

Normal form coordinates for the Benjamin-Ono equation having expansions in terms of pseudo-differential operators

Thomas Kappeler*, Riccardo Montalto†

March 4, 2022

Abstract. Near an arbitrary finite gap potential we construct real analytic, canonical coordinates for the Benjamin-Ono equation on the torus having the following two main properties: (1) up to a remainder term, which is smoothing to any given order, the coordinate transformation is a pseudo-differential operator of order 0 with principal part given by a modified Fourier transform (modification by a phase factor) and (2) the pullback of the Hamiltonian of the Benjamin-Ono is in normal form up to order three and the corresponding Hamiltonian vector field admits an expansion in terms of para-differential operators. Such coordinates are a key ingredient for studying the stability of finite gap solutions of the Benjamin-Ono equation under small, quasi-linear perturbations.

Keywords: Normal form, Benjamin-Ono equation, finite gap potentials, pseudo-differential operators.

MSC 2010: 37K10, 35Q55

Contents

1	Introduction	1
2	The map Ψ_L	10
3	The map Ψ_C	23
4	The BO Hamiltonian in new coordinates	39
5	Summary of the proofs of Theorem 1.1 and Theorem 1.2	52
A	Spectral theory of the Lax operator L_u	54
B	Birkhoff map	56
C	Reversibility structure	57
D	Properties of pseudodifferential and paradifferential calculus	58

1 Introduction

The goal of this paper is to construct canonical coordinates for the Benjamin-Ono (BO) equation on the torus $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$,

$$\partial_t u = \partial_x (|\partial_x| u - u^2), \quad u = u(t, x) \in \mathbb{R}, \quad (1.1)$$

*Supported in part by the Swiss National Science Foundation.

†Supported in part by the Swiss National Science Foundation and INDAM-GNFM.

which are well suited for studying the stability of finite gap solutions of (1.1) under semilinear and quasilinear perturbations. Here $|\partial_x|$ denotes the Fourier multiplier, defined by

$$|\partial_x| : \sum_{n \in \mathbb{Z}} q_n e^{inx} \mapsto \sum_{n \in \mathbb{Z}} |n| q_n e^{inx}.$$

Equation (1.1) was introduced by Benjamin [1] and Davis & Acrivos [2] as a model for internal gravity waves at the interface of two fluids in a special regime. We refer to [21] for a derivation of (1.1) and a comprehensive survey about results of its solutions. Optimal results on the well-posedness of (1.1) on the Sobolev spaces $H^s = \{q \in H_{\mathbb{C}}^s : q \text{ real valued}\}$ have been recently obtained in [4], saying that equation (1.1) is well-posed on H^s for $s > -1/2$ and ill-posed for $s \leq -1/2$. Here, for any $s \in \mathbb{R}$,

$$H_{\mathbb{C}}^s \equiv H^s(\mathbb{T}, \mathbb{C}) := \left\{ q = \sum_{n \in \mathbb{Z}} q_n e^{inx} : \|q\|_s < \infty \right\}, \quad \|q\|_s = \left(\sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} |q_n|^2 \right)^{\frac{1}{2}}, \quad \langle n \rangle := \max\{1, |n|\}. \quad (1.2)$$

To state our main results, we first need to make preliminary considerations and introduce some notations. It is well known that equation (1.1) can be written in Hamiltonian form, $\partial_t u = \partial_x \nabla H^{bo}$ where ∂_x is the Gardner Poisson structure and where by (1.21) below, ∇H^{bo} denotes the L^2 -gradient of the BO Hamiltonian

$$H^{bo}(q) := \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{1}{2} (|\partial_x|^{\frac{1}{2}} q)^2 - \frac{1}{3} q^3 \right) dx, \quad (1.3)$$

whose domain of definition is the energy space $H^{1/2}$. One verifies that $\int_0^{2\pi} u(t, x) dx$ is a prime integral for equation (1.1). Without loss of generality, we restrict our attention to the case where u has zero mean value (cf. Appendix B), i.e., we consider solutions $u(t, x)$ of (1.1) in H_0^s with $s > -1/2$ where for any $s \in \mathbb{R}$,

$$H_0^s = \left\{ q \in H^s : \int_0^{2\pi} q(x) dx = 0 \right\}. \quad (1.4)$$

An important property of equation (1.1) is that it admits Birkhoff coordinates – see Appendix B for a review. It means that there are (complex) coordinates $\zeta_n = \zeta_n(q)$, $n \geq 1$, defined on $L_0^2 \equiv H_0^0$, so that when expressed in these coordinates, equation (1.1) takes the form

$$\dot{\zeta}_n = i \omega_n^{bo} \zeta_n, \quad \forall n \geq 1, \quad (1.5)$$

where ω_n^{bo} , $n \geq 1$, denote the BO frequencies (cf. (1.11) below). To be more precise, introduce for any $s \in \mathbb{R}$ the sequence space,

$$h_{0, \mathbb{C}}^s \equiv h^s(\mathbb{Z} \setminus \{0\}, \mathbb{C}) = \left\{ w = (w_n)_{n \neq 0} \subset \mathbb{C} : \|w\|_s < \infty \right\}, \quad \|w\|_s := \left(\sum_{n \neq 0} |n|^{2s} |w_n|^2 \right)^{\frac{1}{2}}$$

and its real subspace $h_0^s := \{(w_n)_{n \neq 0} \in h_{0, \mathbb{C}}^s : w_{-n} = \overline{w_n} \forall n \geq 1\}$ and define the weighted complex coordinates

$$z_n(q) := \sqrt{n} \zeta_n(q), \quad z_{-n}(q) := \sqrt{n} \overline{\zeta_n}(q), \quad \forall n \geq 1, \quad (1.6)$$

where $\sqrt{\cdot} \equiv \sqrt[+]{\cdot}$ denotes the principal branch of the square root. It then follows from (1.5) that equation (1.1), when expressed in the coordinates z_n , $n \neq 0$, takes the form

$$\dot{z}_n = i \omega_n^{bo} z_n, \quad \forall n \neq 0, \quad \omega_{-n}^{bo} := -\omega_n^{bo}, \quad \forall n \geq 1. \quad (1.7)$$

It follows from Theorem B.1 in Appendix B that the transformation

$$\Phi^{bo} : L_0^2 \equiv H_0^0 \rightarrow h_0^0, \quad q \mapsto (z_n(q))_{n \neq 0}, \quad (1.8)$$

is canonical in the sense that

$$\{z_n, z_{-n}\}(q) = \frac{1}{2\pi} \int_0^{2\pi} \partial_x (\nabla z_n) \nabla z_{-n} dx = -in, \quad \forall n \neq 0,$$

1 whereas the brackets between all other coordinate functions vanish, and has the property that for any $s \geq 0$,
2 its restriction to H_0^s is a real analytic diffeomorphism with range h_0^s , $\Phi^{bo} : H_0^s \rightarrow h_0^s$. Here ∇z_n denotes
3 the L_0^2 -gradient of $z_n : L_0^2 \rightarrow \mathbb{C}$.¹ For notational convenience, we refer to Φ^{bo} as well as to the map Φ of
4 Theorem B.1 as Birkhoff maps. In terms of the coordinates $z_n(q)$, $n \neq 0$, also referred to as complex Birkhoff
5 coordinates, the action variables $I_n(q)$ are defined by

$$I_n(q) := |\zeta_n(q)|^2 = \frac{1}{n} z_n(q) z_{-n}(q) \geq 0, \quad \forall n \geq 1. \quad (1.9)$$

The sequences $I(q) = (I_n(q))_{n \geq 1}$ fill out the whole positive quadrant $\mathcal{Q}_+(\ell^{1,1})$ of $\ell^{1,1}$, where for any $r \geq 0$,

$$\mathcal{Q}_+(\ell^{1,r}) := \{(x_n)_{n \geq 1} \in \ell^{1,r} : x_n \geq 0 \forall n \geq 1\}$$

6 and $\ell^{1,r}$ denotes the weighted ℓ^1 -space,

$$\ell^{1,r} \equiv \ell^{1,r}(\mathbb{N}, \mathbb{R}) := \{(x_n)_{n \geq 1} \subset \mathbb{R} : \sum_{n=1}^{\infty} n^r |x_n| < \infty\}, \quad \mathbb{N} \equiv \mathbb{Z}_{\geq 1}. \quad (1.10)$$

A key feature of the Birkhoff map Φ^{bo} is that the BO Hamiltonian, expressed in the coordinates z_n , $n \neq 0$,

$$H^{bo} \circ \Psi^{bo} : h_0^1 \rightarrow \mathbb{R}, \quad \Psi^{bo} := (\Phi^{bo})^{-1},$$

is a real analytic function \mathcal{H}^{bo} of the actions I alone. More precisely (cf. Theorem B.1, [3, Proposition 8.1]),

$$\mathcal{H}^{bo} : \mathcal{Q}_+(\ell^{1,3}) \rightarrow \mathbb{R}, \quad I = (I_n)_{n \geq 1} \mapsto \sum_{n=1}^{\infty} n^2 I_n - \sum_{n=1}^{\infty} \left(\sum_{k=n}^{\infty} I_k \right)^2.$$

7 The BO frequencies are then defined by (cf. Theorem B.1)

$$\omega_n^{bo}(I) := \partial_{I_n} \mathcal{H}^{bo}(I) = n^2 - 2 \sum_{k=1}^{\infty} \min\{n, k\} I_k, \quad \omega_{-n}^{bo}(I) := -\omega_n^{bo}(I), \quad \forall n \geq 1. \quad (1.11)$$

8 Furthermore, it is well known that the Hamiltonian $H^{mo}(q) := \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2} q(x)^2 dx$, referred to as the moment,
9 is a prime integral of (1.1). When expressed in the coordinates z_n , $n \neq 0$, it is a real analytic function of
10 the actions alone, given by (cf. Theorem A.1(i))

$$H^{mo} \circ \Psi^{bo} = \sum_{n \geq 1} n I_n = \sum_{n \geq 1} z_n z_{-n}. \quad (1.12)$$

11 Finally, the differential $d_0 \Phi^{bo} : L_0^2 \rightarrow h_0^0$ of Φ^{bo} at $q = 0$ and its inverse are given by (cf. Lemma A.3(iii),
12 Theorem B.1 (NF4)),

$$d_0 \Phi^{bo} = -\mathcal{F}, \quad d_0 \Psi^{bo} = (d_0 \Phi^{bo})^{-1} = -\mathcal{F}^{-1}, \quad (1.13)$$

13 where \mathcal{F} denotes the Fourier transform, defined for any $s \in \mathbb{R}$ by

$$\mathcal{F} : H_0^s \rightarrow h_0^s, \quad q \mapsto (q_n)_{n \neq 0}, \quad q_n := \frac{1}{2\pi} \int_0^{2\pi} q(x) e^{-inx} dx, \quad \forall n \neq 0. \quad (1.14)$$

14 For any nonempty, finite subset $S_+ \subseteq \mathbb{N}$, let $S_+^\perp := \mathbb{N} \setminus S_+$ and define

$$N_S := \max S, \quad S := S_+ \cup (-S_+), \quad S^\perp := S_+^\perp \cup (-S_+^\perp). \quad (1.15)$$

We denote by $M_S \subset L_0^2$ the real analytic manifold of S -gap potentials (cf. Appendix A),

$$M_S := \{q \in L_0^2 : z_n(q) = 0 \forall n \in S^\perp\},$$

¹For notational convenience, whenever the context allows, we denote both the L_0^2 -gradient and the L^2 -gradient by ∇ .

and by M_S^o the open subset of M_S , consisting of proper S -gap potentials,

$$M_S^o := \{q \in M_S : z_n(q) \neq 0 \ \forall n \in S\}.$$

- 1 The manifold M_S is contained in $\cap_{s \geq 0} H_0^s$ and hence consists of C^∞ -smooth potentials and M_S^o can be
 2 parametrized by the action-angle coordinates $\theta_S = (\theta_k)_{k \in S_+} \in \mathbb{T}^{S_+}$, $I_S = (I_k)_{k \in S_+} \in \mathbb{R}^{S_+}$,

$$\Psi_S : \mathcal{M}_S^o \rightarrow M_S^o, (\theta_S, I_S) \mapsto \Psi^{bo}(z(\theta_S, I_S)), \quad \mathcal{M}_S^o := \mathbb{T}^{S_+} \times \mathbb{R}_{>0}^{S_+}, \quad (1.16)$$

where $z(\theta_S, I_S) = (z_n(\theta_S, I_S))_{n \neq 0}$ is given by

$$z_{\pm n}(\theta_S, I_S) := \sqrt{n I_n} e^{\pm i \theta_n}, \quad \forall n \in S_+, \quad z_n(\theta_S, I_S) = 0, \quad \forall n \in S^\perp.$$

Introduce for any $s \in \mathbb{R}$,

$$h_{\perp, \mathbb{C}}^s := h^s(S^\perp, \mathbb{C}), \quad h_\perp^s := \{z_\perp = (z_n)_{n \in S^\perp} \in h_{\perp, \mathbb{C}}^s : z_{-n} = \bar{z}_n, \quad \forall n \in S^\perp\},$$

- 3 and the partial Fourier transforms $\mathcal{F}_{N_S}^\pm : L_{\mathbb{C}}^2 \mapsto h_{\perp, \mathbb{C}}^0$, defined for $q \in L_{\mathbb{C}}^2$ and $n \in S^\perp$ by

$$(\mathcal{F}_{N_S}^+[q])_n := \begin{cases} q_n & \text{if } n \geq N_S + 1 \\ 0 & \text{if } n \in S^\perp \setminus [N_S + 1, \infty) \end{cases}, \quad (\mathcal{F}_{N_S}^-[q])_n := \begin{cases} q_n & \text{if } n \leq -N_S - 1 \\ 0 & \text{if } n \in S^\perp \setminus (-\infty, -N_S - 1] \end{cases}. \quad (1.17)$$

- 4 For notational convenience, we denote by $(\mathcal{F}_{N_S}^\pm)^{-1} : h_{\perp, \mathbb{C}}^0 \rightarrow L_{\mathbb{C}}^2$ the maps

$$(\mathcal{F}_{N_S}^+)^{-1} : z_\perp \mapsto \sum_{n \geq N_S + 1} z_n e^{inx}, \quad (\mathcal{F}_{N_S}^-)^{-1} : z_\perp \mapsto \sum_{n \geq N_S + 1} z_{-n} e^{-inx} \quad (1.18)$$

and view $\mathcal{M}_S^o \times h_\perp^s$, $s \in \mathbb{R}$, as a subset of h_\perp^s . The elements of $\mathcal{M}_S^o \times h_\perp^s$ are denoted by

$$\mathfrak{r} := (\theta_S, I_S, z_\perp), \quad \theta_S := (\theta_n)_{n \in S_+}, \quad I_S := (I_n)_{n \in S_+}, \quad z_\perp := (z_n)_{n \in S^\perp}.$$

It is endowed by the standard Poisson bracket (cf. [3, (6.9)]), given by

$$\{I_n, \theta_n\} = 1, \quad \forall n \in S_+, \quad \{z_n, z_{-n}\} = -in, \quad \forall n \in S^\perp,$$

whereas the brackets between all other coordinate functions vanish. For any $s \in \mathbb{R}$ and any given point $\mathfrak{r} \in \mathcal{M}_S^o \times h_\perp^s$, denote by E_s the tangent space of $\mathcal{M}_S^o \times h_\perp^s$. Note that E_s is independent of \mathfrak{r} and given by $E_s = \mathbb{R}^{S_+} \times \mathbb{R}^{S_+} \times h_\perp^s$. Its elements are denoted by $\widehat{\mathfrak{r}} = (\widehat{\theta}_S, \widehat{I}_S, \widehat{z}_\perp)$. Furthermore, for any $k \geq 1$, $\partial_x^{-k} : H_{\mathbb{C}}^s \rightarrow H_{0, \mathbb{C}}^{s+k}$ is the bounded linear operator, defined by

$$\partial_x^{-k}[e^{inx}] = \frac{1}{(in)^k} e^{inx}, \quad \forall n \neq 0, \quad \partial_x^{-k}[1] = 0.$$

With $D := -i\partial_x$, one then obtains for any $n \geq 1$,

$$D^{-k}[e^{inx}] = |\partial_x|^{-k}[e^{inx}], \quad (-D)^{-k}[e^{-inx}] = |\partial_x|^{-k}[e^{-inx}].$$

- 5 Finally, the standard inner products on $L_{\mathbb{C}}^2$ and on $h_{0, \mathbb{C}}^0$ are defined for any $f, g \in L_{\mathbb{C}}^2$ and $z, w \in h_{0, \mathbb{C}}^0$ by

$$\langle f|g \rangle \equiv \langle f|g \rangle_{L_{\mathbb{C}}^2} = \frac{1}{2\pi} \int_0^{2\pi} f(x) \overline{g(x)} dx, \quad \langle z|w \rangle \equiv \langle z|w \rangle_{h_{0, \mathbb{C}}^0} = \sum_{n \neq 0} z_n \bar{w}_n. \quad (1.19)$$

- 6 In addition, we introduce bilinear forms on $L_{\mathbb{C}}^2$ and on $h_{0, \mathbb{C}}^0$, defined for any $f, g \in L_{\mathbb{C}}^2$ and $z, w \in h_{0, \mathbb{C}}^0$ by

$$\langle f, g \rangle \equiv \langle f, g \rangle_{L_{\mathbb{C}}^2} = \frac{1}{2\pi} \int_0^{2\pi} f(x) g(x) dx, \quad \langle z, w \rangle \equiv \langle z, w \rangle_{h_{0, \mathbb{C}}^0} = \sum_{n \neq 0} z_n w_{-n}. \quad (1.20)$$

1 Note that $\langle \cdot, \cdot \rangle$ and $\langle \cdot | \cdot \rangle$ coincide on the real Hilbert spaces L^2 and h_0^0 . In the sequel, restrictions of $\langle \cdot, \cdot \rangle$
2 and $\langle \cdot | \cdot \rangle$ to subspaces and extensions as dual pairings will be denoted in the same way. The gradient of
3 C^1 -functionals $F : L_{\mathbb{C}}^2 \rightarrow \mathbb{C}$ and $G : h_{0,\mathbb{C}}^0 \rightarrow \mathbb{C}$, corresponding to $\langle \cdot, \cdot \rangle$, are denoted by ∇F , respectively ∇G .
4 They are defined by

$$dF[\widehat{f}] = \langle \nabla F, \widehat{f} \rangle, \quad \forall \widehat{f} \in L_{\mathbb{C}}^2, \quad dG[\widehat{z}] = \sum_{n \neq 0} \widehat{z}_n \partial_{z_n} G = \langle \nabla G, \widehat{z} \rangle, \quad \forall \widehat{z} \in h_{0,\mathbb{C}}^0, \quad (1.21)$$

5 where the n th component of ∇G is given by $(\nabla G)_n = \partial_{z_{-n}} G$. Finally, for given Banach spaces Y_1, Y_2 , we
6 denote by $\mathcal{B}(Y_1, Y_2)$ the Banach space of bounded linear operators from Y_1 to Y_2 , endowed with the operator
7 norm.

