text{fin},(1)}|_{\mathcal{H},gg}, \quad (4.137)$$

where the two terms on right-hand side correspond to

$$\Sigma_{\text{DU}}^{(3)}|_{\mathcal{H},gg} \stackrel{\text{def}}{=} - \sum_{(ij)} \langle \bar{S}_{\text{n}} C_{j\text{n}} C_{i\text{m}} \Delta^{(\text{mn})} \Theta_{\text{mn}} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \rangle, \quad (4.138)$$

$$\Sigma_{\text{DU}}^{\text{fin},(1)}|_{\mathcal{H},gg} \stackrel{\text{def}}{=} \left[\left(\frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)} \right)^2 - 1 \right] \times \Sigma_{\text{DU}}^{(3)}|_{\mathcal{H},gg}. \quad (4.139)$$

Once again we note that the unresolved phase space $[dp_{\text{m}}][dp_{\text{n}}]$ does not appear in the above formulas, indicating that collinear operators do not act on it anymore. In addition, we have used $C_{j\text{n}} C_{i\text{m}} \omega^{\text{m}i,\text{n}j} = 1$ to remove the partition functions, and $S_{\text{mn}} \bar{S}_{\text{n}} [\dots] = 0$ and $\bar{S}_{\text{mn}} \bar{S}_{\text{n}} = \bar{S}_{\text{n}}$ in virtue of the argument discussed around Eq. (4.71).

4.4 Analyzing double- and single-collinear contributions: $\mathbf{m}_q \mathbf{n}_g$ case

This section will be structured similarly to Sec. 4.3, but for the case where the two potentially unresolved particles are $\mathbf{m}_q$ and $\mathbf{n}_g$. We present a reorganization of the double- and single-collinear terms $\Sigma_{\text{RR},2\text{c}}|_{\mathcal{H},qq}$ and $\Sigma_{\text{RR},1\text{c}}|_{\mathcal{H},qq}$, defined as in Eqs (4.43) and (4.44), respectively, but without the double-soft limit S_{mn} , and changing the arguments of the F_{LM} function to $\mathbf{m}_q$ and $\mathbf{n}_g$ to indicate the unresolved partons. We will aim to provide all the necessary information; however, the details are largely analogous to those presented in Sec. 4.3 and will therefore be assumed here. We will highlight any substantial

¹²Recall that this is necessary to be able to use the results for $\Sigma_{\text{DU}}^{(2)}|_{\mathcal{H},gg}$ from Ref. [79], see the discussion below Eq. (4.76).

difference with respect to the discussion in Sec. 4.3.

We begin with $\Sigma_{\text{RR},2\text{c}}|_{\mathcal{H},qq}$. To rewrite this term, one can follow the same procedure outlined for $\Sigma_{\text{RR},2\text{c}}|_{\mathcal{H},gg}$ in Subsec. 4.3.3. Here, we present the result,

$$\Sigma_{\text{RR},2\text{c}}|_{\mathcal{H},qq} = \Sigma_{\text{DU}}^{(3)}|_{\mathcal{H},qq} + \Sigma_{\text{DU}}^{\text{fin},(1)}|_{\mathcal{H},qq}, \quad (4.140)$$

where the two terms on the right-hand side correspond to (cf. Eqs (4.138) and (4.139))

$$\Sigma_{\text{DU}}^{(3)}|_{\mathcal{H},qq} \stackrel{\text{def}}{=} - \sum_{(ij)} \langle \bar{S}_n C_{jn} C_{im} \Delta^{(mn)} \Theta_{mn} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_q, \mathbf{n}_g] \rangle, \quad (4.141)$$

$$\Sigma_{\text{DU}}^{\text{fin},(1)}|_{\mathcal{H},qq} \stackrel{\text{def}}{=} \left[\left(\frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)} \right)^2 - 1 \right] \times \Sigma_{\text{DU}}^{(3)}|_{\mathcal{H},qq}. \quad (4.142)$$

Next, we consider $\Sigma_{\text{RR},1\text{c}}|_{\mathcal{H},qq}$, and begin by separating sectors $\theta^{(b)}$ and $\theta^{(d)}$ from the remaining contributions,

$$\Sigma_{\text{RR},1\text{c}}|_{\mathcal{H},qq} = \Sigma_{\text{RR},1\text{c}}^{(a,c,\text{dc})}|_{\mathcal{H},qq} + \Sigma_{\text{RR},1\text{c}}^{(b,d)}|_{\mathcal{H},qq}, \quad (4.143)$$

where

$$\begin{aligned} \Sigma_{\text{RR},1\text{c}}^{(a,c,\text{dc})}|_{\mathcal{H},qq} \stackrel{\text{def}}{=} & \left\langle \bar{S}_n \left[\sum_{(ij)} (C_{im} + C_{jn}) \omega^{mi,nj} + \sum_{i \in \mathcal{H}} (C_{in} \theta^{(a)} + C_{im} \theta^{(c)}) \omega^{mi,ni} \right] \right. \\ & \left. \times [dp_m][dp_n] \Delta^{(mn)} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \right\rangle, \end{aligned} \quad (4.144)$$

and

$$\Sigma_{\text{RR},1\text{c}}^{(b,d)}|_{\mathcal{H},qq} \stackrel{\text{def}}{=} \sum_{i \in \mathcal{H}} \langle \bar{S}_n C_{mn} (\theta^{(b)} + \theta^{(d)}) \omega^{mi,ni} [dp_m][dp_n] \Delta^{(mn)} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \rangle. \quad (4.145)$$

We discuss these two objects in detail in the following two subsections.

4.4.1 $\Sigma_{\text{RR},1\text{c}}^{(a,c,\text{dc})}|_{\mathcal{H},qq}$ contribution

We start by treating $\Sigma_{\text{RR},1\text{c}}^{(a,c,\text{dc})}|_{\mathcal{H},qq}$ from Eq. (4.144). Once again, we follow the steps outlined in Appendix C.2 to obtain an expression for $\Sigma_{\text{RR},1\text{c}}^{(a,c,\text{dc})}|_{\mathcal{H},qq}$ preventing the collinear limits from acting on the phase space, and avoiding restrictions imposed by sectors $\theta^{(a,c)}$. Then, we insert collinear regularizations, getting

$$\Sigma_{\text{RR},1\text{c}}^{(a,c,\text{dc})}|_{\mathcal{H},qq} = \Sigma_{\text{RR},1\text{c}}^{(a,c,\text{dc}),1}|_{\mathcal{H},qq} + \Sigma_{\text{RR},1\text{c}}^{(a,c,\text{dc}),2}|_{\mathcal{H},qq}, \quad (4.146)$$

where

$$\Sigma_{\text{RR},1\text{c}}^{(a,c,\text{dc}),1}|_{\mathcal{H},qq} \stackrel{\text{def}}{=} \left\langle \bar{S}_n \left[\sum_{(ij)} (\bar{C}_{jn} C_{im} + \bar{C}_{im} C_{jn}) \omega^{mi,nj} + \sum_{i \in \mathcal{H}} \left[(\eta_{im}/2)^{-\epsilon} \bar{C}_{im} C_{in} \right. \right. \right.$$

$$+ (\eta_{in}/2)^{-\epsilon} \bar{C}_{in} C_{im} \Big] \omega^{mi,ni} \Big] \Delta^{(mn)} F_{LM}^{\mathcal{H}}[\mathbf{m}_q, \mathbf{n}_g] \Big\rangle, \quad (4.147)$$

$$\begin{aligned} \Sigma_{RR,1c}^{(a,c,dc),2}|_{\mathcal{H},qq} &\stackrel{\text{def}}{=} \frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)} \left\langle \bar{S}_n \left[2 \sum_{(ij)} C_{jn} C_{im} + \sum_{i \in \mathcal{H}} \left[(\eta_{im}/2)^{-\epsilon} C_{im} C_{in} \right. \right. \right. \\ &\quad \left. \left. \left. + (\eta_{in}/2)^{-\epsilon} C_{in} C_{im} \right] \right] \Delta^{(mn)} F_{LM}^{\mathcal{H}}[\mathbf{m}_q, \mathbf{n}_g] \right\rangle. \end{aligned} \quad (4.148)$$

Note that, in writing $\Sigma_{RR,1c}^{(a,c,dc),2}|_{\mathcal{H},qq}$, we used the property

$$C_{jn} C_{im} + C_{im} C_{jn} = 2 C_{jn} C_{im}, \quad \text{with } i \neq j, \quad (4.149)$$

since these two collinear limits are acting on different external legs. More in general, we have

$$\begin{cases} [C_{im}, C_{jn}] = 0, & \text{if } i \neq j, \\ [C_{im}, C_{in}] \neq 0. \end{cases} \quad (4.150)$$

This marks a significant difference from the $\mathbf{m}_g \mathbf{n}_g$ case discussed in Sec. 4.3. where, due to the symmetry between $\mathbf{m}_g$ and $\mathbf{n}_g$, we assumed that the collinear limits C_{im} and C_{in} commute with each other. However, this is no longer true if $\mathbf{m}$ is a quark and $\mathbf{n}$ a gluon, as can easily be seen from the following example:

$$\begin{aligned} C_{im} C_{in} F_{LM}^{ab}[\dots, i_g, \dots | \mathbf{m}_q, \mathbf{n}_g] &\sim P_{qq}(z) P_{gg}(\bar{z}) F_{LM}^{ab}[[i\mathbf{m}\mathbf{n}]_q], \\ C_{in} C_{im} F_{LM}^{ab}[\dots, i_g, \dots | \mathbf{m}_q, \mathbf{n}_g] &\sim P_{qq}(z) P_{gg}(\bar{z}) F_{LM}^{ab}[[i\mathbf{m}\mathbf{n}]_q]. \end{aligned} \quad (4.151)$$

We first consider $\Sigma_{RR,1c}^{(a,c,dc),1}|_{\mathcal{H},qq}$. It is convenient to rewrite it as (cf. Eq. (4.82))

$$\begin{aligned} \Sigma_{RR,1c}^{(a,c,dc),1}|_{\mathcal{H},qq} &= \left\langle \bar{S}_n \left[\sum_{i,j \in \mathcal{H}} (\bar{C}_{jn} C_{im} + \bar{C}_{im} C_{jn}) \omega^{mi,nj} \right. \right. \\ &\quad \left. \left. + \sum_{i \in \mathcal{H}} \left[[(\eta_{im}/2)^{-\epsilon} - 1] \bar{C}_{im} C_{in} + [(\eta_{in}/2)^{-\epsilon} - 1] \bar{C}_{in} C_{im} \right] \right. \right. \\ &\quad \left. \left. \times \omega^{mi,ni} \right] \Delta^{(mn)} F_{LM}^{\mathcal{H}}[\mathbf{m}_q, \mathbf{n}_g] \right\rangle. \end{aligned} \quad (4.152)$$

We are free to replace the NNLO partition functions $\omega^{mi,nj}$ ($i \neq j$) with their NLO counterparts by virtue of Eq. (4.84), which yields

$$\Sigma_{RR,1c}^{(a,c,dc),1}|_{\mathcal{H},qq} = \Sigma_{SU}^{(2)}|_{\mathcal{H},qq} + \Sigma_{SU}^{\text{fin},(1)}|_{\mathcal{H},qq}, \quad (4.153)$$

where the individual contributions read

$$\Sigma_{SU}^{(2)}|_{\mathcal{H},qq} \stackrel{\text{def}}{=} \sum_{i,j \in \mathcal{H}} \left\langle \left[\bar{S}_n \bar{C}_{jn} \omega^{jn} C_{im} + \bar{C}_{im} \omega^{im} \bar{S}_n C_{jn} \right] \Delta^{(mn)} F_{LM}^{\mathcal{H}}[\mathbf{m}_q, \mathbf{n}_g] \right\rangle$$

$$\equiv \sum_{i \in \mathcal{H}} \left\langle \left[\mathcal{O}_{\text{NLO}}^{(n)} C_{im} + \mathcal{O}_{\text{NLO}}^{(m)} \bar{S}_n C_{in} \right] \Delta^{(mn)} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_q, \mathbf{n}_g] \right\rangle, \quad (4.154)$$

$$\begin{aligned} \Sigma_{\text{SU}}^{\text{fin},(1)}|_{\mathcal{H},qq} &\stackrel{\text{def}}{=} \sum_{i \in \mathcal{H}} \left\langle \left[[(\eta_{im}/2)^{-\epsilon} - 1] \bar{C}_{im} \omega_{i||n}^{mi,ni} \bar{S}_n C_{in} \right. \right. \\ &\quad \left. \left. + [(\eta_{in}/2)^{-\epsilon} - 1] \bar{S}_n \bar{C}_{in} \omega_{i||m}^{mi,ni} C_{im} \right] \Delta^{(mn)} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_q, \mathbf{n}_g] \right\rangle \\ &\equiv \sum_{i \in \mathcal{H}} \left\langle \left[\mathcal{O}_{\text{NLO}}^{(i,n)} \omega_{i||m}^{mi,ni} [(\eta_{in}/2)^{-\epsilon} - 1] C_{im} \right. \right. \\ &\quad \left. \left. + \mathcal{O}_{\text{NLO}}^{(i,m)} \omega_{i||n}^{mi,ni} [(\eta_{im}/2)^{-\epsilon} - 1] \bar{S}_n C_{in} \right] \Delta^{(mn)} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_q, \mathbf{n}_g] \right\rangle. \end{aligned} \quad (4.155)$$

We recall that $\mathcal{O}_{\text{NLO}}^{(i,x)}$ and $\mathcal{O}_{\text{NLO}}^{(x)}$ are defined in Eq. (3.31), with the convention that the action of $\mathcal{O}_{\text{NLO}}^{(m)}$ on $F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_q, \mathbf{n}_g]$ omits the soft limit S_m .

Next, we move to $\Sigma_{\text{RR},1c}^{(a,c,\text{dc}),2}|_{\mathcal{H},qq}$ in Eq. (4.148). We isolate the prefactor from the rest of the formula, as follows:

$$\Sigma_{\text{RR},1c}^{(a,c,\text{dc}),2}|_{\mathcal{H},qq} = \Sigma_{\text{DU}}^{(5)}|_{\mathcal{H},qq} + \Sigma_{\text{DU}}^{\text{fin},(3)}|_{\mathcal{H},qq}, \quad (4.156)$$

where

$$\begin{aligned} \Sigma_{\text{DU}}^{(5)}|_{\mathcal{H},qq} &\stackrel{\text{def}}{=} \left\langle \bar{S}_n \left[2 \sum_{(ij)} C_{jn} C_{im} + \sum_{i \in \mathcal{H}} \left[(\eta_{im}/2)^{-\epsilon} C_{im} C_{in} \right. \right. \right. \\ &\quad \left. \left. \left. + (\eta_{in}/2)^{-\epsilon} C_{in} C_{im} \right] \right] \Delta^{(mn)} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_q, \mathbf{n}_g] \right\rangle, \end{aligned} \quad (4.157)$$

$$\Sigma_{\text{DU}}^{\text{fin},(3)}|_{\mathcal{H},qq} \stackrel{\text{def}}{=} \left[\frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)} - 1 \right] \times \Sigma_{\text{DU}}^{(5)}|_{\mathcal{H},qq}. \quad (4.158)$$

These three formulas are the analogs of Eqs (4.92 - 4.94), and $\Sigma_{\text{DU}}^{\text{fin},(3)}|_{\mathcal{H},qq}$ is ϵ -finite by the same argument that followed (4.94).

In conclusion, after all the manipulations, we have rewritten $\Sigma_{\text{RR},1c}^{(a,c,\text{dc})}|_{\mathcal{H},qq}$ as

$$\Sigma_{\text{RR},1c}^{(a,c,\text{dc})}|_{\mathcal{H},qq} = \Sigma_{\text{DU}}^{(5)}|_{\mathcal{H},qq} + \Sigma_{\text{DU}}^{\text{fin},(3)}|_{\mathcal{H},qq} + \Sigma_{\text{SU}}^{(2)}|_{\mathcal{H},qq} + \Sigma_{\text{SU}}^{\text{fin},(1)}|_{\mathcal{H},qq}. \quad (4.159)$$

Table 4.3 summarizes the reference of each contribution.

4.4.2 $\Sigma_{\text{RR},1c}^{(b,d)}|_{\mathcal{H},qq}$ contribution

Here, we discuss the rearrangement of $\Sigma_{\text{RR},1c}^{(b,d)}|_{\mathcal{H},qq}$ from Eq. (4.145). As before, we follow the same steps presented for $\Sigma_{\text{RR},1c}^{(b,d)}|_{\mathcal{H},gg}$, where the manipulations in the $\mathbf{m}_q \mathbf{n}_g$ case are somewhat simpler due to the absence of spin-correlations.

We begin by noting that, similarly to $\Sigma_{\text{RR},1c}^{(b,d)}|_{\mathcal{H},gg}$, the contributions from sectors $\theta^{(b)}$ and $\theta^{(d)}$ are identical. To prove it, in this instance, it is not advantageous to map

Table 4.3: List of the terms that contribute to $\Sigma_{\text{RR},1\text{c}}^{(\text{a,c,dc})}|_{\mathcal{H},qg}$.

Terms that contribute to $\Sigma_{\text{RR},1\text{c}}^{(\text{a,c,dc})} _{\mathcal{H},qg}$	
Name	Reference
$\Sigma_{\text{DU}}^{(5)} _{\mathcal{H},qg}$	Eq. (4.157)
$\Sigma_{\text{DU}}^{\text{fin},(3)} _{\mathcal{H},qg}$	Eq. (4.158)
$\Sigma_{\text{SU}}^{(2)} _{\mathcal{H},qg}$	Eq. (4.154)
$\Sigma_{\text{SU}}^{\text{fin},(1)} _{\mathcal{H},qg}$	Eq. (4.155)

sector $\theta^{(d)}$ onto sector $\theta^{(b)}$, as $F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_q, \mathbf{n}_g]$ lacks symmetry between $\mathbf{m}$ and $\mathbf{n}$. Instead, we observe that the action of C_{mn} on $F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_q, \mathbf{n}_g]$ yields

$$C_{\text{mn}} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_q, \mathbf{n}_g] = \frac{g_{\text{s,b}}^2 \mu_0^{2\epsilon}}{E_{\text{m}} E_{\text{n}} \rho_{\text{mn}}} P_{qq}(z) F_{\text{LM}}^{\mathcal{H}}[[\text{mn}]_q], \quad z = E_{\text{m}}/E_{[\text{mn}]}, \quad (4.160)$$

i.e., a result proportional to the splitting function $P_{qq}(z)$, defined in Eq. (2.84), which has no transverse component $\kappa_{\perp}^{\mu}$. Since the collinear limit of the phase space measure Eq. (4.99) holds for both sectors $\theta^{(b)}$ and $\theta^{(d)}$, and there is no energy ordering, integrating over the phase space for either sector leads to the same outcome. Thus, Eq. (4.145) can be rewritten as

$$\Sigma_{\text{RR},1\text{c}}^{(b,d)}|_{\mathcal{H},qg} = 2 \sum_{i \in \mathcal{H}} \left\langle \bar{S}_{\text{n}} C_{\text{mn}} \theta^{(b)} [\text{d}p_{\text{m}}] [\text{d}p_{\text{n}}] \omega^{\text{mi}, \text{ni}} \Delta^{(\text{mn})} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_q, \mathbf{n}_g] \right\rangle. \quad (4.161)$$

Additionally, the fact that $P_{qq}(z)$ has no transverse component implies that the right-hand side of Eq. (4.160) depends solely on the kinematics of the clustered parton $[\text{mn}]_q$. Hence, after taking the collinear limit C_{mn} on the phase space, as in Eq. (4.99), we can trivially integrate over $\text{d}x_4$, $\text{d}\Omega_a^{(d-3)}$, and $\text{d}\Lambda$, since no spin-correlations arise. As a result, we obtain

$$\begin{aligned} \Sigma_{\text{RR},1\text{c}}^{(b,d)}|_{\mathcal{H},qg} = & - \sum_{i \in \mathcal{H}} [\alpha_{\text{s}}] \mu^{2\epsilon} \frac{N_{\epsilon}^{(b,d)}}{\epsilon} \left\langle \int_0^{E_{\text{max}}} \frac{\text{d}E_{\text{m}}}{E_{\text{m}}^{2\epsilon}} \frac{\text{d}E_{\text{n}}}{E_{\text{n}}^{2\epsilon}} \int [\text{d}\Omega_{[\text{mn}]}^{(d-1)}] \omega_{\text{m}||\text{n}}^{\text{mi}, \text{ni}} \right. \\ & \left. \times \sigma_{i[\text{mn}]}^{-\epsilon} \Delta^{([\text{mn}])} \bar{S}_{\text{n}} P_{qq}(z) F_{\text{LM}}^{\mathcal{H}}[[\text{mn}]_q] \right\rangle, \end{aligned} \quad (4.162)$$

with $\sigma_{ix} = \eta_{ix}/(1 - \eta_{ix})$. Next, we change the variables of integration as $(E_{\text{m}}, E_{\text{n}}) \mapsto (E_{[\text{mn}]}, z)$, and this identifies the two distinct integration regions $\Theta_{(\text{I})}$ and $\Theta_{(\text{II})}$ in (4.108). In contrast with Eq. (4.117), here there is no soft limit $S_{[\text{mn}]}$, so the integral over region $\Theta_{(\text{II})}$ vanishes owing to energy-momentum conservation. Thus, the above equation be-

comes

$$\begin{aligned} \Sigma_{\text{RR},1\text{c}}^{(b,d)}|_{\mathcal{H},qg} &= \sum_{i \in \mathcal{H}} \frac{[\alpha_s]}{\epsilon} N_{\text{m}||\text{n}}(\epsilon) \left\langle \left(\frac{2E_{\text{m}}}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \gamma_{1,q \rightarrow qg}^{22}(L_{\text{m}}) \sigma_{i\text{m}}^{-\epsilon} \right. \\ &\quad \left. \times \omega_{\text{m}||\text{n}}^{mi,ni} \Delta^{(\text{m})} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_q] \right\rangle, \end{aligned} \quad (4.163)$$

where $N_{\text{m}||\text{n}}(\epsilon)$ and $\gamma_{1,q \rightarrow qg}^{22}(L_{\text{m}})$ are given in Eqs (4.105) and (3.124), respectively. According to the argument given in Eqs (3.122 - 3.124), we have

$$\Gamma_{\text{m},q} = \left(\frac{2E_{\text{m}}}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \gamma_{1,q \rightarrow qg}^{22}(L_{\text{m}}), \quad (4.164)$$

which yields

$$\Sigma_{\text{RR},1\text{c}}^{(b,d)}|_{\mathcal{H},qg} = \sum_{i \in \mathcal{H}} \frac{[\alpha_s]}{\epsilon} \left\langle \omega_{\text{m}||\text{n}}^{mi,ni} N_{\text{m}||\text{n}}(\epsilon) \sigma_{i\text{m}}^{-\epsilon} \Delta^{(\text{m})} \Gamma_{\text{m},q} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_q] \right\rangle. \quad (4.165)$$

The above expression contains divergences that arise when the quark m_q goes collinear to a hard parton i . We extract these divergences by introducing collinear operators, and write

$$\Sigma_{\text{RR},1\text{c}}^{(b,d)}|_{\mathcal{H},qg} = \Sigma_{\text{DU}}^{(6)}|_{\mathcal{H},qg} + \Sigma_{\text{SU}}^{(3)}|_{\mathcal{H},qg} + \Sigma_{\text{SU}}^{\text{fin},(2)}|_{\mathcal{H},qg}, \quad (4.166)$$

where the first term in the above formula is double-unresolved, the second term is single-unresolved, and the last one is finite in the limit $\epsilon \rightarrow 0$. They are defined as follows:

$$\Sigma_{\text{DU}}^{(6)}|_{\mathcal{H},qg} \stackrel{\text{def}}{=} \sum_{i \in \mathcal{H}} \frac{[\alpha_s]}{\epsilon} N_{\text{m}||\text{n}}(\epsilon) \left\langle C_{i\text{m}} \sigma_{i\text{m}}^{-\epsilon} \Delta^{(\text{m})} \Gamma_{\text{m},q} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_q] \right\rangle, \quad (4.167)$$

$$\Sigma_{\text{SU}}^{(3)}|_{\mathcal{H},qg} \stackrel{\text{def}}{=} \frac{[\alpha_s]}{\epsilon} \left\langle \mathcal{O}_{\text{NLO}}^{(\text{m})} \Delta^{(\text{m})} \Gamma_{\text{m},q} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_q] \right\rangle, \quad (4.168)$$

$$\Sigma_{\text{SU}}^{\text{fin},(2)}|_{\mathcal{H},qg} \stackrel{\text{def}}{=} \sum_{i \in \mathcal{H}} \frac{[\alpha_s]}{\epsilon} \left\langle \mathcal{O}_{\text{NLO}}^{(i,\text{m})} \omega_{\text{m}||\text{n}}^{mi,ni} [N_{\text{m}||\text{n}}(\epsilon) \sigma_{i\text{m}}^{-\epsilon} - 1] \Delta^{(\text{m})} \Gamma_{\text{m},q} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_q] \right\rangle. \quad (4.169)$$

Notice that in $\Sigma_{\text{SU}}^{\text{fin},(2)}|_{\mathcal{H},qg}$ we replaced the NNLO partition functions $\omega_{\text{m}||\text{n}}^{mi,ni}$ with NLO partition functions ω^{mi} .

4.5 Rearranging single- and double-unresolved terms

In the last two subsections, we have succeeded in writing the single-collinear limits in terms of expressions with either one- or two unresolved partons. In this section, we will combine and rearrange them. Once this is accomplished, the preparatory work will be complete, and the cancellation of singularities between the different contributions can be explored.

Regarding the single-unresolved contributions, we can immediately present the fi-

Table 4.4: List of terms contributing to the single-unresolved contributions in the $m_g n_g$ and $m_q n_g$ cases.

Terms that contribute to $\Sigma_{\text{SU}} _{\mathcal{H},gg}$ and $\Sigma_{\text{SU}} _{\mathcal{H},qg}$					
Name	$m_g n_g$	$m_q n_g$	Name	$m_g n_g$	$m_q n_g$
$\Sigma_{\text{SU}}^{(1)}$	Eq. (4.25)	Eq. (4.66)	$\Sigma_{\text{SU}}^{\text{fin},(1)}$	Eq. (4.89)	Eq. (4.155)
$\Sigma_{\text{SU}}^{(2)}$	Eq. (4.87)	Eq. (4.154)	$\Sigma_{\text{SU}}^{\text{fin},(2)}$	Eq. (4.113)	Eq. (4.169)
$\Sigma_{\text{SU}}^{(3)}$	Eq. (4.115)	Eq. (4.168)	$\Sigma_{\text{SU}}^{\text{fin},(3)}$	Eq. (4.132)	/

nal expressions for both the $m_g n_g$ and $m_q n_g$ cases:

$$\Sigma_{\text{SU}}|_{\mathcal{H},gg} = \sum_{i=1}^3 \Sigma_{\text{SU}}^{(i)}|_{\mathcal{H},gg} + \sum_{i=1}^3 \Sigma_{\text{SU}}^{\text{fin},(i)}|_{\mathcal{H},gg}, \quad (4.170)$$

$$\Sigma_{\text{SU}}|_{\mathcal{H},qg} = \sum_{i=1}^3 \Sigma_{\text{SU}}^{(i)}|_{\mathcal{H},qg} + \sum_{i=1}^2 \Sigma_{\text{SU}}^{\text{fin},(i)}|_{\mathcal{H},qg}. \quad (4.171)$$

The references for these contributions are listed in Table 4.4. The assembly of the double-unresolved contributions requires a bit more work, and we discuss the $m_g n_g$ and $m_q n_g$ cases separately in the following subsections.

4.5.1 The $m_g n_g$ case

We have seen in Sec. 4.3 that the contributions with two unresolved gluons can be written as a sum of eight divergent and five ϵ -finite terms, i.e.,

$$\Sigma_{\text{DU}}|_{\mathcal{H},gg} = \sum_{i=1}^8 \Sigma_{\text{DU}}^{(i)}|_{\mathcal{H},gg} + \sum_{i=1}^5 \Sigma_{\text{DU}}^{\text{fin},(i)}|_{\mathcal{H},gg}, \quad (4.172)$$

where the references of these contribution are listed in Table 4.5. We find that combining three of the divergent contributions, namely $\Sigma_{\text{DU}}^{(3,4,5)}|_{\mathcal{H},gg}$, is helpful for understanding the cancellation of poles. To make the required manipulations more transparent, in $\Sigma_{\text{DU}}^{(4)}$ we write $\bar{C}_{jn}$ as $(\mathbb{1} - C_{jn})$, use the fact that $\sum_{j=1} \omega^{nj} = 1$, and separate the $i \neq j$ and $i = j$ sums. We find

$$\begin{aligned}
& \Sigma_{\text{DU}}^{(3)}|_{\mathcal{H},gg} + \Sigma_{\text{DU}}^{(4)}|_{\mathcal{H},gg} + \Sigma_{\text{DU}}^{(5)}|_{\mathcal{H},gg} \\
&= - \sum_{(ij)} \left\langle \left[\bar{S}_n C_{jn} C_{im} \Theta_{mn} - (\bar{S}_n \Theta_{mn} + \bar{S}_m \Theta_{nm}) C_{jn} C_{im} \right] \Delta^{(mn)} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \right\rangle \\
&+ \sum_{i \in \mathcal{H}} \left\langle S_n \bar{S}_m C_{im} \Delta^{(mn)} \Theta_{nm} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \right\rangle
\end{aligned} \quad (4.173)$$

Table 4.5: The list of terms contributing to the double-unresolved contributions in the $m_g n_g$ and $m_q n_g$ cases.

Terms that contribute to $\Sigma_{\text{DU}} _{\mathcal{H},gg}$ and $\Sigma_{\text{DU}} _{\mathcal{H},qq}$					
Name	$m_g n_g$	$m_q n_g$	Name	$m_g n_g$	$m_q n_g$
$\Sigma_{\text{DU}}^{(1)}$	Eq. (4.24)	Eq. (4.65)	$\Sigma_{\text{DU}}^{\text{fin},(1)}$	Eq. (4.139)	Eq. (4.142)
$\Sigma_{\text{DU}}^{(2)}$	Eq. (4.42)	Eq. (4.67)	$\Sigma_{\text{DU}}^{\text{fin},(2)}$	Eq. (4.90)	/
$\Sigma_{\text{DU}}^{(3)}$	Eq. (4.138)	Eq. (4.141)	$\Sigma_{\text{DU}}^{\text{fin},(3)}$	Eq. (4.94)	Eq. (4.158)
$\Sigma_{\text{DU}}^{(4)}$	Eq. (4.88)	/	$\Sigma_{\text{DU}}^{\text{fin},(4)}$	Eq. (4.129)	/
$\Sigma_{\text{DU}}^{(5)}$	Eq. (4.93)	Eq. (4.157)	$\Sigma_{\text{DU}}^{\text{fin},(5)}$	Eq. (4.133)	/
$\Sigma_{\text{DU}}^{(6)}$	Eq. (4.114)	Eq. (4.167)			
$\Sigma_{\text{DU}}^{(7)}$	Eq. (4.128)	/			
$\Sigma_{\text{DU}}^{(8)}$	Eq. (4.131)	/			

$$\begin{aligned}
& - \sum_{(ij)} \langle S_n \bar{S}_m C_{jn} C_{im} \Delta^{(mn)} \Theta_{nm} F_{\text{LM}}^{\mathcal{H}}[m_g, n_g] \rangle \\
& - \sum_{i \in \mathcal{H}} \left\langle \left[S_n \bar{S}_m C_{in} C_{im} \Theta_{nm} - (\eta_{in}/2)^{-\epsilon} \mathcal{S}(m, n) C_{in} C_{im} \right] \Delta^{(mn)} F_{\text{LM}}^{\mathcal{H}}[m_g, n_g] \right\rangle.
\end{aligned}$$

Combining terms with (ij) sums in the above equation, we obtain

$$\begin{aligned}
& \Sigma_{\text{DU}}^{(3)}|_{\mathcal{H},gg} + \Sigma_{\text{DU}}^{(4)}|_{\mathcal{H},gg} + \Sigma_{\text{DU}}^{(5)}|_{\mathcal{H},gg} \\
& = \sum_{i \in \mathcal{H}} \left\langle S_n \langle \bar{S}_m C_{im} \Delta^{(mn)} \Theta_{nm} F_{\text{LM}}^{\mathcal{H}}[m_g, n_g] \rangle \right\rangle \\
& + \sum_{(ij)} \langle \bar{S}_n \bar{S}_m C_{jn} C_{im} \Delta^{(mn)} \Theta_{nm} F_{\text{LM}}^{\mathcal{H}}[m_g, n_g] \rangle \\
& - \sum_{i \in \mathcal{H}} \left\langle \left[S_n \bar{S}_m C_{in} C_{im} \Theta_{nm} - (\eta_{in}/2)^{-\epsilon} \mathcal{S}(m, n) C_{in} C_{im} \right] \Delta^{(mn)} F_{\text{LM}}^{\mathcal{H}}[m_g, n_g] \right\rangle,
\end{aligned} \tag{4.174}$$

where once again we used $\langle \langle \dots \rangle \rangle$ in the first term on the right-hand side to stress the proper order of limits and integrals. We can further simplify Eq. (4.174). This is achieved by rewriting the sum over (ij) as follows:

$$\begin{aligned}
& \sum_{(ij)} \langle \bar{S}_n \bar{S}_m C_{jn} C_{im} \Delta^{(mn)} \Theta_{nm} F_{\text{LM}}^{\mathcal{H}}[m_g, n_g] \rangle \\
& = \frac{1}{2} \sum_{i,j \in \mathcal{H}} \langle \bar{S}_n \bar{S}_m C_{jn} C_{im} \Delta^{(mn)} F_{\text{LM}}^{\mathcal{H}}[m_g, n_g] \rangle
\end{aligned} \tag{4.175}$$

$$- \frac{1}{2} \sum_{i \in \mathcal{H}} \langle \bar{S}_n \bar{S}_m C_{in} C_{im} \Delta^{(mn)} F_{LM}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \rangle.$$

We now take the last term on right-hand side of the above equation, and combine it with the first term in the last line of Eq. (4.174), which yields

$$- \frac{1}{2} \sum_{i \in \mathcal{H}} \langle (\bar{S}_n \bar{S}_m + 2S_n \bar{S}_m \Theta_{nm}) C_{in} C_{im} \Delta^{(mn)} F_{LM}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \rangle. \quad (4.176)$$

Next, we express the term $2S_n \bar{S}_m \Theta_{nm}$ in Eq. (4.176) as the sum of two identical terms, and exchange $m \leftrightarrow n$ in one of them. As a result, we get

$$\begin{aligned} & - \frac{1}{2} \sum_{i \in \mathcal{H}} \langle (\bar{S}_n \bar{S}_m + S_n \bar{S}_m \Theta_{nm} + S_m \bar{S}_n \Theta_{mn}) C_{in} C_{im} \Delta^{(mn)} F_{LM}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \rangle \\ & + \frac{1}{2} \sum_{i \in \mathcal{H}} \langle S_m \bar{S}_n \Theta_{mn} [C_{in}, C_{im}] \Delta^{(mn)} F_{LM}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \rangle \\ & = - \frac{1}{2} \sum_{i \in \mathcal{H}} \left\langle \left[S(\mathbf{m}, \mathbf{n}) C_{in} C_{im} - S_m \bar{S}_n \Theta_{mn} [C_{in}, C_{im}] \right] \Delta^{(mn)} F_{LM}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \right\rangle, \end{aligned} \quad (4.177)$$

where we recall that $[C_{in}, C_{im}] = C_{in} C_{im} - C_{im} C_{in}$. By carefully assembling all components, we find

$$\begin{aligned} & \Sigma_{DU}^{(3)}|_{\mathcal{H}, gg} + \Sigma_{DU}^{(4)}|_{\mathcal{H}, gg} + \Sigma_{DU}^{(5)}|_{\mathcal{H}, gg} \\ & = \sum_{i \in \mathcal{H}} \left\langle S_n \langle \bar{S}_m C_{im} \Delta^{(mn)} \Theta_{nm} F_{LM}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \rangle \right\rangle \\ & + \frac{1}{2} \sum_{i, j \in \mathcal{H}} \langle \bar{S}_n \bar{S}_m C_{jn} C_{im} \Delta^{(mn)} F_{LM}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \rangle \\ & + \frac{1}{2} \sum_{i \in \mathcal{H}} \langle [2(\eta_{in}/2)^{-\epsilon} - 1] S(\mathbf{m}, \mathbf{n}) C_{in} C_{im} \Delta^{(mn)} F_{LM}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \rangle \\ & + \frac{1}{2} \sum_{i \in \mathcal{H}} \langle S_m \bar{S}_n \Theta_{mn} [C_{in}, C_{im}] \Delta^{(mn)} F_{LM}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \rangle. \end{aligned} \quad (4.178)$$

Finally, we combine this result with the remaining double-unresolved contributions in Eq. (4.172), and we obtain the following final expression:

$$\begin{aligned} \Sigma_{DU}|_{\mathcal{H}, gg} & = \langle F_{VV}^{\mathcal{H}} \rangle + \langle S_{mn} \Theta_{mn} F_{LM}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \rangle + \langle S_m F_{RV}^{\mathcal{H}}[\mathbf{m}_g] \rangle \\ & + \sum_{i \in \mathcal{H}} \left\langle \bar{S}_m C_{im} \Delta^{(m)} \left[F_{RV}^{\mathcal{H}}[\mathbf{m}_g] + \langle S_n \Theta_{mn} F_{LM}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \rangle \right] \right\rangle \\ & + \sum_{i \in \mathcal{H}} \left\langle S_n \langle \bar{S}_m C_{im} \Delta^{(mn)} \Theta_{nm} F_{LM}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \rangle \right\rangle \\ & + \frac{1}{2} \sum_{i, j \in \mathcal{H}} \langle \bar{S}_n \bar{S}_m C_{jn} C_{im} \Delta^{(mn)} F_{LM}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \rangle \end{aligned}$$

$$\begin{aligned}
& + \sum_{i \in \mathcal{H}} \frac{N_{\mathbf{m}||\mathbf{n}}(\epsilon)}{2} \left\langle \bar{S}_{\mathbf{m}} C_{im} \Delta^{(\mathbf{m})} \sigma_{im}^{-\epsilon} \langle (\mathbb{1} - 2\Theta_{\mathbf{mn}} S_{\mathbf{n}}) C_{\mathbf{mn}} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \rangle \right\rangle \\
& + \frac{1}{2} \sum_{i \in \mathcal{H}} \left\langle [2(\eta_{in}/2)^{-\epsilon} - 1] \mathcal{S}(\mathbf{m}, \mathbf{n}) C_{in} C_{im} \Delta^{(\mathbf{mn})} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \right\rangle \quad (4.179) \\
& + \frac{1}{2} \sum_{i \in \mathcal{H}} \langle S_{\mathbf{m}} \bar{S}_{\mathbf{n}} \Theta_{\mathbf{mn}} [C_{in}, C_{im}] \Delta^{(\mathbf{mn})} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \rangle \\
& + [\alpha_s]^2 2^{1+2\epsilon} \delta_g(\epsilon) \left(\frac{2E_{\text{max}}}{\mu} \right)^{-2\epsilon} \left[-\langle I_S(\epsilon) \cdot F_{\text{LM}}^{\mathcal{H}} \rangle \right. \\
& \quad \left. + \frac{(2E_{\text{max}}/\mu)^{-2\epsilon}}{2\epsilon^2} N_c(\epsilon) \sum_{i \in \mathcal{H}} \mathbf{T}_i^2 \langle F_{\text{LM}}^{\mathcal{H}} \rangle \right] \\
& + \Sigma_{\text{DU}}^{(2)}|_{\mathcal{H}, gg} + \Sigma_{\text{DU}}^{(8)}|_{\mathcal{H}, gg} + \sum_{i=1}^5 \Sigma_{\text{DU}}^{\text{fin}, (i)}|_{\mathcal{H}, gg} .
\end{aligned}$$

For completeness, we have also included in this formula the term containing the commutator $[C_{in}, C_{im}]$, which, as we know, must vanish for $\mathbf{m}_g \mathbf{n}_g$.

Eq. (4.179) is completely general, meaning it can be applied to any set of hard partons $\mathcal{H}$, where the potentially unresolved particles are gluons. In the case of the process $\mathcal{A}_0$, the terms in (4.179) account for all and only the double-unresolved contributions required at NNLO. For the process $\mathcal{B}_0$, an analogous formula for the $\mathbf{m}_q \mathbf{n}_g$ case must be added to Eq. (4.179) to determine the NNLO double-unresolved corrections. We describe how to derive this formula in the following subsection.

4.5.2 The $\mathbf{m}_q \mathbf{n}_g$ case

To conclude this chapter, we assemble the double-unresolved contributions with an unresolved quark and gluon. They can be arranged as a sum of five divergent and two ϵ -finite terms, i.e.,

$$\Sigma_{\text{DU}}|_{\mathcal{H}, qg} = \sum_{i=1(\neq 4)}^6 \Sigma_{\text{DU}}^{(i)}|_{\mathcal{H}, qg} + \sum_{i=1,3} \Sigma_{\text{DU}}^{\text{fin}, (i)}|_{\mathcal{H}, qg} , \quad (4.180)$$

where the references of these contribution are listed in Table 4.5. Analogously to the $\mathbf{m}_g \mathbf{n}_g$ case, we combine $\Sigma_{\text{DU}}^{(3,5)}|_{\mathcal{H}, qg}$, as follows:

$$\begin{aligned}
& \Sigma_{\text{DU}}^{(3)}|_{\mathcal{H}, qg} + \Sigma_{\text{DU}}^{(5)}|_{\mathcal{H}, qg} \\
& = - \sum_{(ij)} \langle \bar{S}_{\mathbf{n}} C_{jn} C_{im} \Delta^{(\mathbf{mn})} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_q, \mathbf{n}_g] \rangle + \left\langle \bar{S}_{\mathbf{n}} \left[2 \sum_{(ij)} C_{jn} C_{im} \right. \right. \\
& \quad \left. \left. + \sum_{i \in \mathcal{H}} \left[(\eta_{im}/2)^{-\epsilon} C_{im} C_{in} + (\eta_{in}/2)^{-\epsilon} C_{in} C_{im} \right] \Delta^{(\mathbf{mn})} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_q, \mathbf{n}_g] \right] \right\rangle \quad (4.181) \\
& = \frac{1}{2} \sum_{i,j \in \mathcal{H}} \langle \bar{S}_{\mathbf{n}} (C_{jn} C_{im} + C_{im} C_{jn}) \Delta^{(\mathbf{mn})} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_q, \mathbf{n}_g] \rangle
\end{aligned}$$

$$+ \frac{1}{2} \sum_{i \in \mathcal{H}} \langle [2(\eta_{in}/2)^{-\epsilon} - 1] \bar{S}_n (C_{im} C_{in} + C_{in} C_{im}) \Delta^{(mn)} F_{LM}^{\mathcal{H}}[\mathbf{m}_q, \mathbf{n}_g] \rangle.$$

Note that, in the last line, we substituted $\eta_{im} \mapsto \eta_{in}$. From a mathematical standpoint, the action of $C_{im} C_{in}$ on $F_{LM}^{\mathcal{H}}[\mathbf{m}_q, \mathbf{n}_g]$ produces a factor $1/(\rho_{im} \rho_{in})$. This implies that the double integration over the azimuthal angles remains invariant, whether $\eta_{im}^{-\epsilon}$ or $\eta_{in}^{-\epsilon}$ is used, i.e.

$$\int \frac{[d\eta_{im}]}{\rho_{im}} \int \frac{[d\eta_{in}]}{\rho_{in}} \left(\frac{\eta_{in}}{2} \right)^{-\epsilon} \equiv \int \frac{[d\eta_{im}]}{\rho_{im}} \left(\frac{\eta_{im}}{2} \right)^{-\epsilon} \int \frac{[d\eta_{in}]}{\rho_{in}}. \quad (4.182)$$

Therefore, we combine this expression with the remaining double-unresolved contributions in Eq. (4.180), and we obtain

$$\begin{aligned} \Sigma_{DU}|_{\mathcal{H}, qq} &= \sum_{i \in \mathcal{H}} \left\langle C_{im} \Delta^{(m)} \left[F_{RV}^{\mathcal{H}}[\mathbf{m}_q] + \langle S_n F_{LM}^{\mathcal{H}}[\mathbf{m}_q, \mathbf{n}_g] \rangle \right] \right\rangle \\ &+ \frac{1}{2} \sum_{i, j \in \mathcal{H}} \langle \bar{S}_n (C_{jn} C_{im} + C_{im} C_{jn}) \Delta^{(mn)} F_{LM}^{\mathcal{H}}[\mathbf{m}_q, \mathbf{n}_g] \rangle \\ &+ \sum_{i \in \mathcal{H}} \frac{[\alpha_s]}{\epsilon} N_{m||n}(\epsilon) \langle C_{im} \sigma_{im}^{-\epsilon} \Delta^{(m)} \Gamma_{m,q} F_{LM}^{\mathcal{H}}[\mathbf{m}_q] \rangle \\ &+ \frac{1}{2} \sum_{i \in \mathcal{H}} \langle [2(\eta_{in}/2)^{-\epsilon} - 1] \bar{S}_n (C_{im} C_{in} + C_{in} C_{im}) \Delta^{(mn)} F_{LM}^{\mathcal{H}}[\mathbf{m}_q, \mathbf{n}_g] \rangle \\ &+ \Sigma_{DU}^{(2)}|_{\mathcal{H}, qq} + \sum_{i=1,3} \Sigma_{DU}^{\text{fin},(i)}|_{\mathcal{H}, qq}. \end{aligned} \quad (4.183)$$

This expression we have obtained is conceptually analogous to Eq. (4.179).

From Eqs (4.179) and (4.183), we see that $\Sigma_{DU}|_{\mathcal{H}, qq}$ contains a large number of terms of different physical origin that exhibit IR and collinear singularities, which will cancel when combined with the PDFs renormalization contributions. To simplify the discussion of how this happens, we will identify groups of terms which exhibit shared features. These features include quartic, triple and quadratic correlations of color-charge operators, which originate from exchanges of soft real and virtual gluons, as well as *double-boosted* and *single-boosted* kinematics that are generated by hard-collinear initial-state emissions. We will focus on these different categories in turn, since the cancellation of ϵ -poles has to occur independently for each of them.

Calculation of the NNLO double-unresolved contributions

Contents

5.1	Simplifying virtual and soft corrections	141
5.1.1	Double-virtual contribution	141
5.1.2	Double-soft contribution	142
5.1.3	Soft real-virtual contribution	143
5.1.4	Hard-collinear real-virtual contribution	145
5.1.5	Single-soft hard-collinear double-real contribution	148
5.2	Double-collinear contributions of $\mathcal{O}(\epsilon^{-2})$	150
5.2.1	Double-collinear sector	151
5.2.2	Triple-collinear sector	152
5.3	Computing all remaining double-unresolved contributions	156
5.3.1	Terms of $\mathcal{O}(\epsilon^{-2})$	157
5.3.2	Terms of $\mathcal{O}(\epsilon^{-1})$	159
5.3.3	Terms of $\mathcal{O}(\epsilon^0)$	164

In Ch. 4, we introduced a new reorganization of the terms within the *Nested Soft-Collinear* (NSC) framework without explicitly calculating the individual contributions. This led us to four key formulas, namely Eqs (4.179) and (4.183) for F_{LMS} with a potentially unresolved pair of $m_g n_g$ and $m_q n_g$, respectively, and Eqs (4.170) and (4.171) for the single-unresolved cases. At this point, all that remains is to explicitly compute these formulas.

We will structure the remainder of the thesis as follows. In this chapter, we will focus exclusively on the double-unresolved contributions, namely Eqs (4.179) and (4.183). We will present all the necessary calculations in full detail but will defer the verification of the pole cancellations to Ch. 6. In that chapter, we will also address the single-unresolved contributions, which are algebraically simpler to handle.

As extensively discussed in Ch. 4, for the NNLO corrections to the process $\mathcal{A}_0$ in (3.55), only Eq. (4.179) is required. In contrast, for the process $\mathcal{B}_0$ in (3.56), a combination of both Eqs (4.179) and (4.183) is necessary. In principle, one should address both processes separately. However, we adopt a different approach by first discussing the

process $\mathcal{B}_0$ and subsequently deriving the expression for $\mathcal{A}_0$ as a special case from the final result of the former. As previously noted, our approach treats each particle $i \in \mathcal{H}$ of the hard process independently. The only distinction between $\mathcal{A}_0$ and $\mathcal{B}_0$ lies in the non-symmetric initial and final states of the latter, due to the presence of a final-state quark. Nevertheless, since we will express the result leg by leg, knowing the formula for $\mathcal{B}_0$ enables us to derive that for $\mathcal{A}_0$ *a posteriori*. Moreover, with regard to the initial state, $\mathcal{B}_0$ involves both a gluon and a quark, allowing us to extract all the generalized splitting functions, which can then be applied to the process $\mathcal{A}_0$, where the initial-state particle flavours are not predetermined.

To summarize, the final goal is to compute the following object:

$$\mathcal{P}_{\mathcal{B}_0} [2s_{ab} d\hat{\sigma}_{\mathcal{B}_0}^{\text{NNLO}}] = \mathcal{P}_{\mathcal{B}_0} \left[\Sigma_{\text{DU}}|_{\mathcal{B}_0} + \Sigma_{\text{SU}}|_{\mathcal{B}_0} + 2s_{ab} d\hat{\sigma}_{\mathcal{B}_2}^{\text{pdf}} + 2s_{ab} d\hat{\sigma}_{\mathcal{B}_2}^{\text{int}} \right], \quad (5.1)$$

where we recall that the projector $\mathcal{P}_{\mathcal{B}_0}$, introduced on page 93, selects only those contributions that are proportional to the matrix elements squared of the underlying Born process $\mathcal{B}_0$ plus an arbitrary number of gluons. In Eq. (5.1), $\Sigma_{\text{DU}}|_{\mathcal{B}_0}$ represents the double-unresolved contribution to process $\mathcal{B}_0$, and is given by

$$\Sigma_{\text{DU}}|_{\mathcal{B}_0} \stackrel{\text{def}}{=} \Sigma_{\text{DU}}|_{\mathcal{B}_0,gg} + \Sigma_{\text{DU}}|_{\mathcal{B}_0,qg}, \quad (5.2)$$

where $\Sigma_{\text{DU}}|_{\mathcal{B}_0,gg}$ and $\Sigma_{\text{DU}}|_{\mathcal{B}_0,qg}$ are provided in Eqs (4.179) and (4.183), respectively. Next, $\Sigma_{\text{SU}}|_{\mathcal{B}_0}$ denotes the single-unresolved contribution to the process $\mathcal{B}_0$, and is given by

$$\Sigma_{\text{SU}}|_{\mathcal{B}_0} \stackrel{\text{def}}{=} \Sigma_{\text{SU}}|_{\mathcal{B}_0,gg} + \Sigma_{\text{SU}}|_{\mathcal{B}_0,qg}, \quad (5.3)$$

where $\Sigma_{\text{SU}}|_{\mathcal{B}_0,gg}$ and $\Sigma_{\text{SU}}|_{\mathcal{B}_0,qg}$ are given in Eqs (4.170) and (4.171), respectively. Finally, $d\hat{\sigma}_{\mathcal{B}_2}^{\text{pdf}}$ corresponds to the contribution arising from the PDFs renormalization, while $d\hat{\sigma}_{\mathcal{B}_2}^{\text{int}}$ is the interference piece introduced in Eq. (4.47). In this chapter, we will compute all the components of

$$\mathcal{P}_{\mathcal{B}_0} \left[\Sigma_{\text{DU}}|_{\mathcal{B}_0} + 2s_{ab} d\hat{\sigma}_{\mathcal{B}_2}^{\text{pdf}} + 2s_{ab} d\hat{\sigma}_{\mathcal{B}_2}^{\text{int}} \right]. \quad (5.4)$$

As already observed at NLO, when an unresolved particle is a quark and it goes collinear with an initial-state hard parton, it induces a flavour change of the hard parton. This implies that the single- or double-unresolved corrections no longer project onto the Born process $\mathcal{B}_0$, but rather onto different Born processes. In our specific case, only two such processes are possible, namely $\mathcal{A}_0^{(\bar{q}q)}$ and $\mathcal{A}_0^{(gg)}$, as defined in Eqs (3.110) and (3.111), respectively. Therefore, their NNLO contributions are given by

$$\mathcal{P}_{\mathcal{A}_0^{(\bar{q}q)}} [2s_{ab} d\hat{\sigma}_{\mathcal{B}_0}^{\text{NNLO}}], \quad (5.5)$$

$$\mathcal{P}_{\mathcal{A}_0^{(gg)}} [2s_{ab} d\hat{\sigma}_{\mathcal{B}_0}^{\text{NNLO}}]. \quad (5.6)$$

We discuss these contributions in Subsec. 6.5.2.

5.1 Simplifying virtual and soft corrections

In this subsection, we focus on the color-correlated contributions to the fully-unresolved quantity $\Sigma_{\text{DU}}|_{\mathcal{B}_0}$ defined in Eq. (5.2). To this end, we will examine those terms in $\Sigma_{\text{DU}}|_{\mathcal{B}_0}$ that contain soft limits and/or loop amplitudes. We aim to express the outcomes using the operators I_S , I_V , and I_C , intending to amalgamate these into explicitly finite I_T structures. Specifically, the following contributions need computation:

- i) double-virtual contribution in Subsec. 5.1.1;
- ii) double-soft contribution in Subsec. 5.1.2;
- iii) soft real-virtual contribution in Subsec. 5.1.3;
- iv) hard-collinear real-virtual contribution in Subsec. 5.1.4;
- v) single-soft hard-collinear double-real contribution in Subsec. 5.1.5.

5.1.1 Double-virtual contribution

We begin by considering the double-virtual contribution $\langle F_{\text{VV}}^{\mathcal{B}_0} \rangle$ in $\Sigma_{\text{DU}}|_{\mathcal{B}_0}$, which comes from Eq. (4.179). In this case, we can directly refer to Subsec. 2.2.2. Recall that in that subsection, we proposed a rewriting of the two-loop corrections, specifically (2.51), with the assertion that it would prove useful later in the thesis. That assertion is now realized, as we can use (2.51) to rewrite $\langle F_{\text{VV}}^{\mathcal{B}_0} \rangle$ in the following form:

$$\begin{aligned} \langle F_{\text{VV}}^{\mathcal{B}_0} \rangle &= [\alpha_s]^2 \left\langle \left[\frac{1}{2} I_V^2(\epsilon) - \frac{\Gamma(1-\epsilon)}{e^{\epsilon\gamma_E}} \left(\frac{\beta_0}{\epsilon} I_V(\epsilon) - \left(\frac{\beta_0}{\epsilon} + K \right) I_V(2\epsilon) \right) \right] \cdot F_{\text{LM}}^{\mathcal{B}_0} \right\rangle \\ &\quad + [\alpha_s]^2 \left\langle \left[-\frac{1}{2} [\bar{I}_1(\epsilon), \bar{I}_1^\dagger(\epsilon)] + [\mathcal{H}_{2,\text{cd}} + \mathcal{H}_{2,\text{tc}} + \text{h.c.}] \right] \cdot F_{\text{LM}}^{\mathcal{B}_0} \right\rangle \\ &\quad + [\alpha_s] \langle I_V(\epsilon) \cdot F_{\text{LV},\text{fin}}^{\mathcal{B}_0} \rangle + \langle F_{\text{LV}^2,\text{fin}}^{\mathcal{B}_0} \rangle + \langle F_{\text{VV},\text{fin}}^{\mathcal{B}_0} \rangle. \end{aligned} \quad (5.7)$$

In this formula, $F_{\text{LV}^2,\text{fin}}^{\mathcal{B}_0}$ and $F_{\text{VV},\text{fin}}^{\mathcal{B}_0}$ contain the finite remainders of the one-loop squared and two-loop amplitudes interfered with the tree level, respectively. Here their explicit definition:

$$F_{\text{LV}^2,\text{fin}}^{\mathcal{H}} \equiv F_{\text{LV}^2,\text{fin}}^{ab}[\{i\}_{\mathcal{H}_f}] \stackrel{\text{def}}{=} \left[\frac{\alpha_s(\mu)}{2\pi} \right]^2 \langle \mathcal{M}_1^{\text{fin}} | \mathcal{M}_1^{\text{fin}} \rangle \text{dLips}_X \hat{O}(\{p\}_{\mathcal{H}_f}; p_X), \quad (5.8)$$

$$F_{\text{VV},\text{fin}}^{\mathcal{H}} \equiv F_{\text{VV},\text{fin}}^{ab}[\{i\}_{\mathcal{H}_f}] \stackrel{\text{def}}{=} \left[\frac{\alpha_s(\mu)}{2\pi} \right]^2 2 \text{Re}[\langle \mathcal{M}_0 | \mathcal{M}_2^{\text{fin}} \rangle] \text{dLips}_X \hat{O}(\{p\}_{\mathcal{H}_f}; p_X), \quad (5.9)$$

where $|\mathcal{M}_k\rangle \equiv |\mathcal{M}_k(\{p\}_{\mathcal{H}_f}; p_X)\rangle$, with $k = 0, 1, 2$. Additionally, as we will see, in the general case the commutator in Eq. (5.7) contains triple color-correlated poles.

5.1.2 Double-soft contribution

Next, we consider the double-soft contribution in $\Sigma_{\text{DU}}|_{\mathcal{B}_0}$, which arises from (4.179), and is given by

$$\langle S_{mn} \Theta_{mn} F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g] \rangle. \quad (5.10)$$

We have extensively discussed the double-soft limit in Subsec. 2.4.1. Starting from the expression in Eq. (2.112), it is convenient to divide the double-soft contribution into two terms: one containing explicit *quartic color-correlations* and the other containing (*double*) *color-correlations*, as follows:¹

$$\langle S_{mn} \Theta_{mn} F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g] \rangle = \langle S_{mn} \Theta_{mn} F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g] \rangle_{T^4} + \langle S_{mn} \Theta_{mn} F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g] \rangle_{T^2}. \quad (5.11)$$

The quartic color-correlated component has a simple factorized expression (cf. Eq. (2.124)):

$$\langle S_{mn} \Theta_{mn} F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g] \rangle_{T^4} = [\alpha_s]^2 \left\langle \frac{1}{2} I_S^2(\epsilon) \cdot F_{\text{LM}}^{\mathcal{B}_0} \right\rangle, \quad (5.12)$$

where $I_S(\epsilon)$ is the *soft operator* introduced in (3.62). As for the (double) color-correlated term, it appears to be significantly more complex [80]. However, upon careful inspection, we find that its *poles* can be written in a reasonably simple manner. We obtain

$$\begin{aligned} \langle S_{mn} \Theta_{mn} F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g] \rangle_{T^2} &= [\alpha_s]^2 \left(\frac{C_A}{\epsilon^2} c_1(\epsilon) + \frac{\beta_0}{\epsilon} c_2(\epsilon) + \beta_0 c_3(\epsilon) \right) \langle \tilde{I}_S(2\epsilon) \cdot F_{\text{LM}}^{\mathcal{B}_0} \rangle \\ &\quad + \langle S_{mn} \Theta_{mn} F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g] \rangle_{T^2}^{\text{fin}}, \end{aligned} \quad (5.13)$$

where $c_{1,2,3}$ coefficients read

$$\begin{aligned} c_1(\epsilon) &\stackrel{\text{def}}{=} 1 + \left(\frac{\pi^2}{6} - \frac{32}{9} \right) \epsilon^2 + \left(\frac{217}{27} - \frac{137}{9} \log 2 - 22 \log^2 2 + \frac{11\zeta_3}{2} \right) \epsilon^3, \\ c_2(\epsilon) &\stackrel{\text{def}}{=} 1 + \frac{\pi^2}{3} \epsilon^2, \\ c_3(\epsilon) &\stackrel{\text{def}}{=} 4 \log 2 + 8 \epsilon \log^2 2. \end{aligned} \quad (5.14)$$

We emphasize that $c_{1,2,3}$ do not contain powers of ϵ beyond those shown above. Additionally, in Eq. (5.13) we introduced a new soft operator, namely $\tilde{I}_S(2\epsilon)$, which is defined as

$$\tilde{I}_S(2\epsilon) \stackrel{\text{def}}{=} -\frac{(2E_{\text{max}}/\mu)^{-4\epsilon}}{(2\epsilon)^2} \sum_{(ij)} \eta_{ij}^{-2\epsilon} \tilde{K}_{ij}(\mathbf{T}_i \cdot \mathbf{T}_j), \quad (5.15)$$

where

$$\tilde{K}_{ij}(\epsilon) \stackrel{\text{def}}{=} \frac{\Gamma^2(1-2\epsilon)}{\Gamma(1-4\epsilon)} \eta_{ij}^{1+3\epsilon} {}_2F_1(1+\epsilon, 1+\epsilon, 1-\epsilon, 1-\eta_{ij}). \quad (5.16)$$

We note apparent similarities between $\tilde{I}_S(2\epsilon)$ and $\tilde{K}_{ij}$ with $I_S(2\epsilon)$ and K_{ij} , where the latter is reported in Eq. (2.137). In fact, one can use the following property of the hyper-

¹They corresponds to the first and second terms on the right-hand side of Eq. (2.112), respectively.

geometric functions

$${}_2F_1(a, b, c, z) = (1 - z)^{c-a-b} {}_2F_1(c - a, c - b, c, z), \quad (5.17)$$

to show that

$$\tilde{K}_{ij}(\epsilon) = K_{ij}(2\epsilon) \frac{{}_2F_1(-2\epsilon, -2\epsilon; 1 - \epsilon, 1 - \eta_{ij})}{{}_2F_1(-2\epsilon, -2\epsilon; 1 - 2\epsilon, 1 - \eta_{ij})} = K_{ij}(2\epsilon) + \mathcal{O}(\epsilon^3), \quad (5.18)$$

which yields the following property that relates I_S and $\tilde{I}_S$:

$$\tilde{I}_S(2\epsilon) = I_S(2\epsilon) + \mathcal{O}(\epsilon). \quad (5.19)$$

This relation will be very helpful for demonstrating the cancellation of poles in color-correlated terms.

To conclude, we can write the double-soft contribution in $\Sigma_{\text{DU}}|_{\mathcal{B}_0}$ as

$$\begin{aligned} & \langle S_{mn} \Theta_{mn} F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g] \rangle \\ &= [\alpha_s]^2 \left\langle \left[\frac{1}{2} I_S^2(\epsilon) + \left(\frac{C_A}{\epsilon^2} c_1(\epsilon) + \frac{\beta_0}{\epsilon} c_2(\epsilon) + \beta_0 c_3(\epsilon) \right) \tilde{I}_S(2\epsilon) \right] \cdot F_{\text{LM}}^{\mathcal{B}_0} \right\rangle \\ & \quad + \langle S_{mn} \Theta_{mn} F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g] \rangle_{T^2}^{\text{fin}}. \end{aligned} \quad (5.20)$$

5.1.3 Soft real-virtual contribution

We move to the soft real-virtual contribution in $\Sigma_{\text{DU}}|_{\mathcal{B}_0}$, which once again comes from (4.179) only, and corresponds to

$$\langle S_m F_{\text{RV}}^{\mathcal{B}_1}[\mathbf{m}_g] \rangle. \quad (5.21)$$

We dedicated Subsec. 5.1.3 to discuss the action of the soft limit on $F_{\text{RV}}^{\mathcal{H}}[\mathbf{m}_g]$, and the explicit formula is given in Eq. (2.136). Here, Eq. (2.136) needs to be integrated over the phase space $[dp_m]$ of gluon $\mathbf{m}_g$. The terms in Eq. (2.136) that include $S_{ij}(p_m)$ and $[S_{ij}(p_m)]^{1+\epsilon}$ can be integrated over $[dp_m]$, and they give rise to color-correlated terms that can be arranged in operators $I_S(\epsilon)$ or $\tilde{I}_S(2\epsilon)$, respectively.

More in detail, the first term on the right-hand side of Eq. (2.136) can be written as

$$\begin{aligned} & \left[\frac{\alpha_s(\mu)}{2\pi} \right] 2 \text{Re} [\langle \mathcal{M}_0 | \mathbf{T}_i \cdot \mathbf{T}_j | \mathcal{M}_1 \rangle] \\ &= [\alpha_s] \langle \mathcal{M}_0 | (\mathbf{T}_i \cdot \mathbf{T}_j) \bar{I}_1 + \bar{I}_1^\dagger (\mathbf{T}_i \cdot \mathbf{T}_j) | \mathcal{M}_0 \rangle + \left[\frac{\alpha_s(\mu)}{2\pi} \right] 2 \text{Re} [\langle \mathcal{M}_0 | \mathbf{T}_i \cdot \mathbf{T}_j | \mathcal{M}_1^{\text{fin}} \rangle], \end{aligned} \quad (5.22)$$

where $\bar{I}_1$ is defined in Eq. (2.26). Note that $\bar{I}_1 \sim (\mathbf{T}_k \cdot \mathbf{T}_l)$, which implies

$$[(\mathbf{T}_i \cdot \mathbf{T}_j), \bar{I}_1] \sim [(\mathbf{T}_i \cdot \mathbf{T}_j), (\mathbf{T}_k \cdot \mathbf{T}_l)] \neq 0. \quad (5.23)$$

Therefore, the order of the above operators is important, as their commutator is non-zero. Keeping this subtlety in mind, we integrate Eq. (5.22) over the phase space $[dp_m]$,

and obtain

$$[\alpha_s]^2 \left\langle \left[I_S(\epsilon) \bar{I}_1(\epsilon) + \bar{I}_1^\dagger(\epsilon) I_S(\epsilon) \right] \cdot F_{\text{LM}}^{\mathcal{B}_0} \right\rangle + [\alpha_s] \langle I_S(\epsilon) \cdot F_{\text{LV,fin}}^{\mathcal{H}} \rangle. \quad (5.24)$$

We note that Eqs (5.7) and (5.20) contain the operators $\frac{1}{2} I_V^2$ and $\frac{1}{2} I_S^2$, respectively, and we also observe that the first term in Eq. (5.24) closely resembles a double product between I_V and I_S . To make this an explicit double product, we rewrite $\bar{I}_1$ in terms of the operator I_V as follows:

$$I_S \bar{I}_1 + \bar{I}_1^\dagger I_S = \frac{I_S I_V + I_V I_S}{2} + [I_S, I_-], \quad (5.25)$$

where we have introduced two new operators,

$$I_\pm(\epsilon) \stackrel{\text{def}}{=} \frac{\bar{I}_1(\epsilon) \pm \bar{I}_1^\dagger(\epsilon)}{2}. \quad (5.26)$$

Therefore, Eq. (5.24) becomes

$$[\alpha_s]^2 \left\langle \left[\frac{I_S(\epsilon) I_V(\epsilon) + I_V(\epsilon) I_S(\epsilon)}{2} + [I_S(\epsilon), I_-(\epsilon)] \right] \cdot F_{\text{LM}}^{\mathcal{B}_0} \right\rangle + [\alpha_s] \langle I_S(\epsilon) \cdot F_{\text{LV,fin}}^{\mathcal{B}_0} \rangle. \quad (5.27)$$

The second and third term on the right-hand side of Eq. (2.136) present no further difficulties once integrated over $[dp_m]$. As said, they give rise to contributions proportional to I_S and $\tilde{I}_S$, respectively, and read

$$-[\alpha_s]^2 \frac{\Gamma(1-\epsilon)}{e^{\epsilon\gamma_E}} \frac{\beta_0}{\epsilon} \langle I_S(\epsilon) \cdot F_{\text{LM}}^{\mathcal{B}_0} \rangle - \frac{[\alpha_s]^2}{\epsilon^2} C_A A_K(\epsilon) \langle \tilde{I}_S(2\epsilon) \cdot F_{\text{LM}}^{\mathcal{B}_0} \rangle. \quad (5.28)$$

Finally, the last term on the right-hand side of Eq. (2.136) contains explicit *triple color-correlations*

$$f_{abc} \langle \mathcal{M}_0 | T_k^a T_i^b T_j^c | \mathcal{M}_0 \rangle. \quad (5.29)$$

Integrating this term over $[dp_m]$ is non-trivial. We discuss it at length in Appendix H. In what follows, we will refer to it as the *triple color-correlated real-virtual subtraction term* $I_{\text{tri}}^{\text{RV}}(\epsilon)$, defined in Eq. (H.9), and we use to write the integral over the last term of Eq. (2.136) as

$$[\alpha_s]^2 \langle I_{\text{tri}}^{\text{RV}}(\epsilon) \cdot F_{\text{LM}}^{\mathcal{H}} \rangle. \quad (5.30)$$

To conclude, we combine Eqs (5.27), (5.28) and (5.30), and we write the soft real virtual contribution as

$$\begin{aligned} \langle S_m F_{\text{RV}}^{\mathcal{B}_1} \rangle &= [\alpha_s]^2 \left\langle \frac{1}{2} [I_S(\epsilon) I_V(\epsilon) + I_V(\epsilon) I_S(\epsilon)] \cdot F_{\text{LM}}^{\mathcal{B}_0} \right\rangle \\ &\quad - [\alpha_s]^2 \frac{\Gamma(1-\epsilon)}{e^{\epsilon\gamma_E}} \frac{\beta_0}{\epsilon} \langle I_S(\epsilon) \cdot F_{\text{LM}}^{\mathcal{B}_0} \rangle - \frac{[\alpha_s]^2}{\epsilon^2} C_A A_K(\epsilon) \langle \tilde{I}_S(2\epsilon) \cdot F_{\text{LM}}^{\mathcal{B}_0} \rangle \\ &\quad + [\alpha_s]^2 \left\langle [I_S(\epsilon), I_-(\epsilon)] + I_{\text{tri}}^{\text{RV}}(\epsilon) \right\rangle \cdot F_{\text{LM}}^{\mathcal{B}_0} \end{aligned} \quad (5.31)$$

Similarly to the double-virtual contribution, we will see that the commutator in Eq. (5.31) contains triple color-correlated poles.

5.1.4 Hard-collinear real-virtual contribution

Next, we consider the hard-collinear real-virtual contribution in $\Sigma_{\text{DU}}|_{\mathcal{B}_0}$. In this case, both Eqs (4.179) and (4.183) contribute, as follows:

$$\sum_{i \in \mathcal{H}} \langle \bar{S}_m C_{im} \Delta^{(m)} F_{\text{RV}}^{\mathcal{B}_1}[\mathbf{m}_g] \rangle + \sum_{i \in \mathcal{H}} \mathcal{P}_{\mathcal{B}_0} \langle \bar{S}_m C_{im} \Delta^{(m)} F_{\text{RV}}^{\mathcal{B}_1}[\mathbf{m}_g] \rangle. \quad (5.32)$$

We begin by analyzing the initial-state radiation, focusing specifically on the case where $i = a$. In this scenario, only the term $F_{\text{RV}}^{\mathcal{B}_1}[\mathbf{m}_g]$ in Eq. (5.32) contributes, as the action of C_{am} on $F_{\text{RV}}^{\mathcal{B}_1}[\mathbf{m}_g]$ changes the flavor of the parton a_g , and this contribution is removed by the projector $\mathcal{P}_{\mathcal{B}_0}$. Thus, our attention is directed toward $F_{\text{RV}}^{\mathcal{B}_1}[\mathbf{m}_g]$. It is important to note that we have already discussed the action of the collinear limit on a real-virtual amplitude in detail in Subsec. 2.5.2, where the explicit formula is provided in Eq. (2.146). Here, Eq. (2.146) must be integrated over the phase space $[dp_m]$ of parton $\mathbf{m}_g$. As for the first and second terms on the right-hand side of Eq. (2.146), the expression to be integrated is NLO-like, thus we obtain

$$\begin{aligned} & [\alpha_s]^2 \left\langle \frac{\Gamma_{a,g}}{\epsilon} I_V(\epsilon) \cdot F_{\text{LM}}^{\mathcal{B}_0} \right\rangle + \frac{[\alpha_s]^2}{\epsilon} \langle \mathcal{P}_{gg}^{\text{gen}} \otimes [I_V(\epsilon) \cdot F_{\text{LM}}^{\mathcal{B}_0}] \rangle + \frac{[\alpha_s]}{\epsilon} \langle \mathcal{P}_{gg}^{\text{gen}} \otimes F_{\text{LV,fin}}^{\mathcal{B}_0} \rangle \\ & - [\alpha_s]^2 \frac{\Gamma(1-\epsilon) \beta_0}{e^{\epsilon\gamma_E} \epsilon} \left[\left\langle \frac{\Gamma_{a,g}}{\epsilon} F_{\text{LM}}^{\mathcal{B}_0} \right\rangle + \frac{1}{\epsilon} \langle \mathcal{P}_{gg}^{\text{gen}} \otimes F_{\text{LM}}^{\mathcal{B}_0} \rangle \right] + [\alpha_s] \left\langle \frac{\Gamma_{a,g}}{\epsilon} F_{\text{LV,fin}}^{\mathcal{B}_0} \right\rangle, \end{aligned} \quad (5.33)$$

where $\Gamma_{a,fa}$ and $\mathcal{P}_{\alpha\beta}^{\text{gen}}$ are defined in Eqs (3.68) and (3.69), respectively. We point out that the operator $\Gamma_{a,fa}$ does not involve color-correlations, and hence commutes with I_V . Concerning the last term on the right-hand side of Eq. (2.146), it includes a one-loop splitting function. However, this integral retains a NLO-like structure and can be evaluated using the function $\mathcal{P}_{\alpha\beta}^{(k),1\text{L}}$, defined in Eq. (A.73). The result is given by

$$-\frac{[\alpha_s]^2}{2\epsilon} h_c(\epsilon) \left\langle \left[\left(\frac{2E_a}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \right]^2 \mathcal{P}_{gg}^{(2),1\text{L}} \otimes F_{\text{LM}}^{\mathcal{B}_0} \right\rangle, \quad (5.34)$$

where

$$h_c(\epsilon) \stackrel{\text{def}}{=} \frac{\Gamma^2(1-2\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-3\epsilon)}. \quad (5.35)$$

Eq. (5.34), as written, is not suitable for the explicit cancellation of singularities. Indeed, $\mathcal{P}_{gg}^{1\text{L}}$ contains an anomalous dimension multiplied by a delta function $\delta(1-z)$, which, once integrated over z , yields an elastic contribution that does not combine with the boosted terms. For this reason, emulating the procedure already applied at tree-level, we rewrite $\mathcal{P}_{\alpha\beta}^{(k),1\text{L}}$ as a sum of two contributions via Eq. (A.74), specifically $\Gamma_{a,\alpha}^{(k),1\text{L}}$ and

$\mathcal{P}_{\alpha\beta}^{(k),1L,\text{gen}}$, defined in Eqs (A.75) and (A.76), respectively. We obtain

$$-\frac{[\alpha_s]^2}{\epsilon^2} C_A h_c(\epsilon) \left\langle \frac{\Gamma_{a,g}^{1L}}{2\epsilon} F_{LM}^{\mathcal{B}_0} \right\rangle - \frac{[\alpha_s]^2}{2\epsilon^3} C_A h_c(\epsilon) \langle \mathcal{P}_{gg}^{1L,\text{gen}} \otimes F_{LM}^{\mathcal{B}_0} \rangle. \quad (5.36)$$

All in all, the soft real virtual contribution from leg a corresponds to

$$\begin{aligned} & \langle \bar{S}_m C_{am} \Delta^{(m)} F_{RV}^{\mathcal{B}_1}[\mathbf{m}_g] \rangle \\ &= [\alpha_s]^2 \left\langle \frac{\Gamma_{a,g}}{\epsilon} I_V(\epsilon) \cdot F_{LM}^{\mathcal{B}_0} \right\rangle + \frac{[\alpha_s]^2}{\epsilon} \langle \mathcal{P}_{gg}^{\text{gen}} \otimes [I_V(\epsilon) \cdot F_{LM}^{\mathcal{B}_0}] \rangle + \frac{[\alpha_s]}{\epsilon} \langle \mathcal{P}_{gg}^{\text{gen}} \otimes F_{LV,\text{fin}}^{\mathcal{B}_0} \rangle \\ & - [\alpha_s]^2 \frac{\Gamma(1-\epsilon)}{e^{\epsilon\gamma_E}} \frac{\beta_0}{\epsilon} \left[\left\langle \frac{\Gamma_{a,g}}{\epsilon} F_{LM}^{\mathcal{B}_0} \right\rangle + \frac{1}{\epsilon} \langle \mathcal{P}_{gg}^{\text{gen}} \otimes F_{LM}^{\mathcal{B}_0} \rangle \right] + [\alpha_s] \left\langle \frac{\Gamma_{a,g}}{\epsilon} F_{LV,\text{fin}}^{\mathcal{B}_0} \right\rangle \\ & - \frac{[\alpha_s]^2}{\epsilon^2} C_A h_c(\epsilon) \left\langle \frac{\Gamma_{a,g}^{1L}}{2\epsilon} F_{LM}^{\mathcal{B}_0} \right\rangle - \frac{[\alpha_s]^2}{2\epsilon^3} C_A h_c(\epsilon) \langle \mathcal{P}_{gg}^{1L,\text{gen}} \otimes F_{LM}^{\mathcal{B}_0} \rangle. \end{aligned} \quad (5.37)$$

An analogous expression holds for leg b as well, provided replacing the gluon anomalous dimensions and the generalized splitting functions with quark ones.

We move to final-state radiation, i.e., $i \in \mathcal{H}_f$. In this case, both $F_{RV}^{\mathcal{B}_1}[\mathbf{m}_g]$ and $F_{RV}^{\mathcal{B}_1}[\mathbf{m}_q]$ in Eq. (5.32) contribute. We begin with $F_{RV}^{\mathcal{B}_1}[\mathbf{m}_g]$. The explicit formula of a final-state collinear limit acting on a real-virtual amplitude is reported in Eq. (2.141). Here, we must integrate Eq. (2.141) over the final phase space $[dp_m]$. The same reasoning applied to the initial state holds for the final state as well. The first two terms on the right-hand side of Eq. (2.141) are NLO-like, and thus they can be integrated without further manipulations, yielding

$$\begin{aligned} & [\alpha_s]^2 \left\langle \left[\sum_{i \in \mathcal{H}_f} \frac{\Gamma_{i,f_i \rightarrow f_i g}}{\epsilon} \right] I_V(\epsilon) \cdot F_{LM}^{\mathcal{B}_0} \right\rangle + [\alpha_s] \left\langle \left[\sum_{i \in \mathcal{H}_f} \frac{\Gamma_{i,f_i \rightarrow f_i g}}{\epsilon} \right] F_{LV,\text{fin}}^{\mathcal{B}_0} \right\rangle \\ & - [\alpha_s]^2 \frac{\Gamma(1-\epsilon)}{e^{\epsilon\gamma_E}} \frac{\beta_0}{\epsilon} \left\langle \left[\sum_{i \in \mathcal{H}_f} \frac{\Gamma_{i,f_i \rightarrow f_i g}}{\epsilon} \right] F_{LM}^{\mathcal{B}_0} \right\rangle, \end{aligned} \quad (5.38)$$

with $\Gamma_{i,\alpha \rightarrow \rho\sigma}$ defined in Eq. (3.72). As for the last term on right-hand side of Eq. (2.141), it involves a one-loop splitting function, which can nevertheless be integrated directly over the phase space $[dp_m]$ using formula (A.79) for the one-loop anomalous dimension. We obtain

$$\begin{aligned} & \sum_{i \in \mathcal{H}_f} \frac{[\alpha_s]^2}{2\epsilon} h_c(\epsilon) \left\langle \left[\left(\frac{2E_i}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \right]^2 \cos(\pi\epsilon) \gamma_{z,f_i \rightarrow f_i g}^{33,1L}(\epsilon, L_i) F_{LM}^{\mathcal{B}_0} \right\rangle \\ & = - \frac{[\alpha_s]^2}{\epsilon^2} C_A h_c(\epsilon) \left\langle \left[\sum_{i \in \mathcal{H}_f} \frac{\Gamma_{i,f_i \rightarrow f_i g}^{1L}}{2\epsilon} \right] F_{LM}^{\mathcal{B}_0} \right\rangle. \end{aligned} \quad (5.39)$$

In the second line, we have introduced the one-loop analogue of the generalized final-state anomalous dimension previously encountered at tree-level. The explicit definitions of $\Gamma_{a,\alpha \rightarrow \rho\sigma}^{1L}$ are given in Eq. (A.78).

Next, we consider $F_{RV}^{\mathcal{B}_1}[\mathbf{m}_q]$. We can repeat the same steps as above and obtain the

same results as Eqs (5.38) and (5.39), except that we replace the sum over $\Gamma_{i,f_i \rightarrow f_i g}$ with $\Gamma_{q \rightarrow gq}$, and the sum over $\Gamma_{a,f_i \rightarrow f_i g}^{\text{1L}}$ with $\Gamma_{q \rightarrow gq}^{\text{1L}}$, i.e.

$$[\alpha_s]^2 \left\langle \frac{\Gamma_{q \rightarrow gq}}{\epsilon} I_V(\epsilon) \cdot F_{\text{LM}}^{\mathcal{B}_0} \right\rangle + [\alpha_s] \left\langle \frac{\Gamma_{q \rightarrow gq}}{\epsilon} F_{\text{LV,fin}}^{\mathcal{B}_0} \right\rangle - [\alpha_s]^2 \frac{\Gamma(1-\epsilon)}{e^{\epsilon\gamma_E}} \frac{\beta_0}{\epsilon} \left\langle \frac{\Gamma_{q \rightarrow gq}}{\epsilon} F_{\text{LM}}^{\mathcal{B}_0} \right\rangle - \frac{[\alpha_s]^2}{\epsilon^2} C_A h_c(\epsilon) \left\langle \frac{\Gamma_{q \rightarrow gq}^{\text{1L}}}{2\epsilon} F_{\text{LM}}^{\mathcal{B}_0} \right\rangle. \quad (5.40)$$

What happened to the sum over the index $i \in \mathcal{H}_f$? As argued in Subsec. 3.3.3, after C_{im} acts on $F_{\text{RV}}^{\mathcal{B}_1}[\mathbf{m}_q]$, which has a symmetry factor $1/N!$, we obtain a sum involving N identical copies of the function $F_{\text{LM}}^{gq}[\{g\}_{N-1}, q]$. This sum, therefore, produces a multiplicative factor of N , and this factor, combined with the prefactor $1/N!$ mentioned above, results in the final prefactor $1/(N-1)!$, which can be absorbed into the function $F_{\text{LM}}^{gq}[\{g\}_{N-1}, q]$ to yield $F_{\text{LM}}^{\mathcal{B}_0}$. Indeed, we recall that $F_{\text{LM}}^{\mathcal{B}_0} \equiv F_{\text{LM}}^{gq}[\{g\}_{N-1}, q]/(N-1)!$.

At this point, we combine Eqs (5.38–5.40). When $f_i = g$, no further steps are required. In contrast, if $f_i = q$, we must combine the anomalous dimensions arising from both the $q \rightarrow gq$ and $q \rightarrow qq$ splittings. For the tree-level case, we use the definition in Eq. (3.122) to obtain $\Gamma_q = \Gamma_{q \rightarrow qq} + \Gamma_{q \rightarrow gq}$. Analogously, for the one-loop case, we introduce the *generalized one-loop quark final-state anomalous dimension*

$$\Gamma_{i,q}^{\text{1L}} \stackrel{\text{def}}{=} \Gamma_{i,q \rightarrow qq}^{\text{1L}} + \Gamma_{i,q \rightarrow gq}^{\text{1L}}, \quad i \in \mathcal{H}_{f,q}. \quad (5.41)$$

In what follows, we will always write Γ_q^{1L} since process $\mathcal{B}_0$ contains just one final-state quark.

Therefore, the final-state contribution to the hard-collinear real-virtual piece reads

$$[\alpha_s]^2 \left\langle \left[\sum_{i \in \mathcal{H}_{f,g}} \frac{\Gamma_{i,g \rightarrow gg}}{\epsilon} + \frac{\Gamma_q}{\epsilon} \right] I_V(\epsilon) \cdot F_{\text{LM}}^{\mathcal{B}_0} \right\rangle + [\alpha_s]^2 \left\langle \left[\sum_{i \in \mathcal{H}_{f,g}} \frac{\Gamma_{i,g \rightarrow gg}}{\epsilon} + \frac{\Gamma_q}{\epsilon} \right] F_{\text{LV,fin}}^{\mathcal{B}_0} \right\rangle - [\alpha_s] \frac{\Gamma(1-\epsilon)}{e^{\epsilon\gamma_E}} \frac{\beta_0}{\epsilon} \left\langle \left[\sum_{i \in \mathcal{H}_{f,g}} \frac{\Gamma_{i,g \rightarrow gg}}{\epsilon} + \frac{\Gamma_q}{\epsilon} \right] F_{\text{LM}}^{\mathcal{B}_0} \right\rangle - \frac{[\alpha_s]^2}{\epsilon^2} C_A h_c(\epsilon) \left\langle \left[\sum_{i \in \mathcal{H}_{f,q}} \frac{\Gamma_{i,g \rightarrow gg}^{\text{1L}}}{2\epsilon} + \frac{\Gamma_q^{\text{1L}}}{2\epsilon} \right] F_{\text{LM}}^{\mathcal{B}_0} \right\rangle. \quad (5.42)$$

Finally, to obtain the complete expression for Eq. (5.32), we combine Eqs (5.37), for both legs a and b , and Eq. (5.42). For the tree-level case, the sum of all the generalized anomalous dimensions yields the collinear operator $I_C^{\mathcal{B}_0}$, introduced in Eq. (3.125) in the context of the NLO corrections to the process $\mathcal{B}_0$. Regarding the one-loop case, we can define an operator analogous to $I_C^{\mathcal{B}_0}$, which incorporates the generalized one-loop anomalous dimensions, as follows:

$$\tilde{I}_C^{\mathcal{B}_0}(2\epsilon) \stackrel{\text{def}}{=} \sum_{i=a,b} \frac{\Gamma_{i,f_i}^{\text{1L}}}{2\epsilon} + \sum_{i \in \mathcal{H}_{f,g}} \frac{\Gamma_{i,g \rightarrow gg}^{\text{1L}}}{2\epsilon} + \frac{\Gamma_q^{\text{1L}}}{2\epsilon}. \quad (5.43)$$

Thanks to these operators, we obtain this final expression for Eq. (5.32):

$$\begin{aligned}
& \sum_{i \in \mathcal{H}} \langle \bar{S}_m C_{im} \Delta^{(m)} F_{RV}^{\mathcal{B}_1}[\mathbf{m}_g] \rangle + \sum_{i \in \mathcal{H}} \mathcal{P}_{\mathcal{B}_0} \langle \bar{S}_m C_{im} \Delta^{(m)} F_{RV}^{\mathcal{B}_1}[\mathbf{m}_q] \rangle \\
&= [\alpha_s]^2 \langle I_C(\epsilon) I_V(\epsilon) \cdot F_{LM}^{\mathcal{B}_0} \rangle + [\alpha_s] \langle I_C(\epsilon) \cdot F_{LV, \text{fin}}^{\mathcal{B}_0} \rangle - \frac{[\alpha_s]^2}{\epsilon^2} C_A h_c(\epsilon) \langle \tilde{I}_C(2\epsilon) \cdot F_{LM}^{\mathcal{B}_0} \rangle \\
&\quad - [\alpha_s]^2 \frac{\Gamma(1-\epsilon) \beta_0}{e^{\epsilon \gamma_E}} \frac{1}{\epsilon} \left[\langle I_C(\epsilon) \cdot F_{LM}^{\mathcal{B}_0} \rangle + \frac{1}{\epsilon} \left(\langle \mathcal{P}_{gg}^{\text{gen}} \otimes F_{LM}^{\mathcal{B}_0} \rangle + \langle F_{LM}^{\mathcal{B}_0} \otimes \mathcal{P}_{qq}^{\text{gen}} \rangle \right) \right] \\
&\quad + \frac{[\alpha_s]^2}{\epsilon} \left[\langle \mathcal{P}_{gg}^{\text{gen}} \otimes F_{LM}^{\mathcal{B}_0} \rangle + \langle F_{LM}^{\mathcal{B}_0} \otimes \mathcal{P}_{qq}^{\text{gen}} \rangle \right] + \frac{[\alpha_s]}{\epsilon} \left[\langle \mathcal{P}_{gg}^{\text{gen}} \otimes F_{LV, \text{fin}}^{\mathcal{B}_0} \rangle + \langle F_{LV, \text{fin}}^{\mathcal{B}_0} \otimes \mathcal{P}_{qq}^{\text{gen}} \rangle \right] \\
&\quad - \frac{[\alpha_s]^2}{2\epsilon^3} C_A h_c(\epsilon) \left[\langle \mathcal{P}_{gg}^{1L, \text{gen}} \otimes F_{LM}^{\mathcal{B}_0} \rangle + \langle F_{LM}^{\mathcal{B}_0} \otimes \mathcal{P}_{qq}^{1L, \text{gen}} \rangle \right]. \tag{5.44}
\end{aligned}$$

5.1.5 Single-soft hard-collinear double-real contribution

The final contribution remaining in this section to discuss is the single-soft hard-collinear limit of the double-real term. Once again, both Eqs (4.179) and (4.183) contribute, as follows:

$$\begin{aligned}
& \sum_{i \in \mathcal{H}} \left[\langle \bar{S}_m C_{im} \Delta^{(m)} \langle S_n \Theta_{mn} F_{LM}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g] \rangle \rangle \right. \\
& \quad + \langle S_n \langle \bar{S}_m C_{im} \Delta^{(mn)} \Theta_{nm} F_{LM}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g] \rangle \rangle \\
& \quad \left. + \mathcal{P}_{\mathcal{B}_0} \langle C_{im} \Delta^{(m)} \langle S_n F_{LM}^{\mathcal{B}_2}[\mathbf{m}_q, \mathbf{n}_q] \rangle \rangle \right]. \tag{5.45}
\end{aligned}$$

As we have emphasized repeatedly, these three quantities must be handled with care, ensuring that limits are not interchanged with integrals where such operations are not permitted. The proof of how to compute Eq. (5.45) is rather technical and spans several pages. The details are collected in Appendix F. Here, we just present the final result of Eq. (F.27), which we reproduce below for convenience:

$$\begin{aligned}
& \text{Eq. (5.45)} = \\
&= -[\alpha_s] \frac{(2E_{\text{max}}/\mu)^{-2\epsilon}}{\epsilon^2} \sum_{i \in \mathcal{H}} \sum_{(kl)} \langle \eta_{kl}^{-\epsilon} K_{kl} \bar{S}_m C_{im} \Delta^{(m)} (\mathbf{T}_k \cdot \mathbf{T}_l) \cdot F_{LM}^{\mathcal{B}_1}[\mathbf{m}_g] \rangle \\
&\quad - [\alpha_s] \frac{(2E_{\text{max}}/\mu)^{-2\epsilon}}{\epsilon^2} \sum_{i \in \mathcal{H}} \sum_{(kl)} \mathcal{P}_{\mathcal{B}_0} \left\langle \eta_{kl}^{-\epsilon} K_{kl} (C_{im} \Delta^{(m)} (\mathbf{T}_k \cdot \mathbf{T}_l) \cdot F_{LM}^{\mathcal{B}_1}[\mathbf{m}_q])_{\mathbf{T}_i \mapsto \mathbf{T}_{[im]}} \right\rangle \\
&\quad + \frac{[\alpha_s]}{\epsilon^2} C_A \frac{\Gamma^3(1-\epsilon) \Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} \sum_{i \in \mathcal{H}} \langle (2E_m/\mu)^{-2\epsilon} \eta_{im}^{-\epsilon} \bar{S}_m C_{im} \Delta^{(m)} F_{LM}^{\mathcal{B}_1}[\mathbf{m}_g] \rangle \\
&\quad + [\alpha_s] \frac{(2E_{\text{max}}/\mu)^{-2\epsilon}}{\epsilon^2} \frac{\Gamma^3(1-\epsilon) \Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} \sum_{i \in \mathcal{H}} \mathcal{P}_{\mathcal{B}_0} C_A \langle \eta_{im}^{-\epsilon} C_{im} \Delta^{(m)} F_{LM}^{\mathcal{B}_1}[\mathbf{m}_q] \rangle. \tag{5.46}
\end{aligned}$$

The expression $\mathbf{T}_i \mapsto \mathbf{T}_{[im]}$ signifies that, if $l = i$ (or $k = i$), the color matrix $\mathbf{T}_i$ of parton i must be substituted with that of the clustered parton $[im]$, denoted as $\mathbf{T}_{[im]}$. It is impor-

tant to note that, if parton m is a gluon, then the clustered parton $[im]$ shares the same flavor as parton i , i.e., $\mathbf{T}_{[im]} = \mathbf{T}_i$. However, this equivalence does not hold when m is a quark, which is why we introduced the replacement $\mathbf{T}_i \mapsto \mathbf{T}_{[im]}$.

The final step consists of explicitly calculating the collinear limits in Eq. (5.46). Regarding the first and second terms on the right-hand side of Eq. (5.46), these involve a NLO-like calculation that we have dealt with several times, with the only exception being the presence of additional angular terms in this context. As an example, let us consider the first term. The action of the limits $\bar{S}_m C_{im}$ on $\Delta^{(m)}(\mathbf{T}_k \cdot \mathbf{T}_l) \cdot F_{LM}^{\mathcal{B}_1}[m_g]$ yields

$$\begin{aligned} \langle \bar{S}_m C_{im} \Delta^{(m)}(\mathbf{T}_k \cdot \mathbf{T}_l) \cdot F_{LM}^{\mathcal{B}_1}[m_g] \rangle &= [\alpha_s] \langle I_{C,inc}^{\mathcal{B}_0}(\mathbf{T}_k \cdot \mathbf{T}_l) \cdot F_{LM}^{\mathcal{B}_0} \rangle \\ &+ \frac{[\alpha_s]}{\epsilon} \left[\langle \mathcal{P}_{gg}^{\text{gen}} \otimes [(\mathbf{T}_k \cdot \mathbf{T}_l) \cdot F_{LM}^{\mathcal{B}_0}] \rangle + \langle [(\mathbf{T}_k \cdot \mathbf{T}_l) \cdot F_{LM}^{\mathcal{B}_0}] \otimes \mathcal{P}_{qq}^{\text{gen}} \rangle \right], \end{aligned} \quad (5.47)$$

where, for convenience, we have introduced the notation

$$I_{C,inc}^{(k),\mathcal{B}_0}(\epsilon) \stackrel{\text{def}}{=} \sum_{i=a,b} \frac{\Gamma_{i,f_i}^{(k)}}{\epsilon k/2} + \sum_{i \in \mathcal{H}_t} \frac{\Gamma_{i,f_i \rightarrow f_i g}^{(k)}}{\epsilon k/2}, \quad k = 2, 4. \quad (5.48)$$

The wording *inc* stands for *incomplete*, as we know that the final quark leg lacks the term $\Gamma_{q \rightarrow gq}^{(k)}/\epsilon$ to form the collinear operator $I_C^{\mathcal{B}_0}$. Additionally, we recall that $(\mathbf{T}_k \cdot \mathbf{T}_l)$ commutes with $I_{C,inc}^{\mathcal{B}_0}$. At this point, all the remaining objects which depend on indices k, l form precisely the soft operator $I_S(\epsilon)$. Thus, the first term on the right-hand side of Eq. (5.46) can be written as

$$\begin{aligned} &+ [\alpha_s]^2 \langle I_{C,inc}(\epsilon) I_S(\epsilon) \cdot F_{LM}^{\mathcal{B}_0} \rangle \\ &+ \frac{[\alpha_s]^2}{\epsilon} \left[\langle \mathcal{P}_{gg}^{\text{gen}} \otimes [I_S(\epsilon) \cdot F_{LM}^{\mathcal{B}_0}] \rangle + \langle [I_S(\epsilon) \cdot F_{LM}^{\mathcal{B}_0}] \otimes \mathcal{P}_{qq}^{\text{gen}} \rangle \right]. \end{aligned} \quad (5.49)$$

As for the second term on the right-hand side of Eq. (5.46), one follows the same steps as described above and obtains

$$[\alpha_s]^2 \left\langle \frac{\Gamma_{q \rightarrow gq}}{\epsilon} I_S(\epsilon) \cdot F_{LM}^{\mathcal{B}_0} \right\rangle. \quad (5.50)$$

Therefore, we sum this formula with Eq. (5.49), we notice that

$$I_{C,inc}^{\mathcal{B}_0}(\epsilon) + \frac{\Gamma_{q \rightarrow gq}}{\epsilon} \equiv I_C^{\mathcal{B}_0}(\epsilon), \quad (5.51)$$

and we obtain

$$\begin{aligned} &+ [\alpha_s]^2 \langle I_C(\epsilon) I_S(\epsilon) \cdot F_{LM}^{\mathcal{B}_0} \rangle \\ &+ \frac{[\alpha_s]^2}{\epsilon} \left[\langle \mathcal{P}_{gg}^{\text{gen}} \otimes [I_S(\epsilon) \cdot F_{LM}^{\mathcal{B}_0}] \rangle + \langle [I_S(\epsilon) \cdot F_{LM}^{\mathcal{B}_0}] \otimes \mathcal{P}_{qq}^{\text{gen}} \rangle \right]. \end{aligned} \quad (5.52)$$

The third term on the right-hand side of Eq. (5.46) is, once again, a NLO-like calculation. The only difference arises due to the additional energy factor $(2E_m/\mu)^{-2\epsilon}$.

The hard-collinear integral in this case gives rise to the usual function $\mathcal{P}_{\alpha\beta}^{(k)}$, defined in Eq. (3.67), but with $k = 4$. Consequently, instead of using the operators Γ , $\mathcal{P}_{\alpha\beta}^{\text{gen}}$, and $I_C^{\mathcal{B}_0}$, we will use $\Gamma^{(4)}$, $\mathcal{P}_{\alpha\beta}^{(4),\text{gen}}$, and $I_C^{(4)}\mathcal{B}_0$. Hence, we find

$$\begin{aligned} & + \frac{[\alpha_s]^2}{\epsilon^2} C_A h_c(\epsilon) \langle I_{C,\text{inc}}^{(4)}(\epsilon) \cdot F_{\text{LM}}^{\mathcal{B}_0} \rangle \\ & + \frac{[\alpha_s]^2}{2\epsilon^3} C_A h_c(\epsilon) \left[\langle \mathcal{P}_{gg}^{(4),\text{gen}} \otimes F_{\text{LM}}^{\mathcal{B}_0} \rangle + \langle F_{\text{LM}}^{\mathcal{B}_0} \otimes \mathcal{P}_{qq}^{(4),\text{gen}} \rangle \right], \end{aligned} \quad (5.53)$$

with $h_c(\epsilon)$ given in Eq. (5.35). Finally, the fourth term on the right-hand side of Eq. (5.46) presents no further complications, and reads

$$\frac{[\alpha_s]^2}{\epsilon^2} C_A h_c(\epsilon) \left[\left\langle \frac{\Gamma_{q \rightarrow gq}^{(4)}}{2\epsilon} F_{\text{LM}}^{\mathcal{B}_0} \right\rangle + \langle \sigma_{q \rightarrow gq} F_{\text{LM}}^{\mathcal{B}_0} \rangle \right], \quad (5.54)$$

where

$$\sigma_{i,q \rightarrow gq} \stackrel{\text{def}}{=} \frac{1}{2\epsilon} \left[e^{-2\epsilon L_{\text{max}}} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \Gamma_{i,q \rightarrow gq} - \Gamma_{i,q \rightarrow gq}^{(4)} \right], \quad i \in \mathcal{H}_{f,q}. \quad (5.55)$$

In conclusion, we can write a final expression of Eq. (5.45) by combining Eqs (5.52), (5.53) and (5.54):

$$\begin{aligned} \text{Eq. (5.45)} &= [\alpha_s]^2 \langle I_C(\epsilon) I_S(\epsilon) \cdot F_{\text{LM}}^{\mathcal{B}_0} \rangle + \frac{[\alpha_s]^2}{\epsilon^2} C_A h_c(\epsilon) \left[\langle I_C^{(4)}(\epsilon) \cdot F_{\text{LM}}^{\mathcal{B}_0} \rangle + \langle \sigma_{q \rightarrow gq} F_{\text{LM}}^{\mathcal{B}_0} \rangle \right] \\ &+ \frac{[\alpha_s]^2}{\epsilon} \left[\langle \mathcal{P}_{gg}^{\text{gen}} \otimes [I_S(\epsilon) \cdot F_{\text{LM}}^{\mathcal{B}_0}] \rangle + \langle [I_S(\epsilon) \cdot F_{\text{LM}}^{\mathcal{B}_0}] \otimes \mathcal{P}_{qq}^{\text{gen}} \rangle \right] \\ &+ \frac{[\alpha_s]^2}{2\epsilon^3} C_A h_c(\epsilon) \left[\langle \mathcal{P}_{gg}^{(4),\text{gen}} \otimes F_{\text{LM}}^{\mathcal{B}_0} \rangle + \langle F_{\text{LM}}^{\mathcal{B}_0} \otimes \mathcal{P}_{qq}^{(4),\text{gen}} \rangle \right]. \end{aligned} \quad (5.56)$$

Before concluding this section, we note that the IR structures are already beginning to emerge. Indeed, by examining the first term in each of Eqs (5.7), (5.20), (5.31), (5.44), and (5.56), it becomes evident that we are approaching the expression for the operator $\frac{1}{2}(I_T^{\mathcal{B}_0})^2$. The only remaining component is $\frac{1}{2}(I_C^{\mathcal{B}_0})^2$, which will be generated through the double-collinear limits that will be addressed in the next section.

5.2 Double-collinear contributions of $\mathcal{O}(\epsilon^{-2})$

As previously noted, we are conceptually just one step away from obtaining the operator $\frac{1}{2}(I_T^{\mathcal{B}_0})^2$. The missing piece is the double-collinear contribution $\frac{1}{2}(I_C^{\mathcal{B}_0})^2$. Therefore, this section is entirely dedicated to the calculation of the double-collinear contribution of $\mathcal{O}(\epsilon^{-2})$ in $\Sigma_{\text{DU}}|_{\mathcal{B}_0}$ from Eq. (5.2). As for (4.179), we now address the fourth line on the right-hand side (the first three were already computed in the previous section). Similarly, for Eq. (4.183), we consider the second line on the right-hand side (with the first term

having been previously calculated as well). Thus, we focus on the following expression:

$$\begin{aligned} & \frac{1}{2} \sum_{i,j \in \mathcal{H}} \langle \bar{S}_n \bar{S}_m C_{jn} C_{im} \Delta^{(mn)} F_{LM}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g] \rangle \\ & + \frac{1}{2} \sum_{i,j \in \mathcal{H}} \langle \bar{S}_n (C_{jn} C_{im} + C_{im} C_{jn}) \Delta^{(mn)} F_{LM}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g] \rangle, \end{aligned} \quad (5.57)$$

which we conveniently divide into four distinct contributions as follows:²

$$\mathbb{A}_{dc} \stackrel{\text{def}}{=} \sum_{\substack{i,j \in \mathcal{H} \\ i < j}} \langle \bar{S}_n \bar{S}_m C_{jn} C_{im} \Delta^{(mn)} F_{LM}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g] \rangle, \quad (5.58)$$

$$\mathbb{A}_{tc} \stackrel{\text{def}}{=} \frac{1}{2} \sum_{i \in \mathcal{H}} \langle \bar{S}_m \bar{S}_n C_{im} C_{in} \Delta^{(mn)} F_{LM}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g] \rangle, \quad (5.59)$$

$$\mathbb{B}_{dc} \stackrel{\text{def}}{=} \sum_{(ij)} \mathcal{P}_{\mathcal{B}_0} \langle \bar{S}_n C_{jn} C_{im} \Delta^{(mn)} F_{LM}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g] \rangle, \quad (5.60)$$

$$\mathbb{B}_{tc} \stackrel{\text{def}}{=} \frac{1}{2} \sum_{i \in \mathcal{H}} \mathcal{P}_{\mathcal{B}_0} \langle \bar{S}_n (C_{in} C_{im} + C_{im} C_{in}) \Delta^{(mn)} F_{LM}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g] \rangle, \quad (5.61)$$

where the wordings dc and tc refer to *double-collinear* and *triple-collinear* sectors, respectively. In the following pages, we will calculate each of these contributions step by step. We anticipate that the final result is presented in Eq. (5.85).

5.2.1 Double-collinear sector

We begin with $\mathbb{A}_{dc}$ as defined in Eq. (5.58). This contribution is straightforward to compute, as the operators $\bar{S}_m C_{im}$ and $\bar{S}_n C_{jn}$ acting on $\Delta^{(mn)} F_{LM}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g]$ correspond to two fully decoupled NLO hard-collinear limits, where the diagonal components $i = j$ are omitted. Thus, we obtain

$$\begin{aligned} \mathbb{A}_{dc} = & [\alpha_s]^2 \left\langle \left[\frac{\Gamma_{a,g} \Gamma_{b,q}}{\epsilon^2} + \sum_{i \in \mathcal{H}_f} \frac{(\Gamma_{a,g} + \Gamma_{b,q}) \Gamma_{i,f_i \rightarrow f_i g}}{\epsilon^2} + \sum_{\substack{i,j \in \mathcal{H}_f \\ i < j}} \frac{\Gamma_{i,f_i \rightarrow f_i g} \Gamma_{j,f_j \rightarrow f_j g}}{\epsilon^2} \right] F_{LM}^{\mathcal{B}_0} \right\rangle \\ & + \frac{[\alpha_s]^2}{\epsilon} \left\langle \mathcal{P}_{gg}^{\text{gen}} \otimes \left[\left(\frac{\Gamma_{b,q}}{\epsilon} + \sum_{i \in \mathcal{H}_f} \frac{\Gamma_{i,f_i \rightarrow f_i g}}{\epsilon} \right) \cdot F_{LM}^{\mathcal{B}_0} \right] \right\rangle \\ & + \frac{[\alpha_s]^2}{\epsilon} \left\langle \left[\left(\frac{\Gamma_{a,g}}{\epsilon} + \sum_{i \in \mathcal{H}_f} \frac{\Gamma_{i,f_i \rightarrow f_i g}}{\epsilon} \right) \cdot F_{LM}^{\mathcal{B}_0} \right] \otimes \mathcal{P}_{qq}^{\text{gen}} \right\rangle + \frac{[\alpha_s]^2}{\epsilon^2} \langle \mathcal{P}_{gg}^{\text{gen}} \otimes F_{LM}^{\mathcal{B}_0} \otimes \mathcal{P}_{qq}^{\text{gen}} \rangle. \end{aligned} \quad (5.62)$$

If we now consider $\mathbb{B}_{dc}$ as defined in Eq. (5.60), it produces precisely those terms required to complete all the anomalous dimensions $\Gamma_{q \rightarrow qg}$ present in the expression (5.62),

²We recall that the collinear limits C_{im} and C_{in} commute when applied to $F_{LM}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g]$, but do not commute when applied to $F_{LM}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g]$.

thereby yielding Γ_q for each of them. Therefore, the sum of $\mathbb{A}_{\text{dc}}$ and $\mathbb{B}_{\text{dc}}$ gives

$$\begin{aligned} \mathbb{A}_{\text{dc}} + \mathbb{B}_{\text{dc}} = & [\alpha_s]^2 \left\langle \left[\frac{1}{2} I_C^2 - \frac{\Gamma_{a,g}^2}{2\epsilon^2} - \frac{\Gamma_{b,q}^2}{2\epsilon^2} - \sum_{i \in \mathcal{H}_{f,g}} \frac{\Gamma_{i,g \rightarrow gg}^2}{2\epsilon^2} - \frac{\Gamma_q^2}{2\epsilon^2} \right] F_{\text{LM}}^{\mathcal{B}_0} \right\rangle \\ & + \frac{[\alpha_s]^2}{\epsilon} \left\langle \mathcal{P}_{gg}^{\text{gen}} \otimes \left[\left(I_C - \frac{\Gamma_{a,g}}{\epsilon} \right) \cdot F_{\text{LM}}^{\mathcal{B}_0} \right] \right\rangle \\ & + \frac{[\alpha_s]^2}{\epsilon} \left\langle \left[\left(I_C - \frac{\Gamma_{b,q}}{\epsilon} \right) \cdot F_{\text{LM}}^{\mathcal{B}_0} \right] \otimes \mathcal{P}_{qq}^{\text{gen}} \right\rangle + \frac{[\alpha_s]^2}{\epsilon^2} \langle \mathcal{P}_{gg}^{\text{gen}} \otimes F_{\text{LM}}^{\mathcal{B}_0} \otimes \mathcal{P}_{qq}^{\text{gen}} \rangle. \end{aligned} \quad (5.63)$$

As we will soon see, the terms subtracted from the collinear operator in Eq. (5.63) will be perfectly canceled by the contributions in the triple-collinear sector.

5.2.2 Triple-collinear sector

We now turn to the triple-collinear sector. These contributions are less straightforward to compute, as the action of two distinct collinear limits on the same leg leads to the emergence of two different momentum fractions, z and $\bar{z}$.

We begin with $\mathbb{A}_{\text{tc}}$ as defined in (5.59), specifically with the leg $i = a$. The soft-regulated double-collinear limit yields

$$\begin{aligned} \bar{S}_{\text{m}} \bar{S}_{\text{n}} C_{am} C_{an} \Delta^{(\text{mn})} F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g] = & \frac{[g_{s,b}^2 \mu_0^{2\epsilon}]^2}{E_{\text{m}}^2 E_{\text{n}}^2 \rho_{\text{im}} \rho_{\text{in}}} \\ \times & \left[\frac{P_{gg,i}(\bar{z})}{(1-\bar{z})^{-1}} \frac{P_{gg,i}(z)}{(1-z)^{-1}} \frac{\mathcal{N} F_{\text{LM}}^{(z\bar{z},g)q}[\mathcal{H}_f]}{z\bar{z}} - 2C_A \frac{P_{gg,i}(z)}{(1-z)^{-1}} \frac{\mathcal{N} F_{\text{LM}}^{(z,g)q}[\mathcal{H}_f]}{z} \right. \\ & \left. - 2C_A \frac{P_{gg,i}(w)}{(1-w)^{-1}} \frac{\mathcal{N} F_{\text{LM}}^{(w,g)q}[\mathcal{H}_f]}{w} + 4C_A^2 \mathcal{N} F_{\text{LM}}^{gg}[\mathcal{H}_f] \right], \end{aligned} \quad (5.64)$$

where

$$z = 1 - \frac{E_{\text{n}}}{E_a}, \quad \bar{z} = 1 - \frac{E_{\text{m}}}{z E_a}, \quad w = 1 - \frac{E_{\text{m}}}{E_a}. \quad (5.65)$$

Additionally, $\mathcal{N} = 1/(N-1)!$ is the symmetry factor. Then, Eq. (5.64) must be integrated over the phase space $[dp_{\text{m}}][dp_{\text{n}}]$. The integration over the angular phase space is straightforward. As for the integration over the energies, the second, third and fourth terms on the right-hand side of Eq. (5.64) present no further complications, and correspond to

$$\begin{aligned} & \left\langle 4C_A \int_0^{E_{\text{max}}} \frac{dE_{\text{m}}}{E_{\text{m}}^{1+2\epsilon}} \frac{dE_{\text{n}}}{E_{\text{n}}^{1+2\epsilon}} \left[-\frac{P_{gg,i}(z)}{(1-z)^{-1}} + \delta(1-z) \right] \frac{\mathcal{N} F_{\text{LM}}^{(z,g)q}[\mathcal{H}_f]}{z} \right\rangle \\ = & \left\langle E_a^{-4\epsilon} \left[\frac{C_A}{\epsilon} e^{-2\epsilon L_a} \mathcal{P}_{gg}^{(2)}(z) - \frac{C_A^2}{\epsilon^2} e^{-4\epsilon L_a} \delta(1-z) \right] \otimes F_{\text{LM}}^{\mathcal{B}_0} \right\rangle, \end{aligned} \quad (5.66)$$

where we recall that $L_a = \log(E_{\text{max}}/E_a)$, and the splitting function $\mathcal{P}_{gg}^{(2)}(z)$ is defined in Eq. (A.54). We emphasize that, in order to derive Eq. (5.66), we utilized the fact that the integrals over the second and third terms on the right-hand side of Eq. (5.64) are identical. Additionally, we applied the following reformulation of the definition of $\mathcal{P}_{gg}^{(2)}(z)$ in

Eq. (A.54):

$$\int_0^1 dz \frac{P_{\alpha\beta,i}(z)}{(1-z)^{2\epsilon}} g(z) = \int_0^1 dz \left[\mathcal{P}_{\alpha\beta}^{(2)}(z, E_a) - \delta_{\alpha\beta} \frac{T_\alpha^2}{\epsilon} e^{-2\epsilon L_a} \delta(1-z) \right] g(z). \quad (5.67)$$

Concerning the integral of the first term on the right-hand side of Eq. (5.64), the computation is somewhat more intricate and necessitates a few considerations, which we present in the following *proof*. Its final expression is given by:

$$\left\langle E_a^{-4\epsilon} \left[[\mathcal{P}_{gg}^{(2)} \bar{\otimes} \mathcal{P}_{gg}^{(2)}](z) - \frac{C_A}{\epsilon} (1+z^{-2\epsilon}) e^{-2\epsilon L_a} \mathcal{P}_{gg}^{(2)}(z) + \frac{C_A^2}{\epsilon^2} e^{-4\epsilon L_a} \delta(1-z) \right] \otimes F_{LM}^{B_0} \right\rangle, \quad (5.68)$$

where the $\bar{\otimes}$ -convolution is defined as

$$[f \bar{\otimes} g](z) \stackrel{\text{def}}{=} \int_z^1 \frac{dt}{t} t^{-2\epsilon} f(t) g(z/t). \quad (5.69)$$

Proof:

To show that the integral of the first term on the right-hand side of (5.64) yields Eq. (5.68), first we change the variables of integration from (E_m, E_n) in Eq. (5.65) to $(z, \bar{z})$, as follows:

$$\frac{dE_m}{(1-\bar{z})^{-1} E_m^{1+2\epsilon}} \frac{dE_n}{(1-z)^{-1} E_n^{1+2\epsilon}} = E_a^{-4\epsilon} \frac{d\bar{z}}{(1-\bar{z})^{2\epsilon}} \frac{dz z^{-2\epsilon}}{(1-z)^{2\epsilon}}, \quad \begin{cases} 0 \leq z \leq 1 \\ \bar{z}_{\min} \leq \bar{z} \leq 1, \end{cases} \quad (5.70)$$

with $\bar{z}_{\min} = 1 - E_{\max}/(zE_a)$. Next, we change the variable $\bar{z}$ to $x = z\bar{z}$, where $x \in [x_{\min}, z]$, with $x_{\min} = z\bar{z}_{\min}$. We obtain

$$\begin{aligned} & \int_0^{E_{\max}} \frac{dE_m}{E_m^{1+2\epsilon}} \frac{dE_n}{E_n^{1+2\epsilon}} \frac{P_{gg,i}(\bar{z})}{(1-\bar{z})^{-1}} \frac{P_{gg,i}(z)}{(1-z)^{-1}} \frac{\mathcal{N} F_{LM}^{(z\bar{z}\cdot g)^q}[\mathcal{H}_f]}{z\bar{z}} \\ &= \int_0^1 \frac{dz}{z} z^{-2\epsilon} \frac{P_{gg,i}(z)}{(1-z)^{2\epsilon}} \int_{x_{\min}}^1 dx \frac{P_{gg,i}(x/z)}{(1-x/z)^{2\epsilon}} \frac{\mathcal{N} F_{LM}^{(x\cdot g)^q}[\mathcal{H}_f]}{x}. \end{aligned} \quad (5.71)$$

We note that

$$x_{\min} = z - \frac{E_{\max}}{E_a} = \underbrace{\left(1 - \frac{E_{\max}}{E_a}\right)}_{< 0} + \underbrace{(z-1)}_{< 0}, \quad (5.72)$$

but $F_{LM}^{(x\cdot g)^q}[\mathcal{H}_f]$ has no support for $x < 0$. Thus, the extremes of integration of the integral over x can be set to $x \in [0, 1]$, i.e.,

$$\int_0^1 \frac{dz}{z} z^{-2\epsilon} \frac{P_{gg,i}(z)}{(1-z)^{2\epsilon}} \int_0^1 dx \frac{P_{gg,i}(x/z)}{(1-x/z)^{2\epsilon}} \frac{\mathcal{N} F_{LM}^{(x\cdot g)^q}[\mathcal{H}_f]}{x}. \quad (5.73)$$

At this point, we swap the order of integration, such that $x \in [0, 1]$ and $z \in [x, 1]$, and then, for our convenience, we relabel the integration variables z as t , and x as z . We obtain

$$\int_0^1 dz \left[\int_z^1 \frac{dt}{t} t^{-2\epsilon} \frac{P_{gg,i}(t)}{(1-t)^{2\epsilon}} \frac{P_{gg,i}(z/t)}{(1-z/t)^{2\epsilon}} \right] \frac{\mathcal{N} F_{LM}^{(z\cdot g)^q}[\mathcal{H}_f]}{z}. \quad (5.74)$$

Finally, we use the identity (5.67) to obtain Eq. (5.68).

Therefore, we combine Eqs (5.66) and (5.68) (with also their angular contribution), and find

$$\begin{aligned} \frac{1}{2} \langle \bar{S}_m \bar{S}_n C_{am} C_{an} \Delta^{(mn)} F_{LM}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g] &= \frac{[\alpha_s]^2}{2\epsilon^2} \left\langle \left[\left(\frac{2E_a}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \right]^2 \right. \\ &\times \left. \left[\mathcal{P}_{gg}^{(2)} \bar{\otimes} \mathcal{P}_{gg}^{(2)} \right](z) + \frac{C_A}{\epsilon} (1 - z^{-2\epsilon}) e^{-2\epsilon L_a} \mathcal{P}_{gg}^{(2)}(z) \right] \otimes F_{LM}^{\mathcal{B}_0} \right\rangle. \end{aligned} \quad (5.75)$$

Finally, we exploit the relation (3.67) between $\mathcal{P}_{gg}^{(2)}$, $\Gamma_{a,g}$ and $\mathcal{P}_{gg}^{\text{gen}}$, to obtain the following final result for leg a :

$$\begin{aligned} \frac{1}{2} \langle \bar{S}_m \bar{S}_n C_{am} C_{an} \Delta^{(mn)} F_{LM}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g] &= [\alpha_s]^2 \left\langle \frac{\Gamma_{a,g}^2}{2\epsilon^2} F_{LM}^{\mathcal{B}_0} \right\rangle \\ &+ \frac{[\alpha_s]^2}{2\epsilon^2} \left\langle \left([\mathcal{P}_{gg}^{\text{gen}} \bar{\otimes} \mathcal{P}_{gg}^{\text{gen}}] + G_{gg} \right) \otimes F_{LM}^{\mathcal{B}_0} \right\rangle + \frac{[\alpha_s]^2}{\epsilon} \left\langle \mathcal{P}_{gg}^{\text{gen}} \otimes \left[\frac{\Gamma_{a,g}}{\epsilon} F_{LM}^{\mathcal{B}_0} \right] \right\rangle, \end{aligned} \quad (5.76)$$

where we introduced the splitting function $G_{a,\alpha\beta}$ defined as

$$G_{\alpha\beta}(z, E_a) \stackrel{\text{def}}{=} [\Gamma_{a,\beta} - \Gamma_{z \cdot a, \alpha}] \mathcal{P}_{\alpha\beta}^{\text{gen}}(z, E_a), \quad (5.77)$$

where

$$\Gamma_{z \cdot a, \alpha} \stackrel{\text{def}}{=} \Gamma_{a, \alpha} \big|_{E_a \mapsto z E_a}. \quad (5.78)$$

The result in (5.76) holds identically for leg b , provided that all the gg splittings are replaced with qq splittings.

We turn to the final-state contribution in $\mathbb{A}_{\text{tc}, i}$ i.e., $i \in \mathcal{H}_f$. We note that applying both limits $C_{im} C_{in}$ to F_{LM} and then integrating over $[dp_m][dp_n]$, or applying only C_{in} and integrating over $[dp_n]$, followed by applying C_{im} to this result and integrating over $[dp_m]$, yields the same final expression. We follow the second strategy, and obtain

$$\begin{aligned} &\frac{1}{2} \sum_{i \in \mathcal{H}_f} \langle \bar{S}_m \bar{S}_n C_{im} C_{in} \Delta^{(mn)} F_{LM}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g] \rangle \\ &= \frac{[\alpha_s]}{2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \sum_{i \in \mathcal{H}_f} \left\langle \bar{S}_m C_{im} \Delta^{(m)} (2E_i/\mu)^{-2\epsilon} \gamma_{z, f_i \rightarrow f_i g}^{22}(L_i) F_{LM}^{\mathcal{B}_1}[\mathbf{m}_g] \right\rangle \\ &= [\alpha_s]^2 \sum_{i \in \mathcal{H}_f} \left\langle \left[\frac{\Gamma_{i, f_i \rightarrow f_i g}^2}{2\epsilon^2} + \frac{G_i|_{z, f_i \rightarrow f_i g}}{2\epsilon^2} \right] F_{LM}^{\mathcal{B}_0} \right\rangle, \end{aligned} \quad (5.79)$$

where we introduced the general object

$$\begin{aligned} G_i|_{\tilde{f}(z), \alpha \rightarrow \beta \xi}^{f(z), \alpha \rightarrow \beta \xi} &\stackrel{\text{def}}{=} \left[\left(\frac{2E_i}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \right]^2 \left[\gamma_{f(z), \alpha \rightarrow \beta \xi}^{22}(L_i) + \delta_{\alpha\beta} \frac{T_\alpha^2}{\epsilon} e^{-2\epsilon L_i} \right] \\ &\times \left[\gamma_{\tilde{f}(z), \tilde{\alpha} \rightarrow \tilde{\beta} \tilde{\xi}}^{42}(L_i) - \gamma_{\tilde{f}(z), \tilde{\alpha} \rightarrow \tilde{\beta} \tilde{\xi}}^{22}(L_i) \right]. \end{aligned} \quad (5.80)$$

Note that the definitions (5.77) and (5.80) are a generalization of the initial- and final-state functions G_i introduced in Ref. [81]. At this point, we can combine Eqs (5.76), for both legs a and b , and (5.79), and obtain a final expression for $\mathbb{A}_{tc}$ in Eq. (5.59):

$$\begin{aligned} \mathbb{A}_{tc} = & \frac{[\alpha_s]^2}{2\epsilon^2} \left\langle \left([\mathcal{P}_{gg}^{\text{gen}} \bar{\otimes} \mathcal{P}_{gg}^{\text{gen}}] + G_{gg} \right) \otimes F_{LM}^{\mathcal{B}_0} \right\rangle + \frac{[\alpha_s]^2}{\epsilon} \left\langle \mathcal{P}_{gg}^{\text{gen}} \otimes \left[\frac{\Gamma_{a,g}}{\epsilon} F_{LM}^{\mathcal{B}_0} \right] \right\rangle \\ & + \frac{[\alpha_s]^2}{2\epsilon^2} \left\langle F_{LM}^{\mathcal{B}_0} \otimes \left([\mathcal{P}_{qq}^{\text{gen}} \bar{\otimes} \mathcal{P}_{qq}^{\text{gen}}] + G_{qq} \right) \right\rangle + \frac{[\alpha_s]^2}{\epsilon} \left\langle \left[\frac{\Gamma_{b,q}}{\epsilon} F_{LM}^{\mathcal{B}_0} \right] \otimes \mathcal{P}_{qq}^{\text{gen}} \right\rangle \quad (5.81) \\ & + [\alpha_s]^2 \left\langle \left[\frac{\Gamma_{a,g}^2 + \Gamma_{b,q}^2}{2\epsilon^2} + \sum_{i \in \mathcal{H}_f} \left(\frac{\Gamma_{i,f_i \rightarrow f_i g}^2}{2\epsilon^2} + \frac{G_i|_{z,f_i \rightarrow f_i g}}{2\epsilon^2} \right) \right] F_{LM}^{\mathcal{B}_0} \right\rangle. \end{aligned}$$

Consistent with our notation, no index i appears in the various anomalous dimensions of the above formula, since $\mathcal{B}_0$ contains only one final-state quark.

The next-to-final step is to calculate $\mathbb{B}_{tc}$ in Eq. (5.61). We focus exclusively on the final state, as the initial state is nullified by the projector $\mathcal{P}_{\mathcal{B}_0}$. We recall that, in this case, the collinear operators do not commute. Thus, two separate calculations must be performed: one with $C_{in}C_{im}$ and the other with $C_{im}C_{in}$. There is no conceptual difficulty in this, as the procedure follows the same steps used to obtain Eq. (5.79). After working through the algebraic details, we obtain

$$\begin{aligned} \mathbb{B}_{tc} = & [\alpha_s]^2 \left\langle \frac{\Gamma_{q \rightarrow qq}^2 + 2\Gamma_{q \rightarrow qq} \Gamma_{q \rightarrow gq}}{2\epsilon^2} F_{LM}^{\mathcal{B}_0} \right\rangle \\ & + \frac{[\alpha_s]^2}{2\epsilon^2} \left\langle \left[G|_{z,q \rightarrow qq}^{z,q \rightarrow gq} + G|_{z,q \rightarrow gq}^{z,g \rightarrow gq} + \Gamma_{q \rightarrow gq} (\Gamma_{g \rightarrow gg} - \Gamma_q) \right] F_{LM}^{\mathcal{B}_0} \right\rangle. \quad (5.82) \end{aligned}$$

Finally, we sum $\mathbb{A}_{tc}$ and $\mathbb{B}_{tc}$, and we find the final expression for the triple-collinear sector:

$$\begin{aligned} \mathbb{A}_{tc} + \mathbb{B}_{tc} = & \frac{[\alpha_s]^2}{2\epsilon^2} \left\langle \left([\mathcal{P}_{gg}^{\text{gen}} \bar{\otimes} \mathcal{P}_{gg}^{\text{gen}}] + G_{gg} \right) \otimes F_{LM}^{\mathcal{B}_0} \right\rangle + \frac{[\alpha_s]^2}{\epsilon} \left\langle \mathcal{P}_{gg}^{\text{gen}} \otimes \left[\frac{\Gamma_{a,g}}{\epsilon} F_{LM}^{\mathcal{B}_0} \right] \right\rangle \\ & + \frac{[\alpha_s]^2}{2\epsilon^2} \left\langle F_{LM}^{\mathcal{B}_0} \otimes \left([\mathcal{P}_{qq}^{\text{gen}} \bar{\otimes} \mathcal{P}_{qq}^{\text{gen}}] + G_{qq} \right) \right\rangle + \frac{[\alpha_s]^2}{\epsilon} \left\langle \left[\frac{\Gamma_{b,q}}{\epsilon} F_{LM}^{\mathcal{B}_0} \right] \otimes \mathcal{P}_{qq}^{\text{gen}} \right\rangle \quad (5.83) \\ & + [\alpha_s]^2 \left\langle \left[\frac{\Gamma_{a,g}^2 + \Gamma_{b,q}^2}{2\epsilon^2} + \sum_{i \in \mathcal{H}_{f,g}} \frac{\Gamma_{i,g \rightarrow gg}^2}{2\epsilon^2} + \frac{\Gamma_q^2}{2\epsilon^2} \right] F_{LM}^{\mathcal{B}_0} \right\rangle \\ & + \frac{[\alpha_s]^2}{2\epsilon^2} \left\langle \left[\sum_{i \in \mathcal{H}_{f,g}} G_i|_{z,g \rightarrow gg}^{z,g \rightarrow gq} + \left(G|_{z,q \rightarrow qq}^{1,q \rightarrow gq} + G|_{z,q \rightarrow gq}^{z,g \rightarrow gq} + \Gamma_{q \rightarrow gq} (\Gamma_{g \rightarrow gg} - \Gamma_q) \right) \right] F_{LM}^{\mathcal{B}_0} \right\rangle. \end{aligned}$$

We note that, to get the above formula, we used the definition in Eq. (5.80) to obtain the

function $G|_{z,q \rightarrow qq}^{1,q \rightarrow qq}$, as follows:

$$\begin{aligned}
 G|_{z,q \rightarrow qq}^{z,q \rightarrow qq} + G|_{z,q \rightarrow qq}^{z,q \rightarrow gq} &= \left[\left(\frac{2E}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \right]^2 \overbrace{\left[\gamma_{z,q \rightarrow qq}^{22}(L) + \gamma_{z,q \rightarrow gq}^{22} \right]}^{= \gamma_{1,q \rightarrow qq}^{22}(L)} \\
 &\quad + \frac{\mathbf{T}_q^2}{\epsilon} e^{-2\epsilon L} \left[\gamma_{z,q \rightarrow qq}^{42}(L) - \gamma_{z,q \rightarrow qq}^{22}(L) \right] \\
 &\equiv G|_{z,q \rightarrow qq}^{1,q \rightarrow qq}.
 \end{aligned} \tag{5.84}$$

At this point, we have derived the analytical expression for the double-collinear sector, given in Eq. (5.63), as well as for the triple-collinear sector, which we have just computed in Eq. (5.83). As mentioned at the end of the previous subsection, the sum of these two contributions leads to *clean* collinear operators $I_C^{\mathcal{B}_0}$. In other words, the terms subtracted from the collinear operator in Eq. (5.63) are perfectly canceled when we add the result of Eq. (5.83). Here is the final expression for Eq. (5.57):

$$\begin{aligned}
 \text{Eq. (5.57)} &= [\alpha_s]^2 \left\langle \frac{1}{2} I_C^2 \cdot F_{\text{LM}}^{\mathcal{B}_0} \right\rangle + \frac{[\alpha_s]^2}{\epsilon^2} \langle \mathcal{P}_{gg}^{\text{gen}} \otimes F_{\text{LM}}^{\mathcal{B}_0} \otimes \mathcal{P}_{qq}^{\text{gen}} \rangle \\
 &+ \frac{[\alpha_s]^2}{\epsilon} \left[\langle \mathcal{P}_{gg}^{\text{gen}} \otimes [I_C \cdot F_{\text{LM}}^{\mathcal{B}_0}] \rangle + \langle [I_C \cdot F_{\text{LM}}^{\mathcal{B}_0}] \otimes \mathcal{P}_{qq}^{\text{gen}} \rangle \right] \\
 &+ \frac{[\alpha_s]^2}{2\epsilon^2} \left[\left\langle \left([\mathcal{P}_{gg}^{\text{gen}} \bar{\otimes} \mathcal{P}_{gg}^{\text{gen}}] + G_{gg} \right) \otimes F_{\text{LM}}^{\mathcal{B}_0} \right\rangle + \left\langle F_{\text{LM}}^{\mathcal{B}_0} \otimes \left([\mathcal{P}_{qq}^{\text{gen}} \bar{\otimes} \mathcal{P}_{qq}^{\text{gen}}] + G_{qq} \right) \right\rangle \right] \\
 &+ \frac{[\alpha_s]^2}{2\epsilon^2} \left\langle \left[\sum_{i \in \mathcal{H}_{f,g}} G_i|_{z,g \rightarrow gg}^{z,g \rightarrow gg} + \left(G|_{z,q \rightarrow qq}^{1,q \rightarrow qq} + G|_{z,q \rightarrow gq}^{z,g \rightarrow gg} + \Gamma_{q \rightarrow gq} (\Gamma_{g \rightarrow gg} - \Gamma_q) \right) \right] F_{\text{LM}}^{\mathcal{B}_0} \right\rangle.
 \end{aligned} \tag{5.85}$$

5.3 Computing all remaining double-unresolved contributions

We have reached a point in our path where many terms of $\Sigma_{\text{DU}}|_{\mathcal{B}_0}$ in (5.2) have already been calculated, and we have also started to observe how the I -operators combine with each other. In this section, the goal is to calculate, step by step, all the remaining terms in $\Sigma_{\text{DU}}|_{\mathcal{B}_0}$, plus the PDF renormalization term $d\hat{\sigma}_{\mathcal{B}_2}^{\text{pdf}}$, and the interference piece $d\hat{\sigma}_{\mathcal{B}_2}^{\text{int}}$ introduced in Eq. (5.1). We will structure this section as follows:

- i) in Subsec. 5.3.1, we discuss terms of $\mathcal{O}(\epsilon^{-2})$;
- ii) in Subsec. 5.3.2, we discuss terms of $\mathcal{O}(\epsilon^{-1})$;
- iii) in Subsec. 5.3.3, we discuss terms of $\mathcal{O}(\epsilon^0)$.

5.3.1 Terms of $\mathcal{O}(\epsilon^{-2})$

Considering $\Sigma_{\text{DU}}|_{\mathcal{B}_0}$ in (5.2), there are only two remaining contributions of $\mathcal{O}(\epsilon^{-2})$, namely

$$\sum_{i \in \mathcal{H}} \frac{N_{\mathbf{m}||\mathbf{n}}(\epsilon)}{2} \left\langle \bar{S}_{\mathbf{m}} C_{i\mathbf{m}} \Delta^{(\mathbf{m})} \sigma_{i\mathbf{m}}^{-\epsilon} \langle (\mathbb{1} - 2\Theta_{\mathbf{m}\mathbf{n}} S_{\mathbf{n}}) C_{\mathbf{m}\mathbf{n}} F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g] \rangle \right\rangle, \quad (5.86)$$

$$\sum_{i \in \mathcal{H}} \frac{[\alpha_s]}{\epsilon} N_{\mathbf{m}||\mathbf{n}}(\epsilon) \mathcal{P}_{\mathcal{B}_0} \langle C_{i\mathbf{m}} \sigma_{i\mathbf{m}}^{-\epsilon} \Delta^{(\mathbf{m})} \Gamma_{\mathbf{m},q} F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_q] \rangle, \quad (5.87)$$

where the first arises from Eq. (4.179) and the second from Eq. (4.183).

We begin with Eq. (5.86). First, we compute the inner integral

$$\langle (\mathbb{1} - 2\Theta_{\mathbf{m}\mathbf{n}} S_{\mathbf{n}}) C_{\mathbf{m}\mathbf{n}} F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g] \rangle. \quad (5.88)$$

Note that this formula is similar to an expression at NLO, but it is somewhat unusual: the gluon $\mathbf{m}_g$ is not divergent; rather, it is treated as if it were a hard particle. Additionally, the gluon $\mathbf{n}_g$ features two distinct energy limits between the purely collinear term and the soft-collinear term, that is, $E_{\mathbf{n}} \in [0, E_{\text{max}}]$ in the term that contains $\mathbb{1}$, and $E_{\mathbf{n}} \in [0, E_{\mathbf{m}}]$ in the term that contains $-2\Theta_{\mathbf{m}\mathbf{n}} S_{\mathbf{n}}$. However, despite these differences, Eq. (5.88) can be integrated without issues, and yields

$$\begin{aligned} & \langle (\mathbb{1} - 2\Theta_{\mathbf{m}\mathbf{n}} S_{\mathbf{n}}) C_{\mathbf{m}\mathbf{n}} F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g] \rangle \\ &= \frac{[\alpha_s]^2}{\epsilon} \left\langle \left[\left(\frac{2E_{\mathbf{m}}}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \right] \gamma_{1,g \rightarrow gg}^{22}(0) F_{\text{LM}}^{\mathcal{B}_1}[\mathbf{m}_g] \right\rangle. \end{aligned} \quad (5.89)$$

Notice that (5.89) contains the anomalous dimension $\gamma_{1,g \rightarrow gg}^{22}(0)$ (with the 0 indicating $L = 0$) rather than $\gamma_{z,g \rightarrow gg}^{22}(0)$, because in (5.88) the damping factor is absent, which would have introduced the additional factor of z that is present in $\gamma_{z,g \rightarrow gg}^{22}(0)$. Nevertheless, the following relation holds:

$$\gamma_{1,g \rightarrow gg}^{22}(0) = 2 \gamma_{z,g \rightarrow gg}^{22}(0), \quad (5.90)$$

thus, we can replace this expression into Eq. (5.89). Next, we consider the action of $\bar{S}_{\mathbf{m}} C_{i\mathbf{m}} \Delta^{(\mathbf{m})} \sigma_{i\mathbf{m}}^{-\epsilon}$ on the partial result (5.89). Once again, this is a NLO-like calculation, where the presence of the factor $(2E_{\mathbf{m}}/\mu)^{-2\epsilon}$ implies that we are dealing with the operators $\Gamma^{(4)}$, $\mathcal{P}_{gg}^{(4),\text{gen}}$, etc., instead of the conventional NLO operators. By integrating over the phase space $[dp_{\mathbf{m}}]$, we obtain

$$\begin{aligned} & \sum_{i \in \mathcal{H}} \frac{N_{\mathbf{m}||\mathbf{n}}(\epsilon)}{2} \left\langle \bar{S}_{\mathbf{m}} C_{i\mathbf{m}} \Delta^{(\mathbf{m})} \sigma_{i\mathbf{m}}^{-\epsilon} \langle (\mathbb{1} - 2\Theta_{\mathbf{m}\mathbf{n}} S_{\mathbf{n}}) C_{\mathbf{m}\mathbf{n}} F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g] \rangle \right\rangle \\ &= \frac{[\alpha_s]^2}{\epsilon} N_{\text{sc}}^{(b,d)} \left\langle \gamma_{z,g \rightarrow gg}^{22}(0) \left(I_C^{(4)}(\epsilon) - \frac{\Gamma_{q \rightarrow gq}^{(4)}}{2\epsilon} \right) \cdot F_{\text{LM}}^{\mathcal{B}_0} \right\rangle \\ &+ \frac{[\alpha_s]^2}{2\epsilon^2} N_{\text{sc}}^{(b,d)} \left\langle \gamma_{z,g \rightarrow gg}^{22}(0) \left[\mathcal{P}_{gg}^{(4),\text{gen}} \otimes F_{\text{LM}}^{\mathcal{B}_0} + F_{\text{LM}}^{\mathcal{B}_0} \otimes \mathcal{P}_{qq}^{(4),\text{gen}} \right] \right\rangle, \end{aligned} \quad (5.91)$$

where

$$N_{\text{sc}}^{(b,d)} \stackrel{\text{def}}{=} 2^{2\epsilon} \frac{\Gamma(1+2\epsilon)\Gamma^3(1-2\epsilon)}{\Gamma(1-3\epsilon)\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}. \quad (5.92)$$

Next, we consider the second term in Eq. (5.87). Due to the projector operator $\mathcal{P}_{\mathcal{B}_0}$, only the case $i \in \mathcal{H}_f$ must be calculated. Following the same steps as above, we find

$$\begin{aligned} & \sum_{i \in \mathcal{H}} \frac{[\alpha_s]}{\epsilon} N_{\text{sc}}^{(b,d)}(\epsilon) \mathcal{P}_{\mathcal{B}_0} \langle C_{im} \sigma_{im}^{-\epsilon} \Delta^{(\text{m})} \Gamma_{\text{m},q} F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_q] \rangle \\ &= \frac{[\alpha_s]^2}{\epsilon} N_{\text{sc}}^{(b,d)} \left\langle \left[\gamma_{1,q \rightarrow qq}^{22}(L) \frac{\Gamma_{q \rightarrow qq}^{(4)}}{2\epsilon} + \frac{1}{2\epsilon} H_{q \rightarrow qq} \right] \cdot F_{\text{LM}}^{\mathcal{B}_0} \right\rangle, \end{aligned} \quad (5.93)$$

where we have introduced the final-state object

$$H_{i,q \rightarrow qq} \stackrel{\text{def}}{=} \frac{\mathbf{T}_q^2}{\epsilon} \left[e^{-2\epsilon L_i} \Gamma_{i,q \rightarrow qq}^{(4)} - e^{-2\epsilon L_{\text{max}}} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \Gamma_{i,q \rightarrow qq}^{(2)} \right], \quad (5.94)$$

with $L_{\text{max}} = \log(2E_{\text{max}}/\mu)$. Details on $\gamma_{1,q \rightarrow qq}^{22}(L_i)$ can be found on page 95. These results close the computation of the double-unresolved contributions in $\Sigma_{\text{DU}}|_{\mathcal{B}_0}$ that are potentially of $\mathcal{O}(\epsilon^{-2})$ or more.

Another contribution that starts at $\mathcal{O}(\epsilon^{-2})$ arises from the renormalization of the PDFs. This is a standard result in the literature, and we have outlined the relevant formulas in Appendix A.3.2. At NNLO, this contribution can be derived using Eq. (A.39). By applying this expression, we obtain³

$$\begin{aligned} & \mathcal{P}_{\mathcal{B}_0} [2s_{ab} d\hat{\sigma}_{\mathcal{B}_2}^{\text{pdf}}] = \\ &= \left[\frac{\alpha_s(\mu)}{2\pi} \right] \frac{1}{\epsilon} \left[\langle \mathcal{O}_{\text{NLO}}^{(\text{m})} \Delta^{(\text{m})} [\hat{P}_{gg}^{(0)} \otimes F_{\text{R}}^{\mathcal{B}_1}] \rangle + \langle \hat{P}_{gg}^{(0)} \otimes F_{\text{LV,fin}}^{\mathcal{B}_0} \rangle \right] \\ &+ \left[\frac{\alpha_s(\mu)}{2\pi} \right] \frac{1}{\epsilon} \left[\langle \mathcal{O}_{\text{NLO}}^{(\text{m})} \Delta^{(\text{m})} [F_{\text{R}}^{\mathcal{B}_1} \otimes \hat{P}_{qq}^{(0)}] \rangle + \langle F_{\text{LV,fin}}^{\mathcal{B}_0} \otimes \hat{P}_{qq}^{(0)} \rangle \right] \\ &+ \frac{[\alpha_s]}{\epsilon} \left[\frac{\alpha_s(\mu)}{2\pi} \right] \left[\langle \hat{P}_{gg}^{(0)} \otimes [I_{\text{T}}(\epsilon) \cdot F_{\text{LM}}^{\mathcal{B}_0}] \rangle + \langle [I_{\text{T}}(\epsilon) \cdot F_{\text{LM}}^{\mathcal{B}_0}] \otimes \hat{P}_{qq}^{(0)} \rangle \right] \\ &+ \frac{[\alpha_s]}{\epsilon^2} \left[\frac{\alpha_s(\mu)}{2\pi} \right] \left[\langle [\hat{P}_{gg}^{(0)} \bar{\otimes} \mathcal{P}_{gg}^{\text{gen}}] \otimes F_{\text{LM}}^{\mathcal{B}_0} \rangle + \langle F_{\text{LM}}^{\mathcal{B}_0} \otimes [\hat{P}_{qq}^{(0)} \bar{\otimes} \mathcal{P}_{qq}^{\text{gen}}] \rangle \right. \\ &\quad \left. + \langle \mathcal{P}_{gg}^{\text{gen}} \otimes F_{\text{LM}}^{\mathcal{B}_0} \otimes \hat{P}_{qq}^{(0)} \rangle + \langle \hat{P}_{gg}^{(0)} \otimes F_{\text{LM}}^{\mathcal{B}_0} \otimes \mathcal{P}_{qq}^{\text{gen}} \rangle \right] \\ &+ \left[\frac{\alpha_s(\mu)}{2\pi} \right]^2 \left[\langle \mathbb{P}_{2,gg} \otimes F_{\text{LM}}^{\mathcal{B}_0} \rangle + \langle F_{\text{LM}}^{\mathcal{B}_0} \otimes \mathbb{P}_{2,qq} \rangle \right] \end{aligned} \quad (5.95)$$

with

$$\mathbb{P}_{2,gg}(z) = \frac{[\hat{P}_{gg}^{(0)} \otimes \hat{P}_{gg}^{(0)}](z) - \beta_0 \hat{P}_{gg}^{(0)}(z)}{2\epsilon^2} + \frac{\hat{P}_{gg}^{(1)}(z)}{2\epsilon},$$

³In the following expression, we can exchange the order of the Altarelli-Parisi splitting functions and the $\mathcal{O}_{\text{NLO}}^{(\text{m})}$ operator. In fact, $\hat{P}_{\alpha\beta}^{(0)}$ is independent of any energy variables, and thus can be moved “inside” the fully-resolved operator.

$$\mathbb{P}_{2,q\bar{q}}(z) = \frac{[\hat{P}_{q\bar{q}}^{(0)} \otimes \hat{P}_{q\bar{q}}^{(0)}](z) - \beta_0 \hat{P}_{q\bar{q}}^{(0)}(z)}{2\epsilon^2} + \frac{\hat{P}_{q\bar{q},\tilde{\text{ns}}}^{(1)}(z)}{2\epsilon}. \quad (5.96)$$

A few points are worth emphasizing. First, we have used the *non-singlet* NLO splitting function $\hat{P}_{q\bar{q},\tilde{\text{ns}}}^{(1)}(z)$, defined in Eq. (A.27), from which the contribution of identical quarks has been subtracted. Indeed, in leg b , we have a continuous quark line entering the hard matrix element after the emission of two gluons. Secondly, we recall that $\bar{\otimes}$ and $\otimes$ represent different convolutions, defined in Eqs (5.69) and (A.34), respectively. Finally, in Eq. (5.96), we have neglected off-diagonal convolutions, specifically $[\hat{P}_{gq}^{(0)} \otimes \hat{P}_{q\bar{q}}^{(0)}]$ in $\mathbb{P}_{2,g\bar{q}}$ and $[\hat{P}_{q\bar{q}}^{(0)} \otimes \hat{P}_{gq}^{(0)}]$ in $\mathbb{P}_{2,q\bar{q}}$. These contributions, in fact, combine with other *purely collinear initial-state contributions* arising from channels in which a quark-antiquark pair goes collinear to the same initial-state parton. As an example, a contribution of the form $[\hat{P}_{q\bar{q}}^{(0)} \otimes \hat{P}_{gq}^{(0)}]$ would combine with other purely collinear initial-state contributions, originating from the process

$$a_g + b_q \rightarrow \{g\}_{N-1} + \{q\}_2 + \bar{q} + X, \quad (5.97)$$

when both a quark q and an antiquark $\bar{q}$ go collinear to the initial-state parton b_q . We choose not to discuss these cancellations, as they are conceptually well understood: they correspond to initial-state collinear contributions that combine only with terms originating from the renormalization of the PDFs. They always behave in the same way, regardless of the process under study, and we already compute such terms in the cross sections in Eqs (5.5) and (5.6), which correspond precisely to the off-diagonal initial-state contributions of the process $\mathcal{A}_0$.

5.3.2 Terms of $\mathcal{O}(\epsilon^{-1})$

We dedicate this subsection to computing the remaining collinear contributions of $\mathcal{O}(\epsilon^{-1})$ contained in $\Sigma_{\text{DU}}|_{\mathcal{B}_0}$ in Eq. (5.2), plus the interference piece $d\hat{\sigma}_{\tilde{\mathcal{B}}_2}^{\text{int}}$ in Eq. (5.1). Essentially, we focus on five terms, that are

$$\mathbb{X} \stackrel{\text{def}}{=} \frac{1}{2} \sum_{i \in \mathcal{H}} \left\langle [2(\eta_{in}/2)^{-\epsilon} - 1] \mathcal{S}(\mathbf{m}, \mathbf{n}) C_{in} C_{im} \Delta^{(\text{mn})} F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g] \right\rangle, \quad (5.98)$$

$$\mathbb{Y} \stackrel{\text{def}}{=} \frac{1}{2} \sum_{i \in \mathcal{H}} \left\langle [2(\eta_{in}/2)^{-\epsilon} - 1] \bar{\mathcal{S}}_{\mathbf{n}} (C_{im} C_{in} + C_{in} C_{im}) \Delta^{(\text{mn})} F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_q, \mathbf{n}_g] \right\rangle, \quad (5.99)$$

from Eqs (4.179) and (4.183), respectively, the interference $d\hat{\sigma}_{\tilde{\mathcal{B}}_2}^{\text{int}}$, and $\Sigma_{\text{DU}}^{(2)}|_{\mathcal{B}_0}$ and $\Sigma_{\text{DU}}^{(8)}|_{\mathcal{B}_0}$ from Table 4.5.

We begin with $\mathbb{X}$. According to the definition of $\mathcal{S}(\mathbf{m}, \mathbf{n})$, i.e.,

$$\mathcal{S}(\mathbf{m}, \mathbf{n}) = \bar{\mathcal{S}}_{\mathbf{n}} \Theta_{\mathbf{m}\mathbf{n}} + \bar{\mathcal{S}}_{\mathbf{m}} \Theta_{\mathbf{n}\mathbf{m}}, \quad (5.100)$$

we observe that the expression for $\mathbb{X}$, except for $\mathcal{S}(\mathbf{m}, \mathbf{n})$, is invariant under the relabeling $\mathbf{m} \mapsto \mathbf{n}$ and vice versa (see the argument on page 138 for more details). Therefore, by

using the identities $\Theta_{mn} + \Theta_{nm} = 1$ and $\bar{S}_x + S_x = 1$, we can rewrite $\mathcal{S}(m, n)$ in $\mathbb{X}$ as

$$\begin{aligned}
 \mathbb{X} &= \langle \mathcal{S}(m, n) \dots \rangle = \langle [\bar{S}_n \Theta_{mn} + \bar{S}_m \Theta_{nm}] \dots \rangle = \langle [\bar{S}_n - \bar{S}_n \Theta_{nm} + \bar{S}_m \Theta_{nm}] \dots \rangle \\
 &= \langle [(\bar{S}_m + S_m) \bar{S}_n - (\bar{S}_m + S_m) \bar{S}_n \Theta_{nm} + \bar{S}_m (\bar{S}_n + S_n) \Theta_{nm}] \dots \rangle \\
 &= \langle (\bar{S}_m \bar{S}_n + \textcolor{red}{S}_m \textcolor{red}{\bar{S}}_n - \cancel{\bar{S}_m \bar{S}_n \Theta_{nm}} - \textcolor{red}{S}_m \textcolor{red}{\bar{S}}_n \Theta_{nm} + \cancel{\bar{S}_m \bar{S}_n \Theta_{nm}} + \bar{S}_m S_n \Theta_{nm}) \dots \rangle \quad (5.101) \\
 &= \langle (\bar{S}_m \bar{S}_n + \bar{S}_m S_n - \bar{S}_m S_n \Theta_{mn} + \bar{S}_m S_n \Theta_{nm}) \dots \rangle \\
 &= \langle (\bar{S}_m \bar{S}_n + 2\bar{S}_m S_n \Theta_{nm}) \dots \rangle,
 \end{aligned}$$

where in the terms highlighted in red, we have relabeled m as n and vice versa. Thus, $\mathbb{X}$ becomes

$$\mathbb{X} = \frac{1}{2} \sum_{i \in \mathcal{H}} \left\langle [2(\eta_{in}/2)^{-\epsilon} - 1] (\bar{S}_m \bar{S}_n + 2\bar{S}_m S_n \Theta_{nm}) C_{in} C_{im} \Delta^{(mn)} F_{LM}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g] \right\rangle. \quad (5.102)$$

The contribution from the part of $\mathbb{X}$ containing $\bar{S}_m \bar{S}_n$ is equivalent to the object $\mathbb{A}_{tc}$ defined in (5.59), provided we include the prefactor $\epsilon h_{\text{rdc}}(\epsilon)$, where

$$h_{\text{rdc}}(\epsilon) \stackrel{\text{def}}{=} \frac{1}{\epsilon} \left[\frac{2^\epsilon \Gamma^2(1 - 2\epsilon)}{\Gamma(1 - \epsilon) \Gamma(1 - 3\epsilon)} - 1 \right] \quad (5.103)$$

originates from the angular integral

$$\int_0^1 [d\eta_{im}] \int_0^1 [d\eta_{in}] \frac{1}{\rho_{im} \rho_{in}} [2(\eta_{in}/2)^{-\epsilon} - 1] \equiv \epsilon h_{\text{rdc}}(\epsilon) \overbrace{\int_0^1 [d\eta_{im}] \int_0^1 [d\eta_{in}] \frac{1}{\rho_{im} \rho_{in}}}^{\text{integral in Eq. (5.59)}}. \quad (5.104)$$

The contribution from the second part of $\mathbb{X}$, containing $2\bar{S}_m S_n \Theta_{nm}$, presents no further complications, and here we summarize how to compute it. Starting from $i = a$, we have

$$\bar{S}_m S_n C_{am} C_{an} \Delta^{(mn)} F_{LM}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g] = \frac{[g_{s,b}^2 \mu_0^{2\epsilon}]^2}{E_m^2 E_n^2 \rho_{im} \rho_{in}} 2T_g^2 \bar{S}_z \frac{P_{gg}(z)}{(1-z)^{-1}} \frac{\mathcal{N} F_{LM}^{(z,g)q}[\mathcal{H}_f]}{z}, \quad (5.105)$$

where $\mathcal{N} = 1/(N-1)!$ is the symmetry factor of the F_{LM} function, and $z = 1 - E_m/E_a$. We integrate over the energies, where there is an energy-ordering Θ_{nm} , as follows:

$$\begin{aligned}
 &\int_0^{E_{\max}} \frac{dE_m}{E_m^{1+2\epsilon}} \frac{dE_n}{E_n^{1+2\epsilon}} \Theta_{nm} \bar{S}_z \frac{P_{gg}(z)}{(1-z)^{-1}} \\
 &= \int_0^{E_{\max}} \frac{dE_m}{E_m^{1+2\epsilon}} \frac{E_m^{-2\epsilon} - E_{\max}^{-2\epsilon}}{2\epsilon} \bar{S}_z \frac{P_{gg}(z)}{(1-z)^{-1}} \\
 &= \frac{1}{2\epsilon} \int_{z_{\min}}^1 dz \bar{S}_z \frac{P_{gg}(z)}{(1-z)^{2\epsilon}} \left[\frac{E_a^{-4\epsilon}}{(1-z)^{2\epsilon}} - (E_a E_{\max})^{-2\epsilon} \right] \quad (5.106) \\
 &= \frac{1}{2\epsilon} \int_0^1 dz \left[E_a^{-4\epsilon} \mathcal{P}_{gg}^{(4)}(z, L_a) - (E_a E_{\max})^{-2\epsilon} \mathcal{P}_{gg}^{(2)}(z, L_a) \right],
 \end{aligned}$$

with $\mathcal{P}_{\alpha\beta}^{(k)}$ given in Eq. (A.54). Then, we use the identity (3.67) to obtain

$$\begin{aligned} & \frac{1}{2} \left\langle [2(\eta_{an}/2)^{-\epsilon} - 1] (2\bar{S}_m S_n \Theta_{nm}) C_{an} C_{am} \Delta^{(mn)} F_{LM}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g] \right\rangle \\ &= \frac{[\alpha_s]^2}{2\epsilon} h_{\text{rdc}}(\epsilon) \left[\langle J_{a,g} \cdot F_{LM}^{\mathcal{B}_0} \rangle + \langle M_{gg} \otimes F_{LM}^{\mathcal{B}_0} \rangle \right], \end{aligned} \quad (5.107)$$

where we introduced the two the intial-state objects

$$J_{a,f_a} \stackrel{\text{def}}{=} \frac{2T_{f_a}^2}{\epsilon} \left[e^{-2\epsilon L_{\text{max}}} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \Gamma_{a,f_a} - \Gamma_{a,f_a}^{(4)} \right], \quad (5.108)$$

$$M_{\alpha\alpha}(z, E_a) \stackrel{\text{def}}{=} \frac{2T_{\alpha}^2}{\epsilon} \left[e^{-2\epsilon L_{\text{max}}} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \mathcal{P}_{\alpha\alpha}^{\text{gen}}(z, E_a) - \mathcal{P}_{\alpha\alpha}^{(4),\text{gen}}(z, E_a) \right]. \quad (5.109)$$

As for the final-state radiation, the outcome of the integration over energies is the same as Eq. (5.106), provided replacing $\mathcal{P}_{\alpha\beta}^{(k)}$ with the anomalous dimension $\gamma_{z,f_i \rightarrow f_i g}^{2k}$. Hence, we obtain

$$\begin{aligned} & \frac{1}{2} \sum_{i \in \mathcal{H}_f} \left\langle [2(\eta_{in}/2)^{-\epsilon} - 1] (2\bar{S}_m S_n \Theta_{nm}) C_{in} C_{im} \Delta^{(mn)} F_{LM}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g] \right\rangle \\ &= \frac{[\alpha_s]^2}{2\epsilon} h_{\text{rdc}}(\epsilon) \sum_{i \in \mathcal{H}_f} \langle J_{i,f_i \rightarrow f_i} \cdot F_{LM}^{\mathcal{B}_0} \rangle, \end{aligned} \quad (5.110)$$

where

$$J_{i,f_i \rightarrow f_i g} \stackrel{\text{def}}{=} \frac{2T_{f_i}^2}{\epsilon} \left[e^{-2\epsilon L_{\text{max}}} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \Gamma_{i,f_i \rightarrow f_i g} - \Gamma_{i,f_i \rightarrow f_i g}^{(4)} \right], \quad i \in \mathcal{H}_f. \quad (5.111)$$

Next, we consider $\mathbb{Y}$ in Eq. (5.101). In this case, no further steps are required; $\mathbb{Y}$ corresponds to

$$\mathbb{Y} \equiv \epsilon h_{\text{rdc}}(\epsilon) \mathbb{B}_{\text{tc}}, \quad (5.112)$$

where $\mathbb{B}_{\text{tc}}$ is defined in Eq. (5.61) and explicitly written in Eq. (5.82), with $\epsilon h_{\text{rdc}}(\epsilon)$ originating as in Eq. (5.104). Thus, we combine the results for $\mathbb{X}$ and $\mathbb{Y}$ into a single final expression, which is reported below:

$$\mathbb{X} + \mathbb{Y} = \frac{[\alpha_s]^2}{2\epsilon} h_{\text{rdc}}(\epsilon) \left[\langle \mathcal{P}_{gg}^{\text{rdc}} \otimes F_{LM}^{\mathcal{B}_0} \rangle + \langle F_{LM}^{\mathcal{B}_0} \otimes \mathcal{P}_{qq}^{\text{rdc}} \rangle + \langle R^{\text{rdc}}(\epsilon) \cdot F_{LM}^{\mathcal{B}_0} \rangle \right]. \quad (5.113)$$

In this expression, we introduce a new splitting function,

$$\begin{aligned} \mathcal{P}_{\alpha\beta}^{\text{rdc}}(z, E_a) &\stackrel{\text{def}}{=} G_{\alpha\beta}(z, E_a) + \delta_{\alpha\beta} \left[[\mathcal{P}_{\alpha\alpha}^{\text{gen}} \bar{\otimes} \mathcal{P}_{\alpha\alpha}^{\text{gen}}](z, E_a) + M_{\alpha\alpha}(z, E_a) \right] \\ &\quad + 2\mathcal{P}_{\alpha\beta}^{\text{gen}}(z, E_a) \Gamma_{a,\alpha}(z) + \bar{\delta}_{\alpha\beta} \sum_x [\mathcal{P}_{x\beta}^{\text{gen}} \bar{\otimes} \mathcal{P}_{\alpha x}^{\text{gen}}](z, E_a), \end{aligned} \quad (5.114)$$

with $\bar{\delta}_{\alpha\beta} = 1 - \delta_{\alpha\beta}$, and a z -independent object

$$R^{\text{rdc}}(\epsilon) \stackrel{\text{def}}{=} \sum_{i=a,b} \left(\Gamma_{i,f_i}^2 + J_{i,f_i} \right) + \sum_{i \in \mathcal{H}_{f,g}} \left(\Gamma_{i,g \rightarrow gg}^2 + J_{i,g \rightarrow gg} + G_i \Big|_{z,g \rightarrow gg}^{z,g \rightarrow gg} \right) \\ + \Gamma_q^2 + J_{q \rightarrow qq} + G \Big|_{z,q \rightarrow qq}^{1,q \rightarrow qq} + G \Big|_{z,q \rightarrow qq}^{z,g \rightarrow gg} + \Gamma_{q \rightarrow gq} (\Gamma_{g \rightarrow gg} - \Gamma_q). \quad (5.115)$$

Note that Eq. (5.113) represents the *remainder* of the various manipulations performed to handle the double-collinear contribution in Sec. 5.2. Here, we have collected several terms within the objects $\mathcal{P}_{\alpha\beta}^{\text{rdc}}$ and $R^{\text{rdc}}(\epsilon)$ because, as we will see, these structures will also appear unchanged in certain terms at $\mathcal{O}(\epsilon^0)$.