Theorem 1.1. *Let S_+ be a finite, nonempty subset of \mathbb{N} and let \mathcal{K} be a subset of \mathcal{M}_S^o of the form $\mathbb{T}^{S_+} \times \mathcal{K}_1$ where \mathcal{K}_1 is a compact subset of $\mathbb{R}_{>0}^{S_+}$. Then there exists an open bounded neighbourhood \mathcal{V} of $\mathcal{K} \times \{0\}$ in $\mathcal{M}_S^o \times h_{\perp}^0$ and a canonical real analytic diffeomorphism $\Psi : \mathcal{V} \rightarrow \Psi(\mathcal{V}) \subseteq L_0^2$, satisfying*

$$\Psi(\theta_S, I_S, 0) = \Psi_S(\theta_S, I_S), \quad \forall (\theta_S, I_S, 0) \in \mathcal{V},$$

8 (with Ψ_S given by (1.16)) and having the property that for any $s \in \mathbb{Z}_{\geq 0}$, $\Psi : \mathcal{V} \cap (\mathcal{M}_S^o \times h_{\perp}^s) \rightarrow H_0^s$ is a real
9 analytic diffeomorphism onto its image in H_0^s , so that the following holds:

10 **(AE1)** *For any $\mathfrak{x} = (\theta_S, I_S, z_{\perp}) \in \mathcal{V}$ and any integer $N \geq 1$, $\Psi(\mathfrak{x})$ admits an asymptotic expansion of the
11 form $\Psi_S(\theta_S, I_S) + \mathcal{OP}_N(\mathfrak{x}; \Psi) + \mathcal{R}_N(\mathfrak{x}; \Psi)$ where $\mathcal{OP}_N(\mathfrak{x}; \Psi)$ is given by*

$$\left(-g_{\infty} + \sum_{k=1}^N a_k^+(\mathfrak{x}; \Psi) D^{-k} \right) [(\mathcal{F}_{N_S}^+)^{-1} z_{\perp}] + \left(-\overline{g_{\infty}} + \sum_{k=1}^N a_k^-(\mathfrak{x}; \Psi) (-D)^{-k} \right) [(\mathcal{F}_{N_S}^-)^{-1} z_{\perp}], \quad (1.22)$$

with $g_{\infty}(x) \equiv g_{\infty}(x; q)$ defined by

$$g_{\infty}(x) := e^{D^{-1}q(x)} = e^{i\partial_x^{-1}q(x)}, \quad q(x) := \Psi_S(\theta_S, I_S).$$

The coefficients $a_k^{\pm}(\mathfrak{x}; \Psi)$ and the remainder $\mathcal{R}_N(\mathfrak{x}; \Psi)$ satisfy

$$a_k^-(\mathfrak{x}; \Psi) = \overline{a_k^+(\mathfrak{x}; \Psi)}, \quad \forall k \geq 1, \quad \mathcal{R}_N(\theta_S, I_S, 0; \Psi) = 0,$$

and for any $s \in \mathbb{Z}_{\geq 0}$ and $k \geq 1$,

$$a_k^{\pm}(\cdot; \Psi) : \mathcal{V} \rightarrow H_{\mathbb{C}}^s, \quad \mathcal{R}_N(\cdot; \Psi) : \mathcal{V} \cap (\mathcal{M}_S^o \times h_{\perp}^s) \rightarrow H^{s+N+1},$$

12 are real analytic maps² satisfying the tame estimates of Theorem 1.2 below.

13 **(AE2)** *For any $\mathfrak{x} \in \mathcal{V}$, the transpose $d\Psi(\mathfrak{x})^{\top}$ (with respect to the standard inner products) of the differential
14 $d\Psi(\mathfrak{x}) : E_1 \rightarrow H_0^1$ is a bounded operator $d\Psi(\mathfrak{x})^{\top} : H_0^1 \rightarrow E_1$. For any integer $N \geq 1$, $d\Psi(\mathfrak{x})^{\top}$ admits an
15 expansion of the form $\mathcal{OP}_N(\mathfrak{x}; d\Psi^{\top}) + \mathcal{R}_N(\mathfrak{x}; d\Psi^{\top})$ with*

$$\mathcal{OP}_N(\mathfrak{x}; d\Psi^{\top}) = (0, 0, \mathcal{OP}_N^+(\mathfrak{x}; d\Psi^{\top}) + \mathcal{OP}_N^-(\mathfrak{x}; d\Psi^{\top})), \quad (1.23)$$

where for any $\widehat{q} \in H_0^1$, $\mathcal{OP}_N^{\pm}(\mathfrak{x}; d\Psi^{\top})[\widehat{q}]$ are defined by

$$\begin{aligned} \mathcal{OP}_N^+(\mathfrak{x}; d\Psi^{\top})[\widehat{q}] &= \mathcal{F}_{N_S}^+ \circ \left(-\overline{g_{\infty}} + \sum_{k=1}^N a_k^+(\mathfrak{x}; d\Psi^{\top}) D^{-k} \right) [\widehat{q}] \\ &+ \mathcal{F}_{N_S}^+ \circ \sum_{k=0}^N \mathcal{A}_k^+(\mathfrak{x}; d\Psi^{\top})[\widehat{q}] D^{-k} [(\mathcal{F}_{N_S}^+)^{-1} z_{\perp}], \end{aligned}$$

²In this context, $H_{\mathbb{C}}^s$ is considered as a \mathbb{R} -Hilbert space. This convention for real analytic maps is used throughout the paper.

$$\begin{aligned}\mathcal{OP}_N^-(\mathbf{r}; d\Psi^\top)[\hat{q}] &= \mathcal{F}_{N_S}^- \circ \left(-g_\infty + \sum_{k=1}^N a_k^-(\mathbf{r}; d\Psi^\top)(-D)^{-k} \right) [\hat{q}] \\ &+ \mathcal{F}_{N_S}^- \circ \sum_{k=0}^N \mathcal{A}_k^-(\mathbf{r}; d\Psi^\top)[\hat{q}] (-D)^{-k} [(\mathcal{F}_{N_S}^-)^{-1} z_\perp].\end{aligned}$$

Furthermore, for any $\hat{q} \in H_0^1$,

$$a_k^-(\mathbf{r}; d\Psi^\top) = \overline{a_k^+(\mathbf{r}; d\Psi^\top)}, \quad \forall k \geq 1, \quad \mathcal{A}_k^-(\mathbf{r}; d\Psi^\top)[\hat{q}] = \overline{\mathcal{A}_k^+(\mathbf{r}; d\Psi^\top)[\hat{q}]}, \quad \forall k \geq 0,$$

and for any $s \in \mathbb{N}$,

$$a_k^\pm(\cdot; d\Psi^\top) : \mathcal{V} \rightarrow H_\mathbb{C}^s, \quad \mathcal{A}_k^\pm(\cdot; d\Psi^\top) : \mathcal{V} \rightarrow \mathcal{B}(H_0^1, H_\mathbb{C}^s),$$

$$\mathcal{R}_N(\cdot; d\Psi^\top) : \mathcal{V} \cap (\mathcal{M}_S^o \times h_\perp^s) \rightarrow \mathcal{B}(H_0^s, E_{s+N+1}),$$

are real analytic maps, satisfying the tame estimates of Theorem 1.2 below.

(AE3) The Hamiltonians $\mathcal{H} := H^{bo} \circ \Psi$ (cf. (1.3)) and $\mathcal{H}^{mo} := H^{mo} \circ \Psi$ (cf. (1.12)),

$$\mathcal{H} : \mathcal{V} \cap (\mathcal{M}_S^o \times h_\perp^1) \rightarrow \mathbb{R}, \quad \mathcal{H}^{mo} : \mathcal{V} \cap (\mathcal{M}_S^o \times h_\perp^0) \rightarrow \mathbb{R},$$

are in normal form up to order three. More precisely,

$$\mathcal{H}(\theta_S, I_S, z_\perp) = \mathcal{H}^{bo}(I_S, 0) + \sum_{n \in S_\perp^\pm} \Omega_n(I_S) z_n z_{-n} + \mathcal{P}(\theta_S, I_S, z_\perp),$$

$$\mathcal{H}^{mo}(\theta_S, I_S, z_\perp) = \sum_{n \in S_+} n I_n + \sum_{n \in S_\perp^\pm} z_n z_{-n} + \mathcal{P}^{mo}(\theta_S, I_S, z_\perp),$$

where for any $n \in S^\perp$, $\Omega_n(I_S) := \frac{1}{n} \omega_n^{bo}(I_S, 0)$, and where $\mathcal{P}, \mathcal{P}^{mo} : \mathcal{V} \cap (\mathcal{M}_S^o \times h_\perp^1) \rightarrow \mathbb{R}$ are real analytic and satisfy

$$\mathcal{P}(\theta_S, I_S, z_\perp), \mathcal{P}^{mo}(\theta_S, I_S, z_\perp) = O(\|z_\perp\|_1 \|z_\perp\|_0^2),$$

and where ω_n^{bo} , $n \neq 0$, denote the BO frequencies, introduced in (1.11).

Expansion of $\nabla \mathcal{P}(\mathbf{r})$: for any integer $N \geq 1$, there exists an integer $\sigma_N \geq N$ (loss of regularity) so that the L^2 -gradient $\nabla \mathcal{P}(\mathbf{r})$ of \mathcal{P} with components $\nabla_{\theta_S} \mathcal{P}$, $\nabla_{I_S} \mathcal{P}$, and $\nabla_{z_\perp} \mathcal{P}$ admits an expansion of the form $\nabla \mathcal{P}(\mathbf{r}) = (0, 0, \mathcal{OP}_N(\mathbf{r}; \nabla \mathcal{P})) + \mathcal{R}_N(\mathbf{r}; \nabla \mathcal{P})$,

$$\mathcal{OP}_N(\mathbf{r}; \nabla \mathcal{P}) = \mathcal{F}_{N_S}^+ \circ \left(\sum_{k=0}^N T_{a_k^+(\mathbf{r}; \nabla \mathcal{P})} D^{-k} \right) [(\mathcal{F}_{N_S}^+)^{-1} z_\perp] + \mathcal{F}_{N_S}^- \circ \left(\sum_{k=0}^N T_{a_k^-(\mathbf{r}; \nabla \mathcal{P})} (-D)^{-k} \right) [(\mathcal{F}_{N_S}^-)^{-1} z_\perp]$$

where for any $k \geq 0$, $a_k^-(\mathbf{r}; \nabla \mathcal{P}) = \overline{a_k^+(\mathbf{r}; \nabla \mathcal{P})}$ and for any integer $s \geq 0$,

$$a_k^\pm(\cdot; \nabla \mathcal{P}) : \mathcal{V} \cap (\mathcal{M}_S^o \times h_\perp^{s+\sigma_N}) \rightarrow H_\mathbb{C}^s, \quad \mathcal{R}_N(\cdot; \nabla \mathcal{P}) : \mathcal{V} \cap (\mathcal{M}_S^o \times h_\perp^{s \vee \sigma_N}) \rightarrow E_{s+N+1},$$

are real analytic and satisfy the tame estimates of Theorem 1.2 below. Here $T_{a_k^\pm(\cdot; \nabla \mathcal{P})}$ denotes the operator of para-multiplication with the function $a_k^\pm(\cdot; \nabla \mathcal{P})$ (cf. Appendix D).

Expansion for $\nabla \mathcal{P}^{mo}(\mathbf{r})$: the L^2 -gradient $\nabla \mathcal{P}^{mo}(\mathbf{r})$ admits an expansion similar to the one of $\nabla \mathcal{P}(\mathbf{r})$ with corresponding coefficients $a_k^\pm(\mathbf{r}; \nabla \mathcal{P}^{mo})$ and remainder $\mathcal{R}_N(\mathbf{r}; \nabla \mathcal{P}^{mo})$.

Remark 1.1. It is a remarkable feature of the expansions of Ψ and $d\Psi^\top$, which turns out to be relevant for applications, that for any $k \geq 1$, the coefficients $a_k^\pm(\mathbf{r}; \Psi)$ and $a_k^\pm(\mathbf{r}; d\Psi^\top)$ are C^∞ -smooth functions on \mathbb{T} and that the linear operators $\mathcal{A}_k^\pm(\mathbf{r}; d\Psi^\top)$ are C^∞ -smoothing. These regularity properties of the expansions of Ψ and $d\Psi^\top$ are a consequence of the fact that the map Ψ_L , introduced in (2.3) below as the basic ingredient for the construction of the map Ψ , is given by the linearization of the Birkhoff map in normal direction at finite gap potentials and that such potentials are C^∞ -smooth (actually, even real analytic).

In applications, it is of interest to know whether the coordinate transformation Ψ preserves the reversible structure, defined by the maps $S_{rev} : L_0^2 \rightarrow L_0^2$, $(S_{rev}q)(x) := q(-x)$, and $S_{rev} : \mathcal{M}_S^o \times h_\perp^0 \rightarrow \mathcal{M}_S^o \times h_\perp^0$ where

$$\mathcal{S}_{rev}(\theta_S, I_S, z_\perp) := (\theta_S^{rev}, I_S^{rev}, z_\perp^{rev}), \quad \theta_n^{rev} = -\theta_n, \quad I_n^{rev} = I_n, \quad \forall n \in S_+, \quad z_n^{rev} = z_{-n}, \quad \forall n \in S^\perp. \quad (1.24)$$

Note that for any $s \in \mathbb{R}_{\geq 0}$, $S_{rev} : H_0^s \rightarrow H_0^s$ and $S_{rev} : \mathcal{M}_S^o \times h_\perp^s \rightarrow \mathcal{M}_S^o \times h_\perp^s$ are linear involutions and that without loss of generality, the neighbourhood \mathcal{V} of Theorem 1.1 can be chosen to be invariant under the map \mathcal{S}_{rev} , i.e., $\mathcal{S}_{rev}(\mathcal{V}) = \mathcal{V}$.

Addendum to Theorem 1.1. *The maps $\Psi : \mathcal{V} \rightarrow L_0^2$, $\Psi^{bo} : h_0^0 \rightarrow L_0^2$, and $(\mathcal{F}_{N_S}^\pm)^{-1} : h_{\perp, \mathbb{C}}^0 \rightarrow L_{0, \mathbb{C}}^2$ preserve the reversible structure, i.e.,*

$$\Psi \circ \mathcal{S}_{rev} = \mathcal{S}_{rev} \circ \Psi, \quad \Psi^{bo} \circ \mathcal{S}_{rev} = \mathcal{S}_{rev} \circ \Psi^{bo}, \quad (\mathcal{F}_{N_S}^-)^{-1} \circ \mathcal{S}_{rev} = \mathcal{S}_{rev} \circ (\mathcal{F}_{N_S}^+)^{-1},$$

and so do the maps in the asymptotic expansions **(AE1)** ($\mathfrak{x} \in \mathcal{V}$),

$$a_k^+(\mathcal{S}_{rev}\mathfrak{x}; \Psi) = \mathcal{S}_{rev}(a_k^-(\mathfrak{x}; \Psi)), \quad \mathcal{R}_N(\mathcal{S}_{rev}\mathfrak{x}; \Psi) = \mathcal{S}_{rev}(\mathcal{R}_N(\mathfrak{x}; \Psi)),$$

and the ones in the asymptotic expansions **(AE2)** ($\mathfrak{x} \in \mathcal{V} \cap (\mathcal{M}_S^o \times h_\perp^1)$, $\hat{q} \in H_0^1$),

$$a_k^+(\mathcal{S}_{rev}\mathfrak{x}; d\Psi^\top) = \mathcal{S}_{rev}(a_k^-(\mathfrak{x}; d\Psi^\top)), \quad \mathcal{A}_k^+(\mathcal{S}_{rev}\mathfrak{x}; d\Psi^\top)[\mathcal{S}_{rev}\hat{q}] = \mathcal{S}_{rev}(\mathcal{A}_k^-(\mathfrak{x}; d\Psi^\top)[\hat{q}]),$$

$$\mathcal{R}_N(\mathcal{S}_{rev}\mathfrak{x}; d\Psi^\top)[\mathcal{S}_{rev}\hat{q}] = \mathcal{S}_{rev}(\mathcal{R}_N(\mathfrak{x}; d\Psi^\top)[\hat{q}]).$$

Furthermore, the Hamiltonians H^{bo} , $\mathcal{H} = H^{bo} \circ \Psi$, and \mathcal{P} are reversible, meaning that

$$H^{bo} \circ \mathcal{S}_{rev} = H^{bo}, \quad \mathcal{H} \circ \mathcal{S}_{rev} = \mathcal{H}, \quad \mathcal{P} \circ \mathcal{S}_{rev} = \mathcal{P},$$

and the maps in the asymptotic expansion in **(AE3)** preserve the reversible structure,

$$a_k^+(\mathcal{S}_{rev}\mathfrak{x}; \nabla \mathcal{P}) = \mathcal{S}_{rev}(a_k^-(\mathfrak{x}; \nabla \mathcal{P})), \quad \forall \mathfrak{x} \in \mathcal{V} \cap (\mathcal{M}_S^o \times h_\perp^{1+\sigma_N}),$$

$$\mathcal{R}_N(\mathcal{S}_{rev}\mathfrak{x}; \nabla \mathcal{P}) = \mathcal{S}_{rev}(\mathcal{R}_N(\mathfrak{x}; \nabla \mathcal{P})), \quad \forall \mathfrak{x} \in \mathcal{V} \cap (\mathcal{M}_S^o \times h_\perp^{1 \vee \sigma_N}).$$

Corresponding results hold for the Hamiltonians H^{mo} , $\mathcal{H}^{mo} = H^{mo} \circ \Psi$, and \mathcal{P}^{mo} .

Theorem 1.2 below states tame estimates for the map Ψ as well as the gradient $\nabla \mathcal{P}$ of the remainder term \mathcal{P} in the expansion of \mathcal{H} and the one of the remainder \mathcal{P}^{mo} in the expansion of \mathcal{H}^{mo} . Throughout the paper, the stated estimates for maps hold locally uniformly with respect to their arguments.