Next, we discuss

$$\mathcal{P}_{\mathcal{B}_0} [\Sigma_{\text{DU}}^{(2)}|_{\mathcal{B}_0} + 2s_{ab} d\hat{\sigma}_{\mathcal{B}_2}^{\text{int}}] = \Sigma_{\text{DU}}^{(2)}|_{\mathcal{B}_0,gg} + \mathcal{P}_{\mathcal{B}_0} [\Sigma_{\text{DU}}^{(2)}|_{\mathcal{B}_0,qg} + 2s_{ab} d\hat{\sigma}_{\mathcal{B}_2}^{\text{int}}], \quad (5.116)$$

where $\Sigma_{\text{DU}}^{(2)}|_{\mathcal{B}_0,gg}$ is defined in Eq. (4.42), $\Sigma_{\text{DU}}^{(2)}|_{\mathcal{B}_0,qg}$ is defined similarly to Eq. (4.42) but without the double-soft regulator $\bar{S}_{mn}$, and $d\hat{\sigma}_{\mathcal{B}_2}^{\text{int}}$ is the interference term in Eq. (5.1). We begin by calculating $\Sigma_{\text{DU}}^{(2)}|_{\mathcal{B}_0}$, which is expressed as follows:

$$\Sigma_{\text{DU}}^{(2)}|_{\mathcal{B}_0} = \left\langle \Omega_2 \Delta^{(mn)} \left[\bar{S}_{mn} \bar{S}_n \Theta_{mn} F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g] + \mathcal{P}_{\mathcal{B}_0} \bar{S}_n \Omega_2 \Delta^{(mn)} F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g] \right] \right\rangle, \quad (5.117)$$

where Ω_2 is defined in Eq. (4.37), and corresponds to triple-collinear contribution analyzed in Ref. [79]. We begin with the initial-state radiation, noting that two possible triple-collinear splittings are allowed: $g \rightarrow ggg^*$ for leg a , and $q \rightarrow gq^*$ for leg b . All possible triple-collinear limits that do not project onto $\mathcal{B}_0$ are discarded by the projector $\mathcal{P}_{\mathcal{B}_0}$. Using the results in Table 1 of Ref. [79], we find the following expression:

$$[\alpha_s]^2 \langle (E_a/\mu)^{-4\epsilon} \text{ISR}_2 \otimes F_{\text{LM}}^{\mathcal{B}_0} \rangle + [\alpha_s]^2 \langle (E_b/\mu)^{-4\epsilon} F_{\text{LM}}^{\mathcal{B}_0} \otimes \text{ISR}_1 \rangle. \quad (5.118)$$

Here, we adopt the notation of Ref. [79], where $\text{ISR}_2(z, E_a)$ and $\text{ISR}_1(z, E_b)$ refer to the $g \rightarrow ggg^*$ and $q \rightarrow gq^*$ triple-collinear splittings, respectively. We point out that, in principle, one must also consider prefactors in front of $\text{ISR}_i(z, E)$ coming from the average over colors and spins. Given the general triple-collinear splitting $\beta \rightarrow AB\alpha^*$, the definition of this prefactor corresponds to

$$\text{pref}_{\alpha\beta} \stackrel{\text{def}}{=} \frac{\text{ave}_\beta}{\text{ave}_\alpha}, \quad \text{with} \quad \begin{cases} \text{ave}_q = \frac{1}{2N_c}, \\ \text{ave}_g = \frac{1}{2(1-\epsilon)(N_c^2-1)}. \end{cases} \quad (5.119)$$

In our case, we have $\text{pref}_{gg} = \text{pref}_{qq} = 1$.

Then, we analyze the final-state radiation. The first possible splitting is $g^* \rightarrow ggg$, which occurs when $\mathbf{m}_g$ and $\mathbf{n}_g$ become collinear with any of the $(N-1)$ final-state hard gluons of $F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g]$. Here, $F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g]$ follows an energy-ordering scheme, consistent

with Table 2 of Ref. [79]. We obtain:

$$\sum_{i \in \mathcal{H}_{\ell,g}} [\alpha_s]^2 \langle (E_i/\mu)^{-4\epsilon} \text{FSR}_5 \cdot F_{\text{LM}}^{\mathcal{B}_0} \rangle. \quad (5.120)$$

Next, the second possible splitting is $q^* \rightarrow qgg$, which happens when $\mathbf{m}_g$ and $\mathbf{n}_g$ become collinear with the final-state hard quark q of $F_{\text{LM}}^{\mathcal{B}_0}[\mathbf{m}_g, \mathbf{n}_g]$, or when $\mathbf{m}_q$ and $\mathbf{n}_g$ become collinear with any of the N final-state hard gluons of $F_{\text{LM}}^{\mathcal{B}_0}[\mathbf{m}_q, \mathbf{n}_g]$. It is important to note that this second case is equivalent to considering $\mathbf{m}_q$ and $\mathbf{n}_g$ becoming collinear with only one of the final-state hard gluons in $F_{\text{LM}}^{\mathcal{B}_0}[\mathbf{m}_q, \mathbf{n}_g]$, and then multiplying the result by a factor of N . Therefore, we find

$$[\alpha_s]^2 \langle (E/\mu)^{-4\epsilon} \widetilde{\text{FSR}}_1 \cdot F_{\text{LM}}^{\mathcal{B}_0} \rangle, \quad (5.121)$$

where $\widetilde{\text{FSR}}_1$ is defined in Eq. (G.36). Note that in (5.121), we employed the function $\widetilde{\text{FSR}}_1$ instead of the function FSR_1 , which is presented in Table 2 of Ref. [79]. This choice arises from the fact that FSR_1 is defined to account solely for the contribution

$$\langle \bar{S}_{\text{mn}} \bar{S}_{\text{n}} \Omega_2 \Theta_{\text{mn}} F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g] \rangle, \quad (5.122)$$

which does not correspond to Eq. (5.117). Nevertheless, the additional terms that distinguish (5.122) from (5.117) can be derived without substantial difficulty, as they do not involve purely triple-collinear limits, which would otherwise complicate the phase space integration. The derivation $\widetilde{\text{FSR}}_1$ is discussed in Appendix G.1.

Finally, we consider the interference term $\text{d}\hat{\sigma}_{\widetilde{\mathcal{B}}_2}^{\text{int}}$. A detailed discussion of this contribution is provided in Appendix G.2. Here, we simply note that this term contributes an additional triple-collinear limit to the final-state quark leg, as can be seen from Eq. (G.47), which we reproduce here for convenience:

$$\mathcal{P}_{\mathcal{B}_0} [2s_{ab} \text{d}\hat{\sigma}_{\widetilde{\mathcal{B}}_2}^{\text{int}}] = \frac{[\alpha_s]^2}{2} \langle (E/\mu)^{-4\epsilon} \text{FSR}_3 \cdot F_{\text{LM}}^{\mathcal{B}_0} \rangle. \quad (5.123)$$

The function $\text{FSR}_3(E)$ is reported in Table 2 of Ref. [79].

To conclude, we can combine Eqs (5.118), (5.120), (5.121), (5.123), and write a final expression for Eq. (5.116), which reads

$$\begin{aligned} & \mathcal{P}_{\mathcal{B}_0} [\Sigma_{\text{DU}}^{(2)}|_{\mathcal{B}_0} + 2s_{ab} \text{d}\hat{\sigma}_{\widetilde{\mathcal{B}}_2}^{\text{int}}] \\ &= [\alpha_s]^2 \langle (E_a/\mu)^{-4\epsilon} \text{ISR}_2 \otimes F_{\text{LM}}^{\mathcal{B}_0} \rangle + [\alpha_s]^2 \langle (E_b/\mu)^{-4\epsilon} F_{\text{LM}}^{\mathcal{B}_0} \otimes \text{ISR}_1 \rangle \\ &+ [\alpha_s]^2 \left\langle \left[\sum_{i \in \mathcal{H}_{\ell,g}} (E_i/\mu)^{-4\epsilon} \text{FSR}_5 + (E/\mu)^{-4\epsilon} [\widetilde{\text{FSR}}_1 + \text{FSR}_3] \right] \cdot F_{\text{LM}}^{\mathcal{B}_0} \right\rangle. \end{aligned} \quad (5.124)$$

We close this subsection providing a final expression for $\Sigma_{\text{DU}}^{(8)}$. As reported in Eq. (4.131), $\Sigma_{\text{DU}}^{(8)}$ corresponds to *spin-correlated* contribution $\Sigma_{\text{RR},1\text{c}}^{(b,d),\text{sc},\text{I},1}|_{\mathcal{H},gg}$ in Eq. (E.44). Here is its

explicit expression:

$$\begin{aligned}
\Sigma_{\text{DU}}^{(8)} = & [\alpha_s]^2 N_{\text{sc}}^{(b,d)} \left\langle \left[\frac{\gamma_{\perp, g \rightarrow gg}^{22,r}}{2} \left(I_C^{(4)}(\epsilon) - \frac{\Gamma_{q \rightarrow gg}^{(4)}}{2\epsilon} \right) + \frac{\gamma_{\perp, g \rightarrow gg}^{22}}{4\epsilon} \sum_{i \in \mathcal{H}_f} \Lambda_{i, f_i \rightarrow f_i g}^{\text{sc}} \right] \cdot F_{\text{LM}}^{\mathcal{B}_0} \right\rangle \\
& + \frac{[\alpha_s]^2}{4\epsilon} N_{\text{sc}}^{(b,d)} \left\langle \left[\gamma_{\perp, g \rightarrow gg}^{22,r} \mathcal{P}_{gg}^{(4), \text{gen}} + \gamma_{\perp, g \rightarrow gg}^{22} \mathcal{P}_{gg}^{\text{sc}} \right] \otimes F_{\text{LM}}^{\mathcal{B}_0} \right\rangle \\
& + \frac{[\alpha_s]^2}{4\epsilon} N_{\text{sc}}^{(b,d)} \left\langle F_{\text{LM}}^{\mathcal{B}_0} \otimes \left[\gamma_{\perp, g \rightarrow gg}^{22,r}(\epsilon) \mathcal{P}_{qq}^{(4), \text{gen}} + \gamma_{\perp, g \rightarrow gg}^{22}(\epsilon) \mathcal{P}_{qq}^{\text{sc}} \right] \right\rangle, \tag{5.125}
\end{aligned}$$

where $\gamma_{\perp, g \rightarrow gg}^{22,r}$ and $\gamma_{\perp, g \rightarrow gg}^{22}$ are given in Eq. (A.66), and the prefactor $N_{\text{sc}}^{(b,d)}$ in Eq. (A.8). Additionally, in Eq. (5.125), we have introduced the spin-correlated splitting function

$$\mathcal{P}_{\alpha\alpha}^{\text{sc}}(z, E_a) \stackrel{\text{def}}{=} - \left[\left(\frac{2E_a}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \right]^2 \left[\mathcal{P}_{\alpha\alpha}^{(4), \text{spin}}(z, E_a) - \mathcal{P}_{\alpha\alpha}^{(4)}(z, E_a) \right], \tag{5.126}$$

and the spin-correlated anomalous dimension

$$\Lambda_{i, f_i \rightarrow f_i g}^{\text{sc}} \stackrel{\text{def}}{=} \left[\left(\frac{2E_i}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \right]^2 \left[\gamma_{z, f_i \rightarrow f_i g}^{24, \text{spin}}(\epsilon, L_i) - \gamma_{z, f_i \rightarrow f_i g}^{24}(\epsilon, L_i) \right], \tag{5.127}$$

where $\mathcal{P}_{\alpha\alpha}^{(4), \text{spin}}$ and $\gamma_{z, f_i \rightarrow f_i g}^{nk, \text{spin}}$ are defined in Eqs (A.68) and (A.69), respectively.

5.3.3 Terms of $\mathcal{O}(\epsilon^0)$

This subsection is dedicated to computing the remaining collinear contributions of $\mathcal{O}(\epsilon^0)$ in $\Sigma_{\text{DU}}|_{\mathcal{B}_0}$ in Eq. (5.2). We have already addressed all the divergent parts in Eqs (4.179) and (4.183). The remaining terms, $\Sigma_{\text{DU}}^{\text{fin}, (i)}|_{\mathcal{B}_0, gg}$ and $\Sigma_{\text{DU}}^{\text{fin}, (i)}|_{\mathcal{B}_0, qg}$, are listed in Table 4.5. Some of these are already known: $\Sigma_{\text{DU}}^{\text{fin}, (2)}|_{\mathcal{B}_0, gg}$ is derived in Appendix D.1, with the final expression in Eq. (D.13), while $\Sigma_{\text{DU}}^{\text{fin}, (4)}|_{\mathcal{B}_0, gg}$ and $\Sigma_{\text{DU}}^{\text{fin}, (5)}|_{\mathcal{B}_0, gg}$ appear in Eqs (4.129) and (4.133), respectively. The remaining terms, $\Sigma_{\text{DU}}^{\text{fin}, (1)}|_{\mathcal{B}_0}$ and $\Sigma_{\text{DU}}^{\text{fin}, (3)}|_{\mathcal{B}_0}$, will be computed in this subsection.

We begin with $\Sigma_{\text{DU}}^{\text{fin}, (1)}|_{\mathcal{B}_0}$, where the $\mathbf{m}_g \mathbf{n}_g$ and $\mathbf{m}_q \mathbf{n}_g$ components are defined in Eqs (4.139) and (4.142), respectively. No further manipulations are required, so we directly provide its final expression:

$$\begin{aligned}
\Sigma_{\text{DU}}^{\text{fin}, (1)}|_{\mathcal{B}_0} = & - \left[\left(\frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)} \right)^2 - 1 \right] \sum_{(ij)} \mathcal{P}_{\mathcal{B}_0} \left\langle \bar{S}_n C_{jn} C_{im} \Delta^{(mn)} \right. \\
& \times \left. \left[\Theta_{mn} F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g] + F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_q, \mathbf{n}_g] \right] \right\rangle. \tag{5.128}
\end{aligned}$$

Next, we consider $\Sigma_{\text{DU}}^{\text{fin}, (3)}|_{\mathcal{B}_0}$, with the $\mathbf{m}_g \mathbf{n}_g$ and $\mathbf{m}_q \mathbf{n}_g$ components defined in Eqs (4.94)

and (4.158), respectively. Following the procedure outlined in (5.101), we express it as

$$\begin{aligned}
& \Sigma_{\text{DU}}^{\text{fin},(3)}|_{\mathcal{B}_0} = \\
& = 2 \left[\frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)} - 1 \right] \sum_{(ij)} \mathcal{P}_{\mathcal{B}_0} \left\langle \bar{S}_n C_{jn} C_{im} \Delta^{(\text{mn})} \left[\Theta_{mn} F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g] + F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_q, \mathbf{n}_g] \right] \right\rangle \\
& + \left[\frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)} - 1 \right] \sum_{i \in \mathcal{H}} \left\langle (\eta_{in}/2)^{-\epsilon} (\bar{S}_m \bar{S}_n + 2\bar{S}_m S_n \Theta_{nm}) C_{in} C_{im} \Delta^{(\text{mn})} F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g] \right\rangle \\
& + \left[\frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)} - 1 \right] \sum_{i \in \mathcal{H}} \left\langle (\eta_{in}/2)^{-\epsilon} \bar{S}_n (C_{im} C_{in} + C_{in} C_{im}) \Delta^{(\text{mn})} F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_q, \mathbf{n}_g] \right\rangle. \quad (5.129)
\end{aligned}$$

At this stage, we sum $\Sigma_{\text{DU}}^{\text{fin},(1)}|_{\mathcal{B}_0}$ and $\Sigma_{\text{DU}}^{\text{fin},(3)}|_{\mathcal{B}_0}$. We observe that

$$\begin{aligned}
\left[\left(\frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)} \right)^2 - 1 \right] &= \frac{\pi^2}{3} \epsilon^2 + \mathcal{O}(\epsilon^3), \\
2 \left[\frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)} - 1 \right] &= \frac{\pi^2}{3} \epsilon^2 + \mathcal{O}(\epsilon^3), \quad (5.130)
\end{aligned}$$

thus, the (ij) contributions cancel exactly at $\mathcal{O}(\epsilon^0)$, and can therefore be disregarded. Consequently, we obtain:

$$\begin{aligned}
& \Sigma_{\text{DU}}^{\text{fin},(1)}|_{\mathcal{B}_0} + \Sigma_{\text{DU}}^{\text{fin},(3)}|_{\mathcal{B}_0} = \\
& = \left[\frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)} - 1 \right] \sum_{i \in \mathcal{H}} \left\langle (\eta_{in}/2)^{-\epsilon} (\bar{S}_m \bar{S}_n + 2\bar{S}_m S_n \Theta_{nm}) C_{in} C_{im} \Delta^{(\text{mn})} F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g] \right\rangle \\
& + \left[\frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)} - 1 \right] \sum_{i \in \mathcal{H}} \left\langle (\eta_{in}/2)^{-\epsilon} \bar{S}_n (C_{im} C_{in} + C_{in} C_{im}) \Delta^{(\text{mn})} F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_q, \mathbf{n}_g] \right\rangle. \quad (5.131)
\end{aligned}$$

These two contributions are the same as $\mathbb{X}$ and $\mathbb{Y}$ from Eqs (5.102) and (5.99), respectively, apart from a factor 1/2, and the angular prefactor that here corresponds to

$$\begin{aligned}
& \left[\frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)} - 1 \right] \int_0^1 [d\eta_{im}] \int_0^1 [d\eta_{in}] \frac{(\eta_{in}/2)^{-\epsilon}}{\rho_{im} \rho_{in}} \\
& \equiv \epsilon^2 h_b(\epsilon) \underbrace{\int_0^1 [d\eta_{im}] \int_0^1 [d\eta_{in}] \frac{1}{\rho_{im} \rho_{in}}}_{\text{integral in Eq. (5.59)}}, \quad (5.132)
\end{aligned}$$

where

$$h_b(\epsilon) \stackrel{\text{def}}{=} \frac{2^{\epsilon-1}}{\epsilon^2} \frac{\Gamma^2(1-2\epsilon)}{\Gamma(1-3\epsilon)\Gamma(1-\epsilon)} \left[\frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)} - 1 \right] = \frac{\pi^2}{12} + \mathcal{O}(\epsilon). \quad (5.133)$$

Therefore, we write Eq. (5.131) as

$$\Sigma_{\text{DU}}^{\text{fin},(1)}|_{\mathcal{B}_0} + \Sigma_{\text{DU}}^{\text{fin},(3)}|_{\mathcal{B}_0} = [\alpha_s]^2 h_b(\epsilon) \left[\langle \mathcal{P}_{gg}^{\text{rdc}} \otimes F_{\text{LM}}^{\mathcal{B}_0} \rangle + \langle F_{\text{LM}}^{\mathcal{B}_0} \otimes \mathcal{P}_{qq}^{\text{rdc}} \rangle \right]$$

$$+ \langle R^{\text{rdc}}(\epsilon) \cdot F_{\text{LM}}^{\mathcal{B}_0} \rangle \Big]. \quad (5.134)$$

With this result, the chapter concludes, as all contributions in Eq. (5.4) have been calculated.

Manifest cancellations of IR singularities at NNLO

Contents

6.1	Single-unresolved terms	167
6.2	Double-unresolved triple color-correlated terms	171
6.3	Other double-unresolved color-correlated terms	178
6.4	Analytic cancellation of poles in the double-unresolved terms	181
6.4.1	Double- and single-boosted contributions	183
6.4.2	Elastic pure collinear contribution	184
6.5	Final result	188
6.5.1	Final result for process $\mathcal{B}_0$	188
6.5.2	Final result for processes $\mathcal{A}_0, \mathcal{A}_0^{(\bar{q}q)}, \mathcal{A}_0^{(gg)}$	191

This chapter is dedicated to demonstrating the cancellation of the poles in the cross section (5.1) for the process $\mathcal{B}_0$ defined in Eq. (3.56). There are four main structures within the cross section, and we discuss the cancellation of poles in each of these in turn. First, in Sec. 6.1, we address the cancellation of the poles in the single-unresolved contributions $\Sigma_{\text{SU}}|_{\mathcal{B}_0}$, defined as the sum of Eqs (4.170) and (4.171). Next, in Sections 6.2 and 6.3, we will deal with the cancellations of the poles arising from double-unresolved triple color-correlated terms and other double-unresolved terms containing color-correlations, respectively. Then, in Sec. 6.4, we will thoroughly discuss the *analytic* cancellation of the poles, up to $\mathcal{O}(\epsilon^{-2})$, for all double-unresolved terms, both boosted and elastic, and we briefly comment on the cancellation of the remaining $\mathcal{O}(\epsilon^{-1})$ poles. Finally, Sec. 6.5 is dedicated to the final results of this thesis.

6.1 Single-unresolved terms

We begin our discussion of the IR poles by focusing on the single-unresolved contribution

$$\mathcal{P}_{\mathcal{B}_0} \Sigma_{\text{SU}}|_{\mathcal{B}_0} = \Sigma_{\text{SU}}|_{\mathcal{B}_0, gg} + \mathcal{P}_{\mathcal{B}_0} \Sigma_{\text{SU}}|_{\mathcal{B}_0, qq}, \quad (6.1)$$

where the two terms on the right-hand side are defined in Eqs (4.170) and (4.171), respectively, and the references for the contributions are collected in Table 4.4. We show that the cancellation of poles is *similar* to that in the NLO QCD contribution to the process $\mathcal{B}_0$.

We begin with $\Sigma_{\text{SU}}^{(1)}|_{\mathcal{B}_0}$, where the $\mathbf{m}_g \mathbf{n}_g$ and $\mathbf{m}_q \mathbf{n}_g$ components are defined in Eqs (4.25) and (4.66), respectively. Here, parton $\mathbf{m}$ is resolved, as all the singularities associated with its emission are regulated by the $\mathcal{O}_{\text{NLO}}^{(\mathbf{m})}$ operator. In contrast, the gluon $\mathbf{n}_g$ acts as an unresolved parton in NLO computations. This structure suggests a close relationship between $\Sigma_{\text{SU}}^{(1)}|_{\mathcal{B}_0}$ and the NLO cross section for $(N+1)$ jet production. To make this correspondence explicit, we rewrite $\Sigma_{\text{SU}}^{(1)}|_{\mathcal{B}_0}$ in terms of the virtual and soft operators defined in the phase space for $(N+1)$ final-state partons. We obtain

$$\begin{aligned} \Sigma_{\text{SU}}^{(1)}|_{\mathcal{B}_0} = & [\alpha_s] \langle \mathcal{O}_{\text{NLO}}^{(\mathbf{m})} \Delta^{(\mathbf{m})} [I_V + I_S(E_{\mathbf{m}})] \cdot F_{\text{LM}}^{\mathcal{B}_1}[\mathbf{m}_g] \rangle \\ & + [\alpha_s] \langle \mathcal{O}_{\text{NLO}}^{(\mathbf{m})} \Delta^{(\mathbf{m})} [I_V + I_S(E_{\text{max}})] \cdot F_{\text{LM}}^{\mathcal{B}_1}[\mathbf{m}_q] \rangle \\ & + \langle \mathcal{O}_{\text{NLO}}^{(\mathbf{m})} \Delta^{(\mathbf{m})} (F_{\text{RV,fin}}^{\mathcal{B}_1}[\mathbf{m}_g] + F_{\text{RV,fin}}^{\mathcal{B}_1}[\mathbf{m}_q]) \rangle. \end{aligned} \quad (6.2)$$

In Eq. (6.2), both the I_V and I_S operators act on the $(N+1)$ hard partons of functions $F_{\text{LM}}^{\mathcal{B}_1}[\mathbf{m}_g]$ and $F_{\text{LM}}^{\mathcal{B}_1}[\mathbf{m}_q]$. The soft operator $I_S(E_{\text{max}})$ is defined as in Eq. (3.62), while $I_S(E_{\mathbf{m}})$ follows the same definition, except with the replacement $E_{\text{max}} \mapsto E_{\mathbf{m}}$, due to the energy-ordering $\Theta_{\mathbf{mn}}$ of $F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g]$.

We then address the contributions $\Sigma_{\text{SU}}^{(2)}|_{\mathcal{B}_0}$ and $\Sigma_{\text{SU}}^{(3)}|_{\mathcal{B}_0}$, whose $\mathbf{m}_g \mathbf{n}_g$ and $\mathbf{m}_q \mathbf{n}_g$ components are reported in Eqs (4.87) and (4.154), and Eqs (4.115) and (4.168), respectively. Both these contributions describe soft-subtracted collinear limits; as such they provide either generalized anomalous dimensions (in case of final-state splittings) or generalized anomalous dimensions and splitting functions (in case of initial-state splittings). We aim to assemble these terms to construct the collinear operator I_C for the process with $N+1$ final-state partons. This operator can then be combined with the terms in Eq. (6.2) to produce an IR-finite operator I_T , analogous to the procedure described for the NLO calculation in Ch. 3.

At first glance, this task seems straightforward. Indeed, starting with the $\mathbf{m}_g \mathbf{n}_g$ components, $\Sigma_{\text{SU}}^{(2)}|_{\mathcal{B}_0, gg}$ in Eq. (4.87) contains terms with collinear limits involving N (not $N+1$, as $\mathbf{m}$ is not included in the sum over $\mathcal{H}$ in this equation) final-state partons, while the required collinear limit of the additional parton $\mathbf{m}$ is provided by $\Sigma_{\text{SU}}^{(3)}|_{\mathcal{B}_0, gg}$ in Eq. (4.115). On closer inspection, there appears to be a mismatch between these terms: the final-state collinear operators $C_{i\mathbf{m}}$ acting on $\Delta^{(\mathbf{mn})}$ in Eq. (4.87) produce a factor of $z \Delta^{(\mathbf{m})}$, whereas in Eq. (4.115), the collinear operator $C_{\mathbf{mn}}$ does not act on the damping factor, and thus cannot produce this additional z -factor. The resolution of this apparent problem lies in the fact that the gluon anomalous dimension satisfies the property¹

$$\frac{1}{2} \gamma_{1,g \rightarrow gg}^{22}(L_i) = \gamma_{z,g \rightarrow gg}^{22}(L_i), \quad (6.3)$$

¹This property is not new to us; we already employed it in Eq. (5.90). It follows from the parametrization of the phase space we are using, and from the fact that the splitting function $P_{gg}(z)$ is symmetric under the exchange $z \leftrightarrow 1-z$.

where we recall that $L_i = \log(E_{\max}/E_i)$. Essentially, the additional z -factor plays the same role as the factor of $\frac{1}{2}$ in Eq. (4.115).² Therefore, we obtain

$$\begin{aligned} \Sigma_{\text{SU}}^{(2)}|_{\mathcal{B}_0,gg} + \Sigma_{\text{SU}}^{(3)}|_{\mathcal{B}_0,gg} &= [\alpha_s] \langle \mathcal{O}_{\text{NLO}}^{(\text{m})} \Delta^{(\text{m})} [I_{\text{C,inc}}(E_{\text{m}}) \cdot F_{\text{LM}}^{\mathcal{B}_1}[\mathbf{m}_g]] \rangle \\ &+ [\alpha_s] \langle \mathcal{O}_{\text{NLO}}^{(\text{m})} \Delta^{(\text{m})} [\mathcal{P}_{gg}^{\text{gen}} \otimes F_{\text{LM}}^{\mathcal{B}_1}[\mathbf{m}_g]] \rangle + [\alpha_s] \langle \mathcal{O}_{\text{NLO}}^{(\text{m})} \Delta^{(\text{m})} [F_{\text{LM}}^{\mathcal{B}_1}[\mathbf{m}_g] \otimes \mathcal{P}_{qq}^{\text{gen}}] \rangle. \end{aligned} \quad (6.4)$$

Note that, in Eq. (6.4), the operator $I_{\text{C,inc}}^{\mathcal{B}_1}(E_{\text{m}})$ is defined as in Eq. (5.48), but with the first sum running over a_g and b_q , and the second sum running over the final state partons $\{\{g\}_{N-1}, q, \mathbf{m}_g\}$. We recall that *inc* stands for *incomplete*, as the quark anomalous dimension within it contains only the $q \rightarrow qq$ splitting. Additionally, the energy $E_{\max}$ is replaced by E_{m} in every generalized anomalous dimension. In other words,

$$I_{\text{C,inc}}^{\mathcal{B}_1}(E_{\text{m}}) \stackrel{\text{def}}{=} \left[I_{\text{C,inc}}^{\mathcal{B}_0}(\epsilon) + \frac{\Gamma_{\text{m},g \rightarrow gg}}{\epsilon} \right]_{E_{\max} \mapsto E_{\text{m}}}, \quad (6.5)$$

where $I_{\text{C,inc}}^{\mathcal{B}_0}(\epsilon)$ is defined in Eq. (5.48).

Next, we consider $\Sigma_{\text{SU}}^{(2)}|_{\mathcal{B}_0,qq}$ in Eq. (4.154) and $\Sigma_{\text{SU}}^{(3)}|_{\mathcal{B}_0,qq}$ in Eq. (4.168). This situation is completely analogous to the NLO case, so we report the result directly:

$$\begin{aligned} \mathcal{P}_{\mathcal{B}_0} \left[\Sigma_{\text{SU}}^{(2)}|_{\mathcal{B}_0,qq} + \Sigma_{\text{SU}}^{(3)}|_{\mathcal{B}_0,qq} \right] &= [\alpha_s] \langle \mathcal{O}_{\text{NLO}}^{(\text{m})} \Delta^{(\text{m})} [I_{\text{C}}^{\mathcal{B}_1}(E_{\max}) \cdot F_{\text{LM}}^{\mathcal{B}_1}[\mathbf{m}_q]] \rangle + [\alpha_s] \left\langle \mathcal{O}_{\text{NLO}}^{(\text{m})} \Delta^{(\text{m})} \frac{\Gamma_{q \rightarrow qq}}{\epsilon} F_{\text{LM}}^{\mathcal{B}_1}[\mathbf{m}_q] \right\rangle \\ &+ [\alpha_s] \langle \mathcal{O}_{\text{NLO}}^{(\text{m})} \Delta^{(\text{m})} [\mathcal{P}_{gg}^{\text{gen}} \otimes F_{\text{LM}}^{\mathcal{B}_1}[\mathbf{m}_g]] \rangle + [\alpha_s] \langle \mathcal{O}_{\text{NLO}}^{(\text{m})} \Delta^{(\text{m})} [F_{\text{LM}}^{\mathcal{B}_1}[\mathbf{m}_q] \otimes \mathcal{P}_{qq}^{\text{gen}}] \rangle. \end{aligned} \quad (6.6)$$

Note that, in Eq. (6.6), $I_{\text{C}}^{\mathcal{B}_1}(E_{\max})$ is defined as the standard collinear operator (with $E_{\max}$ instead of E_{m}) and it acts on the hard partons $\{a_g, b_q, \{g\}_N, \mathbf{m}_q\}$.

Combining Eqs (6.4) and (6.6), we obtain:

$$\begin{aligned} \mathcal{P}_{\mathcal{B}_0} \left[\Sigma_{\text{SU}}^{(2)}|_{\mathcal{B}_0} + \Sigma_{\text{SU}}^{(3)}|_{\mathcal{B}_0} \right] &= [\alpha_s] \langle \mathcal{O}_{\text{NLO}}^{(\text{m})} \Delta^{(\text{m})} [I_{\text{C}}(E_{\text{m}}) \cdot F_{\text{LM}}^{\mathcal{B}_1}[\mathbf{m}_g]] \rangle + [\alpha_s] \langle \mathcal{O}_{\text{NLO}}^{(\text{m})} \Delta^{(\text{m})} [I_{\text{C}}(E_{\max}) \cdot F_{\text{LM}}^{\mathcal{B}_1}[\mathbf{m}_q]] \rangle \\ &+ [\alpha_s] \langle \mathcal{O}_{\text{NLO}}^{(\text{m})} \Delta^{(\text{m})} [\mathcal{P}_{gg}^{\text{gen}} \otimes F_{\text{R}}^{\mathcal{B}_1}] \rangle + [\alpha_s] \langle \mathcal{O}_{\text{NLO}}^{(\text{m})} \Delta^{(\text{m})} [F_{\text{R}}^{\mathcal{B}_1} \otimes \mathcal{P}_{qq}^{\text{gen}}] \rangle, \end{aligned} \quad (6.7)$$

where $F_{\text{R}}^{\mathcal{B}_1} = F_{\text{LM}}^{\mathcal{B}_1}[\mathbf{m}_g] + F_{\text{LM}}^{\mathcal{B}_1}[\mathbf{m}_q]$, and

$$I_{\text{C}}^{\mathcal{B}_1}(E_{\text{m}}) \stackrel{\text{def}}{=} I_{\text{C,inc}}^{\mathcal{B}_1}(E_{\text{m}}) + \frac{\Gamma_{q \rightarrow qq}}{\epsilon}. \quad (6.8)$$

At this point, we sum Eqs (6.2) and (6.7), define the following operators for processes with $N + 1$ partons,

$$I_{\text{T}}^{\mathcal{B}_1}(E_{\text{m}}) \stackrel{\text{def}}{=} I_{\text{V}} + I_{\text{S}}(E_{\text{m}}) + I_{\text{C}}^{\mathcal{B}_1}(E_{\text{m}}), \quad (6.9)$$

²More explicitly, the factor z introduces an additional factor of $\frac{1}{2}$ when integrating over the final-state phase space.

$$I_T^{\mathcal{B}_1}(E_{\max}) \stackrel{\text{def}}{=} I_V + I_S(E_{\max}) + I_C^{\mathcal{B}_1}(E_{\max}), \quad (6.10)$$

and get the following expression:

$$\begin{aligned} \mathcal{P}_{\mathcal{B}_0} \left[\Sigma_{\text{SU}}^{(1)}|_{\mathcal{B}_0} + \Sigma_{\text{SU}}^{(2)}|_{\mathcal{B}_0} + \Sigma_{\text{SU}}^{(3)}|_{\mathcal{B}_0} \right] &= \langle \mathcal{O}_{\text{NLO}}^{(\text{m})} \Delta^{(\text{m})} (F_{\text{RV},\text{fin}}^{\mathcal{B}_1}[\mathbf{m}_g] + F_{\text{RV},\text{fin}}^{\mathcal{B}_1}[\mathbf{m}_q]) \rangle \cdot \\ &+ [\alpha_s] \langle \mathcal{O}_{\text{NLO}}^{(\text{m})} \Delta^{(\text{m})} [I_T(E_{\text{m}}) \cdot F_{\text{LM}}^{\mathcal{B}_1}[\mathbf{m}_g]] \rangle + [\alpha_s] \langle \mathcal{O}_{\text{NLO}}^{(\text{m})} \Delta^{(\text{m})} [I_T(E_{\max}) \cdot F_{\text{LM}}^{\mathcal{B}_1}[\mathbf{m}_q]] \rangle \quad (6.11) \\ &+ [\alpha_s] \langle \mathcal{O}_{\text{NLO}}^{(\text{m})} \Delta^{(\text{m})} [\mathcal{P}_{gg}^{\text{gen}} \otimes F_{\text{R}}^{\mathcal{B}_1}] \rangle + [\alpha_s] \langle \mathcal{O}_{\text{NLO}}^{(\text{m})} \Delta^{(\text{m})} [F_{\text{R}}^{\mathcal{B}_1} \otimes \mathcal{P}_{qq}^{\text{gen}}] \rangle. \end{aligned}$$

We emphasize that the $\mathcal{O}_{\text{NLO}}^{(\text{m})}$ operator does not commute with the I_T operators or the splitting function $\mathcal{P}_{\alpha\beta}^{\text{gen}}$. Indeed the latter depends on the energy of parton m , which is sensitive to the action of the soft limit encoded in $\mathcal{O}_{\text{NLO}}^{(\text{m})}$.

We note that Eq. (6.11) still contains hard-collinear singularities related to initial state emissions. To remove them, we need to add the PDF renormalization contribution proportional to the $\mathcal{O}_{\text{NLO}}^{(\text{m})}$ operator, which we read from Eq. (5.95) and which corresponds to

$$\left[\frac{\alpha_s(\mu)}{2\pi} \right] \frac{1}{\epsilon} \left[\langle \mathcal{O}_{\text{NLO}}^{(\text{m})} \Delta^{(\text{m})} [\hat{\mathcal{P}}_{gg}^{(0)} \otimes F_{\text{R}}^{\mathcal{B}_1}] \rangle + \langle \mathcal{O}_{\text{NLO}}^{(\text{m})} \Delta^{(\text{m})} [F_{\text{R}}^{\mathcal{B}_1} \otimes \hat{\mathcal{P}}_{qq}^{(0)}] \rangle \right]. \quad (6.12)$$

Combining this formula with Eq. (6.11), we obtain

$$\begin{aligned} \mathcal{P}_{\mathcal{B}_0} \Sigma_{\text{SU}}^{\text{fin},(4)}|_{\mathcal{B}_0} &\stackrel{\text{def}}{=} \langle \mathcal{O}_{\text{NLO}}^{(\text{m})} \Delta^{(\text{m})} (F_{\text{RV},\text{fin}}^{\mathcal{B}_1}[\mathbf{m}_g] + F_{\text{RV},\text{fin}}^{\mathcal{B}_1}[\mathbf{m}_q]) \rangle \cdot \\ &+ [\alpha_s] \langle \mathcal{O}_{\text{NLO}}^{(\text{m})} \Delta^{(\text{m})} [I_T(E_{\text{m}}) \cdot F_{\text{LM}}^{\mathcal{B}_1}[\mathbf{m}_g]] \rangle + [\alpha_s] \langle \mathcal{O}_{\text{NLO}}^{(\text{m})} \Delta^{(\text{m})} [I_T(E_{\max}) \cdot F_{\text{LM}}^{\mathcal{B}_1}[\mathbf{m}_q]] \rangle \quad (6.13) \\ &+ [\alpha_s] \langle \mathcal{O}_{\text{NLO}}^{(\text{m})} \Delta^{(\text{m})} [\mathcal{P}_{gg}^{\text{NLO}} \otimes F_{\text{R}}^{\mathcal{B}_1}] \rangle + [\alpha_s] \langle \mathcal{O}_{\text{NLO}}^{(\text{m})} \Delta^{(\text{m})} [F_{\text{R}}^{\mathcal{B}_1} \otimes \mathcal{P}_{qq}^{\text{NLO}}] \rangle, \end{aligned}$$

where the ϵ -finite splitting functions $\mathcal{P}_{\alpha\beta}^{\text{NLO}}$ were introduced in the context of NLO corrections, and are defined in Eq. (I.3). As expected, Eq. (6.13) contains a generalized version of the ϵ -finite operator I_T given in Eq. (3.81). Note also that, as we mentioned at the beginning of this section, Eq. (6.13) contains almost exactly the NLO contribution to the $(N+1)$ -jet production cross section. The only missing piece is the fully-regulated term with up to $N+2$ resolved jets.

In addition to $\Sigma_{\text{SU}}^{\text{fin},(4)}|_{\mathcal{B}_0}$, there are other contributions with $N+1$ resolved final-state partons that are explicitly ϵ -finite (see Table 4.4). We combine all these contributions into a single quantity that we will refer to as

$$2s_{ab} \, \text{d}\hat{\sigma}_{\mathcal{B}_0}^{\text{SU}} = \sum_{i=1}^4 \mathcal{P}_{\mathcal{B}_0} \Sigma_{\text{SU}}^{\text{fin},(i)}|_{\mathcal{B}_0}, \quad (6.14)$$

where $\Sigma_{\text{SU}}^{\text{fin},(i)}|_{\mathcal{B}_0}$ must be intended as the sum of the $\mathbf{m}_g \mathbf{n}_g$ and $\mathbf{m}_q \mathbf{n}_g$ components. The computation of these terms involves NLO-like steps that have been encountered several times throughout this thesis. Therefore, we present their final expressions, already

expanded to $\mathcal{O}(\epsilon^0)$:

$$\begin{aligned} \mathcal{P}_{\mathcal{B}_0} \Sigma_{\text{SU}}^{\text{fin},(1)}|_{\mathcal{B}_0} &= [\alpha_s] \left\langle \mathcal{O}_{\text{NLO}}^{(a,\mathbf{m})} \omega_{a||\mathbf{n}}^{\mathbf{m}a,\mathbf{n}a} \log\left(\frac{\eta_{a\mathbf{m}}}{2}\right) \Delta^{(\mathbf{m})} [\hat{P}_{gg}^{(0)} \otimes F_{\mathbf{R}}^{\mathcal{B}_1}] \right\rangle \\ &\quad + [\alpha_s] \left\langle \mathcal{O}_{\text{NLO}}^{(b,\mathbf{m})} \omega_{b||\mathbf{n}}^{\mathbf{m}b,\mathbf{n}b} \log\left(\frac{\eta_{b\mathbf{m}}}{2}\right) \Delta^{(\mathbf{m})} [F_{\mathbf{R}}^{\mathcal{B}_1} \otimes \hat{P}_{qq}^{(0)}] \right\rangle \\ &\quad - \sum_{i \in \mathcal{H}} [\alpha_s] \left\langle \mathcal{O}_{\text{NLO}}^{(i,\mathbf{m})} \omega_{i||\mathbf{n}}^{\mathbf{m}i,\mathbf{n}i} \log\left(\frac{\eta_{i\mathbf{m}}}{2}\right) \Delta^{(\mathbf{m})} \left[\left(\delta_{ig} \Gamma_{i,g \rightarrow gg}(E_{\mathbf{m}}) \right. \right. \right. \\ &\quad \left. \left. \left. + \delta_{iq} \Gamma_q(E_{\mathbf{m}}) \right) F_{\text{LM}}^{\mathcal{B}_1}[\mathbf{m}_g] + \Gamma_{i,g \rightarrow gg} F_{\text{LM}}^{\mathcal{B}_1}[\mathbf{m}_q] \right] \right\rangle, \end{aligned} \quad (6.15)$$

$$\begin{aligned} \mathcal{P}_{\mathcal{B}_0} \Sigma_{\text{SU}}^{\text{fin},(2)}|_{\mathcal{B}_0} &= - \sum_{i \in \mathcal{H}} [\alpha_s] \left\langle \mathcal{O}_{\text{NLO}}^{(i,\mathbf{m})} \omega_{\mathbf{m}||\mathbf{n}}^{\mathbf{m}i,\mathbf{n}i} \log\left(\frac{\eta_{i\mathbf{m}}}{4(1-\eta_{i\mathbf{m}})}\right) \Delta^{(\mathbf{m})} \left[\gamma_g F_{\text{LM}}^{\mathcal{B}_1}[\mathbf{m}_g] \right. \right. \\ &\quad \left. \left. + (\gamma_q + 2L_{\mathbf{m}} T_q^2) F_{\text{LM}}^{\mathcal{B}_1}[\mathbf{m}_q] \right] \right\rangle, \end{aligned} \quad (6.16)$$

$$\begin{aligned} \mathcal{P}_{\mathcal{B}_0} \Sigma_{\text{SU}}^{\text{fin},(3)}|_{\mathcal{B}_0} &= \sum_{i \in \mathcal{H}} \frac{[\alpha_s]}{2} \gamma_{\perp,g \rightarrow gg}^{22} \langle \mathcal{O}_{\text{NLO}}^{(\mathbf{m})} \omega_{\mathbf{m}||\mathbf{n}}^{\mathbf{m}i,\mathbf{n}i} (r_i^\mu r_i^\nu + g^{\mu\nu}) \Delta^{(\mathbf{m})} F_{\text{LM}}^{\mathcal{B}_1}[\mathbf{m}_g] \rangle \\ &\quad + \sum_{i \in \mathcal{H}} \frac{[\alpha_s]}{2} \gamma_{\perp,g \rightarrow gg}^{22,\text{r}} \langle \mathcal{O}_{\text{NLO}}^{(\mathbf{m})} \omega_{\mathbf{m}||\mathbf{n}}^{\mathbf{m}i,\mathbf{n}i} \Delta^{(\mathbf{m})} F_{\text{LM}}^{\mathcal{B}_1}[\mathbf{m}_g] \rangle. \end{aligned} \quad (6.17)$$

A few comments are in order. In Eq. (6.15), the additional argument $E_{\mathbf{m}}$ in the gluon and quark generalized anomalous dimensions corresponds to the substitution $E_{\text{max}} \mapsto E_{\mathbf{m}}$, as discussed. In Eq. (6.16), we employed the expansion (3.126); the term proportional to γ_g does not include the factor $2L_{\mathbf{m}} T_g^2$ because the energy ordering imposes the replacement $E_{\text{max}} \mapsto E_{\mathbf{m}}$, from which we derive $L_{\mathbf{m}} = \log(E_{\text{max}}/E_{\mathbf{m}}) \mapsto \log(E_{\mathbf{m}}/E_{\mathbf{m}}) = 0$. Finally, in Eq. (6.17), $\gamma_{\perp,g \rightarrow gg}^{22}$ and $\gamma_{\perp,g \rightarrow gg}^{22,\text{r}}$ are provided in Eq. (A.66).

Before concluding, we point out that, as in the NLO case at the end of Subsec. 3.3.3, we commit a small abuse of notation here. Since we have applied the projector $\mathcal{P}_{\mathcal{B}_0}$, strictly speaking, in all these formulas we should exclude the subtraction of those initial-state collinear limits which, when applied to $F_{\text{LM}}^{\mathcal{B}_1}[\mathbf{m}_q]$, are set to zero by the projector $\mathcal{P}_{\mathcal{B}_0}$. We choose to leave this implicit in writing these formulas.

6.2 Double-unresolved triple color-correlated terms

Having demonstrated how ϵ -poles in single-unresolved terms cancel, we continue with the discussion of poles in the double-unresolved contribution $\Sigma_{\text{DU}}|_{\mathcal{B}_0}$. We begin with the investigation of ϵ -poles that involve matrix elements of triple correlators of color-charge operators

$$f_{abc} \langle \mathcal{M}_0 | T_k^a T_i^b T_j^c | \mathcal{M}_0 \rangle. \quad (6.18)$$

Such terms vanish for processes with three or fewer partons at tree-level, but are non-zero in general. In our case, divergent triple color-correlated terms are contained in

the second line of the double-virtual contribution (5.7),³ and in the last line of soft real-virtual contribution (5.31), which we collect as follows:

$$\begin{aligned} \Sigma_{\text{DU}}^{\text{tri}}|_{\mathcal{B}_0} \stackrel{\text{def}}{=} [\alpha_s]^2 \left\langle \left[-\frac{1}{2} [\bar{I}_1(\epsilon), \bar{I}_1^\dagger(\epsilon)] + \mathcal{H}_{2,\text{tc}} + \mathcal{H}_{2,\text{tc}}^\dagger \right] \cdot F_{\text{LM}}^{\mathcal{B}_0} \right\rangle \\ + [\alpha_s]^2 \left\langle [I_S(\epsilon), I_-(\epsilon)] + I_{\text{tri}}^{\text{RV}}(\epsilon) \right\rangle \cdot F_{\text{LM}}^{\mathcal{B}_0}, \end{aligned} \quad (6.19)$$

where $\mathcal{H}_{2,\text{tc}}$ and $I_{\text{tri}}^{\text{RV}}(\epsilon)$ are given in Eqs (2.38) and (H.9), respectively. Note that triple color-correlations arise in two distinct ways: triple color-correlators are *explicit* in $\mathcal{H}_{2,\text{tc}}$ and $I_{\text{tri}}^{\text{RV}}(\epsilon)$, and *implicit* in the *commutators* of various I -operators. Hence, it is necessary to write these commutators explicitly, in order to be able to combine them with $\mathcal{H}_{2,\text{tc}}$ and $I_{\text{tri}}^{\text{RV}}(\epsilon)$.

Therefore, let us compute the commutators in Eq. (6.19). We find it convenient to rewrite them as follows:

$$-\frac{1}{2} [\bar{I}_1, \bar{I}_1^\dagger] + [I_S, I_-] = -[I_+, I_-] + [2I_+ + I_S, I_-], \quad (6.20)$$

where we recall that the operators $I_\pm$ are defined in Eq. (5.26). We then combine the commutators with the operator $\mathcal{H}_{2,\text{tc}}$, and express $\Sigma_{\text{DU}}^{\text{tri}}|_{\mathcal{B}_0}$ as

$$\Sigma_{\text{DU}}^{\text{tri}}|_{\mathcal{B}_0} = [\alpha_s]^2 \left\langle [I_{\text{tri}}^{(\text{cc})}(\epsilon) + I_{\text{tri}}^{\text{RV}}(\epsilon)] \cdot F_{\text{LM}}^{\mathcal{B}_0} \right\rangle, \quad (6.21)$$

where

$$I_{\text{tri}}^{(\text{cc})} \stackrel{\text{def}}{=} -[I_+, I_-] + [2I_+ + I_S, I_-] + \mathcal{H}_{2,\text{tc}} + \mathcal{H}_{2,\text{tc}}^\dagger. \quad (6.22)$$

We start by computing $[I_+, I_-]$. We rewrite the operator $\bar{I}_1$ as (see Eq. (2.26))⁴

$$\bar{I}_1 = -\frac{1}{2} \sum_{i \in \mathcal{H}} \mathcal{V}_i^{\text{sing}}(\epsilon) + \frac{1}{2} \sum_{(ij)} \frac{\mathcal{V}_i^{\text{sing}}(\epsilon)}{\mathbf{T}_i^2} (\mathbf{T}_i \cdot \mathbf{T}_j) (e^{i\pi\lambda_{ij}\epsilon} e^{\epsilon L_{ij}} - 1), \quad (6.24)$$

where

$$\mathcal{V}_i^{\text{sing}}(\epsilon) = \frac{\mathbf{T}_i^2}{\epsilon^2} + \frac{\gamma_i}{\epsilon}, \quad L_{ij} = \log(\mu^2/s_{ij}), \quad s_{ij} = 2p_i \cdot p_j, \quad (6.25)$$

and $\lambda_{ij} = 1$ if both i and j are either incoming or outgoing, and $\lambda_{ij} = 0$ otherwise. Since we are interested in commutators of I -operators, in general the only non-vanishing contributions come from color-correlated terms. Therefore, the first term on the right-hand side of Eq. (6.24) is irrelevant, and only the term with the $(\mathbf{T}_i \cdot \mathbf{T}_j)$ product can play

³We neglect the term $\mathcal{H}_{2,\text{cd}}$ since it does not contain triple color-correlations.

⁴Here we use the color conservation property

$$\sum_{\substack{j \in \mathcal{H} \\ j \neq i}} \mathbf{T}_i \cdot \mathbf{T}_j = -\mathbf{T}_i^2. \quad (6.23)$$

a role. Hence, we define

$$\bar{I}_1^{(\text{cc})}(\epsilon) \stackrel{\text{def}}{=} \frac{1}{2} \sum_{(ij)} \frac{\mathcal{V}_i^{\text{sing}}(\epsilon)}{\mathbf{T}_i^2} (\mathbf{T}_i \cdot \mathbf{T}_j) (e^{i\pi\lambda_{ij}\epsilon} e^{\epsilon L_{ij}} - 1), \quad (6.26)$$

and we use this operator instead of $\bar{I}_1$ to treat the commutators in Eq. (6.22). To this end, we compute the color-correlated versions of $I_{\pm}$ using $\bar{I}_1^{(\text{cc})}$. We obtain

$$I_+^{(\text{cc})}(\epsilon) \stackrel{\text{def}}{=} \frac{1}{2} \sum_{(ij)} \frac{\mathcal{V}_i^{\text{sing}}(\epsilon)}{\mathbf{T}_i^2} (\mathbf{T}_i \cdot \mathbf{T}_j) (e^{\epsilon L_{ij}} \cos(\pi\lambda_{ij}\epsilon) - 1), \quad (6.27)$$

$$I_-^{(\text{cc})}(\epsilon) \stackrel{\text{def}}{=} \frac{i}{2} \sum_{(ij)} \frac{\mathcal{V}_i^{\text{sing}}(\epsilon)}{\mathbf{T}_i^2} (\mathbf{T}_i \cdot \mathbf{T}_j) e^{\epsilon L_{ij}} \sin(\pi\lambda_{ij}\epsilon), \quad (6.28)$$

and we point out that

$$[I_+, I_-] = [I_+^{(\text{cc})}, I_-^{(\text{cc})}]. \quad (6.29)$$

Next, we expand these two operators in powers of ϵ up to $\mathcal{O}(\epsilon^2)$, which yields:

$$I_+^{(\text{cc})}(\epsilon) = \frac{1}{2} \sum_{(ij)} (\mathbf{T}_i \cdot \mathbf{T}_j) \left[\frac{L_{ij}}{\epsilon} + \delta_{ij}^+ + \epsilon \xi_{ij}^+ \right] + \mathcal{O}(\epsilon^2), \quad (6.30)$$

$$I_-^{(\text{cc})}(\epsilon) = \frac{i\pi}{2} \sum_{(ij)} (\mathbf{T}_i \cdot \mathbf{T}_j) \left[\frac{\lambda_{ij}}{\epsilon} + \delta_{ij}^- + \epsilon \xi_{ij}^- \right] + \mathcal{O}(\epsilon^2), \quad (6.31)$$

where⁵

$$\begin{aligned} \delta_{ij}^+ &\stackrel{\text{def}}{=} \frac{1}{2} L_{ij}^2 + \frac{\gamma_i}{\mathbf{T}_i^2} L_{ij} - \frac{\pi^2}{2} \lambda_{ij}^2, \\ \delta_{ij}^- &\stackrel{\text{def}}{=} L_{ij} \lambda_{ij} + \frac{\gamma_i}{\mathbf{T}_i^2} \lambda_{ij}, \\ \xi_{ij}^+ &\stackrel{\text{def}}{=} \frac{1}{6} L_{ij}^3 + \frac{\gamma_i}{2\mathbf{T}_i^2} L_{ij}^2 - \frac{\pi^2}{2} \left(L_{ij} + \frac{\gamma_i}{\mathbf{T}_i^2} \right) \lambda_{ij}^2, \\ \xi_{ij}^- &\stackrel{\text{def}}{=} \frac{1}{2} L_{ij}^2 \lambda_{ij} + \frac{\gamma_i}{\mathbf{T}_i^2} \lambda_{ij} L_{ij} - \frac{\pi^2}{6} \lambda_{ij}^3. \end{aligned} \quad (6.32)$$

We note that the objects shown in Eq. (6.32) are sufficient to compute the poles in the triple color-correlated contributions to $\Sigma_{\text{DU}}|_{\mathcal{B}_0}$.

We can now proceed with the calculation of the commutator $[I_+, I_-]$. Since it involves objects such as $[\mathbf{T}_i \cdot \mathbf{T}_j, \mathbf{T}_k \cdot \mathbf{T}_l]$, it is convenient to report the following general relation: given two operators A and B defined as

$$A = \sum_{(ij)} a_{ij} (\mathbf{T}_i \cdot \mathbf{T}_j), \quad B = \sum_{(ij)} b_{ij} (\mathbf{T}_i \cdot \mathbf{T}_j), \quad (6.33)$$

⁵Notice that $\lambda_{ij}^n = \lambda_{ij} \forall n \geq 1$.

their commutator reads

$$[A, B] = i \sum_{(ijk)} (a_{jk} + a_{kj}) (b_{ij} + b_{ji}) F^{(kij)}, \quad F^{(kij)} \stackrel{\text{def}}{=} f_{abc} T_k^a T_i^b T_j^c. \quad (6.34)$$

Note that in the above equation, we introduced the handy notation (ijk) to label triplets with different i, j and k in the sum.

Proof:

Let A and B be the operators defined in Eq. (6.33). Their commutator

$$[A, B] = \sum_{(ij)} \sum_{(kl)} a_{ij} b_{kl} [T_i \cdot T_j, T_k \cdot T_l] \quad (6.35)$$

requires the expression for $[T_i \cdot T_j, T_k \cdot T_l]$, which we can write as follows (cf. Subsec. 1.2.1):

$$\begin{aligned} [T_i \cdot T_j, T_k \cdot T_l] &= [T_i^a T_j^a, T_k^b T_l^b] = T_i^a [T_j^a, T_k^b T_l^b] + [T_i^a, T_k^b T_l^b] T_j^a \\ &= T_i^a T_k^b [T_j^a, T_l^b] + T_i^a [T_j^a, T_k^b] T_l^b + T_k^b [T_i^a, T_l^b] T_j^a + [T_i^a, T_k^b] T_l^b T_j^a \\ &= i f_{abc} [\delta_{jl} T_i^a T_k^b T_j^c + \delta_{jk} T_i^a T_j^c T_l^b + \delta_{il} T_k^b T_i^c T_j^a + \delta_{ik} T_i^c T_l^b T_j^a], \end{aligned} \quad (6.36)$$

where we used the property $[T_j^a, T_l^b] = \delta_{jl} i f_{abc} T_j^c$.

The first term on the right-hand side of (6.36) can already be written as $\delta_{jl} i F^{(ikj)}$, and thus requires no further manipulation. The same does not apply to the second term, as the color matrices $T_i^a T_j^c T_l^b$ are not in the desired order, since we would like them written as $T_i^a T_l^b T_j^c$. Starting from $T_i^a T_j^c T_l^b$, if $j \neq l$, the matrices T_j^c and T_l^b correspond to different particles, and thus can be swapped. However, if $j = l$, we cannot swap them, as they refer to the same particle. Nevertheless, we obtain

$$i f_{abc} T_i^a T_j^c T_j^b = 0, \quad (6.37)$$

because the contraction involves an antisymmetric tensor in indices b, c , i.e., f_{abc} , and a symmetric one, $T_j^c T_j^b$. Therefore,

$$\delta_{jk} i f_{abc} T_i^a T_j^c T_l^b = \delta_{jk} i f_{abc} T_i^a T_l^b T_j^c \equiv \delta_{jk} i F^{(ilj)}. \quad (6.38)$$

Additionally, from the above steps we see that $F^{(kij)}$ is fully-antisymmetric in k, i, j indices.

We can treat the third and fourth terms on the right-hand side of (6.36) in the same way, and we obtain

$$\begin{aligned} [A, B] &= i \sum_{(ij)} \sum_{(kl)} a_{ij} b_{kl} [\delta_{jl} F^{(ikj)} + \delta_{jk} F^{(ilj)} + \delta_{il} F^{(jki)} + \delta_{ik} F^{(jli)}] \\ &= i \sum_{(ijk)} [a_{kj} b_{ij} F^{(kij)} + a_{ij} b_{jk} F^{(ikj)} + a_{ij} b_{ki} F^{(jki)} + a_{ij} b_{ik} F^{(jki)}]. \end{aligned} \quad (6.39)$$

Finally, we relabel the indices of each of the four contributions so as to obtain $F^{(kij)}$, which yields

$$\begin{aligned} [A, B] &= i \sum_{(ijk)} [a_{kj} b_{ij} F^{(kij)} + a_{kj} b_{ji} F^{(kij)} + a_{jk} b_{ij} F^{(kij)} + a_{jk} b_{ji} F^{(kij)}] \\ &= i \sum_{(ijk)} (a_{jk} + a_{kj}) (b_{ij} + b_{ji}) F^{(kij)}. \end{aligned} \quad (6.40)$$

Eq. (6.34) can be used to compute the commutator $[I_+, I_-]$, replacing $I_\pm$ with their color-correlated analogues $I_\pm^{(\text{cc})}$, as discussed above. We find

$$\begin{aligned} [I_+^{(\text{cc})}, I_-^{(\text{cc})}] = & -\frac{\pi}{2} \sum_{(ijk)} F^{(kij)} \left[\frac{2L_{jk} \lambda_{ij}}{\epsilon^2} + \frac{1}{\epsilon} [L_{jk}(\delta_{ij}^- + \delta_{ji}^-) + \lambda_{ij}(\delta_{jk}^+ + \delta_{kj}^+)] \right. \\ & + \frac{1}{2} (\delta_{jk}^+ + \delta_{kj}^+) (\delta_{ij}^- + \delta_{ji}^-) + L_{jk}(\xi_{ij}^- + \xi_{ji}^-) + \lambda_{ij}(\xi_{jk}^+ + \xi_{kj}^+) \left. \right] \\ & + \mathcal{O}(\epsilon). \end{aligned} \quad (6.41)$$

We turn to the second commutator in Eq. (6.22), i.e., $[2I_+ + I_S, I_-]$. To compute it, we extract the color-correlated contributions to I_S . Proceeding along the same lines as in the derivation of $I_\pm^{(\text{cc})}$, we obtain

$$I_S^{(\text{cc})}(\epsilon) = \sum_{(ij)} (\mathbf{T}_i \cdot \mathbf{T}_j) \left[\frac{\log \eta_{ij}}{\epsilon} + \phi_{ij} + \epsilon \psi_{ij} \right] + \mathcal{O}(\epsilon^2), \quad (6.42)$$

where

$$\begin{aligned} \phi_{ij} &\stackrel{\text{def}}{=} -2L_{\max} \log \eta_{ij} - \frac{1}{2} \log^2 \eta_{ij} - \text{Li}_2(1 - \eta_{ij}), \\ \psi_{ij} &\stackrel{\text{def}}{=} 2L_{\max}^2 \log \eta_{ij} + \left(L_{\max} - \frac{1}{2} \log(1 - \eta_{ij}) \right) \log^2 \eta_{ij} + \frac{1}{6} \log^3 \eta_{ij} \\ &\quad + 2L_{\max} \text{Li}_2(1 - \eta_{ij}) - \text{Li}_3(1 - \eta_{ij}) - \text{Li}_3(\eta_{ij}), \end{aligned} \quad (6.43)$$

where we recall that $L_{\max} = \log(2E_{\max}/\mu)$. Note that $\phi_{ij} = \phi_{ji}$ and $\psi_{ij} = \psi_{ji}$. Then, we sum $I_S^{(\text{cc})}$ with $2I_+^{(\text{cc})}$, getting

$$2I_+^{(\text{cc})} + I_S^{(\text{cc})} = \sum_{(ij)} (\mathbf{T}_i \cdot \mathbf{T}_j) \left[\frac{L_{ij} + \log \eta_{ij}}{\epsilon} + (\delta_{ij}^+ + \phi_{ij}) + \epsilon(\xi_{ij}^+ + \psi_{ij}) \right] + \mathcal{O}(\epsilon^2). \quad (6.44)$$

However, calling $\tilde{L}_i = \log(2E_i/\mu)$, we have

$$L_{ij} + \log \eta_{ij} = -\tilde{L}_i - \tilde{L}_j, \quad (6.45)$$

which implies, using color conservation, that the first term on the right-hand side of (6.44) is not color-correlated, and thus can be neglected. Therefore, we keep the following expression:

$$2I_+^{(\text{cc})} + I_S^{(\text{cc})} = \sum_{(ij)} (\mathbf{T}_i \cdot \mathbf{T}_j) \left[(\delta_{ij}^+ + \phi_{ij}) + \epsilon(\xi_{ij}^+ + \psi_{ij}) \right] + \mathcal{O}(\epsilon^2). \quad (6.46)$$

With this representation at hand, by using Eq. (6.34) we can calculate the second color-

correlated commutator required in Eq. (6.22), i.e.,

$$\begin{aligned} [2I_+^{(\text{cc})} + I_S^{(\text{cc})}, I_-^{(\text{cc})}] = & -\frac{\pi}{2} \sum_{(ijk)} F^{(kij)} \left[\frac{2\lambda_{ij}}{\epsilon} (\delta_{jk}^+ + \delta_{kj}^+ + 2\phi_{jk}) \right. \\ & + (\delta_{ij}^- + \delta_{ji}^-) (\delta_{jk}^+ + \delta_{kj}^+ + 2\phi_{jk}) + 2\lambda_{ij} (\xi_{jk}^+ + \xi_{kj}^+ + 2\psi_{jk}) \left. \right] \\ & + \mathcal{O}(\epsilon). \end{aligned} \quad (6.47)$$

The last contribution to be analyzed in Eq. (6.22) is $\mathcal{H}_{2,\text{tc}} + \mathcal{H}_{2,\text{tc}}^\dagger$. In Subsec. 2.2.2, we quoted that $\mathcal{H}_{2,\text{tc}}$ can be written as [150, 151]

$$\mathcal{H}_{2,\text{tc}} = \frac{1}{2\epsilon} [\Gamma, C], \quad (6.48)$$

where Γ and C are defined as

$$\Gamma \stackrel{\text{def}}{=} \frac{1}{2} \sum_{(ij)} \mathbf{T}_i \cdot \mathbf{T}_j (L_{ij} + i\pi\lambda_{ij}), \quad C \stackrel{\text{def}}{=} \frac{1}{2} \sum_{(ij)} \mathbf{T}_i \cdot \mathbf{T}_j (\delta_{ij}^+ + i\pi\delta_{ij}^-), \quad (6.49)$$

Once again using Eq. (6.34), we express $\mathcal{H}_{2,\text{tc}}$ as

$$\mathcal{H}_{2,\text{tc}} = \frac{i}{4\epsilon} \sum_{(ijk)} F^{(kij)} (L_{jk} + i\pi\lambda_{jk}) \left[(\delta_{ij}^+ + \delta_{ji}^+) + i\pi(\delta_{ij}^- + \delta_{ji}^-) \right], \quad (6.50)$$

thus the sum $\mathcal{H}_{2,\text{tc}} + \mathcal{H}_{2,\text{tc}}^\dagger$ gives

$$\mathcal{H}_{2,\text{tc}} + \mathcal{H}_{2,\text{tc}}^\dagger = -\frac{\pi}{2\epsilon} \sum_{(ijk)} F^{(kij)} \left[L_{jk} (\delta_{ij}^- + \delta_{ji}^-) + \lambda_{jk} (\delta_{ij}^+ + \delta_{ji}^+) \right]. \quad (6.51)$$

We can now combine the three triple color-correlated terms in Eqs (6.41), (6.47) and (6.51) to obtain the final expression for the operator $I_{\text{tri}}^{(\text{cc})}$ in Eq. (6.22), which reads

$$\begin{aligned} I_{\text{tri}}^{(\text{cc})}(\epsilon) = & \frac{\pi}{2} \sum_{(ijk)} F^{(kij)} \left[\frac{2\lambda_{ij} L_{kj}}{\epsilon^2} - \frac{4\lambda_{ij} \phi_{jk}}{\epsilon} + \frac{1}{2} (\delta_{ij}^- + \delta_{ji}^-) (\delta_{jk}^+ + \delta_{kj}^+ + 4\phi_{jk}) \right. \\ & \left. + L_{jk} (\xi_{ij}^- + \xi_{ji}^-) - \lambda_{ij} (\xi_{jk}^+ + \xi_{kj}^+ + 4\psi_{jk}) \right] + \mathcal{O}(\epsilon). \end{aligned} \quad (6.52)$$

The calculation of $\Sigma_{\text{DU}}^{\text{tri}}|_{\mathcal{B}_0}$ in Eq. (6.21) requires us to compute $I_{\text{tri}}^{\text{RV}}$ up to finite terms in ϵ . Such a calculation is non-trivial; we describe it in Appendix H, and the result for $I_{\text{tri}}^{\text{RV}}$ is given in Eq. (H.9). Once $I_{\text{tri}}^{\text{RV}}$ is computed, it is then possible to show that $\Sigma_{\text{DU}}^{\text{tri}}|_{\mathcal{B}_0}$ is free of ϵ -poles. To do it, we need to rewrite Eq. (6.52) to make the role of the factors λ_{ij} clearer. We recall that λ_{ij} are phase factors that distinguish between time-like and space-like processes. In fact, $\lambda_{ij} = 1$ if partons i and j are both either incoming or outgoing, and zero otherwise. Furthermore, important simplifications in Eq. (6.52) occur because ϕ_{jk} and L_{jk} are symmetric and $F_{\text{LM}}^{(kij)}$ is antisymmetric with respect to $k \leftrightarrow j$ exchange.

Thus, for a process with only outgoing (or only incoming) partons, we have $\lambda_{ij} = 1 \forall i, j$, and hence the triple color-correlated poles in Eq. (6.52) vanish. A similar analysis shows that ϵ -poles in $I_{\text{tri}}^{\text{RV}}$ also vanish if all resolved partons are in the final state.

To understand what happens in processes where both incoming and outgoing partons are present, it is convenient to write λ_{ij} in the following way:

$$\lambda_{ij} = 1 - \delta_{ia} - \delta_{ib} - \delta_{ja} - \delta_{jb} + 2\delta_{ia}\delta_{jb} + 2\delta_{ib}\delta_{ja}, \quad (6.53)$$

where a and b label initial-state partons. We have already argued that the first term on the right-hand side provides a vanishing contribution to Eq. (6.52). Terms in Eq. (6.53) that depend on the index i only also do not contribute since they do not break the $k \leftrightarrow j$ (anti)symmetry. The terms that depend on the index j also vanish. Indeed, we have

$$\begin{aligned} \sum_{(ijk)} F^{(kij)} \mathcal{A}_{kj} \mathcal{C}_j &= \sum_{(jk)} \sum_{\substack{i \in \mathcal{H} \\ i \neq j, k}} f_{abc} \mathcal{A}_{kj} \mathcal{C}_j T_k^a T_i^b T_j^c \\ &= - \sum_{(jk)} f_{abc} \mathcal{A}_{kj} \mathcal{C}_j T_k^a T_j^c (T_j^b + T_k^b) \\ &= 0, \end{aligned} \quad (6.54)$$

where $\mathcal{A}_{kj}$ stands for L_{kj} or $\phi_{kj} + \phi_{jk}$, and $\mathcal{C}_j$ is any object that depends solely on the index j . Note that, to move from the first to the second line, we have used color conservation

$$\sum_{\substack{i \in \mathcal{H} \\ i \neq j, k}} T_i^b = -(T_j^b + T_k^b), \quad (6.55)$$

and, in addition, the identity (6.37) to prove that the second line vanishes. Finally, in Eq. (6.52), we write L_{jk} as in Eq. (6.45) and, using the same arguments as above, we see that the terms proportional to $\tilde{L}_j$ and $\tilde{L}_k$ do not contribute to $I_{\text{tri}}^{(\text{cc})}$. The only pieces that remain include $\log \eta_{jk}$ and the final two terms of Eq. (6.53). Combining all these results, we finally arrive at an expression for the triple color-correlated poles:

$$\begin{aligned} I_{\text{tri}}^{(\text{cc})}(\epsilon) &= - \sum_{k \in \mathcal{H}_f} F_{\text{LM}}^{(kab)} \left\{ \frac{2\pi}{\epsilon^2} \log \frac{\eta_{bk}}{\eta_{ak}} + \frac{2\pi}{\epsilon} \left[\log^2 \eta_{ak} - \log^2 \eta_{bk} + 2L_{\text{max}}^2 \log \left(\frac{\eta_{ak}}{\eta_{bk}} \right) \right. \right. \\ &\quad \left. \left. + 2\text{Li}_2(1 - \eta_{ak}) - 2\text{Li}_2(1 - \eta_{bk}) \right] \right\} + \mathcal{O}(\epsilon^0). \end{aligned} \quad (6.56)$$

Comparing this result with the expression for $I_{\text{tri}}^{\text{RV}}$ in Eq. (H.15), we find that their poles are equal and opposite in sign.⁶ This verifies the cancellation of ϵ -poles in triple color-correlated contributions for process $\mathcal{B}_0$.

⁶One could equally well understand this as the $\mathcal{O}(\epsilon^{-2})$ poles cancelling between $I_{\text{tri}}^{\text{RV}}$ and the commutator terms of $I_{\text{tri}}^{(\text{cc})}$ in Eq. (6.22), leaving a simple pole which cancels against the contribution originating from the double-virtual amplitude in Eq. (6.51).

6.3 Other double-unresolved color-correlated terms

We continue with the discussion of divergent *elastic* (or *unboosted*) contributions to the double-unresolved piece $\Sigma_{\text{DU}}|_{\mathcal{B}_0}$ that contain *double* and *quartic color-correlated* matrix elements squared. As these contributions must involve either a loop amplitude or a soft limit, we are interested in those terms in Eqs (4.128), (5.7), (5.20), (5.31), (5.44), (5.56), that contain either I_V or I_S operators, or both. We denote such a collection of terms as $\Sigma_{\text{DU}}^{\text{el,inc}}|_{\mathcal{B}_0}$, where the term *inc* stands for *incomplete*, because many other elastic contributions are missing. Additionally, since we aim to reconstruct the operator I_T , it is worth considering the terms proportional to I_C and $\frac{1}{2}I_C^2$ in Eqs (5.44) and (5.85), respectively, even though they are not color-correlated. We obtain

$$\begin{aligned} \Sigma_{\text{DU}}^{\text{el,inc}}|_{\mathcal{B}_0} &\stackrel{\text{def}}{=} [\alpha_s]^2 \left\langle \left[\frac{1}{2} I_T^2(\epsilon) + \tilde{c}_\epsilon K I_V(2\epsilon) \right] \cdot F_{\text{LM}}^{\mathcal{B}_0} \right\rangle \\ &+ [\alpha_s]^2 \frac{\beta_0}{\epsilon} \left\langle \left[\tilde{c}_\epsilon I_V(2\epsilon) + c_2(\epsilon) \tilde{I}_S(2\epsilon) - \tilde{c}_\epsilon I_T(\epsilon) \right] \cdot F_{\text{LM}}^{\mathcal{B}_0} \right\rangle \\ &+ [\alpha_s]^2 \left\langle \left[C_A \left(\frac{c_1(\epsilon)}{\epsilon^2} - \frac{A_K(\epsilon)}{\epsilon^2} \right) + \beta_0 c_3(\epsilon) \right] \tilde{I}_S(2\epsilon) \cdot F_{\text{LM}}^{\mathcal{B}_0} \right\rangle \\ &+ [\alpha_s]^2 2^{1+2\epsilon} \delta_g(\epsilon) \left(\frac{2E_{\text{max}}}{\mu} \right)^{-2\epsilon} \left[- \langle I_S(\epsilon) \cdot F_{\text{LM}}^{\mathcal{B}_0} \rangle \right. \\ &\quad \left. + \frac{(2E_{\text{max}}/\mu)^{-2\epsilon}}{2\epsilon^2} N_c(\epsilon) \sum_{i \in \mathcal{H}} \mathbf{T}_i^2 \langle F_{\text{LM}}^{\mathcal{B}_0} \rangle \right], \end{aligned} \quad (6.57)$$

where

$$\tilde{c}_\epsilon \stackrel{\text{def}}{=} \frac{\Gamma(1-\epsilon)}{e^{\epsilon\gamma_E}} = 1 + \frac{\pi^2}{12} \epsilon^2 + \mathcal{O}(\epsilon^3), \quad (6.58)$$

the expressions of the objects K , $c_{1,2,3}(\epsilon)$, $A_K(\epsilon)$, $N_c(\epsilon)$ are listed in Appendix A.1, and $\delta_g(\epsilon)$ is reported in Eq. (A.65). As mentioned, the individual terms in Eq. (6.57) contain all the double and quartic color-correlated poles that appear in the cross section for the process $\mathcal{B}_0$ at NNLO. Our goal in this subsection is to show that these poles cancel, so that color-correlations only appear at $\mathcal{O}(\epsilon^0)$. To achieve this, we perform the following manipulations on Eq. (6.57).

In the first line on the right-hand side of Eq. (6.57), we rewrite the operator $\tilde{c}_\epsilon K I_V(2\epsilon)$ as

$$\begin{aligned} \tilde{c}_\epsilon K I_V(2\epsilon) &= \tilde{c}_\epsilon K [I_T(2\epsilon) - I_S(2\epsilon) - I_C(2\epsilon)] \\ &= K I_T^{(0)} - \tilde{c}_\epsilon K [I_S(2\epsilon) + I_C(2\epsilon)] + \mathcal{O}(\epsilon), \end{aligned} \quad (6.59)$$

where we have used

$$\tilde{c}_\epsilon K I_T(2\epsilon) = [1 + \mathcal{O}(\epsilon^2)] K [I_T^{(0)} + \mathcal{O}(\epsilon)] = K I_T^{(0)} + \mathcal{O}(\epsilon). \quad (6.60)$$

We recall that $I_T^{(n)}$, with $n = 0, 1, 2, \dots$, corresponds to the coefficient of ϵ^n in the expansion

sion of the operator $I_T(\epsilon)$ in powers of ϵ , i.e.,⁷

$$I_T(\epsilon) = \sum_{n=0}^{\infty} \epsilon^n I_T^{(n)}. \quad (6.61)$$

In the second line on the right-hand side of Eq. (6.57), we write

$$\begin{aligned} & \frac{\beta_0}{\epsilon} \left[\tilde{c}_\epsilon I_V(2\epsilon) + c_2(\epsilon) \tilde{I}_S(2\epsilon) - \tilde{c}_\epsilon I_T(\epsilon) \right] \\ &= \frac{\beta_0}{\epsilon} \left[(\tilde{c}_\epsilon - 1) I_V(2\epsilon) + c_2(\epsilon) \tilde{I}_S(2\epsilon) - I_S(2\epsilon) - I_C(2\epsilon) \right] + \frac{\beta_0}{\epsilon} [I_T(2\epsilon) - \tilde{c}_\epsilon I_T(\epsilon)], \end{aligned} \quad (6.62)$$

and we note that

$$\begin{aligned} & \frac{\beta_0}{\epsilon} [I_T(2\epsilon) - \tilde{c}_\epsilon I_T(\epsilon)] \\ &= \frac{\beta_0}{\epsilon} [I_T^{(0)} + 2\epsilon I_T^{(1)} - I_T^{(0)} - \epsilon I_T^{(1)} + \mathcal{O}(\epsilon^2)] \\ &= \beta_0 I_T^{(1)} + \mathcal{O}(\epsilon). \end{aligned} \quad (6.63)$$

In the third line on the right-hand side of Eq. (6.57), from the definitions of $A_K(\epsilon)$ and $c_{1,3}(\epsilon)$ given in Eqs (2.138) and (5.14), respectively, we note that

$$C_A \left(\frac{c_1(\epsilon) - A_K(\epsilon)}{\epsilon^2} \right) = \mathcal{O}(\epsilon^0), \quad c_3(\epsilon) = \mathcal{O}(\epsilon^0), \quad (6.64)$$

which implies that the operator $\tilde{I}_S(2\epsilon)$ can be replaced with $I_S(\epsilon)$, by virtue of the property

$$\tilde{I}_S(2\epsilon) = I_S(2\epsilon) + \mathcal{O}(\epsilon). \quad (6.65)$$

Finally, in the fourth line on the right-hand side of Eq. (6.57), we add and subtract the following quantity:

$$[\alpha_s]^2 2^{2+2\epsilon} \delta_g(\epsilon) \langle I_S(2\epsilon) \cdot F_{LM}^{B_0} \rangle. \quad (6.66)$$

All in all, Eq. (6.57) becomes (up to corrections of $\mathcal{O}(\epsilon)$)

$$\begin{aligned} & \Sigma_{DU}^{\text{el,inc}}|_{B_0} \stackrel{\text{def}}{=} [\alpha_s]^2 \left\langle \left[\frac{1}{2} (I_T^{(0)})^2 + K I_T^{(0)} + \beta_0 I_T^{(1)} \right] \cdot F_{LM}^{B_0} \right\rangle \\ & + [\alpha_s]^2 \frac{\beta_0}{\epsilon} \left\langle \left[(\tilde{c}_\epsilon - 1) I_V(2\epsilon) + c_2(\epsilon) \tilde{I}_S(2\epsilon) - I_S(2\epsilon) \right] \cdot F_{LM}^{B_0} \right\rangle \\ & + [\alpha_s]^2 \left[C_A \left(\frac{c_1(\epsilon)}{\epsilon^2} - \frac{A_K(\epsilon)}{\epsilon^2} \right) - \tilde{c}_\epsilon K + \beta_0 c_3(\epsilon) - 2^{2+2\epsilon} \delta_g(\epsilon) \right] \langle I_S(2\epsilon) \cdot F_{LM}^{B_0} \rangle \\ & + 2^{1+2\epsilon} [\alpha_s]^2 \delta_g(\epsilon) \left\langle \left[(2E_{\text{max}}/\mu)^{-2\epsilon} \left(\frac{(2E_{\text{max}}/\mu)^{-2\epsilon}}{2\epsilon^2} N_c(\epsilon) \sum_{i \in \mathcal{H}} \mathbf{T}_i^2 - I_S(\epsilon) \right) \right] \cdot F_{LM}^{B_0} \right\rangle \end{aligned} \quad (6.67)$$

⁷Note that, in the expansion of $I_T(k\epsilon)$, the coefficient $I_T^{(0)}$ is independent on the parameter k . This is why the $\mathcal{O}(\epsilon^0)$ of $I_T(2\epsilon)$ gives $I_T^{(0)}$ in Eq. (6.59).

$$+ 2I_S(2\epsilon) \left] \cdot F_{LM}^{\mathcal{B}_0} \right\rangle - [\alpha_s]^2 \frac{\beta_0}{\epsilon} \langle I_C(2\epsilon) \cdot F_{LM}^{\mathcal{B}_0} \rangle.$$

Let us analyze this formula. The first line on the right-hand side of Eq. (6.67) is explicitly of $\mathcal{O}(\epsilon^0)$. Regarding the second line, the combination of Eq. (6.65) and the relations

$$\tilde{c}_\epsilon - 1 = \mathcal{O}(\epsilon^2), \quad c_2(\epsilon) = 1 + \mathcal{O}(\epsilon^2), \quad (6.68)$$

yields⁸

$$\frac{\beta_0}{\epsilon} \left[(\tilde{c}_\epsilon - 1) I_V(2\epsilon) + c_2(\epsilon) \tilde{I}_S(2\epsilon) - I_S(2\epsilon) \right] \propto \frac{I_S^{(-2)}}{\epsilon} + \mathcal{O}(\epsilon^0), \quad (6.69)$$

where $I_S^{(-2)}$ is the coefficient of ϵ^{-2} in the expansion of the operator $I_S(\epsilon)$ in powers of ϵ . We recall that

$$I_S^{(-2)} = \sum_{i \in \mathcal{H}} \mathbf{T}_i^2, \quad (6.70)$$

thus the entire second line in Eq. (6.67), does not contain any singular terms with color-correlations. The same applies to the fourth line on the right-hand side of Eq. (6.67), since

$$\delta_g(\epsilon) = \mathcal{O}(\epsilon^0), \quad N_c(\epsilon) = 1 + \mathcal{O}(\epsilon^3), \quad (6.71)$$

which result in

$$(2E_{\max}/\mu)^{-2\epsilon} \left(\frac{(2E_{\max}/\mu)^{-2\epsilon}}{2\epsilon^2} N_c(\epsilon) \sum_{i \in \mathcal{H}} \mathbf{T}_i^2 - I_S(\epsilon) \right) + 2I_S(2\epsilon) = \mathcal{O}(\epsilon^0). \quad (6.72)$$

Next, the expression in square brackets in the third line on the right-hand side of Eq. (6.67) behaves as

$$C_A \left(\frac{c_1(\epsilon)}{\epsilon^2} - \frac{A_K(\epsilon)}{\epsilon^2} \right) - \tilde{c}_\epsilon K + \beta_0 c_3(\epsilon) - 2^{2+2\epsilon} \delta_g(\epsilon) = \mathcal{O}(\epsilon). \quad (6.73)$$

This term multiplies $I_S(2\epsilon)$, which is $\mathcal{O}(\epsilon^{-2})$. As a result, the pole in the third line is also proportional to $I_S^{(-2)}$, meaning it is free of color-correlations. Finally, the last term in the last line of Eq. (6.67) is proportional to I_C , which by definition does not contain color-correlations.

To summarize, the analyses performed in the current and previous sections prove the cancellation of *all* color-correlated divergent terms in a generic process $\mathcal{B}_0$. The remaining divergences in the double-unresolved contribution are *not* color-correlated and, instead, are proportional to the squares of color charges or anomalous dimensions of the external partons. These are related to collinear emissions and we continue with their analysis in the next section.

⁸Note that $I_S^{(-2)} = -I_V^{(-2)}$.

6.4 Analytic cancellation of poles in the double-unresolved terms

In this section, we provide a comprehensive summary of *all* the double-unresolved terms from Eq. (5.4) that we have computed throughout Ch. 4, Ch. 5, and the preceding sections of the current chapter. First, we gather a preliminary formula for Eq. (5.4), in which all the poles are still explicitly present. Subsequently, we will systematically discuss the analytic cancellation of all remaining poles that have not yet been explicitly cancelled (in other words, those poles multiplying matrix elements that do not include color-correlations). The cancellation of the $\mathcal{O}(\epsilon^{-3})$ and $\mathcal{O}(\epsilon^{-2})$ poles is performed using the singularity structures that we have identified, and we discuss this in some detail. On the other hand, the cancellation of the $\mathcal{O}(\epsilon^{-1})$ poles is performed analytically using *Mathematica*, and we do not present this in any depth. This will prepare the ground for the true final formula for the $\mathcal{B}_0$ process, which will be presented in the following subsection.

We express the contribution of Eq. (5.4), projected onto the Born-level process $\mathcal{B}_0$, as follows:

$$\begin{aligned} \mathcal{P}_{\mathcal{B}_0} & \left[\Sigma_{\text{DU}}|_{\mathcal{B}_0} + 2s_{ab} d\hat{\sigma}_{\mathcal{B}_2}^{\text{pdf}} + 2s_{ab} d\hat{\sigma}_{\tilde{\mathcal{B}}_2}^{\text{int}} \right] \\ & = \Sigma_{\text{DU}}^{\text{db}}|_{\mathcal{B}_0} + \Sigma_{\text{DU}}^{\text{sb},a}|_{\mathcal{B}_0} + \Sigma_{\text{DU}}^{\text{sb},b}|_{\mathcal{B}_0} + \Sigma_{\text{DU}}^{\text{tri}}|_{\mathcal{B}_0} + \Sigma_{\text{DU}}^{\text{el,pc}}|_{\mathcal{B}_0} + \Sigma_{\text{DU}}^{\text{el,r}}|_{\mathcal{B}_0}, \end{aligned} \quad (6.74)$$

where

- i) $\Sigma_{\text{DU}}^{\text{db}}|_{\mathcal{B}_0}$ contains all the *double-boosted* contributions;
- ii) $\Sigma_{\text{DU}}^{\text{sb},a}|_{\mathcal{B}_0}$ contains all the contributions that are *single-boosted* on leg a ;
- iii) $\Sigma_{\text{DU}}^{\text{sb},b}|_{\mathcal{B}_0}$ contains all the contributions that are *single-boosted* on leg b ;
- iv) $\Sigma_{\text{DU}}^{\text{tri}}|_{\mathcal{B}_0}$ is the contributions with *triple color-correlations* introduced in Eq. (6.21);
- v) $\Sigma_{\text{DU}}^{\text{el,pc}}|_{\mathcal{B}_0}$ contains all the *elastic pure collinear* contributions;
- vi) $\Sigma_{\text{DU}}^{\text{el,r}}|_{\mathcal{B}_0}$ contains all the *remaining elastic* contributions.

Below, we provide the formulas for these objects, where the condition $n_f = 0$ is understood:

$$\begin{aligned} \Sigma_{\text{DU}}^{\text{db}}|_{\mathcal{B}_0} & = \frac{[\alpha_s]^2}{\epsilon^2} \langle \mathcal{P}_{gg}^{\text{gen}} \otimes F_{\text{LM}}^{\mathcal{B}_0} \otimes \mathcal{P}_{qq}^{\text{gen}} \rangle + \frac{1}{\epsilon^2} \left[\frac{\alpha_s(\mu)}{2\pi} \right]^2 \langle \hat{P}_{gg}^{(0)} \otimes F_{\text{LM}}^{\mathcal{B}_0} \otimes \hat{P}_{qq}^{(0)} \rangle \\ & + \frac{[\alpha_s]}{\epsilon^2} \left[\frac{\alpha_s(\mu)}{2\pi} \right] \left[\langle \hat{P}_{gg}^{(0)} \otimes F_{\text{LM}} \otimes \mathcal{P}_{qq}^{\text{gen}} \rangle + \langle \mathcal{P}_{gg}^{\text{gen}} \otimes F_{\text{LM}} \otimes \hat{P}_{qq}^{(0)} \rangle \right], \end{aligned} \quad (6.75)$$

$$\Sigma_{\text{DU}}^{\text{sb},a}|_{\mathcal{B}_0} = \frac{[\alpha_s]^2}{2\epsilon^3} \left\langle \left[C_A h_c(\epsilon) [\mathcal{P}_{gg}^{(4),\text{gen}} - \mathcal{P}_{gg}^{1\text{L},\text{gen}}] + \epsilon G_{gg} \right] \otimes F_{\text{LM}}^{\mathcal{B}_0} \right\rangle$$

$$\begin{aligned}
& + \frac{1}{\epsilon^2} \left\langle \left\{ \frac{[\alpha_s]^2}{2} N_{\text{sc}}^{(b,d)} \gamma_{z,g \rightarrow gg}^{22}(0) \mathcal{P}_{gg}^{(4),\text{gen}} - [\alpha_s] \left[\frac{\alpha_s(\mu)}{2\pi} \right] \beta_0 \mathcal{P}_{gg}^{\text{gen}} - \left[\frac{\alpha_s(\mu)}{2\pi} \right]^2 \frac{\beta_0}{2} \hat{P}_{gg}^{(0)} \right\} \otimes F_{\text{LM}}^{\mathcal{B}_0} \right\rangle \\
& + \frac{1}{\epsilon^2} \left\langle \left\{ \frac{[\alpha_s]^2}{2} [\mathcal{P}_{gg}^{\text{gen}} \bar{\otimes} \mathcal{P}_{gg}^{\text{gen}}] + [\alpha_s] \left[\frac{\alpha_s(\mu)}{2\pi} \right] [\hat{P}_{gg}^{(0)} \bar{\otimes} \mathcal{P}_{gg}^{\text{gen}}] + \frac{1}{2} \left[\frac{\alpha_s(\mu)}{2\pi} \right]^2 [\hat{P}_{gg}^{(0)} \otimes \hat{P}_{gg}^{(0)}] \right\} \otimes F_{\text{LM}}^{\mathcal{B}_0} \right\rangle \\
& \dots \\
& + [\alpha_s]^2 \left[\frac{h_{\text{rdc}}(\epsilon)}{2\epsilon} + h_b(\epsilon) \right] \langle \mathcal{P}_{gg}^{\text{rdc}} \otimes F_{\text{LM}}^{\mathcal{B}_0} \rangle + \frac{[\alpha_s]^2}{4\epsilon} N_{\text{sc}}^{(b,d)} \left\langle \left[\gamma_{\perp,g \rightarrow gg}^{22,\text{r}} \mathcal{P}_{gg}^{(4),\text{gen}} + \gamma_{\perp,g \rightarrow gg}^{22} \mathcal{P}_{gg}^{\text{sc}} \right] \otimes F_{\text{LM}}^{\mathcal{B}_0} \right\rangle \\
& + \frac{1}{\epsilon} \left\langle \left\{ \left[\frac{\alpha_s(\mu)}{2\pi} \right] \hat{P}_{gg}^{(0)} + [\alpha_s] \mathcal{P}_{gg}^{\text{gen}} \right\} \otimes \left[I_{\text{T}}(\epsilon) \cdot ([\alpha_s] F_{\text{LM}}^{\mathcal{B}_0} + F_{\text{LV,fin}}^{\mathcal{B}_0}) \right] \right\rangle \\
& + \left[\frac{\alpha_s(\mu)}{2\pi} \right]^2 \frac{1}{2\epsilon} \langle \hat{P}_{gg}^{(1)} \otimes F_{\text{LM}}^{\mathcal{B}_0} \rangle + [\alpha_s]^2 \langle E_a^{-2\epsilon} \text{ISR}_2 \otimes F_{\text{LM}}^{\mathcal{B}_0} \rangle + [\alpha_s]^2 \langle \mathcal{P}_{gg}^{\mathcal{W}} \otimes [\mathcal{W}_a^{\text{a}||\text{n,fin}} \cdot F_{\text{LM}}^{\mathcal{B}_0}] \rangle, \tag{6.76}
\end{aligned}$$

$$\begin{aligned}
\Sigma_{\text{DU}}^{\text{el,pc}}|_{\mathcal{B}_0} &= \frac{[\alpha_s]^2}{\epsilon^2} \left\langle \left\{ C_A h_c(\epsilon) [I_C^{(4)}(\epsilon) - \tilde{I}_C(2\epsilon) + \sigma_{q \rightarrow gg}] + \frac{1}{2} \left[\sum_{i \in \mathcal{H}_{\text{f},g}} G_i|_{z,g \rightarrow gg}^{z,g \rightarrow gg} + (G|_{z,q \rightarrow qq}^{1,q \rightarrow qq} \right. \right. \right. \\
& + \left. \left. \left. G|_{z,q \rightarrow gg}^{z,g \rightarrow gg} + \Gamma_{q \rightarrow gg} (\Gamma_{g \rightarrow gg} - \Gamma_q) \right] \right\} + \frac{N_{\text{sc}}^{(b,d)}}{2} [\gamma_{1,q \rightarrow qq}^{22}(L) \Gamma_{q \rightarrow gg}^{(4)} + H_{q \rightarrow gg}] \right\} \cdot F_{\text{LM}}^{\mathcal{B}_0} \rangle \\
& + \frac{[\alpha_s]^2}{\epsilon} \left\langle \left[N_{\text{sc}}^{(b,d)} \gamma_{z,g \rightarrow gg}^{22}(0) \left(I_C^{(4)}(\epsilon) - \frac{\Gamma_{q \rightarrow gg}^{(4)}}{2\epsilon} \right) - \beta_0 I_C(2\epsilon) \right] \cdot F_{\text{LM}}^{\mathcal{B}_0} \right\rangle \\
& \dots \\
& + [\alpha_s]^2 \left[\frac{h_{\text{rdc}}(\epsilon)}{2\epsilon} + h_b(\epsilon) \right] \langle R^{\text{rdc}}(\epsilon) \cdot F_{\text{LM}}^{\mathcal{B}_0} \rangle - [\alpha_s]^2 \tilde{c}_\epsilon K \langle I_C(2\epsilon) \cdot F_{\text{LM}}^{\mathcal{B}_0} \rangle \\
& + [\alpha_s]^2 N_{\text{sc}}^{(b,d)} \left\langle \left[\frac{\gamma_{\perp,g \rightarrow gg}^{22,\text{r}}}{2} \left(I_C^{(4)}(\epsilon) - \frac{\Gamma_{q \rightarrow gg}^{(4)}}{2\epsilon} \right) + \frac{\gamma_{\perp,g \rightarrow gg}^{22}}{4\epsilon} \sum_{i \in \mathcal{H}_{\text{f}}} \Lambda_{i,f_i \rightarrow f_i,g}^{\text{sc}} \right] \cdot F_{\text{LM}}^{\mathcal{B}_0} \right\rangle \\
& + [\alpha_s]^2 \left\langle \left[\sum_{i \in \mathcal{H}_{\text{f},g}} (E_i/\mu)^{-4\epsilon} \text{FSR}_5 + (E/\mu)^{-4\epsilon} [\widetilde{\text{FSR}}_1 + \text{FSR}_3] \right] \cdot F_{\text{LM}}^{\mathcal{B}_0} \right\rangle, \tag{6.77}
\end{aligned}$$

$$\begin{aligned}
\Sigma_{\text{DU}}^{\text{el,r}}|_{\mathcal{B}_0} &= [\alpha_s]^2 \frac{\beta_0}{\epsilon} \left\langle \left[(\tilde{c}_\epsilon - 1) I_V(2\epsilon) + c_2(\epsilon) \tilde{I}_S(2\epsilon) - I_S(2\epsilon) \right] \cdot F_{\text{LM}}^{\mathcal{B}_0} \right\rangle \\
& + [\alpha_s]^2 \left[C_A \left(\frac{c_1(\epsilon)}{\epsilon^2} - \frac{A_K(\epsilon)}{\epsilon^2} \right) - \tilde{c}_\epsilon K + \beta_0 c_3(\epsilon) + 2^{2+2\epsilon} \delta_g(\epsilon) \right] \langle I_S(2\epsilon) \cdot F_{\text{LM}}^{\mathcal{B}_0} \rangle \\
& + 2^{1+2\epsilon} [\alpha_s]^2 \delta_g(\epsilon) \left\langle \left[(2E_{\text{max}}/\mu)^{-2\epsilon} \left(\frac{(2E_{\text{max}}/\mu)^{-2\epsilon}}{2\epsilon^2} N_c(\epsilon) \sum_{i \in \mathcal{H}} \mathbf{T}_i^2 - I_S(\epsilon) \right) + 2I_S(2\epsilon) \right] \cdot F_{\text{LM}}^{\mathcal{B}_0} \right\rangle \\
& + [\alpha_s]^2 \left\langle \left[\frac{1}{2} (I_{\text{T}}^{(0)})^2 + K I_{\text{T}}^{(0)} + \beta_0 I_{\text{T}}^{(1)} \right] \cdot F_{\text{LM}}^{\mathcal{B}_0} \right\rangle + [\alpha_s] \langle I_{\text{T}}^{(0)} \cdot F_{\text{LV,fin}}^{\mathcal{B}_0} \rangle \\
& + [\alpha_s]^2 \sum_{i \in \mathcal{H}} \left\langle \left[\bar{\delta}_{ia} \bar{\delta}_{ib} \gamma_{z,f_i \rightarrow f_i,g}^{\mathcal{W}} \mathcal{W}_i^{i||\text{n,fin}} + \delta_g^{(0)} \mathcal{W}_i^{\text{m}||\text{n,fin}} + \delta_g^\perp \mathcal{W}_i^{(i)} \right] \cdot F_{\text{LM}}^{\mathcal{B}_0} \right\rangle \\
& + \langle \Sigma_{\text{mn}} \Theta_{\text{mn}} F_{\text{LM}}^{\mathcal{B}_2} [\mathbf{m}_g, \mathbf{n}_g] \rangle_{T^2}^{\text{fin}} + \langle F_{\text{LV,fin}}^{\mathcal{B}_0} \rangle + \langle F_{\text{VV,fin}}^{\mathcal{B}_0} \rangle. \tag{6.78}
\end{aligned}$$

The dotted lines in Eqs (6.76) and (6.77) separate the contributions that are potentially of $\mathcal{O}(\epsilon^{-3})$ and $\mathcal{O}(\epsilon^{-2})$ (above) from those that are explicitly of $\mathcal{O}(\epsilon^{-1})$ or $\mathcal{O}(\epsilon^0)$ (below).

Before discussing the cancellation of the poles, a few comments are in order. First, we have not provided the explicit expression for $\Sigma_{\text{DU}}^{\text{sb},b}|_{\mathcal{B}_0}$, as it is identical to $\Sigma_{\text{DU}}^{\text{sb},a}|_{\mathcal{B}_0}$, up to the replacement of all the gg splitting functions with qq splitting functions, and the triple-

collinear splitting function ISR_2 with ISR_1 .⁹ Next, as mentioned, the definition of $\Sigma_{\text{DU}}^{\text{tri}}|_{\mathcal{B}_0}$ can be found in Eq. (6.21) so we do not repeat it here. Finally, the fourth and fifth lines of Eq. (6.78) have already been expanded in powers of ϵ , and the terms shown correspond to contributions of $\mathcal{O}(\epsilon^0)$. We also note that $\tilde{\delta}_{ij} = 1 - \delta_{ij}$, and that $\delta_g^{(0)}$ corresponds to the $\mathcal{O}(\epsilon^0)$ coefficient of $\delta_g(\epsilon)$.

As indicated earlier, we can systematically demonstrate how cancellations occur for poles of $\mathcal{O}(\epsilon^{-3})$ and $\mathcal{O}(\epsilon^{-2})$ without resorting to cumbersome algebra. We will do so for Eqs (6.75) and (6.76) in the next subsection, and for Eq. (6.77) in the subsection thereafter. As already shown, the $\Sigma_{\text{DU}}^{\text{tri}}|_{\mathcal{B}_0}$ is ϵ -finite, while $\Sigma_{\text{DU}}^{\text{el,r}}|_{\mathcal{B}_0}$ contains poles of $\mathcal{O}(\epsilon^{-1})$. We will briefly comment on the cancellation of all $\mathcal{O}(\epsilon^{-1})$ poles at the end of this section.

6.4.1 Double- and single-boosted contributions

We begin with the double-boosted contribution $\Sigma_{\text{DU}}^{\text{db}}|_{\mathcal{B}_0}$ in Eq. (6.75). In this case, the cancellation mechanism is entirely analogous to that already demonstrated at NLO in Ch. 3, only it applies to both initial-state legs simultaneously. Therefore, we employ the property

$$\left[\frac{\alpha_s(\mu)}{2\pi} \right] = [\alpha_s] + \mathcal{O}(\epsilon^2) \quad (6.79)$$

and the following general expansion:

$$\left[\frac{\alpha_s(\mu)}{2\pi} \right] \hat{P}_{\alpha\beta}^{(0)}(z) + [\alpha_s] \mathcal{P}_{\alpha\beta}^{\text{gen}}(z, E_a) = [\alpha_s] \epsilon \mathcal{P}_{\alpha\beta}^{\text{NLO}}(z, E_a) + \mathcal{O}(\epsilon^2), \quad (6.80)$$

where the functions $\mathcal{P}_{\alpha\beta}^{\text{NLO}}$ are defined in Eq. (I.3). We then obtain

$$\Sigma_{\text{DU}}^{\text{db}}|_{\mathcal{B}_0} = [\alpha_s]^2 \langle \mathcal{P}_{gg}^{\text{NLO}} \otimes F_{\text{LM}}^{\mathcal{B}_0} \otimes \mathcal{P}_{qq}^{\text{NLO}} \rangle. \quad (6.81)$$

which is clearly ϵ -finite.

We proceed to examine the cancellation of singularities of $\mathcal{O}(\epsilon^{-3})$ and $\mathcal{O}(\epsilon^{-2})$ in the expression for $\Sigma_{\text{DU}}^{\text{sb},a}|_{\mathcal{B}_0}$ in Eq. (6.76). Since the cancellations occur similarly for both the $\Sigma_{\text{DU}}^{\text{sb},a}|_{\mathcal{B}_0}$ and $\Sigma_{\text{DU}}^{\text{sb},b}|_{\mathcal{B}_0}$ terms, we will refer generically to the $\alpha\alpha$ splitting, where $\alpha = g$ for the former and $\alpha = q$ for the latter. We begin with the first line of Eq. (6.76). Using the definitions of $\mathcal{P}_{\alpha\beta}^{(4),\text{gen}}$ and $\mathcal{P}_{\alpha\beta}^{1\text{L},\text{gen}}$ given in Eqs (A.57) and (A.76), respectively, and expanding them up to $\mathcal{O}(\epsilon)$, we obtain

$$\mathcal{P}_{\alpha\alpha}^{(k),\text{gen}}(z, E) = -\hat{P}_{\alpha\alpha}^{(0)}(z) + \mathcal{O}(\epsilon), \quad \mathcal{P}_{\alpha\alpha}^{1\text{L},\text{gen}}(z, E) = -\hat{P}_{\alpha\alpha}^{(0)}(z) + \mathcal{O}(\epsilon), \quad (6.82)$$

which implies that the $\mathcal{O}(\epsilon^{-3})$ pole in Eq. (6.75) vanishes. Furthermore, by expanding both the above two functions and the object $\epsilon G_{\alpha\alpha}(z, E)$ up to $\mathcal{O}(\epsilon^2)$, which gives

$$\begin{aligned} C_A h_c(\epsilon) [\mathcal{P}_{\alpha\alpha}^{(4),\text{gen}}(z, E) - \mathcal{P}_{\alpha\alpha}^{1\text{L},\text{gen}}(z, E)] &= 2\mathbf{T}_\alpha^2 \epsilon \log(z) \hat{P}_{\alpha\alpha}^{(0)}(z) + \mathcal{O}(\epsilon^2), \\ \epsilon G_{\alpha\alpha}(z, E) &= -2\mathbf{T}_\alpha^2 \epsilon \log(z) \hat{P}_{\alpha\alpha}^{(0)}(z) + \mathcal{O}(\epsilon^2), \end{aligned} \quad (6.83)$$

⁹We recall that $\text{ISR}_2(z, E_a)$ and $\text{ISR}_1(z, E_b)$ were introduced in Ref. [79], and they refer to the $g \rightarrow ggg^*$ and $q \rightarrow gqg^*$ triple-collinear splittings, respectively.

we find that the $\mathcal{O}(\epsilon^{-2})$ pole in the first line of Eq. (6.76) also vanishes. Next, we consider the second line on the right-hand side of Eq. (6.76). In this case, we employ the relations in Eqs (6.79) and (6.82), along with the expansions

$$\gamma_{z,g \rightarrow gg}^{22}(0) = \frac{11}{6} C_A + \mathcal{O}(\epsilon), \quad N_{sc}^{(b,d)} = 1 + \mathcal{O}(\epsilon), \quad (6.84)$$

and the fact that $\beta_0 = \frac{11}{6} C_A$ in our setup, to observe that the $\mathcal{O}(\epsilon^{-2})$ pole in the second line vanishes. Finally, we analyze the third line on the right-hand side of Eq. (6.76). First, using the definitions of the $\bar{\otimes}$ - and $\otimes$ -convolutions given in Eqs (5.69) and (A.34), respectively, we write

$$[\hat{P}_{\alpha\beta}^{(0)} \otimes \hat{P}_{\gamma\delta}^{(0)}](z) = [\hat{P}_{\alpha\beta}^{(0)} \bar{\otimes} \hat{P}_{\gamma\delta}^{(0)}](z) + \mathcal{O}(\epsilon), \quad (6.85)$$

since

$$[\hat{P}_{\alpha\beta}^{(0)} \otimes \hat{P}_{\gamma\delta}^{(0)}](z) - [\hat{P}_{\alpha\beta}^{(0)} \bar{\otimes} \hat{P}_{\gamma\delta}^{(0)}](z) = \underbrace{\int_z^1 dt (1 - t^{-2\epsilon}) \hat{P}_{\alpha\beta}^{(0)}(t) \hat{P}_{\gamma\delta}^{(0)}(z/t)}_{\mathcal{O}(\epsilon)} = \mathcal{O}(\epsilon). \quad (6.86)$$

Then, we substitute Eq. (6.85) into the third line of Eq. (6.76), and using the expansions (6.79), (6.80) and (6.82), we find

$$\begin{aligned} & \frac{[\alpha_s]^2}{2\epsilon^2} [\mathcal{P}_{\alpha\beta}^{\text{gen}} \bar{\otimes} \mathcal{P}_{\gamma\delta}^{\text{gen}}] + \frac{[\alpha_s]}{\epsilon^2} \left[\frac{\alpha_s(\mu)}{2\pi} \right] [\hat{P}_{\alpha\beta}^{(0)} \bar{\otimes} \mathcal{P}_{\gamma\delta}^{\text{gen}}] + \frac{1}{2\epsilon^2} \left[\frac{\alpha_s(\mu)}{2\pi} \right]^2 [\hat{P}_{\alpha\beta}^{(0)} \otimes \hat{P}_{\gamma\delta}^{(0)}] \\ &= \frac{[\alpha_s]}{\epsilon^2} \left[\left(\frac{[\alpha_s]}{2} \mathcal{P}_{\alpha\beta}^{\text{gen}} + \left[\frac{\alpha_s(\mu)}{2\pi} \right] \hat{P}_{\alpha\beta}^{(0)} \right) \bar{\otimes} \mathcal{P}_{\gamma\delta}^{\text{gen}} \right] + \frac{[\alpha_s]^2}{2\epsilon^2} [\hat{P}_{\alpha\beta}^{(0)} \bar{\otimes} \hat{P}_{\gamma\delta}^{(0)}] + \mathcal{O}(\epsilon^{-1}) \quad (6.87) \\ &= \frac{[\alpha_s]^2}{\epsilon^2} \left[\left(\frac{1}{2} \hat{P}_{\alpha\beta}^{(0)} + \mathcal{O}(\epsilon) \right) \bar{\otimes} \left(-\hat{P}_{\gamma\delta}^{(0)} + \mathcal{O}(\epsilon) \right) \right] + \frac{[\alpha_s]^2}{2\epsilon^2} [\hat{P}_{\alpha\beta}^{(0)} \bar{\otimes} \hat{P}_{\gamma\delta}^{(0)}] + \mathcal{O}(\epsilon^{-1}) \\ &= \mathcal{O}(\epsilon^{-1}), \end{aligned}$$

for any flavor-pair $\alpha\beta$ and $\gamma\delta$. Consequently, also the third line of Eq. (6.76) is of $\mathcal{O}(\epsilon^{-1})$.

Finally, as mentioned above, the last three lines on the right-hand side of Eq. (6.76) are at most of $\mathcal{O}(\epsilon^{-1})$, since all the objects therein are by definition of $\mathcal{O}(\epsilon^0)$, with the exception of ISR_2 (or ISR_1 in the qq case), which is of $\mathcal{O}(\epsilon^{-1})$ but does not have an explicit $1/\epsilon$ factor in front of it. Therefore, we conclude that the $\mathcal{O}(\epsilon^{-2})$ poles in the single-boosted contributions $\Sigma_{\text{DU}}^{\text{sb},a}|_{\mathcal{B}_0}$ and $\Sigma_{\text{DU}}^{\text{sb},b}|_{\mathcal{B}_0}$ vanish. Once again, we have demonstrated this in a fairly simple way, and we defer the discussion of the remaining $1/\epsilon$ poles to the end of the section.

6.4.2 Elastic pure collinear contribution

Here we discuss the cancellation of the poles of $\mathcal{O}(\epsilon^{-3})$ and $\mathcal{O}(\epsilon^{-2})$ in the elastic pure collinear contribution $\Sigma_{\text{DU}}^{\text{el,pc}}|_{\mathcal{B}_0}$ in Eq. (6.77).

We begin with the first and second lines on the right-hand side of Eq. (6.77), which

seemingly contain poles of $\mathcal{O}(\epsilon^{-3})$ owing to the operators $I_C^{(4)}$ and $\tilde{I}_C$. Indeed, while these are of $\mathcal{O}(\epsilon^{-1})$, all the remaining objects are, by construction of $\mathcal{O}(\epsilon^0)$. In this regard, we recall the definitions of the collinear operators:

$$I_C^{(k)}(\epsilon) \stackrel{\text{def}}{=} \sum_{i=a,b} \frac{\Gamma_{i,f_i}^{(k)}}{\epsilon k/2} + \sum_{i \in \mathcal{H}_{f,g}} \frac{\Gamma_{i,g \rightarrow gg}^{(k)}}{\epsilon k/2} + \sum_{i \in \mathcal{H}_{f,q}} \frac{\Gamma_{i,q}^{(k)}}{\epsilon k/2}, \quad (6.88)$$

$$\tilde{I}_C(2\epsilon) \stackrel{\text{def}}{=} \sum_{i=a,b} \frac{\Gamma_{i,f_i}^{1L}}{2\epsilon} + \sum_{i \in \mathcal{H}_{f,g}} \frac{\Gamma_{i,g \rightarrow gg}^{1L}}{2\epsilon} + \sum_{i \in \mathcal{H}_{f,q}} \frac{\Gamma_{i,q}^{1L}}{2\epsilon}, \quad (6.89)$$

where the expressions of the objects within $I_C^{(k)}(\epsilon)$ and $\tilde{I}_C(2\epsilon)$ are reported in Eqs (A.56), (A.58), (A.75), (A.78), and (A.60). Recall that for the process $\mathcal{B}_0$, there is only one final-state quark, thus $\mathcal{H}_{f,q} = \{q\}$. Moreover, in Eq. (6.77), the elastic contributions are always grouped within collinear operators such as Eq. (6.88) or Eq. (6.89). However, in the following, we will discuss the cancellations leg-by-leg, i.e., by separating the initial-state contributions from the final-state gluon radiation and the final-state quark radiation. This is because, as we will see, each leg behaves independently, and the poles of different legs do not mix with one another. Therefore, starting from Eqs (6.88) and (6.89), we observe that:

$$\begin{aligned} \Gamma_{i,f_i}^{(4)}(\epsilon) - \Gamma_{i,f_i}^{1L}(\epsilon) &= \mathcal{O}(\epsilon^2), & i \in \{a, b\}, \\ \Gamma_{i,g \rightarrow gg}^{(4)}(\epsilon) - \Gamma_{i,g \rightarrow gg}^{1L}(\epsilon) &= \mathcal{O}(\epsilon), & i \in \mathcal{H}_{f,g}, \\ \Gamma_q^{(4)}(\epsilon) - \Gamma_q^{1L}(\epsilon) &= \mathcal{O}(\epsilon), & i \in \mathcal{H}_{f,q}. \end{aligned} \quad (6.90)$$

which yields

$$I_C^{(4)}(\epsilon) - \tilde{I}_C(2\epsilon) = \mathcal{O}(\epsilon^0). \quad (6.91)$$

By adding the fact that $h_c(\epsilon) = 1 + \mathcal{O}(\epsilon^3)$, we find

$$\frac{[\alpha_s]^2}{\epsilon^2} C_A h_c(\epsilon) [I_C^{(4)}(\epsilon) - \tilde{I}_C(2\epsilon)] = \mathcal{O}(\epsilon^{-2}), \quad (6.92)$$

that is, the $\mathcal{O}(\epsilon^{-3})$ poles in the first and second lines of Eq. (6.77) vanish. In Eq. (6.92), if we only consider the initial-state radiation, using the first relation in Eq. (6.90), we find that the $\mathcal{O}(\epsilon^{-2})$ pole also vanishes, as

$$\frac{[\alpha_s]^2}{\epsilon^2} C_A h_c(\epsilon) \left[\frac{\Gamma_{i,f_i}^{(4)}(\epsilon)}{2\epsilon} - \frac{\Gamma_{i,f_i}^{1L}(\epsilon)}{2\epsilon} \right] = \mathcal{O}(\epsilon^{-1}), \quad i \in \{a, b\}. \quad (6.93)$$

We move to the final-state radiation in the first and second lines of Eq. (6.77), and begin by considering the terms involving collinear singularities with respect to the final-state gluon, i.e., $i \in \mathcal{H}_{f,g}$. In this case, the single-soft plus hard-collinear real-virtual con-

tribution, as well as the double-collinear contribution, read respectively (see Eq. (5.80)):

$$\begin{aligned} \frac{C_A}{\epsilon^2} h_c(\epsilon) \left[\frac{\Gamma_{i,g}^{(4)}(\epsilon)}{2\epsilon} - \frac{\Gamma_{i,g}^{1L}(\epsilon)}{2\epsilon} \right] &= \frac{C_A^2}{\epsilon^2} \left(\frac{\pi^2}{3} - \frac{65}{72} \right) + \mathcal{O}(\epsilon^{-1}), \\ \frac{1}{2\epsilon^2} G_i|_{z,g \rightarrow gg} &= -\frac{C_A^2}{\epsilon^2} \left(\frac{\pi^2}{3} - \frac{65}{72} \right) + \mathcal{O}(\epsilon^{-1}). \end{aligned} \quad (6.94)$$

Thus, the $\mathcal{O}(\epsilon^{-2})$ pole vanishes. By contrast, the pole arising from collinear singularity with respect to the final state quark, namely the $i \in \mathcal{H}_{f,q}$ case, is more complex because the cancellation of the $\mathcal{O}(\epsilon^{-2})$ pole occurs only by combining the first two lines of Eq. (6.77) with the third. For this reason, we now turn to the third line of Eq. (6.77), discussing initial-state singularities, $i = \{a, b\}$, and gluon final-state singularities $i \in \mathcal{H}_{f,g}$. Thus, in these cases, it turns out that the cancellation is straightforward. Indeed, since in general

$$\begin{aligned} \Gamma_{i,f_i}^{(4)}(\epsilon) - \Gamma_{i,f_i}^{(2)}(2\epsilon) &= \mathcal{O}(\epsilon^2), & i \in \{a, b\}, \\ \Gamma_{i,g \rightarrow gg}^{(4)}(\epsilon) - \Gamma_{i,g \rightarrow gg}^{(2)}(2\epsilon) &= \mathcal{O}(\epsilon), & i \in \mathcal{H}_{f,g}, \\ \Gamma_q^{(4)}(\epsilon) - \Gamma_q^{(2)}(2\epsilon) &= \mathcal{O}(\epsilon), & i \in \mathcal{H}_{f,q}, \end{aligned} \quad (6.95)$$

and, in addition, since the expansions in Eq. (6.84) hold and $\beta_0 = \frac{11}{6} C_A$ in our setup, we find

$$\frac{[\alpha_s]^2}{\epsilon} \left[N_{sc}^{(b,d)} \gamma_{z,g \rightarrow gg}^{22}(0) \frac{\Gamma_{i,f_i}^{(4)}(\epsilon)}{2\epsilon} - \beta_0 \frac{\Gamma_{i,f_i}^{(2)}(2\epsilon)}{2\epsilon} \right] = \mathcal{O}(\epsilon^{-1}), \quad i \in \{a, b\}, \quad (6.96)$$

for the initial-state radiation, and an analogous expression for the gluon final-state radiation, which states that the $\mathcal{O}(\epsilon^{-2})$ poles vanish. Thus, for the poles arising from the collinear singularities $i = \{a, b\}$ and $i \in \mathcal{H}_{f,g}$, the first two lines of Eq. (6.77), when considered together, and the third line of Eq. (6.77), are each of $\mathcal{O}(\epsilon^{-1})$ without requiring any further combination among them.

We now turn to the case $i \in \mathcal{H}_{f,q}$, and consider the terms present in the first three lines of Eq. (6.77). Starting from the first and second lines, the single-soft plus hard-collinear real-virtual contribution reads¹⁰

$$\begin{aligned} &\frac{C_A}{\epsilon^2} h_c(\epsilon) \left[\frac{\Gamma_q^{(4)}(\epsilon)}{2\epsilon} - \frac{\Gamma_q^{1L}(\epsilon)}{2\epsilon} + \sigma_{q \rightarrow gq} \right] \\ &= \frac{1}{\epsilon^2} \left[C_A C_F \left(\frac{29}{18} + \frac{4}{3} L \right) + C_F^2 \left(\frac{\pi^2}{3} - \frac{5}{4} \right) \right] + \mathcal{O}(\epsilon^{-1}), \end{aligned} \quad (6.97)$$

¹⁰Recall that $L = \log(E_{\max}/E)$, where E labels the energy of the final-state quark. In contrast, E_i in $L_i = \log(E_{\max}/E_i)$ labels the energy of the final-state gluon i , with $i \in \mathcal{H}_{f,g}$.

while the double-collinear contribution corresponds to

$$\begin{aligned}\frac{1}{2\epsilon^2} G|_{z,q \rightarrow qq} &= \frac{C_F^2}{\epsilon^2} \left(\frac{85}{36} - \frac{\pi^2}{3} \right) + \mathcal{O}(\epsilon^{-1}), \\ \frac{1}{2\epsilon^2} G|_{z,q \rightarrow gg} &= -\frac{C_A C_F}{\epsilon^2} \left(\frac{29}{18} \right) + \mathcal{O}(\epsilon^{-1}), \\ \frac{1}{2\epsilon^2} \Gamma_{q \rightarrow gq} (\Gamma_{g \rightarrow gg} - \Gamma_q) &= \frac{1}{\epsilon^2} \left[-C_A C_F \left(\frac{11}{9} + \frac{4}{3} L \right) + C_F^2 \left(1 + \frac{4}{3} L \right) \right] + \mathcal{O}(\epsilon^{-1}),\end{aligned}\tag{6.98}$$

and the contribution proportional to $N_{\text{sc}}^{(b,d)}$ is

$$\begin{aligned}\frac{N_{\text{sc}}^{(b,d)}}{2\epsilon^2} \gamma_{1,q \rightarrow qq}^{22}(L) \Gamma_{q \rightarrow gq}^{(4)} &= -\frac{C_F^2}{\epsilon^2} \left(1 + \frac{4}{3} L \right) + \mathcal{O}(\epsilon^{-1}), \\ \frac{N_{\text{sc}}^{(b,d)}}{2\epsilon^2} H_{q \rightarrow gq} &= -\frac{C_F^2}{\epsilon^2} \left(\frac{10}{9} \right) + \mathcal{O}(\epsilon^{-1}).\end{aligned}\tag{6.99}$$

Combining the above, the first and second lines of Eq. (6.77) for $i \in \mathcal{H}_{f,q}$ leads to

$$1^{\text{st}} + 2^{\text{nd}} \text{ lines in Eq. (6.77)} = -\frac{11}{9} \frac{C_A C_F}{\epsilon^2} + \mathcal{O}(\epsilon^{-1}).\tag{6.100}$$

This result perfectly combines with the third line of Eq. (6.77), which, using Eqs (6.84) and (6.95) becomes

$$3^{\text{rd}} \text{ line in Eq. (6.77)} = \frac{11}{9} \frac{C_A C_F}{\epsilon^2} + \mathcal{O}(\epsilon^{-1}).\tag{6.101}$$

This demonstrates that, in the case $i \in \mathcal{H}_{f,q}$, the first two lines of Eq. (6.77), when considered together, and the third line of Eq. (6.77), contain a remainder of $\mathcal{O}(\epsilon^{-2})$ proportional to $\beta_0 = \frac{11}{6} C_A$, such that their sum results in $\mathcal{O}(\epsilon^{-1})$.

As mentioned previously, the contributions in the last three lines of Eq. (6.77) are at most of $\mathcal{O}(\epsilon^{-1})$. We have therefore confirmed that all $\mathcal{O}(\epsilon^{-3})$ and $\mathcal{O}(\epsilon^{-2})$ poles in the elastic pure collinear contribution $\Sigma_{\text{DU}}^{\text{el,pc}}|_{\mathcal{B}_0}$ cancel.

Before concluding this section, we briefly discuss the cancellation of the $\mathcal{O}(\epsilon^{-1})$ poles in $\Sigma_{\text{DU}}|_{\mathcal{B}_0}$ from Eq. (6.74). In principle, one could organize the cancellation following the procedure applied for the higher-order poles. However, this approach becomes significantly more cumbersome. For this reason, we simply note that the cancellation of the remaining $\mathcal{O}(\epsilon^{-1})$ poles has been verified through a straightforward, albeit tedious, algebraic computation. We emphasize that this computation is performed with the number N of final-state jets treated as a parameter, and thus, the result holds for an *arbitrary* number of final-state gluons in the process $\mathcal{B}_0$. All the necessary elements to confirm this cancellation are provided within the main body of this thesis and the relevant appendices.

Having cancelled all the poles, we can safely take the $\epsilon \rightarrow 0$ limit, yielding a finite result for the NNLO contribution to the cross section for the process $\mathcal{B}_0$. This result is presented in the following section.

6.5 Final result

In this section, we present the main result of this thesis: the *finite* NNLO QCD contribution to the partonic cross section of the process $\mathcal{B}_0$ in Eq. (3.56), namely $d\hat{\sigma}_{\mathcal{B}_0}^{\text{NNLO}}$ defined in Eq. (5.1). As explained throughout Part II, we arrive at this result by considering the double-real contributions from processes $\mathcal{B}_2$ and $\tilde{\mathcal{B}}_2$ in Eqs (4.48) and (4.49), the real-virtual contributions from process $\mathcal{B}_1$ in Eq. (3.103), and the double-virtual contributions from process $\mathcal{B}_0$. Therefore, our calculation does not treat contributions proportional to n_f , which require an analogous calculation for processes where one or more gluons split into $q\bar{q}$ pairs. It follows that every formula presented in what follows must be understood with $n_f = 0$. Lastly, we recall that the projector $\mathcal{P}_{\mathcal{B}_0}$, introduced on page 93, has been applied to $d\hat{\sigma}_{\mathcal{B}_0}^{\text{NNLO}}$, so as to select only those contributions that are proportional to the matrix elements squared of the underlying Born process $\mathcal{B}_0$ plus an arbitrary number of gluons.

A key feature of our approach is the organization of the subtraction terms - as far as possible - into iterations of NLO-like structures, which mitigates the proliferation of subtraction terms that typically complicates NNLO calculations. As a result, the *finite* NNLO remainder can be expressed in a remarkably compact form. Additionally, the structure of our result is highly versatile. It can be thought of as a Lego-like construction, where the functions we introduce serve as building blocks that can be assembled in various ways. In fact, from our main formula, we will derive as a special case its analogous expression for the process $\mathcal{A}_0$ from Eq. (3.55). We recall that the NNLO corrections to the process $\mathcal{A}_0$ represent the main result of Ref. [81]. In this thesis, by following the same approach of the final-state partons, we have introduced an additional final-state quark, thereby breaking the symmetry between final-state gluons, and also an asymmetric initial state. Nonetheless, the method we have employed in this thesis is identical to that in Ref. [81], merely adapted to a more general case. This serves to illustrate the versatility of the new approach to NSC.

6.5.1 Final result for process $\mathcal{B}_0$

We first present the final expression for $d\hat{\sigma}_{\mathcal{B}_0}^{\text{NNLO}}$ in Eq. (5.1). We split this object into fully resolved, single-unresolved and double-unresolved contributions, with $N+2$, $N+1$ and N resolved final-state partons, respectively, and write

$$\mathcal{P}_{\mathcal{B}_0} [2s_{ab} d\hat{\sigma}_{\mathcal{B}_0}^{\text{NNLO}}] = 2s_{ab} [d\hat{\sigma}_{\mathcal{B}_0}^{\text{FR}} + d\hat{\sigma}_{\mathcal{B}_0}^{\text{SU}} + d\hat{\sigma}_{\mathcal{B}_0}^{\text{DU}}]. \quad (6.102)$$

The first term on the right-hand side of (6.102) is the finite, fully-regulated contribution discussed in Eqs (4.41) and (4.67). It corresponds to

$$2s_{ab} d\hat{\sigma}_{\text{FR}}^{\mathcal{B}_0} = \langle \bar{S}_{mn} \bar{S}_n \Omega_1 \Delta^{(mn)} (\Theta_{mn} F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g] + F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_q, \mathbf{n}_g]) \rangle, \quad (6.103)$$

where the operator Ω_1 is defined in Eq. (4.36), and the two F_{LM} functions are given in Eq. (4.62). As in the NLO case, there is a small abuse of notation here. Since we have applied the projector $\mathcal{P}_{\mathcal{B}_0}$, strictly speaking, in Eq. (6.103) we should exclude the sub-

traction of those initial-state collinear limits which, when applied to $F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_q, \mathbf{n}_g]$, are set to zero by the projector $\mathcal{P}_{\mathcal{B}_0}$. We choose to leave this implicit in the writing of Eq. (6.103), acknowledging the abuse of notation. If we were to avoid this notation, we would need to split the contribution into several terms, explicitly indicating which ones are nullified by $\mathcal{P}_{\mathcal{B}_0}$. However, this would provide no concrete advantage, as we write Eq. (6.103) mainly as a reminder that this object needs to be implemented in numerical code.

The second term on the right-hand side of Eq. (6.102) is the single-unresolved cross section $d\hat{\sigma}_{\mathcal{B}_0}^{\text{SU}}$ reported in Eq. (6.14). Notice that the same discussion about our abuse of notation holds for this term too.

The third term on the right-hand side of Eq. (6.102) corresponds instead to the finite-remainder of the double-unresolved contribution (6.74). It is convenient to write it as the sum of the following quantities:

$$d\hat{\sigma}_{\mathcal{B}_0}^{\text{DU}} = d\hat{\sigma}_{\mathcal{B}_0, \text{db}}^{\text{DU}} + d\hat{\sigma}_{\mathcal{B}_0, \text{sb}, a}^{\text{DU}} + d\hat{\sigma}_{\mathcal{B}_0, \text{sb}, b}^{\text{DU}} + d\hat{\sigma}_{\mathcal{B}_0, \text{el}}^{\text{DU}}, \quad (6.104)$$

where

- i) $d\hat{\sigma}_{\mathcal{B}_0, \text{db}}^{\text{DU}}$ contains all the *double-boosted* contributions;
- ii) $d\hat{\sigma}_{\mathcal{B}_0, \text{sb}, a}^{\text{DU}}$ contains all the contributions that are *single-boosted* on leg a ;
- iii) $d\hat{\sigma}_{\mathcal{B}_0, \text{sb}, b}^{\text{DU}}$ contains all the contributions that are *single-boosted* on leg b ;
- iv) $d\hat{\sigma}_{\mathcal{B}_0, \text{el}}^{\text{DU}}$ contains all the *elastic* contributions.

We present each contribution separately, using several functions that we collect in Appendix I. The double-boosted contribution is described by the very simple expression (cf. Eq. (6.81))

$$2s_{ab} d\hat{\sigma}_{\mathcal{B}_0, \text{db}}^{\text{DU}} = [\alpha_s]^2 \langle \mathcal{P}_{gg}^{\text{NLO}} \otimes F_{\text{LM}}^{\mathcal{B}_0} \otimes \mathcal{P}_{qq}^{\text{NLO}} \rangle, \quad (6.105)$$

where $\mathcal{P}_{\alpha\beta}^{\text{NLO}}$ is the finite remainder of NLO splitting functions, and can be found in Eq. (I.3).

The expression for the single-boosted contributions on legs a and b is slightly more complex and corresponds to

$$\begin{aligned} 2s_{ab} d\hat{\sigma}_{\mathcal{B}_0, \text{sb}, a}^{\text{DU}} = & [\alpha_s]^2 \left[\langle \mathcal{P}_{gg}^{\text{NLO}} \otimes [I_{\text{T}}^{(0)} \cdot F_{\text{LM}}^{\mathcal{B}_0}] \rangle + \langle \mathcal{P}_{gg}^{\mathcal{W}} \otimes [\mathcal{W}_a^{\|\mathbf{n}, \text{fin}} \cdot F_{\text{LM}}^{\mathcal{B}_0}] \rangle \right. \\ & \left. + \langle \mathcal{P}_{gg}^{\text{NNLO}} \otimes F_{\text{LM}}^{\mathcal{B}_0} \rangle + \langle \mathcal{P}_{gg}^{\text{NLO}} \otimes F_{\text{LV}, \text{fin}}^{\mathcal{B}_0} \rangle \right], \end{aligned} \quad (6.106)$$

$$\begin{aligned} 2s_{ab} d\hat{\sigma}_{\mathcal{B}_0, \text{sb}, b}^{\text{DU}} = & [\alpha_s]^2 \left[\langle [I_{\text{T}}^{(0)} \cdot F_{\text{LM}}^{\mathcal{B}_0}] \otimes \mathcal{P}_{qq}^{\text{NLO}} \rangle + \langle [\mathcal{W}_a^{\|\mathbf{n}, \text{fin}} \cdot F_{\text{LM}}^{\mathcal{B}_0}] \otimes \mathcal{P}_{qq}^{\mathcal{W}} \rangle \right. \\ & \left. + \langle F_{\text{LM}}^{\mathcal{B}_0} \otimes \mathcal{P}_{qq}^{\text{NNLO}} \rangle + \langle F_{\text{LV}, \text{fin}}^{\mathcal{B}_0} \otimes \mathcal{P}_{qq}^{\text{NLO}} \rangle \right]. \end{aligned} \quad (6.107)$$

We recall that $I_{\text{T}}^{(n)}$, with $n = 0, 1, 2, \dots$, corresponds to the coefficient of ϵ^n in the expansion of the operator $I_{\text{T}}(\epsilon)$ in powers of ϵ (cf. Eq. (3.131)), and they are listed in Appendix A.6. The objects $\mathcal{W}_i^{\|\mathbf{n}, \text{fin}}$, with $i = a, b$, are partition-dependent functions defined in Eq. (I.12), $\mathcal{P}_{\alpha\alpha}^{\mathcal{W}}$ and the NNLO splitting functions $\mathcal{P}_{\alpha\beta}^{\text{NNLO}}$ are reported in Eq. (I.5) and

Eqs (I.6 - I.11), respectively, and $F_{\text{LV},\text{fin}}^{\mathcal{B}_0}$ is the finite remainder of the one-loop virtual amplitude.

Finally, the elastic contribution reads

$$\begin{aligned}
2s_{ab} d\hat{\sigma}_{\mathcal{B}_0,\text{el}}^{\text{DU}} &= \\
&= [\alpha_s]^2 \langle [I_{\text{cc}}^{\text{fin}} + I_{\text{tri}}^{\text{fin}} + I_{\text{unc}}^{\text{fin}}] \cdot F_{\text{LM}}^{\mathcal{B}_0} \rangle \\
&+ [\alpha_s]^2 \sum_{i \in \mathcal{H}} \left\langle \left[\theta_{\mathcal{H}_f} \gamma_{z,f_i \rightarrow f_{ig}}^{\mathcal{W}} \mathcal{W}_i^{i||n,\text{fin}} + \delta_g^{(0)} \mathcal{W}_i^{m||n,\text{fin}} + \delta_g^\perp \mathcal{W}_r^{(i)} \right] \cdot F_{\text{LM}}^{\mathcal{B}_0} \right\rangle \\
&+ [\alpha_s] \langle I_T^{(0)} \cdot F_{\text{LV},\text{fin}}^{\mathcal{B}_0} \rangle + \langle S_{mn} \Theta_{mn} F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g] \rangle_{T^2}^{\text{fin}} + \langle F_{\text{LV}^2,\text{fin}}^{\mathcal{B}_0} \rangle + \langle F_{\text{VV},\text{fin}}^{\mathcal{B}_0} \rangle.
\end{aligned} \tag{6.108}$$

In the first line of Eq. (6.108), we have the combination of a double color-correlated contribution $I_{\text{cc}}^{\text{fin}}$, a triple color-correlated component $I_{\text{tri}}^{\text{fin}}$, and a color-uncorrelated part $I_{\text{unc}}^{\text{fin}}$. They are defined in Eqs (I.17), (I.19), (I.26), respectively. Note that $I_{\text{unc}}^{\text{fin}}$ for process $\mathcal{B}_0$ reads

$$I_{\text{unc}}^{\text{fin}}|_{\mathcal{B}_0} = D_{c,g}^{\text{in}}(E_a) + D_{c,q}^{\text{in}}(E_b) + \sum_{i \in \mathcal{H}_{f,g}} D_{c,g}^{\text{fin}}(E_i) + D_{c,q}^{\text{fin}}(E), \tag{6.109}$$

where these objects are collected in Eqs (I.27 - I.30). In the second line of Eq. (6.108), we have $\theta_{\mathcal{H}_f} = 1$ if $i \in \mathcal{H}_f$ and $\theta_{\mathcal{H}_f} = 0$ otherwise, and the constants $\delta_g^{(0)}$ and $\delta_g^\perp$ are reported in Eq. (I.16). Additionally, $\gamma_{z,f_i \rightarrow f_{ig}}^{\mathcal{W}}$ is defined in Eq. (I.32), while the partition-dependent functions $\mathcal{W}_i^{i||n,\text{fin}}$, $\mathcal{W}_i^{m||n,\text{fin}}$, $\mathcal{W}_r^{(i)}$, are collected in Eqs (I.12 - I.14). In the third line of Eq. (6.108), the term $\langle S_{mn} \Theta_{mn} F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g] \rangle_{T^2}^{\text{fin}}$ refers to the finite remainder of the double-soft integrated subtraction term. It can be extracted from Ref. [80], and its explicit expression is reported in Eq. (I.33). Finally, $\langle F_{\text{LV}^2,\text{fin}}^{\mathcal{B}_0} \rangle$ and $\langle F_{\text{VV},\text{fin}}^{\mathcal{B}_0} \rangle$ are the process-dependent finite remainders of virtual amplitudes.

The result presented here can be directly utilized to implement the finite remainder of the NNLO QCD corrections to the process $\mathcal{B}_0$ in a computer code. In principle, this implementation can be achieved for an *arbitrary* number N of final-state particles. Furthermore, it is important to highlight that the structure of the final result is well-suited for numerical implementation. Specifically, the parameter N , which controls the final-state multiplicity, appears in only a limited number of terms. Indeed, the splitting functions that arise in the boosted contributions are universal and depend solely on the flavor of the external partons and their energies. In the elastic contribution, the final-state multiplicity only affects the upper limit of the sum over partons. As a result, implementing the color-uncorrelated elastic terms in a numerical code for any N is relatively straightforward. The implementation of contributions involving color-correlations (such as $I_T^{(0)}$) could be less trivial, as these require evaluating color-correlated matrix elements for high-multiplicity processes. Nevertheless, even in such cases, a numerical implementation for a given N should remain manageable, employing techniques such as color ordering [164, 165].

6.5.2 Final result for processes $\mathcal{A}_0, \mathcal{A}_0^{(\bar{q}q)}, \mathcal{A}_0^{(gg)}$

At the beginning of Ch. 5, we made the statement that the NNLO corrections for the process $\mathcal{A}_0$ can be derived effortlessly from those for the process $\mathcal{B}_0$. This is primarily due to two reasons. First, the process $\mathcal{B}_0$ consists of two contributions, which we have denoted as $m_g n_g$ and $m_q n_g$, whereas the process $\mathcal{A}_0$ includes only the $m_g n_g$ contribution, making it a subset of $\mathcal{B}_0$. Second, the definitions of all the functions involved in the result for $\mathcal{B}_0$, including those with color-correlations, depend, in a very clear way, on each parton, such that isolating and removing any of them is straightforward. As a result, since $\mathcal{A}_0$ does not contain a quark in the final state, all terms related to the final-state quark are simply "switched off" in these color-uncorrelated operators.

Thus, we can express the final result for $d\hat{\sigma}_{\mathcal{A}_0}^{\text{NNLO}}$ as follows:

$$2s_{ab} d\hat{\sigma}_{\mathcal{A}_0}^{\text{NNLO}} = 2s_{ab} [d\hat{\sigma}_{\mathcal{A}_0}^{\text{FR}} + d\hat{\sigma}_{\mathcal{A}_0}^{\text{SU}} + d\hat{\sigma}_{\mathcal{A}_0}^{\text{DU}}]. \quad (6.110)$$

In Eq. (6.110), the finite, fully-regulated contribution reads

$$2s_{ab} d\hat{\sigma}_{\mathcal{A}_0}^{\text{FR}} = \langle \bar{S}_{mn} \bar{S}_n \Omega_1 \Delta^{(mn)} \Theta_{mn} F_{LM}^{\mathcal{A}_2} [m_g, n_g] \rangle, \quad (6.111)$$

and the single-unresolved contribution coincides with the single-unresolved cross section given in Eq. (6.14), with the adjustments of removing the contributions from the final-state quark and replacing the gg splittings with aa splittings and qq splittings with bb , resulting in:

$$2s_{ab} d\hat{\sigma}_{\mathcal{A}_0}^{\text{SU}} = \sum_{i=1}^4 \Sigma_{\text{SU}}^{\text{fin},(i)}|_{\mathcal{A}_0}, \quad (6.112)$$

where

$$\begin{aligned} \Sigma_{\text{SU}}^{\text{fin},(1)}|_{\mathcal{A}_0} &= [\alpha_s] \left\langle \mathcal{O}_{\text{NLO}}^{(a,m)} \omega_{a||n}^{ma,na} \log \left(\frac{\eta_{am}}{2} \right) \Delta^{(m)} [\hat{P}_{aa}^{(0)} \otimes F_{LM}^{\mathcal{A}_1} [m_g]] \right\rangle \\ &\quad + [\alpha_s] \left\langle \mathcal{O}_{\text{NLO}}^{(b,m)} \omega_{b||n}^{mb,nb} \log \left(\frac{\eta_{bm}}{2} \right) \Delta^{(m)} [F_{LM}^{\mathcal{A}_1} [m_g] \otimes \hat{P}_{bb}^{(0)}] \right\rangle \\ &\quad - \sum_{i \in \mathcal{H}} [\alpha_s] \left\langle \mathcal{O}_{\text{NLO}}^{(i,m)} \omega_{i||n}^{mi,ni} \log \left(\frac{\eta_{im}}{2} \right) \Delta^{(m)} \Gamma_{i,g \rightarrow gg}(E_m) F_{LM}^{\mathcal{A}_1} [m_g] \right\rangle, \end{aligned} \quad (6.113)$$

$$\Sigma_{\text{SU}}^{\text{fin},(2)}|_{\mathcal{A}_0} = - \sum_{i \in \mathcal{H}} [\alpha_s] \gamma_g \left\langle \mathcal{O}_{\text{NLO}}^{(i,m)} \omega_{m||n}^{mi,ni} \log \left(\frac{\eta_{im}}{4(1-\eta_{im})} \right) \Delta^{(m)} F_{LM}^{\mathcal{A}_1} [m_g] \right\rangle, \quad (6.114)$$

$$\begin{aligned} \Sigma_{\text{SU}}^{\text{fin},(3)}|_{\mathcal{A}_0} &= \sum_{i \in \mathcal{H}} \frac{[\alpha_s]}{2} \gamma_{\perp, g \rightarrow gg}^{22} \langle \mathcal{O}_{\text{NLO}}^{(m)} \omega_{m||n}^{mi,ni} (r_i^\mu r_i^\nu + g^{\mu\nu}) \Delta^{(m)} F_{LM}^{\mathcal{A}_1} [m_g] \rangle \\ &\quad + \sum_{i \in \mathcal{H}} \frac{[\alpha_s]}{2} \gamma_{\perp, g \rightarrow gg}^{22,r} \langle \mathcal{O}_{\text{NLO}}^{(m)} \omega_{m||n}^{mi,ni} \Delta^{(m)} F_{LM}^{\mathcal{A}_1} [m_g] \rangle. \end{aligned} \quad (6.115)$$

$$\Sigma_{\text{SU}}^{\text{fin},(4)}|_{\mathcal{A}_0} = [\alpha_s] \left[\langle \mathcal{O}_{\text{NLO}}^{(m)} \Delta^{(m)} [\mathcal{P}_{aa}^{\text{NLO}} \otimes F_{LM}^{\mathcal{A}_1} [m_g]] \rangle + \langle \mathcal{O}_{\text{NLO}}^{(m)} \Delta^{(m)} [F_{LM}^{\mathcal{A}_1} [m_g] \otimes \mathcal{P}_{bb}^{\text{NLO}}] \rangle \right]$$

$$+ \langle \mathcal{O}_{\text{NLO}}^{(\text{m})} \Delta^{(\text{m})} F_{\text{RV,fin}}^{\mathcal{A}_1}[\mathbf{m}_g] \rangle + [\alpha_s] \langle \mathcal{O}_{\text{NLO}}^{(\text{m})} \Delta^{(\text{m})} [I_{\text{T}}(E_{\text{m}}) \cdot F_{\text{LM}}^{\mathcal{A}_1}[\mathbf{m}_g]] \rangle. \quad (6.116)$$

Finally, the double-unresolved contribution can be split into double-boosted, single-boosted and elastic terms

$$\text{d}\hat{\sigma}_{\mathcal{A}_0}^{\text{DU}} = \text{d}\hat{\sigma}_{\mathcal{A}_0,\text{db}}^{\text{DU}} + \text{d}\hat{\sigma}_{\mathcal{A}_0,\text{sb},a}^{\text{DU}} + \text{d}\hat{\sigma}_{\mathcal{A}_0,\text{sb},b}^{\text{DU}} + \text{d}\hat{\sigma}_{\mathcal{A}_0,\text{el}}^{\text{DU}}, \quad (6.117)$$

where

$$2s_{ab} \text{d}\hat{\sigma}_{\mathcal{A}_0,\text{db}}^{\text{DU}} = [\alpha_s]^2 \langle \mathcal{P}_{aa}^{\text{NLO}} \otimes F_{\text{LM}}^{\mathcal{A}_0} \otimes \mathcal{P}_{bb}^{\text{NLO}} \rangle, \quad (6.118)$$

$$2s_{ab} \text{d}\hat{\sigma}_{\mathcal{A}_0,\text{sb},a}^{\text{DU}} = [\alpha_s]^2 \left[\langle \mathcal{P}_{aa}^{\text{NLO}} \otimes [I_{\text{T}}^{(0)} \cdot F_{\text{LM}}^{\mathcal{A}_0}] \rangle + \langle \mathcal{P}_{aa}^{\mathcal{W}} \otimes [\mathcal{W}_a^{a||\text{n,fin}} \cdot F_{\text{LM}}^{\mathcal{A}_0}] \rangle \right. \\ \left. + \langle \mathcal{P}_{aa}^{\text{NNLO}} \otimes F_{\text{LM}}^{\mathcal{A}_0} \rangle + \langle \mathcal{P}_{aa}^{\text{NLO}} \otimes F_{\text{LV,fin}}^{\mathcal{A}_0} \rangle \right], \quad (6.119)$$

$$2s_{ab} \text{d}\hat{\sigma}_{\mathcal{A}_0,\text{sb},b}^{\text{DU}} = [\alpha_s]^2 \left[\langle [I_{\text{T}}^{(0)} \cdot F_{\text{LM}}^{\mathcal{A}_0}] \otimes \mathcal{P}_{bb}^{\text{NLO}} \rangle + \langle [\mathcal{W}_a^{a||\text{n,fin}} \cdot F_{\text{LM}}^{\mathcal{A}_0}] \otimes \mathcal{P}_{bb}^{\mathcal{W}} \rangle \right. \\ \left. + \langle F_{\text{LM}}^{\mathcal{A}_0} \otimes \mathcal{P}_{bb}^{\text{NNLO}} \rangle + \langle F_{\text{LV,fin}}^{\mathcal{A}_0} \otimes \mathcal{P}_{bb}^{\text{NLO}} \rangle \right]. \quad (6.120)$$

$$2s_{ab} \text{d}\hat{\sigma}_{\mathcal{A}_0,\text{el}}^{\text{DU}} = [\alpha_s]^2 \langle [I_{\text{cc}}^{\text{fin}} + I_{\text{tri}}^{\text{fin}} + I_{\text{unc}}^{\text{fin}}] \cdot F_{\text{LM}}^{\mathcal{A}_0} \rangle \\ + [\alpha_s]^2 \sum_{i \in \mathcal{H}} \left\langle \left[\theta_{\mathcal{H}_f} \gamma_{z,g \rightarrow gg}^{\mathcal{W}} \mathcal{W}_i^{i||\text{n,fin}} + \delta_g^{(0)} \mathcal{W}_i^{\text{m}||\text{n,fin}} + \delta_g^\perp \mathcal{W}_r^{(i)} \right] \cdot F_{\text{LM}}^{\mathcal{A}_0} \right\rangle \\ + [\alpha_s] \langle I_{\text{T}}^{(0)} \cdot F_{\text{LV,fin}}^{\mathcal{A}_0} \rangle + \langle S_{\text{mn}} \Theta_{\text{mn}} F_{\text{LM}}^{\mathcal{A}_2}[\mathbf{m}_g, \mathbf{n}_g] \rangle_{T^2}^{\text{fin}} + \langle F_{\text{LV}^2,\text{fin}}^{\mathcal{A}_0} \rangle + \langle F_{\text{VV,fin}}^{\mathcal{A}_0} \rangle. \quad (6.121)$$

In Eq. (6.121), the operator $I_{\text{unc}}^{\text{fin}}$ reads

$$I_{\text{unc}}^{\text{fin}}|_{\mathcal{A}_0} = D_{c,f_a}^{\text{in}}(E_a) + D_{c,f_b}^{\text{in}}(E_b) + \sum_{i \in \mathcal{H}_{f,g}} D_{c,g}^{\text{fn}}(E_i). \quad (6.122)$$

The results of the general computation presented here can be compared with those obtained for two specific values of N . The case $N = 0$ corresponds to the Drell-Yan process, while the case $N = 1$ corresponds to V + jet production, restricted to gluonic final state. It is well-known that, in both cases, the correlators of color-charge operators can be expressed through Casimir operators. For example, in the process $q_a + \bar{q}_b \rightarrow g_1 + V$, we have

$$\mathbf{T}_a \cdot \mathbf{T}_b = \frac{C_A}{2} - C_F, \quad \mathbf{T}_a \cdot \mathbf{T}_1 = \mathbf{T}_b \cdot \mathbf{T}_1 = -\frac{C_A}{2}. \quad (6.123)$$

Using such expressions, it is straightforward to replace all products of color-charge operators in Eq. (6.110) with the corresponding Casimir operators. Moreover, one can easily verify that the partition functions defined for a general N reduce to structures already employed in earlier computations. From the definition in Eq. (B.12), it follows that

$\Delta^{(ij)} = 1$ for the Drell-Yan process, while

$$\Delta^{(mn)} = \frac{p_{\perp,1}}{p_{\perp,1} + p_{\perp,m} + p_{\perp,n}}, \quad (6.124)$$

for the $V + \text{jet}$ partitioning. Similarly, it is straightforward to observe that the partition functions, defined for generic N , match those used in Refs [23, 65] for $N = 0$ and in Ref. [166] for $N = 1$. We have reproduced the analytic results for the finite NNLO remainders for Drell-Yan production reported in Ref. [65] by starting from Eq. (6.110) and setting $N = 0$. We have also verified that, upon setting $N = 1$, the general formulas reproduce the results of a dedicated computation of the NNLO QCD corrections to the process $q_a + \bar{q}_b \rightarrow g_1 + V$ that we performed earlier. Although that computation was also based on the NSC subtraction scheme, it was organized differently, with an emphasis on separately integrating the various subtraction terms over unresolved phase spaces *before* combining and simplifying them. The two approaches are sufficiently independent to provide a valuable check of the general- N formula reported in this subsection.

Finally, we present the results for the two expressions Eqs (5.5) and (5.6), which correspond to the processes $\mathcal{A}_0^{(\bar{q}q)}$ and $\mathcal{A}_0^{(gg)}$, respectively. As repeatedly emphasized, we have applied the projector $\mathcal{P}_{\mathcal{B}_0}$ throughout the computations detailed in Ch. 5. This ensured that all the single- and double-unresolved contributions in Eqs (4.171) and (4.183), involving an unresolved quark collinear to an initial-state leg, were set to zero by the action of $\mathcal{P}_{\mathcal{B}_0}$. However, by following the steps discussed in Ch. 5 and in the first part of this chapter, such contributions can be calculated without difficulty. The fully-resolved and single-unresolved contributions are *embedded* in Eqs (6.103) and (6.14), respectively, as a result of the notation that we adopted (see the discussion after each of these equations). As for the double-unresolved contributions, they correspond to:

$$d\hat{\sigma}_{\mathcal{A}_0^{(\bar{q}q)}}^{\text{DU}} = d\hat{\sigma}_{\mathcal{A}_0^{(\bar{q}q)},\text{db}}^{\text{DU}} + d\hat{\sigma}_{\mathcal{A}_0^{(\bar{q}q)},\text{sb},a}^{\text{DU}}, \quad (6.125)$$

$$d\hat{\sigma}_{\mathcal{A}_0^{(gg)}}^{\text{DU}} = d\hat{\sigma}_{\mathcal{A}_0^{(gg)},\text{db}}^{\text{DU}} + d\hat{\sigma}_{\mathcal{A}_0^{(gg)},\text{sb},b}^{\text{DU}}, \quad (6.126)$$

where

$$2s_{ab} d\hat{\sigma}_{\mathcal{A}_0^{(\bar{q}q)},\text{db}}^{\text{DU}} = [\alpha_s]^2 \langle \mathcal{P}_{qg}^{\text{NLO}} \otimes F_{\text{LM}}^{\mathcal{A}_0^{(\bar{q}q)}} \otimes \mathcal{P}_{qg}^{\text{NLO}} \rangle, \quad (6.127)$$

$$2s_{ab} d\hat{\sigma}_{\mathcal{A}_0^{(gg)},\text{db}}^{\text{DU}} = [\alpha_s]^2 \langle \mathcal{P}_{gg}^{\text{NLO}} \otimes F_{\text{LM}}^{\mathcal{A}_0^{(gg)}} \otimes \mathcal{P}_{gg}^{\text{NLO}} \rangle, \quad (6.128)$$

$$\begin{aligned} 2s_{ab} d\hat{\sigma}_{\mathcal{A}_0^{(\bar{q}q)},\text{sb},a}^{\text{DU}} &= [\alpha_s]^2 \left[\langle \mathcal{P}_{qg}^{\text{NLO}} \otimes [I_{\text{T}}^{(0)} \cdot F_{\text{LM}}^{\mathcal{A}_0^{(\bar{q}q)}}] \rangle + \langle \mathcal{P}_{qg}^{\text{NNLO}} \otimes F_{\text{LM}}^{\mathcal{A}_0^{(\bar{q}q)}} \rangle \right. \\ &\quad \left. + \langle \mathcal{P}_{qg}^{\text{NLO}} \otimes F_{\text{LV},\text{fin}}^{\mathcal{A}_0^{(\bar{q}q)}} \rangle \right], \end{aligned} \quad (6.129)$$

$$\begin{aligned} 2s_{ab} d\hat{\sigma}_{\mathcal{A}_0^{(gg)},\text{sb},b}^{\text{DU}} &= [\alpha_s]^2 \left[\langle [I_{\text{T}}^{(0)} \cdot F_{\text{LM}}^{\mathcal{A}_0^{(gg)}}] \otimes \mathcal{P}_{gg}^{\text{NLO}} \rangle + \langle F_{\text{LM}}^{\mathcal{A}_0^{(gg)}} \otimes \mathcal{P}_{gg}^{\text{NNLO}} \rangle \right. \\ &\quad \left. + \langle F_{\text{LV},\text{fin}}^{\mathcal{A}_0^{(gg)}} \otimes \mathcal{P}_{gg}^{\text{NLO}} \rangle \right]. \end{aligned} \quad (6.130)$$

The relevant definitions are collected in Appendix [I](#).

Conclusions

In this thesis, we investigated the application of the Nested Soft-Collinear (NSC) subtraction scheme to treat the infra-red (IR) divergences in the computation of the NNLO QCD corrections for a hadron-hadron scattering process that produces an *arbitrary* number N of partons and a colorless object X in the final state, where we assume that *one* of these final-state partons is a quark, and *all* additional radiated partons are gluons, i.e.,

$$\mathcal{B}_0 : \quad g_a + q_b \rightarrow \{g\}_{N-1} + q + X.$$

This thesis constitutes an extension of the work presented in Ref. [81], where a generalization of the NSC method was presented for the first time. While in Ref. [81] this approach was utilized to study the process

$$\mathcal{A}_0 : \quad a + b \rightarrow \{g\}_N + X$$

in the specific case $a = q$ and $b = \bar{q}$, thus with both the initial and final states being symmetric, here this generalization of the NSC scheme is presented in an even more comprehensive form, to compute a more complicated process, $\mathcal{B}_0$, which, in contrast, is asymmetric in both the initial and final states.

We have divided the thesis into two parts. *Part I* is dedicated to fundamental aspects of QCD and IR divergences, and it is written with a somewhat pedagogical motivation. It consists of two chapters. In Ch. 1, we reviewed the fundamental principles of QCD and introduced the problem of IR divergences in higher-order perturbative calculations. In Ch. 2, we discussed the sources of IR divergences in QCD necessary for the computation of cross sections at NLO and NNLO. Specifically, we examined the structure of divergences in one-loop and two-loop amplitudes, the behavior of tree-level and one-loop amplitudes in the soft and collinear limits, and the behavior of tree-level amplitudes in the double-soft and triple-collinear limits. This allowed us to provide universal building blocks in perturbative QCD, which serve as the starting point for analyzing in detail why it is necessary to introduce IR regularization schemes.

The second part of the thesis, *Part II*, is entirely focused on the NSC subtraction scheme and consists of four chapters. In Ch. 3, we thoroughly discussed the philosophy of the NSC/FKS approach at NLO and provided a detailed comparison between the

standard NSC method and the generalization developed in this thesis. Specifically, we first introduced the notation that was employed throughout the rest of the thesis, and then selected a sample process to compute its NLO corrections using the standard NSC approach. This allowed us to observe the limitations of the standard approach, which we summarize as follows:

- i) Color-correlated contributions cannot be trivially simplified for Born-level processes involving $n \geq 4$ partons, i.e., for non-trivial color configurations.
- ii) *Boosted* and *elastic* contributions are mixed, and this implies that divergences must be canceled by summing all the contributions together. Therefore, one has to deal with large expressions where it is necessary to separate the elastic pieces from the boosted ones and distinguish the contributions containing color-correlations from those that do not. This procedure is process-dependent.
- iii) For each process, deriving real corrections, virtual corrections, and those from the renormalization of the PDFs, expanding in ϵ , verifying pole cancellation, and obtaining a finite expression is a time-consuming and inefficient procedure.

Aware of these challenges, we introduced a generalization of the NSC scheme, with the central role played by the I -operators. We identified recurring building blocks, namely, I_V , I_S , I_C , and I_T operators, which proved to be extremely useful and versatile in describing IR divergences at NLO. We leveraged the properties of these operators to compute the NLO corrections for the $\mathcal{A}_0$ and $\mathcal{B}_0$ processes, outlining in detail a procedure that is (almost) process-independent.

We began Ch. 4 by presenting an analysis similar to that of Ch. 3, but at NNLO. We started by discussing the physical principles underlying the NSC scheme at NNLO, highlighting its strengths, and then focused on the aspects that hinder a generalization of this subtraction scheme. In addition to the points mentioned earlier, we emphasized two more:

- iv) There is a issue of efficiency in computing processes with a varying number N of hard jets. Assuming we have the NNLO result for a process like $\mathcal{A}_0$ or $\mathcal{B}_0$ for a certain N , and we want the result for $N + 1$, the calculation must be almost entirely repeated.
- v) In addition to double color-correlated contributions, which already appear at NLO, at NNLO one also encounters quartic and triple color-correlated terms. Without a general procedure, the cancellation of poles in these terms becomes highly process-dependent.

Following these considerations, we introduced a new organizational principle that allowed us to employ the I -operators, introduced in Ch. 3, to tackle these challenges. After showing that the NNLO corrections for the process $\mathcal{A}_0$ can be derived from those of the process $\mathcal{B}_0$, in the remainder of the chapter, we carefully reorganized the single- and double-unresolved terms of $\mathcal{B}_0$, laying the groundwork for the introduction of the I -operators.