Theorem 1.2. *Let $N, l \in \mathbb{N}$. Then under the same assumptions as in Theorem 1.1, the following estimates hold:*

(Est1) *For any $\mathfrak{x} = (\theta_S, I_S, z_\perp) \in \mathcal{V}$, $k \geq 1$, $\hat{\mathfrak{x}}_1, \dots, \hat{\mathfrak{x}}_l \in E_0$, $s \in \mathbb{Z}_{\geq 0}$,*

$$\|a_k^\pm(\mathfrak{x}; \Psi)\|_s \lesssim_{s,k} 1, \quad \|d^l a_k^\pm(\mathfrak{x}; \Psi)[\hat{\mathfrak{x}}_1, \dots, \hat{\mathfrak{x}}_l]\|_s \lesssim_{s,k,l} \prod_{j=1}^l \|\hat{\mathfrak{x}}_j\|_0.$$

Similarly, for any $\mathfrak{x} \in \mathcal{V} \cap (\mathcal{M}_S^o \times h_\perp^s)$, $\hat{\mathfrak{x}}_1, \dots, \hat{\mathfrak{x}}_l \in E_s$, $s \in \mathbb{Z}_{\geq 0}$,

$$\|\mathcal{R}_N(\mathfrak{x}; \Psi)\|_{s+N+1} \lesssim_{s,N} \|z_\perp\|_s,$$

$$\|d^l \mathcal{R}_N(\mathfrak{x}; \Psi)[\hat{\mathfrak{x}}_1, \dots, \hat{\mathfrak{x}}_l]\|_{s+N+1} \lesssim_{s,N,l} \sum_{j=1}^l \|\hat{\mathfrak{x}}_j\|_s \prod_{i \neq j} \|\hat{\mathfrak{x}}_i\|_0 + \|z_\perp\|_s \prod_{j=1}^l \|\hat{\mathfrak{x}}_j\|_0.$$

(Est2) For any $\mathfrak{x} = (\theta_S, I_S, z_\perp) \in \mathcal{V} \cap (\mathcal{M}_S^o \times h_\perp^1)$, $k \geq 1$, $\widehat{\mathfrak{x}}_1, \dots, \widehat{\mathfrak{x}}_l \in E_1$, $s \in \mathbb{N}$,

$$\begin{aligned} \|a_k^\pm(\mathfrak{x}; d\Psi^\top)\|_s &\lesssim_{s,k} 1 + \|z_\perp\|_1, & \|d^l a_k^\pm(\mathfrak{x}; d\Psi^\top)[\widehat{\mathfrak{x}}_1, \dots, \widehat{\mathfrak{x}}_l]\|_s &\lesssim_{s,k,l} \prod_{j=1}^l \|\widehat{\mathfrak{x}}_j\|_1, \\ \|\mathcal{A}_k^\pm(\mathfrak{x}; d\Psi^\top)\|_s &\lesssim_{s,k} 1 + \|z_\perp\|_1, & \|d^l \mathcal{A}_k^\pm(\mathfrak{x}; d\Psi^\top)[\widehat{\mathfrak{x}}_1, \dots, \widehat{\mathfrak{x}}_l]\|_s &\lesssim_{s,k,l} \prod_{j=1}^l \|\widehat{\mathfrak{x}}_j\|_1. \end{aligned}$$

Similarly, for any $\mathfrak{x} \in \mathcal{V} \cap (\mathcal{M}_S^o \times h_\perp^s)$, $\widehat{\mathfrak{x}}_1, \dots, \widehat{\mathfrak{x}}_l \in E_s$, $\widehat{q} \in H_0^s$, $s \in \mathbb{N}$,

$$\begin{aligned} \|\mathcal{R}_N(\mathfrak{x}; d\Psi^\top)[\widehat{q}]\|_{s+N+1} &\lesssim_{s,N} \|\widehat{q}\|_s + \|z_\perp\|_s \|\widehat{q}\|_1, \\ \|d^l(\mathcal{R}_N(\mathfrak{x}; d\Psi^\top)[\widehat{q}])[\widehat{\mathfrak{x}}_1, \dots, \widehat{\mathfrak{x}}_l]\|_{s+N+1} &\lesssim_{s,N,l} \|\widehat{q}\|_s \prod_{j=1}^l \|\widehat{\mathfrak{x}}_j\|_1 + \|\widehat{q}\|_1 \sum_{j=1}^l \|\widehat{\mathfrak{x}}_j\|_s \prod_{i \neq j} \|\widehat{\mathfrak{x}}_i\|_1 + \|\widehat{q}\|_1 \|z_\perp\|_s \prod_{j=1}^l \|\widehat{\mathfrak{x}}_j\|_1. \end{aligned}$$

(Est3) For any $s \in \mathbb{Z}_{\geq 0}$, $\mathfrak{x} = (\theta_S, I_S, z_\perp) \in \mathcal{V} \cap (\mathcal{M}_S^o \times h_\perp^{s+\sigma_N})$, $\|z_\perp\|_{\sigma_N} \leq 1$, $1 \leq k \leq N$, $\widehat{\mathfrak{x}}_1, \dots, \widehat{\mathfrak{x}}_l \in E_{s+\sigma_N}$,

$$\begin{aligned} \|a_k^\pm(\mathfrak{x}; \nabla \mathcal{P})\|_s &\lesssim_{s,k} \|z_\perp\|_{s+\sigma_N}, \\ \|d^l a_k^\pm(\mathfrak{x}; \nabla \mathcal{P})[\widehat{\mathfrak{x}}_1, \dots, \widehat{\mathfrak{x}}_l]\|_s &\lesssim_{s,k,l} \sum_{j=1}^l \|\widehat{\mathfrak{x}}_j\|_{s+\sigma_N} \prod_{n \neq j} \|\widehat{\mathfrak{x}}_n\|_{\sigma_N} + \|z_\perp\|_{s+\sigma_N} \prod_{j=1}^l \|\widehat{\mathfrak{x}}_j\|_{\sigma_N}. \end{aligned}$$

For any $s \in \mathbb{Z}_{\geq 0}$, $\mathfrak{x} \in \mathcal{V} \cap (\mathcal{M}_S^o \times h_\perp^{s \vee \sigma_N})$ with $\|z_\perp\|_{\sigma_N} \leq 1$, $\widehat{\mathfrak{x}} \in E_{s \vee \sigma_N}$,

$$\begin{aligned} \|\mathcal{R}_N(\mathfrak{x}; \nabla \mathcal{P})\|_{s+N+1} &\lesssim_{s,N} \|z_\perp\|_{\sigma_N} \|z_\perp\|_{s \vee \sigma_N}, \\ \|d \mathcal{R}_N(\mathfrak{x}; \nabla \mathcal{P})[\widehat{\mathfrak{x}}]\|_{s+N+1} &\lesssim_{s,N} \|z_\perp\|_{\sigma_N} \|\widehat{\mathfrak{x}}\|_{s \vee \sigma_N} + \|z_\perp\|_{s \vee \sigma_N} \|\widehat{\mathfrak{x}}\|_{\sigma_N}. \end{aligned}$$

If in addition $\widehat{\mathfrak{x}}_1, \dots, \widehat{\mathfrak{x}}_l \in E_{s \vee \sigma_N}$, $l \geq 2$, then

$$\|d^l \mathcal{R}_N(\mathfrak{x}; \nabla \mathcal{P})[\widehat{\mathfrak{x}}_1, \dots, \widehat{\mathfrak{x}}_l]\|_{s+N+1} \lesssim_{s,N,l} \sum_{j=1}^l \|\widehat{\mathfrak{x}}_j\|_{s \vee \sigma_N} \prod_{n \neq j} \|\widehat{\mathfrak{x}}_n\|_{\sigma_N} + \|z_\perp\|_{s \vee \sigma_N} \prod_{j=1}^l \|\widehat{\mathfrak{x}}_j\|_{\sigma_N}.$$

1 Corresponding estimates hold for the maps $a_k^\pm(\cdot; \nabla \mathcal{P}^{mo})$ and $\mathcal{R}_N(\cdot; \nabla \mathcal{P}^{mo})$.

2 Here and throughout the paper, the meaning of the symbol \lesssim with various subindices is the standard one.
3 So for example, the estimate $\|a_k^\pm(\mathfrak{x}; \nabla \mathcal{P})\|_s \lesssim_{s,k} \|z_\perp\|_{s+\sigma_N}$ in (Est3) means that for any $1 \leq k \leq N$, there
4 exists a constant $C(s, k) > 0$, depending on s and k , so that $\|a_k^\pm(\mathfrak{x}; \nabla \mathcal{P})\|_s \leq C(s, k) \|z_\perp\|_{s+\sigma_N}$ for any \mathfrak{x} as
5 indicated in the statement of (Est3), i.e., for any $\mathfrak{x} = (\theta_S, I_S, z_\perp) \in \mathcal{V} \cap (\mathcal{M}_S^o \times h_\perp^{s+\sigma_N})$ with $\|z_\perp\|_{\sigma_N} \leq 1$.

Remark 1.2. In the case $S_+ = \emptyset$, one has $M_0 = \{0\}$, $S^\perp = \mathbb{Z} \setminus \{0\}$, and $h_\perp^0 = h_0^0$. Since the differential $d\Phi^{bo}$ at $q = 0$ is given by $-\mathcal{F}: L_0^2 \rightarrow h_0^0$ (cf. (1.13)), one easily verifies that in this case, the map Ψ is given by $-\mathcal{F}^{-1}$, hence linear and defined on all of h_0^0 . (According to Section 2, $\Psi = \Psi_L \circ \Psi_C$ and in the case at hand, $\Psi_L = -\mathcal{F}^{-1}$, $\Psi_C = \text{Id.}$) As a consequence, $d\Psi^\top = -\mathcal{F}$. Furthermore, $H^{bo} \circ \Psi: h_0^1 \rightarrow \mathbb{R}$ is given by

$$H^{bo} \circ \Psi(z) = \sum_{n \geq 1} \Omega_n^{bo}(0) z_n z_{-n} + \mathcal{P}(z), \quad \Omega_n^{bo}(0) = n, \quad \forall n \geq 1,$$

and $\mathcal{P}(z) = \frac{1}{2\pi} \int_0^{2\pi} -\frac{1}{3} (\mathcal{F}^{-1} z)^3 dx$. Hence by the Bony decomposition (cf. Lemma D.1(i)), the gradient $\nabla \mathcal{P}$ admits for any $N \geq 1$, $z \in h_0^{s+\sigma_N}$ with $s \geq 0$ and $\sigma_N := N + 2$, an expansion of the form

$$\nabla \mathcal{P}(z) = 2T_{\mathcal{F}^{-1}z}[\mathcal{F}^{-1}z] + \mathcal{R}^{(B)}(z),$$

where for any $s \geq 0$, the Bony remainder $\mathcal{R}^{(B)}: h_0^{s \vee \sigma_N} \mapsto h_0^{s+N+1}$ is real analytic and satisfies the estimate $\|\mathcal{R}^{(B)}(z)\|_{s+N+1} \lesssim_{s,N} \|z\|_{s \vee \sigma_N} \|z\|_{\sigma_N}$. Furthermore,

$$\mathcal{H}^{mo}(z) = H^{mo} \circ \Psi(z) = \sum_{n \geq 1} z_n z_{-n}.$$

1 *Applications.* The Birkhoff coordinates are well suited to study the initial value problem of (1.1) (cf. [4] and
2 references therein). Using the arguments, developed in the case of the Korteweg-de Vries (KdV) equation
3 (cf. e.g. [14], [18]), one can obtain KAM type results for semilinear perturbations of (1.1). However, when
4 equation (1.1) is expressed in Birkhoff coordinates, various features of the BO equation and its perturbations
5 such as being pseudo-differential equations, get lost. On the other hand, due to the expansions (AE1) –
6 (AE3), the coordinates of Theorem 1.1 allow to preserve the essence of such features and in the form
7 stated turn out to be well suited to study quasilinear perturbations of the BO equation as well as questions
8 of stability of finite gap solutions. We plan to investigate the stability of finite gap solutions of the BO
9 equation under quasi-linear Hamiltonian perturbations

$$\partial_t u = \partial_x (|\partial_x| u - u^2 + \varepsilon \nabla F(u)), \quad x \in \mathbb{T}, \quad (1.25)$$

10 where $0 < \varepsilon < 1$ is a small parameter and ∇F the L^2 -gradient of Hamiltonian F such as

$$F(u) := \frac{1}{2\pi} \int_0^{2\pi} f(u(x), |\partial_x|^{1/2} u(x)) dx, \quad f \in \mathcal{C}^\infty(\mathbb{R} \times \mathbb{R}, \mathbb{R}). \quad (1.26)$$

11 Our goal is to use Theorem 1.1 and Theorem 1.2 to prove that for ε small enough, there is a large set of finite
12 gap solutions of (1.1) with the property that for any initial data, ε -close to one of them, the corresponding
13 solution of the perturbed equation exists for large time intervals and stays ε -close to the finite gap solution
14 considered.

Outline of the construction of the coordinates of Theorem 1.1. In [18], Kuksin presents a general scheme for
proving KAM-type theorems for integrable PDEs in one space dimension such as the KdV or the sine-Gordon
(sG) equations, which possess a Lax pair formulation and admit finite dimensional integrable subsystems
foliated by invariant tori. The starting point is to construct local canonical coordinates, suitable to apply
KAM methods. Expanding on work of Krichever [17], Kuksin considers bounded, finite dimensional inte-
grable subsystems of such a PDE which admit action-angle coordinates. The latter are complemented by
infinitely many coordinates whose construction is based on a set of time periodic solutions, referred to as
Floquet solutions of the PDE, obtained by linearizing the PDE under consideration along a solution evolving
in the integrable subsystem. It turns out that the resulting coordinate transformation is typically not
symplectic. Extending arguments of Moser and Weinstein to the given infinite dimensional setup (see [18],
Lemma 1.4 and Section 1.7), he constructs a second coordinate transformation so that the composition of
the two transformations become symplectic. In our previous work [12], we follow the general scheme of the
construction in [18] to construct canonical coordinates for the KdV equation near finite dimensional invariant
tori, which have the property that they admit an expansion as described in Theorem 1.1 and Theorem 1.2.
Following the arguments developed of [12], we construct the transformation Ψ as the composition of $\Psi_L \circ \Psi_C$
of two transformations. The transformation Ψ_L is given by the Taylor expansion of the Birkhoff map Ψ^{bo} of
order one in the normal direction z_\perp around $(z_S, 0)$,

$$\Psi^{bo}(z_S, 0) + d\Psi^{bo}(z_S, 0)[(0, z_\perp)],$$

15 where we write z as (z_S, z_\perp) with $z_S = (z_n)_{n \in S}$ and $z_\perp = (z_n)_{n \in S^\perp}$. The neighbourhood \mathcal{V} of $\mathcal{K} \times \{0\}$
16 (cf. Theorem 1.1) is chosen sufficiently small so that by the inverse function theorem, Ψ_L is a real analytic
17 diffeomorphism onto its image. Using that Ψ_L is given in terms of the Birkhoff map Ψ^{bo} and taking advantage
18 of features of the spectral theory of the Lax operator, we prove that Ψ_L admits a high frequency expansion
19 and tame estimates corresponding to the ones of Theorems 1.1, 1.2. In a second step we establish the
20 corresponding results for the symplectic corrector Ψ_C .

21 *Comments.* (i) In view of the definition of Ψ_L , the map Ψ can be considered as a symplectic version of the
22 Taylor expansion of Ψ^{bo} of order 1 in normal directions at points in $\mathcal{M}_S^o \times \{0\}$ and hence as a locally defined
23 symplectic approximation of Ψ^{bo} . In the special case $N = 1$, Theorem 1.1 implies that

$$-g_\infty \cdot (\mathcal{F}_{N_S}^+)^{-1}[z_\perp] - \overline{g_\infty} \cdot (\mathcal{F}_{N_S}^-)^{-1}[z_\perp] \quad (1.27)$$

is a high frequency approximation of Ψ . More precisely,

$$\Psi(\theta_S, I_S, z_\perp) + g_\infty \cdot (\mathcal{F}_{N_S}^+)^{-1}[z_\perp] + \overline{g_\infty} \cdot (\mathcal{F}_{N_S}^-)^{-1}[z_\perp]$$

maps $\mathcal{V} \cap (\mathcal{M}_S^0 \times h_\perp^s)$ into H^{s+1} for any $s \geq 0$, i.e., it is one-smoothing. In [9], such a property has been established for the differential of Ψ^{bo} and the one of Φ^{bo} .³ In contrast, Theorem 1.1 says that for the map Ψ , a much stronger property holds: up to a remainder term, which is $(N+1)$ -smoothing, Ψ is a (nonlinear) pseudo-differential operator acting on $\mathcal{F}_\perp^{-1}(h_\perp^0)$.

(ii) In comparison with the corresponding study for the KdV equation, the main differences in the case of the Benjamin-Ono equation stems from the facts that the Lax operator of the Benjamin-Ono equation is a pseudo-differential operator (cf. (A.1) in Appendix A) whereas the one of the KdV equation is the differential operator $-\frac{d^2}{dx^2} + u$ (cf. e.g. [14, Section 1]), and that the high frequency approximation of the Birkhoff map of the Benjamin-Ono equation, given in [8, Theorem 3], is a 'quasi-linear' perturbation of the Fourier transform, involving the function g_∞ and its complex conjugate (cf. also (1.27)), whereas the corresponding one of the KdV equation is given by the Fourier transform and hence independent of u (cf. [12, Theorem A.1 (B5)]).

Notations. The standard inner product on $L_\mathbb{C}^2 \equiv H_\mathbb{C}^0$ and the corresponding norm are defined by

$$\langle f|g \rangle \equiv \langle f|g \rangle_{L_\mathbb{C}^2} = \frac{1}{2\pi} \int_0^{2\pi} f(x) \overline{g(x)} dx, \quad \|f\| = \langle f|f \rangle^{1/2}, \quad \forall f, g \in L_\mathbb{C}^2.$$

Restrictions of this inner product to various subspaces and extensions as dual pairings will be denoted in the same way.

Let h and g be real valued functions, depending on various variables. In addition, h might depend on parameters α, \dots . The notation $h \lesssim_{\alpha, \dots} g$ means that h satisfies an estimate of the form $h \leq Cg$ where the constant $C > 0$ only depends on the parameters α, \dots .

Throughout the paper, whenever possible, we will use for the asymptotic expansions of maps such as $\mathfrak{x} \mapsto \Psi(\mathfrak{x})$ (cf. Theorem 1.1(AE1)) or $\mathfrak{x} \mapsto d\Psi(\mathfrak{x})^\top$ (cf. Theorem 1.1(AE2)) the same type of notation as introduced in the statement of Theorem 1.1. For coefficients in such an expansion, which are operators, we use the upper case letter \mathcal{A} and write \mathcal{A}_k for the k th coefficient, whereas for coefficients, which are functions (or operators, defined as multiplication by a function), we use the lower case letter a and write a_k for the k th coefficient. The map, which is expanded, is indicated as an *argument* of the coefficient \mathcal{A}_k and respectively, a_k (cf. also Remark 2.2).

Organization. The maps Ψ_L and Ψ_C are studied in Section 2 and Section 3, respectively. The expansion of the BO Hamiltonian in the new coordinates is treated in Section 4 and a summary of the proofs of Theorem 1.1 and Theorem 1.2 is given in Section 5. In Appendix A - Appendix C, we present results needed in the main body of the paper and in Appendix D we review material from the pseudo-differential and para-differential calculus.

Acknowledgements. We would like to thank the referees for their careful reading of the article and for their suggestions of how to improve the exposition of the paper. Thomas Kappeler is supported by the Swiss National Science Foundation and Riccardo Montalto by INDAM-GNFM.

2 The map Ψ_L

In this section we define and study the map Ψ_L described in Section 1. First let us introduce some more notation. For $S \subset \mathbb{Z}$ finite as in (1.15), let

$$h_S = \{z_S = (z_n)_{n \in S} \in (\mathbb{C}^*)^S : z_{-n} = \bar{z}_n \ \forall n \in S_+\}, \quad \mathbb{C}^* := \mathbb{C} \setminus \{0\}, \quad (2.1)$$

endowed with the norm $\|z_S\| = (\sum_{n \in S} |z_n|^2)^{1/2}$. Recall that $N_S = \max\{n : n \in S\}$. For notational convenience, for any $s \in \mathbb{Z}$, we identify $h_S \times h_\perp^s$ with the subset $\{z = (z_n)_{n \neq 0} \in h_0^s : z_n \neq 0 \ \forall n \in S\}$ of h_0^s and write (z_S, z_\perp) for $z \in h_0^s$.

The restriction of the Birkhoff map Φ^{bo} to the space M_S^0 of proper S -gap potentials yields a real analytic diffeomorphism, $\Phi^{bo}|_{M_S^0} : M_S^0 \rightarrow h_S$ (cf. Corollary B.1 in Appendix B). We endow h_S with the pull back

³In the case of the KdV equation (cf. [15], [19]) and the defocusing NLS equation (cf. [16]), such type of results have been proved for the corresponding differentials of the Birkhoff maps (and their inverses), as well as for the Birkhoff maps themselves.

of the standard Poisson structure on h_0^0 by the natural embedding $h_S \hookrightarrow h_0^0$, where the standard Poisson structure is the one for which $\{z_n, z_{-n}\} = -in$ for any $n \neq 0$ and the Poisson brackets among all the other coordinates vanish. By Corollary B.1, M_S^0 is a real analytic submanifold of \mathcal{U}_{N_S} where \mathcal{U}_{N_S} is the set of finite gap potentials q with the property that $\gamma_{N_S}(q) > 0$ and $\gamma_n(q) = 0$ for any $n \geq N_S + 1$. By Appendix A, any $u \in \mathcal{U}_{N_S}$ is of the form

$$u(x) = \sum_{j=1}^{N_S} \left(\frac{1 - r_j^2}{1 - 2r_j \cos(x + \alpha_j) + r_j^2} - 1 \right), \quad 0 < r_j < 1, \quad 0 \leq \alpha_j < 2\pi, \quad \forall 1 \leq j \leq N_S. \quad (2.2)$$

Consider the partial linearization of the inverse Birkhoff map $\Psi^{bo} := (\Phi^{bo})^{-1}$, defined as

$$\Psi_L : h_S \times h_\perp^0 \rightarrow L_0^2, \quad (z_S, z_\perp) \mapsto \Psi^{bo}(z_S, 0) + \Psi_1(z_S)[z_\perp], \quad \Psi_1(z_S)[z_\perp] := d_\perp \Psi^{bo}(z_S, 0)[z_\perp], \quad (2.3)$$

where $d_\perp \Psi^{bo}(z_S, 0)$ denotes the Fréchet derivative of the map $z_\perp \mapsto \Psi^{bo}(z_S, z_\perp)$, evaluated at the point $(z_S, 0)$. By Theorem B.1, Ψ_L is a real analytic map.

Proposition 2.1. *The map Ψ_L has the following properties:*

(i) *For any $z_S \in h_S$,*

$$\Psi_L(z_S, 0) = \Psi^{bo}(z_S, 0), \quad d\Psi_L(z_S, 0) = d\Psi^{bo}(z_S, 0).$$

(ii) *For any compact subset $\mathcal{K} \subseteq h_S$ there exists an open neighborhood \mathcal{V} of $\mathcal{K} \times \{0\}$ in $h_S \times h_\perp^0$ so that for any integer $s \geq 0$, the restriction $\Psi_L|_{\mathcal{V} \cap h_0^s}$ is a map $\mathcal{V} \cap h_0^s \rightarrow H_0^s$ which is a real analytic diffeomorphism onto its image. The neighborhood \mathcal{V} is chosen of the form $\mathcal{V}_S \times \mathcal{V}_\perp$ where \mathcal{V}_S is an open, bounded neighborhood of \mathcal{K} in h_S and \mathcal{V}_\perp is an open ball in h_\perp^0 of sufficiently small radius, centered at zero.*

(iii) *For any $z = (z_S, z_\perp) \in \mathcal{V}$ and $\hat{z} = (\hat{z}_S, \hat{z}_\perp) \in h_S \times h_\perp^0$,*

$$d\Psi_L(z)[\hat{z}] = d\Psi_L(z_S, 0)[\hat{z}] + d_S(d_\perp \Psi^{bo}(z_S, 0)[z_\perp])[\hat{z}_S]. \quad (2.4)$$

For any $z_S \in \mathcal{V}_S$, the linear map $d\Psi_L(z_S, 0) = d\Psi^{bo}(z_S, 0)$ is canonical and for any $z_\perp \in \mathcal{V}_\perp$, we denote by $d_S(d_\perp \Psi^{bo}(z_S, 0)[z_\perp])$ the Fréchet derivative of the map $\mathcal{V}_S \rightarrow L_0^2$, $w_S \mapsto d_\perp \Psi^{bo}(w_S, 0)[z_\perp]$ at $w_S = z_S$.

Proof. (i) The stated formulas follow from the definition of Ψ_L . (ii) The claimed statements follow from Theorem B.1 and the inverse function theorem by arguing as in the proof of the corresponding results for the defocusing NLS equation in [11, Proposition 3.1]. Item (iii) is proved in a straightforward way. \square

For any $z = (z_S, z_\perp) \in \mathcal{V} = \mathcal{V}_S \times \mathcal{V}_\perp$ and $n \neq 0$, let $q := \Psi^{bo}(z_S, 0)$ and

$$W_n(x) \equiv W_n(x, q) := (d_q \Phi^{bo})^{-1}[e^{(n)}], \quad W_{-n}(x) \equiv W_{-n}(x, q) := (d_q \Phi^{bo})^{-1}[e^{(-n)}], \quad (2.5)$$

where $e^{(k)}$, $k \neq 0$, are the standard basis elements in the sequence space $h_{0,\mathbb{C}}^0$, $e^{(k)} = (\delta_{k,j})_{j \neq 0}$. (Here we extended $(d_u \Phi^{bo})^{-1} : h_0^0 \rightarrow L_0^2$ as a \mathbb{C} -linear map $h_{0,\mathbb{C}}^0 \rightarrow L_{0,\mathbb{C}}^2$.) Then $\Psi_1(z_S)[z_\perp] = \Psi_L(z_S, z_\perp) - q$ can be written as

$$\Psi_1(z_S)[z_\perp](x) = \sum_{n \in S^\perp} z_n W_n(x, q). \quad (2.6)$$

In a next step, we want to analyze $\Psi_1(z_S)[z_\perp]$ further. Consider the Hamiltonian vector fields $\partial_x \nabla z_n$, $n \neq 0$, corresponding to the Hamiltonians given by the Birkhoff coordinates $z_n : L_0^2 \rightarrow \mathbb{C}, u \mapsto z_n(u)$. Since Φ^{bo} is canonical in the sense that $\{z_n, z_{-n}\} = -in$ for any $n \neq 0$ and the brackets among all the other coordinates vanish, it follows that for any $u \in L_0^2$ and $n \neq 0$,

$$d_u \Phi^{bo}[\partial_x \nabla z_n] = X_{z_n},$$

where X_{z_n} is the constant Hamiltonian vector field on $h_{0,\mathbb{C}}^0$ given by (cf. (1.21))

$$X_{z_n} = -ine^{(-n)}.$$

1 Hence for any $n \neq 0$,

$$(d_u \Phi^{bo})^{-1}[e^{(n)}] = \frac{1}{in} \partial_x \nabla z_{-n}. \quad (2.7)$$

We need to explicitly compute $\partial_x \nabla z_n$ at $q := \Psi^{bo}(z_S, 0)$, $z_S \in \mathcal{V}_S$, for $|n| \geq N_S + 1$ (cf. (2.1)). It requires some elements of the spectral theory of the Lax operator L_u , recorded in Appendix A. The operator L_u , $u \in L^2$, is given by $D - T_u$ and acts on the Hardy space L_+^2 . Here $D = -i\partial_x$ and T_u is the Toeplitz operator $f \mapsto \Pi(uf)$ with symbol u . The eigenvalues are listed in increasing order and denoted by $\lambda_n(u)$, $n \geq 0$. The corresponding L^2 -normalized eigenfunctions $f_n(\cdot, u) \in H_+^1$ with the normalization conditions in (A.2) form an orthonormal basis of L_+^2 . First let us consider the case $n \geq N_S + 1$. For any $n \geq N_S + 1$, the eigenvalue $\lambda_n : L^2 \rightarrow \mathbb{R}$ (cf. Theorem A.1(iv)) and the corresponding eigenfunction $f_n : L^2 \rightarrow H_+^1$ (cf. Corollary A.1(iii)) of the Lax operator are real analytic functions. Denote by $\delta\lambda_n$ and δf_n the variation of λ_n and respectively, f_n at q in direction $u \in L^2$,

$$\delta\lambda_n = \frac{d}{d\varepsilon}|_{\varepsilon=0} \lambda_n(q + \varepsilon u), \quad \delta f_n = \frac{d}{d\varepsilon}|_{\varepsilon=0} f_n(\cdot, q + \varepsilon u).$$

Since f_n is L^2 -normalized, $\langle \delta f_n | f_n \rangle = i\xi_n$ for some $\xi_n \in \mathbb{R}$ and hence (cf. [3, Section 5])

$$\delta f_n = i\xi_n f_n + \sum_{\ell \neq n} \langle \delta f_n | f_\ell \rangle f_\ell.$$

The real numbers ξ_n , $n \geq 1$, are determined by the normalization conditions (A.2) of f_n , $n \geq 0$. The derivative of $(D - T_{q+\varepsilon u} - \lambda_n(q + \varepsilon u))f_n(\cdot, q + \varepsilon u) = 0$ with respect to ε at $\varepsilon = 0$ can then be computed as

$$(-T_u - \delta\lambda_n)f_n + (D - T_q - \lambda_n)\delta f_n = 0.$$

Since $\delta\lambda_n = -\langle u f_n | f_n \rangle$ and $T_u f_n = \sum_{\ell \geq 0} \langle u f_n | f_\ell \rangle f_\ell$ one infers that

$$\delta f_n = i\xi_n f_n + (D - T_q - \lambda_n)^{-1} \sum_{\ell \neq n} \langle u f_n | f_\ell \rangle f_\ell = i\xi_n f_n + \sum_{\ell \neq n} \frac{\langle u f_n | f_\ell \rangle}{\lambda_\ell - \lambda_n} f_\ell.$$

By Lemma A.3(ii), one has for any $n \geq N_S + 1$,

$$\lambda_n \equiv \lambda(q) = n, \quad f_n(x) \equiv f_n(x, q) = e^{inx} g_\infty(x, q), \quad g_\infty(x) \equiv g_\infty(x, q) = e^{i\partial_x^{-1} q(x)}.$$

2 Since q is a S -gap potential and hence $\langle 1 | f_\ell \rangle = 0$ for any $\ell \in S_+^\perp$, it then follows that

$$1 = \sum_{\ell \in S_0} \langle 1 | f_\ell \rangle f_\ell, \quad S_0 := S_+ \cup \{0\}, \quad (2.8)$$

and

$$\langle 1 | \delta f_n \rangle = \sum_{\ell \in S_0} \overline{\langle u f_n | f_\ell \rangle} \frac{\langle 1 | f_\ell \rangle}{\lambda_\ell - n} = \langle u, e^{-inx} \overline{g_\infty} \sum_{\ell \in S_0} \frac{\langle 1 | f_\ell \rangle}{\lambda_\ell - n} f_\ell \rangle.$$

Recall that by (1.6)

$$z_n(q) = \sqrt{n} \zeta_n(q), \quad z_{-n}(q) = \sqrt{n} \overline{\zeta_n}(q), \quad \forall n \geq 1,$$

where by (B.1), $\zeta_n(q) = \frac{\langle 1 | f_n(\cdot, q) \rangle}{\sqrt{\kappa_n(q)}}$ and $\kappa_n(q)$ is defined in Lemma A.1. Hence $z_n = n \frac{\langle 1 | f_n \rangle}{\sqrt{n\kappa_n}}$ for any $n \geq 1$ and in view of the definition of the gradient ∇z_n , one then concludes that

$$\partial_x \nabla z_n(\cdot, q) = -\frac{1}{\sqrt{n\kappa_n}} \sum_{\ell \in S_0} \frac{\langle 1 | f_\ell \rangle}{1 - \frac{\lambda_\ell}{n}} \partial_x (e^{-inx} \overline{g_\infty} f_\ell), \quad \forall n \geq N_S + 1.$$

3 Taking into account that $z_{-n} = \overline{z_n}$ and $\partial_x = iD$ we are led to the following

Lemma 2.1. For any $q \in M_S^0$ and $n \geq N_S + 1$,

$$\begin{aligned}\partial_x \nabla z_n(\cdot, q) &= -e^{-inx} \frac{1}{\sqrt{n\kappa_n}} \sum_{\ell \in S_0} \frac{\langle 1|f_\ell \rangle}{1 - \frac{\lambda_\ell}{n}} (-in \overline{g_\infty} f_\ell + iD(\overline{g_\infty} f_\ell)), \\ \partial_x \nabla z_{-n}(\cdot, q) &= -e^{inx} \frac{1}{\sqrt{n\kappa_n}} \sum_{\ell \in S_0} \frac{\overline{\langle 1|f_\ell \rangle}}{1 - \frac{\lambda_\ell}{n}} (in g_\infty \overline{f_\ell} + iD(g_\infty \overline{f_\ell})).\end{aligned}$$

Hence by (2.5) and (2.7),

$$\begin{aligned}W_{-n}(x, q) &= \frac{1}{-in} \partial_x \nabla z_n(\cdot, q) = -e^{-inx} \frac{1}{\sqrt{n\kappa_n}} \sum_{\ell \in S_0} \frac{\langle 1|f_\ell \rangle}{1 - \frac{\lambda_\ell}{n}} (\overline{g_\infty} f_\ell + \frac{1}{n} q \overline{g_\infty} f_\ell - \frac{1}{n} \overline{g_\infty} Df_\ell), \\ W_n(x, q) &= \frac{1}{in} \partial_x \nabla z_{-n}(\cdot, q) = -e^{inx} \frac{1}{\sqrt{n\kappa_n}} \sum_{\ell \in S_0} \frac{\overline{\langle 1|f_\ell \rangle}}{1 - \frac{\lambda_\ell}{n}} (g_\infty \overline{f_\ell} + \frac{1}{n} q g_\infty \overline{f_\ell} - \frac{1}{n} g_\infty \overline{Df_\ell}).\end{aligned}$$

Next we want to show that $\Psi_L(z)$ admits an expansion of the type stated in Theorem 1.1. It is convenient to introduce

$$V_S := \Psi^{bo}(\mathcal{V}_S \times \{0\}).$$

1 First note that being a finite gap potential, $q \in V_S$ is C^∞ -smooth and so is $W_n(\cdot, q)$ for any $n \in S^\perp$.

2

3 **Theorem 2.1.** For any $q \in V_S$, $N \in \mathbb{N}$, $W_{\pm n}(x, q)$, $n \geq N_S + 1$, have the expansion as $n \rightarrow \infty$

$$W_{\pm n}(x, q) = -e^{\pm inx} \left(\sum_{k=0}^N \frac{W_k^{ae, \pm}(x, q)}{n^k} + \frac{\mathcal{R}_N^{W_{\pm n}}(x, q)}{n^{N+1}} \right) \quad (2.9)$$

where for any $1 \leq k \leq N$, $W_k^{ae, -}(x, q) = \overline{W_k^{ae, +}(x, q)}$ and $W_k^{ae, +}(x) \equiv W_k^{ae, +}(x, q)$ is given by

$$W_k^{ae, +}(x) := g_\infty(x) \sum_{\ell \in S_0} c_{\ell, k} \overline{\langle 1|f_\ell \rangle f_\ell(x)} + q(x) g_\infty(x) \sum_{\ell \in S_0} c_{\ell, k-1} \overline{\langle 1|f_\ell \rangle f_\ell(x)} - g_\infty(x) \sum_{\ell \in S_0} c_{\ell, k-1} \overline{\langle 1|f_\ell \rangle Df_\ell(x)}.$$

4 The coefficients $c_{\ell, k}$ are real valued constants and for any $\ell \in S_0$ (cf. (2.8)) have the following properties:

5 (i) for $k = -1, 0$, one has $c_{\ell, -1} = 0$ and $c_{\ell, 0} = 1$, implying that

$$W_0^{ae, +} = g_\infty, \quad W_0^{ae, -} = \overline{g_\infty}; \quad (2.10)$$

6 (ii) for any $k \geq 1$, $c_{\ell, k}$ is a polynomial of degree k in the gap lengths $\gamma_j(q)$, $j \in S_+$.

Finally, the remainders $\mathcal{R}_N^{W_{\pm n}}(x) \equiv \mathcal{R}_N^{W_{\pm n}}(x, q)$ satisfy $\mathcal{R}_N^{W_{-n}}(x) = \overline{\mathcal{R}_N^{W_n}(x)}$ and $\mathcal{R}_N^{W_n}(x)$ is given by

$$\mathcal{R}_N^{W_n}(x) := g_\infty(x) \sum_{\ell \in S_0} r_{\ell, n}^N \overline{\langle 1|f_\ell \rangle f_\ell(x)} + q(x) g_\infty(x) \sum_{\ell \in S_0} r_{\ell, n}^{N-1} \overline{\langle 1|f_\ell \rangle f_\ell(x)} - g_\infty(x) \sum_{\ell \in S_0} r_{\ell, n}^{N-1} \overline{\langle 1|f_\ell \rangle Df_\ell(x)},$$

7 where for any $\ell \in S_0$, the coefficients $r_{\ell, n}^N$ are real, bounded functions of $\gamma_j(q)$, $j \in S_+$, independent of x .