In Ch. 5, we incorporated the most complicated calculations of this thesis for the NNLO corrections to the process $\mathcal{B}_0$. We described in meticulous detail how the double-virtual, real-virtual, and double-real contributions, as well as those related to the renormalization of the PDFs, can be entirely expressed through the I -operators I_V , I_S , I_C , and I_T . We showed that much of the complexity of NNLO computations, which is related to the interplay of soft and collinear singularities, can be mitigated by organizing the calculation to fully expose the iterative nature of these finite contributions. In this way, we demonstrated that although NNLO corrections are, de facto, much more complex than those at NLO, the IR structures observed at these two orders are closely related.

Finally, in Ch. 6, we collected all the results from Ch. 3 and Ch. 5, and addressed the cancellation of IR singularities. Specifically, we dealt with the *analytic* cancellation of the poles in the single-unresolved contributions, as well as in the double-unresolved contributions containing quartic, triple, and double color-correlated terms. We then thoroughly discussed the analytic cancellation of the poles, up to $\mathcal{O}(\epsilon^{-2})$, for the color-uncorrelated double-boosted, single-boosted, and elastic terms; we showed that, through the approach described in the previous chapters, this task can be accomplished in a straightforward way. Next, we briefly commented on the cancellation of the remaining $\mathcal{O}(\epsilon^{-1})$ poles. In the end, we presented the main result of this thesis, namely the NNLO QCD cross section for the process $\mathcal{B}_0$, expressed in a way that each term is manifestly finite. This result not only has a very compact expression compared to the complexity of the process it describes, but it also exhibits a fully general structure, easily adaptable to other processes. As proof of this, starting from the result for $\mathcal{B}_0$, we derived, without further manipulations, the NNLO QCD cross section for the process $\mathcal{A}_0$. In doing so, we not only confirmed the results previously presented in Ref. [81], but also extended them to the case of a gg initial state.

While this thesis represents a significant step forward in the application of the NSC subtraction scheme to NNLO QCD corrections, several avenues for future research remain. First, the results we presented for the processes $\mathcal{A}_0$ and $\mathcal{B}_0$ do not consider corrections proportional to n_f , and indeed, in our formulas, we assumed $n_f = 0$. We note that the approach described in this thesis can be used as it is for the calculation of n_f corrections, thus it is worthwhile to complete the NLO and NNLO corrections to processes $\mathcal{A}_0$ and $\mathcal{B}_0$ with their corresponding n_f contributions, effectively completing the β_0 coefficients.

Once this task is accomplished, the next step will be to apply this general NSC approach to a process with a *fully arbitrary final state*, that is, with any number of gluons and quarks of any flavour. Such a goal is clearly ambitious; however, in light of the insights learned from the study presented in this thesis, we understand that it is more of a *combinatorial* problem rather than a physical one. The process $\mathcal{A}_0$ already contains, from a physical standpoint, the situation with the maximum complexity of soft and collinear divergences. With the process $\mathcal{B}_0$, we also learned how to break the initial-state and final-state symmetry, and we saw how the anomalous dimensions of the final-state splittings $q^* \rightarrow qg$ and $q^* \rightarrow gq$ recombine. Once the analogous behavior for the final-state splittings $g^* \rightarrow gg$ and $g^* \rightarrow q\bar{q}$ is demonstrated (which can be done by calculating the n_f corrections), what remains is to understand how to keep track of the combina-

torics of quark flavors during the calculation of NNLO corrections. We are confident that such a task is achievable. This will complete the construction of a fully local and fully analytic subtraction scheme capable of handling the NNLO QCD corrections to any hadroproduction process. Such an achievement should greatly enhance our ability to provide highly-precise theoretical predictions for LHC production processes, hopefully shedding more light on the SM and beyond.

Appendices

Constants, angular integrals, splitting functions, anomalous dimensions and fundamental operators

Contents

A.1 Useful constants	201
A.2 Collection of simple angular integrals	203
A.3 Altarelli-Parisi splitting functions and PDFs renormalization	205
A.3.1 Altarelli-Parisi splitting functions	205
A.3.2 PDFs renormalization	206
A.4 Standard tree-level and one-loop splitting functions	208
A.4.1 Tree-level splitting functions	208
A.4.2 One-loop splitting functions	209
A.5 Generalized splittings and anomalous dimensions	211
A.5.1 Tree-level	211
A.5.2 Spin-correlations	212
A.5.3 One-loop	214
A.6 Definitions of I-operators, commutators, and expansions	215

In this Appendix, we provide a collection of formulas that are utilized throughout the main text of this thesis. Please note that we will adopt the following recurrent notation:

$$\bar{z} = 1 - z, \quad \mathcal{D}_n(z) = \left[\frac{\log^n(1-z)}{1-z} \right]_+,$$

$$\tilde{L}_i = \log\left(\frac{2E_i}{\mu}\right), \quad L_i = \log\left(\frac{E_{\max}}{E_i}\right), \quad L_{\max} = \log\left(\frac{2E_{\max}}{\mu}\right). \quad (\text{A.1})$$

A.1 Useful constants

In this section, we collect definitions of useful constants. As in Ref. [8], we denote the color-charge operators with T_i . Squares of these matrices are the Casimir operators of

the corresponding representations of $SU(3)$, i.e.

$$\begin{aligned} \mathbf{T}_q^2 = \mathbf{T}_{\bar{q}}^2 = C_F, \quad C_F &= \frac{N_c^2 - 1}{2N_c}, \\ \mathbf{T}_g^2 = C_A, \quad C_A &= N_c, \end{aligned} \quad (\text{A.2})$$

where $N_c = 3$ is the number of colors. Quark and gluon anomalous dimensions read

$$\gamma_q = \frac{3}{2}C_F, \quad \gamma_g = \frac{11}{6}C_A - \frac{2}{3}T_R n_f, \quad (\text{A.3})$$

where $T_R = 1/2$ and n_f is the number of massless quark flavors.

The strong coupling constant is renormalized in the $\overline{\text{MS}}$ scheme, i.e.

$$\alpha_{s,b} \mu_0^{2\epsilon} = \alpha_s(\mu) \mu^{2\epsilon} S_\epsilon Z[\alpha_s(\mu)], \quad (\text{A.4})$$

where $S_\epsilon = (4\pi)^{-\epsilon} e^{\epsilon\gamma_E}$,

$$Z[\alpha_s] = 1 - \frac{\beta_0}{\epsilon} \left(\frac{\alpha_s}{2\pi} \right) + \left(\frac{\beta_0^2}{\epsilon^2} - \frac{\beta_1}{2\epsilon} \right) \left(\frac{\alpha_s}{2\pi} \right)^2 + \mathcal{O}(\alpha_s^3), \quad (\text{A.5})$$

and

$$\beta_0 = \frac{11}{6}C_A - \frac{2}{3}T_R n_f \equiv \gamma_g, \quad \beta_1 = \frac{17}{6}C_A^2 - \frac{5}{3}C_A T_R n_f - C_F T_R n_f. \quad (\text{A.6})$$

We also define the coupling

$$[\alpha_s] \stackrel{\text{def}}{=} \left[\frac{\alpha_s(\mu)}{2\pi} \right] \frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}. \quad (\text{A.7})$$

In the main text of this paper we encounter a number of angular integrals, for which we introduce the following normalization constants:

$$\begin{aligned} N_\epsilon^{(b,d)} &\stackrel{\text{def}}{=} \frac{\Gamma(1-\epsilon)\Gamma(1+2\epsilon)}{\Gamma(1+\epsilon)} = 1 + \frac{\pi^2}{3}\epsilon^2 + \mathcal{O}(\epsilon^3), \\ N_{m||n}(\epsilon) &\stackrel{\text{def}}{=} 2^{2\epsilon} \frac{\Gamma(1+2\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1+\epsilon)\Gamma(1-\epsilon)} = 1 + 2\epsilon \log 2 + \frac{1}{2}\epsilon^2(\pi^2 + 4\log^2 2) + \mathcal{O}(\epsilon^3), \\ N_c(\epsilon) &\stackrel{\text{def}}{=} -\frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)} + \frac{2\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} = 1 + \mathcal{O}(\epsilon^3), \\ N_{sc}^{(b,d)} &\stackrel{\text{def}}{=} 2^{2\epsilon} \frac{\Gamma(1+2\epsilon)\Gamma^3(1-2\epsilon)}{\Gamma(1-3\epsilon)\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)} = 1 + 2\epsilon \log 2 + \frac{1}{3}\epsilon^2(\pi^2 + 6\log^2 2) + \mathcal{O}(\epsilon^3). \end{aligned} \quad (\text{A.8})$$

To describe virtual corrections, we have used the convention of Refs [82, 155]

$$\lambda_{ij} \stackrel{\text{def}}{=} \begin{cases} +1 & i \text{ and } j \text{ are both incoming or outgoing} \\ 0 & \text{otherwise,} \end{cases}$$

$$\kappa_{ij} \stackrel{\text{def}}{=} (\lambda_{ij} - \lambda_{im} - \lambda_{jm}) = \begin{cases} +1 & i \text{ and } j \text{ are both incoming or outgoing} \\ -1 & \text{otherwise} . \end{cases} \quad (\text{A.9})$$

For double-virtual amplitudes, we have used the following constants [82]:

$$K = \left(\frac{67}{18} - \frac{\pi^2}{6} \right) C_A - \frac{10}{9} T_R n_f ,$$

$$c_\epsilon = \frac{e^{-\epsilon\gamma_E} \Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)} = 1 + \frac{\pi^2}{4} \epsilon^2 + \frac{7}{3} \zeta_3 \epsilon^3 + \mathcal{O}(\epsilon^4) . \quad (\text{A.10})$$

$$\tilde{c}_\epsilon = \frac{\Gamma(1-\epsilon)}{e^{\epsilon\gamma_E}} = 1 + \frac{\pi^2}{12} \epsilon^2 + \mathcal{O}(\epsilon^3) . \quad (\text{A.11})$$

To describe integrated double-soft limits, we have introduced

$$c_1(\epsilon) \stackrel{\text{def}}{=} 1 + \left(\frac{\pi^2}{6} - \frac{32}{9} \right) \epsilon^2 + \left(\frac{217}{27} - \frac{137}{9} \log 2 - 22 \log^2 2 + \frac{11\zeta_3}{2} \right) \epsilon^3 ,$$

$$c_2(\epsilon) \stackrel{\text{def}}{=} 1 + \frac{\pi^2}{3} \epsilon^2 , \quad (\text{A.12})$$

$$c_3(\epsilon) \stackrel{\text{def}}{=} 4 \log 2 + 8 \epsilon \log^2 2 .$$

We emphasize that $c_{1,2,3}$ do not contain powers of ϵ beyond those shown above.

To compute soft and collinear limits of the real-virtual contributions, we used

$$A_K(\epsilon) = \frac{\Gamma^3(1+\epsilon) \Gamma^5(1-\epsilon)}{\Gamma(1+2\epsilon) \Gamma^2(1-2\epsilon)} = 1 - \frac{\pi^2}{3} \epsilon^2 + \mathcal{O}(\epsilon^3) ,$$

$$h_c(\epsilon) = \frac{\Gamma^2(1-2\epsilon) \Gamma(1+\epsilon)}{\Gamma(1-3\epsilon)} = 1 + \mathcal{O}(\epsilon^3) . \quad (\text{A.13})$$

A.2 Collection of simple angular integrals

Throughout the thesis we make use of various integrals over the angles of unresolved partons. We summarize some of the useful formulas here. First, we define the normalized element of the solid angle in $(d-1)$ - and $(d-2)$ -dimensions as

$$[d\Omega_m^{(d-1)}] \stackrel{\text{def}}{=} \frac{d\Omega_m^{(d-1)}}{2(2\pi)^{d-1}} , \quad [d\Omega_m^{(d-2)}] \stackrel{\text{def}}{=} \frac{d\Omega_m^{(d-2)}}{2(2\pi)^{d-1}} , \quad (\text{A.14})$$

and the normalized integral over the azimuthal angles θ_{im} between partons i and m as

$$\int_{-1}^{+1} d \cos \theta_{ij} (1 - \cos^2 \theta_{ij})^{d/2-2} = \int_0^1 \frac{2 d\eta_{ij}}{[4\eta_{ij}(1-\eta_{ij})]^\epsilon} \stackrel{\text{def}}{=} \int [d\eta_{ij}] , \quad (\text{A.15})$$

with

$$\rho_{ij} \stackrel{\text{def}}{=} 1 - \cos \theta_{ij} , \quad 0 \leq \rho_{ij} \leq 2 , \quad (\text{A.16})$$

$$\eta_{ij} \stackrel{\text{def}}{=} \frac{\rho_{ij}}{2}, \quad 0 \leq \eta_{ij} \leq 1, \quad (\text{A.17})$$

It follows that

$$\int [\text{d}\Omega_{\text{m}}^{(d-1)}] \equiv \int [\text{d}\eta_{iq}] \int [\text{d}\Omega_{\text{m}}^{(d-2)}]. \quad (\text{A.18})$$

We report some important general integrals over the solid angle:

$$\begin{aligned} \int \frac{[\text{d}\Omega_{\text{m}}^{(d-1)}]}{[\Omega^{(d-2)}]} \frac{1}{\rho_{im}^\alpha \rho_{jm}^\beta} &= 2^{1-(\alpha+\beta)-2\epsilon} \frac{\Gamma(1-\epsilon-\alpha)\Gamma(1-\epsilon-\beta)}{\Gamma(2-(\alpha+\beta)-2\epsilon)} \\ &\times {}_2F_1(\alpha, \beta, 1-\epsilon, 1-\eta_{ij}), \end{aligned} \quad (\text{A.19})$$

$$\int \frac{[\text{d}\Omega_{\text{m}}^{(d-1)}]}{[\Omega^{(d-2)}]} \frac{1}{(\rho_{im} + \rho_{jm})^\alpha} = 2^{1-\alpha-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)} {}_2F_1\left(\frac{\alpha}{2}, \frac{1}{2} + \frac{\alpha}{2}, \frac{3}{2} - \epsilon, 1 - \eta_{ij}\right), \quad (\text{A.20})$$

where

$$[\Omega^{(d-2)}] \stackrel{\text{def}}{=} \int [\text{d}\Omega^{(d-2)}] = \frac{(4\pi)^\epsilon}{8\pi^2 \Gamma(1-\epsilon)} \quad (\text{A.21})$$

corresponds to the integral over the $(d-2)$ -dimensional solid angle $[\text{d}\Omega_{\text{m}}^{(d-2)}]$. From formula (A.19), useful integrals can be derived:

$$\begin{aligned} \int \frac{[\text{d}\Omega_{\text{m}}^{(d-1)}]}{[\Omega^{(d-2)}]} \frac{\rho_{ij}}{\rho_{im} \rho_{jm}} &= -\frac{2^{1-2\epsilon}}{\epsilon} \eta_{ij}^{-\epsilon} K_{ij}, \\ \int \frac{[\text{d}\Omega_{\text{m}}^{(d-1)}]}{[\Omega^{(d-2)}]} \frac{1}{\rho_{im}} &= -\frac{2^{-2\epsilon}}{\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}, \\ \int \frac{[\text{d}\Omega_{\text{m}}^{(d-1)}]}{[\Omega^{(d-2)}]} \left(\frac{\rho_{im}}{4}\right)^{-\epsilon} \frac{1}{\rho_{im}} &= -\frac{2^{-2\epsilon}}{2\epsilon} \frac{2^\epsilon \Gamma(1-\epsilon) \Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}, \end{aligned} \quad (\text{A.22})$$

where K_{ij} is defined as

$$\begin{aligned} K_{ij} &\stackrel{\text{def}}{=} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \eta_{ij}^{1+\epsilon} {}_2F_1(1, 1, 1-\epsilon, 1-\eta_{ij}) \\ &= 1 + \left[\text{Li}_2(1-\eta_{ij}) - \frac{\pi^2}{6} \right] \epsilon^2 + \mathcal{O}(\epsilon^3). \end{aligned} \quad (\text{A.23})$$

Other integrals that we require involve the collinear limits acting on the angular phase space measure. Such integrals can be computed using the phase space parametrization described in Appendix C. Here, we present two examples that appear frequently:

$$\int [C_{im} \text{d}\Omega_{\text{m}}^{(d-1)}] \frac{1}{\rho_{im}} = -\frac{2^{-2\epsilon}}{\epsilon} \left[\frac{1}{8\pi^2} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \right], \quad (\text{A.24})$$

and

$$\int [C_{im} \text{d}\Omega_{\text{m}}^{(d-1)}] \frac{1}{\rho_{im}} \Theta\left(\eta_{im} < \frac{\eta_{ik}}{2}\right) = -\frac{1}{\epsilon} \left[\frac{1}{8\pi^2} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \right] \rho_{ik}^{-\epsilon}. \quad (\text{A.25})$$

A.3 Altarelli-Parisi splitting functions and PDFs renormalization

A.3.1 Altarelli-Parisi splitting functions

List of leading-order Altarelli-Parisi splitting functions [87]:

$$\begin{aligned}
 \hat{P}_{qq}^{(0)}(z) &\stackrel{\text{def}}{=} C_F \left[2\mathcal{D}_0(z) - (1+z) + \frac{3}{2}\delta(1-z) \right], \\
 \hat{P}_{qg}^{(0)}(z) &\stackrel{\text{def}}{=} T_R \left[(1-z)^2 + z^2 \right], \\
 \hat{P}_{gq}^{(0)}(z) &\stackrel{\text{def}}{=} C_F \left[\frac{1+(1-z)^2}{z} \right], \\
 \hat{P}_{gg}^{(0)}(z) &\stackrel{\text{def}}{=} 2C_A \left[\mathcal{D}_0(z) + z(1-z) + \frac{1}{z} - 2 \right] + \beta_0 \delta(1-z).
 \end{aligned} \tag{A.26}$$

List of one-loop Altarelli-Parisi splitting functions [87]:

$$\begin{aligned}
 \hat{P}_{qq,\tilde{\text{ns}}}^{(1)}(z) &\stackrel{\text{def}}{=} C_A C_F \left[\frac{\pi^2}{6}(1+z) - \frac{62}{9}z - \frac{19}{18} + \left(\frac{67}{9} - \frac{\pi^2}{3} \right) \mathcal{D}_0(z) \right. \\
 &\quad + \frac{2+11z^2}{6(1-z)} \log z - \frac{1+z^2}{1-z} \text{Li}_2(1-z) + \delta(1-z) \left(\frac{17}{24} + \frac{11}{18}\pi^2 - 3\zeta_3 \right) \Big] \\
 &\quad + C_F^2 \left[3 - 2z - 2\frac{1+z^2}{1-z} \log(1-z) \log z + 2\log z + \frac{1+3z^2}{2(1-z)} \log^2 z \right. \\
 &\quad \left. + 2\frac{1+z^2}{1-z} \text{Li}_2(1-z) + \delta(1-z) \left(\frac{3}{8} - \frac{\pi^2}{2} + 6\zeta_3 \right) \right],
 \end{aligned} \tag{A.27}$$

$$\begin{aligned}
 \hat{P}_{qg}^{(1)} &\stackrel{\text{def}}{=} \frac{C_F T_R}{2} \left\{ 4 - 9z - (1-4z) \log z - (1-2z) \log^2 z + 4\log(1-z) \right. \\
 &\quad \left. + \left[2\log^2 \left(\frac{1-z}{z} \right) - 4\log \left(\frac{1-z}{z} \right) - \frac{2\pi^2}{3} + 10 \right] p_{qg}(z) \right\} \\
 &\quad + \frac{C_A T_R}{2} \left\{ \frac{182}{9} + \frac{14z}{9} + \frac{40}{9z} + \left(\frac{136z}{3} - \frac{38}{3} \right) \log z - 4\log(1-z) \right. \\
 &\quad - (2+8z) \log^2 z + 2p_{qg}(-z) S_2(z) + \left[-\log^2 z + \frac{44\log z}{3} \right. \\
 &\quad \left. \left. - 2\log^2(1-z) + 4\log[1-z] + \frac{\pi^2}{3} - \frac{218}{9} \right] p_{qg}(z) \right\},
 \end{aligned} \tag{A.28}$$

$$\begin{aligned}
 \hat{P}_{gq}^{(1)} &\stackrel{\text{def}}{=} C_F^2 \left[-\frac{5}{2} - \frac{7z}{2} + \left(2 + \frac{7z}{2} \right) \log z - \left(1 - \frac{z}{2} \right) \log^2 z - 2z \log(1-z) \right. \\
 &\quad \left. - \left[3\log(1-z) + \log^2(1-z) \right] p_{gq}(z) \right]
 \end{aligned}$$

$$\begin{aligned}
& + C_A C_F \left\{ \frac{28}{9} + \frac{65z}{18} + \frac{44z^2}{9} - \left(12 + 5z + \frac{8z^2}{3} \right) \log z + (4+z) \log^2 z \right. \\
& + 2z \log(1-z) + p_{gq}(-z) S_2(z) + \left[\frac{1}{2} - 2 \log(1-z) \log z + \frac{1}{2} \log^2 z \right. \\
& + \left. \left. \frac{11}{3} \log(1-z) + \log^2(1-z) - \frac{\pi^2}{6} \right] p_{gq}(z) \right\} \\
& + C_F T_R n_f \left[-\frac{4z}{3} - \left(\frac{20}{9} + \frac{4}{3} \log(1-z) \right) p_{gq}(z) \right], \tag{A.29}
\end{aligned}$$

$$\begin{aligned}
\hat{P}_{gg}^{(1)} \stackrel{\text{def}}{=} & C_A T_R n_f \left[2 - 2z - \frac{20}{9} \left(-2 + \mathcal{D}_0(z) + \frac{1}{z} + (1-z)z \right) + \frac{26}{9} \left(-\frac{1}{z} + z^2 \right) \right. \\
& \left. - \frac{4}{3} (1+z) \log z \right] \\
& + C_F T_R n_f \left[-16 + \frac{4}{3z} + 8z + \frac{20z^2}{3} - (6+10z) \log z - (2+2z) \log^2 z \right] \\
& + C_A^2 \left[\frac{27(1-z)}{2} + \frac{67}{9} \left(-\frac{1}{z} + z^2 \right) - \left(\frac{25}{3} - \frac{11z}{3} + \frac{44z^2}{3} \right) \log z \right. \\
& + 4(1+z) \log^2 z + \left(-2 + \mathcal{D}_0(z) + \frac{1}{z} + (1-z)z \right) \left(\frac{67}{9} - \frac{\pi^2}{3} + \log^2 z \right. \\
& - 4 \log(1-z) \log z \left. \right) + 2 \left(-2 - \frac{1}{z} + \frac{1}{1+z} - z(1+z) \right) \left(-\frac{\pi^2}{6} + \frac{\log^2 z}{2} \right. \\
& \left. \left. - 2 \log(1+z) \log z - 2 \text{Li}_2(-z) \right) \right] \\
& + \delta(1-z) \left[-\frac{4}{3} C_A T_R n_f - C_F T_R n_f + C_A^2 \left(\frac{8}{3} + 3\zeta_3 \right) \right]. \tag{A.30}
\end{aligned}$$

Regarding the $\hat{P}_{qq}^{(1)}$ splitting, we point out that we need the *non-singlet* splitting function, from which the contribution of identical quarks has been subtracted. This is what we termed $\hat{P}_{qq,\widetilde{\text{ns}}}^{(1)}$.

A.3.2 PDFs renormalization

Collinear divergences associated with initial-state QCD radiation are removed by a redefinition of parton distribution functions (PDFs), which *run* through the DGLAP equations (1.56). In the $\overline{\text{MS}}$ renormalization scheme, this amounts to the replacement

$$f_{\alpha,b} \mapsto \sum_j \mathbb{P}_{\alpha\beta} \otimes f_\beta(\mu), \tag{A.31}$$

where the sum over j runs over all possible flavors, and we use the shorthand notation

$$\mathbb{P}_{\alpha\beta} \stackrel{\text{def}}{=} \delta_{\alpha\beta} + \left[\frac{\alpha_s(\mu)}{2\pi} \right] \mathbb{P}_{1,\alpha\beta} + \left[\frac{\alpha_s(\mu)}{2\pi} \right]^2 \mathbb{P}_{2,\alpha\beta} + \mathcal{O}(\alpha_s^3), \tag{A.32}$$

with

$$\begin{aligned}\mathbb{P}_{1,\alpha\beta}(z) &\stackrel{\text{def}}{=} \frac{\hat{P}_{\alpha\beta}^{(0)}(z)}{\epsilon}, \\ \mathbb{P}_{2,\alpha\beta}(z) &\stackrel{\text{def}}{=} \frac{\sum_y [\hat{P}_{\alpha y}^{(0)} \otimes \hat{P}_{y\beta}^{(0)}](z) - \beta_0 \hat{P}_{\alpha\beta}^{(0)}(z)}{2\epsilon^2} + \frac{\hat{P}_{\alpha\beta}^{(1)}(z)}{2\epsilon}.\end{aligned}\quad (\text{A.33})$$

The $\otimes$ -product in the definition of $\mathbb{P}_{2,\alpha\beta}$ is the standard convolution¹:

$$[f \otimes g](z) \stackrel{\text{def}}{=} \int_z^1 \frac{dt}{t} f(t) g\left(\frac{z}{t}\right). \quad (\text{A.34})$$

Starting from the UV-renormalized cross section

$$d\bar{\sigma}_{ab} = d\bar{\sigma}_{ab}^{\text{LO}} + d\bar{\sigma}_{ab}^{\text{NLO}} + d\bar{\sigma}_{ab}^{\text{NNLO}}, \quad (\text{A.35})$$

and using Eq. (A.31), we find that the UV- and PDF-renormalized cross section $d\hat{\sigma}_{ab}$ corresponds to

$$d\hat{\sigma}_{ab} = \sum_{x,y} \mathbb{P}_{xa} \otimes d\bar{\sigma}_{xy} \otimes \mathbb{P}_{yb}. \quad (\text{A.36})$$

Order by order, Eq. (A.36) reads

$$\begin{aligned}d\hat{\sigma}_{ab}^{\text{LO}} &= d\bar{\sigma}_{ab}^{\text{LO}}, \\ d\hat{\sigma}_{ab}^{\text{NLO}} &= d\bar{\sigma}_{ab}^{\text{NLO}} + d\hat{\sigma}_{ab}^{\text{pdf}}|_{\text{NLO}}, \\ d\hat{\sigma}_{ab}^{\text{NNLO}} &= d\bar{\sigma}_{ab}^{\text{NNLO}} + d\hat{\sigma}_{ab}^{\text{pdf}}|_{\text{NNLO}},\end{aligned}\quad (\text{A.37})$$

where

$$d\hat{\sigma}_{ab}^{\text{pdf}}|_{\text{NLO}} = \left[\frac{\alpha_s(\mu)}{2\pi} \right] \sum_x \left[\mathbb{P}_{1,xa} \otimes d\bar{\sigma}_{xb}^{\text{LO}} + d\bar{\sigma}_{ax}^{\text{LO}} \otimes \mathbb{P}_{1,xb} \right], \quad (\text{A.38})$$

$$\begin{aligned}d\hat{\sigma}_{ab}^{\text{pdf}}|_{\text{NNLO}} &= \left[\frac{\alpha_s(\mu)}{2\pi} \right] \sum_x \left[\mathbb{P}_{1,xa} \otimes d\bar{\sigma}_{xb}^{\text{NLO}} + d\bar{\sigma}_{ax}^{\text{NLO}} \otimes \mathbb{P}_{1,xb} \right] \\ &\quad + \left[\frac{\alpha_s(\mu)}{2\pi} \right]^2 \sum_{x,y} \left[\mathbb{P}_{1,xa} \otimes d\bar{\sigma}_{xy}^{\text{LO}} \otimes \mathbb{P}_{1,yb} + \mathbb{P}_{2,xa} \otimes d\bar{\sigma}_{xb}^{\text{LO}} \right. \\ &\quad \left. + d\bar{\sigma}_{ax}^{\text{LO}} \otimes \mathbb{P}_{2,xb} \right].\end{aligned}\quad (\text{A.39})$$

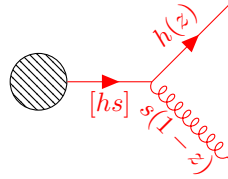
We stress that, in Eqs (A.38) and (A.39), $d\bar{\sigma}_{ab}^{\text{LO}}$ and $d\bar{\sigma}_{ab}^{\text{NLO}}$ indicate the LO and NLO cross sections without their PDF renormalization, involving initial states a and b , while the indices x and y can take values $x, y = g, q, \bar{q}$.

¹Note the distinction from the $\otimes$ -convolution in Eq. (5.69)

A.4 Standard tree-level and one-loop splitting functions

A.4.1 Tree-level splitting functions

Here we collect some important definitions presented in Subsec. 2.3.2. We consider the final-state splitting process $[hs] \rightarrow h(z) \times s(1-z)$, where z represents the momentum fraction involved in the collinear splitting. In this context, $[hs]$ denotes the *mother (clustered) particle*, while h and s are the two particles produced in the splitting process, taking longitudinal fractions z and $1-z$ of the incoming momentum, respectively. Additionally, we have $E_h = z E_{[hs]}$ and $E_s = (1-z) E_{[hs]}$. We denote the averaged final-state splitting functions as $P_{[hs]h}(z)$, i.e.,



$$\mapsto P_{[hs]h}(z). \quad (\text{A.40})$$

Here the list of their explicit definitions:

$$\begin{aligned} P_{qq}(z) &\stackrel{\text{def}}{=} C_F \left[\frac{1+z^2}{1-z} - \epsilon(1-z) \right], \\ P_{qg}(z) &\stackrel{\text{def}}{=} C_F \left[\frac{1+(1-z)^2}{z} - \epsilon z \right] \equiv P_{qg}(1-z), \\ P_{gq}(z) &\stackrel{\text{def}}{=} T_R \left[1 - \frac{2z(1-z)}{1-\epsilon} \right], \\ P_{gg}(z) &\stackrel{\text{def}}{=} 2C_A \left[\frac{z}{1-z} + \frac{1-z}{z} + z(1-z) \right]. \end{aligned} \quad (\text{A.41})$$

Spin-unaveraged splitting functions for gq and gg splitting read

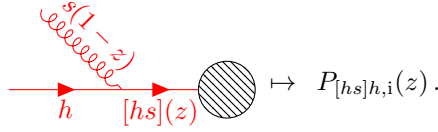
$$\begin{aligned} P_{gq}^{\mu\nu}(z) &\stackrel{\text{def}}{=} T_R \left[-g^{\mu\nu} - 4z(1-z)\kappa_\perp^\mu \kappa_\perp^\nu \right], \\ P_{gg}^{\mu\nu}(z) &\stackrel{\text{def}}{=} 2C_A \left[-g^{\mu\nu} \left(\frac{1-z}{z} + \frac{z}{1-z} \right) + 2(1-\epsilon)z(1-z)\kappa_\perp^\mu \kappa_\perp^\nu \right], \end{aligned} \quad (\text{A.42})$$

with

$$\kappa_\perp^\mu = \frac{k_\perp^\mu}{\sqrt{-k_\perp^2}}, \quad \kappa_\perp^2 = -1. \quad (\text{A.43})$$

Regarding the initial-state radiation, we consider the initial-state splitting process $h \rightarrow [hs](z) \times s(1-z)$, where h is the incoming parton, and $[hs]$ and a radiate from it

with a momentum fraction z and $1 - z$, respectively, i.e.,



$$\mapsto P_{[hs]h,i}(z). \quad (\text{A.44})$$

The list of the averaged initial-state splitting functions:

$$\begin{aligned} P_{qq,i}(z) &\stackrel{\text{def}}{=} -zP_{qq}(1/z) = P_{qq}(z), \\ P_{qg,i}(z) &\stackrel{\text{def}}{=} \left[\frac{2N_c}{2(1-\epsilon)(N_c^2-1)} \right] zP_{qg}(1/z) = P_{qg}(z), \\ P_{gq,i}(z) &\stackrel{\text{def}}{=} \left[\frac{2(1-\epsilon)(N_c^2-1)}{2N_c} \right] zP_{gq}(1/z) = P_{gq}(z), \\ P_{gg,i}(z) &\stackrel{\text{def}}{=} -zP_{gg}(1/z) = P_{gg}(z). \end{aligned} \quad (\text{A.45})$$

A.4.2 One-loop splitting functions

Here we report some important definitions presented in Subsec. 2.5.2, so as to make this appendix as self-contained as possible. We assume the same notation as the tree-level case.

The list of spin-averaged one-loop final-state splitting functions $P_{[hs]h}^{1\text{L}}(z)$ [156]²:

$$\begin{aligned} P_{qq}^{1\text{L}}(z) &\stackrel{\text{def}}{=} P_{qq}(z) \left[C_A \frac{a(z)}{2} + (2C_F - C_A) \frac{\tilde{b}(z)}{2} \right] + \tilde{P}_{qq}^{\text{new}}(z) \left[\frac{C_A - C_F}{1 - 2\epsilon} \right], \\ P_{qg}^{1\text{L}}(z) &\stackrel{\text{def}}{=} P_{qg}(z) \left[C_A \frac{b(z)}{2} + (2C_F - C_A) \frac{\tilde{a}(z)}{2} \right] + \tilde{P}_{qg}^{\text{new}}(z) \left[\frac{C_A - C_F}{1 - 2\epsilon} \right] \\ &\equiv P_{qq}^{1\text{L}}(1 - z), \\ P_{gg}^{1\text{L}}(z) &\stackrel{\text{def}}{=} P_{gg}(z) \left\{ C_A \frac{a(z) + \tilde{b}(z)}{2} + \frac{C_A - C_F}{\epsilon^2} \left[\frac{2}{1 - 2\epsilon} - \epsilon \right] - [n_f - C_A(1 - \epsilon)] \right. \\ &\quad \left. \times \left[\frac{2(1 - \epsilon)}{\epsilon(1 - 2\epsilon)(3 - 2\epsilon)} \right] \right\}, \\ P_{gg}^{1\text{L}}(z) &\stackrel{\text{def}}{=} P_{gg}(z) \left[C_A \frac{a(z) + \tilde{b}(z)}{2} \right] + \tilde{P}_{gg}^{\text{new}}(z) \left[\frac{n_f - C_A(1 - \epsilon)}{(1 - \epsilon)(1 - 2\epsilon)(3 - 2\epsilon)} \right], \end{aligned} \quad (\text{A.46})$$

with $P_{\alpha\beta}$ listed in Eq. (A.41), and where

$$\begin{aligned} a(z) &= (1 - z)F_1(1 - z), & \tilde{a}(z) &= a(z) + \frac{2}{\epsilon^2}, \\ b(z) &= zF_1(z) \equiv a(1 - z), & \tilde{b}(z) &= b(z) + \frac{2}{\epsilon^2} \equiv \tilde{a}(1 - z), \end{aligned} \quad (\text{A.47})$$

²Here we extract an overall normalization factor w.r.t. Ref. [156].

$$F_1(z) = \frac{2}{z\epsilon^2} \left[-\Gamma(1-\epsilon)\Gamma(1+\epsilon)z^{-\epsilon}(1-z)^\epsilon - 1 + (1-z)^\epsilon {}_2F_1(\epsilon, \epsilon, 1+\epsilon, z) \right],$$

and

$$\begin{aligned}\tilde{P}_{qq}^{\text{new}}(z) &= C_F [1 - \epsilon(1-z)], \\ \tilde{P}_{qg}^{\text{new}}(z) &= C_F [1 - \epsilon z], \\ \tilde{P}_{gg}^{\text{new}}(z) &= -C_A \left[\frac{1 - 2z(1-z)\epsilon}{1-\epsilon} \right].\end{aligned}\tag{A.48}$$

The list of spin-averaged one-loop initial-state splitting functions $P_{[hs]h}^{1L}(z)$, which can be derived by the final-state ones through crossing symmetry:

$$\begin{aligned}P_{qq,i}^{1L}(z) &\stackrel{\text{def}}{=} \left\{ -\frac{C_A}{\epsilon^2} \left[\frac{\Gamma^2(1-\epsilon)\Gamma^2(1+\epsilon)}{\Gamma(1-2\epsilon)\Gamma(1+2\epsilon)} (1-z)^{-\epsilon} + 2 \sum_{n=1}^{\infty} \epsilon^{2n} \text{Li}_{2n}(1-z) \right] \right. \\ &\quad \left. + \frac{2C_F}{\epsilon^2} \sum_{n=1}^{\infty} \epsilon^n \text{Li}_n(1-z) \right\} (1-z)^{-\epsilon} P_{qq,i}(z) \\ &\quad + \left[\frac{C_A - C_F}{1-2\epsilon} \right] (1-z)^{-\epsilon} \tilde{P}_{qq,i}^{\text{new}}(z),\end{aligned}\tag{A.49}$$

$$\begin{aligned}P_{gg,i}^{1L}(z) &\stackrel{\text{def}}{=} \left\{ \frac{C_A}{\epsilon^2} \left[\frac{\Gamma^2(1-\epsilon)\Gamma^2(1+\epsilon)}{\Gamma(1-2\epsilon)\Gamma(1+2\epsilon)} (1-z)^{-\epsilon} - 2 + 2 \sum_{n=1}^{\infty} \epsilon^{2n} \text{Li}_{2n}(1-z) \right] \right. \\ &\quad \left. - \frac{2C_F}{\epsilon^2} \left[\frac{\Gamma^2(1-\epsilon)\Gamma^2(1+\epsilon)}{\Gamma(1-2\epsilon)\Gamma(1+2\epsilon)} (1-z)^{-\epsilon} - 1 + \sum_{n=1}^{\infty} (-\epsilon)^n \text{Li}_n(1-z) \right] \right\} \\ &\quad \times (1-z)^{-\epsilon} P_{gg,i}(z) + \left[\frac{C_A - C_F}{1-2\epsilon} \right] (1-z)^{-\epsilon} \tilde{P}_{gg,i}^{\text{new}}(z),\end{aligned}\tag{A.50}$$

$$\begin{aligned}P_{gq,i}^{1L}(z) &\stackrel{\text{def}}{=} \left\{ \frac{C_A}{\epsilon^2} \left[-\frac{\Gamma^2(1-\epsilon)\Gamma^2(1+\epsilon)}{\Gamma(1-2\epsilon)\Gamma(1+2\epsilon)} (1-z)^{-\epsilon} + 2 \sum_{n=1}^{\infty} \epsilon^{2n-1} \text{Li}_{2n-1}(1-z) \right] \right. \\ &\quad \left. + \frac{C_A - C_F}{\epsilon^2} \left[\frac{2}{1-2\epsilon} - \epsilon \right] - [n_f - C_A(1-\epsilon)] \left[\frac{2(1-\epsilon)}{\epsilon(1-2\epsilon)(3-2\epsilon)} \right] \right\} \\ &\quad \times (1-z)^{-\epsilon} P_{gq,i}(z),\end{aligned}\tag{A.51}$$

$$\begin{aligned}P_{gg,i}^{1L}(z) &\stackrel{\text{def}}{=} \frac{C_A}{\epsilon^2} \left[-\frac{\Gamma^2(1-\epsilon)\Gamma^2(1+\epsilon)}{\Gamma(1-2\epsilon)\Gamma(1+2\epsilon)} (1-z)^{-\epsilon} + 2 \sum_{n=1}^{\infty} \epsilon^{2n-1} \text{Li}_{2n-1}(1-z) \right] \\ &\quad \times (1-z)^{-\epsilon} P_{gg,i}(z) + \left[\frac{n_f - C_A(1-\epsilon)}{(1-\epsilon)(1-2\epsilon)(3-2\epsilon)} \right] (1-z)^{-\epsilon} \tilde{P}_{gg,i}^{\text{new}}(z),\end{aligned}\tag{A.52}$$

with $P_{\alpha\beta,i}$ given in Eq. (A.45), and

$$\tilde{P}_{qq,i}^{\text{new}}(z) \stackrel{\text{def}}{=} -C_F [z + \epsilon(1-z)],$$

$$\begin{aligned}\tilde{P}_{qg,i}^{\text{new}}(z) &\stackrel{\text{def}}{=} T_R \left(\frac{z-\epsilon}{1-\epsilon} \right), \\ \tilde{P}_{gg,i}^{\text{new}}(z) &\stackrel{\text{def}}{=} \frac{C_A}{1-\epsilon} \left[z + 2\epsilon \left(\frac{1}{z} - 1 \right) \right].\end{aligned}\tag{A.53}$$

A.5 Generalized splittings and anomalous dimensions

A.5.1 Tree-level

For the configurations where a final-state parton of flavor α becomes collinear to an initial-state parton, say a of flavor f_a , we require convolutions of the type

$$\begin{aligned}\int_0^1 dz \mathcal{P}_{f_a f_a}^{(k)}(z, E_a) g(z) &\stackrel{\text{def}}{=} \int_0^1 dz \left[(1-S_z) \left[(1-z)^{-k\epsilon} P_{f_a f_a, i}(z) \right] \right. \\ &\quad \left. - 2T_{f_a}^2 \frac{1-e^{-k\epsilon L_a}}{k\epsilon} \delta(1-z) \right] g(z), \quad \text{if } \alpha = g, \\ \int_0^1 dz \mathcal{P}_{qg}^{(k)}(z) g(z) &\stackrel{\text{def}}{=} \int_0^1 dz \left[(1-z)^{-k\epsilon} P_{qg, i}(z) \right] g(z), \quad \text{if } \begin{cases} \alpha = q, \\ f_a = g, \end{cases} \\ \int_0^1 dz \mathcal{P}_{gq}^{(k)}(z) g(z) &\stackrel{\text{def}}{=} \int_0^1 dz \left[(1-z)^{-k\epsilon} P_{gq, i}(z) \right] g(z), \quad \text{if } \begin{cases} \alpha = q, \\ f_a = q, \end{cases}\end{aligned}\tag{A.54}$$

where $k = 2$ at NLO and $k = 4$ at NNLO, and $P_{\alpha\beta, i}$ are given in Eq. (A.45). Throughout the thesis, we always drop the index (k) when $k = 2$ to lighten the notation. Additionally, we point out that $\mathcal{P}_{\alpha\beta}^{(k)}(z, E_a)$ in Eq. (A.54) are energy-independent if $\alpha \neq \beta$.

It is worth rewriting the above splitting function as

$$-\left[\left(\frac{2E_a}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \right]^{\frac{k}{2}} \mathcal{P}_{\alpha\beta}^{(k)}(z, E_a) = \Gamma_{a,\alpha}^{(k)} \delta(1-z) \delta_{\alpha\beta} + \mathcal{P}_{\alpha\beta}^{(k), \text{gen}}(z, E_a), \tag{A.55}$$

with

$$\Gamma_{a,\alpha}^{(k)} \stackrel{\text{def}}{=} \left[\left(\frac{2E_a}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \right]^{\frac{k}{2}} \left[\gamma_\alpha + 2T_\alpha^2 \frac{1-e^{-k\epsilon L_a}}{k\epsilon} \right], \tag{A.56}$$

$$\mathcal{P}_{\alpha\beta}^{(k), \text{gen}}(z, E_a) \stackrel{\text{def}}{=} \left[\left(\frac{2E_a}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \right]^{\frac{k}{2}} \left[-\hat{P}_{\alpha\beta}^{(0)}(z) + \epsilon \mathcal{P}_{\alpha\beta}^{(k), \text{fin}}(z) \right]. \tag{A.57}$$

Here, $\hat{P}_{\alpha\beta}^{(0)}$ are the Altarelli-Parisi splitting functions given in Eq. (A.26), while $\mathcal{P}_{\alpha\beta}^{(k), \text{fin}}(z)$ corresponds to ϵ -expansion of $-\mathcal{P}_{\alpha\beta}^{(k)}(z, E_a = 0)$ starting from $\mathcal{O}(\epsilon)$.

If the unresolved final-state parton goes collinear to another final-state parton, say i

of flavour f_i , the generalized final-state anomalous dimensions read

$$\Gamma_{i,\alpha \rightarrow \rho\sigma}^{(k)} \stackrel{\text{def}}{=} \left[\left(\frac{2E_i}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \right]^{\frac{k}{2}} \gamma_{z,\alpha \rightarrow \rho\sigma}^{2k}(L_i), \quad \alpha \rightarrow \rho\sigma = \begin{cases} q \rightarrow qq \\ q \rightarrow gq \\ g \rightarrow q\bar{q} \\ g \rightarrow gg, \end{cases} \quad (\text{A.58})$$

where the standard final-state anomalous dimensions correspond to

$$\begin{aligned} \gamma_{g(z),f_i \rightarrow f_i g}^{nk}(L_i) &\stackrel{\text{def}}{=} - \int_0^1 dz (\mathbb{1} - S_z) \left[z^{-n\epsilon} (1-z)^{-k\epsilon} g(z) P_{f_i f_i}(z) \right] \\ &\quad + 2T_{f_i}^2 \frac{1 - e^{-k\epsilon L_i}}{k\epsilon} g(1), \\ \gamma_{g(z),q \rightarrow gq}^{nk} &\stackrel{\text{def}}{=} - \int_0^1 dz \left[z^{-n\epsilon} (1-z)^{-k\epsilon} g(z) P_{gq}(z) \right], \\ \gamma_{g(z),g \rightarrow q\bar{q}}^{nk} &\stackrel{\text{def}}{=} - \int_0^1 dz \left[z^{-n\epsilon} (1-z)^{-k\epsilon} g(z) P_{gq}(z) \right]. \end{aligned} \quad (\text{A.59})$$

The definitions of the splittings $P_{\alpha\beta}$ are provided in Eq. (A.41). Using these definitions and setting $k = 2$, we can also define the generalized quark final-state anomalous dimension:

$$\Gamma_{i,q} \stackrel{\text{def}}{=} \Gamma_{i,q \rightarrow qq} + \Gamma_{i,q \rightarrow gq}. \quad (\text{A.60})$$

A.5.2 Spin-correlations

To describe the spin-correlated component arising from sectors $\theta^{(b)}$ and $\theta^{(d)}$ defined in Eq. (4.28), we report the following integrals over the energy:

$$\delta_g^{\text{sa}}(\epsilon) \stackrel{\text{def}}{=} \frac{N_\epsilon^{(b,d)} E_{\text{max}}^{4\epsilon}}{2} \int_{E_{\text{max}}}^{2E_{\text{max}}} \frac{dE_{[\text{mn}]}}{E_{[\text{mn}]^{1+4\epsilon}}} \int_{1-\xi}^{\xi} dz [z(1-z)]^{-2\epsilon} P_{gg}(z), \quad (\text{A.61})$$

$$\delta_g^{\perp,r}(\epsilon) \stackrel{\text{def}}{=} \frac{N_\epsilon^{(b,d)} E_{\text{max}}^{4\epsilon}}{2} \int_{E_{\text{max}}}^{2E_{\text{max}}} \frac{dE_{[\text{mn}]}}{E_{[\text{mn}]^{1+4\epsilon}}} \int_{1-\xi}^{\xi} dz [z(1-z)]^{-2\epsilon} \epsilon P_{gg}^{\perp,r}(z), \quad (\text{A.62})$$

$$\delta_g^{\perp} \stackrel{\text{def}}{=} \frac{N_\epsilon^{(b,d)} E_{\text{max}}^{4\epsilon}}{2} \int_{E_{\text{max}}}^{2E_{\text{max}}} \frac{dE_{[\text{mn}]}}{E_{[\text{mn}]^{1+4\epsilon}}} \int_{1-\xi}^{\xi} dz [z(1-z)]^{-2\epsilon} P_{gg}^{\perp}(z), \quad (\text{A.63})$$

where $\xi = E_{\text{max}}/E_{[\text{mn}]}$, and

$$\delta_g(\epsilon) \stackrel{\text{def}}{=} \delta_g^{\text{sa}}(\epsilon) + \delta_g^{\perp,r}(\epsilon). \quad (\text{A.64})$$

Here the explicit expression of $\delta_g(\epsilon)$ and $\delta_g^{\perp}$:

$$\delta_g(\epsilon) \stackrel{\text{def}}{=} C_A \left[-\frac{131}{72} + \frac{11}{6} \log 2 + \frac{\pi^2}{6} + \epsilon \left(4\zeta_3 + \frac{11\pi^2}{18} - \frac{1541}{216} - \frac{\log 2}{6} \right) \right]$$

$$\begin{aligned}
 & + \epsilon^2 \left(\frac{77}{6} \zeta_3 + \frac{7\pi^4}{45} + \frac{125\pi^2}{216} - \frac{9607}{324} + \frac{11\pi^2}{18} \log 2 + \log 2 \right) \Bigg], \quad (\text{A.65}) \\
 \delta_g^\perp & \stackrel{\text{def}}{=} C_A \left(\frac{13}{36} - \frac{\log 2}{3} \right).
 \end{aligned}$$

Additionally, we also have the following integrals over z :

$$\gamma_{\perp, g \rightarrow ff}^{22} \stackrel{\text{def}}{=} - \int_0^1 dz \frac{P_{ff}^\perp(z)}{[z(1-z)]^{2\epsilon}}, \quad \gamma_{\perp, g \rightarrow ff}^{22, \text{r}} \stackrel{\text{def}}{=} - \int_0^1 dz \frac{P_{ff}^{\perp, \text{r}}(z)}{[z(1-z)]^{2\epsilon}}, \quad (\text{A.66})$$

where $f = q, g$. In all the above integrals, the following splittings are required:

$$\begin{aligned}
 P_{gg}^\perp(z) & \stackrel{\text{def}}{=} 4C_A(1-\epsilon)z(1-z), & P_{gg}^{\perp, \text{r}}(z) & \stackrel{\text{def}}{=} 2C_Az(1-z)(1-2\epsilon), \\
 P_{gq}^\perp(z) & \stackrel{\text{def}}{=} -4T_Rz(1-z), & P_{gq}^{\perp, \text{r}}(z) & \stackrel{\text{def}}{=} -2T_Rz(1-z)\frac{1-2\epsilon}{1-\epsilon}.
 \end{aligned} \quad (\text{A.67})$$

Note that, in order to compute the spin-correlated contribution, the following splitting function and anomalous dimension are also required:

$$\begin{aligned}
 \int_0^1 dz \mathcal{P}_{fafa}^{(k), \text{spin}}(z, E_a) g(z) & \stackrel{\text{def}}{=} \int_0^1 dz \left[(1-S_z) \left[(1-z)^{-k\epsilon} P_{fafa}^{\text{spin}}(z) \right] \right. \\
 & \quad \left. - 2T_{fa}^2 \frac{1 - e^{-k\epsilon L_a}}{k\epsilon} \delta(1-z) \right] g(z), \quad (\text{A.68})
 \end{aligned}$$

$$\begin{aligned}
 \gamma_{g(z), fi \rightarrow fi g}^{nk, \text{spin}}(L_i) & \stackrel{\text{def}}{=} - \int_0^1 dz (1-S_z) \left[z^{-n\epsilon} (1-z)^{-k\epsilon} g(z) P_{fi fi}^{\text{spin}}(z) \right] \\
 & \quad + 2T_{fi}^2 \frac{1 - e^{-k\epsilon L_i}}{k\epsilon} g(1), \quad (\text{A.69})
 \end{aligned}$$

where

$$\begin{aligned}
 P_{gg}^{\text{spin}}(z) & \stackrel{\text{def}}{=} 2C_A \left[\frac{z}{1-z} + \frac{(1-z)/z + z(1-z)}{2(1-\epsilon)} \right], \\
 P_{qq}^{\text{spin}}(z) & \stackrel{\text{def}}{=} \frac{C_F}{2} \frac{(1+z)^2}{1-z}.
 \end{aligned} \quad (\text{A.70})$$

In addition, we also introduce the spin-correlated splitting function

$$\mathcal{P}_{\alpha\alpha}^{\text{sc}}(z, E_a) \stackrel{\text{def}}{=} - \left[\left(\frac{2E_a}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \right]^2 \left[\mathcal{P}_{\alpha\alpha}^{(4), \text{spin}}(z, E_a) - \mathcal{P}_{\alpha\alpha}^{(4)}(z, E_a) \right], \quad (\text{A.71})$$

and the spin-correlated anomalous dimension

$$\Lambda_{i, fi \rightarrow fi g}^{\text{sc}} \stackrel{\text{def}}{=} \left[\left(\frac{2E_i}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \right]^2 \left[\gamma_{z, fi \rightarrow fi g}^{24, \text{spin}}(\epsilon, L_i) - \gamma_{z, fi \rightarrow fi g}^{24}(\epsilon, L_i) \right]. \quad (\text{A.72})$$

A.5.3 One-loop

While computing the real-virtual contributions, convolutions analogous to those in Eq. (A.54) emerge when a final-state parton α of flavor f_α becomes collinear to an initial-state parton, denoted as a with flavor f_a . The expressions are as follows:

$$\begin{aligned}
 \int_0^1 dz \mathcal{P}_{f_a f_a}^{(k),1L}(z, E_{f_a}) g(z) &\stackrel{\text{def}}{=} \int_0^1 dz \left[(1 - S_z) \left[(1 - z)^{-k\epsilon} P_{f_a f_a, i}^{1L}(z) \right] \right. \\
 &\quad \left. + 2C_A \mathbf{T}_{f_a}^2 \frac{1 - e^{-(2+k)\epsilon L_1}}{(2+k)\epsilon^2} \pi \cot(\pi\epsilon) \delta(1 - z) \right] g(z), \quad \text{if } f_\alpha = g, \\
 \int_0^1 dz \mathcal{P}_{qg}^{(k),1L}(z, E_{f_a}) g(z) &\stackrel{\text{def}}{=} \int_0^1 dz \left[(1 - z)^{-k\epsilon} P_{qg, i}^{1L}(z) \right] g(z), \quad \text{if } \begin{cases} f_\alpha = q, \\ f_a = q, \end{cases} \\
 \int_0^1 dz \mathcal{P}_{gq}^{(k),1L}(z, E_{f_a}) g(z) &\stackrel{\text{def}}{=} \int_0^1 dz \left[(1 - z)^{-k\epsilon} P_{gq, i}^{1L}(z) \right] g(z), \quad \text{if } \begin{cases} f_\alpha = q, \\ f_a = g. \end{cases}
 \end{aligned} \tag{A.73}$$

where $P_{\alpha\beta}^{1L}$ are listed in Eqs (A.49 - A.52). We rewrite $\mathcal{P}_{\alpha\beta}^{(k),1L}$ in analogy with Eq. (A.55), getting ($f_a = g, q$)

$$\begin{aligned}
 \frac{\epsilon^2}{C_A} \left[\left(\frac{2E_a}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1 - \epsilon)}{\Gamma(1 - 2\epsilon)} \right]^k \mathcal{P}_{f_a f_a}^{(k),1L}(z, E_a) &\stackrel{\text{def}}{=} \left[\Gamma_{a, f_a}^{(k),1L} \delta(1 - z) + \mathcal{P}_{f_a f_a}^{(k),1L, \text{gen}}(z, E_a) \right], \\
 \frac{\epsilon^2}{C_A} \left[\left(\frac{2E_a}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1 - \epsilon)}{\Gamma(1 - 2\epsilon)} \right]^k \mathcal{P}_{qg}^{(k),1L}(z, E_a) &\stackrel{\text{def}}{=} \mathcal{P}_{qg}^{(k),1L, \text{gen}}(z, E_a), \\
 \frac{\epsilon^2}{2C_F - C_A} \left[\left(\frac{2E_a}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1 - \epsilon)}{\Gamma(1 - 2\epsilon)} \right]^k \mathcal{P}_{gq}^{(k),1L}(z, E_a) &\stackrel{\text{def}}{=} \mathcal{P}_{gq}^{(k),1L, \text{gen}}(z, E_a),
 \end{aligned} \tag{A.74}$$

where

$$\Gamma_{a, f_a}^{(k),1L} \stackrel{\text{def}}{=} \left[\left(\frac{2E_a}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1 - \epsilon)}{\Gamma(1 - 2\epsilon)} \right]^k \left[\gamma_{f_a} + 2\mathbf{T}_{f_a}^2 \frac{1 - e^{-(2+k)\epsilon L_a}}{(2+k)} \pi \frac{\cos(\pi\epsilon)}{\sin(\pi\epsilon)} \right], \tag{A.75}$$

$$\mathcal{P}_{f_a f_a}^{(k),1L, \text{gen}}(z, E_a) \stackrel{\text{def}}{=} \left[\left(\frac{2E_a}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1 - \epsilon)}{\Gamma(1 - 2\epsilon)} \right]^k \left[-\hat{P}_{f_a f_a}^{(0)}(z) + \epsilon \mathcal{P}_{f_a f_a}^{(k),1L, \text{fin}}(z) \right]. \tag{A.76}$$

In Eq. (A.76), $\mathcal{P}_{\alpha\alpha}^{(k),1L, \text{fin}}(z)$ corresponds to ϵ -expansion, starting from $\mathcal{O}(\epsilon)$, of

$$\frac{\epsilon^2}{C_A} \mathcal{P}_{\alpha\alpha}^{(k),1L}(z, E_a = 0). \tag{A.77}$$

In Eq. (A.76), functions $\mathcal{P}_{\alpha\beta}^{(k),1L, \text{gen}}$ are finite in ϵ and can be extracted from Refs [154, 156, 167].

If the unresolved final-state parton α goes collinear to a final-state parton, say i of flavour f_i , the generalized final-state anomalous dimensions read

$$\begin{aligned}
 \Gamma_{i,q \rightarrow qg}^{(k),1L} &\stackrel{\text{def}}{=} - \left[\left(\frac{2E_i}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \right]^k \frac{\epsilon^2 \cos(\pi\epsilon)}{C_A} \gamma_{z,q \rightarrow qg}^{3(k+1),1L}(L_i), \\
 \Gamma_{i,q \rightarrow gq}^{(k),1L} &\stackrel{\text{def}}{=} - \left[\left(\frac{2E_i}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \right]^k \frac{\epsilon^2 \cos(\pi\epsilon)}{C_A} \gamma_{z,q \rightarrow gq}^{3(k+1),1L}, \\
 \Gamma_{i,g \rightarrow gg}^{(k),1L} &\stackrel{\text{def}}{=} - \left[\left(\frac{2E_i}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \right]^k \frac{\epsilon^2 \cos(\pi\epsilon)}{C_A} \gamma_{z,g \rightarrow gg}^{3(k+1),1L}(L_i), \\
 \Gamma_{i,g \rightarrow q\bar{q}}^{(k),1L} &\stackrel{\text{def}}{=} - \left[\left(\frac{2E_i}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \right]^k \frac{\epsilon^2 \cos(\pi\epsilon)}{2C_F - C_A} \gamma_{z,g \rightarrow q\bar{q}}^{3(k+1),1L}.
 \end{aligned} \tag{A.78}$$

We wrote the above definitions in terms of the usual one-loop anomalous dimensions

$$\begin{aligned}
 \gamma_{g(z),f_i \rightarrow f_i g}^{n(k+1),1L}(L_i) &\stackrel{\text{def}}{=} - \int_0^1 dz (1 - S_z) \left[z^{-n\epsilon} (1-z)^{-(k+1)\epsilon} g(z) P_{f_i f_i}^{1L}(z) \right] \\
 &\quad - 2C_A \mathbf{T}_{f_i}^2 \frac{1 - e^{-(2+k)\epsilon L_i}}{(2+k)} \frac{\pi}{\epsilon^2 \sin(\pi\epsilon)} g(1), \\
 \gamma_{f(z),q \rightarrow gq}^{n(k+1),1L} &\stackrel{\text{def}}{=} - \int_0^1 dz \left[z^{-n\epsilon} (1-z)^{-(k+1)\epsilon} g(z) P_{gq}^{1L}(z) \right], \\
 \gamma_{f(z),g \rightarrow q\bar{q}}^{n(k+1),1L} &\stackrel{\text{def}}{=} - \int_0^1 dz \left[z^{-n\epsilon} (1-z)^{-(k+1)\epsilon} g(z) P_{gq}^{1L}(z) \right],
 \end{aligned} \tag{A.79}$$

where the one-loop final-state splitting functions $P_{\alpha\beta}^{1L}$ are listed in Eq. (A.46). Note that, although the above definitions are provided for any value of k , we consistently set $k = 2$ in the real-virtual contributions. Additionally, similar to the expressions in Eq. (A.60), we define the one-loop case as follows:

$$\Gamma_{i,q}^{1L} \stackrel{\text{def}}{=} \Gamma_{i,q \rightarrow qg}^{1L} + \Gamma_{i,q \rightarrow gq}^{1L}. \tag{A.80}$$

A.6 Definitions of I -operators, commutators, and expansions

Throughout this thesis, we have used virtual, soft, and collinear operators to encode singularities, and have made use of various relations between them. For the reader's convenience, we list these definitions and relations here.

We begin with Catani's operator [82]

$$I_1(\epsilon) \stackrel{\text{def}}{=} \frac{1}{2} \frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)} \sum_{(ij)} \frac{\mathcal{V}_i^{\text{sing}}(\epsilon)}{\mathbf{T}_i^2} \mathbf{T}_i \cdot \mathbf{T}_j \left(\frac{\mu^2}{2p_i \cdot p_j} \right)^\epsilon e^{i\pi\lambda_{ij}\epsilon}, \tag{A.81}$$

where

$$\mathcal{V}_i^{\text{sing}}(\epsilon) = \frac{\mathbf{T}_i^2}{\epsilon^2} + \frac{\gamma_i}{\epsilon}, \quad (\text{A.82})$$

γ_i are the anomalous dimensions (A.3), and λ_{ij} constants are defined in Eq. (A.9). We find it convenient to modify the normalization slightly, yielding

$$\bar{I}_1(\epsilon) \stackrel{\text{def}}{=} \frac{1}{2} \sum_{(ij)} \frac{\mathcal{V}_i^{\text{sing}}(\epsilon)}{\mathbf{T}_i^2} \mathbf{T}_i \cdot \mathbf{T}_j \left(\frac{\mu^2}{2p_i \cdot p_j} \right)^\epsilon e^{i\pi\lambda_{ij}\epsilon}, \quad (\text{A.83})$$

from which we define the operators for amplitudes-squared

$$I_\pm(\epsilon) \stackrel{\text{def}}{=} \frac{\bar{I}_1(\epsilon) \pm \bar{I}_1^\dagger(\epsilon)}{2}, \quad I_V(\epsilon) \stackrel{\text{def}}{=} \bar{I}_1(\epsilon) + \bar{I}_1^\dagger(\epsilon) \equiv 2I_+(\epsilon). \quad (\text{A.84})$$

The Laurent expansion for $I_V(\epsilon)$ reads

$$I_V(\epsilon) = \sum_{n=-2}^{\infty} \epsilon^n I_V^{(n)}, \quad (\text{A.85})$$

where

$$I_V^{(-2)} = - \sum_{i \in \mathcal{H}} \mathbf{T}_i^2, \quad (\text{A.86})$$

$$I_V^{(-1)} = \sum_{(ij)} L_{ij} (\mathbf{T}_i \cdot \mathbf{T}_j) - \sum_{i \in \mathcal{H}} \gamma_i. \quad (\text{A.87})$$

Here, $L_{ij} = \log(\mu^2/s_{ij})$, with $s_{ij} = 2p_i \cdot p_j$.

The soft operator is equal to

$$I_S(\epsilon) \stackrel{\text{def}}{=} - \frac{(2E_{\text{max}}/\mu)^{-2\epsilon}}{\epsilon^2} \sum_{(ij)} \eta_{ij}^{-\epsilon} K_{ij} (\mathbf{T}_i \cdot \mathbf{T}_j), \quad (\text{A.88})$$

with K_{ij} given in Eq. (A.23). The Laurent expansion of I_S reads

$$I_S(\epsilon) = \sum_{n=-2}^{\infty} \epsilon^n I_S^{(n)}, \quad (\text{A.89})$$

and we require the following terms in the above expansion

$$I_S^{(-2)} = \sum_{i \in \mathcal{H}} \mathbf{T}_i^2, \quad (\text{A.90})$$

$$I_S^{(-1)} = \sum_{(ij)} (\mathbf{T}_i \cdot \mathbf{T}_j) \log \eta_{ij} - 2L_{\text{max}} \sum_{i \in \mathcal{H}} \mathbf{T}_i^2, \quad (\text{A.91})$$

$$\begin{aligned}
 I_S^{(0)} = & - \sum_{(ij)} (\mathbf{T}_i \cdot \mathbf{T}_j) \left[2L_{\max} \log \eta_{ij} + \frac{1}{2} \log^2 \eta_{ij} + \text{Li}_2(1 - \eta_{ij}) \right] \\
 & + \sum_{i \in \mathcal{H}} \mathbf{T}_i^2 \left[2L_{\max}^2 - \frac{\pi^2}{6} \right], \tag{A.92}
 \end{aligned}$$

$$\begin{aligned}
 I_S^{(1)} = & \sum_{(ij)} (\mathbf{T}_i \cdot \mathbf{T}_j) \left[2L_{\max}^2 \log \eta_{ij} + \left(L_{\max} - \frac{1}{2} \log(1 - \eta_{ij}) \right) \log^2 \eta_{ij} \right. \\
 & + \frac{1}{6} \log^3 \eta_{ij} + 2L_{\max} \text{Li}_2(1 - \eta_{ij}) - \text{Li}_3(1 - \eta_{ij}) - \text{Li}_3(\eta_{ij}) \left. \right] \\
 & - \sum_{i \in \mathcal{H}} \mathbf{T}_i^2 \left[L_{\max} \left(\frac{4}{3} L_{\max}^2 - \frac{\pi^2}{3} \right) + 3\zeta_3 \right]. \tag{A.93}
 \end{aligned}$$

The computation of the soft contributions requires a variant of the soft operator $I_S(\epsilon)$, i.e.,

$$\tilde{I}_S(2\epsilon) \stackrel{\text{def}}{=} - \frac{(2E_{\max}/\mu)^{-4\epsilon}}{(2\epsilon)^2} \sum_{(ij)} \eta_{ij}^{-2\epsilon} \tilde{K}_{ij}(\mathbf{T}_i \cdot \mathbf{T}_j), \tag{A.94}$$

with

$$\tilde{K}_{ij}(\epsilon) \stackrel{\text{def}}{=} \frac{\Gamma^2(1 - 2\epsilon)}{\Gamma(1 - 4\epsilon)} \eta_{ij}^{1+3\epsilon} {}_2F_1(1 + \epsilon, 1 + \epsilon, 1 - \epsilon, 1 - \eta_{ij}). \tag{A.95}$$

The following property relates I_S and $\tilde{I}_S$:

$$\tilde{I}_S(2\epsilon) = I_S(2\epsilon) + \mathcal{O}(\epsilon). \tag{A.96}$$

We also require an ϵ -expansion for $\tilde{I}_S$. Given Eq. (A.96), the first three coefficients $\tilde{I}_S^{(n)}$ with $n = -2, -1, 0$ can be directly obtained from those of $I_S(\epsilon)$, up to a rescaling by factors of $1/4$, $1/2$ and 1 , respectively. The coefficient at $\mathcal{O}(\epsilon)$ reads

$$\begin{aligned}
 \tilde{I}_S^{(1)} = & \sum_{(ij)} (\mathbf{T}_i \cdot \mathbf{T}_j) \left[\left(2L_{\max} - \frac{3}{2} \log(1 - \eta_{ij}) \right) \log^2 \eta_{ij} + \frac{1}{3} \log^3 \eta_{ij} \right. \\
 & + \left(\frac{\pi^2}{6} + 4L_{\max}^2 - \text{Li}_2(1 - \eta_{ij}) \right) \log \eta_{ij} \\
 & + 4L_{\max} \text{Li}_2(1 - \eta_{ij}) - \text{Li}_3(1 - \eta_{ij}) - 3\text{Li}_3(\eta_{ij}) \left. \right] \\
 & + \left[\frac{2}{3} L_{\max} \left(\pi^2 - 4L_{\max}^2 \right) - 7\zeta_3 \right] \sum_{i \in \mathcal{H}} \mathbf{T}_i^2. \tag{A.97}
 \end{aligned}$$

Moving to collinear limits, we define the hard-collinear operator as

$$I_C^{(k)}(\epsilon) \stackrel{\text{def}}{=} \sum_{i=a,b} \frac{\Gamma_{i,f_i}^{(k)}}{\epsilon k/2} + \sum_{i \in \mathcal{H}_{f,g}} \frac{\Gamma_{i,g \rightarrow gg}^{(k)}}{\epsilon k/2} + \sum_{i \in \mathcal{H}_{f,q}} \frac{\Gamma_{i,q}^{(k)}}{\epsilon k/2}, \tag{A.98}$$

where generalized initial- and final-state anomalous dimensions are given in Eqs (A.56) and (A.58), respectively. To treat the hard-collinear limits of the real-virtual matrix element we have introduced

$$\tilde{I}_C(2\epsilon) \stackrel{\text{def}}{=} \sum_{i=a,b} \frac{\Gamma_{i,f_i}^{1L}}{2\epsilon} + \sum_{i \in \mathcal{H}_{f,g}} \frac{\Gamma_{i,g \rightarrow gg}^{1L}}{2\epsilon} + \sum_{i \in \mathcal{H}_{f,q}} \frac{\Gamma_{i,q}^{1L}}{2\epsilon}, \quad (\text{A.99})$$

where generalized one-loop initial- and final-state anomalous dimensions are given in Eqs (A.75) and (A.78), respectively. We note that the following relations hold:

$$\begin{aligned} \tilde{I}_C(2\epsilon) &= I_C(2\epsilon) + \mathcal{O}(\epsilon^0), \\ I_C^{(4)}(\epsilon) &= I_C(2\epsilon) + \mathcal{O}(\epsilon^0). \end{aligned} \quad (\text{A.100})$$

Furthermore, we have used the ϵ -finite operators I_T defined as

$$I_T(\epsilon) \stackrel{\text{def}}{=} I_V(\epsilon) + I_S(\epsilon) + I_C(\epsilon), \quad (\text{A.101})$$

to simplify the NLO and NNLO calculations. Its expansion in powers of ϵ reads

$$I_T(\epsilon) = \sum_{n=0}^{\infty} \epsilon^n I_T^{(n)}, \quad (\text{A.102})$$

with expansion coefficients given by

$$\begin{aligned} I_T^{(0)} &= - \sum_{(ij)} (\mathbf{T}_i \cdot \mathbf{T}_j) \left[\left(2L_{\max} + \frac{1}{2} \log \eta_{ij} \right) \log \eta_{ij} - \frac{1}{2} L_{ij} \left(L_{ij} + \frac{2\gamma_i}{\mathbf{T}_i^2} \right) \right. \\ &\quad \left. + \text{Li}_2(1 - \eta_{ij}) + \frac{\pi^2}{2} \lambda_{ij} \right] + \sum_{i \in \mathcal{H}} \mathbf{T}_i^2 \left[2L_{\max}^2 - \frac{\pi^2}{6} - (2\tilde{L}_i \tilde{\gamma}_i^{(0)} - \tilde{\gamma}_i^{(1)}) \frac{\theta_{\mathcal{H}_f}}{\mathbf{T}_i^2} \right. \\ &\quad \left. - 2 \left(L_i^2 + 2L_i \tilde{L}_i + \tilde{L}_i \frac{\gamma_i}{\mathbf{T}_i^2} \right) \bar{\theta}_{\mathcal{H}_f} \right], \end{aligned} \quad (\text{A.103})$$

$$\begin{aligned} I_T^{(1)} &= \sum_{(ij)} (\mathbf{T}_i \cdot \mathbf{T}_j) \left[\frac{1}{6} (L_{ij}^3 + \log^3 \eta_{ij}) + 2L_{\max}^2 \log \eta_{ij} - \frac{\pi^2}{2} \lambda_{ij} L_{ij} \right. \\ &\quad \left. + \left(L_{\max} - \frac{1}{2} \log(1 - \eta_{ij}) \right) \log^2 \eta_{ij} + 2L_{\max} \text{Li}_2(1 - \eta_{ij}) - \text{Li}_3(\eta_{ij}) \right. \\ &\quad \left. - \text{Li}_3(1 - \eta_{ij}) + \frac{\gamma_i}{2\mathbf{T}_i^2} (L_{ij}^2 - \pi^2 \lambda_{ij}) \right] + \sum_{i \in \mathcal{H}} \mathbf{T}_i^2 \left\{ -\frac{4}{3} L_{\max}^3 + \frac{\pi^2}{3} L_{\max} - 3\zeta_3 \right. \\ &\quad \left. + \left[\left(2\tilde{L}_i^2 - \frac{\pi^2}{6} \right) \tilde{\gamma}_i^{(0)} - 2\tilde{L}_i \tilde{\gamma}_i^{(1)} + \tilde{\gamma}_i^{(2)} \right] \frac{\theta_{\mathcal{H}_f}}{\mathbf{T}_i^2} + \left[\frac{4}{3} L_i^3 + 4L_i^2 \tilde{L}_i + 4L_i \tilde{L}_i^2 \right. \right. \\ &\quad \left. \left. - \frac{\pi^2}{3} L_i + \frac{\gamma_i}{\mathbf{T}_i^2} \left(2\tilde{L}_i^2 - \frac{\pi^2}{6} \right) \right] \bar{\theta}_{\mathcal{H}_f} \right\}. \end{aligned} \quad (\text{A.104})$$

In these definitions, we have used

$$\theta_{\mathcal{H}_f} \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } i \in \mathcal{H}_f \\ 0 & \text{otherwise,} \end{cases} \quad \bar{\theta}_{\mathcal{H}_f} \stackrel{\text{def}}{=} 1 - \theta_{\mathcal{H}_f}, \quad (\text{A.105})$$

and also the coefficients $\tilde{\gamma}_i^{(n)}$, which corresponds to the expansion

$$\tilde{\gamma}_i(\epsilon) = \sum_{n=0}^{\infty} \epsilon^n \tilde{\gamma}_i^{(n)}, \quad (\text{A.106})$$

where (cf. Eq. (A.59))

$$\begin{aligned} \tilde{\gamma}_g(\epsilon) &\stackrel{\text{def}}{=} \gamma_{z,g \rightarrow gg}^{22}(L_i), \\ \tilde{\gamma}_q(\epsilon) &\stackrel{\text{def}}{=} \gamma_{z,q \rightarrow qg}^{22}(L_i) + \gamma_{z,q \rightarrow gq}^{22}. \end{aligned} \quad (\text{A.107})$$

Partitions at NLO and NNLO for an arbitrary number of final-state particles

To handle the infra-red (IR) singularities in processes with a large number of final-state particles, we require partitions that separate between resolved and potentially unresolved partons. To construct these partitions, consider a process that at leading order (LO) involves two partons in the initial state, N partons in the final state, and an arbitrary colorless final state, i.e.,

$$f_a(p_a) + f_b(p_b) \rightarrow f_1(p_1) + \dots + f_N(p_N) + X. \quad (\text{B.1})$$

We define $\mathcal{H} = \{a, b, 1, \dots, N\}$ as the set of $N + 2$ initial- and final-state hard partons, and $\mathcal{H}_f = \{1, \dots, N\}$ as the set of N final-state hard partons.

At next-to-leading order (NLO), an additional particle is introduced in the final state to account for the real-emission process. We denote the corresponding set of final-state partons as

$$\psi_{N+1} \stackrel{\text{def}}{=} \{f_1, f_2, \dots, f_N, f_{N+1}\}. \quad (\text{B.2})$$

In principle, any of these final-state partons can become unresolved. Suppose we want to describe a situation where this occurs with a parton i . We then write the set of $N + 1$ partons as

$$\psi_{N+1} \stackrel{\text{def}}{=} \{i, \psi_N^{(i)}\}, \quad \text{with} \quad \psi_N^{(i)} = \psi_{N+1} \setminus \{i\}, \quad (\text{B.3})$$

where we introduce the function

$$d^{(i)} \stackrel{\text{def}}{=} \prod_{k \in \psi_N^{(i)}} p_{k,\perp} \prod_{\substack{l, m \in \psi_N^{(i)} \\ l < m}} (1 - \cos \theta_{lm}), \quad (\text{B.4})$$

with $p_{k,\perp}$ being the transverse momentum of parton k .¹ These functions are then used to construct the partitions

$$\Delta^{(i)} \stackrel{\text{def}}{=} \frac{d^{(i)}}{\sum_{j \in \psi_{N+1}} d^{(j)}}, \quad i \in \psi_{N+1}. \quad (\text{B.5})$$

¹We note that in the case of only one hard jet, say k , $d^{(i)}$ reduces to $p_{k,\perp}$.

It follows from their definition that the functions $\Delta^{(i)}$ provide a partition of unity

$$\sum_{i \in \psi_{N+1}} \Delta^{(i)} = 1. \quad (\text{B.6})$$

It is straightforward to determine the action of soft and collinear operators on the partition functions. In the soft limit of parton k , described by the operator S_k , we find

$$S_k \Delta^{(i)} = \delta_{ki}. \quad (\text{B.7})$$

In the limit where partons l and m become collinear, we have

$$C_{lm} \Delta^{(i)} = \begin{cases} 0, & \text{if } l, m \neq i, \\ 1, & \text{if } l = i, m \in \{a, b\}, \\ z_{m,i}, & \text{if } l = i, m \in \psi_N, \end{cases} \quad (\text{B.8})$$

where $z_{m,i} = E_m/(E_m + E_i)$, and we assume that partons a and b are in the initial state. The limits obtained by interchanging the assignments of l and m follow naturally from the above formulas and are not shown for this reason.

A new element required for NNLO computations is the double-real emission process. To construct the corresponding partition functions, we consider an extended set of final-state partons

$$\psi_{N+2} \stackrel{\text{def}}{=} \{f_1, f_2, \dots, f_{N+1}, f_{N+2}\}. \quad (\text{B.9})$$

Two of these final-state partons can become unresolved, and we assume that this happens with partons i and j . We then write

$$\psi_{N+2} = \{(i, j), \psi_N^{(ij)}\}, \quad (\text{B.10})$$

define functions $d^{(ij)}$ as follows:

$$d^{(ij)} \stackrel{\text{def}}{=} \prod_{k \in \psi_N^{(ij)}} p_{k,\perp} \prod_{\substack{l, m \in \psi_N^{(ij)} \\ l < m}} (1 - \cos \theta_{lm}), \quad (\text{B.11})$$

and use them to construct the NNLO partitions

$$\Delta^{(ij)} \stackrel{\text{def}}{=} \frac{d^{(ij)}}{\sum_{(lm) \in \psi_{N+2}} d^{(lm)}}. \quad (\text{B.12})$$

Similar to the NLO case, the functions $\Delta^{(ij)}$ provide a partition of unity

$$\sum_{\substack{i, j \in \psi_{N+2} \\ i < j}} \Delta^{(ij)} = 1. \quad (\text{B.13})$$

For the NNLO computation, we require the double-soft (S_{lm}), the single-soft (S_i),

the collinear (C_{lk}) and the triple-collinear ($C_{lk,m}$) limits of the partition functions $\Delta^{(ij)}$. The double-soft limit reads

$$S_{lm} \Delta^{(ij)} = \delta_{(ij),(lm)}, \quad (\text{B.14})$$

where the Kronecker delta indicates that the unordered pair (ij) should coincide with the unordered pair (lm) for this limit to be different from zero. The single-soft limit is

$$S_l \Delta^{(ij)} = \begin{cases} 0, & l \neq i, l \neq j, \\ \Delta^{(j)}, & l = i, \\ \Delta^{(i)}, & l = j, \end{cases} \quad (\text{B.15})$$

where $\Delta^{(i)}$ and $\Delta^{(j)}$ in the above formulas are NLO partitions constructed for sets $\psi_{N+1} = \{j, \psi_N^{(ij)}\}$ and $\psi_{N+1} = \{i, \psi_N^{(ij)}\}$, respectively.

Next, we consider the collinear limits. We find

$$C_{lk} \Delta^{(ij)} = \begin{cases} 0, & l \neq i, j, k \neq i, j, \\ \Delta^{(k')}, & l \in \{a, b\}, k \in \{i, j\}, \\ \Delta^{([ij])}, & \{l, k\} = \{i, j\}, \\ z_{k,i} \Delta^{(j)}, & l = i, k \neq j, \end{cases} \quad (\text{B.16})$$

where $k' = i$ if $k = j$ and $k' = j$ if $k = i$, and $[ij]$ represents the *clustered particle* whose four-momentum is given by $p_{[ij]} = (1 + E_j/E_i) p_i$, and the function $\Delta^{([ij])}$ is constructed from the set $\psi_{N+1} = \{[ij], \psi_N^{(ij)}\}$. In the final line of Eq. (B.16), the Δ -function is constructed using the transverse momentum of the clustered particle $[kl]$.

It is instructive to explain how the last formula in (B.16) is derived, since the other formulas in that equation can be computed in a similar way. To describe the collinear $i \parallel k$ limit, where k is a final state particle, we write $\Delta^{(ij)}$ as follows

$$\Delta^{(ij)} = \frac{d^{(ij)}}{d^{(ij)} + d^{(ik)} + \sum_{m \neq k, j} d^{(im)} + d^{(kj)} + \sum_{m \neq i, j} d^{(km)} + \sum_{m, n \neq i, k} d^{(mn)}}. \quad (\text{B.17})$$

We now study what happens to the various entries in the above formula when the relevant limit is taken. First, we note that the numerator $d^{(ij)}$ does not contain i but contains k . We replace $p_{\perp, k}$ with $p_{\perp, [ik]}$ and write the resulting expression as

$$C_{ik} d^{(ij)} = \frac{E_k}{E_k + E_i} d^{(j)} = z_{k,i} d^{(j)}, \quad (\text{B.18})$$

where $d^{(j)}$ is constructed using the list $\{j, \psi_{N+2}^{(ij)}(k \rightarrow [ki])\}$. The various entries in the denominator of Eq. (B.17) behave as follows

$$C_{ik} d^{(ik)} = d^{([ik])}, \quad C_{ik} d^{(kj)} = z_{i,k} d^{(j)},$$

$$C_{ik} \sum_{m \neq k, j}^N d^{(im)} = z_{k,i} \sum_{m \neq k, j}^N d^{(m)}, \quad C_{ik} \sum_{m \neq i, j}^N d^{(km)} = z_{i,k} \sum_{m \neq i, j}^N d^{(m)}. \quad (\text{B.19})$$

Therefore

$$C_{ik} \Delta^{(ij)} = z_{k,i} \frac{d^{(j)}}{d^{([ik])} + d^{(j)} + \sum_{m \neq j, i, k} d^{(m)}} = z_{k,i} \Delta^{(j)}, \quad (\text{B.20})$$

with $\Delta^{(j)}$ being a NLO partition where partons i and k that appear in the original list of partons are clustered together.

Finally, formulas for triple-collinear limits can be derived in a similar way. We find that the only non-vanishing limits are

$$C_{kij} \Delta^{(ij)} = \begin{cases} 1, & k \in \{a, b\} \\ z_{k,ij} \Delta^{(ij)}, & k \in \{1, 2, \dots\}, \end{cases} \quad (\text{B.21})$$

where $z_{k,ij} = E_k / (E_k + E_i + E_j)$.

In addition to Δ -partitions, which allow us to separate resolved and potentially unresolved partons, we require angular partition functions ω . These functions are supposed to define possible collinear singular directions between unresolved partons. Below we give an example of how such functions can be designed.

We begin with the construction of these angular partition functions at NLO. To this end, we consider a situation where parton m is potentially unresolved, so that $\psi_{N+1} = \{m, \psi_N^{(m)}\}$. We define the quantities

$$g_{kl} = \rho_{kl}^{-1}, \quad (\text{B.22})$$

and use them to write the function ω^{mi} as

$$\omega^{mi} \stackrel{\text{def}}{=} \frac{g_{im}}{\sum_{j \in \psi_N^{(m)}} g_{jm}}, \quad i \in \psi_N^{(m)}. \quad (\text{B.23})$$

Since

$$\sum_{i \in \psi_N^{(m)}} \omega^{mi} = 1, \quad C_{km} \omega^{mi} = \delta_{ki}, \quad (\text{B.24})$$

the functions ω^{mi} possess the required properties to be used as angular partitions in NLO computations.

We continue with the discussion of the NNLO case, where partons m and n are potentially unresolved and the remaining N hard partons are described by the set $\psi_N^{(mn)}$. We proceed as follows. First, we employ the NLO partitions to construct a partition of unity as

$$1 = \sum_{(ij)} \omega^{mi} \omega^{nj} + \sum_{i, j \in \mathcal{H}} \delta_{ij} \omega^{mi} \omega^{nj}, \quad (\text{B.25})$$

where (ij) means $i, j \in \mathcal{H}$, with $i \neq j$. The two sums on the right-hand side are almost the right partitions for double- and triple-collinear limits except for the fact that the collinear

$m \parallel n$ singularity is present in both terms of this formula. However, we would like to move it into the triple-collinear partition. To achieve this, we introduce yet another partition of unity which involves ρ_{mn} , ρ_{im} and ρ_{jn} only, and write

$$1 = \frac{\rho_{mn}}{d_{mnij}} + \frac{\rho_{im} + \rho_{jn}}{d_{mnij}}, \quad (\text{B.26})$$

where

$$d_{mnij} \stackrel{\text{def}}{=} \rho_{mn} + \rho_{im} + \rho_{jn}. \quad (\text{B.27})$$

We now employ these expressions to define the double-collinear partition

$$\omega^{mi,nj} \stackrel{\text{def}}{=} \omega^{mi} \omega^{nj} \frac{\rho_{mn}}{d_{mnij}}, \quad i \neq j, \quad (\text{B.28})$$

and the triple-collinear partition

$$\omega^{mi,ni} \stackrel{\text{def}}{=} \omega^{mi} \omega^{ni} + \omega^{ni} \sum_{\substack{j \in \mathcal{H} \\ j \neq i}} \frac{\rho_{jm} \omega^{nj}}{d_{mnji}} + \omega^{mi} \sum_{\substack{j \in \mathcal{H} \\ j \neq i}} \frac{\rho_{jn} \omega^{nj}}{d_{mnij}}. \quad (\text{B.29})$$

It is easy to check that the following identity holds

$$1 = \sum_{(ij)} \omega^{mi,nj} + \sum_{i \in \mathcal{H}} \omega^{mi,ni}. \quad (\text{B.30})$$

The partitions constructed in Eqs (B.28) and (B.29) satisfy all the properties that we need for NNLO QCD computations. In particular, each partition selects a minimal number of collinear singularities and satisfies the following relations

$$\begin{aligned} C_{im} \omega^{mi,nj} &= \omega_{i||m}^{mi,nj} \stackrel{\text{def}}{=} \lim_{\rho_{im} \rightarrow 0} \omega^{mi,nj}, & C_{jn} \omega^{mi,nj} &= \omega_{j||n}^{mi,nj} \stackrel{\text{def}}{=} \lim_{\rho_{jn} \rightarrow 0} \omega^{mi,nj}, \\ C_{mn} \omega^{mi,nj} &= \delta_{ij} \omega_{m||n}^{mi,nj} \stackrel{\text{def}}{=} \lim_{\rho_{mn} \rightarrow 0} \omega^{mi,nj}, \\ C_{im} C_{mn} \omega^{mi,nj} &= \delta_{ij}, & C_{jn} C_{mn} \omega^{mi,nj} &= \delta_{ij}, \\ C_{im} C_{jn} \omega^{mk,nl} &= \delta_{ik} \delta_{jl}, & C_{im} C_{in} \omega^{mj,nj} &= \delta_{ij}, \\ C_{mn,i} \omega^{mi,nj} &= \delta_{ij}. \end{aligned} \quad (\text{B.31})$$

We note that these relations are important for simplifying the required subtraction terms. The partitions in Eqs (B.28) and (B.29) correspond to those defined in Eq. (B.14) in Ref. [23] when we restrict them to the case of color-singlet production, i.e., $N = 0$.

Phase-space parametrization and collinear limits

Contents

C.1 Phase-space parametrizations for unresolved partons	227
C.2 Action of the collinear operators on the phase space	230

C.1 Phase-space parametrizations for unresolved partons

In this section we describe phase-space parametrizations for two unresolved partons that naturally achieve the angular sectoring required for NNLO computations [19]. We recall that there are two distinct kinematic configurations that require different parametrizations. The first is a triple-collinear configuration which requires a genuine NNLO parametrization to describe strongly-ordered collinear limits. The second is the case where the two partons are emitted by different hard legs and can be described by two independent NLO-like parametrizations. In both cases, we begin by separating the energy and the angular parts of the phase space and write

$$[dp_m][dp_n] = (dE_m E_m^{1-2\epsilon}) (dE_n E_n^{1-2\epsilon}) [d\Omega_{mn}^{(d-1)}], \quad (C.1)$$

where

$$[d\Omega_{mn}^{(d-1)}] \stackrel{\text{def}}{=} [d\Omega_m^{(d-1)}][d\Omega_n^{(d-1)}], \quad [d\Omega_i^{(d-1)}] \stackrel{\text{def}}{=} \frac{d\Omega_i^{(d-1)}}{2(2\pi)^{d-1}}. \quad (C.2)$$

We first focus on the triple-collinear sectors and assume that the unresolved partons m and n are emitted by a hard parton i , with $i \in \mathcal{H}$. It is convenient to choose the momentum of parton i as the reference direction. We then write

$$\begin{aligned} p_m^\mu &= E_m (t^\mu + \cos \theta_{im} e_i^\mu + \sin \theta_{im} b^\mu), \\ p_n^\mu &= E_n (t^\mu + \cos \theta_{in} e_i^\mu + \sin \theta_{in} (\cos \phi_{mn} b^\mu + \sin \phi_{mn} a^\mu)), \end{aligned} \quad (C.3)$$

where

$$t^\mu = (1, \mathbf{0}), \quad e_i^\mu = (0, \mathbf{n}_i), \quad p_i^\mu = E_i (t^\mu + e_i^\mu). \quad (C.4)$$

Here n_i is a unit vector in $(d-1)$ spatial dimensions, and a^μ and b^ν are d -dimensional unit vectors such that

$$t \cdot a = e_i \cdot a = t \cdot b = e_i \cdot b = a \cdot b = 0. \quad (\text{C.5})$$

We can use this parametrization to express the angular part of the phase space as [19]

$$\begin{aligned} [d\Omega_{\text{mn}}^{(d-1)}] &= \frac{d\Omega_b^{(d-2)} d\Omega_a^{(d-3)}}{2^{6\epsilon} (2\pi)^{2d-2}} [\eta_{\text{im}}(1-\eta_{\text{im}})]^{-\epsilon} [\eta_{\text{in}}(1-\eta_{\text{in}})]^{-\epsilon} \\ &\times \frac{|\eta_{\text{im}} - \eta_{\text{in}}|^{1-2\epsilon}}{D^{1-2\epsilon}} \frac{d\eta_{\text{im}} d\eta_{\text{in}} d\lambda}{[\lambda(1-\lambda)]^{\frac{1}{2}+\epsilon}}, \end{aligned} \quad (\text{C.6})$$

where

$$D = \eta_{\text{im}} + \eta_{\text{in}} - 2\eta_{\text{im}}\eta_{\text{in}} + 2(2\lambda - 1)\sqrt{\eta_{\text{im}}\eta_{\text{in}}(1-\eta_{\text{im}})(1-\eta_{\text{in}})}. \quad (\text{C.7})$$

The variable λ parametrizes the dependence on the azimuthal angle ϕ_{mn} through the relation

$$\sin^2 \phi_{\text{mn}} = 4\lambda(1-\lambda) \frac{|\eta_{\text{im}} - \eta_{\text{in}}|^2}{D^2}. \quad (\text{C.8})$$

The phase space can be split into four different sectors that we will refer to as (a), (b), (c), (d). The following parametrizations are chosen for each of the four sectors

$$a) \eta_{\text{im}} = x_3, \quad \eta_{\text{in}} = x_3 x_4 / 2, \quad (\text{C.9})$$

$$b) \eta_{\text{im}} = x_3, \quad \eta_{\text{in}} = x_3(1 - x_4/2), \quad (\text{C.10})$$

$$c) \eta_{\text{im}} = x_3 x_4 / 2, \quad \eta_{\text{in}} = x_3, \quad (\text{C.11})$$

$$d) \eta_{\text{im}} = x_3(1 - x_4/2), \quad \eta_{\text{in}} = x_3, \quad (\text{C.12})$$

with $0 \leq x_{3,4} \leq 1$. We use them to obtain explicit expressions for

$$[d\Omega_{\text{mn}}^{(i)}] \stackrel{\text{def}}{=} [d\Omega_{\text{mn}}^{(d-1)}] \theta^{(i)}, \quad i = a, b, c, d, \quad (\text{C.13})$$

with $\theta^{(i)}$ defined in Eq. (4.28). It turns out that the angular phase spaces for sectors (a) and (c) and for sectors (b) and (d) are identical. For sectors (a) and (c) we find

$$\begin{aligned} [d\Omega_{\text{mn}}^{(a,c)}] &= \left[\frac{1}{8\pi^2} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \right]^2 \left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \right] \frac{[d\Omega_b^{(d-2)}]}{[\Omega_b^{(d-2)}]} \frac{[d\Omega_a^{(d-3)}]}{[\Omega_a^{(d-3)}]} \\ &\times \frac{dx_3}{x_3^{1+2\epsilon}} \frac{dx_4}{x_4^{1+2\epsilon}} \frac{d\lambda}{\pi[\lambda(1-\lambda)]^{\frac{1}{2}+\epsilon}} \left(256 F_\epsilon^{(a,c)} \right)^{-\epsilon} 4F_0^{(a,c)} x_3^2 x_4, \end{aligned} \quad (\text{C.14})$$

where

$$\begin{aligned} F_\epsilon^{(a,c)} &= \frac{(1-x_3)(1-x_3 x_4/2)(1-x_4/2)^2}{4[N(x_3, x_4/2, \lambda)]^2}, \\ F_0^{(a,c)} &= \frac{1-x_4/2}{2N(x_3, x_4/2, \lambda)}. \end{aligned} \quad (\text{C.15})$$

For sectors (b) and (d), we obtain

$$[\mathrm{d}\Omega_{\mathrm{mn}}^{(b,d)}] = \left[\frac{1}{8\pi^2} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \right]^2 \left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \right] \frac{[\mathrm{d}\Omega_b^{(d-2)}]}{[\Omega_b^{(d-2)}]} \frac{[\mathrm{d}\Omega_a^{(d-3)}]}{[\Omega_a^{(d-3)}]} \\ \times \frac{\mathrm{d}x_3}{x_3^{1+2\epsilon}} \frac{\mathrm{d}x_4}{x_4^{1+2\epsilon}} \frac{\mathrm{d}\lambda}{\pi[\lambda(1-\lambda)]^{\frac{1}{2}+\epsilon}} \left(256 F_\epsilon^{(b,d)} \right)^{-\epsilon} 4 F_0^{(b,d)} x_3^2 x_4^2, \quad (\text{C.16})$$

where

$$F_\epsilon^{(b,d)} = \frac{(1-x_3)(1-x_4/2)(1-x_3(1-x_4/2))}{4[N(x_3, 1-x_4/2, \lambda)]^2}, \\ F_0^{(b,d)} = \frac{1}{4N(x_3, 1-x_4/2, \lambda)}. \quad (\text{C.17})$$

The function $N(x_3, x_4, \lambda)$ introduced in the above equations reads

$$N(x_3, x_4, \lambda) = 1 + x_4(1-2x_3) - 2(1-2\lambda)\sqrt{x_4(1-x_3)(1-x_3x_4)}. \quad (\text{C.18})$$

To simplify the subtraction terms, we need particular collinear limits of the unresolved phase space. To obtain those, we note that the following identities hold

$$\lim_{x_4 \rightarrow 0} F_\epsilon^{(a,c)} = \frac{1-x_3}{2}, \quad \lim_{x_4 \rightarrow 0} F_0^{(a,c)} = \frac{1}{2}, \\ \lim_{x_4 \rightarrow 0} F_\epsilon^{(b,d)} = \frac{1}{64\lambda^2}, \quad \lim_{x_4 \rightarrow 0} F_0^{(b,d)} = \frac{1}{16\lambda(1-x_3)}. \quad (\text{C.19})$$

The $x_4 \rightarrow 0$ limit corresponds to the $\mathbf{n} \parallel i$ and $\mathbf{m} \parallel i$ collinear limits in sectors (a) and (c), respectively, and to the $\mathbf{m} \parallel \mathbf{n}$ limit in sectors (b) and (d). The singular quantities in sectors (a) and (c) are $\eta_{i\mathbf{n}}$ and $\eta_{i\mathbf{m}}$, respectively, and they are given in Eqs (C.9) and (C.11). For sectors (b) and (d), the limit of the corresponding singular variable is more complex. It reads

$$\lim_{x_4 \rightarrow 0} \eta_{\mathrm{mn}} = \lim_{x_4 \rightarrow 0} \frac{x_3 x_4^2}{4N(x_3, 1-x_4, \lambda)} = \frac{x_3 x_4^2}{16\lambda(1-x_3)} \equiv \bar{\eta}_{\mathrm{mn}}. \quad (\text{C.20})$$

The phase-space parametrization is significantly simpler for the double-collinear partitions. Consider the case when parton $\mathbf{m}$ is collinear to parton i and parton $\mathbf{n}$ to parton j , with $i \neq j$. We parametrize the momenta $p_{\mathbf{m}}^\mu$ and $p_{\mathbf{n}}^\mu$ using the momenta of partons i and j respectively, i.e.,

$$p_{\mathbf{m}}^\mu = E_{\mathbf{m}}(t^\mu + \cos \theta_{i\mathbf{m}} e_i^\mu + \sin \theta_{i\mathbf{m}} b_{\mathbf{m}}^\mu), \\ p_{\mathbf{n}}^\mu = E_{\mathbf{n}}(t^\mu + \cos \theta_{j\mathbf{n}} e_j^\mu + \sin \theta_{j\mathbf{n}} b_{\mathbf{n}}^\mu), \quad (\text{C.21})$$

and set

$$\eta_{i\mathbf{m}} = x_3, \quad \eta_{j\mathbf{n}} = x_4. \quad (\text{C.22})$$

We then write the angular phase space for the double-collinear partition $[d\Omega_{mn}^{dc}]$ as

$$[d\Omega_{mn}^{dc}] \stackrel{\text{def}}{=} [d\Omega_m^{(d-1)}][d\Omega_n^{(d-1)}], \quad (\text{C.23})$$

where

$$\begin{aligned} [d\Omega_m^{(d-1)}] &= \left[\frac{1}{8\pi^2} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \right] 2^{4-4\epsilon} \frac{[d\Omega_m^{(d-2)}]}{[\Omega^{(d-2)}]} \frac{dx_3}{x_3^{1+\epsilon}} (1-x_3)^{-\epsilon} x_3, \\ [d\Omega_n^{(d-1)}] &= \left[\frac{1}{8\pi^2} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \right] 2^{4-4\epsilon} \frac{[d\Omega_n^{(d-2)}]}{[\Omega^{(d-2)}]} \frac{dx_4}{x_4^{1+\epsilon}} (1-x_4)^{-\epsilon} x_4. \end{aligned} \quad (\text{C.24})$$

C.2 Action of the collinear operators on the phase space

In our definitions of the angular terms $\Omega_{1,\dots,4}$ in Eqs (4.36 - 4.39), the collinear operators act on the phase space of the two unresolved partons. As we have seen, it is useful to rewrite the subtraction terms in such a way that these operators do not act on the phase-space measure. We have quoted the results in the main text of this thesis without deriving them, see e.g. Eqs (4.78) and (4.79). The goal of this section is to provide the omitted details.

We begin by considering the double-collinear partitioning with a collinear operator C_{im} ; an example can be found in the first term on the right-hand side of Eq. (4.74). Since in the double-collinear parametrization of the phase space the collinear limit $i \parallel m$ is controlled by the variable x_3 (see Eq. (C.22)), we find

$$\begin{aligned} \int C_{im} \frac{[d\Omega_{mn}^{dc}]}{\rho_{im}} [\dots] &= \left[\frac{1}{8\pi^2} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \right] 2^{3-4\epsilon} \int \frac{[d\Omega_m^{(d-2)}]}{[\Omega^{(d-2)}]} [d\Omega_n^{(d-1)}] \int_0^1 \frac{dx_3}{x_3^{1+\epsilon}} C_{im} [\dots] \\ &= - \left[\frac{1}{8\pi^2} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \right] \frac{2^{3-4\epsilon}}{\epsilon} \int \frac{[d\Omega_m^{(d-2)}]}{[\Omega^{(d-2)}]} [d\Omega_n^{(d-1)}] C_{im} [\dots], \end{aligned} \quad (\text{C.25})$$

where $[\dots]$ stands for generic non-singular contributions whose exact form is not relevant for the following discussion. If we repeat the above steps without acting with C_{im} on the phase space, we find

$$\begin{aligned} &\int \frac{[d\Omega_{mn}^{dc}]}{\rho_{im}} C_{im} [\dots] \\ &= \left[\frac{1}{8\pi^2} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \right] 2^{3-4\epsilon} \int \frac{[d\Omega_m^{(d-2)}]}{[\Omega^{(d-2)}]} [d\Omega_n^{(d-1)}] \int_0^1 \frac{dx_3}{x_3^{1+\epsilon}} (1-x_3)^{-\epsilon} C_{im} [\dots] \\ &= - \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \left[\frac{1}{8\pi^2} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \right] \frac{2^{3-4\epsilon}}{\epsilon} \int \frac{[d\Omega_m^{(d-2)}]}{[\Omega^{(d-2)}]} [d\Omega_n^{(d-1)}] C_{im} [\dots]. \end{aligned} \quad (\text{C.26})$$

Comparing the two formulas, we conclude that

$$\int C_{im} \frac{[d\Omega_{mn}^{dc}]}{\rho_{im}} [\dots] = \frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)} \int \frac{[d\Omega_{mn}^{dc}]}{\rho_{im}} C_{im} [\dots]. \quad (C.27)$$

We can use the above relation when rewriting Eq. (4.136) as Eq. (4.137). Since in this case we have two collinear operators $C_{jn}C_{im}$, we need to apply it twice, i.e.,

$$\int C_{jn}C_{im} \frac{[d\Omega_{mn}^{dc}]}{\rho_{im}\rho_{jn}} [\dots] = \left[\frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)} \right]^2 \int \frac{[d\Omega_{mn}^{dc}]}{\rho_{im}\rho_{jn}} C_{jn}C_{im} [\dots], \quad (C.28)$$

so that Eq. (4.136) becomes

$$\Sigma_{RR,2c}|\mathcal{H},gg\rangle = - \left[\frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)} \right]^2 \sum_{(ij)} \langle \bar{S}_{mn} \bar{S}_n C_{jn} C_{im} \omega^{mi,nj} \Delta^{(mn)} \Theta_{mn} F_{LM}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \rangle. \quad (C.29)$$

We stress that the absence of the phase space $[dp_m][dp_n]$ in the above equation indicates that the collinear operators $C_{jn}C_{im}$ no longer act on it. Notice that Eq. (C.28) has been used to obtain also Eq. (4.140).

Similar formulas can also be derived for the triple-collinear partitions that involve sector $\theta^{(c)}$. As an example, we discuss the second term on the right-hand side of Eq. (4.74). In this case, the collinear limit $i \parallel m$ corresponds to the $x_4 \rightarrow 0$ limit in the phase space parametrization in Eq. (C.14). We use Eq. (C.19) to compute this limit and find

$$\int C_{im} \frac{[d\Omega_{mn}^{(c)}]}{\rho_{im}} [\dots] = \frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)} \int \frac{[d\Omega_{mn}^{(d-1)}]}{\rho_{im}} (\eta_{in}/2)^{-\epsilon} C_{im} [\dots], \quad (C.30)$$

where the integration over the angular variables of parton m on the right hand side of Eq. (C.30) is *not restricted* to sector $\theta^{(c)}$ anymore. It follows from the above discussion that Eq. (4.74) can be rewritten as

$$\begin{aligned} \Sigma_{RR,1c}^{(a,c,dc)} &= \frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)} \left\langle \mathcal{S}(\mathbf{m}, \mathbf{n}) \left[\sum_{(ij)} C_{im} \omega^{mi,nj} \right. \right. \\ &\quad \left. \left. + \sum_{i \in \mathcal{H}} (\eta_{in}/2)^{-\epsilon} C_{im} \omega^{mi,ni} \right] \Delta^{(mn)} F_{LM}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \right\rangle. \end{aligned} \quad (C.31)$$

This expression is the starting point to obtain Eqs (4.78) and (4.79).

Finally, we perform similar manipulations for sector $\theta^{(b)}$, where the collinear limit of interest is $m \parallel n$. This limit corresponds to $x_4 \rightarrow 0$ in the phase space parametrization given in Eq. (C.16). Using Eq. (C.19), we find

$$\begin{aligned} \int C_{mn} \frac{[d\Omega_{mn}^{(b,d)}]}{\rho_{mn}} [\dots] &= \left[\frac{1}{8\pi^2} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \right] N_\epsilon^{(b,d)} \int [d\Omega_{mn}^{(d-1)}] \\ &\quad \times \eta_{i[mn]}^{-\epsilon} (1 - \eta_{i[mn]})^\epsilon d\Lambda \frac{d\Omega_a^{(d-3)}}{[\Omega^{(d-3)}]} \frac{dx_4}{x_4^{1+2\epsilon}} C_{mn} [\dots]. \end{aligned} \quad (C.32)$$

The normalization constants $N_\epsilon^{(b,d)}$ that appear in Eq. (C.32) can be found in Eq. (A.8), while $[d\Omega_{[mn]}^{(d-1)}]$ is the (exact) angular phase space of the clustered parton $[mn]$, whose momentum $p_{[mn]}^\mu = p_m^\mu + p_n^\mu$ must be computed in the strict collinear limit. Furthermore we have introduced a new variable Λ such that

$$d\Lambda = \frac{\Gamma(1+\epsilon)\Gamma(1-\epsilon)}{\Gamma(1+2\epsilon)\Gamma(1-2\epsilon)} \frac{\lambda^{-1/2+\epsilon}(1-\lambda)^{-1/2-\epsilon}}{\pi} d\lambda. \quad (C.33)$$

We note that the action of the operator C_{mn} on the matrix element squared is non-trivial because it can lead to integrands that depend on the parameter λ and the transverse vector a^μ . This phenomenon, known as *spin correlations*, is discussed in the next appendix. Here we consider only those terms for which the action of C_{mn} in Eq. (4.98) does not lead to such terms. In this case, we can integrate over x_4 , the directions of a^μ , and the azimuthal variable Λ using

$$\int d\Lambda = 1. \quad (C.34)$$

Comparing the result with the one that is obtained when the collinear operator C_{mn} does not act on the phase space, we find

$$\int C_{mn} \frac{[d\Omega_{mn}^{(b,d)}]}{\rho_{mn}} [\dots] = 2^{2\epsilon-1} \frac{\Gamma(1+2\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1+\epsilon)\Gamma(1-\epsilon)} \int \frac{[d\Omega_{mn}^{(d-1)}]}{\rho_{mn}} C_{mn} [\dots], \quad (C.35)$$

where the integration over the angular variables of partons on the right-hand side is unrestricted. We use this relation in Eq. (4.104) and the analysis that follows.

As we just mentioned, the action of the collinear operator C_{mn} on matrix elements may result in a limit that depends on λ and a^μ . In such cases Eq. (C.35), cannot be used. To understand how to proceed, we write (see Appendix E)

$$C_{mn} F_{LM}^{\mathcal{H}}[m_g, n_g] = \frac{g_{s,b}^2 \mu_0^{2\epsilon}}{E_n E_m \rho_{mn}} \left[-g^{\mu\nu} P_{gg}^{(0)}(z) + P_{gg}^\perp(z) \kappa_{\perp,(b)}^\mu \kappa_{\perp,(b)}^\nu \right] F_{LM,\mu\nu}^{\mathcal{H}}[[mn]_g], \quad (C.36)$$

where vector $\kappa_{\perp,(b)}^\mu$ is a unit space-like vector which is orthogonal to p_m^μ

$$\kappa_{\perp,(b)} \cdot p_m = 0. \quad (C.37)$$

Using the phase space parametrization for sector $\theta^{(b)}$, we can write this vector as

$$\kappa_{\perp,(b)}^\mu = a^\mu \sqrt{1-\lambda} + r_{i,(b)}^\mu \sqrt{\lambda}, \quad (C.38)$$

where vectors a^μ and b^μ were introduced in Eq. (C.3), and $r_{i,(b)}^\mu$ is the auxiliary spacelike vector ($r_{i,(b)} \cdot r_{i,(b)} = -1$) defined as

$$r_{i,(b)}^\mu \stackrel{\text{def}}{=} \sin \theta_{im} e_i^\mu - \cos \theta_{im} b^\mu. \quad (C.39)$$

The momentum of the clustered parton $[mn]_g$ is aligned with momentum p_m^μ , which does

not depend on λ and a^μ . Since $F_{\text{LM},\mu\nu}^{\mathcal{H}}[[\text{mn}]_g]$ is independent of λ and a^μ , we can integrate over $d\Omega_a^{(d-3)}$ and $d\Lambda$. Specifically, we need to calculate

$$\langle \kappa_{\perp,(b)}^\mu \kappa_{\perp,(b)}^\nu \rangle = \int d\Lambda \frac{d\Omega_a^{(d-3)}}{\Omega^{(d-3)}} \kappa_{\perp,(b)}^\mu \kappa_{\perp,(b)}^\nu. \quad (\text{C.40})$$

To compute this integral, we use Eq. (C.34) together with

$$\begin{aligned} \int d\Lambda \frac{d\Omega_a^{(d-3)}}{\Omega^{(d-3)}} a^\mu &= 0, & \int d\Lambda \frac{d\Omega_a^{(d-3)}}{\Omega^{(d-3)}} a^\mu a^\nu &= -\frac{g_{\perp,(d-3)}^{\mu\nu}}{d-3}, \\ \int d\Lambda \lambda &= \frac{1+2\epsilon}{2}, & \int d\Lambda (1-\lambda) &= \frac{1-2\epsilon}{2}, \end{aligned} \quad (\text{C.41})$$

and find

$$\begin{aligned} \langle \kappa_{\perp,(b)}^\mu \kappa_{\perp,(b)}^\nu \rangle &= -\frac{g_{\perp,(d-3)}^{\mu\nu}}{2} + \frac{1+2\epsilon}{2} r_{i,(b)}^\mu r_{i,(b)}^\nu \\ &= \frac{1}{2} \left[-g_{\perp,(d-3)}^{\mu\nu} + r_{i,(b)}^\mu r_{i,(b)}^\nu \right] + \epsilon r_{i,(b)}^\mu r_{i,(b)}^\nu \\ &\equiv -\frac{g_{\perp,(d-2)}^{\mu\nu}}{2} + \epsilon r_{i,(b)}^\mu r_{i,(b)}^\nu. \end{aligned} \quad (\text{C.42})$$

We then obtain

$$\langle \kappa_{\perp,(b)}^\mu \kappa_{\perp,(b)}^\nu \rangle F_{\text{LM},\mu\nu} = \frac{1}{2} F_{\text{LM}} + \epsilon r_{i,(b)}^\mu r_{i,(b)}^\nu F_{\text{LM},\mu\nu}, \quad (\text{C.43})$$

where we used

$$-g_{\perp,(d-2)}^{\mu\nu} F_{\text{LM},\mu\nu} = -g^{\mu\nu} F_{\text{LM},\mu\nu} = F_{\text{LM}}, \quad (\text{C.44})$$

as allowed by the transversality of scattering amplitudes.

Partition-dependent contributions

Contents

D.1 Derivation of $\mathcal{W}_i^{i n, \text{fin}}$	235
D.1.1 How $\mathcal{W}_i^{i n, \text{fin}}$ emerges in $\Sigma_{\text{DU}}^{\text{fin}, (2)} _{\mathcal{H}, gg}$	235
D.1.2 Explicit expression of $\mathcal{W}_i^{i n, \text{fin}}$	237
D.2 Derivation of $\mathcal{W}_i^{m n, \text{fin}}$	238
D.2.1 How $\mathcal{W}_i^{m n, \text{fin}}$ emerges in $\Sigma_{\text{RR}, 1c}^{(b, d), \text{sa}, \text{II}} _{\mathcal{H}, gg}$	238
D.2.2 Explicit expression of $\mathcal{W}_i^{m n, \text{fin}}$	241

This appendix is dedicated to discussing the double-unresolved contributions in Eq. (4.179), whose explicit expressions depend on the partition functions ω^{m_i, n_j} introduced in Appendix B.

D.1 Derivation of $\mathcal{W}_i^{i||n, \text{fin}}$

D.1.1 How $\mathcal{W}_i^{i||n, \text{fin}}$ emerges in $\Sigma_{\text{DU}}^{\text{fin}, (2)}|_{\mathcal{H}, gg}$

In this subsection, we focus on the derivation of the partition-dependent function $\mathcal{W}_i^{i||n, \text{fin}}$. We begin by considering the term $\Sigma_{\text{DU}}^{\text{fin}, (2)}|_{\mathcal{H}, gg}$ in Eq. (4.90), which we report here for convenience:

$$\begin{aligned} \Sigma_{\text{DU}}^{\text{fin}, (2)}|_{\mathcal{H}, gg} &= \sum_{i \in \mathcal{H}} \Sigma_{\text{DU}}^{\text{fin}, (2, i)}|_{\mathcal{H}, gg} \\ &= \sum_{i \in \mathcal{H}} \left\langle [(\eta_{im}/2)^{-\epsilon} - 1] \bar{C}_{im} S_m \langle \bar{S}_n C_{in} \omega^{m_i, n_i} \Delta^{(mn)} \Theta_{mn} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \rangle \right\rangle. \end{aligned} \quad (\text{D.1})$$

We proceed by evaluating Eq. (D.1) step by step, starting from the initial-state leg $i = a$. After the action of the operators $\bar{S}_n C_{in}$, we obtain

$$\begin{aligned} \Sigma_{\text{DU}}^{\text{fin}, (2, a)}|_{\mathcal{H}, gg} &= -\frac{[\alpha_s]}{\epsilon} 2^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \left\langle \int_0^{E_m} \frac{dE_n}{E_n^{1+2\epsilon}} \bar{S}_z \frac{P_{aa}(z)}{(1-z)^{-1}} \bar{C}_{am} [(\eta_{am}/2)^{-\epsilon} - 1] \right. \\ &\quad \left. \times \omega_{a||n}^{ma, na} \Delta^{(m)} S_m \frac{F_{\text{LM}}^{(z \cdot a)b}[\mathcal{H}_f|\mathbf{m}_g]}{z} \right\rangle, \end{aligned} \quad (\text{D.2})$$

and the subsequent action of the soft limit S_m on F_{LM} yields

$$S_m F_{\text{LM}}^{(z \cdot a)b}[\mathcal{H}_f | m_g] = -\frac{g_{s,b}^2 \mu_0^{2\epsilon}}{E_m^2} \sum_{(kl)} \frac{\rho_{kl}}{\rho_{km} \rho_{lm}} (\mathbf{T}_k \cdot \mathbf{T}_l) \cdot F_{\text{LM}}^{(z \cdot a)b}[\mathcal{H}_f] / N_g!, \quad (\text{D.3})$$

where (kl) stands for $k, l \in \mathcal{H}$, with $k \neq l$, and $1/N_g!$ is the symmetry factor of the LO function $F_{\text{LM}}^{(z \cdot a)b}[\mathcal{H}_f]$. Hence, we obtain

$$\begin{aligned} \Sigma_{\text{DU}}^{\text{fin}, (2, a)} |_{\mathcal{H}, gg} &= \frac{[\alpha_s]^2}{\epsilon} \left(\frac{2E_a}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \left\langle \bar{C}_{am} \left[-\sum_{(kl)} \int \frac{[d\Omega_m^{(d-1)}]}{[\Omega^{(d-2)}]} \right. \right. \\ &\quad \times \left. \left. [(\eta_{am}/2)^{-\epsilon} - 1] \frac{\rho_{kl}}{\rho_{km} \rho_{lm}} \omega_{a||n}^{\text{ma}, \text{na}}(\mathbf{T}_k \cdot \mathbf{T}_l) \right] \right. \\ &\quad \times (-1) \int_0^{E_{\text{max}}} \frac{dE_m}{E_m^{1+2\epsilon}} \int_{z_{m/a}}^1 dz \bar{S}_z \frac{P_{aa}(z)}{(1-z)^{2\epsilon}} \times \left. \frac{F_{\text{LM}}^{(z \cdot a)b}[\mathcal{H}_f] / N_g!}{z} \right\rangle, \end{aligned} \quad (\text{D.4})$$

where $z_{m/a} = 1 - E_m/E_a$. Then, we swap the order of integration as

$$\int_0^{E_{\text{max}}} dE_m \int_{z_{m/a}}^1 dz = \int_{z_{\text{min}}}^1 dz \int_{(1-z)E_a}^{E_{\text{max}}} dE_m, \quad z_{\text{min}} = 1 - \frac{E_{\text{max}}}{E_a}, \quad (\text{D.5})$$

we integrate over dE_m , and we get

$$\begin{aligned} \Sigma_{\text{DU}}^{\text{fin}, (2, a)} |_{\mathcal{H}, gg} &= \frac{[\alpha_s]^2}{\epsilon^2} \left(\frac{2E_a}{\mu} \right)^{-4\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \\ &\quad \times \left\langle \bar{C}_{am} \left[-\epsilon 2^{2\epsilon} \sum_{(kl)} \int \frac{[d\Omega_m^{(d-1)}]}{[\Omega^{(d-2)}]} [(\eta_{am}/2)^{-\epsilon} - 1] \frac{\rho_{kl}}{\rho_{km} \rho_{lm}} \omega_{a||n}^{\text{ma}, \text{na}}(\mathbf{T}_k \cdot \mathbf{T}_l) \right] \right. \\ &\quad \times (-1) \int_{z_{\text{min}}}^1 dz \bar{S}_z \frac{1}{2\epsilon} \left[\frac{P_{aa}(z)}{(1-z)^{4\epsilon}} - e^{-2\epsilon L_a} \frac{P_{aa}(z)}{(1-z)^{2\epsilon}} \right] \times \left. \frac{F_{\text{LM}}^{(z \cdot a)b}[\mathcal{H}_f] / N_g!}{z} \right\rangle. \end{aligned}$$

At this point, first we use the definitions of $\mathcal{P}_{\alpha\beta}^{(k)}$ in Eq. (3.69) to write the integral over dz as

$$\begin{aligned} & - \int_{z_{\text{min}}}^1 dz \bar{S}_z \frac{1}{2\epsilon} \left[\frac{P_{gg}(z)}{(1-z)^{4\epsilon}} - e^{-2\epsilon L_a} \frac{P_{gg}(z)}{(1-z)^{2\epsilon}} \right] \times \frac{F_{\text{LM}}^{(z \cdot a)b}[\mathcal{H}_f] / N_g!}{z} \\ &= - \int_0^1 dz \frac{1}{2\epsilon} \left[\mathcal{P}_{gg}^{(4)}(z, E_a) - e^{-2\epsilon L_a} \mathcal{P}_{gg}^{(2)}(z, E_a) \right] \times \frac{F_{\text{LM}}^{(z \cdot a)b}[\mathcal{H}_f] / N_g!}{z}. \end{aligned} \quad (\text{D.6})$$

Then, we introduce the splitting function

$$\mathcal{P}_{\alpha\alpha}^{\mathcal{W}}(z, E_a) \stackrel{\text{def}}{=} -\frac{1}{2\epsilon} \left(\frac{2E_a}{\mu} \right)^{-4\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \left[\mathcal{P}_{\alpha\alpha}^{(4)}(z, E_a) - e^{-2\epsilon L_a} \mathcal{P}_{\alpha\alpha}^{(2)}(z, E_a) \right], \quad (\text{D.7})$$

and, finally, we define the partition-dependent function

$$\mathcal{W}_i^{i||n} \stackrel{\text{def}}{=} -\epsilon 2^{2\epsilon} \sum_{(kl)} \int \frac{[\mathrm{d}\Omega_{\mathbf{m}}^{(d-1)}]}{[\Omega^{(d-2)}]} [(\eta_{im}/2)^{-\epsilon} - 1] \frac{\rho_{kl}}{\rho_{km}\rho_{lm}} \omega_{i||n}^{mi,ni}(\mathbf{T}_k \cdot \mathbf{T}_l). \quad (\text{D.8})$$

We point out that the expansion of $\mathcal{W}_a^{a||n}$ in powers of ϵ returns a series of $\mathcal{O}(\epsilon)$, and the operator $\bar{C}_{am}$ is also of $\mathcal{O}(\epsilon)$ once applied to $\mathcal{W}_a^{a||n}$. It follows that the expression $\bar{C}_{am}\mathcal{W}_i^{i||n}$ is of $\mathcal{O}(\epsilon^2)$. Therefore, we define a new partition-dependent function, i.e.,

$$\mathcal{W}_i^{i||n,\text{fin}} \stackrel{\text{def}}{=} \frac{1}{\epsilon^2} \bar{C}_{im} \mathcal{W}_i^{i||n} = \sum_{(kl)} \int \frac{\mathrm{d}\Omega_{\mathbf{m}}^{(3)}}{2\pi} \bar{C}_{im} \left[\log(\eta_{im}/2) \frac{\rho_{kl}}{\rho_{km}\rho_{lm}} \omega_{i||n}^{mi,ni} \right] (\mathbf{T}_k \cdot \mathbf{T}_l), \quad (\text{D.9})$$

which is ϵ -finite by construction. Notice that, in the definition of $\mathcal{W}_i^{i||n,\text{fin}}$, we neglect contributions of $\mathcal{O}(\epsilon)$ and set the spacetime dimension to $d = 4$. Thus, to conclude, we rewrite Eq. (D.4) as

$$\Sigma_{\text{DU}}^{\text{fin},(2,a)}|_{\mathcal{H},gg} = [\alpha_s]^2 \langle \mathcal{P}_{aa}^{\mathcal{W}} \otimes [\mathcal{W}_a^{a||n,\text{fin}} \cdot F_{\text{LM}}^{\mathcal{H}}] \rangle. \quad (\text{D.10})$$

This expression holds for leg b as well.

Now we consider the case of final-state radiation, i.e., $i \in \mathcal{H}_f$. The steps are analogous to those described above, with the splitting functions $\mathcal{P}_{\alpha\beta}^{(k)}$ replaced by the anomalous dimensions $\gamma_{z,f_i \rightarrow f_i g}^{2k}$, defined in Eq. (A.59), where f_i is the flavour of parton i . Thus, we find

$$\Sigma_{\text{DU}}^{\text{fin},(2,i)}|_{\mathcal{H},gg} = [\alpha_s]^2 \langle \gamma_{f_i \rightarrow f_i g}^{\mathcal{W}} [\mathcal{W}_i^{i||n,\text{fin}} \cdot F_{\text{LM}}^{\mathcal{H}}] \rangle, \quad (\text{D.11})$$

where

$$\gamma_{f_i \rightarrow f_i g}^{\mathcal{W}}(L_i) \stackrel{\text{def}}{=} \frac{1}{2\epsilon} \left(\frac{2E_i}{\mu} \right)^{-4\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \left[\gamma_{z,f_i \rightarrow f_i g}^{24}(L_i) - e^{-2\epsilon L_i} \gamma_{z,f_i \rightarrow f_i g}^{22}(L_i) \right]. \quad (\text{D.12})$$

Combining both initial- and final-state radiation, we obtain

$$\begin{aligned} \Sigma_{\text{DU}}^{\text{fin},(2)}|_{\mathcal{H},gg} &= [\alpha_s]^2 \left[\langle \mathcal{P}_{aa}^{\mathcal{W}} \otimes [\mathcal{W}_a^{a||n,\text{fin}} \cdot F_{\text{LM}}^{\mathcal{H}}] \rangle + \langle [\mathcal{W}_b^{b||n,\text{fin}} \cdot F_{\text{LM}}^{\mathcal{H}}] \otimes \mathcal{P}_{bb}^{\mathcal{W}} \rangle \right] \\ &\quad + \sum_{i \in \mathcal{H}} [\alpha_s]^2 \langle \gamma_{f_i \rightarrow f_i g}^{\mathcal{W}} [\mathcal{W}_i^{i||n,\text{fin}} \cdot F_{\text{LM}}^{\mathcal{H}}] \rangle. \end{aligned} \quad (\text{D.13})$$

D.1.2 Explicit expression of $\mathcal{W}_i^{i||n,\text{fin}}$

We present an explicit expression for $\mathcal{W}_i^{i||n,\text{fin}}$ as defined in Eq. (D.9). By expressing $\bar{C}_{im}$ as $1 - C_{im}$, we can rewrite Eq. (D.9) as follows:

$$\mathcal{W}_i^{i||n,\text{fin}} = \sum_{\substack{(kl) \\ k,l \neq i}} \int \frac{\mathrm{d}\Omega_{\mathbf{m}}^{(3)}}{2\pi} (1 - C_{im}) \left[\log(\eta_{im}/2) \frac{\rho_{kl}}{\rho_{km}\rho_{lm}} \omega_{i||n}^{mi,ni} \right] (\mathbf{T}_k \cdot \mathbf{T}_l)$$

$$+ 2 \sum_{\substack{k \in \mathcal{H} \\ k \neq i}} \int \frac{d\Omega_{\mathbf{m}}^{(3)}}{2\pi} (1 - C_{im}) \left[\log(\eta_{im}/2) \frac{\rho_{ki}}{\rho_{km}\rho_{im}} \omega_{i||\mathbf{n}}^{\mathbf{m}i,ni} \right] (\mathbf{T}_k \cdot \mathbf{T}_i). \quad (\text{D.14})$$

When considering the action of the collinear limit C_{im} , we find

$$C_{im} \left[\log(\eta_{im}/2) \frac{\rho_{kl}}{\rho_{km}\rho_{lm}} \omega_{i||\mathbf{n}}^{\mathbf{m}i,ni} \right] = 0, \quad \text{if } k, l \neq i, \quad (\text{D.15})$$

as it does not generate collinear singularities. Similarly, we have

$$C_{im} \left[\log(\eta_{im}/2) \frac{\rho_{ki}}{\rho_{km}\rho_{im}} \omega_{i||\mathbf{n}}^{\mathbf{m}i,ni} \right] = \frac{\log(\eta_{im}/2)}{\rho_{im}}, \quad \text{if } k \neq i, \quad (\text{D.16})$$

since $\rho_{ki} = \rho_{km}$ when $i \parallel \mathbf{m}$. Thus, $\mathcal{W}_i^{\mathbf{m}||\mathbf{n},\text{fin}}$ can be expressed as

$$\begin{aligned} \mathcal{W}_i^{\mathbf{m}||\mathbf{n},\text{fin}} &= \sum_{\substack{(kl) \\ k, l \neq i}} \int \frac{d\Omega_{\mathbf{m}}^{(3)}}{2\pi} \log(\eta_{im}/2) \frac{\rho_{kl}}{\rho_{km}\rho_{lm}} \omega_{i||\mathbf{n}}^{\mathbf{m}i,ni} (\mathbf{T}_k \cdot \mathbf{T}_l) \\ &+ 2 \sum_{\substack{k \in \mathcal{H} \\ k \neq i}} \int \frac{d\Omega_{\mathbf{m}}^{(3)}}{2\pi} \frac{\log(\eta_{im}/2)}{\rho_{im}} \left[\frac{\rho_{ki}}{\rho_{km}} \omega_{i||\mathbf{n}}^{\mathbf{m}i,ni} - 1 \right] (\mathbf{T}_k \cdot \mathbf{T}_i). \end{aligned} \quad (\text{D.17})$$

D.2 Derivation of $\mathcal{W}_i^{\mathbf{m}||\mathbf{n},\text{fin}}$

D.2.1 How $\mathcal{W}_i^{\mathbf{m}||\mathbf{n},\text{fin}}$ emerges in $\Sigma_{\text{RR},1c}^{(b,d),\text{sa},\text{II}}|_{\mathcal{H},gg}$

In this subsection, we focus on the derivation of the partition-dependent function $\mathcal{W}_i^{\mathbf{m}||\mathbf{n},\text{fin}}$ and discuss how it emerges in $\Sigma_{\text{RR},1c}^{(b,d),\text{sa},\text{II}}|_{\mathcal{H},gg}$.

We start from Eq. (E.47), which we report here for convenience:

$$\begin{aligned} \Sigma_{\text{RR},1c}^{(b,d),\text{sa},\text{II}}|_{\mathcal{H},gg} &= -[\alpha_s]^2 \frac{\delta_g^{\text{sa}}(\epsilon) (E_{\text{max}}/\mu)^{-4\epsilon}}{\epsilon} \sum_{i \in \mathcal{H}} \sum_{(kl)} \int \frac{[d\Omega_{\mathbf{m}}^{(d-1)}]}{[\Omega^{(d-2)}]} \\ &\times \left\langle \sigma_{i[\mathbf{mn}]}^{-\epsilon} \omega_{\mathbf{m}||\mathbf{n}}^{\mathbf{m}i,ni} \frac{\rho_{kl}}{\rho_{k[\mathbf{mn}]} \rho_{l[\mathbf{mn}]}} (\mathbf{T}_k \cdot \mathbf{T}_l) \cdot F_{\text{LM}}^{\mathcal{H}} \right\rangle. \end{aligned} \quad (\text{D.18})$$

We relabel $[\mathbf{mn}]$ as $\mathbf{m}$, we introduce the partition-dependent function

$$\mathcal{W}_i^{\mathbf{m}||\mathbf{n}} \stackrel{\text{def}}{=} -\epsilon 2^{2\epsilon} \sum_{(kl)} \int \frac{[d\Omega_{\mathbf{m}}^{(d-1)}]}{[\Omega^{(d-2)}]} \sigma_{im}^{-\epsilon} \omega_{\mathbf{m}||\mathbf{n}}^{\mathbf{m}i,ni} \frac{\rho_{kl}}{\rho_{km} \rho_{lm}} (\mathbf{T}_k \cdot \mathbf{T}_l), \quad (\text{D.19})$$

and we use it to rewrite Eq. (D.18) as

$$\Sigma_{\text{RR},1c}^{(b,d),\text{sa},\text{II}}|_{\mathcal{H},gg} = [\alpha_s]^2 2^{2\epsilon} \delta_g^{\text{sa}}(\epsilon) \frac{(2E_{\text{max}}/\mu)^{-4\epsilon}}{\epsilon^2} \sum_{i \in \mathcal{H}} \langle \mathcal{W}_i^{\mathbf{m}||\mathbf{n}} \cdot F_{\text{LM}}^{\mathcal{H}} \rangle. \quad (\text{D.20})$$

Notice that $\mathcal{W}_i^{\mathbf{m}||\mathbf{n}}$ in Eq. (D.20) must be expanded in powers of ϵ up to $\mathcal{O}(\epsilon^2)$ in order to obtain $\Sigma_{\text{RR},1\text{c}}^{(b,d),\text{sa},\text{II}}|_{\mathcal{H},gg}$ up to $\mathcal{O}(\epsilon^0)$. We also observe that, due to the partition functions $\omega_{\mathbf{m}||\mathbf{n}}^{mi,ni}$ in the definition (D.18), the only allowed collinear singularities correspond to the kinematic configurations where $i \parallel \mathbf{m}$. To isolate these divergences, we insert the identity

$$\sigma_{i\mathbf{m}}^{-\epsilon} = \overline{C}_{i\mathbf{m}}(\sigma_{i\mathbf{m}}^{-\epsilon} - 1) - C_{i\mathbf{m}}(1 - \sigma_{i\mathbf{m}}^{-\epsilon}) + 1 \quad (\text{D.21})$$

on the right-hand side of Eq. (D.19), i.e.,

$$\begin{aligned} \mathcal{W}_i^{\mathbf{m}||\mathbf{n}} \stackrel{\text{def}}{=} & -\epsilon 2^{2\epsilon} \sum_{(kl)} \int \frac{[\text{d}\Omega_{\mathbf{m}}^{(d-1)}]}{[\Omega^{(d-2)}]} \left[\overline{C}_{i\mathbf{m}}(\sigma_{i\mathbf{m}}^{-\epsilon} - 1) + 1 - C_{i\mathbf{m}}(1 - \sigma_{i\mathbf{m}}^{-\epsilon}) \right] \\ & \times \omega_{\mathbf{m}||\mathbf{n}}^{mi,ni} \frac{\rho_{kl}}{\rho_{k\mathbf{m}} \rho_{l\mathbf{m}}} (\mathbf{T}_k \cdot \mathbf{T}_l). \end{aligned} \quad (\text{D.22})$$

The first term in (D.22) is already of $\mathcal{O}(\epsilon^2)$. Thus, we extract from it the coefficient that multiplies ϵ^2 and define it as

$$\mathcal{W}_i^{\mathbf{m}||\mathbf{n},\text{fin}} \stackrel{\text{def}}{=} \sum_{(kl)} \int \frac{\text{d}\Omega_{\mathbf{m}}^{(3)}}{2\pi} \overline{C}_{i\mathbf{m}} \left[\log \left(\frac{\eta_{i\mathbf{m}}}{1 - \eta_{i\mathbf{m}}} \right) \frac{\rho_{kl}}{\rho_{k\mathbf{m}} \rho_{l\mathbf{m}}} \omega_{\mathbf{m}||\mathbf{n}}^{mi,ni} \right] (\mathbf{T}_k \cdot \mathbf{T}_l). \quad (\text{D.23})$$

Notice that $\mathcal{W}_i^{\mathbf{m}||\mathbf{n},\text{fin}}$ is ϵ -finite and can therefore be evaluated in $d = 4$ dimensions. We substitute (D.23) in Eq. (D.20) and we obtain the first contribution:

$$[\alpha_s]^2 2^{2\epsilon} \delta_g^{\text{sa}}(\epsilon) \left(\frac{2E_{\text{max}}}{\mu} \right)^{-4\epsilon} \sum_{i \in \mathcal{H}} \langle \mathcal{W}_i^{\mathbf{m}||\mathbf{n},\text{fin}} \cdot F_{\text{LM}}^{\mathcal{H}} \rangle. \quad (\text{D.24})$$

Next, in the second term of (D.22), only the partition function $\omega_{\mathbf{m}||\mathbf{n}}^{mi,ni}$ depends on the index i . Hence, we can sum over this index, use the property

$$\sum_{i \in \mathcal{H}} \omega_{\mathbf{m}||\mathbf{n}}^{mi,ni} = 1, \quad (\text{D.25})$$

and obtain

$$\begin{aligned} & \frac{(2E_{\text{max}}/\mu)^{-4\epsilon}}{\epsilon^2} \left[-\epsilon 2^{2\epsilon} \sum_{i \in \mathcal{H}} \sum_{(kl)} \int \frac{[\text{d}\Omega_{\mathbf{m}}^{(d-1)}]}{[\Omega^{(d-2)}]} \omega_{\mathbf{m}||\mathbf{n}}^{mi,ni} \frac{\rho_{kl}}{\rho_{k\mathbf{m}} \rho_{l\mathbf{m}}} (\mathbf{T}_k \cdot \mathbf{T}_l) \right] \\ & = - \left(\frac{2E_{\text{max}}}{\mu} \right)^{-2\epsilon} \times 2 \left[- \frac{(2E_{\text{max}}/\mu)^{-2\epsilon}}{\epsilon^2} \sum_{(kl)} \eta_{kl}^{-\epsilon} K_{kl}(\mathbf{T}_k \cdot \mathbf{T}_l) \right] \\ & = - \left(\frac{2E_{\text{max}}}{\mu} \right)^{-2\epsilon} \times 2I_S(\epsilon), \end{aligned} \quad (\text{D.26})$$

where in the second line we used the integral in Eq. (A.22). We replace this expression

in Eq. (D.20) and we get the second contribution:

$$-[\alpha_s]^2 2^{1+2\epsilon} \delta_g^{\text{sa}}(\epsilon) \left(\frac{2E_{\text{max}}}{\mu} \right)^{-2\epsilon} \langle I_S(\epsilon) \cdot F_{\text{LM}}^{\mathcal{H}} \rangle. \quad (\text{D.27})$$

Finally, to compute the contribution of the third term in (D.22), we note that

$$C_{im} (1 - \sigma_{im}^{-\epsilon}) \omega_{\text{m||n}}^{\text{mi,ni}} \frac{\rho_{kl}}{\rho_{km} \rho_{lm}} = (1 - \eta_{im}^{-\epsilon}) \frac{1}{\rho_{im}} (\delta_{ik} + \delta_{il}). \quad (\text{D.28})$$

By using this expression in Eq. (D.22), it is possible to sum over l in the term proportional to δ_{ik} , as well as over k in the term proportional to δ_{il} , by applying the color conservation

$$\sum_{k \in \mathcal{H}} \mathbf{T}_k = 0. \quad (\text{D.29})$$

Thus, we have

$$\begin{aligned} & \epsilon 2^{2\epsilon} \sum_{(kl)} \int \frac{[\text{d}\Omega_{\text{m}}^{(d-1)}]}{[\Omega^{(d-2)}]} C_{im} (1 - \sigma_{im}^{-\epsilon}) \sigma_{im}^{-\epsilon} \omega_{\text{m||n}}^{\text{mi,ni}} \frac{\rho_{kl}}{\rho_{km} \rho_{lm}} (\mathbf{T}_k \cdot \mathbf{T}_l) \\ &= \epsilon 2^{1+2\epsilon} \int \frac{[\text{d}\Omega_{\text{m}}^{(d-1)}]}{[\Omega^{(d-2)}]} \frac{1 - \eta_{im}^{-\epsilon}}{\rho_{im}} \sum_{\substack{l \in \mathcal{H} \\ l \neq i}} (\mathbf{T}_i \cdot \mathbf{T}_l) \\ &= N_c(\epsilon) \mathbf{T}_i^2, \end{aligned} \quad (\text{D.30})$$

where

$$N_c(\epsilon) \stackrel{\text{def}}{=} \frac{2\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} - \frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)} = 1 + \mathcal{O}(\epsilon^3). \quad (\text{D.31})$$

We replace the above formula in Eq. (D.20) and we obtain the third contribution:

$$[\alpha_s]^2 2^{1+2\epsilon} \delta_g^{\text{sa}}(\epsilon) \frac{(2E_{\text{max}}/\mu)^{-4\epsilon}}{2\epsilon^2} N_c(\epsilon) \sum_{i \in \mathcal{H}} \mathbf{T}_i^2 \langle F_{\text{LM}}^{\mathcal{H}} \rangle. \quad (\text{D.32})$$

At this point, we combine Eqs (D.24), (D.27), and (D.32), and rewrite Eq. (D.18) as

$$\begin{aligned} \Sigma_{\text{RR,1c}}^{(b,d),\text{sa},\Pi}|_{\mathcal{H},gg} &= [\alpha_s]^2 2^{2\epsilon} \delta_g^{\text{sa}}(\epsilon) \left(\frac{2E_{\text{max}}}{\mu} \right)^{-4\epsilon} \sum_{i \in \mathcal{H}} \langle \mathcal{W}_i^{\text{m||n,fin}} \cdot F_{\text{LM}}^{\mathcal{H}} \rangle \\ &+ [\alpha_s]^2 2^{1+2\epsilon} \delta_g^{\text{sa}}(\epsilon) \left(\frac{2E_{\text{max}}}{\mu} \right)^{-2\epsilon} \\ &\times \left[-\langle I_S(\epsilon) \cdot F_{\text{LM}}^{\mathcal{H}} \rangle + \frac{(2E_{\text{max}}/\mu)^{-4\epsilon}}{2\epsilon^2} N_c(\epsilon) \sum_{i \in \mathcal{H}} \mathbf{T}_i^2 \langle F_{\text{LM}}^{\mathcal{H}} \rangle \right]. \end{aligned} \quad (\text{D.33})$$

D.2.2 Explicit expression of $\mathcal{W}_i^{\text{m}\parallel\text{n},\text{fin}}$

In Eq. (D.23), we introduced the partition-dependent function $\mathcal{W}_i^{\text{m}\parallel\text{n},\text{fin}}$. Similarly to $\mathcal{W}_i^{i\parallel\text{n},\text{fin}}$, we aim to provide an explicit expression for this quantity. To achieve this, we first express $\bar{C}_{im}$ as $\mathbb{1} - C_{im}$ and use it to rewrite Eq. (D.23) as follows:

$$\begin{aligned} \mathcal{W}_i^{\text{m}\parallel\text{n},\text{fin}} = & \sum_{\substack{(kl) \\ k,l \neq i}} \int \frac{d\Omega_{\text{m}}^{(3)}}{2\pi} (\mathbb{1} - C_{im}) \left[\log \left(\frac{\eta_{im}}{1 - \eta_{im}} \right) \frac{\rho_{kl}}{\rho_{km}\rho_{lm}} \omega_{\text{m}\parallel\text{n}}^{\text{m}i,\text{n}i} \right] (\mathbf{T}_k \cdot \mathbf{T}_l) \\ & + 2 \sum_{\substack{k \in \mathcal{H} \\ k \neq i}} \int \frac{d\Omega_{\text{m}}^{(3)}}{2\pi} (\mathbb{1} - C_{im}) \left[\log \left(\frac{\eta_{im}}{1 - \eta_{im}} \right) \frac{\rho_{ki}}{\rho_{km}\rho_{im}} \omega_{\text{m}\parallel\text{n}}^{\text{m}i,\text{n}i} \right] (\mathbf{T}_k \cdot \mathbf{T}_i). \end{aligned} \quad (\text{D.34})$$

Next, by considering the action of the collinear limit, we obtain:

$$\begin{aligned} C_{im} \left[\log \left(\frac{\eta_{im}}{1 - \eta_{im}} \right) \frac{\rho_{kl}}{\rho_{km}\rho_{lm}} \omega_{\text{m}\parallel\text{n}}^{\text{m}i,\text{n}i} \right] &= 0, & \text{if } k, l \neq i, \\ C_{im} \left[\log \left(\frac{\eta_{im}}{1 - \eta_{im}} \right) \frac{\rho_{ki}}{\rho_{km}\rho_{im}} \omega_{\text{m}\parallel\text{n}}^{\text{m}i,\text{n}i} \right] &= \frac{\log \eta_{im}}{\rho_{im}}, & \text{if } k \neq i. \end{aligned} \quad (\text{D.35})$$

Thus, $\mathcal{W}_i^{\text{m}\parallel\text{n},\text{fin}}$ can be written as

$$\begin{aligned} \mathcal{W}_i^{\text{m}\parallel\text{n},\text{fin}} = & \sum_{\substack{(kl) \\ k,l \neq i}} \int \frac{d\Omega_{\text{m}}^{(3)}}{2\pi} \log \left(\frac{\eta_{im}}{1 - \eta_{im}} \right) \frac{\rho_{kl}}{\rho_{km}\rho_{lm}} \omega_{\text{m}\parallel\text{n}}^{\text{m}i,\text{n}i} (\mathbf{T}_k \cdot \mathbf{T}_l) \\ & + 2 \sum_{\substack{k \in \mathcal{H} \\ k \neq i}} \int \frac{d\Omega_{\text{m}}^{(3)}}{2\pi} \frac{1}{\rho_{im}} \left[\log \left(\frac{\eta_{im}}{1 - \eta_{im}} \right) \frac{\rho_{ki}}{\rho_{km}} \omega_{\text{m}\parallel\text{n}}^{\text{m}i,\text{n}i} - \log \eta_{im} \right] (\mathbf{T}_k \cdot \mathbf{T}_i). \end{aligned} \quad (\text{D.36})$$

Spin correlations

Contents

E.1	Definition of the spin-correlated contribution	243
E.2	Integration of the spin-correlated contribution over the physical and unphysical regions	247
E.2.1	The physical region	247
E.2.2	The unphysical region	251

E.1 Definition of the spin-correlated contribution

In this appendix, we discuss the double-real contributions where the so-called *spin correlations* appear. These effects arise in sectors $\theta^{(b)}$ and $\theta^{(d)}$, defined in Eq. (4.28), in the limits when gluons $\mathbf{m}_g$ and $\mathbf{n}_g$ become collinear to each other. Specifically, we begin the discussion from Eq. (4.98), which we report here for convenience:

$$\begin{aligned} \Sigma_{\text{RR},1\text{c}}^{(b,d)}|_{\mathcal{H},gg} &= \sum_{i \in \mathcal{H}} \left\langle \bar{S}_{\text{mn}} (\mathbb{1} - 2S_{\text{n}} \Theta_{\text{mn}}) C_{\text{mn}} \theta^{(b)} [\text{d}p_{\text{m}}] [\text{d}p_{\text{n}}] \omega^{\text{m}i,\text{n}i} \right. \\ &\quad \left. \times \Delta^{(\text{mn})} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \right\rangle. \end{aligned} \quad (\text{E.1})$$

As outlined at the beginning of Subsec. 4.3.2, Eq. (E.1) already accounts for the contributions of both sectors $\theta^{(b)}$ and $\theta^{(d)}$.

To make this appendix self-contained, we start by considering the $\mathbf{m}_g \parallel \mathbf{n}_g$ limit, which is described by the following expression:

$$\begin{aligned} C_{\text{mn}} \Delta^{(\text{mn})} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] &= \frac{g_{\text{s,b}}^2 \mu_0^{2\epsilon}}{E_{\text{m}} E_{\text{n}} \rho_{\text{mn}}} \Delta^{([\text{mn}])} P_{gg}^{\mu\nu}(z) F_{\text{LM},\mu\nu}^{\mathcal{H}}[[\text{mn}]_g] \\ &= \frac{g_{\text{s,b}}^2 \mu_0^{2\epsilon}}{E_{\text{m}} E_{\text{n}} \rho_{\text{mn}}} \Delta^{([\text{mn}])} \left[P_{gg}^{(0)}(z) F_{\text{LM}}^{\mathcal{H}}[[\text{mn}]_g] + P_{gg}^{\perp}(z) \kappa_{\perp,(b)}^{\mu} \kappa_{\perp,(b)}^{\nu} F_{\text{LM},\mu\nu}^{\mathcal{H}}[[\text{mn}]_g] \right]. \end{aligned} \quad (\text{E.2})$$

Here, we are using the parametrization introduced in Appendix C.1. Specifically, $z =$

$E_m/(E_m + E_n)$, $P_{gg}^{\mu\nu}$ is reported in Eq. (A.42), $P_{gg}^{(0)}$ and $P_{gg}^\perp(z)$ read

$$\begin{aligned} P_{gg}^{(0)} &\stackrel{\text{def}}{=} 2C_A \left(\frac{z}{1-z} + \frac{1-z}{z} \right), \\ P_{gg}^\perp(z) &\stackrel{\text{def}}{=} 4C_A(1-\epsilon)z(1-z), \end{aligned} \quad (\text{E.3})$$

and $\kappa_{\perp,(b)}^\mu$ is the normalized transverse vector

$$\kappa_{\perp,(b)}^\mu \stackrel{\text{def}}{=} \frac{k_{\perp,(b)}^\mu}{\sqrt{-k_{\perp,(b)}^2}} \quad (\text{E.4})$$

defined through the decomposition

$$p_n^\mu = \alpha p_m^\mu + \beta \bar{p}_m^\mu + k_{\perp,(b)}^\mu, \quad (\text{E.5})$$

with $\bar{p}_m^\mu = (p_m^0, -\mathbf{p}_m)$ and $k_{\perp,(b)} \cdot p_m = k_{\perp,(b)} \cdot \bar{p}_m = 0$. The four-momentum of the clustered parton $[\text{mn}]_g$ is equal to

$$p_{[\text{mn}]}^\mu = E_{[\text{mn}]} n_m^\mu = \frac{E_{[\text{mn}]}}{E_m} p_m^\mu, \quad E_{[\text{mn}]} = E_m + E_n, \quad (\text{E.6})$$

where the vector n_m^μ is a light-like vector defined as $n_m^\mu = p_m^\mu/E_m$. Moreover, we note that the hard matrix element squared appears in Eq. (E.2) in two distinct ways: once as $F_{\text{LM}}^{\mathcal{H}}[[\text{mn}]_g]$ and once as $F_{\text{LM},\mu\nu}^{\mathcal{H}}[[\text{mn}]_g]$, where the open spin indices refer to the clustered parton. The relation between the two contributions reads

$$F_{\text{LM}}^{\mathcal{H}}[[\text{mn}]_g] = \sum_{\lambda_{[\text{mn}]}} \varepsilon_\mu^{\lambda_{[\text{mn}]}} (\varepsilon_\nu^{\lambda_{[\text{mn}]}})^* F_{\text{LM},\mu\nu}^{\mathcal{H}}[[\text{mn}]_g] = -g_{\mu\nu} F_{\text{LM},\mu\nu}^{\mathcal{H}}[[\text{mn}]_g], \quad (\text{E.7})$$

where the sum runs over the physical polarizations of the clustered gluon $[\text{mn}]_g$ and the last step follows from the transversality of $F_{\text{LM},\mu\nu}$.

To proceed further, we consider the action of the collinear limit C_{mn} on the angular phase space $[\text{d}\Omega_{\text{mn}}^{(b,d)}]$, parametrized in Eq. (C.16). The result is written in Eq. (C.32), that we report here for completeness:

$$\begin{aligned} \int C_{\text{mn}} \frac{[\text{d}\Omega_{\text{mn}}^{(b,d)}]}{\rho_{\text{mn}}} [\dots] &= \left[\frac{1}{8\pi^2} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \right] N_\epsilon^{(b,d)} \int [\text{d}\Omega_{[\text{mn}]}^{(d-1)}] \\ &\times \eta_{i[\text{mn}]}^{-\epsilon} (1 - \eta_{i[\text{mn}]})^\epsilon \text{d}\Lambda \frac{\text{d}\Omega_a^{(d-3)}}{[\Omega^{(d-3)}]} \frac{\text{d}x_4}{x_4^{1+2\epsilon}} C_{\text{mn}}[\dots]. \end{aligned} \quad (\text{E.8})$$

Therefore, considering both Eqs (E.2) and (E.8), we integrate over $\text{d}x_4$, $\text{d}\Omega_a^{(d-3)}$ and $\text{d}\Lambda$,

as described on page 233, and we obtain

$$\begin{aligned} \Sigma_{\text{RR},1\text{c}}^{(b,d)}|_{\mathcal{H},gg} = & - \sum_{i \in \mathcal{H}} [\alpha_s] \mu^{2\epsilon} \frac{N_\epsilon^{(b,d)}}{2\epsilon} \left\langle \int_0^{E_{\text{max}}} \frac{dE_{\text{m}}}{E_{\text{m}}^{2\epsilon-1}} \frac{dE_{\text{n}}}{E_{\text{n}}^{2\epsilon-1}} \int [d\Omega_{[\text{mn}}^{(d-1)}] \sigma_{i[\text{mn}}^{-\epsilon} \right. \\ & \times \omega_{\text{m}||\text{n}}^{\text{mi},\text{ni}} \bar{S}_{\text{mn}} (\mathbb{1} - 2S_{\text{n}} \Theta_{\text{mn}}) \Delta^{([\text{mn}])} \frac{1}{E_{\text{m}} E_{\text{n}}} \left[P_{gg}^{(b,d)}(z) F_{\text{LM}}^{\mathcal{H}}[[\text{mn}]_g] \right. \\ & \left. \left. + \epsilon P_{gg}^\perp(z) (r_{i,(b)}^\mu r_{i,(b)}^\nu) F_{\text{LM},\mu\nu}^{\mathcal{H}}[[\text{mn}]_g] \right] \right\rangle, \end{aligned} \quad (\text{E.9})$$

where $r_{i,(b)}^\mu$ is defined in Eq. (C.39), $\sigma_{ij} = \eta_{ij}(1 - \eta_{ij})^{-1}$, and

$$P_{gg}^{(b,d)}(z) \stackrel{\text{def}}{=} P_{gg}^{(0)}(z) + \frac{P_{gg}^\perp(z)}{2} = 2C_{\text{A}} \left[\frac{z}{1-z} + \frac{1-z}{z} + z(1-z)(1-\epsilon) \right]. \quad (\text{E.10})$$

In Eq. (E.9), we find it convenient to split the terms proportional to $F_{\text{LM},\mu\nu}^{\mathcal{H}}[[\text{mn}]_g]$ in such a way that the coefficient of $F_{\text{LM}}^{\mathcal{H}}[[\text{mn}]_g]$ in Eq. (E.9) is the spin-averaged splitting function $P_{gg}(z)$ in Eq. (A.41). Hence, we write

$$\begin{aligned} & P_{gg}^{(b,d)}(z) F_{\text{LM}}^{\mathcal{H}}[[\text{mn}]_g] + \epsilon P_{gg}^\perp(z) (r_{i,(b)}^\mu r_{i,(b)}^\nu) F_{\text{LM},\mu\nu}^{\mathcal{H}}[[\text{mn}]_g] \\ & \equiv P_{gg}(z) F_{\text{LM}}^{\mathcal{H}}[[\text{mn}]_g] + \epsilon \left[P_{gg}^\perp(z) (r_{i,(b)}^\mu r_{i,(b)}^\nu + g^{\mu\nu}) - P_{gg}^{\perp,\text{r}}(z) g^{\mu\nu} \right] F_{\text{LM},\mu\nu}^{\mathcal{H}}[[\text{mn}]_g], \end{aligned} \quad (\text{E.11})$$

with

$$P_{gg}^{\perp,\text{r}}(z) \stackrel{\text{def}}{=} 2C_{\text{A}} z(1-z)(1-2\epsilon), \quad (\text{E.12})$$

which yields

$$\begin{aligned} \Sigma_{\text{RR},1\text{c}}^{(b,d)}|_{\mathcal{H},gg} = & - \sum_{i \in \mathcal{H}} [\alpha_s] \mu^{2\epsilon} \frac{N_\epsilon^{(b,d)}}{2\epsilon} \left\langle \int_0^{E_{\text{max}}} \frac{dE_{\text{m}}}{E_{\text{m}}^{2\epsilon-1}} \frac{dE_{\text{n}}}{E_{\text{n}}^{2\epsilon-1}} \int [d\Omega_{[\text{mn}}^{(d-1)}] \sigma_{i[\text{mn}}^{-\epsilon} \right. \\ & \times \omega_{\text{m}||\text{n}}^{\text{mi},\text{ni}} \bar{S}_{\text{mn}} (\mathbb{1} - 2S_{\text{n}} \Theta_{\text{mn}}) \Delta^{([\text{mn}])} \frac{1}{E_{\text{m}} E_{\text{n}}} \left[P_{gg}(z) F_{\text{LM}}^{\mathcal{H}}[[\text{mn}]_g] \right. \\ & \left. \left. + \epsilon \left[P_{gg}^\perp(z) (r_{i,(b)}^\mu r_{i,(b)}^\nu + g^{\mu\nu}) - P_{gg}^{\perp,\text{r}}(z) g^{\mu\nu} \right] F_{\text{LM},\mu\nu}^{\mathcal{H}}[[\text{mn}]_g] \right] \right\rangle. \end{aligned} \quad (\text{E.13})$$

Following the discussion in Subsec. 4.3.2, we decompose Eq. (E.13) into two terms,

$$\Sigma_{\text{RR},1\text{c}}^{(b,d)}|_{\mathcal{H},gg} = \Sigma_{\text{RR},1\text{c}}^{(b,d),\text{sa}}|_{\mathcal{H},gg} + \Sigma_{\text{RR},1\text{c}}^{(b,d),\text{sc}}|_{\mathcal{H},gg}. \quad (\text{E.14})$$

The first term on the right-hand side is the *spin-averaged* contribution. It depends on the spin-averaged splitting function P_{gg} and yields the leading divergent part of $\Sigma_{\text{RR},1\text{c}}^{(b,d)}|_{\mathcal{H},gg}$, with its Laurent series beginning at $\mathcal{O}(\epsilon^{-2})$. The second term is the *spin-correlated* contribution: it encompasses all terms in Eq. (E.13) proportional to $F_{\text{LM},\mu\nu}^{\mathcal{H}}[[\text{mn}]_g]$. Since this contribution is suppressed by a factor of ϵ , the spin-correlated part is less divergent than the spin-averaged one; its Laurent series starts at $\mathcal{O}(\epsilon^{-1})$. In Subsec. 4.3.2, we discuss the

term $\Sigma_{\text{RR},1c}^{(b,d),\text{sa}}|_{\mathcal{H},gg}$ in detail. In this appendix, instead, we focus on the term $\Sigma_{\text{RR},1c}^{(b,d),\text{sc}}|_{\mathcal{H},gg}$, i.e.,

$$\begin{aligned} \Sigma_{\text{RR},1c}^{(b,d),\text{sc}}|_{\mathcal{H},gg} \stackrel{\text{def}}{=} & - \sum_{i \in \mathcal{H}} [\alpha_s] \mu^{2\epsilon} \frac{N_\epsilon^{(b,d)}}{2} \left\langle \int_0^{E_{\text{max}}} \frac{dE_m}{E_m^{2\epsilon-1}} \frac{dE_n}{E_n^{2\epsilon-1}} \int [d\Omega_{[mn]}^{(d-1)}] \sigma_{i[mn]}^{-\epsilon} \omega_{m||n}^{m,i,n} \right. \\ & \times \bar{S}_{mn} (\mathbb{1} - 2S_n \Theta_{mn}) \Delta^{([mn])} \frac{1}{E_m E_n} \left[P_{gg}^\perp(z) (r_{i,(b)}^\mu r_{i,(b)}^\nu + g^{\mu\nu}) \right. \\ & \left. \left. - P_{gg}^{\perp,r}(z) g^{\mu\nu} \right] F_{\text{LM},\mu\nu}^{\mathcal{H}}[[mn]_g] \right\rangle. \end{aligned} \quad (\text{E.15})$$

Note that the subtraction term containing $-2S_n \Theta_{mn}$ in Eq. (E.15) gives a vanishing contribution, since both the splitting functions $P_{gg}^\perp(z)$ and $P_{gg}^{\perp,r}(z)$ vanish under the action of the soft limit S_n . In contrast, the double-soft operator S_{mn} can “pass through” distributions that depend on z , since by definition the ratio E_m/E_n is fixed when S_{mn} acts. Furthermore, since $F_{\text{LM},\mu\nu}^{\mathcal{H}}[[mn]_g]$ depends on the clustered parton $[mn]_g$, S_{mn} acts as the single-soft limit $S_{[mn]}$. It follows that

$$\begin{aligned} \Sigma_{\text{RR},1c}^{(b,d),\text{sc}}|_{\mathcal{H},gg} \stackrel{\text{def}}{=} & - \sum_{i \in \mathcal{H}} [\alpha_s] \mu^{2\epsilon} \frac{N_\epsilon^{(b,d)}}{2} \left\langle \int_0^{E_{\text{max}}} \frac{dE_m}{E_m^{2\epsilon}} \frac{dE_n}{E_n^{2\epsilon}} \int [d\Omega_{[mn]}^{(d-1)}] \sigma_{i[mn]}^{-\epsilon} \omega_{m||n}^{m,i,n} \right. \\ & \times \left[P_{gg}^\perp(z) (r_i^\mu r_i^\nu + g^{\mu\nu}) - P_{gg}^{\perp,r}(z) g^{\mu\nu} \right] \\ & \left. \times \bar{S}_{[mn]} \Delta^{([mn])} F_{\text{LM},\mu\nu}^{\mathcal{H}}[[mn]_g] \right\rangle, \end{aligned} \quad (\text{E.16})$$

where we relabeled $r_{i,(b)}^\mu$ as r_i^μ to lighten the notation.