Proof. The claimed results follow from Lemma 2.1. Indeed, for any $n \geq N_S + 1$, one has by the product representation of κ_n (cf. Lemma A.1),

$$n\kappa_n = \frac{n}{\lambda_n - \lambda_0} \prod_{j \neq n} (1 - \frac{\gamma_j}{\lambda_j - \lambda_n}) = \frac{n}{n - \lambda_0} \prod_{j \in S_+} (1 + \frac{\gamma_j}{n - \lambda_j}) = \frac{1}{1 - \frac{\lambda_0}{n}} \prod_{j \in S_+} (1 + \frac{\gamma_j}{n} \frac{1}{1 - \frac{\lambda_j}{n}}).$$

For any $\ell \in S_0$, one then obtains an expansion of the form

$$(n\kappa_n)^{-1/2} \frac{1}{1 - \frac{\lambda_\ell}{n}} = 1 + \sum_{k=1}^N \frac{c_{\ell,k}}{n^k} + \frac{r_{\ell,n}^N}{n^{N+1}}$$

where the coefficients $c_{\ell,k}$, $1 \leq k \leq N$, and the remainder $r_{\ell,n}^N \frac{1}{n^{N+1}}$ can be explicitly computed, using that for any $a \in \mathbb{R}$ with $|a| < 1$ and any $N \geq 1$

$$\frac{1}{1-a} = 1 + \sum_{k=1}^N a^k + a^{N+1} \frac{1}{1-a},$$

$$(1-a)^{-1/2} = 1 + \sum_{k=1}^N (-1)^k \binom{-1/2}{k} a^k + a^{N+1} C_N \int_0^1 (1-t)^N (1-ta)^{-1/2-N-1} dt,$$

1 with C_N being a combinatorial constant. Furthermore, we set $c_{\ell,-1} := 0$ and $c_{\ell,0} := 1$ for any $\ell \in S_0$. Since
2 for any $j \in S_0$, one has $\lambda_j = j - \sum_{m=j+1}^{N_S} \gamma_m$, the claimed expansions for $W_{\pm n}$ with the stated formulas for
3 $W_k^{ae,\pm}$ and the ones for $\mathcal{R}_N^{W_{\pm n}}$ then follow from the formulas for $W_{\pm n}$ of Lemma 2.1. The stated properties
4 of $c_{\ell,k}$, $-1 \leq k \leq N$, and $r_{\ell,n}^N$ can be verified in a straightforward way. The identities (2.10) follow from the
5 fact that $1 = \sum_{\ell \in S_0} \langle 1 | f_\ell \rangle f_\ell$ (cf. (2.8)). \square

For notational convenience, in the sequel, we will view $W_k^{ae,\pm}(\cdot, q)$ and $\mathcal{R}_N^{W_{\pm n}}(\cdot, q)$ as functions of z_S ,

$$W_k^{ae,\pm}(\cdot, z_S) \equiv W_k^{ae,\pm}(\cdot, \Psi^{bo}(z_S, 0)), \quad \mathcal{R}_N^{W_{\pm n}}(\cdot, z_S) \equiv \mathcal{R}_N^{W_{\pm n}}(\cdot, \Psi^{bo}(z_S, 0)).$$

6 Theorem 2.1, combined with Lemma A.2(ii), has then the following immediate two applications.

Corollary 2.1. *For any $s \in \mathbb{Z}_{\geq 0}$ the following holds:*

- (i) *for any $k \geq 0$, the maps $W_k^{ae,\pm} : \mathcal{V}_S \rightarrow H_{\mathbb{C}}^s$, $z_S \mapsto W_k^{ae,\pm}(\cdot, z_S)$ are real analytic;*
- (ii) *for any $N \geq 1$ and $n \geq N_S + 1$, the maps $\mathcal{R}_N^{W_{\pm n}} : \mathcal{V}_S \rightarrow H_{\mathbb{C}}^s$, $z_S \mapsto \mathcal{R}_N^{W_{\pm n}}(\cdot, z_S)$ are real analytic and satisfy for any $m \geq 0$,*

$$\sup_{\substack{0 \leq x \leq 1 \\ n \in S^\perp}} |\partial_x^m \mathcal{R}_N^{W_{\pm n}}(x, z_S)| \leq C_{N,m}.$$

7 The constants $C_{N,m}$ can be chosen locally uniformly for $z_S \in \mathcal{V}_S$.

The second application concerns the linear operator $\Psi_1(z_S) : h_\perp^0 \rightarrow L_0^2$, given for $z_S \in \mathcal{V}_S$ by

$$\widehat{z}_\perp \mapsto \Psi_1(z_S)[\widehat{z}_\perp] = \sum_{n \in S^\perp} \widehat{z}_n W_n(\cdot, z_S) = \sum_{|n| \in [1, N_S] \setminus S_+} \widehat{z}_n W_n(\cdot, z_S) + \sum_{|n| \geq N_S+1} \widehat{z}_n W_n(\cdot, z_S).$$

Note that for any $s \in \mathbb{Z}_{\geq 0}$, the restriction of $\Psi_1(z_S)$ to h_\perp^s is a bounded linear operator $h_\perp^s \rightarrow H_0^s$. Recall that in (1.17), we introduced the partial Fourier transforms $\mathcal{F}_{N_S}^+ : L_{\mathbb{C}}^2 \mapsto h_{\perp, \mathbb{C}}^0$ and $\mathcal{F}_{N_S}^- : L_{\mathbb{C}}^2 \mapsto h_{\perp, \mathbb{C}}^0$, and in (1.18) the partial inverses $(\mathcal{F}_{N_S}^\pm)^{-1} : h_{\perp, \mathbb{C}}^0 \rightarrow L_{\mathbb{C}}^2$ of the Fourier transform. Using that

$$\sum_{n > N_S} \widehat{z}_n \frac{1}{n^k} e^{inx} = D^{-k} (\mathcal{F}_{N_S}^+)^{-1} [\widehat{z}_\perp], \quad \sum_{n > N_S} \widehat{z}_{-n} \frac{1}{n^k} e^{-inx} = (-D)^{-k} (\mathcal{F}_{N_S}^-)^{-1} [\widehat{z}_\perp],$$

8 Theorem 2.1 implies the following two corollaries.

Corollary 2.2. *For any $z_S \in \mathcal{V}_S$, up to a remainder, the operator $\Psi_1(z_S) : h_\perp^0 \rightarrow L_0^2$ is a pseudo-differential operator of order 0. More precisely, $\Psi_1(z_S)$ has an expansion to any order $N \geq 1$ of the form $\mathcal{OP}_N(z_S; \Psi_1) + \mathcal{R}_N(z_S; \Psi_1)$ where*

$$\mathcal{OP}_N(z_S; \Psi_1) := \left(-g_\infty + \sum_{k=1}^N a_k^+(z_S; \Psi_1) D^{-k} \right) \circ (\mathcal{F}_{N_S}^+)^{-1} + \left(-\overline{g_\infty} + \sum_{k=1}^N a_k^-(z_S; \Psi_1) (-D)^{-k} \right) \circ (\mathcal{F}_{N_S}^-)^{-1}$$

1 and

$$a_k^+(z_S; \Psi_1) := -W_k^{ae,+}(\cdot, z_S), \quad a_k^-(z_S; \Psi_1) := -W_k^{ae,-}(\cdot, z_S) = \overline{a_k^+(z_S; \Psi_1)}, \quad \forall k \geq 1, \quad (2.11)$$

$$\mathcal{R}_N(z_S; \Psi_1)[\widehat{z}_\perp](x) := - \sum_{|n| \in [1, N_S] \setminus S_+} \widehat{z}_n W_n(x, z_S) - \sum_{|n| > N_S} \widehat{z}_n \frac{\mathcal{R}_N^{W_n}(x, z_S)}{|n|^{N+1}} e^{inx}.$$

For any $s \geq 0$, the restriction of $\mathcal{R}_N(z_S; \Psi_1)$ to h_\perp^s defines a bounded linear operator $h_\perp^s \rightarrow H^{s+N+1}$ and the map

$$\mathcal{V}_S \rightarrow \mathcal{B}(h_\perp^s, H^{s+N+1}), \quad z_S \mapsto \mathcal{R}_N(z_S; \Psi_1),$$

2 is real analytic. Correspondingly, the map $\Psi_L : \mathcal{V} \rightarrow L_0^2$, defined in Proposition 2.1, admits an expansion of
3 the form $\Psi^{bo}(z_S, 0) + \mathcal{OP}_N(z; \Psi_L) + \mathcal{R}_N(z; \Psi_L)$ where $\mathcal{OP}_N(z; \Psi_L)$ is given by

$$\left(-g_\infty + \sum_{k=1}^N a_k^+(z_S; \Psi_L) D^{-k} \right) [(\mathcal{F}_{N_S}^+)^{-1} z_\perp] + \left(-\overline{g_\infty} + \sum_{k=1}^N a_k^-(z_S; \Psi_L) (-D)^{-k} \right) [(\mathcal{F}_{N_S}^-)^{-1} z_\perp] \quad (2.12)$$

4 with

$$a_k^\pm(z_S; \Psi_L) := a_k^\pm(z_S; \Psi_1), \quad \forall k \geq 1, \quad \mathcal{R}_N(z; \Psi_L) := \mathcal{R}_N(z_S; \Psi_1)[z_\perp]. \quad (2.13)$$

5 In particular, one has $a_k^-(z_S; \Psi_L) = \overline{a_k^+(z_S; \Psi_L)}$.

Corollary 2.3. For any $z = (z_S, z_\perp) \in \mathcal{V}$, $d\Psi_1(z) : h_0^0 \rightarrow L^2$, given for any $\widehat{z} = (\widehat{z}_S, \widehat{z}_\perp) \in h_0^0$ by

$$d\Psi_1(z)[\widehat{z}] = d_S(\Psi_1(z_S)[z_\perp])[\widehat{z}_S] + \Psi_1(z_S)[\widehat{z}_\perp]$$

admits an expansion of the form $\mathcal{OP}_N(z; d\Psi_1) + \mathcal{R}_N(z; d\Psi_1)$, where the pseudo-differential operator $\mathcal{OP}_N(z; d\Psi_1)$, when written as a 1×2 matrix,

$$(\mathcal{OP}_N(z; d\Psi_1)^S \quad \mathcal{OP}_N(z; d\Psi_1)^\perp), \quad \mathcal{OP}_N(z; d\Psi_1)^S : h_S \rightarrow L^2, \quad \mathcal{OP}_N(z; d\Psi_1)^\perp : h_\perp^0 \rightarrow L^2,$$

is given by

$$\begin{aligned} \mathcal{OP}_N(z; d\Psi_1)^\perp &= \left(-g_\infty + \sum_{k=1}^N a_k^+(z_S; d\Psi_1) D^{-k} \right) \circ (\mathcal{F}_{N_S}^+)^{-1} + \left(-\overline{g_\infty} + \sum_{k=1}^N a_k^-(z_S; d\Psi_1) (-D)^{-k} \right) \circ (\mathcal{F}_{N_S}^-)^{-1}, \\ \mathcal{OP}_N(z; d\Psi_1)^S &= -d_S g_\infty[\cdot] \cdot [(\mathcal{F}_{N_S}^+)^{-1} z_\perp] + \sum_{k=1}^N \mathcal{A}_k^+(z_S; d\Psi_1)[\cdot] \cdot D^{-k} [(\mathcal{F}_{N_S}^+)^{-1} z_\perp] \\ &\quad - d_S \overline{g_\infty}[\cdot] \cdot [(\mathcal{F}_{N_S}^-)^{-1} z_\perp] + \sum_{k=1}^N \mathcal{A}_k^-(z_S; d\Psi_1)[\cdot] \cdot (-D)^{-k} [(\mathcal{F}_{N_S}^-)^{-1} z_\perp], \end{aligned}$$

where for any $k \geq 1$, $a_k^\pm(z_S; d\Psi_1) = a_k^\pm(z_S; \Psi_1)$, and $\mathcal{A}_k^\pm(z_S; d\Psi_1)$ are the following linear operators,

$$\mathcal{A}_k^\pm(z_S; d\Psi_1) : h_S \rightarrow L_{\mathbb{C}}^2, \quad \widehat{z}_S \mapsto d_S a_k^\pm(z_S; \Psi_1)[\widehat{z}_S].$$

Furthermore, for any $s \geq 0$, $k \geq 1$, the maps

$$\mathcal{V}_S \mapsto \mathcal{B}(h_S, H_{\mathbb{C}}^s), \quad z_S \mapsto \mathcal{A}_k^\pm(z_S; d\Psi_1), \quad \mathcal{V} \cap h_0^s \rightarrow \mathcal{B}(h_0^s, H^{s+N+1}), \quad z \mapsto \mathcal{R}_N(z; d\Psi_1),$$

6 are real analytic. As a consequence, $a_k^-(z_S; d\Psi_1) = \overline{a_k^+(z_S; d\Psi_1)}$ and $\mathcal{A}_k^-(z_S; d\Psi_1)[z_S] = \overline{\mathcal{A}_k^+(z_S; d\Psi_1)[z_S]}$.

7 **Remark 2.1.** (i) Note that for any $N \geq 1$ and $s \in \mathbb{R}$, $\mathcal{OP}_N(z_S; \Psi_1) : h_\perp^s \rightarrow H^s$ and $\mathcal{R}_N(z_S; \Psi_1) : h_\perp^s \rightarrow$
8 H^{s+N+1} are bounded linear operators.

9 (ii) The fact that up to a remainder term, $\Psi_L(z_S, \cdot)$ is given by a pseudo-differential operator of order 0,
10 acting on the scale of Hilbert spaces h_\perp^s , $s \in \mathbb{Z}_{\geq 0}$, shows that the differential of the Birkhoff map $z \mapsto \Psi^{bo}(z)$
11 at a finite gap potential, has distinctive features. These features are the starting point for the construction
12 of the coordinates of Theorem 1.1.

Remark 2.2. As already mentioned in the paragraph 'Notations' at the end of Section 1, whenever possible, we will use the same type of notation for the coefficients of the expansion of maps such as $z_S \mapsto \Psi_1(z_S)$ (cf. Corollary 2.2) or $z \mapsto d\Psi_1(z)$ (cf. Corollary 2.3) as introduced in Corollary 2.2 and respectively Corollary 2.3. For coefficients in such an expansion, which are operators, we use the upper case letter \mathcal{A} and write \mathcal{A}_k for the k th coefficient, whereas for coefficients, which are functions (or operators, defined as multiplication by a function), we use the lower case letter a and write a_k for the k th coefficient. The map, which is expanded, is indicated as an argument of the coefficient \mathcal{A}_k and respectively, a_k .

A straightforward application of Corollary 2.2 yields an expansion of the transpose $\Psi_1(z_S)^\top$ of the operator $\Psi_1(z_S)$. First note that the transpose $(\mathcal{F}_{N_S}^+)^{-\top}$ of $(\mathcal{F}_{N_S}^+)^{-1}$ with respect to the standard inner products in L_0^2 and h_\perp^0 is given by $\mathcal{F}_{N_S}^+$, i.e., for any $\hat{q} \in L^2$,

$$\begin{aligned} \langle (\mathcal{F}_{N_S}^+)^{-1}[z_\perp] | \hat{q} \rangle &= \frac{1}{2\pi} \int_0^{2\pi} \sum_{n > N_S} z_n e^{inx} \hat{q}(x) dx = \sum_{n > N_S} z_n \frac{1}{2\pi} \int_0^{2\pi} \hat{q}(x) e^{inx} dx \\ &= \sum_{n > N_S} z_n \hat{q}_{-n} = \sum_{n > N_S} z_n \overline{\hat{q}_n} = \langle z_\perp | \mathcal{F}_{N_S}^+ \hat{q} \rangle. \end{aligned}$$

Similarly, one computes the transpose $(\mathcal{F}_{N_S}^-)^{-\top}$ of $(\mathcal{F}_{N_S}^-)^{-1}$. For later reference we record

$$(\mathcal{F}_{N_S}^+)^{-\top} = \mathcal{F}_{N_S}^+, \quad (\mathcal{F}_{N_S}^-)^{-\top} = \mathcal{F}_{N_S}^-. \quad (2.14)$$

Corollary 2.4. For any $z_S \in \mathcal{V}_S$ and $N \in \mathbb{N}$, $\Psi_1(z_S)^\top : L_0^2 \rightarrow h_\perp^0$, $\hat{q} \mapsto (\langle W_{-n}(\cdot, z_S) | \hat{q} \rangle)_{n \in S^\perp}$ has an expansion of the form $\mathcal{OP}_N(z_S; \Psi_1^\top) + \mathcal{R}_N(z_S; \Psi_1^\top)$ with

$$\mathcal{OP}_N(z_S; \Psi_1^\top) := \mathcal{F}_{N_S}^+ \circ \left(-\overline{g_\infty} + \sum_{k=1}^N a_k^+(z_S; \Psi_1^\top) D^{-k} \right) + \mathcal{F}_{N_S}^- \circ \left(-g_\infty + \sum_{k=1}^N a_k^-(z_S; \Psi_1^\top) (-D)^{-k} \right). \quad (2.15)$$

For any $k \geq 1$, $a_k^-(z_S; \Psi_1^\top) = \overline{a_k^+(z_S; \Psi_1^\top)}$ and for any $s \geq 0$, the coefficients $\mathcal{V}_S \rightarrow H_\mathbb{C}^s$, $z_S \mapsto a_k^\pm(z_S; \Psi_1^\top)$, $k \geq 1$, and the remainder $\mathcal{V}_S \rightarrow \mathcal{B}(H^s, h_\perp^{s+N+1})$, $z_S \mapsto \mathcal{R}_N(z_S; \Psi_1^\top)$, are real analytic.

Corresponding properties hold for the map

$$\mathcal{V} \rightarrow \mathcal{B}(L_0^2, h_0^0), \quad z \mapsto d\Psi_L(z)^\top.$$

For any $z \in \mathcal{V}$ and $N \in \mathbb{N}$, $d\Psi_L(z)^\top$ has an expansion of the form $\mathcal{OP}_N(z_S; d\Psi_L^\top) + \mathcal{R}_N(z; d\Psi_L^\top)$ where $\mathcal{OP}_N(z_S; d\Psi_L^\top)$ is given by

$$\left(0, \mathcal{F}_{N_S}^+ \circ \left(-\overline{g_\infty} + \sum_{k=1}^N a_k^+(z_S; d\Psi_L^\top) D^{-k} \right) + \mathcal{F}_{N_S}^- \circ \left(-g_\infty + \sum_{k=1}^N a_k^-(z_S; d\Psi_L^\top) (-D)^{-k} \right) \right) \quad (2.16)$$

with $a_k^\pm(z_S; d\Psi_L^\top) = a_k^\pm(z_S; \Psi_1^\top)$ and where for any integer $s \geq 0$,

$$\mathcal{V} \cap h_0^s \rightarrow \mathcal{B}(L^2, h_0^{s+N+1}), \quad z \mapsto \mathcal{R}_N(z; d\Psi_L^\top)$$

is real analytic. In particular, one has $a_k^-(z_S; d\Psi_L^\top) = \overline{a_k^+(z_S; d\Psi_L^\top)}$.