Next, we change the integration variables from (E_m, E_n) to $(E_{[mn]}, z)$. As described in Subsec. 4.3.2, due to the constraints on the energies of m_g and n_g , the energy of the clustered parton, $E_{[mn]}$, may exceed E_{max} , potentially reaching values up to $2E_{\text{max}}$. Consequently, the two regions of integration, $\Theta_{(\text{I})}$ and $\Theta_{(\text{II})}$, remain defined:

$$\Theta_{(\text{I})} \stackrel{\text{def}}{=} \begin{cases} 0 \leq E_{[mn]} \leq E_{\text{max}} \\ 0 \leq z \leq 1, \end{cases} \quad \Theta_{(\text{II})} \stackrel{\text{def}}{=} \begin{cases} E_{\text{max}} \leq E_{[mn]} \leq 2E_{\text{max}} \\ z \leq \frac{E_{\text{max}}}{E_{[mn]}} \\ z \geq 1 - \frac{E_{\text{max}}}{E_{[mn]}}. \end{cases} \quad (\text{E.17})$$

The details are provided in Subsec. 2.3.2. We will refer to $\Theta_{(\text{I})}$ and $\Theta_{(\text{II})}$ as the *physical* and *unphysical* regions, respectively. We name the region $\Theta_{(\text{II})}$ as *unphysical* because the energy $E_{[mn]}$ exceeds the maximal energy a parton can attain in a physical process, i.e., E_{max} . Nevertheless, $\Theta_{(\text{II})}$ still yields a non-zero contribution when the soft limit is applied, since the parton $[mn]_g$ no longer appears in the matrix element (more precisely, it no longer appears in the momentum-conserving δ -function within $F_{\text{LM}}^{\mathcal{H}}$). Hence, from a mathematical standpoint, there is no factor that would nullify the integral over $E_{[mn]}$.

Therefore, we express $\Sigma_{\text{RR},1c}^{(b,d),\text{sc}}|_{\mathcal{H},gg}$ as the sum of two terms,

$$\Sigma_{\text{RR},1c}^{(b,d),\text{sc}}|_{\mathcal{H},gg} = \Sigma_{\text{RR},1c}^{(b,d),\text{sc,I}}|_{\mathcal{H},gg} + \Sigma_{\text{RR},1c}^{(b,d),\text{sc,II}}|_{\mathcal{H},gg}, \quad (\text{E.18})$$

where $\Sigma_{\text{RR},1\text{c}}^{(b,d),\text{sc,I}}|_{\mathcal{H},gg}$ and $\Sigma_{\text{RR},1\text{c}}^{(b,d),\text{sc,II}}|_{\mathcal{H},gg}$ correspond to Eq. (E.16), integrated over the regions $\Theta_{\text{(I)}}$ and $\Theta_{\text{(II)}}$, respectively. We will compute these two contributions in the following section.

E.2 Integration of the spin-correlated contribution over the physical and unphysical regions

E.2.1 The physical region

We begin with discussing $\Sigma_{\text{RR},1\text{c}}^{(b,d),\text{sc,I}}|_{\mathcal{H},gg}$, that, according to the steps described on page 50, reads

$$\begin{aligned} \Sigma_{\text{RR},1\text{c}}^{(b,d),\text{sc,I}}|_{\mathcal{H},gg} &\stackrel{\text{def}}{=} - \sum_{i \in \mathcal{H}} [\alpha_s] \mu^{2\epsilon} \frac{N_\epsilon^{(b,d)}}{2} \left\langle \int_0^{E_{\text{max}}} \frac{dE_{[\text{mn}]}}{E_{[\text{mn}]}^{4\epsilon-1}} \int [\text{d}\Omega_{[\text{mn}]}^{(d-1)}] \sigma_{i[\text{mn}]}^{-\epsilon} \omega_{\text{m}||\text{n}}^{\text{mi},\text{ni}} \right. \\ &\quad \times \int_0^1 dz [z(1-z)]^{-2\epsilon} \left[P_{gg}^\perp(z) (r_i^\mu r_i^\nu + g^{\mu\nu}) - P_{gg}^{\perp,\text{r}}(z) g^{\mu\nu} \right] \\ &\quad \left. \times \bar{S}_{\text{mn}} \Delta^{([\text{mn}])} F_{\text{LM},\mu\nu}^{\mathcal{H}}[[\text{mn}]_g] \right\rangle. \end{aligned} \quad (\text{E.19})$$

We integrate over dz , we rename $E_{[\text{mn}]}$ as E_{m} , and we obtain

$$\begin{aligned} \Sigma_{\text{RR},1\text{c}}^{(b,d),\text{sc,I}}|_{\mathcal{H},gg} &= \sum_{i \in \mathcal{H}} [\alpha_s] \frac{N_\epsilon^{(b,d)}}{2} \left\langle (E_{\text{m}}/\mu)^{-2\epsilon} \sigma_{i\text{m}}^{-\epsilon} \omega_{\text{m}||\text{n}}^{\text{mi},\text{ni}} \left[\gamma_{\perp,g \rightarrow gg}^{22} (r_i^\mu r_i^\nu + g^{\mu\nu}) \right. \right. \\ &\quad \left. \left. - \gamma_{\perp,g \rightarrow gg}^{22,\text{r}} g^{\mu\nu} \right] \bar{S}_{\text{m}} \Delta^{(\text{m})} F_{\text{LM},\mu\nu}^{\mathcal{H}}[\text{m}_g] \right\rangle, \end{aligned} \quad (\text{E.20})$$

where

$$\gamma_{\perp,g \rightarrow gg}^{22} \stackrel{\text{def}}{=} - \int_0^1 dz \frac{P_{gg}^\perp(z)}{[z(1-z)]^{2\epsilon}}, \quad \gamma_{\perp,g \rightarrow gg}^{22,\text{r}} \stackrel{\text{def}}{=} - \int_0^1 dz \frac{P_{gg}^{\perp,\text{r}}(z)}{[z(1-z)]^{2\epsilon}}. \quad (\text{E.21})$$

Eq. (E.20) is soft-regulated, the only singularity left there is $[\text{mn}] \parallel i$. To regularize and extract this collinear divergence, we insert the identity $\mathbb{1} = \bar{C}_{i\text{m}} + C_{i\text{m}}$ into Eq. (E.20), and write

$$\Sigma_{\text{RR},1\text{c}}^{(b,d),\text{sc,I}}|_{\mathcal{H},gg} = \Sigma_{\text{RR},1\text{c}}^{(b,d),\text{sc,I,1}}|_{\mathcal{H},gg} + \Sigma_{\text{RR},1\text{c}}^{(b,d),\text{sc,I,2}}|_{\mathcal{H},gg}, \quad (\text{E.22})$$

where

$$\begin{aligned} \Sigma_{\text{RR},1\text{c}}^{(b,d),\text{sc,I,1}}|_{\mathcal{H},gg} &\stackrel{\text{def}}{=} \sum_{i \in \mathcal{H}} [\alpha_s] \frac{N_\epsilon^{(b,d)}}{2} \left\langle \bar{S}_{\text{m}} C_{i\text{m}} (E_{\text{m}}/\mu)^{-2\epsilon} \sigma_{i\text{m}}^{-\epsilon} \left[\gamma_{\perp,g \rightarrow gg}^{22} (r_i^\mu r_i^\nu + g^{\mu\nu}) \right. \right. \\ &\quad \left. \left. - \gamma_{\perp,g \rightarrow gg}^{22,\text{r}} g^{\mu\nu} \right] \Delta^{(\text{m})} F_{\text{LM},\mu\nu}^{\mathcal{H}}[\text{m}_g] \right\rangle, \end{aligned} \quad (\text{E.23})$$

$$\Sigma_{\text{RR},1\text{c}}^{(b,d),\text{sc,I,2}}|_{\mathcal{H},gg}$$

$$\begin{aligned}
 &\stackrel{\text{def}}{=} \sum_{i \in \mathcal{H}} [\alpha_s] \frac{N_\epsilon^{(b,d)}}{2} \gamma_{\perp, g \rightarrow gg}^{22} \langle \mathcal{O}_{\text{NLO}}^{(\mathbf{m})} \omega_{\mathbf{m} \parallel \mathbf{n}}^{m_i, n_i} \sigma_{i\mathbf{m}}^{-\epsilon} (E_{\mathbf{m}}/\mu)^{-2\epsilon} (r_i^\mu r_i^\nu + g^{\mu\nu}) \Delta^{(\mathbf{m})} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_g] \rangle \\
 &+ \sum_{i \in \mathcal{H}} [\alpha_s] \frac{N_\epsilon^{(b,d)}}{2} \gamma_{\perp, g \rightarrow gg}^{22, \text{r}} \langle \mathcal{O}_{\text{NLO}}^{(\mathbf{m})} \omega_{\mathbf{m} \parallel \mathbf{n}}^{m_i, n_i} \sigma_{i\mathbf{m}}^{-\epsilon} (E_{\mathbf{m}}/\mu)^{-2\epsilon} \Delta^{(\mathbf{m})} F_{\text{LM}}^{\mathcal{H}}[\mathbf{m}_g] \rangle. \quad (\text{E.24})
 \end{aligned}$$

The object $\Sigma_{\text{RR}, 1\text{c}}^{(b,d), \text{sc}, \text{I}, 1}|_{\mathcal{H}, gg}$ represents a fully regulated single-unresolved contribution, which is finite in the limit $\epsilon \rightarrow 0$ and can be numerically integrated in four space-time dimensions.

In contrast, $\Sigma_{\text{RR}, 1\text{c}}^{(b,d), \text{sc}, \text{I}, 1}|_{\mathcal{H}, gg}$ will contain a $1/\epsilon$ pole once we integrate over the unresolved parton $\mathbf{m}_g$. To perform this integration, we must evaluate the soft and collinear limits of $r_i^\mu r_i^\nu F_{\text{LM}, \mu\nu}^{\mathcal{H}}[[\mathbf{m}\mathbf{n}]_g]$, which we have not previously encountered in this thesis. This necessitates revisiting the construction of the vectors r_i^μ . We recall that, following Appendix C, the angular parametrization uses the direction of parton i as a reference axis, such that

$$p_{\mathbf{m}}^\mu = E_{\mathbf{m}} (t^\mu + \cos \theta_{i\mathbf{m}} e_i^\mu + \sin \theta_{i\mathbf{m}} b^\mu), \quad (\text{E.25})$$

$$r_i^\mu = \sin \theta_{i\mathbf{m}} e_i^\mu - \cos \theta_{i\mathbf{m}} b^\mu, \quad (\text{E.26})$$

where

$$t^\mu = (1, \mathbf{0}), \quad e_i^\mu = (0, \mathbf{n}_i), \quad p_i^\mu = E_i (t^\mu + e_i^\mu). \quad (\text{E.27})$$

Here $\mathbf{n}_i$ is a unit vector in $(d-1)$ spatial dimensions, and a^μ and b^μ are d -dimensional unit vectors such that

$$t \cdot a = e_i \cdot a = t \cdot b = e_i \cdot b = a \cdot b = 0. \quad (\text{E.28})$$

According to this parametrization, we have

$$r_i^2 = -1, \quad p_{\mathbf{m}} \cdot r_i = 0, \quad (\text{E.29})$$

thus r_i^μ is a valid polarization vector of gluon $\mathbf{m}_g$. Therefore, we can write a general expression for the soft limits $S_{\mathbf{m}}$ of spin-correlated amplitudes-squared (cf. Ref. [152]):

$$S_{\mathbf{m}} r_i^\mu r_i^\nu F_{\text{LM}, \mu\nu}^{\mathcal{H}}[\mathbf{m}_g] = -g_{s,b}^2 \mu_0^{2\epsilon} \sum_{k,l \in \mathcal{H}} \frac{(p_k \cdot r_i)(p_l \cdot r_i)}{(p_k \cdot p_{\mathbf{m}})(p_l \cdot p_{\mathbf{m}})} (\mathbf{T}_k \cdot \mathbf{T}_l) \cdot F_{\text{LM}}^{\mathcal{H}}. \quad (\text{E.30})$$

We also need to consider the limit $C_{i\mathbf{m}}$ of this expression, which develops singularities arising from two contributions in the sum: first from $k = i, l = i$, and second from $k = i, l \neq i$ or $k \neq i, l = i$.

We begin with the first case, $k = i, l = i$, and write

$$\frac{(p_i \cdot r_i)(p_i \cdot r_i)}{(p_i \cdot p_{\mathbf{m}})(p_i \cdot p_{\mathbf{m}})} = \frac{1}{E_{\mathbf{m}}^2} \frac{\sin^2 \theta_{i\mathbf{m}}}{(1 - \cos \theta_{i\mathbf{m}})^2} = \frac{1}{E_{\mathbf{m}}^2} \frac{(2 - \rho_{i\mathbf{m}})}{\rho_{i\mathbf{m}}}, \quad (\text{E.31})$$

where we used the explicit parametrization of momenta p_i^μ and $p_{\mathbf{m}}^\mu$ and the vector r_i^μ .

The collinear limit of the term in Eq. (E.31) with $k = i, l = i$ therefore reads

$$-g_{s,b}^2 \mu_0^{2\epsilon} C_{im} \frac{(p_i \cdot r_i)(p_i \cdot r_i)}{(p_i \cdot p_m)(p_i \cdot p_m)} (\mathbf{T}_i \cdot \mathbf{T}_i) \cdot F_{LM}^{\mathcal{H}} = -\frac{2g_{s,b}^2 \mu_0^{2\epsilon}}{E_m^2 \rho_{im}} \mathbf{T}_i^2 F_{LM}^{\mathcal{H}}. \quad (\text{E.32})$$

Next, we consider terms with $k = i$ and $l \neq i$, i.e.,

$$-g_{s,b}^2 \mu_0^{2\epsilon} \sum_{\substack{l \in \mathcal{H} \\ l \neq i}} \frac{(p_i \cdot r_i)(p_l \cdot r_i)}{(p_i \cdot p_m)(p_l \cdot p_m)} (\mathbf{T}_i \cdot \mathbf{T}_l) \cdot F_{LM}^{\mathcal{H}}. \quad (\text{E.33})$$

Since $p_i \cdot r_i \sim \sin \theta_{im}$ and $p_i \cdot p_m \sim (1 - \cos \theta_{im})$, and all other factors in the above expression are regular in the limit $\theta_{im} \rightarrow 0$, we conclude that this contribution is actually integrable in the collinear limit $m \parallel i$. The same conclusion holds for the case with $k \neq i$ and $l = i$. Hence, we state that

$$C_{im} S_m r_i^\mu r_i^\nu F_{LM,\mu\nu}^{\mathcal{H}}[\mathbf{m}_g] = -\frac{2g_{s,b}^2 \mu_0^{2\epsilon}}{E_m^2 \rho_{im}} \mathbf{T}_i^2 F_{LM}^{\mathcal{H}}. \quad (\text{E.34})$$

We notice that this expression coincides with the soft-collinear limit without spin-correlations, $C_{im} S_m F_{LM}^{\mathcal{H}}[\mathbf{m}_g]$, thus we obtain the following statement:

$$C_{im} S_m (r_i^\mu r_i^\nu + g^{\mu\nu}) F_{LM,\mu\nu}^{\mathcal{H}}[\mathbf{m}_g] = 0. \quad (\text{E.35})$$

We use this cancellation to simplify $\Sigma_{RR,1c}^{(b,d),sc,I,1}|_{\mathcal{H},gg}$ in Eq. (E.23), and write it as

$$\begin{aligned} \Sigma_{RR,1c}^{(b,d),sc,I,1}|_{\mathcal{H},gg} &= \sum_{i \in \mathcal{H}} [\alpha_s] \frac{N_\epsilon^{(b,d)}}{2} \left\langle \left[\gamma_{\perp,g \rightarrow gg}^{22} C_{im} (r_i^\mu r_i^\nu + g^{\mu\nu}) \right. \right. \\ &\quad \left. \left. - \gamma_{\perp,g \rightarrow gg}^{22,r} g^{\mu\nu} \bar{S}_m C_{im} \right] (E_m/\mu)^{-2\epsilon} \sigma_{im}^{-\epsilon} \Delta^{(m)} F_{LM,\mu\nu}^{\mathcal{H}}[\mathbf{m}_g] \right\rangle. \end{aligned} \quad (\text{E.36})$$

In Eq. (E.36), the term proportional to $\gamma_{\perp,g \rightarrow gg}^{22,r}$ suggests a NLO-like calculation, provided that the functions $\mathcal{P}_{\alpha\beta}^{(4),\text{gen}}$ and $I_C^{(4)}$ are employed, due to the presence of the additional energy factor $(E_m/\mu)^{-2\epsilon}$. As for the behavior of C_{im} on $(r_i^\mu r_i^\nu + g^{\mu\nu}) F_{LM,\mu\nu}^{\mathcal{H}}[\mathbf{m}_g]$, the following formula holds:

$$C_{im} (r_i^\mu r_i^\nu + g^{\mu\nu}) F_{LM,\mu\nu}^{\mathcal{H}}[\mathbf{m}_g] = \frac{g_{s,b}^2 \mu_0^{2\epsilon}}{p_i \cdot p_m} \cdot \begin{cases} P_{f_i f_i}^{\text{spin}}(z) F_{LM}^{(z \cdot a)b}[\mathcal{H}_f]/z, & i = a, \\ P_{f_i f_i}^{\text{spin}}(z) F_{LM}^{a(z \cdot b)}[\mathcal{H}_f]/z, & i = b, \\ P_{f_i f_i}^{\text{spin}}(z) F_{LM}^{ab}[\dots[\mathbf{m}]_{f_i}], & i \in \mathcal{H}_f, \end{cases} \quad (\text{E.37})$$

where

$$P_{gg}^{\text{spin}}(z) \stackrel{\text{def}}{=} 2C_A \left[\frac{z}{1-z} + \frac{(1-z)/z + z(1-z)}{2(1-\epsilon)} \right],$$

$$P_{qq}^{\text{spin}}(z) \stackrel{\text{def}}{=} \frac{C_F}{2} \frac{(1+z)^2}{1-z}. \quad (\text{E.38})$$

Therefore, thanks to this formula, we can compute Eq. (E.36) without further manipulations, and we obtain

$$\begin{aligned} \Sigma_{\text{RR},1c}^{(b,d),\text{sc},\text{I},1} |_{\mathcal{H},gg} &= \frac{[\alpha_s]^2}{4\epsilon} \frac{2^{2\epsilon} \Gamma(1-\epsilon) \Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)} N_\epsilon^{(b,d)} \left[\sum_{i \in \mathcal{H}_f} \left\langle (2E_i/\mu)^{-4\epsilon} \right. \right. \\ &\times \left[\left(\gamma_{\perp,g \rightarrow gg}^{22,r} - \gamma_{\perp,g \rightarrow gg}^{22} \right) \gamma_{z,f_i \rightarrow f_i,g}^{24}(L_i) + \gamma_{\perp,g \rightarrow gg}^{22} \gamma_{z,f_i \rightarrow f_i,g}^{24,\text{spin}}(L_i) \right] F_{\text{LM}}^{\mathcal{H}} \Big\rangle, \\ &+ \left\langle (2E_i/\mu)^{-4\epsilon} \left[\left(\gamma_{\perp,g \rightarrow gg}^{22,r} - \gamma_{\perp,g \rightarrow gg}^{22} \right) \mathcal{P}_{gg}^{(4)} + \gamma_{\perp,g \rightarrow gg}^{22} \mathcal{P}_{gg}^{(4),\text{spin}} \right] \otimes F_{\text{LM}}^{\mathcal{H}} \right\rangle \\ &+ \left. \left\langle (2E_i/\mu)^{-4\epsilon} F_{\text{LM}}^{\mathcal{H}} \otimes \left[\left(\gamma_{\perp,g \rightarrow gg}^{22,r} - \gamma_{\perp,g \rightarrow gg}^{22} \right) \mathcal{P}_{qq}^{(4)} + \gamma_{\perp,g \rightarrow gg}^{22} \mathcal{P}_{qq}^{(4),\text{spin}} \right] \right\rangle \right], \end{aligned} \quad (\text{E.39})$$

where

$$\begin{aligned} \int_0^1 dz \mathcal{P}_{fa fa}^{(k),\text{spin}}(z, E_a) g(z) &\stackrel{\text{def}}{=} \int_0^1 dz \left[(1 - S_z) \left[(1 - z)^{-k\epsilon} P_{fa fa}^{\text{spin}}(z) \right] \right. \\ &\quad \left. - 2T_{fa}^2 \frac{1 - e^{-k\epsilon L_a}}{k\epsilon} \delta(1 - z) \right] g(z), \end{aligned} \quad (\text{E.40})$$

$$\begin{aligned} \gamma_{g(z),f_i \rightarrow f_i,g}^{nk,\text{spin}}(L_i) &\stackrel{\text{def}}{=} - \int_0^1 dz (\mathbb{1} - S_z) \left[z^{-n\epsilon} (1 - z)^{-k\epsilon} g(z) P_{fi fi}^{\text{spin}}(z) \right] \\ &\quad + 2T_{fi}^2 \frac{1 - e^{-k\epsilon L_i}}{k\epsilon} g(1). \end{aligned} \quad (\text{E.41})$$

In order to obtain a more compact writing, it is convenient to introduce the spin-correlated splitting function

$$\mathcal{P}_{\alpha\alpha}^{\text{sc}}(z, E_a) \stackrel{\text{def}}{=} - \left[\left(\frac{2E_a}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \right]^2 \left[\mathcal{P}_{\alpha\alpha}^{(4),\text{spin}}(z, E_a) - \mathcal{P}_{\alpha\alpha}^{(4)}(z, E_a) \right], \quad (\text{E.42})$$

and the spin-correlated anomalous dimension

$$\Lambda_{i,f_i \rightarrow f_i,g}^{\text{sc}} \stackrel{\text{def}}{=} \left[\left(\frac{2E_i}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \right]^2 \left[\gamma_{z,f_i \rightarrow f_i,g}^{24,\text{spin}}(\epsilon, L_i) - \gamma_{z,f_i \rightarrow f_i,g}^{24}(\epsilon, L_i) \right], \quad (\text{E.43})$$

thanks to which we can express $\Sigma_{\text{RR},1c}^{(b,d),\text{sc},\text{I},1} |_{\mathcal{H},gg}$ as

$$\begin{aligned} \Sigma_{\text{RR},1c}^{(b,d),\text{sc},\text{I},1} |_{\mathcal{H},gg} &= \\ &= [\alpha_s]^2 N_{\text{sc}}^{(b,d)} \left\langle \left[\frac{\gamma_{\perp,g \rightarrow gg}^{22,r}}{2} \left(I_C^{(4)}(\epsilon) - \frac{\Gamma_{q \rightarrow gg}^{(4)}}{2\epsilon} \right) + \frac{\gamma_{\perp,g \rightarrow gg}^{22}}{4\epsilon} \sum_{i \in \mathcal{H}_f} \Lambda_{i,f_i \rightarrow f_i,g}^{\text{sc}} \right] \cdot F_{\text{LM}}^{\mathcal{B}_0} \right\rangle \\ &+ \frac{[\alpha_s]^2}{4\epsilon} N_{\text{sc}}^{(b,d)} \left\langle \left[\gamma_{\perp,g \rightarrow gg}^{22,r} \mathcal{P}_{gg}^{(4),\text{gen}} + \gamma_{\perp,g \rightarrow gg}^{22} \mathcal{P}_{gg}^{\text{sc}} \right] \otimes F_{\text{LM}}^{\mathcal{B}_0} \right\rangle \end{aligned} \quad (\text{E.44})$$

$$+ \frac{[\alpha_s]^2}{4\epsilon} N_{\text{sc}}^{(b,d)} \left\langle F_{\text{LM}}^{\mathcal{B}_0} \otimes \left[\gamma_{\perp, g \rightarrow gg}^{22,r}(\epsilon) \mathcal{P}_{qq}^{(4),\text{gen}} + \gamma_{\perp, g \rightarrow gg}^{22}(\epsilon) \mathcal{P}_{qq}^{\text{sc}} \right] \right\rangle,$$

with

$$N_{\text{sc}}^{(b,d)} \stackrel{\text{def}}{=} 2^{2\epsilon} \frac{\Gamma(1+2\epsilon)\Gamma^3(1-2\epsilon)}{\Gamma(1-3\epsilon)\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}. \quad (\text{E.45})$$

E.2.2 The unphysical region

We turn to the contribution in Eq. (E.18) which is integrated over the unphysical region $\Theta_{(\text{II})}$, namely $\Sigma_{\text{RR},1\text{c}}^{(b,d),\text{sc},\text{II}}|_{\mathcal{H},gg}$. It reads

$$\begin{aligned} \Sigma_{\text{RR},1\text{c}}^{(b,d),\text{sc},\text{II}}|_{\mathcal{H},gg} &\stackrel{\text{def}}{=} - \sum_{i \in \mathcal{H}} [\alpha_s] \mu^{2\epsilon} \frac{N_{\epsilon}^{(b,d)}}{2} \left\langle \int_{E_{\text{max}}}^{2E_{\text{max}}} \frac{dE_{[\text{mn}]}}{E_{[\text{mn}]}^{4\epsilon-1}} \int [d\Omega_{[\text{mn}]}^{(d-1)}] \sigma_{i[\text{mn}]}^{-\epsilon} \omega_{\text{m}||\text{n}}^{\text{mi},\text{ni}} \right. \\ &\quad \times \int_{1-\xi}^{\xi} dz [z(1-z)]^{-2\epsilon} \left[P_{gg}^{\perp}(z) (r_i^{\mu} r_i^{\nu} + g^{\mu\nu}) - P_{gg}^{\perp,r}(z) g^{\mu\nu} \right] \\ &\quad \left. \times \bar{S}_{\text{mn}} \Delta^{([\text{mn}])} F_{\text{LM},\mu\nu}^{\mathcal{H}}[[\text{mn}]_g] \right\rangle, \quad \xi = \frac{E_{\text{max}}}{E_{[\text{mn}]}}. \end{aligned} \quad (\text{E.46})$$

We follow the same steps outlined in Subsec. 4.3.2 to compute $\Sigma_{\text{RR},1\text{c}}^{(b,d),\text{sa},\text{II}}|_{\mathcal{H},gg}$ in Eq. (4.116). Thus, we substitute $\bar{S}_{[\text{mn}]}$ with $-S_{[\text{mn}]}$, because, as previously stated, $F_{\text{LM}}^{\mathcal{H}}[[\text{mn}]_g]$ has zero support when the energy of the clustered parton exceeds E_{max} . Next, we apply the soft limit to $F_{\text{LM},\mu\nu}^{\mathcal{H}}[[\text{mn}]_g]$, as described in Eq. (E.30), and integrate over $dE_{[\text{mn}]}$ and dz . The result is

$$\begin{aligned} \Sigma_{\text{RR},1\text{c}}^{(b,d),\text{sc},\text{II}}|_{\mathcal{H},gg} &= -[\alpha_s]^2 \frac{(E_{\text{max}}/\mu)^{-4\epsilon}}{\epsilon} \sum_{i \in \mathcal{H}} \sum_{k,l \in \mathcal{H}} \int \frac{[d\Omega_{[\text{mn}]}^{(d-1)}]}{[\Omega^{(d-2)}]} \left\langle \sigma_{i[\text{mn}]}^{-\epsilon} \omega_{\text{m}||\text{n}}^{\text{mi},\text{ni}} \right. \\ &\quad \times \left[\delta_g^{\perp} (r_i^{\mu} r_i^{\nu} + g^{\mu\nu}) - \frac{\delta_g^{\perp,r}(\epsilon)}{\epsilon} g^{\mu\nu} \right] \frac{n_{k,\mu} n_{l,\nu}}{(n_k \cdot n_{[\text{mn}]}) (n_l \cdot n_{[\text{mn}]})} (\mathbf{T}_k \cdot \mathbf{T}_l) \cdot F_{\text{LM}}^{\mathcal{H}} \Big\rangle, \end{aligned} \quad (\text{E.47})$$

where

$$\delta_g^{\perp} \stackrel{\text{def}}{=} \frac{N_{\epsilon}^{(b,d)} E_{\text{max}}^{4\epsilon}}{2} \int_{E_{\text{max}}}^{2E_{\text{max}}} \frac{dE_{[\text{mn}]}}{E_{[\text{mn}]}^{1+4\epsilon}} \int_{1-\xi}^{\xi} dz [z(1-z)]^{-2\epsilon} P_{gg}^{\perp}(z), \quad (\text{E.48})$$

$$\delta_g^{\perp,r}(\epsilon) \stackrel{\text{def}}{=} \frac{N_{\epsilon}^{(b,d)} E_{\text{max}}^{4\epsilon}}{2} \int_{E_{\text{max}}}^{2E_{\text{max}}} \frac{dE_{[\text{mn}]}}{E_{[\text{mn}]}^{1+4\epsilon}} \int_{1-\xi}^{\xi} dz [z(1-z)]^{-2\epsilon} \epsilon P_{gg}^{\perp,r}(z). \quad (\text{E.49})$$

Regarding the term proportional to $\delta_g^{\perp}$, we rename the label $[\text{mn}]$ as m , we introduce the partition-dependent function

$$\mathcal{W}_{\text{r}}^{(i)} \stackrel{\text{def}}{=} \sum_{k,l \in \mathcal{H}} \int \frac{[d\Omega_{\text{m}}^{(d-1)}]}{[\Omega^{(d-2)}]} \sigma_{i\text{m}}^{-\epsilon} \omega_{\text{m}||\text{n}}^{\text{mi},\text{ni}} (r_i^{\mu} r_i^{\nu} + g^{\mu\nu}) \frac{n_{k,\mu} n_{k,\nu}}{(n_k \cdot n_{\text{m}}) (n_l \cdot n_{\text{m}})} (\mathbf{T}_k \cdot \mathbf{T}_l), \quad (\text{E.50})$$

and we obtain

$$[\alpha_s]^2 \delta_g^\perp \left(\frac{E_{\max}}{\mu} \right)^{-4\epsilon} \sum_{i \in \mathcal{H}} \langle \mathcal{W}_r^{(i)} \cdot F_{\text{LM}}^{\mathcal{H}} \rangle. \quad (\text{E.51})$$

We point out that the function $\mathcal{W}_r^{(i)}$ is finite in ϵ because the pole arising from the term proportional to $r_i^\mu r_i^\nu$ cancels with that arising from the $g^{\mu\nu}$ term. This can be understood as follows: the most singular contribution affecting the term proportional to $r_i^\mu r_i^\nu$ stems from the combination $k = l = i$, since the partition functions damp all other potential collinear configurations. In this case, the singularity is proportional to $2T_i^2 / \rho_{i[\text{mn}]}$, as we already saw in Eq. (E.32). On the other hand, the singularity proportional to $g^{\mu\nu}$ can only arise when $k = i, l \neq i$ and $k \neq i, l = 1$, given that $n_k^2 = n_l^2 = 0$. We can then isolate the divergent ratio $1/(n_i \cdot n_{[\text{mn}]})$ and sum over colors, obtaining precisely $-2T_i^2 / \rho_{i[\text{mn}]}$.

Next, we consider the term proportional to $\delta_g^{\perp, r}(\epsilon)$. Since $n_k^2 = n_l^2 = 0$, we can write this term as

$$\begin{aligned} & - [\alpha_s]^2 \frac{\delta_g^{\perp, r}(\epsilon) (E_{\max}/\mu)^{-4\epsilon}}{\epsilon} \sum_{i \in \mathcal{H}} \sum_{(kl)} \int \frac{[d\Omega_{[\text{mn}]}^{(d-1)}]}{[\Omega^{(d-2)}]} \left\langle \sigma_{i[\text{mn}]}^{-\epsilon} \omega_{\text{m}|\text{n}}^{\text{mi}, \text{ni}} \right. \\ & \times \left. \frac{\rho_{kl}}{\rho_{k[\text{mn}]} \rho_{l[\text{mn}]}} (\mathbf{T}_k \cdot \mathbf{T}_l) \cdot F_{\text{LM}}^{\mathcal{H}} \right\rangle. \end{aligned} \quad (\text{E.52})$$

Notice that this object is the same as $\Sigma_{\text{RR}, 1\text{c}}^{(b,d), \text{sa}, \text{II}}|_{\mathcal{H}, gg}$ in Eq. (D.18), provided the replacement $\delta_g^{\text{sa}}(\epsilon) \mapsto \delta_g^{\perp, r}(\epsilon)$. Therefore, the final expression for (E.52) can be read from Eq. (D.33), and then combined with (E.51), which yields

$$\begin{aligned} \Sigma_{\text{RR}, 1\text{c}}^{(b,d), \text{sc}, \text{II}}|_{\mathcal{H}, gg} &= [\alpha_s]^2 2^{1+2\epsilon} \delta_g^{\perp, r}(\epsilon) \left(\frac{2E_{\max}}{\mu} \right)^{-2\epsilon} \\ & \times \left[- \langle I_S(\epsilon) \cdot F_{\text{LM}}^{\mathcal{H}} \rangle + \frac{(2E_{\max}/\mu)^{-4\epsilon}}{2\epsilon^2} N_c(\epsilon) \sum_{i \in \mathcal{H}} T_i^2 \langle F_{\text{LM}}^{\mathcal{H}} \rangle \right] \\ & + [\alpha_s]^2 2^{2\epsilon} \delta_g^{\perp, r}(\epsilon) \left(\frac{2E_{\max}}{\mu} \right)^{-4\epsilon} \sum_{i \in \mathcal{H}} \langle \mathcal{W}_i^{\text{m}|\text{n}, \text{fin}} \cdot F_{\text{LM}}^{\mathcal{H}} \rangle \\ & + [\alpha_s]^2 \delta_g^\perp \left(\frac{E_{\max}}{\mu} \right)^{-4\epsilon} \sum_{i \in \mathcal{H}} \langle \mathcal{W}_r^{(i)} \cdot F_{\text{LM}}^{\mathcal{H}} \rangle. \end{aligned} \quad (\text{E.53})$$

We recall that $\mathcal{W}_i^{\text{m}|\text{n}, \text{fin}}$ is defined in Eq. (D.23).

Details on the single-soft hard-collinear double-real contribution

In this appendix, we discuss the details related to the single-soft hard-collinear double-real contribution presented in Subsec. 5.1.5. Specifically, we provide a general derivation that allows rewriting Eq. (5.45) as Eq. (5.46). As we have emphasized repeatedly, the quantities in Eq. (5.45) must be handled with care, ensuring that limits are not interchanged with integrals where such operations are not permitted.

We begin with Eq. (5.45), which we reproduce below for convenience:

$$\begin{aligned} \sum_{i \in \mathcal{H}} & \left[\left\langle \bar{S}_m C_{im} \Delta^{(m)} \langle S_n \Theta_{mn} F_{LM}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g] \rangle \right\rangle \right. \\ & + \left\langle S_n \langle \bar{S}_m C_{im} \Delta^{(mn)} \Theta_{nm} F_{LM}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g] \rangle \right\rangle \\ & \left. + \mathcal{P}_{\mathcal{B}_0} \left\langle C_{im} \Delta^{(m)} \langle S_n F_{LM}^{\mathcal{B}_2}[\mathbf{m}_q, \mathbf{n}_g] \rangle \right\rangle \right]. \end{aligned} \quad (\text{F.1})$$

In the first term on the right-hand side of Eq. (F.1), the action of the soft limit S_n yields

$$\begin{aligned} & \sum_{i \in \mathcal{H}} \left\langle C_{im} \Delta^{(m)} \langle S_n F_{LM}^{\mathcal{B}_2}[\mathbf{m}_q, \mathbf{n}_g] \rangle \right\rangle \\ & = -\frac{[\alpha_s]}{\epsilon^2} \sum_{i \in \mathcal{H}} \sum_{\substack{(kl) \in \\ \{\mathcal{H}, \mathbf{m}\}}} \left\langle \bar{S}_m C_{im} (2E_m/\mu)^{-2\epsilon} \Delta^{(m)} \eta_{kl}^{-\epsilon} K_{kl}(\mathbf{T}_k \cdot \mathbf{T}_l) \cdot F_{LM}^{\mathcal{B}_1}[\mathbf{m}_g] \right\rangle, \end{aligned} \quad (\text{F.2})$$

where the angular-dependent object K_{kl} is defined in Eq. (2.68). Here, $\{\mathcal{H}, \mathbf{m}\}$ comprises particles from $\mathcal{H} \cup \mathbf{m}$, with $\mathcal{H}$ representing the list of hard partons and $\mathbf{m}$ indicating the unresolved parton. Note that, in Eq. (F.2), we first apply the soft limit S_n , then integrate over the phase space of the parton $\mathbf{n}$, and finally apply the operators $\bar{S}_m C_{im}$ to this result. Had we applied $\bar{S}_m C_{im}$ *before* integrating over the phase space of $\mathbf{n}$, we would have obtained a different expression.

A similar outcome to Eq. (F.2) is expected for the term in the third line on the right-hand side of Eq. (F.1), with E_m replaced by $E_{\max}$ due to the absence of energy ordering,

i.e.

$$\begin{aligned} & \sum_{i \in \mathcal{H}} \mathcal{P}_{\mathcal{B}_0} \left\langle C_{im} \Delta^{(m)} \left\langle S_n F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_q, \mathbf{n}_g] \right\rangle \right\rangle \\ &= -[\alpha_s] \frac{(2E_{\text{max}}/\mu)^{-2\epsilon}}{\epsilon^2} \sum_{i \in \mathcal{H}} \sum_{(kl) \in \{\mathcal{H}, \mathbf{m}\}} \mathcal{P}_{\mathcal{B}_0} \left\langle C_{im} \Delta^{(m)} \eta_{kl}^{-\epsilon} K_{kl} (\mathbf{T}_k \cdot \mathbf{T}_l) \cdot F_{\text{LM}}^{\mathcal{B}_1}[\mathbf{m}_q] \right\rangle. \end{aligned} \quad (\text{F.3})$$

Finally, the second term on the right-hand side of Eq. (F.1) can be computed straightforwardly. It reads

$$\begin{aligned} & \sum_{i \in \mathcal{H}} \left\langle S_n \left\langle \bar{S}_m C_{im} \Delta^{(mn)} \Theta_{nm} F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g] \right\rangle \right\rangle \\ &= -\frac{[\alpha_s]}{\epsilon^2} \sum_{i \in \mathcal{H}} \sum_{(kl)} \left\langle \left[(2E_{\text{max}}/\mu)^{-2\epsilon} - (2E_m/\mu)^{-2\epsilon} \right] \eta_{kl}^{-\epsilon} K_{kl} \right. \\ & \quad \times \bar{S}_m C_{im} \Delta^{(m)} (\mathbf{T}_k \cdot \mathbf{T}_l) \cdot F_{\text{LM}}^{\mathcal{B}_1}[\mathbf{m}_g] \left. \right\rangle, \end{aligned} \quad (\text{F.4})$$

where we recall that the standalone (kl) means $k, l \in \mathcal{H}$ and $k \neq l$.

Before combining Eqs (F.2 - F.4), we have to understand the behavior of collinear limit C_{im} on the angular-dependent objects $\eta_{kl}^{-\epsilon} K_{kl}$ in Eqs (F.2) and (F.3). To do so, we first introduce a simplified notation for the upcoming equations, distinct from the one used elsewhere in this thesis. We represent the F_{LM} functions as:

$$F_{\text{LM}}^{(kl)} \stackrel{\text{def}}{=} (\mathbf{T}_k \cdot \mathbf{T}_l) \cdot F_{\text{LM}}^{ab}[\mathcal{H}_f], \quad F_{\text{LM}}^{(kl)}[\mathbf{m}] \stackrel{\text{def}}{=} (\mathbf{T}_k \cdot \mathbf{T}_l) \cdot F_{\text{LM}}^{ab}[\mathcal{H}_f|\mathbf{m}], \quad (\text{F.5})$$

thus we imply all particles from the Born process. Additionally, we do not specify the flavor of parton $\mathbf{m}$ yet, since the following equations hold for both an unresolved gluon or quark. The goal consists in computing explicitly the following object:

$$\sum_{\substack{(kl) \in \\ \{\mathcal{H}, \mathbf{m}\}}} C_{im} \eta_{kl}^{-\epsilon} K_{kl} \Delta^{(m)} F_{\text{LM}}^{(kl)}[\mathbf{m}]. \quad (\text{F.6})$$

We start with the following rearrangement:

$$\begin{aligned} & \sum_{\substack{(kl) \in \\ \{\mathcal{H}, \mathbf{m}\}}} C_{im} \eta_{kl}^{-\epsilon} K_{kl} \Delta^{(m)} F_{\text{LM}}^{(kl)}[\mathbf{m}] \\ &= \sum_{k \in \{\mathcal{H}, \mathbf{m}\}} C_{im} \left[\sum_{\substack{l \in \mathcal{H} \\ l \neq k}} \eta_{kl}^{-\epsilon} K_{kl} \Delta^{(m)} F_{\text{LM}}^{(kl)}[\mathbf{m}] + \eta_{k\mathbf{m}}^{-\epsilon} K_{k\mathbf{m}} \Delta^{(m)} F_{\text{LM}}^{(k\mathbf{m})}[\mathbf{m}] \right] \\ &= \underbrace{\sum_{(kl)} \eta_{kl}^{-\epsilon} K_{kl} C_{im} \Delta^{(m)} F_{\text{LM}}^{(kl)}[\mathbf{m}]}_{\stackrel{\text{def}}{=} \mathbb{F}} + \underbrace{\sum_{k \in \mathcal{H}} 2 C_{im} \eta_{k\mathbf{m}}^{-\epsilon} K_{k\mathbf{m}} \Delta^{(m)} F_{\text{LM}}^{(k\mathbf{m})}[\mathbf{m}]}_{\stackrel{\text{def}}{=} \mathbb{S}}, \end{aligned} \quad (\text{F.7})$$

where in the last line we used the property

$$C_{im}\eta_{kl}^{-\epsilon}K_{kl}[\dots] = \eta_{kl}^{-\epsilon}K_{kl}C_{im}[\dots], \quad \text{with } k, l \neq m, \quad (\text{F.8})$$

and the symmetry of $\eta_{km}^{-\epsilon}$, K_{km} , and $(\mathbf{T}_k \cdot \mathbf{T}_m)$ under the swap $k \leftrightarrow m$. Note that we have introduced two terms, which we refer to as $\mathbb{F}$ and $\mathbb{S}$. We now rewrite these objects as follows:

$$\begin{aligned} \mathbb{F} &= \sum_{k \in \mathcal{H}} \sum_{\substack{l \in \mathcal{H} \\ l < k}} 2 \eta_{kl}^{-\epsilon} K_{kl} C_{im} \Delta^{(m)} F_{\text{LM}}^{(kl)}[\mathbf{m}] \\ &= \sum_{\substack{k \in \mathcal{H} \\ k \neq i}} 2 \eta_{ki}^{-\epsilon} K_{ki} C_{im} \Delta^{(m)} F_{\text{LM}}^{(ki)}(\mathbf{m}) + \sum_{\substack{k \in \mathcal{H} \\ k \neq i}} \sum_{\substack{l \in \mathcal{H} \\ l < k, \neq i}} 2 \eta_{kl}^{-\epsilon} K_{kl} C_{im} \Delta^{(m)} F_{\text{LM}}^{(kl)}[\mathbf{m}] \\ &\stackrel{\text{def}}{=} \mathbb{F}_a + \mathbb{F}_b, \end{aligned} \quad (\text{F.9})$$

$$\begin{aligned} \mathbb{S} &= \sum_{\substack{k \in \mathcal{H} \\ k \neq i}} 2 C_{im} \eta_{km}^{-\epsilon} K_{km} F_{\text{LM}}^{(km)}(\mathbf{m}) + 2 C_{im} \eta_{im}^{-\epsilon} K_{im} \Delta^{(m)} F_{\text{LM}}^{(im)}[\mathbf{m}] \\ &= \sum_{\substack{k \in \mathcal{H} \\ k \neq i}} 2 \eta_{ki}^{-\epsilon} K_{ki} C_{im} \Delta^{(m)} F_{\text{LM}}^{(km)}(\mathbf{m}) + 2 \frac{\Gamma^3(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} \eta_{im}^{-\epsilon} C_{im} \Delta^{(m)} F_{\text{LM}}^{(im)}[\mathbf{m}] \\ &\stackrel{\text{def}}{=} \mathbb{S}_a + \mathbb{S}_b, \end{aligned} \quad (\text{F.10})$$

where we exploited the relations

$$\begin{aligned} C_{im} \eta_{km}^{-\epsilon} K_{km}[\dots] &= \eta_{ik}^{-\epsilon} K_{ik} C_{im}[\dots], \quad k \neq i, \\ C_{im} \eta_{im}^{-\epsilon} K_{im}[\dots] &= \frac{\Gamma^3(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} \eta_{im}^{-\epsilon} C_{im}. \end{aligned} \quad (\text{F.11})$$

Next, we sum $\mathbb{F}_a$ and $\mathbb{S}_a$:

$$\begin{aligned} \mathbb{F}_a + \mathbb{S}_a &= \sum_{\substack{k \in \mathcal{H} \\ k \neq i}} 2 \eta_{ki}^{-\epsilon} K_{ki} C_{im} \Delta^{(m)} \left[F_{\text{LM}}^{(ki)}[\mathbf{m}] + F_{\text{LM}}^{(km)}[\mathbf{m}] \right] \\ &= \sum_{\substack{k \in \mathcal{H} \\ k \neq i}} 2 \eta_{ki}^{-\epsilon} K_{ki} C_{im} \Delta^{(m)} F_{\text{LM}}^{(k[i\mathbf{m}])}[\mathbf{m}], \end{aligned} \quad (\text{F.12})$$

where $\mathbf{T}_{[i\mathbf{m}]}$ labels the color matrix of the clustered parton $[i\mathbf{m}]$, i.e.

$$\mathbf{T}_{[i\mathbf{m}]} \stackrel{\text{def}}{=} \mathbf{T}_i + \mathbf{T}_m \quad \Rightarrow \quad \mathbf{T}_k \cdot \mathbf{T}_i + \mathbf{T}_k \cdot \mathbf{T}_m = \mathbf{T}_k \cdot \mathbf{T}_{[i\mathbf{m}]} \quad (\text{F.13})$$

Then, we recombine $\mathbb{F}_a + \mathbb{S}_a$ with $\mathbb{F}_b$:

$$\mathbb{F}_a + \mathbb{S}_a + \mathbb{F}_b = \sum_{(kl)} \eta_{kl}^{-\epsilon} K_{kl} (C_{im} \Delta^{(m)} F_{\text{LM}}^{(kl)}[\mathbf{m}])_{\mathbf{T}_i \mapsto \mathbf{T}_{[i\mathbf{m}]}} \quad (\text{F.14})$$

The expression $T_i \mapsto T_{[im]}$ signifies that, if $l = i$ (or $k = i$), the color matrix T_i of parton i must be substituted with that of the clustered parton $[im]$, denoted as $T_{[im]}$. It is important to note that, if parton m is a gluon, then the clustered parton $[im]$ shares the same flavor as parton i , i.e., $T_{[im]} = T_i$. However, this equivalence does not hold when m is a quark, which is why we introduced the replacement $T_i \mapsto T_{[im]}$.

To proceed, we consider $\mathbb{S}_b$, i.e.

$$\mathbb{S}_b = 2 \frac{\Gamma^3(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} \eta_{im}^{-\epsilon} C_{im} \Delta^{(m)} F_{LM}^{(im)}[m]. \quad (\text{F.15})$$

Note that

$$T_{[im]}^2 = (T_i + T_m)^2 = T_i^2 + T_m^2 + 2T_i \cdot T_m, \quad (\text{F.16})$$

which implies

$$T_i \cdot T_m = -\frac{T_i^2 + T_m^2 - T_{[im]}^2}{2}. \quad (\text{F.17})$$

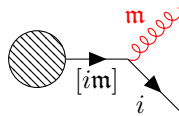
Hence, we rewrite $\mathbb{S}_b$ as

$$\mathbb{S}_b = -\frac{\Gamma^3(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} \left[T_i^2 + T_m^2 - T_{[im]}^2 \right] \eta_{im}^{-\epsilon} C_{im} \Delta^{(m)} F_{LM}[m]. \quad (\text{F.18})$$

After applying the collinear limit C_{im} , F_{LM} will depend on the clustered parton $[im]$ (in the initial state, $[im]$ corresponds to the parton entering the hard scattering process). for the initial state's collinear limit, the clustered parton should be intended through an analytical continuation). Depending on the flavour of partons i , m , $[im]$, the color factor

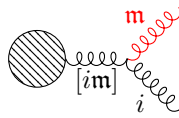
$$T_i^2 + T_m^2 - T_{[im]}^2 \quad (\text{F.19})$$

can assume different values. Specifically, if parton m is a gluon, as previously mentioned, then $T_{[im]} = T_i$, resulting in



$\Rightarrow T_i^2 + T_m^2 - T_{[im]}^2 = T_m^2 = C_A,$

(F.20)

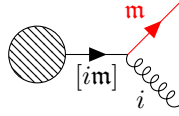


$\Rightarrow T_i^2 + T_m^2 - T_{[im]}^2 = T_m^2 = C_A,$

(F.21)

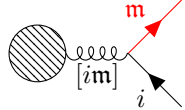
that is, the color factor (F.19) is always C_A . In contrast, if parton m is a quark, different scenarios are possible:

i) If parton i is a gluon, then $[im]$ must be a quark, i.e., $T_{[im]} = T_m$. It follows that



$$\Rightarrow T_i^2 + T_m^2 - T_{[im]}^2 = T_i^2 = C_A. \quad (\text{F.22})$$

ii) If parton i is an anti-quark, then $[im]$ must be a gluon. In this case,



$$\Rightarrow T_i^2 + T_m^2 - T_{[im]}^2 = 2C_F - C_A. \quad (\text{F.23})$$

Therefore, if m is a quark, it is convenient to introduce the following object:

$$\tilde{C}_x \stackrel{\text{def}}{=} \begin{cases} C_A, & x = q/\bar{q}, \\ 2C_F - C_A, & x = g, \end{cases} \quad (\text{F.24})$$

through which we can express $\mathbb{S}_b$ in this compact expression:

$$\mathbb{S}_b = -\frac{\Gamma^3(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} \left[\delta_{mg} C_A + \delta_{qm} \tilde{C}_{[im]} \right] \eta_{im}^{-\epsilon} C_{im} \Delta^{(m)} F_{LM}[m]. \quad (\text{F.25})$$

To conclude the proof, we combine Eqs (F.14) and (F.25) to obtain the final expression of Eq. (F.6), which corresponds to

$$\begin{aligned} \sum_{\substack{(kl) \in \\ \{\mathcal{H}, m\}}} C_{im} \eta_{kl}^{-\epsilon} K_{kl} \Delta^{(m)} F_{LM}^{(kl)}[m] &= \sum_{(kl)} \eta_{kl}^{-\epsilon} K_{kl} (C_{im} \Delta^{(m)} F_{LM}^{(kl)}[m])_{T_i \mapsto T_{[im]}} \\ &- \frac{\Gamma^3(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} \left[\delta_{mg} C_A + \delta_{qm} \tilde{C}_{[im]} \right] \eta_{im}^{-\epsilon} C_{im} \Delta^{(m)} F_{LM}[m]. \end{aligned} \quad (\text{F.26})$$

We note that (F.26) is completely general and holds for any list of hard particles, and regardless of whether the unresolved particle m is a gluon or a quark.

At this point, we can substitute this result into Eqs (F.2) and (F.3), and then combine them with Eq. (F.4). This yields the following expression for Eq. (F.1):

Eq. (F.1) =

$$\begin{aligned} &= -[\alpha_s] \frac{(2E_{\text{max}}/\mu)^{-2\epsilon}}{\epsilon^2} \sum_{i \in \mathcal{H}} \sum_{(kl)} \langle \eta_{kl}^{-\epsilon} K_{kl} \bar{S}_m C_{im} \Delta^{(m)} (T_k \cdot T_l) \cdot F_{LM}^{\mathcal{B}_1}[m_g] \rangle \\ &- [\alpha_s] \frac{(2E_{\text{max}}/\mu)^{-2\epsilon}}{\epsilon^2} \sum_{i \in \mathcal{H}} \sum_{(kl)} \mathcal{P}_{\mathcal{B}_0} \langle \eta_{kl}^{-\epsilon} K_{kl} (C_{im} \Delta^{(m)} (T_k \cdot T_l) \cdot F_{LM}^{\mathcal{B}_1}[m_q])_{T_i \mapsto T_{[im]}} \rangle \\ &+ \frac{[\alpha_s]}{\epsilon^2} C_A \frac{\Gamma^3(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} \sum_{i \in \mathcal{H}} \langle (2E_m/\mu)^{-2\epsilon} \eta_{im}^{-\epsilon} \bar{S}_m C_{im} \Delta^{(m)} F_{LM}^{\mathcal{B}_1}[m_g] \rangle \end{aligned}$$

$$+ [\alpha_s] \frac{(2E_{\text{max}}/\mu)^{-2\epsilon}}{\epsilon^2} \frac{\Gamma^3(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} \sum_{i \in \mathcal{H}} \mathcal{P}_{\mathcal{B}_0} C_A \langle \eta_{im}^{-\epsilon} C_{im} \Delta^{(\text{m})} F_{\text{LM}}^{\mathcal{B}_1}[\mathbf{m}_q] \rangle. \quad (\text{F.27})$$

Notice that, in our case, the condition $\tilde{C}_{[im]} = C_A$ always holds, since there are no $g \rightarrow q\bar{q}$ splittings in the final state, and the scenario where $\mathbf{m}_q$ is radiated from an initial-state parton is excluded by the projector $\mathcal{P}_{\mathcal{B}_0}$. However, this condition no longer applies when projecting onto the Born processes $\mathcal{A}_0^{(\bar{q}q)}$ or $\mathcal{A}_0^{(gg)}$.

Triple-collinear limit for the final-state quark leg

Contents

G.1 Derivation of the triple-collinear splitting function for $q^* \rightarrow g g q$	259
G.2 Singularities arising from interferences	265

G.1 Derivation of the triple-collinear splitting function for $q^* \rightarrow g g q$

In Subsec. 5.3.2, we discussed the triple-collinear contribution contained in $\Sigma_{\text{DU}}^{(2)}|_{\mathcal{B}_0}$ from Eq. (5.117), that we report here for convenience:

$$\Sigma_{\text{DU}}^{(2)}|_{\mathcal{B}_0} = \left\langle \Omega_2 \Delta^{(\text{mn})} \left[\bar{S}_{\text{mn}} \bar{S}_{\text{n}} \Theta_{\text{mn}} F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g] + \mathcal{P}_{\mathcal{B}_0} \bar{S}_{\text{n}} \Delta^{(\text{mn})} F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_q, \mathbf{n}_g] \right] \right\rangle, \quad (\text{G.1})$$

where Ω_2 corresponds to triple-collinear contribution

$$\begin{aligned} \Omega_2 &\equiv \sum_{i \in \mathcal{H}} \Omega_2^{(i)} \\ &\stackrel{\text{def}}{=} \sum_{i \in \mathcal{H}} \left[\bar{C}_{i\text{n}} \theta^{(a)} + \bar{C}_{\text{mn}} \theta^{(b)} + \bar{C}_{i\text{m}} \theta^{(c)} + \bar{C}_{\text{mn}} \theta^{(d)} \right] [\text{d}p_{\text{m}}][\text{d}p_{\text{n}}] C_{\text{mn},i} \omega^{\text{mi}, \text{ni}}. \end{aligned} \quad (\text{G.2})$$

If $i = a, b$, the triple-collinear limit can be explicitly written using the results presented in Table 1 of Ref. [79]. Similarly, when considering the final-state splitting $g^* \rightarrow g g g$, this process can be described by the function $\text{FSR}_5(E_i)$, as shown in Table 2 of Ref. [79]. However, this does not apply to the final-state splitting $q^* \rightarrow q g g$, as the function $\text{FSR}_1(E)$ reported in Table 2 of Ref. [79] accounts solely for the contribution

$$\text{TC}_{\text{old}} = \left\langle \bar{S}_{\text{mn}} \bar{S}_{\text{n}} \Omega_2^{(q)} \Theta_{\text{mn}} F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g] \right\rangle, \quad (\text{G.3})$$

which differs from (G.1). Nevertheless, the additional terms that distinguish (G.1) from (G.3) can be derived without substantial difficulty, as they do not involve purely triple-

collinear limits, which would otherwise complicate the phase space integration. We dedicate this appendix to discuss how to derive these terms.

The $q^* \rightarrow q g g$ splitting occurs when $\mathbf{m}_g$ and $\mathbf{n}_g$ become collinear with the final-state hard quark q in $F_{\text{LM}}^{\mathcal{B}_0}[\mathbf{m}_g, \mathbf{n}_g]$, or when $\mathbf{m}_q$ and $\mathbf{n}_g$ become collinear with any of the N final-state hard gluons in $F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_q, \mathbf{n}_g]$. It is important to note that this second case is equivalent to considering $\mathbf{m}_q$ and $\mathbf{n}_g$ becoming collinear with only one of the N final-state hard gluons in $F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_q, \mathbf{n}_g]$, and then multiplying the result by a factor of N . Thus, the contribution associated with the $q^* \rightarrow q g g$ splitting is given by

$$\text{TC}_{gg+qg} = \text{TC}_{gg} + \text{TC}_{qg}, \quad (\text{G.4})$$

where

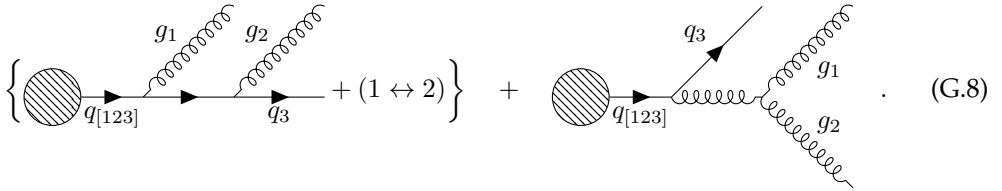
$$\text{TC}_{gg} \stackrel{\text{def}}{=} \langle \bar{S}_{\text{mn}} \bar{S}_{\text{n}} \Omega_2^{(q)} \Delta^{(\text{mn})} \Theta_{\text{mn}} F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_g, \mathbf{n}_g] \rangle, \quad (\text{G.5})$$

$$\text{TC}_{qg} \stackrel{\text{def}}{=} \langle \bar{S}_{\text{n}} \Omega_2^{(g)} \Delta^{(\text{mn})} \Theta_{\text{mn}} N F_{\text{LM}}^{\mathcal{B}_2}[\mathbf{m}_q, \mathbf{n}_g] \rangle. \quad (\text{G.6})$$

Our objective is, therefore, to compute

$$\Delta \text{TC}_{gg+qg} \stackrel{\text{def}}{=} \text{TC}_{gg+qg} - \text{TC}_{\text{old}}, \quad (\text{G.7})$$

which accounts for the discrepancy between Eqs (G.1) and (G.3). To achieve this, we first introduce the following notation from Ref. [152]. We denote the two gluons as g_1, g_2 , and the quark as q_3 , corresponding to the $q^* \rightarrow g_1 g_2 q_3$ splitting



$$\left\{ \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} + (1 \leftrightarrow 2) \right\} + \text{Diagram 3}. \quad (\text{G.8})$$

Then, we introduce the three parameters

$$z_k \stackrel{\text{def}}{=} \sum_{i=1}^3 \frac{E_k}{E_1 + E_2 + E_3}, \quad z_1 + z_2 + z_3 = 1, \quad (\text{G.9})$$

and we write the triple-collinear limit for the case of the $[i\text{mn}]_q^* \rightarrow i_q \mathbf{m}_g \mathbf{n}_g$ splitting as follows:

$$C_{gg,q} F_{\text{LM}}^{gq}[\dots, i_q | \mathbf{m}_g, \mathbf{n}_g] = \left(\frac{2g_{\text{s,b}}^2 \mu_0^{2\epsilon}}{s_{i\text{mn}}} \right)^2 P_{ggq}(z_1, z_2, z_3) \mathcal{N} F_{\text{LM}}^{gq}[\{g\}_{N-1}, [i\text{mn}]_q], \quad (\text{G.10})$$

where $s_{i\text{mn}} = (p_i + p_{\text{m}} + p_{\text{n}})^2$, and $\mathcal{N} = 1/(N-1)!$ is the symmetry factor of F_{LM} . To

lighten the notation, we define

$$F_{\text{LM}}[\dots, [\text{imn}]_q] \stackrel{\text{def}}{=} \frac{1}{(N-1)!} F_{\text{LM}}^{gq}[\{g\}_{N-1}, [\text{imn}]_q]. \quad (\text{G.11})$$

Notice that the expression of $\mathcal{P}_{ggq}(z_1, z_2|z_3)$ does not change if we consider the $[\text{imn}]_q^* \rightarrow i_g \mathbf{m}_q \mathbf{n}_g$ case, thus WLOG we write

$$C_{\mathbf{mn},i} F_{\text{LM}}^{gq}[\dots, i|\mathbf{m}, \mathbf{n}] = \mathcal{P}_{ggq}(z_1, z_2|z_3) F_{\text{LM}}[\dots, [123]_q]. \quad (\text{G.12})$$

By definition, the first two arguments of $\mathcal{P}_{ggq}$ always correspond to gluons, while the third refers to the quark. Consequently, $\mathcal{P}_{ggq}(z_1, z_2|z_3)$ is symmetric under the exchange $z_1 \leftrightarrow z_2$, but not under the exchanges $z_1 \leftrightarrow z_3$ or $z_2 \leftrightarrow z_3$. Additionally, for our convenience, we rewrite $\Omega_2^{(i)}$ in Eq. (G.2) as

$$\Omega_2^{(i)} = (\mathbb{1} - \mathfrak{C}(\mathbf{m}, \mathbf{n}|i)) C_{\mathbf{mn},i} \omega^{\mathbf{m}i, \mathbf{n}i}, \quad (\text{G.13})$$

where

$$\mathfrak{C}(\mathbf{m}, \mathbf{n}|i) \stackrel{\text{def}}{=} \left[C_{i\mathbf{n}} \theta^{(a)} + C_{\mathbf{mn}} \theta^{(b)} + C_{i\mathbf{m}} \theta^{(c)} + C_{\mathbf{mn}} \theta^{(d)} \right] [\text{dp}_{\mathbf{m}}][\text{dp}_{\mathbf{n}}]. \quad (\text{G.14})$$

Note that $\mathfrak{C}(\mathbf{m}, \mathbf{n}|i)$ is symmetric under the exchange $\mathbf{m} \leftrightarrow \mathbf{n}$, but not under the exchanges $\mathbf{m} \leftrightarrow i$ or $\mathbf{n} \leftrightarrow i$.

Following these preliminaries, we proceed by considering TC_{gg} in (G.5). In this expression, q_3 represents the hard parton, while g_1 and g_2 are the potentially unresolved partons. Thus, we identify $i = q_3$, $\mathbf{m} = g_1$, and $\mathbf{n} = g_2$. Unlike in (G.12), here the triple-collinear limit also applies to $\Delta^{(\mathbf{mn})}$, introducing an additional factor:¹

$$C_{\mathbf{mn},i} \Delta^{(\mathbf{mn})} = z_i \stackrel{\text{def}}{=} \frac{E_i}{E_i + E_{\mathbf{m}} + E_{\mathbf{n}}}. \quad (\text{G.15})$$

Note that z_i corresponds to the z -factor of the hard parton, so in TC_{gg} , we have $z_i = z_3$. Therefore, we can express TC_{gg} as

$$\text{TC}_{gg} = \langle \bar{S}_{12} \bar{S}_2 (\mathbb{1} - \mathfrak{C}(1, 2|3)) z_3 \mathcal{P}_{ggq}(z_1, z_2|z_3) \Theta_{12} F_{\text{LM}}[\dots, [\text{imn}]_q] \rangle. \quad (\text{G.16})$$

Next, we discuss the action of the soft and collinear operators on z_3 . First, we note that $\mathfrak{C}(1, 2|3) z_3[\dots] = z_3 \mathfrak{C}(1, 2|3)[\dots]$, since $\mathfrak{C}(1, 2|3)$ acts on angles, while z_3 is an energy-related term. Secondly, we observe that

$$S_{12} z_3 = \lim_{\substack{E_1, E_2 \rightarrow 0 \\ E_1/E_2 = \text{const}}} \frac{E_3}{E_1 + E_2 + E_3} = 1, \quad S_{12} z_3[\dots] = S_{12}[\dots], \quad (\text{G.17})$$

and, lastly,

$$S_2 z_3[\dots] = S_2[\dots] - S_2(1 - z_3)[\dots], \quad (\text{G.18})$$

¹In Appendix B, this z_i factor is denoted as $z_{i,\mathbf{mn}}$. For simplicity, we omit the two extra indices here.

since

$$S_2 z_3 = \lim_{E_2 \rightarrow 0} \frac{E_3}{E_1 + E_2 + E_3} = \frac{E_3}{E_1 + E_3}. \quad (\text{G.19})$$

With this in mind, we have

$$\begin{aligned} & \bar{S}_{12} \bar{S}_2 (\mathbb{1} - \mathfrak{C}(1, 2|3)) z_3 [\dots] \\ &= \bar{S}_{12} \bar{S}_2 (\mathbb{1} - \mathfrak{C}(1, 2|3)) [\dots] - \bar{S}_2 (1 - z_3) (\mathbb{1} - \mathfrak{C}(1, 2|3)) [\dots], \end{aligned} \quad (\text{G.20})$$

which leads to

$$\text{TC}_{gg} = \text{TC}_{\text{old}} - \langle \bar{S}_2 (1 - z_3) (\mathbb{1} - \mathfrak{C}(1, 2|3)) \mathcal{P}_{ggq}(z_1, z_2|z_3) \Theta_{12} F_{\text{LM}}[\dots, [123]_q] \rangle, \quad (\text{G.21})$$

where

$$\text{TC}_{\text{old}} \equiv \langle \bar{S}_{12} \bar{S}_2 (\mathbb{1} - \mathfrak{C}(1, 2|3)) \mathcal{P}_{ggq}(z_1, z_2|z_3) \Theta_{12} F_{\text{LM}}[\dots, [123]_q] \rangle. \quad (\text{G.22})$$

We now consider TC_{gg} from (G.6). Here, the hard parton is $i = g_1$, while the unresolved partons are $\mathfrak{m} = q_3$ and $\mathfrak{n} = g_2$, respectively. Eq. (G.12) still holds, but the action of the triple-collinear limit on $\Delta^{(\mathfrak{mn})}$ now yields $z_i = z_1$. By repeating the steps outlined above concerning the action of the single-soft and $\mathfrak{C}(\mathfrak{m}, \mathfrak{n}|i)$ operators, we obtain

$$\text{TC}_{gg} = \langle \bar{S}_2 z_1 (\mathbb{1} - \mathfrak{C}(3, 2|1)) \mathcal{P}_{ggq}(z_1, z_2|z_3) F_{\text{LM}}[\dots, [123]_q] \rangle. \quad (\text{G.23})$$

Notice that the parametrization of the single-collinear subtraction in Eqs (G.21) and (G.23) differs, as $\mathfrak{C}(3, 2|1) \neq \mathfrak{C}(1, 2|3)$.

At this point, by combining Eqs (G.21) and (G.23), we derive the following expression for ΔTC_{gg+qg} :

$$\begin{aligned} \Delta \text{TC}_{gg+qg} &= - \langle (\mathbb{1} - S_2) (1 - z_3) (\mathbb{1} - \mathfrak{C}(1, 2|3)) \mathcal{P}_{ggq}(z_1, z_2|z_3) \Theta_{12} F_{\text{LM}}[\dots, [123]_q] \rangle \\ &\quad + \langle (\mathbb{1} - S_2) z_1 (\mathbb{1} - \mathfrak{C}(3, 2|1)) \mathcal{P}_{ggq}(z_1, z_2|z_3) F_{\text{LM}}[\dots, [123]_q] \rangle. \end{aligned} \quad (\text{G.24})$$

In the first term on the right-hand side of (G.24), we focus on the part proportional to $\mathbb{1}(1 - z_3)[\dots]$: notice that $1 - z_3 = z_1 + z_2$, so using the symmetry of $\mathcal{P}_{ggq}$ in its first two entries, we can write

$$\begin{aligned} & - \langle (1 - z_3) \mathcal{P}_{ggq}(z_1, z_2|z_3) \Theta_{12} F_{\text{LM}}[\dots, [123]_q] \rangle \\ &= - \langle z_1 \mathcal{P}_{ggq}(z_1, z_2|z_3) F_{\text{LM}}[\dots, [123]_q] \rangle. \end{aligned} \quad (\text{G.25})$$

This term cancels against the part proportional to $\mathbb{1}$ of the second expression on the right-hand side of Eq. (G.24). Hence, in Eq. (G.24) we have no pure triple-collinear leftover, and its new expression becomes

$$\begin{aligned} \Delta \text{TC}_{gg+qg} &= \left\langle \left[S_2 + \mathfrak{C}(1, 2|3) - S_2 \mathfrak{C}(1, 2|3) \right] (1 - z_3) \mathcal{P}_{ggq}(z_1, z_2|z_3) \Theta_{12} F_{\text{LM}}[\dots, [123]_q] \right\rangle \\ &\quad - \left\langle \left[S_2 + \mathfrak{C}(3, 2|1) - S_2 \mathfrak{C}(3, 2|1) \right] z_1 \mathcal{P}_{ggq}(z_1, z_2|z_3) F_{\text{LM}}[\dots, [123]_q] \right\rangle. \end{aligned} \quad (\text{G.26})$$

At this point, we separate the *strongly-ordered* contributions (i.e., the terms proportional to the $\mathfrak{C}$ -operators) from the remaining pieces. Regarding the terms where only S_2 acts, given that $1 - z_3 = z_1 + z_2$, $S_2 z_2 = 0$, and $\Theta_{12} + \Theta_{21} = 1$, we obtain

$$\begin{aligned} & \langle S_2 [(1 - z_3)\Theta_{12} - z_1] \mathcal{P}_{ggq}(z_1, z_2|z_3) F_{\text{LM}}[\dots, [123]_q] \rangle \\ &= - \langle S_2 z_1 \mathcal{P}_{ggq}(z_1, z_2|z_3) \Theta_{21} F_{\text{LM}}[\dots, [123]_q] \rangle, \end{aligned} \quad (\text{G.27})$$

where this term is explicitly soft-regulated due to the energy-ordering enforced by Θ_{21} . As for the strongly-ordered terms, we exchange the labels 1 as 3 in the last line of Eq. (G.26), which maps $\mathfrak{C}(3, 2|1)$ in $\mathfrak{C}(1, 2|3)$ and $z_1 \mathcal{P}_{ggq}(z_1, z_2|z_3)$ in $z_3 \mathcal{P}_{ggq}(z_3, z_2|z_1)$. All in all, we rewrite Eq. (G.26) as

$$\begin{aligned} \Delta \text{TC}_{gg+qq} &= - \langle S_2 z_1 \mathcal{P}_{ggq}(z_1, z_2|z_3) \Theta_{21} F_{\text{LM}}[\dots, [imn]_q] \rangle \\ &+ \langle \bar{S}_2 \mathfrak{C}(1, 2|3) [(1 - z_3) \mathcal{P}_{ggq}(z_1, z_2|z_3) \Theta_{12} - z_3 \mathcal{P}_{ggq}(z_3, z_2|z_1)] \\ &\times F_{\text{LM}}[\dots, [123]_q] \rangle. \end{aligned} \quad (\text{G.28})$$

Finally, it is worthwhile to revert to the standard notation involving $\{i, m, n\}$. Notice that the definition of $\mathfrak{C}$ in (G.14) specifies that the first two arguments of $\mathfrak{C}$ correspond to partons m and n , respectively. This leads to the following expression:

$$\begin{aligned} \Delta \text{TC}_{gg+qq} &= - \langle S_n z_m \mathcal{P}_{ggq}(z_m, z_n|z_i) \Theta_{nm} F_{\text{LM}}^{qq}[\dots, [imn]_q] \rangle \\ &+ \langle \bar{S}_n \mathfrak{C}(m, n|i) [(1 - z_i) \mathcal{P}_{ggq}(z_m, z_n|z_i) \Theta_{mn} - z_i \mathcal{P}_{ggq}(z_i, z_n|z_m)] \\ &\times F_{\text{LM}}[\dots, [imn]_q] \rangle. \end{aligned} \quad (\text{G.29})$$

where

$$z_{i,m,n} \stackrel{\text{def}}{=} \frac{E_{i,m,n}}{(E_i + E_m + E_n)}, \quad z_i + z_m + z_n = 1. \quad (\text{G.30})$$

If desired, Eq. (G.29) can also be rewritten in a form that is explicitly of $\mathcal{O}(\epsilon^{-1})$. To show it, we start by considering the second term on the right-hand side of Eq. (G.29), specifically the following contribution:

$$\langle (\mathbb{1} - S_n) \mathfrak{C}(m, n|i) (1 - z_i) \mathcal{P}_{ggq}(z_m, z_n|z_i) \Theta_{mn} F_{\text{LM}}[\dots, [imn]_q] \rangle. \quad (\text{G.31})$$

This term is made of a pure collinear contribution and a soft-collinear one, which is subtracted from the former. Since $1 - z_i = z_m + z_n$, in the pure collinear contribution, we have $\langle [\dots](z_m + z_n) \Theta_{mn} \rangle$, where $[\dots]$ is symmetric in m and n . Therefore, we can write

$$\begin{aligned} \langle [\dots](z_m + z_n) \Theta_{mn} \rangle &= \langle [\dots] z_m \Theta_{mn} \rangle + \langle [\dots] z_n \Theta_{mn} \rangle \\ &= \langle [\dots] z_m (\Theta_{mn} + \Theta_{nm}) \rangle \\ &= \langle [\dots] z_m \rangle, \end{aligned} \quad (\text{G.32})$$

where in the term in red we relabelled m as n and vice versa. As for the soft-collinear

contribution, we have $S_n z_n = 0$, which yields

$$\begin{aligned} & - \langle S_n \mathfrak{C}(\mathbf{m}, \mathbf{n}|i)(z_m + z_n) \mathcal{P}_{ggq}(z_m, z_n|z_i) \Theta_{mn} F_{\text{LM}}[\dots, [imn]_q] \rangle \\ & = - \langle S_n \mathfrak{C}(\mathbf{m}, \mathbf{n}|i) z_m \mathcal{P}_{ggq}(z_m, z_n|z_i) F_{\text{LM}}[\dots, [imn]_q] \rangle \\ & \quad + \langle S_n \mathfrak{C}(\mathbf{m}, \mathbf{n}|i) z_n \mathcal{P}_{ggq}(z_m, z_n|z_i) \Theta_{nm} F_{\text{LM}}[\dots, [imn]_q] \rangle. \end{aligned} \quad (\text{G.33})$$

Replacing both Eqs (G.32) and (G.33) in Eq. (G.29), we obtain

$$\begin{aligned} \Delta \text{TC}_{gg+qq} & = - \langle S_n (\mathbb{1} - \mathfrak{C}(\mathbf{m}, \mathbf{n}|i)) z_m \mathcal{P}_{ggq}(z_m, z_n|z_i) \Theta_{nm} F_{\text{LM}}[\dots, [imn]_q] \rangle \\ & \quad + \langle \bar{S}_n \mathfrak{C}(\mathbf{m}, \mathbf{n}|i) [z_m \mathcal{P}_{ggq}(z_m, z_n|z_i) - z_i \mathcal{P}_{ggq}(z_i, z_n|z_m)] F_{\text{LM}}[\dots, [imn]_q] \rangle, \end{aligned} \quad (\text{G.34})$$

which is explicitly of $\mathcal{O}(\epsilon^{-1})$.

The formula in Eq. (G.34) can be computed analytically. It is a fairly simple calculation, which does not involve advanced techniques. However, it involves a lot of algebra, and for this reason we just report the final result. Indeed, once Eq. (G.34) has been computed, we have all the information to obtain the triple-collinear contribution TC_{gg+qq} in Eq. (G.4). We write it as

$$\text{TC}_{gg+qq} = [\alpha_s]^2 \langle (E/\mu)^{-4\epsilon} \widetilde{\text{FSR}}_1 \cdot F_{\text{LM}}^{\mathcal{B}_0} \rangle, \quad (\text{G.35})$$

where the function $\widetilde{\text{FSR}}_1$ is defined as

$$\begin{aligned} \widetilde{\text{FSR}}_1(E_i) & \stackrel{\text{def}}{=} \frac{1}{\epsilon} \left\{ C_A C_F \left[L_i \left(\frac{\pi^2}{6} - \frac{32}{9} - \frac{7}{3} \log 2 \right) + \frac{19\zeta_3}{8} + \frac{\pi^2}{8} - \frac{503}{54} + \frac{11 \log^2 2}{2} \right. \right. \\ & \quad \left. \left. + \left(\frac{\pi^2}{3} - \frac{85}{36} \right) \log 2 \right] + C_F^2 \left[\left(4L_i + \frac{155}{24} + \frac{\pi^2}{3} \right) \log 2 - 2\zeta_3 + \frac{31}{16} \right] \right\} \\ & \quad + C_A C_F \left\{ L_i^2 \left(6 \log 2 - \frac{\pi^2}{3} + \frac{64}{9} \right) + L_i \left[\frac{11\zeta_3}{2} - \frac{22\pi^2}{9} + \frac{383}{54} - \frac{31}{3} \log^2 2 + \left(\frac{49}{9} - \frac{2\pi^2}{3} \right. \right. \right. \\ & \quad \left. \left. \left. - \frac{16}{3} \tilde{L}_i \right) \log 2 \right] - \tilde{L}_i \left(\frac{1}{3} + \frac{14}{9} \log 2 \right) - 2\text{Li}_4(1/2) + \frac{25\zeta_3}{24} + \left(\frac{161\pi^2}{36} - \frac{13\zeta_3}{4} - \frac{71}{9} \right) \right. \\ & \quad \left. \times \log 2 + \frac{173\pi^4}{480} - \frac{6743}{144} - \frac{709\pi^2}{432} - \frac{1}{12} \log^4 2 - \frac{176 \log^3 2}{9} - \left(\frac{14}{9} + \frac{11\pi^2}{12} \right) \log^2 2 \right\} \\ & \quad + C_F^2 \left\{ -4L_i^2 \log 2 + L_i \left(\frac{8\pi^2}{9} + \frac{10 \log^2 2}{3} + \frac{46 \log 2}{3} - 16 \tilde{L}_i \log 2 \right) - \frac{64}{3} \tilde{L}_i \log 2 \right. \\ & \quad \left. + \left(16\zeta_3 - \frac{7\pi^2}{9} + \frac{3941}{72} \right) \log 2 + \frac{38\pi^2}{27} - \frac{7\pi^4}{30} + \frac{715}{32} + \left(\frac{121}{144} - \frac{7\pi^2}{6} \right) \log^2 2 \right\}, \end{aligned} \quad (\text{G.36})$$

with $L_i = \log(E_{\text{max}}/E_i)$ and $\tilde{L}_i = \log(2E_i/\mu)$.

G.2 Singularities arising from interferences

In this section we discuss the singularities that arise as a result of interference amplitudes. We begin by looking at process

$$\mathcal{Y} : \quad a_g + b_q \rightarrow \{g\}_{N-1} + q + q' \bar{q}' + X, \quad (\text{G.37})$$

where the quark-antiquark pair $q' \bar{q}'$ may have any flavors. Notice that one of the final-state quarks is connected to the initial state quark, the other quark-antiquark pair arises from a gluon splitting. We split process $\mathcal{Y}$ into a process where q and q' have *different* flavors, and one in which they have the *same* flavors:

$$\mathcal{Y}_{\text{df}} : \quad a_g + b_q \rightarrow \{g\}_{N-1} + q + q' \bar{q}' + X, \quad q' \neq q, \quad (\text{G.38})$$

$$\mathcal{Y}_{\text{sf}} : \quad a_g + b_q \rightarrow \{g\}_{N-1} + q + q' \bar{q}' + X, \quad q' = q, \quad (\text{G.39})$$

although we find it convenient to keep the notation $q' \bar{q}'$ throughout. Notice that process $\mathcal{Y}_{\text{sf}}$ coincides with process $\tilde{\mathcal{B}}_2$ defined in Eq. (4.49).

We write the amplitude for the different-flavor process as

$$\mathcal{M}_{\text{df}} \stackrel{\text{def}}{=} \mathcal{M}(p_X; p_a, p_b, \{p_g\}_{N-1}, p_q, p_{q'}, p_{\bar{q}'}) \equiv \mathcal{M}(p_q | p_{q'}, p_{\bar{q}'}), \quad (\text{G.40})$$

where we adopt the convention that the final arguments are the momenta of the quark-antiquark pair arising from gluon splitting. Additionally, we suppressed the arguments of the color-singlet momentum, the initial-state momenta and the final-state gluon momenta for brevity. We then square this amplitude and integrate it over the final-state phase space, obtaining (cf. Sec. 3.1)

$$\langle F_{\text{LM}}^{gq}[\{g\}_{N-1}, q, q', \bar{q}'] \rangle_{\text{df}} = \int |\mathcal{M}_{\text{df}}|^2 \text{dLips}_X \hat{O}(\{p\}_{\mathcal{H}_f}; p_X) \quad (\text{G.41})$$

for a single quark flavor $q' \neq q$. The amplitude for the same flavor process is

$$\mathcal{M}_{\text{sf}} \stackrel{\text{def}}{=} \mathcal{M}(p_q | p_{q'}, p_{\bar{q}'}) + \mathcal{M}(p_{q'} | p_q, p_{\bar{q}'}). \quad (\text{G.42})$$

Upon squaring this object and integrating over the final-state phase space, we have

$$\begin{aligned} \langle F_{\text{LM}}^{gq}[\{g\}_{N-1}, q, q', \bar{q}'] \rangle_{\text{sf}} &= \int |\mathcal{M}_{\text{sf}}|^2 \text{dLips}_X \hat{O}(\{p\}_{\mathcal{H}_f}; p_X) \\ &= \langle F_{\text{LM}}^{gq}[\{g\}_{N-1}, q, q', \bar{q}'] \rangle_{\text{df}} + \langle F_{\text{LM}}^{gq}[\{g\}_{N-1}, q, q', \bar{q}'] \rangle_{\text{int}} \end{aligned} \quad (\text{G.43})$$

where

$$\begin{aligned} \langle F_{\text{LM}}^{gq}[\{g\}_{N-1}, q, q', \bar{q}'] \rangle_{\text{int}} &\stackrel{\text{def}}{=} \int 2 \text{Re}[\mathcal{M}^*(p_q | p_{q'}, p_{\bar{q}'}) \mathcal{M}(p_{q'} | p_q, p_{\bar{q}'})] \\ &\quad \times \text{dLips}_X \hat{O}(\{p\}_{\mathcal{H}_f}; p_X). \end{aligned} \quad (\text{G.44})$$

Multiplying the different-flavor result by the $(n_f - 1)$ flavors for q' with $q' \neq q$, and then adding the same-flavor results, we have

$$2s_{ab} d\tilde{\sigma}_{\tilde{B}_2}^{\text{int}} = \frac{n_f}{(N-1)!} \langle F_{\text{LM}}^{gq}[\{g\}_{N-1}, q, q', \bar{q}'] \rangle_{\text{df}} + \frac{1}{2!(N-1)!} \langle F_{\text{LM}}^{gq}[\{g\}_{N-1}, q, q', \bar{q}'] \rangle_{\text{int}}. \quad (\text{G.45})$$

Since we parametrically set $n_f = 0$ throughout this thesis, we can discard the first term, and focus our attention on the interference term only.

Let us consider a configuration in which one of the final state quarks becomes collinear to any other parton in the process. In that case, one of the two interfering amplitudes will develop an integrable singularity, while the other amplitude will remain finite. Thus, integrating over the phase space will not produce a pole in $1/\epsilon$. Similarly, we will not have poles due to double-soft configurations. Indeed, the only configuration involving the quarks that can lead to a pole in the interference term is the triple-collinear limit, $C_{qq'\bar{q}'}$. With this in mind, we can introduce damping factors defined in Appendix B:

$$\begin{aligned} \mathcal{P}_{B_0} [2s_{ab} d\tilde{\sigma}_{\tilde{B}_2}^{\text{int}}] &= \sum_{(ij)} \mathcal{P}_{B_0} \left\langle \frac{\Delta^{(ij)}}{2!(N-1)!} F_{\text{LM}}^{gq}[\{g\}_{N-1}, q, q', \bar{q}'] \right\rangle_{\text{int}} \\ &= \mathcal{P}_{B_0} \left\langle \Delta^{(\text{mn})} \left[F_{\text{LM}}^{\tilde{B}_2}[\mathbf{m}_{q'}, \mathbf{n}_{\bar{q}'}] + F_{\text{LM}}^{\tilde{B}_2}[\mathbf{m}_q, \mathbf{n}_{\bar{q}'}] + F_{\text{LM}}^{\tilde{B}_2}[\mathbf{m}_{q'}, \mathbf{n}_q] \right] \right\rangle_{\text{int}}, \end{aligned} \quad (\text{G.46})$$

where, as usual, the symmetry factors have been reabsorbed within the functions $F_{\text{LM}}^{\tilde{B}_2}$. Since we are interested in those limits that project this process onto B_0 , in Eq. (G.46) we have immediately discarded the option $(i, j) = (g, g)$. Similarly, and recalling that the only singular limit involving the quark is the triple collinear one, $C_{qq'\bar{q}'}$, we have also discarded the option $(i, j) = (q, g)$.

Therefore, from Eq. (G.46) we extract the final-state triple-collinear limit which takes place in the $q^* \rightarrow qq\bar{q}$ splitting process. The final result reads

$$\mathcal{P}_{B_0} [2s_{ab} d\tilde{\sigma}_{\tilde{B}_2}^{\text{int}}] = \frac{[\alpha_s]^2}{2} \langle (E/\mu)^{-4\epsilon} \text{FSR}_3 \cdot F_{\text{LM}}^{B_0} \rangle, \quad (\text{G.47})$$

where the function $\text{FSR}_3(E)$ is reported in Table 2 of Ref. [79].

Triple color-correlated contributions to real-virtual corrections

In this appendix we discuss the computation of the triple color-correlated component arising from the integrated soft limit of the real-virtual contribution.¹ The relevant factorization formula in the soft limit is given in Eq. (2.136), and we are interested in the final term

$$S_{\mathbf{m}}^{\text{tri}} F_{\text{RV}}(\mathbf{m}) = -[\alpha_s] \frac{4\pi \Gamma(1+\epsilon) \Gamma^3(1-\epsilon)}{\epsilon \Gamma(1-2\epsilon)} \sum_{(ijk)} \kappa_{ij} S_{ki}(p_{\mathbf{m}}) [S_{ij}(p_{\mathbf{m}})]^{1+\epsilon} F_{\text{LM}}^{(kij)}, \quad (\text{H.1})$$

where (ijk) stands for $i, j, k \in \mathcal{H}$, with $i \neq j \neq k$, and

$$F_{\text{LM}}^{(kij)} \stackrel{\text{def}}{=} f_{abc} \langle \mathcal{M}_0 | T_k^a T_i^b T_j^c | \mathcal{M}_0 \rangle \quad (\text{H.2})$$

indicates the triple color-correlated matrix element. The phase factor κ_{ij} is reported in Eq. (2.137), and the eikonal factor S_{ij} in Eq. (2.54). Here we just recall that κ_{ij} is completely symmetric under the exchange $i \leftrightarrow j$ and (obviously) is independent of k . Moreover, we point out that in (H.1), we include all particles from the Born process, while maintaining that the subsequent discussion remains entirely independent of their specifics.

We begin by pointing out that the triple color-correlated matrix element gives a non-zero contribution only when there are at least four colored particles in the Born-level process (cf. Sec. 2.1). Indeed, with three colored particles one can use color conservation to obtain the following identity:

$$f_{abc} T_1^a T_2^b T_3^c | \mathcal{M}_0 \rangle = -f_{abc} T_1^a T_2^b (T_1^c + T_2^c) | \mathcal{M}_0 \rangle = 0. \quad (\text{H.3})$$

Our goal is to integrate Eq. (H.1) over the phase space of the soft gluon with mo-

¹This Appendix has been included in this thesis exactly as it appears in Ref. [81], as it represents the only contribution in Part II that was not directly calculated by the author of this thesis.

mentum p_m . We begin by integrating over the energy E_m and obtain

$$\begin{aligned} \langle S_m^{\text{tri}} F_{\text{RV}}(m) \rangle &= -[\alpha_s]^2 \frac{4\pi^{3-\epsilon} 2^\epsilon \Gamma(1+\epsilon) \Gamma^4(1-\epsilon)}{\epsilon^2 \Gamma(1-2\epsilon)} \left(\frac{4E_{\text{max}}^2}{\mu^2} \right)^{-2\epsilon} \\ &\times \sum_{(ijk)} \left\langle \kappa_{ij} G^{kij} F_{\text{LM}}^{(kij)} \right\rangle. \end{aligned} \quad (\text{H.4})$$

In Eq. (H.4), we have introduced the object

$$G^{kij} \stackrel{\text{def}}{=} \int \frac{d\Omega_m^{(d-1)}}{2(2\pi)^{d-1}} \frac{\rho_{ki}}{\rho_{km}\rho_{im}} \left(\frac{\rho_{ij}}{\rho_{im}\rho_{jm}} \right)^\epsilon, \quad (\text{H.5})$$

which is a function of the angular variables ρ_{ij} , ρ_{ik} and ρ_{jk} . We note that, since κ_{ij} and $F_{\text{LM}}^{(kij)}$ are a symmetric and a fully anti-symmetric tensors, respectively, then only the anti-symmetric part of G^{kij} , namely $G^{kij} - G^{kji}$, can contribute to the sum, whereas the symmetric part drops out. We will use this result when writing intermediate expressions for $\langle S_m^{\text{tri}} F_{\text{RV}}(m) \rangle$.

To perform the remaining integration over the soft-gluon angle, we employ the *Mellin-Barnes representation* of d -dimensional angular integrals presented in Ref. [168], and write the integral as

$$\begin{aligned} G^{kij} &= \int \frac{d\Omega_m^{(d-1)}}{2(2\pi)^{d-1}} \frac{\rho_{ki}}{\rho_{km}\rho_{im}} \left(\frac{\rho_{ij}}{\rho_{im}\rho_{jm}} \right)^\epsilon \\ &= \rho_{ki}(\rho_{ij})^\epsilon \int_{-i\infty}^{+i\infty} \frac{dz_{ij} dz_{jk} dz_{ki}}{(2\pi i)^3} \frac{\pi^{-2+\epsilon}}{2^{4+2\epsilon}} \Gamma(-z_{ij}) \Gamma(-z_{ki}) \Gamma(-z_{jk}) \\ &\quad \times \Gamma(1+\epsilon+z_{ij}+z_{ki}) \Gamma(-1-3\epsilon-z_{ij}-z_{ki}-z_{jk}) \Gamma(\epsilon+z_{ij}+z_{jk}) \\ &\quad \times \Gamma(1+z_{ki}+z_{jk}) \frac{1}{\Gamma(-4\epsilon)\Gamma(\epsilon)\Gamma(1+\epsilon)} (\eta_{ij})^{z_{ij}} (\eta_{ki})^{z_{ki}} (\eta_{jk})^{z_{jk}}. \end{aligned} \quad (\text{H.6})$$

In the above equation, we have introduced the three complex Mellin-Barnes variables z_{ij} , z_{ki} , z_{jk} , and $\eta_{ij} = \rho_{ij}/2$. The integration contour has to be chosen in such a way that the poles of $\Gamma(\dots + x)$ are separated from the poles of $\Gamma(\dots - x)$, with x being a generic integration variable. In order to resolve the singularity structure in ϵ , we employ the packages `MBresolve` [169] and `MB` [170], which allow us to express our original integral as a linear combination of integrals that can be safely expanded in ϵ under the integration sign, and whose integration contours are straight vertical lines in the complex plane. Upon applying this procedure we find that it is possible to express the function G^{kij} up to $\mathcal{O}(\epsilon^0)$ in terms of classical and *generalized polylogarithms* (GPLs) [171, 172] up to weight three. It is convenient to write the final result for the angular integral as

$$\begin{aligned} G^{kij} &= \int \frac{d\Omega_m^{(d-1)}}{2(2\pi)^{d-1}} \frac{\rho_{ki}}{\rho_{km}\rho_{im}} \left(\frac{\rho_{ij}}{\rho_{im}\rho_{jm}} \right)^\epsilon \\ &= -\frac{\epsilon^2}{4\pi^2} \left[\frac{2^{-\epsilon} \pi^\epsilon \Gamma(1-\epsilon)}{\Gamma(1-4\epsilon)\Gamma^2(1+\epsilon)} \right] \bar{G}^{kij}, \end{aligned} \quad (\text{H.7})$$

where

$$\begin{aligned} \overline{G}^{kij} \stackrel{\text{def}}{=} & \frac{3}{4\epsilon^3} + \frac{1}{2\epsilon^2} \left[\log(\eta_{ij}) - 3\log(\eta_{ik}) - \log(\eta_{jk}) \right] + \frac{1}{\epsilon} \left[\frac{1}{2} \log^2(\eta_{ij}) \right. \\ & + \log(\eta_{ij}) \left[-\log(\eta_{ik}) - \log(\eta_{jk}) \right] + \log(\eta_{ik}) \left[\log(\eta_{jk}) - 2\log(1 - \eta_{ik}) \right] \\ & \left. - 2\text{Li}_2(\eta_{ik}) + \frac{3}{2} \log^2(\eta_{ik}) + \frac{1}{2} \log^2(\eta_{jk}) + \pi^2 \right] + \mathcal{O}(\epsilon^0). \end{aligned} \quad (\text{H.8})$$

Note that the ϵ -dependent prefactor in Eq. (H.7) starts at $\mathcal{O}(\epsilon^2)$, so that the whole angular integral is effectively $\mathcal{O}(\epsilon^{-1})$, as expected. Inserting the result for G^{kij} into Eq. (H.4), we get the following final result for the triple color-correlated contribution to the real-virtual counterterm:

$$\begin{aligned} \langle S_{\text{m}}^{\text{tri}} F_{\text{RV}}(\mathbf{m}) \rangle &= [\alpha_s]^2 \langle I_{\text{tri}}^{\text{RV}} \cdot F_{\text{LM}}^{\mathcal{H}} \rangle \\ &\stackrel{\text{def}}{=} [\alpha_s]^2 \frac{\pi \Gamma^5(1 - \epsilon)}{\Gamma(1 - 2\epsilon) \Gamma(1 - 4\epsilon) \Gamma(1 + \epsilon)} \left(\frac{4E_{\text{max}}^2}{\mu^2} \right)^{-2\epsilon} \sum_{(ijk)} \langle \kappa_{ij} \overline{G}^{kij} F_{\text{LM}}^{(kij)} \rangle. \end{aligned} \quad (\text{H.9})$$

To proceed further, it is convenient to split the function $\overline{G}^{kij}$ into contributions using its symmetry properties under the $i \leftrightarrow j$ permutations. We write

$$\overline{G}^{kij} = \overline{G}_{\text{s}}^{kij} + \overline{G}_{\text{r}}^{kij}, \quad (\text{H.10})$$

with

$$\overline{G}_{\text{s}}^{kij} \stackrel{\text{def}}{=} \frac{3}{4} \frac{1}{\epsilon^3} + \frac{1}{2\epsilon^2} \log\left(\frac{\eta_{ij}}{\eta_{jk}\eta_{ik}}\right) + \frac{1}{\epsilon} \left[\frac{2\pi^2}{3} + \frac{1}{2} \log^2\left(\frac{\eta_{ij}}{\eta_{jk}\eta_{ik}}\right) \right] + \mathcal{O}(\epsilon^0), \quad (\text{H.11})$$

and

$$\overline{G}_{\text{r}}^{kij} \stackrel{\text{def}}{=} -\frac{\log \eta_{ik}}{\epsilon^2} + \frac{1}{\epsilon} \left[\log^2 \eta_{ik} + 2\text{Li}_2(1 - \eta_{ik}) \right] + \mathcal{O}(\epsilon^0). \quad (\text{H.12})$$

The function $\overline{G}_{\text{s}}^{kij}$ is symmetric under $i \leftrightarrow j$ permutations; hence, it does not contribute to $\langle S_{\text{m}}^{\text{tri}} F_{\text{RV}}(\mathbf{m}) \rangle$ and can be dropped. Note also that the function $\overline{G}_{\text{r}}^{kij}$, up to $\mathcal{O}(\epsilon^{-1})$, is symmetric under the $i \leftrightarrow k$ permutation. It follows that $\langle S_{\text{m}}^{\text{tri}} F_{\text{RV}}(\mathbf{m}) \rangle$ is free of poles for processes with a color-singlet initial state, as in this case we have $\kappa_{ij} = -1$. For a hadron collider process with two incoming a, b partons, and any number of outgoing partons, the coefficient κ_{ij} reads

$$\kappa_{ij} = -1 + 2\delta_{ia}\delta_{jb} + 2\delta_{ib}\delta_{ja}, \quad (\text{H.13})$$

from which it follows that

$$\begin{aligned} \sum_{(ijk)} \langle \kappa_{ij} \overline{G}^{kij} F_{\text{LM}}^{(kij)} \rangle &= \sum_{(ijk)} \langle \kappa_{ij} \overline{G}_{\text{r}}^{kij} F_{\text{LM}}^{(kij)} \rangle \\ &= 2 \sum_{k \in \mathcal{H}_{\text{f}}} \langle (\overline{G}_{\text{r}}^{kab} - \overline{G}_{\text{r}}^{kba}) F_{\text{LM}}^{(kab)} \rangle. \end{aligned} \quad (\text{H.14})$$

Here, $\mathcal{H}_f$ corresponds to the list of final-state hard partons. Using this result together with Eqs (H.9) and (H.12), we obtain the final formula for the poles in the triple color-correlated contribution to the soft limit of the real-virtual corrections:

$$\begin{aligned} \langle S_{\mathbf{m}}^{\text{tri}} F_{\text{RV}}(\mathbf{m}) \rangle &= [\alpha_s]^2 \sum_{k \in \mathcal{H}_f} \left\langle F_{\text{LM}}^{(kab)} \left\{ \frac{2\pi}{\epsilon^2} \log \frac{\eta_{bk}}{\eta_{ak}} + \frac{2\pi}{\epsilon} \left[\log^2 \eta_{ak} - \log^2 \eta_{bk} \right. \right. \right. \\ &\quad \left. \left. + 2L_{\text{max}}^2 \log \left(\frac{\eta_{ak}}{\eta_{bk}} \right) + 2\text{Li}_2(1 - \eta_{ak}) - 2\text{Li}_2(1 - \eta_{bk}) \right] \right. \\ &\quad \left. \left. + \mathcal{O}(\epsilon^0) \right\} \right\rangle, \end{aligned} \quad (\text{H.15})$$

with $L_{\text{max}} = \log(2E_{\text{max}}/\mu)$.

We now present the formula for the $\mathcal{O}(\epsilon^0)$ terms of Eq. (H.8). We exploit once again the symmetry properties of the triple color-correlated contribution under the exchange of $i \leftrightarrow j$ indices and therefore only present results for the antisymmetric part. The result reads

$$\begin{aligned} \overline{G}_{\text{r,fin}}^{kij} &= \text{Li}_2(\eta_{ij}) \log \left(\frac{\eta_{ik}}{\eta_{jk}} \right) - \text{Li}_2(\eta_{ik}) \log \left(\frac{\eta_{jk}}{\eta_{ij}\eta_{ik}} \right) + \text{Li}_2(\eta_{jk}) \log \left(\frac{\eta_{ik}}{\eta_{ij}\eta_{jk}} \right) \\ &\quad + \log(\eta_{ik}) \text{Li}_2 \left(-\frac{\eta_{ik} - \eta_{jk}}{1 - \eta_{ik}} \right) + \log(\eta_{ik}) \text{Li}_2 \left(-\frac{\eta_{ik} - \eta_{jk}}{\eta_{jk}} \right) + 3\text{Li}_3(1 - \eta_{ik}) \\ &\quad - 3\text{Li}_3(1 - \eta_{jk}) + \text{Li}_2 \left(\frac{1 - \eta_{jk}}{1 - \eta_{ik}} \right) \log(\eta_{ik}\eta_{jk}) + \text{Li}_2 \left(\frac{\eta_{ik}}{\eta_{jk}} \right) \log(\eta_{ik}\eta_{jk}) \\ &\quad - \log(\eta_{jk}) \text{Li}_2 \left(-\frac{\eta_{jk} - \eta_{ik}}{\eta_{ik}} \right) - \log(\eta_{jk}) \text{Li}_2 \left(-\frac{\eta_{jk} - \eta_{ik}}{1 - \eta_{jk}} \right) + \text{Li}_3(\eta_{ik}) \\ &\quad - \text{Li}_3(\eta_{jk}) + \log^2(\eta_{ik}) \left[\frac{1}{2} \log \left(\frac{1 - \eta_{jk}}{\eta_{ij}} \right) + \log \left(\frac{\eta_{jk} - \eta_{ik}}{\eta_{jk}} \right) \right] \\ &\quad + \log(\eta_{ik}) \left[-\frac{1}{2} \log^2(\eta_{ij}) + \log(1 - \eta_{ij}) \log(\eta_{ij}) + \frac{1}{2} \log^2 \left(\frac{1 - \eta_{jk}}{1 - \eta_{ik}} \right) \right. \\ &\quad \left. + \frac{1}{2} \log^2 \left(\frac{\eta_{ik}}{\eta_{jk}} \right) + \log(1 - \eta_{jk}) \log(\eta_{jk}(\eta_{jk} - \eta_{ik})) + \log^2(\eta_{jk}) - \frac{13\pi^2}{6} \right] \\ &\quad + \log(1 - \eta_{jk}) \left[-\log(\eta_{jk}) \log \left(\frac{\eta_{ij}}{\eta_{jk} - \eta_{ik}} \right) - \log^2(\eta_{jk}) - \frac{\pi^2}{6} \right] \\ &\quad + \log(1 - \eta_{ik}) \left[\log(\eta_{ik}) \left[\log \left(\frac{\eta_{ij}}{\eta_{jk} - \eta_{ik}} \right) - \log(1 - \eta_{jk}) \right] + \log^2(\eta_{ik}) \right. \\ &\quad \left. - \log(\eta_{jk}) \log(\eta_{jk} - \eta_{ik}) - \frac{\log^2(\eta_{jk})}{2} + \frac{\pi^2}{6} \right] + \frac{1}{2} \log(\eta_{ij}) \log^2(\eta_{jk}) \\ &\quad + \frac{1}{2} \log^2(\eta_{ij}) \log(\eta_{jk}) - \log(1 - \eta_{ij}) \log(\eta_{ij}) \log(\eta_{jk}) - \frac{1}{3} \log^3 \left(\frac{\eta_{ik}}{\eta_{jk}} \right) \\ &\quad - \frac{1}{2} \log \left(\frac{1 - \eta_{jk}}{1 - \eta_{ik}} \right) \log^2 \left(\frac{\eta_{ik}}{\eta_{jk}} \right) - \frac{1}{2} \log^2 \left(\frac{1 - \eta_{jk}}{1 - \eta_{ik}} \right) \log \left(\frac{\eta_{ik}}{\eta_{jk}} \right) \\ &\quad - \log^2(\eta_{jk}) \log(\eta_{jk} - \eta_{ik}) + \frac{2}{3} \pi^2 \log \left(\frac{\eta_{ik}}{\eta_{jk}} \right) + \log \left(\frac{\eta_{ik}}{1 - \eta_{ik}} \right) \end{aligned} \quad (\text{H.16})$$

$$\begin{aligned}
& \times \left[\frac{\pi^2}{6} - \log(1 - \eta_{jk}) \log(\eta_{jk}) \right] - \frac{1}{2} \log^3(\eta_{ik}) + \log^2(1 - \eta_{ik}) \log(\eta_{ik}) \\
& + \frac{\log^3(\eta_{jk})}{2} - \log^2(1 - \eta_{jk}) \log(\eta_{jk}) + \frac{3}{2} \pi^2 \log(\eta_{jk}) - \frac{1}{6} \pi^2 \log \left(\frac{\eta_{jk}}{1 - \eta_{jk}} \right) \\
& + \log(\eta_{ij}) \left[G(\tilde{\eta}_{ik}, w^+, 1) - G(\tilde{\eta}_{jk}, w^+, 1) + G(\tilde{\eta}_{ik}, w^-, 1) - G(\tilde{\eta}_{jk}, w^-, 1) \right] \\
& - G(\tilde{\eta}_{ik}, w^+, \tilde{\eta}_{jk}, 1) + G(\tilde{\eta}_{jk}, w^+, \tilde{\eta}_{ik}, 1) + G(\tilde{\eta}_{ik}, w^+, 1, 1) - G(\tilde{\eta}_{ik}, w^+, \tilde{\eta}_{ik}, 1) \\
& - G(\tilde{\eta}_{jk}, w^+, 1, 1) + G(\tilde{\eta}_{jk}, w^+, \tilde{\eta}_{jk}, 1) - G(\tilde{\eta}_{ik}, w^-, \tilde{\eta}_{jk}, 1) + G(\tilde{\eta}_{jk}, w^-, \tilde{\eta}_{ik}, 1) \\
& + G(\tilde{\eta}_{ik}, w^-, 1, 1) - G(\tilde{\eta}_{ik}, w^-, \tilde{\eta}_{ik}, 1) - G(\tilde{\eta}_{jk}, w^-, 1, 1) + G(\tilde{\eta}_{jk}, w^-, \tilde{\eta}_{jk}, 1),
\end{aligned}$$

where we defined

$$w^\pm \stackrel{\text{def}}{=} \frac{2 - \eta_{ij} - \eta_{ik} - \eta_{jk} \pm \sqrt{(\eta_{ij}\eta_{ik} - \eta_{jk})^2 - 4\eta_{ik}\eta_{jk}(1 - \eta_{ij})}}{2(\eta_{ik}\eta_{jk} - \eta_{ik} - \eta_{jk} + 1)}, \quad (\text{H.17})$$

and

$$\tilde{\eta}_{ab} \stackrel{\text{def}}{=} \frac{1}{1 - \eta_{ab}}. \quad (\text{H.18})$$

The expression in Eq. (H.16) is well defined in the region $\eta_{ik} < \eta_{jk}$. However, this is sufficient to cover the entire phase space since the other region can be obtained by swapping indices i and j . Thanks to the antisymmetry of the result under such an exchange, this only amounts to an overall sign change.

Collection of functions used in the final result

Contents

I.1	Splitting functions	273
I.2	Partition-dependent objects and relevant constants	279
I.3	Elastic operators	280

In this appendix we collect all the functions that are necessary to write the final results for the NNLO QCD contribution to the partonic cross sections reported in Sec. 6.5. For the reader's convenience, we attempt to make this appendix as self-contained as possible. We use the following notations:

$$\begin{aligned} \bar{z} &= 1 - z, & \mathcal{D}_n(z) &= \left[\frac{\log^n(1-z)}{1-z} \right]_+, \\ \tilde{L}_i &= \log\left(\frac{2E_i}{\mu}\right), & L_i &= \log\left(\frac{E_{\max}}{E_i}\right), & L_{\max} &= \log\left(\frac{2E_{\max}}{\mu}\right). \end{aligned} \quad (\text{I.1})$$

I.1 Splitting functions

We collect all the splitting functions that have been introduced to present the double- and single-boosted results in Eqs (6.105 - 6.107). We proceed in order of appearance, starting from

$$\mathcal{P}_{\alpha\beta}^{\text{NLO}}(z, E_i) \stackrel{\text{def}}{=} 2\tilde{L}_i \hat{P}_{\alpha\beta}^{(0)}(z) + \mathcal{P}_{\alpha\beta}^{\text{fin}}(z), \quad (\text{I.2})$$

where $\hat{P}_{\alpha\beta}^{(0)}$ is the Altarelli-Parisi splitting function given in Eq. (A.26), and $\mathcal{P}_{\alpha\beta}^{\text{fin}}$ is reported in Eq. (A.57). The $\epsilon \rightarrow 0$ limits of the splittings in Eq. (I.2) read

$$\begin{aligned} \mathcal{P}_{qq}^{\text{NLO}}(z, E_i) &= 4C_F \left[\mathcal{D}_1(z) + \frac{\bar{z}}{4} - \frac{1+z}{2} \left(\tilde{L}_i + \log(\bar{z}) \right) + \left(\mathcal{D}_0(z) + \frac{3}{4}\delta(\bar{z}) \right) \tilde{L}_i \right], \\ \mathcal{P}_{qg}^{\text{NLO}}(z, E_i) &= 2T_R \left[\bar{z}z + (1-2z+2z^2) \left(\tilde{L}_i + \log(\bar{z}) \right) \right], \\ \mathcal{P}_{gg}^{\text{NLO}}(z, E_i) &= 2C_F \left[\frac{z}{2} + \frac{(2-2z+z^2)}{z} \left(\tilde{L}_i + \log(\bar{z}) \right) \right], \end{aligned} \quad (\text{I.3})$$

$$\mathcal{P}_{gg}^{\text{NLO}}(z, E_i) = 4C_A \left[\mathcal{D}_1(z) + \left(\frac{1}{z} + z - z^2 - 2 \right) (\tilde{L}_i + \log(\bar{z})) + \left(\mathcal{D}_0(z) + \frac{11}{12} \delta(\bar{z}) \right) \tilde{L}_i \right].$$

Next, we have

$$\mathcal{P}_{\alpha\alpha}^{\mathcal{W}}(z, E_i) \stackrel{\text{def}}{=} -\frac{1}{2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \left(\frac{2E_i}{\mu} \right)^{-4\epsilon} \left[\mathcal{P}_{\alpha\alpha}^{(4)}(z, L_i) - e^{-2\epsilon L_i} \mathcal{P}_{\alpha\alpha}^{(2)}(z, L_i) \right]. \quad (\text{I.4})$$

Taking the limit $\epsilon \rightarrow 0$, we obtain

$$\mathcal{P}_{qq}^{\mathcal{W}}(z, E_i) = 2C_F \left[\mathcal{D}_1(z) + \frac{1+z}{2} [L_i - \log(\bar{z})] + L_i \left(\frac{\delta(\bar{z})}{2} L_i - \mathcal{D}_0(z) \right) \right], \quad (\text{I.5})$$

$$\mathcal{P}_{gg}^{\mathcal{W}}(z, E_i) = 2C_A \left[\mathcal{D}_1(z) - \left(\frac{1}{z} + z - z^2 - 2 \right) [L_i - \log(\bar{z})] + L_i \left(\frac{\delta(\bar{z})}{2} L_i - \mathcal{D}_0(z) \right) \right].$$

Furthermore, we use $\mathcal{P}_{\alpha\beta}^{\text{NNLO}}$ splitting functions. Here are the four possible combinations:

$$\mathcal{P}_{qq}^{\text{NNLO}}(z, E_i) \stackrel{\text{def}}{=} C_F^2 P_{qq}^{\text{NNLO,a}}(z, E_i) + C_F C_A P_{qq}^{\text{NNLO,na}}(z, E_i), \quad (\text{I.6})$$

with

$$\begin{aligned} P_{qq}^{\text{NNLO,a}}(z, E_i) &\stackrel{\text{def}}{=} 2\tilde{L}_i^2 \left[8\mathcal{D}_1(z) + 6\mathcal{D}_0(z) - \frac{(3z^2+1)\log(z)}{\bar{z}} - 4(z+1)\log(\bar{z}) \right. \\ &\quad \left. - z - 5 \right] + \tilde{L}_i \left[24\mathcal{D}_2(z) + 12\mathcal{D}_1(z) - \frac{8\pi^2}{3}\mathcal{D}_0(z) + \frac{8\text{Li}_2(\bar{z})}{\bar{z}} - (1+3z^2) \frac{\log^2(z)}{\bar{z}} \right. \\ &\quad \left. + 4(1+z+z^2) \frac{\log(z)}{\bar{z}} - \log(\bar{z}) \left(\frac{8z^2\log(z)}{\bar{z}} + 2(5+z) \right) - 12(1+z)\log^2(\bar{z}) \right. \\ &\quad \left. + \frac{4\pi^2}{3}(z+1) + 9 - 7z \right] + 8\mathcal{D}_3(z) - \frac{8\pi^2}{3}\mathcal{D}_1(z) + 16\zeta_3\mathcal{D}_0(z) - 2(5+3z^2) \frac{\text{Li}_3(z)}{\bar{z}} \\ &\quad - (5-3z^2) \frac{\text{Li}_3(\bar{z})}{\bar{z}} + \frac{\log(\bar{z})}{\bar{z}} \left[(7-z^2)\text{Li}_2(\bar{z}) - 6(1+z^2)\log^2(z) + \frac{4\pi^2}{3}(1-z^2) \right. \\ &\quad \left. + (7-2z+7z^2)\log(z) + \bar{z} \left(6 - \frac{9}{2}z \right) - 4(z+1)\bar{z}\log^2(\bar{z}) \right] + \frac{\log(z)}{\bar{z}} \left[\left(\frac{5}{2} - \frac{9z}{2} \right) \bar{z} \right. \\ &\quad \left. - 2(1+z^2) \left(\text{Li}_2(\bar{z}) - \frac{5\pi^2}{6} \right) \right] + 3\bar{z} \left(\text{Li}_2(\bar{z}) + \frac{2\pi^2}{9} \right) + \left(\frac{5}{4} + \frac{13}{12}z^2 \right) \frac{\log^3(z)}{\bar{z}} \\ &\quad + \frac{2\zeta_3(1+7z^2)}{\bar{z}} - \log^2(\bar{z}) \left[2\bar{z} - \left(\frac{5}{2} - \frac{3z^2}{2} \right) \frac{\log(z)}{\bar{z}} \right] + 8z + \frac{z}{2}\log^2(z) - 9 \\ &\quad \left. + \delta(\bar{z}) \left[\left(\frac{9}{2} - \frac{4\pi^2}{3} \right) \tilde{L}_i^2 + \left(16\zeta_3 + \frac{9}{2}\log 2 \right) \tilde{L}_i + \frac{3\pi^2}{16} - \frac{\pi^4}{45} - \frac{9}{16}\log^2 2 \right], \quad (\text{I.7}) \end{aligned}$$

$$\begin{aligned} P_{qq}^{\text{NNLO,na}}(z, E_i) &= -\frac{11}{3} \tilde{L}_i^2 \left[2\mathcal{D}_0(z) - 1 - z \right] + \tilde{L}_i \left[\left(\frac{134}{9} - \frac{2\pi^2}{3} \right) \mathcal{D}_0(z) - \frac{44}{3}\mathcal{D}_1(z) \right. \\ &\quad \left. + 2(1+z^2) \frac{\text{Li}_2(z)}{\bar{z}} + \left[\frac{2}{3} + \frac{11}{3}z^2 + 2(1+z^2)\log(\bar{z}) \right] \frac{\log(z)}{\bar{z}} + \frac{22}{3}(z+1)\log(\bar{z}) \right] \end{aligned}$$

$$\begin{aligned}
& + \frac{2\pi^2}{3} - \frac{52}{9} - \frac{91z}{9} \Big] - \frac{22}{3} \mathcal{D}_2(z) + \left(\frac{134}{9} - \frac{2\pi^2}{3} \right) \mathcal{D}_1(z) + \left[9\zeta_3 - \frac{208}{27} + \frac{11\pi^2}{6} \right. \\
& - \frac{2\log 2}{3} \Big] \mathcal{D}_0(z) - \frac{(1+6z+19z^2)}{6} \frac{\text{Li}_2(z)}{\bar{z}} + \frac{2\log 2}{3} + \frac{(1+z^2)}{4\bar{z}} \left[2\text{Li}_3(z) - 8\text{Li}_3(\bar{z}) \right. \\
& - 2(\log(z) - 2\log(\bar{z}))\text{Li}_2(z) - \log^2(z)\log(\bar{z}) + 4\log(z)\log^2(\bar{z}) \Big] + \frac{2+11z^2}{8\bar{z}} \log^2(z) \\
& + \frac{11}{3}(1+z)\log^2(\bar{z}) + \frac{20-57z-49z^2}{36\bar{z}} \log(z) - \log \bar{z} \left[\frac{52}{9} + \frac{173}{18}z - \frac{1}{2}(1-z)\log(z) \right. \\
& - \frac{\pi^2}{6} \frac{1+3z^2}{\bar{z}} \Big] - \frac{5-4z^2}{\bar{z}} \zeta_3 + \frac{\pi^2}{36} \frac{12z+49z^2-35}{\bar{z}} + \frac{563}{108} + \frac{197}{108}z + \delta(\bar{z}) \left\{ L_{\max}^2 \left[\frac{64}{9} \right. \right. \\
& - \frac{\pi^2}{3} + \frac{22\log 2}{3} \Big] + \tilde{L}_i^2 \left(\frac{\pi^2}{3} - \frac{227}{18} - \frac{22}{3}\log 2 \right) + L_{\max} \left[\frac{11\zeta_3}{2} - \frac{22\pi^2}{9} + \frac{383}{54} \right. \\
& - \frac{77}{3}\log^2 2 - \frac{125\log 2}{9} \Big] + \tilde{L}_i \left[\frac{263}{6} - 7\zeta_3 - \frac{7\pi^2}{9} + \frac{11\log^2 2}{3} + \left(\frac{224}{9} - \frac{4\pi^2}{3} \right) \log 2 \right] \\
& - 2\text{Li}_4(1/2) + \frac{22\log^3 2}{9} + \zeta_3 \left(\frac{217}{8} + \frac{25\log 2}{4} \right) + \frac{211\pi^4}{1440} - \frac{1561}{36} - \frac{103\pi^2}{432} - \frac{\log^4 2}{12} \\
& \left. + \left(\frac{15\pi^2}{4} - \frac{284}{9} \right) \log 2 + \left(\frac{5\pi^2}{12} - \frac{415}{36} \right) \log^2 2 \right\}, \tag{I.8}
\end{aligned}$$

$$\begin{aligned}
\mathcal{P}_{qg}^{\text{NNLO}}(z, E_i) & \stackrel{\text{def}}{=} C_A T_R \left\{ \frac{2\log 2(1-z)^3}{3z} + \frac{13}{6} [2(z-1)z+1] \log^3(1-z) + \left(3z + \frac{1}{6} \right) \right. \\
& \times \log^3(z) + [(2z(z+1)+1)] \log^3(z+1) + \left(-\frac{62z^2}{3} + 16z + (8(z-1)z+4) \log(1-z) \right. \\
& + (16z+4) \log(z) + \frac{8}{3z} + 2 \Big) \tilde{L}_i^2 + \frac{1}{54z} \left[z^2(129-970z) + 447z + 6\pi^2 [z^2(53z+27) + 6z \right. \\
& + 4] + 340 \Big] + \log^2(1-z) \left(15z - \frac{59z^2}{3} + (2z^2-18z-3) \log(z) + \frac{8}{3z} + 2 \right) - \log(z+1) \\
& \times \frac{\pi^2}{2} (2z^2+2z+1) + \log^2(z) \left[11z^2+17z - \left(z^2+z+\frac{1}{2} \right) \log(z+1) + \frac{11}{4} + \frac{8}{3z} \right) \\
& + \log(z) \left[- (6z^2+6z+3) \log^2(z+1) - \frac{2(z+1)(19z^2+2z+4) \log(z+1)}{3z} + \frac{1}{9} \left(68z^2 \right. \right. \\
& + 3(105+4\pi^2)z + 75 + \frac{64}{z} \Big) \Big] + \log(1-z) \left[\frac{1}{18} [z(399-320z) - 3\pi^2(6z(3z-7)+1) \right. \\
& - 156] + \frac{52}{9z} + \log(z) \left(\frac{22z^2}{3} - 8z - (8z^2+8z+4) \log(z+1) - \frac{4}{3z} + 2 \right) \Big] + \left[\log^2(1-z) \right. \\
& \times [12(z-1)z+6] + \frac{4}{3} \left(24z-31z^2+3+\frac{4}{z} \right) \log(1-z) + (12z+2) \log^2(z) + \frac{1}{9z} \\
& \times [52-2z(z(71z+6\pi^2(2z-5)-93)+39)] + \log(z) \left(\frac{2}{3} \left(44z^2+48z+15+\frac{8}{z} \right) \right. \\
& \left. \left. + (-8z(z+1)-4) \log(z+1) \right) \right] \tilde{L}_i + \left[-\frac{2(z+1)(z(19z+2)+4)}{3z} + (-8z(z+1)-4) \right.
\end{aligned}$$

$$\begin{aligned}
& \times \log(1-z) + (-2(z-3)z-1)\log(z) + (-8z(z+1)-4)\tilde{L}_i \Big] \text{Li}_2(-z) + \left[-22z^2 \right. \\
& + 4\log(z)z - 40z + (2(z-17)z-7)\log(1-z) - 8(4z+1)\tilde{L}_i - \frac{20}{3z} - 8 \Big] \text{Li}_2(z) \\
& + (2z(9z-17)+1)\text{Li}_3(1-z) + (-2z(5z+13)-5)\text{Li}_3(-z) - 4(8z+1)\text{Li}_3(z) \\
& - 6(2z(z+1)+1)\text{Li}_3\left(\frac{z}{z+1}\right) + (-4z(z+1)-2)\text{Li}_3\left(1-z^2\right) + (2z(5z+8)+9)\zeta_3 \Big\} \\
& + C_F T_R \left\{ \frac{11}{6}(2(z-1)z+1)\log^3(1-z) + \left[z(21-17z) - \left(z^2 - 5z + \frac{5}{2} \right) \log(z) \right. \right. \\
& - 7 \Big] \log^2(1-z) + \left[- \left(9(z-1)z + \frac{9}{2} \right) \log^2(z) + (14(z-1)z+6)\log(z) \right. \\
& + \frac{1}{6} \left(3z(88z-109) + \pi^2(2(1-5z)z-1) + 96 \right) \Big] \log(1-z) + \left(\frac{7z^2}{3} - \frac{5z}{2} + \frac{5}{4} \right) \log^3(z) \\
& + \left(3z^2 - \frac{5z}{2} - \frac{3}{8} \right) \log^2(z) + \left(-4(z-2)z + (16(z-1)z+8)\log(1-z) + (-8z^2 \right. \\
& + 4z-2)\log(z) - 1 \Big) \tilde{L}_i^2 + \frac{1}{12} \left(-588z^2 + 765z + \pi^2(34z^2 - 42z + 11) - 207 \right) \\
& + \left(\frac{35z}{4} - 8z^2 + \frac{4}{3}\pi^2(2(z-1)z+1) + 1 \right) \log(z) + \frac{1}{3}\pi^2(2(z-1)z+1)L_{\max} \\
& + \log(2) \left[(-4(z-1)z-2)\tilde{L}_i^2 + \left((-4(z-1)z-2)\log(z) + (4(z-1)z+2)[\tilde{L}_i + \log(z)] \right) \right. \\
& \times \tilde{L}_i \Big] + \tilde{L}_i \left((12(z-1)z+6)\log^2(1-z) + \left((-8(z-1)z-4)[\tilde{L}_i + \log(z)] - 4(z-1) \right. \right. \\
& \times (5z-2) \Big) \log(1-z) + (-4z^2+2z-1)\log^2(z) + z(36z-47) - \frac{1}{3}\pi^2(6z^2-2z+1) \\
& + (4z-3)(4z-1)\log(z) + 4(z-1)z[\tilde{L}_i + \log(z)] + 16 \Big) + \left((-2(z-5)z-5)\log(1-z) \right. \\
& + (2(z-1)z+1)\log(z) + (8z-4)\tilde{L}_i + 3 \Big) \text{Li}_2(z) + (2(9-5z)z-9)\text{Li}_3(1-z) \\
& \left. + (2(9-7z)z-9)\text{Li}_3(z) + (4z(8z-9)+18)\zeta_3 \right\}, \tag{I.9}
\end{aligned}$$

$$\begin{aligned}
\mathcal{P}_{gq}^{\text{NNLO}}(z, E_i) & \stackrel{\text{def}}{=} C_F^2 \left\{ \frac{13(z^2-2z+2)\log^3(1-z)}{6z} + \left(2(z-2)\log(z) - \frac{27z^2-58z+42}{4z} \right) \right. \\
& \times \log^2(1-z) - \frac{(14\pi^2z^2+9z^2-28\pi^2z+138z+20\pi^2-108)\log(1-z)}{6z} + \frac{2-z}{12}\log^3(z) \\
& + \frac{1}{8}(17z+20)\log^2(z) + \left(-z-2(z-2)\log(z) + 4 + \frac{4(z^2-2z+2)\log(1-z)}{z} \right) \tilde{L}_i^2 \\
& \left. + \frac{27\pi^2z^2-87z^2-22\pi^2z+438z+30\pi^2-324}{12z} + \frac{1}{4}(5z-3)\log(z) + \frac{\pi^2(z^2-2z+2)L_{\max}}{3z} \right\}
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{6(z^2 - 2z + 2) \log^2(1 - z)}{z} - \frac{4(z - 1)(2z - 3) \log(1 - z)}{z} + (2 - z) \log^2(z) + (5z + 8) \right. \\
& \times \log(z) - \frac{7\pi^2 z^2 + 12z^2 - 14\pi^2 z + 45z + 10\pi^2 - 30}{3z} \Big) \tilde{L}_i + \left(-5z + 4(z - 2) \log(1 - z) \right. \\
& + 4(z - 2) \tilde{L}_i - 8 \Big) \text{Li}_2(z) + 4(z - 2) \text{Li}_3(1 - z) + 2(z - 2) \text{Li}_3(z) + \frac{2(3z^2 - 6z + 8) \zeta_3}{z} \Big\} \\
& + C_A C_F \left\{ -\frac{2 \log(2)(1 - z)^3}{3z} + \frac{11(z^2 - 2z + 2) \log^3(1 - z)}{6z} - \frac{z + 18}{6} \log^3(z) \right. \\
& + \left(\frac{32z^3 + 43z^2 + 214z - 270}{12z} + \frac{(9z^2 + 30z + 2) \log(z)}{2z} \right) \log^2(1 - z) + \left(\log(z + 1) \right. \\
& \times \frac{(z^2 + 2z + 2)}{2z} - \frac{24z^3 + 99z^2 + 132z + 176}{12z} \Big) \log^2(z) + \left(\frac{2(z - 1)(4z^2 + 7z + 31)}{3z} \right. \\
& + \frac{4(z^2 - 2z + 2) \log(1 - z)}{z} - \frac{8(z^2 + z + 1) \log(z)}{z} \Big) \tilde{L}_i^2 + \left[-\frac{9(z^2 - 2z + 2) \log^2(z)}{2z} \right. \\
& + \left(\frac{4(z^2 + 2z + 2) \log(z + 1)}{z} - \frac{4z^3 - 25z^2 + 68z - 66}{3z} \right) \log(z) \\
& - \frac{-176z^3 + 24\pi^2 z^2 - 33z^2 + 72\pi^2 z + 84z - 136}{18z} \Big] \log(1 - z) + \left[\frac{3(z^2 + 2z + 2) \log^2(z + 1)}{z} \right. \\
& + \frac{(8z^3 + 15z^2 + 48z + 44) \log(z + 1)}{3z} + \frac{12\pi^2 z^2 - 88z^3 - 240z^2 - 48\pi^2 z - 357z + 24\pi^2}{18z} \\
& - \frac{280}{9z} \Big] \log(z) + \frac{\pi^2(z^2 + 2z + 2) \log(z + 1)}{2z} + \left[\frac{6(z^2 - 2z + 2) \log^2(1 - z)}{z} \right. \\
& + \left(\frac{4(4z^3 + 6z^2 + 24z - 31)}{3z} - \frac{4(z^2 - 2z + 2) \log(z)}{z} \right) \log(1 - z) + \left(\log(z + 1) \right. \\
& \times \frac{4(z^2 + 2z + 2)}{z} - \frac{4(4z^3 + 8z^2 + 35z + 11)}{3z} \Big) \log(z) - \frac{2(3z^2 + 2z + 4) \log^2(z)}{z} \\
& - \frac{-88z^3 + 12\pi^2 z^2 - 71z^2 + 36\pi^2 z + 118z - 144}{9z} \Big] \tilde{L}_i + \left(\frac{8z^3 + 15z^2 + 48z + 44}{3z} \right. \\
& + \frac{4(z^2 + 2z + 2) \log(1 - z)}{z} + \frac{4(z^2 + 2z + 2) \tilde{L}_i}{z} - \frac{(3z^2 - 2z + 6) \log(z)}{z} \Big) \text{Li}_2(-z) \\
& + \left(\frac{2(6z^3 + 24z^2 + 36z + 55)}{3z} + 8(z + 4) \log(1 - z) + 8(z + 4) \tilde{L}_i - \frac{(z^2 + 2z + 2) \log(z)}{z} \right) \\
& \times \text{Li}_2(z) + \frac{4(z^2 + 6z - 2) \text{Li}_3(1 - z)}{z} + \frac{(13z^2 + 10z + 26) \text{Li}_3(-z)}{z} + \frac{(5z^2 + 30z + 2)}{z} \\
& \times \text{Li}_3(z) + \frac{6(z^2 + 2z + 2)}{z} \text{Li}_3\left(\frac{z}{z + 1}\right) + \frac{2(z^2 + 2z + 2)}{z} \text{Li}_3(1 - z^2) + \frac{2(2z^2 - 21z + 8) \zeta_3}{z} \\
& - \frac{(z^2 + 2z + 2) \log^3(z + 1)}{z} - \frac{24\pi^2 z^3 - 164z^3 + 72\pi^2 z^2 - 77z^2 + 144\pi^2 z - 869z + 39\pi^2}{27z} \\
& + \frac{1225}{27z} \Big\}, \tag{I.10}
\end{aligned}$$

$$\begin{aligned}
\mathcal{P}_{gg}^{\text{NNLO}}(z, E_i) \stackrel{\text{def}}{=} C_A^2 \Bigg\{ & \delta(\bar{z}) \left\{ -\frac{1}{12} \log^4 2 + \frac{22}{9} \log^3 2 + \left(\frac{11}{12} \log(z) + \frac{5\pi^2}{12} - \frac{623}{48} \right) \right. \\
& \times \log^2 2 + \left(-\frac{11}{6} \log^2(z) + \frac{121}{36} \log(z) + \frac{143\pi^2}{36} + \frac{25}{4} \zeta_3 - \frac{1835}{54} \right) \log 2 + \frac{179\pi^4}{1440} \\
& + \frac{31\pi^2}{216} + \frac{121}{36} \log^2(z) + \left(\frac{22}{3} \log 2 - \frac{\pi^2}{3} + \frac{64}{9} \right) L_{\text{max}}^2 + \left(-\frac{22}{3} \log 2 - \pi^2 \right. \\
& - \frac{44}{3} \log(z) - \frac{64}{9} \Big) \tilde{L}_i^2 + \frac{11}{36} \pi^2 \log(z) + \left(-\frac{77}{3} \log^2 2 - \frac{125}{9} \log 2 + \frac{11}{2} \zeta_3 - \frac{22\pi^2}{9} \right. \\
& + \frac{383}{54} \Big) L_{\text{max}} - 2\text{Li}_4(1/2) + \frac{715}{24} \zeta_3 + \tilde{L}_i \left[\frac{11}{3} \log^2 2 + \left(-\frac{22}{3} \log(z) - \frac{4\pi^2}{3} + \frac{613}{18} \right) \right. \\
& \times \log 2 + \frac{121}{9} \log(z) + 9\zeta_3 - \frac{22}{3} \log^2(z) - \frac{11\pi^2}{9} + \frac{2629}{54} \Big] - \frac{11 \log^3(z)}{9} - \frac{15653}{324} \Big\} \\
& + \left\{ -\frac{8(z^3 - z^2 + 2z - 1) \log^3(\bar{z})}{z} + \log^2(\bar{z}) \left[\frac{2(55z^3 - 47z^2 + 58z - 55)}{3z} \right. \right. \\
& + \frac{(z^4 - 18z^3 + 3z^2 + 14z + 1) \log(z)}{\bar{z}z} \Big] + \left[-\frac{9(z^2 - z + 1)^2 \log^2(z)}{\bar{z}z} \right. \\
& + \left(\frac{8 \log(z+1)(z^2 + z + 1)^2}{z(z+1)} + \frac{2(33z^4 - 67z^3 + 90z^2 - 67z + 33)}{3\bar{z}z} \right) \log(z) \\
& + \frac{2\pi^2(6z^4 - 6z^3 - 8z^2 - z - 4)}{3z(z+1)} - \frac{536z^3 - 337z^2 + 602z - 536}{18z} \Big] \log(\bar{z}) \\
& + \frac{(7z^4 + 2z^3 + 9z^2 - 2z - 9) \log^3(z)}{3\bar{z}(z+1)} + \log^2(z) \left[-\frac{132z^3 + 159z^2 + 99z + 176}{12z} \right. \\
& + \frac{(z^2 + z + 1)^2 \log(z+1)}{z(z+1)} \Big] + \left[\frac{2(33z^3 - 25z^2 + 14z - 33)}{3z} + \frac{22}{3} \mathcal{D}_0(z) + 16\mathcal{D}_1(z) \right. \\
& - \frac{16(z^3 - z^2 + 2z - 1) \log(\bar{z})}{z} - \frac{8(z^4 - 4z^3 + 3z^2 + 1) \log(z)}{\bar{z}z} \Big] \tilde{L}_i^2 + 8\mathcal{D}_3(z) + \left(\frac{134}{9} \right. \\
& - \frac{10\pi^2}{3} \Big) \mathcal{D}_1(z) + \frac{2032z^3 - 1423z^2 + 1658z - 2059}{27z} + \left[\frac{2(z^3 - 3z^2 + 4z - 1)}{3z} \right. \\
& - \frac{2\mathcal{D}_0(z)}{3} \Big] \log 2 + \left[\frac{6(z^2 + z + 1)^2 \log^2(z+1)}{z(z+1)} + \frac{4(z+1)(11z^2 + z + 11) \log(z+1)}{3z} \right. \\
& + \frac{536z^4 + 969z^3 - 708z^2 + 335z - 1120}{36\bar{z}z} + \frac{4\pi^2(2z^5 - z^4 + 2z^3 + z^2 - 2z + 1)}{3\bar{z}z(z+1)} \Big] \\
& \times \log(z) + \frac{\pi^2(z^2 + z + 1)^2 \log(z+1)}{z(z+1)} + \left[-\frac{24(z^3 - z^2 + 2z - 1) \log^2(\bar{z})}{z} \right. \\
& + \left(\frac{16(z-1)(11z^2 + 2z + 11)}{3z} - \frac{8(z^2 - z + 1)^2 \log(z)}{\bar{z}z} \right) \log(\bar{z}) + 24\mathcal{D}_2(z) \\
& + \left(\frac{134}{9} - \frac{10\pi^2}{3} \right) \mathcal{D}_0(z) + \frac{2\pi^2(6z^4 - 6z^3 - 8z^2 - z - 4)}{3z(z+1)} + \left(\frac{8 \log(z+1)(z^2 + z + 1)^2}{z(z+1)} \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{2(66z^4 - 51z^3 + 66z^2 - 37z - 22)}{3\bar{z}z} \Big) \log(z) - \frac{268z^3 - 167z^2 + 301z - 268}{9z} \\
& - \frac{4(z^5 - 4z^4 - z^3 + 4z^2 + z + 2) \log^2(z)}{\bar{z}z(z+1)} \Big] \tilde{L}_i + \text{Li}_2(-z) \Big[\frac{8 \log(\bar{z})(z^2 + z + 1)^2}{z(z+1)} \\
& + \frac{8\tilde{L}_i(z^2 + z + 1)^2}{z(z+1)} + \frac{4(z+1)(11z^2 + z + 11)}{3z} - \frac{2(z^5 - 3z^4 + z^3 + 3z^2 - z + 3) \log(z)}{\bar{z}z(z+1)} \Big] \\
& + \Big[- \frac{2 \log(z)(z^2 + z + 1)^2}{z(z+1)} + \frac{2(33z^3 + 49z^2 + 25z + 55)}{3z} + 32(z+1) \log(\bar{z}) \\
& + 32(z+1) \tilde{L}_i \Big] \text{Li}_2(z) + \frac{12(z^2 + z + 1)^2 \text{Li}_3\left(\frac{z}{z+1}\right)}{z(z+1)} + \frac{4(z^2 + z + 1)^2 \text{Li}_3(1 - z^2)}{z(z+1)} \\
& + \frac{(42z^5 - 16z^4 + 58z^3 - 9z^2 - 83z + 32)\zeta_3}{\bar{z}z(z+1)} + \mathcal{D}_0(z) \left(25\zeta_3 + \frac{11\pi^2}{6} - \frac{208}{27} \right) \\
& - \frac{22\mathcal{D}_2(z)}{3} - \frac{\pi^2(165z^3 - 31z^2 + 164z - 33)}{18z} - \frac{2(z^2 + z + 1)^2 \log^3(z+1)}{z(z+1)} \\
& - \frac{8(z^4 - 2z^3 - 5z^2 - 2z + 1) \text{Li}_3\bar{z}}{z(z+1)} - \frac{2(5z^5 + 13z^4 + 5z^3 - 13z^2 - 5z - 13) \text{Li}_3(-z)}{\bar{z}z(z+1)} \\
& - \frac{2(7z^5 + 9z^4 + 15z^3 - 9z^2 - 15z - 1) \text{Li}_3(z)}{\bar{z}z(z+1)} \Big\} \Big\}. \tag{I.11}
\end{aligned}$$

I.2 Partition-dependent objects and relevant constants

Boosted and elastic contributions involve certain partition-dependent functions, which have emerged throughout the NNLO calculations. We provide below a list of their definitions:

$$\mathcal{W}_i^{i||n, \text{fin}} \stackrel{\text{def}}{=} \sum_{(kl)} \int \frac{d\Omega_{\mathbf{m}}^{(3)}}{2\pi} \bar{C}_{i\mathbf{m}} \left[\log(\eta_{i\mathbf{m}}/2) \frac{\rho_{kl}}{\rho_{k\mathbf{m}}\rho_{l\mathbf{m}}} \omega_{i||n}^{\mathbf{m}i, \mathbf{n}i} \right] (\mathbf{T}_k \cdot \mathbf{T}_l), \tag{I.12}$$

$$\mathcal{W}_i^{\mathbf{m}||n, \text{fin}} \stackrel{\text{def}}{=} \sum_{(kl)} \int \frac{d\Omega_{\mathbf{m}}^{(3)}}{2\pi} \bar{C}_{i\mathbf{m}} \left[\log\left(\frac{\eta_{i\mathbf{m}}}{1 - \eta_{i\mathbf{m}}}\right) \frac{\rho_{kl}}{\rho_{k\mathbf{m}}\rho_{l\mathbf{m}}} \omega_{\mathbf{m}||n}^{\mathbf{m}i, \mathbf{n}i} \right] (\mathbf{T}_k \cdot \mathbf{T}_l). \tag{I.13}$$

$$\mathcal{W}_{\mathbf{r}}^{(i)} \stackrel{\text{def}}{=} \sum_{k, l \in \mathcal{H}} \int \frac{[d\Omega_{\mathbf{m}}^{(d-1)}]}{[\Omega^{(d-2)}]} \sigma_{i\mathbf{m}}^{-\epsilon} \omega_{\mathbf{m}||n}^{\mathbf{m}i, \mathbf{n}i} (r_i^\mu r_i^\nu + g^{\mu\nu}) \frac{n_{k, \mu} n_{k, \nu}}{(n_k \cdot n_{\mathbf{m}})(n_l \cdot n_{\mathbf{m}})} (\mathbf{T}_k \cdot \mathbf{T}_l). \tag{I.14}$$

Furthermore, we use the object

$$\theta_{\mathcal{H}_{\text{f}}} \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } i \in \mathcal{H}_{\text{f}} \\ 0 & \text{otherwise,} \end{cases} \tag{I.15}$$

and the two quantities

$$\delta_g^{(0)} = C_A \left(-\frac{131}{72} + \frac{\pi^2}{6} + \frac{11}{6} \log 2 \right), \quad \delta_g^\perp = C_A \left(\frac{13}{36} - \frac{\log 2}{3} \right). \quad (\text{I.16})$$

I.3 Elastic operators

We collect all the functions that have been introduced to present the elastic contribution in Eq. (6.108). We proceed in order of appearance, starting with the operator that collects color-correlated contributions:

$$\begin{aligned} I_{\text{cc}}^{\text{fin}} = & \frac{1}{2} (I_{\text{T}}^{(0)})^2 + K I_{\text{T}}^{(0)} + C_A \left[\frac{11}{6} \left(I_{\text{T}}^{(1)} + \tilde{I}_{\text{S}}^{(1)} - 2I_{\text{S}}^{(1)} + \frac{\pi^2}{24} I_{\text{V}}^{(-1)} \right) \right. \\ & + I_{\text{S}}^{(-1)} \left[\left(\frac{2\pi^2}{3} - \frac{131}{18} + \frac{22}{3} \log 2 \right) L_{\text{max}} - \frac{17\zeta_3}{4} + \frac{1975}{108} - \frac{11}{12} \pi^2 \right. \\ & \left. \left. - 11 \log^2 2 - \frac{2}{3} \pi^2 \log 2 \right] + I_{\text{S}}^{(0)} \left(\frac{\pi^2}{3} - \frac{131}{36} + \frac{11 \log 2}{3} \right) \right], \end{aligned} \quad (\text{I.17})$$

where

$$K = \left(\frac{67}{18} - \frac{\pi^2}{6} \right) C_A - \frac{10}{9} T_{\text{R}} n_{\text{f}}, \quad (\text{I.18})$$

and $I_{\text{S}}^{(n)}$, $\tilde{I}_{\text{S}}^{(n)}$, $I_{\text{V}}^{(n)}$, $I_{\text{T}}^{(n)}$ are the coefficients of the n -th power in the ϵ -expansion of the corresponding operators reported in Appendix A.6.

The finite part of the triple color-correlated operator is given by

$$I_{\text{tri}}^{\text{fin}} = I_{\text{tri}}^{(\text{cc}),\text{fin}} + I_{\text{tri}}^{\text{RV},\text{fin}}. \quad (\text{I.19})$$

In (I.19), $I_{\text{tri}}^{(\text{cc}),\text{fin}}$ corresponds to the $\mathcal{O}(\epsilon^0)$ contributions of the operator $I_{\text{tri}}^{(\text{cc})}$ in Eq. (6.52), and reads

$$\begin{aligned} I_{\text{tri}}^{(\text{cc}),\text{fin}} \stackrel{\text{def}}{=} & \frac{\pi}{2} \sum_{(ijk)} F^{(kij)} \left[\frac{1}{2} (\delta_{ij}^- + \delta_{ji}^-) (\delta_{jk}^+ + \delta_{kj}^+ + 4\phi_{jk}) + L_{jk} (\xi_{ij}^- + \xi_{ji}^-) \right. \\ & \left. - \lambda_{ij} (\xi_{jk}^+ + \xi_{kj}^+ + 4\psi_{jk}) \right], \end{aligned} \quad (\text{I.20})$$

where

$$\begin{aligned} \delta_{ij}^+ & \stackrel{\text{def}}{=} \frac{1}{2} L_{ij}^2 + \frac{\gamma_i}{T_i^2} L_{ij} - \frac{\pi^2}{2} \lambda_{ij}^2, \\ \delta_{ij}^- & \stackrel{\text{def}}{=} L_{ij} \lambda_{ij} + \frac{\gamma_i}{T_i^2} \lambda_{ij}, \end{aligned} \quad (\text{I.21})$$

$$\xi_{ij}^+ \stackrel{\text{def}}{=} \frac{1}{6} L_{ij}^3 + \frac{\gamma_i}{2T_i^2} L_{ij}^2 - \frac{\pi^2}{2} \left(L_{ij} + \frac{\gamma_i}{T_i^2} \right) \lambda_{ij}^2,$$

$$\xi_{ij}^- \stackrel{\text{def}}{=} \frac{1}{2} L_{ij}^2 \lambda_{ij} + \frac{\gamma_i}{T_i^2} \lambda_{ij} L_{ij} - \frac{\pi^2}{6} \lambda_{ij}^3. \quad (\text{I.22})$$

$$\begin{aligned} \phi_{ij} &\stackrel{\text{def}}{=} -2L_{\max} \log \eta_{ij} - \frac{1}{2} \log^2 \eta_{ij} - \text{Li}_2(1 - \eta_{ij}), \\ \psi_{ij} &\stackrel{\text{def}}{=} 2L_{\max}^2 \log \eta_{ij} + \left(L_{\max} - \frac{1}{2} \log(1 - \eta_{ij}) \right) \log^2 \eta_{ij} + \frac{1}{6} \log^3 \eta_{ij} \\ &\quad + 2L_{\max} \text{Li}_2(1 - \eta_{ij}) - \text{Li}_3(1 - \eta_{ij}) - \text{Li}_3(\eta_{ij}). \end{aligned} \quad (\text{I.23})$$

The second term on the right-hand side of (I.19) corresponds to the $\mathcal{O}(\epsilon^0)$ contribution of the operator $I_{\text{tri}}^{\text{RV}}$ in Eq. (H.9). It is defined as

$$I_{\text{tri}}^{\text{RV,fin}} \stackrel{\text{def}}{=} \sum_{(ijk)} \kappa_{ij} \bar{G}_{\text{r,fin}}^{kij} F^{(kij)}, \quad (\text{I.24})$$

where

$$\kappa_{ij} = -1 + 2\delta_{ia}\delta_{jb} + 2\delta_{ib}\delta_{ja}, \quad F^{(kij)} = f_{abc} T_k^a T_i^b T_j^c, \quad (\text{I.25})$$

and $\bar{G}_{\text{r,fin}}^{kij}$ is written in Eq. (H.16).

The operator $I_{\text{unc}}^{\text{fin}}$ collects color-uncorrelated contributions. It reads

$$\begin{aligned} I_{\text{unc}}^{\text{fin}} &\stackrel{\text{def}}{=} I_{\text{S}}^{(-2)} C_{\text{A}} \left\{ \left(\frac{2\pi^2}{3} - \frac{131}{18} + \frac{22}{3} \log 2 \right) L_{\max}^2 - \frac{935}{72} \zeta_3 + \frac{9607}{324} + \left(\frac{1433}{108} - 8\zeta_3 \right. \right. \\ &\quad \left. \left. - \frac{11\pi^2}{6} \right) \log 2 - \frac{199\pi^4}{1440} - \frac{21\pi^2}{32} - \frac{11}{3} \log^3 2 + \left(\frac{143}{36} - \frac{\pi^2}{3} \right) \log^2 2 \right\} \\ &\quad + \sum_{i \in \mathcal{H}} D_{\text{c},fi}(E_i). \end{aligned} \quad (\text{I.26})$$

In (I.26), if $i \in \{a, b\}$ (initial-state radiation), we have

$$\begin{aligned} D_{\text{c},g}^{\text{in}}(E_i) &= C_{\text{A}}^2 \left\{ L_i \left[\left(\frac{2\pi^2}{3} - \frac{128}{9} - \frac{44}{3} \log 2 \right) \tilde{L}_i - 16\zeta_3 + \frac{802}{27} + \frac{11}{3} \log^2 2 + \left(\frac{131}{9} \right. \right. \right. \\ &\quad \left. \left. - \frac{4\pi^2}{3} \right) \log 2 \right] + \left(\frac{\pi^2}{3} - \frac{64}{9} - \frac{22}{3} \log 2 \right) L_i^2 + \left(\frac{11\pi^2}{6} - \frac{352}{27} - \frac{121}{6} \log 2 \right) \tilde{L}_i \\ &\quad \left. - \frac{44}{3} \zeta_3 + \frac{4411}{162} - \frac{121}{432} \pi^2 + \frac{605}{144} \log^2 2 + \left(\frac{1441}{108} - \frac{11\pi^2}{9} \right) \log 2 \right\}, \end{aligned} \quad (\text{I.27})$$

$$\begin{aligned} D_{\text{c},q}^{\text{in}}(E_i) &= C_{\text{A}} C_{\text{F}} \left\{ L_i \left[\left(\frac{2\pi^2}{3} - \frac{128}{9} - \frac{44}{3} \log 2 \right) \tilde{L}_i - 16\zeta_3 + \frac{802}{27} + \frac{11}{3} \log^2 2 \right. \right. \\ &\quad \left. \left. + \left(\frac{131}{9} - \frac{4}{3} \pi^2 \right) \log 2 \right] + \left(\frac{\pi^2}{3} - \frac{64}{9} - \frac{22}{3} \log 2 \right) L_i^2 + \left(\frac{3\pi^2}{2} - \frac{32}{3} - 11 \log 2 \right) \tilde{L}_i \right. \\ &\quad \left. - 12\zeta_3 + \frac{401}{18} + \frac{11}{4} \log^2 2 + \left(\frac{131}{12} - \pi^2 \right) \log 2 \right\} \end{aligned}$$

$$+ C_F^2 \left(\frac{9}{16} \log^2 2 - \frac{9}{2} \log 2 \tilde{L}_i - \frac{3\pi^2}{16} \right), \quad (\text{I.28})$$

if $i \in \mathcal{H}_f$ (final-state radiation), we have

$$\begin{aligned} D_{c,g}^{\text{fn}}(E_i) = C_A^2 \left\{ \left[-\frac{21\zeta_3}{2} - \frac{22\pi^2}{9} + \frac{1987}{54} - 22 \log^2 2 + \left(\frac{2}{3} - \frac{4\pi^2}{3} \right) \log 2 \right] L_i \right. \\ + \left(-\frac{15}{2} \zeta_3 - \frac{11\pi^2}{6} + \frac{1010}{27} - 22 \log^2 2 - \frac{4\pi^2}{3} \log 2 \right) \tilde{L}_i - 2\text{Li}_4(1/2) + \frac{583}{24} \zeta_3 \\ + \left(\frac{25}{4} \zeta_3 + \frac{11\pi^2}{4} - \frac{713}{36} \right) \log 2 + \frac{47\pi^4}{160} + \frac{259\pi^2}{432} - \frac{40201}{648} - \frac{\log^4 2}{12} + \frac{22}{9} \log^3 2 \\ \left. + \left(\frac{5\pi^2}{12} - \frac{79}{9} \right) \log^2 2 \right\}, \quad (\text{I.29}) \end{aligned}$$

$$\begin{aligned} D_{c,q}^{\text{fn}}(E_i) = C_A C_F \left\{ L_i \left[-\frac{16}{3} \log 2 \tilde{L}_i - \frac{21}{2} \zeta_3 - \frac{22\pi^2}{9} + \frac{1987}{54} - \frac{50}{3} \log^2 2 \right. \right. \\ + \left. \left(\frac{2}{3} - \frac{4\pi^2}{3} \right) \log 2 \right] + \left[\frac{9}{2} \zeta_3 - \frac{53\pi^2}{18} + \frac{4159}{108} - 22 \log^2 2 - \left(\frac{14}{9} + \frac{4\pi^2}{3} \right) \log 2 \right] \tilde{L}_i \\ - 2\text{Li}_4(1/2) + \frac{491}{24} \zeta_3 + \left(\frac{25}{4} \zeta_3 + \frac{11\pi^2}{4} - \frac{701}{36} \right) \log 2 + \frac{103\pi^4}{1440} + \frac{643\pi^2}{432} - \frac{1695}{32} \\ - \frac{\log^4 2}{12} + \frac{22 \log^3 2}{9} + \left(\frac{5\pi^2}{12} - \frac{65}{9} \right) \log^2 2 \left. \right\} \\ + C_F^2 \left[L_i \left(-16 \log 2 \tilde{L}_i + \frac{4\pi^2}{9} + 16 \log^2 2 \right) + \left(-12\zeta_3 + \pi^2 - \frac{3}{4} - \frac{64}{3} \log 2 \right) \tilde{L}_i \right. \\ \left. + \frac{9\zeta_3}{2} + \frac{2\pi^4}{9} - \frac{271}{32} - \frac{59\pi^2}{108} + \frac{64}{3} \log^2 2 \right]. \quad (\text{I.30}) \end{aligned}$$

The function $\gamma_{z,fi \rightarrow f_i g}^{\mathcal{W}}$ is a combination of anomalous dimensions. It is defined as

$$\gamma_{fi \rightarrow f_i g}^{\mathcal{W}}(L_i) \stackrel{\text{def}}{=} \frac{1}{2\epsilon} \left(\frac{2E_i}{\mu} \right)^{-4\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \left[\gamma_{z,fi \rightarrow f_i g}^{24}(L_i) - e^{-2\epsilon L_i} \gamma_{z,fi \rightarrow f_i g}^{22}(L_i) \right], \quad (\text{I.31})$$

and, in the limit $\epsilon \rightarrow 0$, it reads for quarks and gluons, respectively:

$$\begin{aligned} \gamma_{z,q \rightarrow qg}^{\mathcal{W}} &= C_F \left[\frac{121}{36} + L_i \left(\frac{17}{6} + L_i \right) \right], \\ \gamma_{z,g \rightarrow gg}^{\mathcal{W}} &= C_A \left[\frac{203}{72} + L_i \left(\frac{11}{6} + L_i \right) \right]. \quad (\text{I.32}) \end{aligned}$$

The term $\langle S_{mn} \Theta_{mn} F_{LM}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \rangle_{T^2}^{\text{fn}}$ refers to the finite remainder of the double-soft integrated subtraction term. It can be extracted from Ref. [80], and its explicit expression reads

$$\langle S_{mn} \Theta_{mn} F_{LM}^{\mathcal{H}}[\mathbf{m}_g, \mathbf{n}_g] \rangle_{T^2}^{\text{fn}} =$$

$$\begin{aligned}
&= \left[\frac{\alpha_s(\mu)}{2\pi} \right]^2 \sum_{(ij)}^{N_p} C_A \left\langle \left\{ -\frac{\text{Si}_2(2\delta_{ij})}{6 \tan(\delta_{ij})} - \frac{11}{3} \text{Ci}_3(2\delta_{ij}) - 2G_{-1,0,0,1}(\eta_{ij}) \right. \right. \\
&+ \frac{7}{2} G_{0,1,0,1}(\eta_{ij}) - \frac{5}{24} \log^4(\eta_{ij}) - \frac{1}{12} \log^4(1 + \eta_{ij}) + \frac{1}{2} \log(1 - \eta_{ij}) \log^3(\eta_{ij}) - \left[\frac{5\pi^2}{12} \right. \\
&+ \left. \left. \frac{11}{12} \log(1 - \eta_{ij}) + \frac{7}{4} \log^2(1 - \eta_{ij}) \right] \log^2(\eta_{ij}) + \frac{\pi^2}{12} \log^2(1 + \eta_{ij}) - \frac{7}{4} \text{Li}_2(\eta_{ij})^2 \right. \\
&+ 3\text{Li}_4(\eta_{ij}) - 5\text{Li}_4\left(1 - \frac{1}{\eta_{ij}}\right) - 5\text{Li}_4(1 - \eta_{ij}) - 2\text{Li}_4\left(\frac{1}{1 + \eta_{ij}}\right) + \text{Li}_4\left(\frac{1 - \eta_{ij}}{1 + \eta_{ij}}\right) \\
&- \text{Li}_4\left(-\frac{1 - \eta_{ij}}{1 + \eta_{ij}}\right) - \frac{1}{2} \text{Li}_4(1 - \eta_{ij}^2) - \text{Li}_2(\eta_{ij}) \left[\log^2(\eta_{ij}) + \frac{11}{6} \log(\eta_{ij}) + \frac{1 + 2\pi^2}{12} \right. \\
&+ \left. \left. \frac{11 \log 2}{3} \right] + \text{Li}_2(-\eta_{ij}) \left[2 \log(1 - \eta_{ij}) \log(\eta_{ij}) + 2\text{Li}_2(1 - \eta_{ij}) - \frac{\pi^2}{3} \right] \right. \\
&- 2 \log(1 - \eta_{ij}) \text{Li}_3(-\eta_{ij}) + 2\text{Li}_3(1 - \eta_{ij}) \left(\log(1 + \eta_{ij}) - \log(\eta_{ij}) \right) + \text{Li}_3(\eta_{ij}) \left[\frac{11}{6} \right. \\
&+ \left. \left. 2 \log(\eta_{ij}) - 2 \log(\eta_{ij} + 1) - 7 \log(1 - \eta_{ij}) \right] + \log 2 \left[-\frac{11}{3} \log(1 - \eta_{ij}) \log(\eta_{ij}) \right. \right. \\
&+ \left. \left. \frac{21}{4} \zeta_3 + \frac{33\pi^2 - 868}{108} \right] + \frac{11}{2} \zeta_3 \log(1 - \eta_{ij}) - \frac{7}{4} \zeta_3 \log(1 + \eta_{ij}) + 6\text{Li}_4\left(\frac{1}{2}\right) \right. \\
&+ \log(\eta_{ij}) \left[\left(\pi^2 - \frac{1}{12} \right) \log(1 - \eta_{ij}) + 2\zeta_3 - \frac{1}{6} \right] - \frac{11}{24} \zeta_3 + \frac{137\pi^2}{432} - \frac{17\pi^4}{160} + \frac{649}{162} \\
&+ \left. \left. \frac{\log^4 2}{4} - \frac{11 \log^3 2}{9} - \frac{137 + 9\pi^2}{36} \log^2 2 \right\} (\mathbf{T}_i \cdot \mathbf{T}_j) \cdot F_{\text{LM}}^{\mathcal{H}} \right\rangle. \tag{I.33}
\end{aligned}$$

In this formula, we have used $\delta_{ij} = \theta_{ij}/2$, where θ_{ij} is the opening angle between momenta of partons i and j . The Clausen functions are defined as

$$\text{Ci}_n(z) = \frac{\text{Li}_n(e^{iz}) + \text{Li}_n(e^{-iz})}{2}, \quad \text{Si}_n(z) = \frac{\text{Li}_n(e^{iz}) - \text{Li}_n(e^{-iz})}{2i}, \tag{I.34}$$

and $G_{a_1, a_2, \dots, a_m}(x)$ are the standard Goncharov polylogarithms.

List of Publications

- [81] Federica Devoto, Kirill Melnikov, Raoul Röntsch, Chiara Signorile-Signorile, and Davide Maria Tagliabue. “A fresh look at the nested soft-collinear subtraction scheme: NNLO QCD corrections to N-gluon final states in $q\bar{q}$ annihilation”. In: *JHEP* 02 (2024), p. 016. DOI: [10.1007/JHEP02\(2024\)016](https://doi.org/10.1007/JHEP02(2024)016). arXiv: [2310.17598](https://arxiv.org/abs/2310.17598) [hep-ph].

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