Remark 2.3. We record that for any $N \geq 1$ and $s \in \mathbb{R}$, the operators $\mathcal{OP}_N(z_S; \Psi_1^\top) : H^s \rightarrow h_\perp^s$ and $\mathcal{R}_N(z_S; \Psi_1^\top) : H^s \rightarrow h_\perp^{s+N+1}$ are bounded.

Proof. Let $N \geq 1$ be given. By Corollary 2.2, $\Psi_1(z_S) = \mathcal{OP}_N(z_S; \Psi_1) + \mathcal{R}_N(z_S; \Psi_1)$ where for any $z_S \in \mathcal{V}_S$,

$$\mathcal{OP}_N(z_S; \Psi_1) = \left(-g_\infty + \sum_{k=1}^N a_k^+(z_S; \Psi_1) D^{-k} \right) \circ (\mathcal{F}_{N_S}^+)^{-1} + \left(-\overline{g_\infty} + \sum_{k=1}^N a_k^-(z_S; \Psi_1) (-D)^{-k} \right) \circ (\mathcal{F}_{N_S}^-)^{-1}.$$

Taking into account (2.11) and (2.14), $\mathcal{OP}_N(z_S; \Psi_1)^\top$ is given by

$$\begin{aligned} & (\mathcal{F}_{N_S}^+)^{-\top} \circ \left(-\overline{g_\infty} + \sum_{k=1}^N D^{-k} \circ a_k^-(z_S; \Psi_1) \right) + (\mathcal{F}_{N_S}^-)^{-\top} \circ \left(-g_\infty + \sum_{k=1}^N (-D)^{-k} \circ a_k^+(z_S; \Psi_1) \right) \\ &= \mathcal{F}_{N_S}^+ \circ \left(-\overline{g_\infty} + \sum_{k=1}^N D^{-k} \circ a_k^-(z_S; \Psi_1) \right) + \mathcal{F}_{N_S}^- \circ \left(-g_\infty + \sum_{k=1}^N (-D)^{-k} \circ a_k^+(z_S; \Psi_1) \right). \end{aligned}$$

By Lemma D.2, $\mathcal{F}_{N_S}^+ \circ D^{-k} \circ a_k^-(z_S; \Psi)$, $k \geq 1$, has an expansion of the form

$$\mathcal{F}_{N_S}^+ \circ \left(a_k^-(z_S; \Psi_1) D^{-k} + \sum_{j=1}^{N-k} C_j^+(k) \partial_x^j a_k^-(z_S; \Psi_1) D^{-k-j} \right) + \mathcal{R}_{N,k,0}^{\psi do}(a_k^-)$$

where $C_j^+(k)$, $j \geq 1$, are combinatorial constants and $\mathcal{R}_{N,k,0}^{\psi do}(a_k^-)$ is a remainder term, which can be obtained from Lemma D.2. Similarly, one gets for $\mathcal{F}_{N_S}^- \circ (-D)^{-k} \circ a_k^+(z_S; \Psi_1)$ the expansion

$$\mathcal{F}_{N_S}^- \circ \left(a_k^+(z_S; \Psi_1) (-D)^{-k} + \sum_{j=1}^{N-k} C_j^-(k) \partial_x^j a_k^+(z_S; \Psi_1) (-D)^{-k-j} \right) + \mathcal{R}_{N,k,0}^{\psi do}(a_k^+).$$

For any $k \geq 1$, $j \geq 1$, one has $C_j^-(k) = \overline{C_j^+(k)}$ and $\mathcal{R}_{N,k,0}^{\psi do}(a_k^-) = \overline{\mathcal{R}_{N,k,0}^{\psi do}(a_k^+)}$. In conclusion, $\mathcal{OP}_N(z_S; \Psi_1)^\top$ has an expansion of the form

$$\mathcal{F}_{N_S}^+ \circ \left(-\overline{g_\infty} + \sum_{k=1}^N a_k^+(z_S; \Psi_1^\top) D^{-k} \right) + \mathcal{F}_{N_S}^- \circ \left(-g_\infty + \sum_{k=1}^N a_k^-(z_S; \Psi_1^\top) (-D)^{-k} \right) + \mathcal{R}_N^{(1)}(z_S),$$

where the coefficients $a_k^\pm(z_S; \Psi_1^\top)$ have the claimed properties and where for any $s \geq 0$, the remainder $\mathcal{V}_S \rightarrow \mathcal{B}(H^s, h_\perp^{s+N+1})$, $z_S \mapsto \mathcal{R}_N^{(1)}(z_S)$, is real analytic. By Corollary 2.2, $\mathcal{R}_N(z_S; \Psi_1)^\top[\hat{q}]$, $\hat{q} \in H^s$, equals

$$\left(\left(\frac{1}{2\pi} \int_0^{2\pi} \hat{q}(x) W_{-n}(x, z_S) dx \right)_{|n| \in [1, N_S] \setminus S_+}, \left(\frac{1}{|n|^{N+1}} \frac{1}{2\pi} \int_0^{2\pi} \hat{q}(x) \mathcal{R}_N^{W-n}(x, z_S) e^{inx} dx \right)_{|n| > N_S} \right) \in h_\perp^{s+N+1},$$

- 1 implying that for any $s \geq 0$, $\mathcal{R}_N(z_S; \Psi_1)^\top : H^s \rightarrow h_\perp^{s+N+1}$ is bounded, and the map $\mathcal{V}_S \rightarrow \mathcal{B}(H^s, h_\perp^{s+N+1})$,
- 2 $z_S \mapsto \mathcal{R}_N(z_S; \Psi_1)^\top$, is real analytic. Setting $\mathcal{R}_N(z_S; \Psi_1^\top) := \mathcal{R}_N^{(1)}(z_S) + \mathcal{R}_N(z_S; \Psi_1)^\top$, one infers that
- 3 $\Psi_1(z_S)^\top$ admits an expansion of the form $\mathcal{OP}_N(z_S; \Psi_1^\top) + \mathcal{R}_N(z_S; \Psi_1^\top)$ with the claimed properties.

Since $d\Psi_L(z)[\hat{z}]$ is given by the formula

$$d\Psi_L(z)[\hat{z}] = (d_S \Psi^{bo}(z_S, 0) + d_S(\Psi_1(z_S)[z_\perp]))[\hat{z}_S] + \Psi_1(z_S)[\hat{z}_\perp],$$

- 4 one verifies in a straightforward way the claimed expansion of $d\Psi_L(z)^\top$ from the one of $\Psi_1(z_S)^\top$. \square

5 Next we discuss the properties of the functions $W_n(\cdot, q)$, $n \neq 0$, and the map $\Psi_L(z)$ with regard to the
6 reversible structures, defined by the maps S_{rev} and \mathcal{S}_{rev} , introduced in Section 1. For notational convenience,
7 we also denote by \mathcal{S}_{rev} the involution $h_0^0 \rightarrow h_0^0$, $(z_n)_{n \neq 0} \mapsto (z_{-n})_{n \neq 0}$, (cf. (1.24)), and sometimes write q_*
8 for $\mathcal{S}_{rev} q(x) = q(-x)$, $q \in L^2$. By Lemma A.2, for any $q \in L^2$,

$$\lambda_n(q_*) = \lambda_n(q), \quad f_n(x, q_*) = \overline{f_n(-x, q)}, \quad \forall n \geq 0, \quad g_\infty(x, q_*) = \overline{g_\infty(-x, q)}, \quad (2.17)$$

9 and

$$\gamma_n(q_*) = \gamma_n(q), \quad \kappa_n(q_*) = \kappa_n(q), \quad \forall n \geq 1. \quad (2.18)$$

10 As a consequence, the manifolds M_S and M_S^g are left invariant by \mathcal{S}_{rev} . Without loss of generality, we will
11 assume in the sequel that the neighborhood \mathcal{V} is invariant under \mathcal{S}_{rev} .

Addendum to Theorem 2.1 (i) For any $z_S \in \mathcal{V}_S$, one has

$$S_{rev}\Psi^{bo}(z_S, 0) = \Psi^{bo}(S_{rev}(z_S, 0)), \quad W_n(x, S_{rev}q) = W_{-n}(-x, q), \quad \forall n \neq 0, x \in \mathbb{R},$$

1 where $q = \Psi^{bo}(z_S, 0)$. Furthermore, for any $k \geq 1$

$$W_k^{ae,+}(x, S_{rev}z_S) = W_k^{ae,-}(-x, z_S), \quad W_k^{ae,-}(x, S_{rev}z_S) = W_k^{ae,+}(-x, z_S), \quad (2.19)$$

2 and any $N \geq 1$ and $|n| > N_S$

$$\mathcal{R}_N^{W_n}(x; S_{rev}z_S) = \mathcal{R}_N^{W_{-n}}(-x; z_S). \quad (2.20)$$

(ii) For any $z = (z_S, z_\perp) \in \mathcal{V}$, $x \in \mathbb{R}$, and $k \geq 1$,

$$(\Psi_1(S_{rev}z_S)[S_{rev}z_\perp])(x) = (\Psi_1(z_S)[z_\perp])(-x),$$

$$a_k^+(S_{rev}z_S; \Psi_1)(x) = a_k^-(z_S; \Psi_1)(-x), \quad a_k^-(S_{rev}z_S; \Psi_1)(x) = a_k^+(z_S; \Psi_1)(-x)$$

3 As a consequence, for any $z \in \mathcal{V}$ and $N \geq 1$, one has $\mathcal{R}_N(S_{rev}z; \Psi_1)(x) = \mathcal{R}_N(z; \Psi_1)(-x)$ and in turn

$$(\Psi_L(S_{rev}z))(x) = (\Psi_L(z))(-x), \quad a_k^\pm(S_{rev}z_S; \Psi_L)(x) = a_k^\mp(z_S; \Psi_L)(-x), \quad \forall k \geq 1, \quad (2.21)$$

as well as $\mathcal{R}_N(S_{rev}z; \Psi_L)(x) = \mathcal{R}_N(z; \Psi_L)(-x)$.

(iii) For any $z_S \in h_S$ and $\hat{q} \in L_0^2$, one has $\Psi_1(S_{rev}z_S)^\top [S_{rev}\hat{q}] = S_{rev}(\Psi_1(z_S)^\top [\hat{q}])$. Furthermore, for any $k \geq 1$ and $N \geq 1$,

$$a_k^+(S_{rev}z_S; \Psi_1^\top)(x) = a_k^-(z_S; \Psi_1^\top)(-x), \quad a_k^-(S_{rev}z_S; \Psi_1^\top)(x) = a_k^+(z_S; \Psi_1^\top)(-x),$$

and

$$\mathcal{R}_N(S_{rev}z_S; \Psi_1^\top)[S_{rev}\hat{q}] = S_{rev}(\mathcal{R}_N(z_S; \Psi_1^\top)[\hat{q}]).$$

4

Proof of Addendum to Theorem 2.1 (i) By Proposition C.1, it follows that for any $z \in \mathcal{V}$,

$$S_{rev}\Psi^{bo}(z) = \Psi^{bo}(S_{rev}(z)).$$

5 Hence, for any $z_S \in \mathcal{V}_S$ and $q = \Psi^{bo}(z_S, 0)$, one has for any $n \neq 0$ (cf. definition (2.5)),

$$W_n(x, S_{rev}q) = W_{-n}(-x, q) = \overline{W_n(-x, q)}. \quad (2.22)$$

6 The identities (2.19) follow from (2.17) - (2.18) and the definition of $W_k^{ae,\pm}$ in Theorem 2.1. Combining
7 (2.19) and (2.22), one infers (2.20).

8 (ii) By item (i) one has for any $z_S \in \mathcal{V}_S$ and $q = \Psi^{bo}(z_S, 0)$,

$$(\Psi_1(S_{rev}z_S)[S_{rev}z_\perp])(x) = \sum_{n \in S^\perp} z_{-n} W_n(x, S_{rev}q) = \sum_{n \in S^\perp} z_{-n} W_{-n}(-x, q) = (\Psi_1(z_S)[z_\perp])(-x) \quad (2.23)$$

9 as well as $W_k^{ae,+}(x, S_{rev}z_S) = W_k^{ae,-}(-x, z_S)$ for any $k \geq 1$. Hence by the definition (2.11),

$$a_k^+(S_{rev}z_S; \Psi_1)(x) = a_k^-(z_S; \Psi_1)(-x), \quad a_k^-(S_{rev}z_S; \Psi_1)(x) = a_k^+(z_S; \Psi_1)(-x). \quad (2.24)$$

10 Combining (2.23) and (2.24) then yields $(\mathcal{R}_N(S_{rev}z_S; \Psi_1)[S_{rev}z_\perp])(x) = (\mathcal{R}_N(z_S; \Psi_1)[z_\perp])(-x)$. By (2.3),
11 the claimed identities (2.21) then follow.

12 (iii) Recall that for any $z_S \in h_S$, $\hat{q} \in L_0^2$, one has $\Psi_1(z_S)^\top [\hat{q}] = (\langle W_{-n}(\cdot, q), \hat{q} \rangle)_{n \in S^\perp}$. It then follows from
13 item (i) that

$$\Psi_1(S_{rev}z_S)^\top [S_{rev}\hat{q}] = S_{rev}(\Psi_1(z_S)^\top [\hat{q}]). \quad (2.25)$$

Taking into account the definition (2.11) of the coefficients $a_k^\pm(z_S; \Psi_1)$ in the expansion for $\Psi_1(z_S)$, the construction of the coefficients $a_k^\pm(z_S; \Psi_1^\top)$ in the expansion of $\Psi_1(z_S)^\top$ in the proof of Corollary 2.4 (cf. also Lemma D.2), and the identities $S_{rev} \circ \mathcal{F}_{N_S}^+ = \mathcal{F}_{N_S}^- \circ S_{rev}$, $S_{rev} \circ \mathcal{F}_{N_S}^- = \mathcal{F}_{N_S}^+ \circ S_{rev}$ one infers that

$$a_k^+(S_{rev}z_S; \Psi_1^\top)(x) = a_k^-(z_S; \Psi_1^\top)(-x), \quad a_k^-(S_{rev}z_S; \Psi_1^\top)(x) = a_k^+(z_S; \Psi_1^\top)(-x), \quad \forall k \geq 1.$$

1 When combined with (2.25), one then arrives at $\mathcal{R}_N(\mathcal{S}_{rev} z_S; \Psi_1^\top)[\mathcal{S}_{rev} \hat{q}] = \mathcal{S}_{rev}(\mathcal{R}_N(z_S; \Psi_1^\top)[\hat{q}])$. \square

In the remaining part of this section we describe the pull back $\Psi_L^* \Lambda_G$ of the symplectic form Λ_G by the map Ψ_L , defined in Proposition 2.1. The symplectic form Λ_G is the one defined by the Gardner Poisson structure and is given by

$$\Lambda_G[\hat{u}, \hat{v}] = \langle \hat{u}, \partial_x^{-1} \hat{v} \rangle = \frac{1}{2\pi} \int_0^{2\pi} \hat{u}(x) \partial_x^{-1} \hat{v}(x) dx, \quad \forall \hat{u}, \hat{v} \in L_0^2.$$

2 Note that $\Lambda_G = d\lambda_G$ where the one form λ_G , defined on L_0^2 , is given by

$$\lambda_G(u)[\hat{v}] = \langle u, \partial_x^{-1} \hat{v} \rangle = \frac{1}{2\pi} \int_0^{2\pi} u(x) \partial_x^{-1} \hat{v}(x) dx, \quad \forall u, \hat{v} \in L_0^2. \quad (2.26)$$

To compute the pull back of Λ_G by Ψ_L , note that for any $z = (z_S, z_\perp) \in \mathcal{V} = \mathcal{V}_S \times \mathcal{V}_\perp$, the derivative $d\Psi_L(z) : h_S \times h_\perp^0 \rightarrow L_0^2$, when written in 1×2 matrix form, is given by (cf (2.6))

$$d\Psi_L(z) = d\Psi_L(z_S, 0) + \begin{pmatrix} d_S(\Psi_1(z_S)[z_\perp]) & 0 \end{pmatrix} = \begin{pmatrix} d_S \Psi^{bo}(z_S, 0) & \Psi_1(z_S) \end{pmatrix} + \begin{pmatrix} d_S(\Psi_1(z_S)[z_\perp]) & 0 \end{pmatrix}. \quad (2.27)$$

For any $\hat{z} = (\hat{z}_S, \hat{z}_\perp)$, $\hat{w} = (\hat{w}_S, \hat{w}_\perp) \in h_S \times h_\perp^0$ one has

$$\begin{aligned} (\Psi_L^* \Lambda_G)(z)[\hat{z}, \hat{w}] &= \Lambda_G[d\Psi_L(z)[\hat{z}], d\Psi_L(z)[\hat{w}]] = \langle d\Psi_L(z)[\hat{z}], \partial_x^{-1} d\Psi_L(z)[\hat{w}] \rangle \\ &= \langle d\Psi_L(z_S, 0)[\hat{z}] + d_S(\Psi_1(z_S)[z_\perp])[\hat{z}_S], \partial_x^{-1}(d\Psi_L(z_S, 0)[\hat{w}] + \partial_x^{-1}(d_S(\Psi_1(z_S)[z_\perp])[\hat{w}_S])) \rangle. \end{aligned}$$

Since by construction, $d\Psi_L(z_S, 0) : h_0^0 \rightarrow L_0^2$ is symplectic, one has

$$(\Psi_L^* \Lambda_G)(z_S, 0) = \Lambda$$

3 where Λ is the symplectic form on h_0^0 ,

$$\Lambda[\hat{z}, \hat{w}] := \langle \hat{z}, J^{-1} \hat{w} \rangle = \sum_{n \neq 0} \frac{1}{i(-n)} \hat{z}_n \hat{w}_{-n}, \quad \forall \hat{z}, \hat{w} \in h_0^0, \quad (2.28)$$

4 and J^{-1} denotes the inverse of the diagonal operator J , acting on the scale of Hilbert spaces h_0^s , $s \in \mathbb{R}$,

$$J : h_0^{s+1} \rightarrow h_0^s : (z_n)_{n \neq 0} \mapsto (in z_n)_{n \neq 0}. \quad (2.29)$$

Note that the Poisson bracket, associated to J , is defined for functionals $F, G : h_0^0 \rightarrow \mathbb{R}$ with sufficiently decaying gradients,

$$\{F, G\}(z) = \langle J \nabla F, \nabla G \rangle(z) = \sum_{n \neq 0} in (\nabla F)_n (\nabla G)_{-n} = \sum_{n \neq 0} in \partial_{z_{-n}} F \cdot \partial_{z_n} G.$$

5 (Here we used that for any $n \neq 0$, $(\nabla F)_n = \partial_{z_{-n}} F$.) We remark that $\langle J \hat{z}, \hat{w} \rangle = \langle J \hat{z} | \hat{w} \rangle$.

6 The two form Λ is exact, $\Lambda = d\lambda$, where λ is the following one form on h_0^0 ,

$$\lambda(z)[\hat{w}] := \langle z, J^{-1} \hat{w} \rangle = \sum_{n \neq 0} \frac{1}{-in} z_n \hat{w}_{-n}, \quad \forall z, \hat{w} \in h_0^0. \quad (2.30)$$

7 Altogether one concludes that

$$(\Psi_L^* \Lambda_G)(z)[\hat{z}, \hat{w}] = \Lambda[\hat{z}, \hat{w}] + \Lambda_L(z)[\hat{z}, \hat{w}] \quad (2.31)$$

where

$$\begin{aligned} \Lambda_L(z)[\hat{z}, \hat{w}] &= \langle d_S(\Psi_1(z_S)[z_\perp])[\hat{z}_S], \partial_x^{-1}(d_S(\Psi_1(z_S)[z_\perp])[\hat{w}_S]) \rangle \\ &\quad + \langle d\Psi_L(z_S, 0)[\hat{z}], \partial_x^{-1}(d_S(\Psi_1(z_S)[z_\perp])[\hat{w}_S]) \rangle + \langle d_S(\Psi_1(z_S)[z_\perp])[\hat{z}_S], \partial_x^{-1}(d\Psi_L(z_S, 0)[\hat{w}]) \rangle. \end{aligned}$$

Using the standard inner products (cf. (1.19)) and the definition $\Psi_L(z_S, z_\perp) = \Psi^{bo}(z_S, 0) + \Psi_1(z_S)[z_\perp]$, $\Lambda_L(z)[\widehat{z}, \widehat{w}]$ can be written in the form

$$\Lambda_L(z)[\widehat{z}, \widehat{w}] = \langle \widehat{z} | \mathcal{L}(z)[\widehat{w}] \rangle, \quad (2.32)$$

where $\mathcal{L}(z) : h_S \times h_\perp^0 \rightarrow h_S \times h_\perp^0$ is given by

$$\mathcal{L}(z) = \begin{pmatrix} \mathcal{L}_S^S(z) & \mathcal{L}_S^\perp(z) \\ \mathcal{L}_\perp^S(z) & 0 \end{pmatrix} \quad (2.33)$$

with $\mathcal{L}_S^S(z) : h_S \rightarrow h_S$, $\mathcal{L}_S^\perp(z) : h_\perp^0 \rightarrow h_S$, and $\mathcal{L}_\perp^S : h_S \rightarrow h_\perp^0$ given by

$$\begin{aligned} \mathcal{L}_S^S(z) &:= (d_S \Psi_1(z_S)[z_\perp])^\top \circ \partial_x^{-1} (d_S \Psi_1(z_S)[z_\perp]) + (d_S \Psi^{bo}(z_S, 0))^\top \circ \partial_x^{-1} (d_S \Psi_1(z_S)[z_\perp]) \\ &\quad + (d_S \Psi_1(z_S)[z_\perp])^\top \circ \partial_x^{-1} d_S \Psi^{bo}(z_S, 0), \\ \mathcal{L}_S^\perp(z) &:= (d_S \Psi_1(z_S)[z_\perp])^\top \circ \partial_x^{-1} \Psi_1(z_S), \quad \mathcal{L}_\perp^S(z) := \Psi_1(z_S)^\top \circ \partial_x^{-1} (d_S \Psi_1(z_S)[z_\perp]), \end{aligned} \quad (2.34)$$

where we recall that $(\cdot)^\top$ denotes the transpose of an operator (such as the ones considered above) with respect to the standard inner products defined in (1.19). For any $z = (z_S, z_\perp) \in \mathcal{V}$, the operators $\mathcal{L}(z)$, $\mathcal{L}_S^S(z)$, $\mathcal{L}_S^\perp(z)$, and $\mathcal{L}_\perp^S(z)$ are bounded. In the sequel, we will often write the operators defined in (2.34) in the following way

$$\begin{aligned} \mathcal{L}_S^S(z)[\widehat{z}_S] &= (\langle \partial_x^{-1} d_S(\Psi_1(z_S)[z_\perp])[\widehat{z}_S] | \partial_{z_n} \Psi_1(z_S)[z_\perp] \rangle)_{n \in S} + (\langle \partial_x^{-1} d_S(\Psi_1(z_S)[z_\perp])[\widehat{z}_S] | \partial_{z_n} \Psi^{bo}(z_S, 0) \rangle)_{n \in S} \\ &\quad + (\langle \partial_x^{-1} d_S(\Psi^{bo}(z_S, 0))[\widehat{z}_S] | \partial_{z_n} \Psi_1(z_S)[z_\perp] \rangle)_{n \in S}, \\ \mathcal{L}_S^\perp(z)[\widehat{z}_\perp] &= (\langle \partial_x^{-1} \Psi_1(z_S)[\widehat{z}_\perp] | \partial_{z_n} \Psi_1(z_S)[z_\perp] \rangle)_{n \in S}, \\ \mathcal{L}_\perp^S(z)[\widehat{z}_S] &= \Psi_1(z_S)^\top \partial_x^{-1} d_S(\Psi_1(z_S)[z_\perp])[\widehat{z}_S] = (\langle \partial_x^{-1} d_S(\Psi_1(z_S)[z_\perp])[\widehat{z}_S] | W_n(\cdot, q) \rangle)_{n \in S^\perp}, \end{aligned} \quad (2.35)$$

where $q = \Psi^{bo}(z_S, 0)$. It follows from (2.31) that $\mathcal{L}(z)$ is skew-adjoint, $\mathcal{L}(z)^\top = -\mathcal{L}(z)$. As a consequence, one has

$$\begin{pmatrix} \mathcal{L}_S^S(z)^\top & \mathcal{L}_\perp^S(z)^\top \\ \mathcal{L}_S^\perp(z)^\top & 0 \end{pmatrix} = \begin{pmatrix} -\mathcal{L}_S^S(z) & -\mathcal{L}_S^\perp(z) \\ -\mathcal{L}_\perp^S(z) & 0 \end{pmatrix}. \quad (2.36)$$

The operators $\mathcal{L}_S^S(z)$, $\mathcal{L}_S^\perp(z)$, and $\mathcal{L}_\perp^S(z)$ satisfy the following properties.

Lemma 2.2. (i) *The maps*

$$\mathcal{V} \rightarrow \mathcal{B}(h_S, h_S), z \mapsto \mathcal{L}_S^S(z), \quad \mathcal{V} \rightarrow \mathcal{B}(h_\perp^0, h_S), z \mapsto \mathcal{L}_S^\perp(z),$$

are real analytic. Furthermore, the following estimates hold: for any $z = (z_S, z_\perp) \in \mathcal{V}$, $\widehat{z}_S \in h_S$, and $\widehat{z}_1, \dots, \widehat{z}_l \in h_\perp^0$, $l \geq 1$,

$$\|\mathcal{L}_S^S(z)[\widehat{z}_S]\| \lesssim \|\widehat{z}_S\| \|z_\perp\|_0, \quad \|d^l(\mathcal{L}_S^S(z)[\widehat{z}_S])[\widehat{z}_1, \dots, \widehat{z}_l]\| \lesssim_l \|\widehat{z}_S\| \prod_{j=1}^l \|\widehat{z}_j\|_0.$$

If in addition, $\widehat{z}_\perp \in h_\perp^0$,

$$\|\mathcal{L}_S^\perp(z)[\widehat{z}_\perp]\| \lesssim \|z_\perp\|_0 \|\widehat{z}_\perp\|_0, \quad \|d^l(\mathcal{L}_S^\perp(z)[\widehat{z}_\perp])[\widehat{z}_1, \dots, \widehat{z}_l]\| \lesssim_l \|z_\perp\|_0 \prod_{j=1}^l \|\widehat{z}_j\|_0.$$

(ii) For any $z = (z_S, z_\perp) \in \mathcal{V}$, $\widehat{z}_S \in h_S$, and arbitrary order $N \geq 1$,

$$\mathcal{L}_\perp^S(z)[\widehat{z}_S] = \mathcal{OP}_N(z_S; \mathcal{L}_\perp^S)[\widehat{z}_S] + \mathcal{RN}(z; \mathcal{L}_\perp^S)[\widehat{z}_S]$$

1 with $\mathcal{OP}_N(z_S; \mathcal{L}_\perp^S)[\widehat{z}_S]$ given by

$$\mathcal{F}_{N_S}^+ \circ \sum_{k=1}^N \mathcal{A}_k^+(z_S; \mathcal{L}_\perp^S)[\widehat{z}_S] \cdot D^{-k}[(\mathcal{F}_{N_S}^+)^{-1} z_\perp] + \mathcal{F}_{N_S}^- \circ \sum_{k=1}^N \mathcal{A}_k^-(z_S; \mathcal{L}_\perp^S)[\widehat{z}_S] \cdot (-D)^{-k}[(\mathcal{F}_{N_S}^-)^{-1} z_\perp], \quad (2.37)$$

where for any $s \geq 0$, $k \geq 1$, the maps

$$\mathcal{V}_S \mapsto \mathcal{B}(h_S, H_\mathbb{C}^s), \quad z_S \mapsto \mathcal{A}_k^\pm(z_S; \mathcal{L}_\perp^S), \quad \mathcal{V} \cap h_0^s \rightarrow \mathcal{B}(h_S, h_\perp^{s+N+1}), \quad z \mapsto \mathcal{R}_N(z; \mathcal{L}_\perp^S),$$

are real analytic and $\mathcal{A}_k^-(z_S; \mathcal{L}_\perp^S)[\widehat{z}_S] = \overline{\mathcal{A}_k^+(z_S; \mathcal{L}_\perp^S)[\widehat{z}_S]}$. For any $\widehat{z}_S \in h_S$, $\mathcal{A}_1^\pm(z_S; \mathcal{L}_\perp^S)[\widehat{z}_S]$ are given by

$$\mathcal{A}_1^+(z_S; \mathcal{L}_\perp^S)[\widehat{z}_S] = -i \overline{g_\infty} \cdot d_S g_\infty[\widehat{z}_S], \quad \mathcal{A}_1^-(z_S; \mathcal{L}_\perp^S)[\widehat{z}_S] = i g_\infty \cdot d_S \overline{g_\infty}[\widehat{z}_S].$$

In particular, the operator $\mathcal{L}_\perp^S(z)$ is one smoothing. More precisely, for any $s \geq 0$,

$$\mathcal{V} \cap h_0^s \rightarrow \mathcal{B}(h_S, h_\perp^{s+1}), \quad z \mapsto \mathcal{L}_\perp^S(z)$$

is real analytic. The coefficients $\mathcal{A}_k^\pm(z_S; \mathcal{L}_\perp^S)$ are independent of z_\perp and satisfy for any $s \geq 0$, $z_S \in \mathcal{V}_S$, $\widehat{z}_S \in h_S$, $\widehat{z}_1, \dots, \widehat{z}_l \in h_0^0$, $l \geq 1$, the following estimates

$$\|\mathcal{A}_k^\pm(z_S; \mathcal{L}_\perp^S)[\widehat{z}_S]\|_s \lesssim_{s,k} \|\widehat{z}_S\|, \quad \|d^l(\mathcal{A}_k^\pm(z_S; \mathcal{L}_\perp^S)[\widehat{z}_S])[\widehat{z}_1, \dots, \widehat{z}_l]\|_s \lesssim_{s,k,l} \|\widehat{z}_S\| \prod_{j=1}^l \|\widehat{z}_j\|_0.$$

Furthermore, for any $s \in \mathbb{Z}_{\geq 0}$, $z = (z_S, z_\perp) \in \mathcal{V} \cap h_0^s$, $\widehat{z}_S \in h_S$, and $\widehat{z}_1, \dots, \widehat{z}_l \in h_0^s$, $l \geq 1$, $\mathcal{R}_N(z; \mathcal{L}_\perp^S)[\widehat{z}_S]$ satisfies $\|\mathcal{R}_N(z; \mathcal{L}_\perp^S)[\widehat{z}_S]\|_{s+N+1} \lesssim_{s,N} \|\widehat{z}_S\| \|z_\perp\|_s$ and

$$\|d^l(\mathcal{R}_N(z; \mathcal{L}_\perp^S)[\widehat{z}_S])[\widehat{z}_1, \dots, \widehat{z}_l]\|_{s+N+1} \lesssim_{s,N,l} \|\widehat{z}_S\| \cdot \left(\sum_{j=1}^l \|\widehat{z}_j\|_s \prod_{i \neq j} \|\widehat{z}_i\|_0 + \|z_\perp\|_s \prod_{j=1}^l \|\widehat{z}_j\|_0 \right).$$

- 2 (iii) As a consequence, for any integer $s \geq 0$, the map $\mathcal{V} \cap h_0^s \rightarrow \mathcal{B}(h_0^0, h_0^{s+1}), z \mapsto \mathcal{L}(z)$ is real analytic.
 3 Furthermore, for any $z = (z_S, z_\perp) \in \mathcal{V} \cap h_0^s$ and $\widehat{z} \in h_0^0$, it satisfies the estimates

$$\|\mathcal{L}(z)[\widehat{z}]\|_{s+1} \leq C(s; \mathcal{L}) \|\widehat{z}\|_0 \|z_\perp\|_s \quad (2.38)$$

- 4 and if in addition $\widehat{z}_1, \dots, \widehat{z}_l \in h_0^s$, $l \geq 1$, one has

$$\|d^l(\mathcal{L}(z)[\widehat{z}])[\widehat{z}_1, \dots, \widehat{z}_l]\|_{s+1} \leq C(s, l; \mathcal{L}) \cdot \|\widehat{z}\|_0 \cdot \left(\sum_{j=1}^l \|\widehat{z}_j\|_s \prod_{i \neq j} \|\widehat{z}_i\|_0 + \|z_\perp\|_s \prod_{j=1}^l \|\widehat{z}_j\|_0 \right) \quad (2.39)$$

- 5 for some constants $C(s; \mathcal{L}) \geq 1$, $C(s, l; \mathcal{L}) \geq 1$.

Remark 2.4. (i) The coefficients $\mathcal{A}_1^\pm(z_S; \mathcal{L}_\perp^S)$ can be computed more explicitly. First note that for any $n \geq 1$, $\partial_x^{-1} e^{inx} = -i D^{-1} e^{inx}$ and $\partial_x^{-1} e^{-inx} = i(-D)^{-1} e^{-inx}$ and that by definition, $g_\infty = e^{i\partial_x^{-1} q}$ with $q = \Psi^{bo}(z_S, 0)$ real valued. Hence for any $n \geq 1$,

$$\overline{g_\infty} \cdot d_S g_\infty[\widehat{z}_S] \cdot \partial_x^{-1} e^{inx} = \overline{g_\infty} g_\infty i \partial_x^{-1} (d_S \Psi^{bo}(z_S, 0)[\widehat{z}_S]) \cdot (-i) D^{-1} e^{inx} = \partial_x^{-1} (d_S \Psi^{bo}(z_S, 0)[\widehat{z}_S]) \cdot D^{-1} e^{inx},$$

yielding

$$\mathcal{A}_1^+(z_S; \mathcal{L}_\perp^S)[\widehat{z}_S] = \partial_x^{-1} (d_S \Psi^{bo}(z_S, 0)[\widehat{z}_S]), \quad \mathcal{A}_1^-(z_S; \mathcal{L}_\perp^S)[\widehat{z}_S] = \partial_x^{-1} (d_S \Psi^{bo}(z_S, 0)[\widehat{z}_S]).$$

- 6 Hence $\mathcal{A}_1^+(z_S; \mathcal{L}_\perp^S)[\widehat{z}_S] = \mathcal{A}_1^-(z_S; \mathcal{L}_\perp^S)[\widehat{z}_S]$ and $\mathcal{A}_1^\pm(z_S; \mathcal{L}_\perp^S)[\widehat{z}_S]$ are real valued.

- 7 (ii) Recall that by Remark 2.1(i), $\partial_x^{-1} \Psi_1(z_S) : h_\perp^{-1} \rightarrow H_0^0$ is a bounded linear operator for any $z \in \mathcal{V}$. Since
 8 $\partial_{z_n} \Psi_1(z_S)[z_\perp] \in H_0^0$ for any $n \in S$, it then follows that $\mathcal{L}_S^\perp(z) : h_\perp^{-1} \rightarrow h_S$ and in turn $\mathcal{L}(z) : h_0^{-1} \rightarrow h_0^0$ are
 9 bounded linear operators. Estimates, corresponding to the ones for $\mathcal{L}_S^\perp(z)$ and $\mathcal{L}(z)$ of Lemma 2.2, continue
 10 to hold, when these operators are extended to h_\perp^{-1} and, respectively, h_0^{-1} .

Proof. Item (i) is verified in a straightforward way and item (iii) is a direct consequence of item (i) and (ii). It remains to verify the statements of item (ii). Recall that by definition,

$$\mathcal{L}_\perp^S(z)[\widehat{z}_S] = \Psi_1(z_S)^\top \partial_x^{-1} d_S(\Psi_1(z_S)[z_\perp])[\widehat{z}_S].$$

By Corollary 2.4, $\Psi_1(z_S)^\top$ has the expansion $\mathcal{OP}_N(z_S; \Psi_1^\top) + \mathcal{R}_N(z_S; \Psi_1^\top)$ with

$$\mathcal{OP}_N(z_S; \Psi_1^\top) := \mathcal{F}_{N_S}^+ \circ \left(-\overline{g_\infty} + \sum_{k=1}^N a_k^+(z_S; \Psi_1^\top) D^{-k} \right) + \mathcal{F}_{N_S}^- \circ \left(g_\infty + \sum_{k=1}^N a_k^-(z_S; \Psi_1^\top) (-D)^{-k} \right), \quad (2.40)$$

where for any $k \geq 1$, $a_k^-(z_S; \Psi_1^\top) = \overline{a_k^+(z_S; \Psi_1^\top)}$ and for any $s \geq 0$, $k \geq 1$,

$$\mathcal{V}_S \rightarrow H_{\mathbb{C}}^s, \quad z_S \mapsto a_k^\pm(z_S; \Psi_1^\top), \quad \mathcal{V}_S \rightarrow \mathcal{B}(H^s, h_\perp^{s+N+1}), \quad z_S \mapsto \mathcal{R}_N(z_S; \Psi_1^\top),$$

are real analytic. By Corollary 2.3, $d_S(\Psi_1(z_S)[z_\perp])[\widehat{z}_S]$ admits the expansion

$$\begin{aligned} & -d_S g_\infty[\widehat{z}_S] \cdot (\mathcal{F}_{N_S}^+)^{-1} z_\perp + \sum_{k=1}^N \mathcal{A}_k^+(z_S; d\Psi_1)[\widehat{z}_S] \cdot D^{-k} [(\mathcal{F}_{N_S}^+)^{-1} z_\perp] \\ & -d_S \overline{g_\infty}[\widehat{z}_S] \cdot (\mathcal{F}_{N_S}^-)^{-1} z_\perp + \sum_{k=1}^N \mathcal{A}_k^-(z_S; d\Psi_1)[\widehat{z}_S] \cdot (-D)^{-k} [(\mathcal{F}_{N_S}^+)^{-1} z_\perp], \end{aligned}$$

where for any $k \geq 1$, $\mathcal{A}_k^\pm(z_S; d\Psi_1)$ are the following linear operators,

$$\mathcal{A}_k^\pm(z_S; d\Psi_1) : h_S \rightarrow L_{\mathbb{C}}^2, \quad \widehat{z}_S \mapsto d_S a_k^\pm(z_S; \Psi_1)[\widehat{z}_S].$$

- 1 One has It then follows from the expansion of the composition $\partial_x^{-n} \circ a \partial_x^{-\ell}$ (cf. Lemma D.2) and the
2 smoothing properties of Hankel operators (cf. Corollary D.1) that $\mathcal{L}_\perp^S(z)[\widehat{z}_S]$ admits an expansion with the
3 stated properties. \square

Finally, we discuss the properties of the symplectic forms Λ_G , Λ , and $\Psi_L^* \Lambda_G$ with respect to the reversible structures introduced in Section 1. First note that for any $\widehat{u}, \widehat{v} \in L_0^2$,

$$(\mathcal{S}_{rev}^* \Lambda_G)[\widehat{u}, \widehat{v}] = \Lambda_G[\mathcal{S}_{rev} \widehat{u}, \mathcal{S}_{rev} \widehat{v}] = \frac{1}{2\pi} \int_0^{2\pi} \widehat{u}(-x) \partial_x^{-1} (\widehat{v}(-x)) dx = -\Lambda_G[\widehat{u}, \widehat{v}]$$

and similarly, for any $\widehat{z}, \widehat{w} \in h_0^0$,

$$(\mathcal{S}_{rev}^* \Lambda)[\widehat{z}, \widehat{w}] = \Lambda[\mathcal{S}_{rev} \widehat{z}, \mathcal{S}_{rev} \widehat{w}] = \sum_{n \neq 0} \frac{1}{-in} \widehat{z}_{-n} \widehat{w}_n = -\Lambda[\widehat{z}, \widehat{w}].$$

Since by (2.21) $\mathcal{S}_{rev} \Psi_L = \Psi_L \mathcal{S}_{rev}$, the pullback $\mathcal{S}_{rev}^* \Psi_L^* \Lambda_G$ can then be computed as

$$(\mathcal{S}_{rev}^* \Psi_L^*) \Lambda_G = \Psi_L^* (\mathcal{S}_{rev}^* \Lambda_G) = -\Psi_L^* \Lambda_G.$$

By (2.31) it then follows that the two form Λ_L , introduced in (2.31), satisfies

$$\mathcal{S}_{rev}^* \Lambda_L = -\Lambda_L.$$

- 4 In view of the definitions (2.32) - (2.33) one then concludes that the operators $\mathcal{L}_S^S(z)$, $\mathcal{L}_S^\perp(z)$, and $\mathcal{L}_\perp^S(z)$
5 have the following symmetry properties.

Addendum to Lemma 2.2 For any $z = (z_S, z_\perp) \in h_S \times h_\perp^0$ and any $\widehat{z}_S \in h_S$, $\widehat{z}_\perp \in h_\perp^0$,

$$\mathcal{L}_S^S(\mathcal{S}_{rev} z)[\mathcal{S}_{rev} \widehat{z}_S] = -\mathcal{S}_{rev} (\mathcal{L}_S^S(z)[\widehat{z}_S]),$$

$$\mathcal{L}_S^\perp(\mathcal{S}_{rev} z)[\mathcal{S}_{rev} \widehat{z}_\perp] = -\mathcal{S}_{rev} (\mathcal{L}_S^\perp(z)[\widehat{z}_\perp]), \quad \mathcal{L}_\perp^S(\mathcal{S}_{rev} z)[\mathcal{S}_{rev} \widehat{z}_S] = -\mathcal{S}_{rev} (\mathcal{L}_\perp^S(z)[\widehat{z}_S]).$$

By (2.37) it then follows that

$$\begin{aligned} \mathcal{A}_k^+(\mathcal{S}_{rev} z; \mathcal{L}_\perp^S)[\mathcal{S}_{rev} \widehat{z}_S](x) &= -\mathcal{A}_k^-(z_S; \mathcal{L}_\perp^S)[\widehat{z}_S](-x), \\ \mathcal{A}_k^-(\mathcal{S}_{rev} z; \mathcal{L}_\perp^S)[\mathcal{S}_{rev} \widehat{z}_S](x) &= -\mathcal{A}_k^+(z_S; \mathcal{L}_\perp^S)[\widehat{z}_S](-x), \\ \mathcal{R}_N(\mathcal{S}_{rev} z; \mathcal{L}_\perp^S)[\mathcal{S}_{rev} \widehat{z}_S] &= -\mathcal{S}_{rev} (\mathcal{R}_N(z; \mathcal{L}_\perp^S)[\widehat{z}_S]). \end{aligned}$$

3 The map Ψ_C

In this section we construct the symplectic corrector Ψ_C . Our approach is based on a well known method of Moser and Weinstein, implemented for an infinite dimensional setup in [18] (cf. also [11]). We begin by briefly outlining the construction. At the end of Section 2, we introduce the symplectic forms Λ and $\Psi_L^* \Lambda_G$ (cf. (2.32)). They are defined on $\mathcal{V} = \mathcal{V}_S \times \mathcal{V}_\perp$ and are related as follows ($z \in \mathcal{V}$, $\widehat{z}, \widehat{w} \in h_0^0$),

$$\Psi_L^* \Lambda_G(z)[\widehat{z}, \widehat{w}] = \Lambda[\widehat{z}, \widehat{w}] + \Lambda_L(z)[\widehat{z}, \widehat{w}], \quad \Lambda[\widehat{z}, \widehat{w}] = \langle \widehat{z} | J^{-1} \widehat{w} \rangle, \quad \Lambda_L(z)[\widehat{z}, \widehat{w}] = \langle \widehat{z} | \mathcal{L}(z)[\widehat{w}] \rangle, \quad (3.1)$$

where J^{-1} is the inverse of the diagonal operator J , defined by (2.29) and $\mathcal{L}(z)$ the operator defined by (2.33). Our candidate for Ψ_C is $\Psi_X^{0,1}$ where $X \equiv X(\tau, z)$ is a non-autonomous vector field, defined for $z \in \mathcal{V}$ and $0 \leq \tau \leq 1$, so that $(\Psi_X^{0,1})^*(\Psi_L^* \Lambda_G) = \Lambda$. The flow $\Psi_X^{\tau_0, \tau}$, corresponding to the vector field X , is required to be well defined on a neighborhood \mathcal{V}' (cf. Lemma 3.3 below) for $0 \leq \tau_0, \tau \leq 1$, and to satisfy the standard normalization conditions $\Psi_X^{\tau_0, \tau_0}(z) = z$ for any $z \in \mathcal{V}'$ and $0 \leq \tau_0 \leq 1$. To find X with the desired properties, introduce the one parameter family of two forms,

$$\Lambda_\tau(z) := \Lambda + \tau \Lambda_L(z), \quad 0 \leq \tau \leq 1.$$

Note that Λ_τ is a closed two form. Indeed, since Λ_G is such a two form, so is $\Psi_L^* \Lambda_G$ and in turn $\Lambda_L = \Psi_L^* \Lambda_G - \Lambda$. Furthermore, $\Lambda_0 = \Lambda$, $\Lambda_1 = \Psi_L^* \Lambda_G$, and $(\Psi_X^{0,0})^* \Lambda_0 = \Lambda_0$. The desired identity $(\Psi_X^{0,1})^* \Lambda_1 = \Lambda_0$ then follows if one can show that $(\Psi_X^{0,\tau})^* \Lambda_\tau$ is independent of τ , i.e., that $\partial_\tau((\Psi_X^{0,\tau})^* \Lambda_\tau) = 0$. Since $(\Psi_X^{0,\tau})^* \Lambda_\tau = (\Psi_X^{0,\tau})^* \Lambda + \tau (\Psi_X^{0,\tau})^* \Lambda_L$, it follows that

$$\partial_\tau((\Psi_X^{0,\tau})^* \Lambda_\tau) = (\Psi_X^{0,\tau})^*(\mathfrak{L}_X \Lambda_\tau + \Lambda_L),$$

where \mathfrak{L}_X denotes the Lie derivative with respect to the vector field X . Using that Λ_τ is a closed two form, one infers from Cartan's formula that

$$\mathfrak{L}_X \Lambda_\tau = d(\iota_X \Lambda_\tau) + \iota_X d\Lambda_\tau = d(\iota_X \Lambda_\tau),$$

where ι_X denotes the interior product of a form with the vector field X . Thus, the equation $\partial_\tau((\Psi_X^{0,\tau})^* \Lambda_\tau) = 0$ can be written as

$$(\Psi_X^{0,\tau})^*(\Lambda_L + d(\Lambda_\tau[X(\tau, \cdot), \cdot])) = 0.$$

Hence we need to choose the vector field $X(\tau, z)$ in such a way that

$$\Lambda_L(z) - d(\Lambda_\tau(z)[\cdot, X(\tau, z)]) = 0. \quad (3.2)$$

Note that

$$\Lambda_\tau(z)[\cdot, X(\tau, z)] = \langle \cdot | (J^{-1} + \tau \mathcal{L}(z))[X(\tau, z)] \rangle = \langle \cdot | J^{-1} \mathcal{L}_\tau(z)[X(\tau, z)] \rangle \quad (3.3)$$

where

$$\mathcal{L}_\tau(z) := \text{Id} + \tau J \mathcal{L}(z). \quad (3.4)$$

We remark that by Lemma 2.2, the operator $\mathcal{L}_\tau(z) : h_0^0 \rightarrow h_0^0$ is bounded for any $0 \leq \tau \leq 1$ and $z \in \mathcal{V}$.

In order to find a vector field X , satisfying (3.2), we rewrite the two form $\Lambda_L(z)$ as the differential of a suitably chosen one form. First note that since $\Lambda_G = d\lambda_G$ (cf. (2.26)) and $\Lambda = d\lambda$ (cf. (2.30)), one has

$$\Lambda_L = \Psi_L^* \Lambda_G - \Lambda = d(\lambda_1 - \lambda_0), \quad \lambda_1 := \Psi_L^* \lambda_G, \quad \lambda_0 := \lambda.$$

Furthermore, $\mathcal{L}(z_S, 0) = 0$ by (2.35) and hence $\Lambda_L(z_S, 0) = 0$ for any $z_S \in \mathcal{V}_S$. It then follows by the Poincaré Lemma (cf. e.g. [11, Appendix 1]) that $d\lambda_L = \Lambda_L$ where λ_L is the one form on \mathcal{V} , obtained by the cone construction,

$$\lambda_L(z)[\widehat{z}] := \int_0^1 \Lambda_L(z_S, t z_\perp)[(0, z_\perp), (\widehat{z}_S, t \widehat{z}_\perp)] dt = - \int_0^1 \Lambda_L(z_S, t z_\perp)[(\widehat{z}_S, t \widehat{z}_\perp), (0, z_\perp)] dt.$$

By the definition of the operator $\mathcal{L}(z)$ (cf. (3.1)), it then follows that

$$\lambda_L(z)[\widehat{z}] = - \int_0^1 \langle (\widehat{z}_S, t\widehat{z}_\perp) | \mathcal{L}(z_S, tz_\perp)[(0, z_\perp)] \rangle dt \stackrel{(2.33)}{=} - \int_0^1 \langle \widehat{z}_S | \mathcal{L}_S^\perp(z_S, tz_\perp)[z_\perp] \rangle dt.$$

Since $\mathcal{L}_S^\perp(z_S, tz_\perp) = t\mathcal{L}_S^\perp(z_S, z_\perp)$ (cf. (2.35)), the latter integral can be computed explicitly,

$$\lambda_L(z)[\widehat{z}] = -\langle \widehat{z} | \mathcal{E}(z) \rangle, \quad \mathcal{E}(z) := (\mathcal{E}_S(z), 0) \in h_S \times h_\perp^0, \quad z \in \mathcal{V}, \quad \widehat{z} \in h_0^0, \quad (3.5)$$

where

$$\mathcal{E}_S : \mathcal{V} \rightarrow h_S, \quad z \mapsto \mathcal{E}_S(z) := \frac{1}{2} \mathcal{L}_S^\perp(z)[z_\perp]. \quad (3.6)$$

Combining (3.3) - (3.5), equation (3.2) reads

$$d\langle \cdot | \mathcal{E}(z) + J^{-1}\mathcal{L}_\tau(z)[X(\tau, z)] \rangle = 0.$$

In view of the definition (3.4) of $\mathcal{L}_\tau(z)$, we now choose X to be a solution of

$$J\mathcal{E}(z) + (\text{Id} + \tau J\mathcal{L}(z))[X(\tau, z)] = 0, \quad \forall z \in \mathcal{V}, \quad 0 \leq \tau \leq 1. \quad (3.7)$$

To solve the latter equation for X , we need to invert $\text{Id} + \tau J\mathcal{L}(z)$. To this end define the standard projections

$$\Pi_S : h_S \times h_\perp^0 \rightarrow h_S \times h_\perp^0, (\widehat{z}_S, \widehat{z}_\perp) \mapsto (\widehat{z}_S, 0), \quad \Pi_\perp : h_S \times h_\perp^0 \rightarrow h_S \times h_\perp^0, (\widehat{z}_S, \widehat{z}_\perp) \mapsto (0, \widehat{z}_\perp), \quad (3.8)$$

the standard inclusions

$$\iota_S : h_S \rightarrow h_S \times h_\perp^0, \quad \widehat{z}_S \mapsto (\widehat{z}_S, 0), \quad \iota_\perp : h_\perp^0 \rightarrow h_S \times h_\perp^0, \quad \widehat{z}_\perp \mapsto (0, \widehat{z}_\perp), \quad (3.9)$$

and the maps

$$\pi_S : h_S \times h_\perp^0 \rightarrow h_S, \quad z = (z_S, z_\perp) \mapsto z_S, \quad \pi_\perp : h_S \times h_\perp^0 \rightarrow h_\perp^0, \quad z = (z_S, z_\perp) \mapsto z_\perp. \quad (3.10)$$

Furthermore, let

$$J_S := \pi_S J \iota_S, \quad J_\perp := \pi_\perp J \iota_\perp, \quad \text{Id}_S := \pi_S \text{Id} \iota_S, \quad \text{Id}_\perp := \pi_\perp \text{Id} \iota_\perp. \quad (3.11)$$

By the definition (2.33) of $\mathcal{L}(z)$, one has $\text{Id} + \tau J\mathcal{L}(z) = A + \tau B$ where

$$A := \begin{pmatrix} \text{Id}_S & 0 \\ \tau A_{21} & \text{Id}_\perp \end{pmatrix}, \quad A_{21} := J_\perp \mathcal{L}_S^\perp(z),$$

$$B := \begin{pmatrix} B_{11} & B_{12} \\ 0 & 0 \end{pmatrix}, \quad B_{11} := J_S \mathcal{L}_S^\perp(z), \quad B_{12} := J_S \mathcal{L}_S^\perp(z). \quad (3.12)$$

Note that A is invertible with $A^{-1} = \begin{pmatrix} \text{Id}_S & 0 \\ -\tau A_{21} & \text{Id}_\perp \end{pmatrix}$ and hence $\text{Id} + \tau J\mathcal{L}(z) = CA$, where

$$C := \text{Id} + \tau BA^{-1} = \begin{pmatrix} C_{11} & \tau B_{12} \\ 0 & \text{Id}_\perp \end{pmatrix}, \quad C_{11} := \text{Id}_S + \tau B_{11} - \tau^2 B_{12} A_{21}. \quad (3.13)$$

By Lemma 2.2 it follows after shrinking the ball \mathcal{V}_\perp , if needed, that $C_{11} : h_S \rightarrow h_S$ is invertible. As a consequence, C and in turn CA are invertible. One then obtains the following formula for $(\text{Id} + \tau J\mathcal{L}(z))^{-1}$,

$$(\text{Id} + \tau J\mathcal{L}(z))^{-1} = A^{-1}C^{-1}, \quad C^{-1} = \begin{pmatrix} C_{11}^{-1} & -\tau C_{11}^{-1} B_{12} \\ 0 & \text{Id}_\perp \end{pmatrix} \quad (3.14)$$

or, in more explicit terms,

$$(\text{Id} + \tau J\mathcal{L}(z))^{-1} = \begin{pmatrix} C_{11}^{-1} & -\tau C_{11}^{-1} B_{12} \\ -\tau J_\perp \mathcal{L}_S^\perp(z) C_{11}^{-1} & \text{Id}_\perp + \tau^2 J_\perp \mathcal{L}_S^\perp(z) C_{11}^{-1} B_{12} \end{pmatrix}, \quad (3.15)$$

1 with $C_{11} : h_S \rightarrow h_S$ and $B_{12} : h_\perp^0 \rightarrow h_S$ given as above. In view of (3.7) and (3.14), we then define for any
2 $0 \leq \tau \leq 1$ and $z \in \mathcal{V}$ the vector field $X(\tau, z)$ as follows,

$$X(\tau, z) := -(\text{Id} + \tau J\mathcal{L}(z))^{-1}[J\mathcal{E}(z)] = -A^{-1}C^{-1} \begin{pmatrix} J_S \mathcal{E}_S(z) \\ 0 \end{pmatrix} = -A^{-1} \begin{pmatrix} C_{11}^{-1} J_S \mathcal{E}_S(z) \\ 0 \end{pmatrix} \quad (3.16)$$

3 or, in more explicit terms, $X(\tau, z) = (X_S(\tau, z), X_\perp(\tau, z))$, where

$$\begin{aligned} X_S(\tau, z) &= -(\text{Id}_S + \tau J_S \mathcal{L}_S^S(z) - \tau^2 J_S \mathcal{L}_S^\perp(z) J_\perp \mathcal{L}_\perp^S(z))^{-1} [J_S \mathcal{E}_S(z)], \\ X_\perp(\tau, z) &= -\tau J_\perp \mathcal{L}_\perp^S(z) [X_S(\tau, z)]. \end{aligned} \quad (3.17)$$

4 Note that by (3.6) and (2.35), $\mathcal{E}(z)$ and hence $\lambda_L(z)$ are quadratic expressions in z_\perp . By Lemma 2.2, one
5 obtains the following estimates for $\mathcal{E}(z)$:

Lemma 3.1. *The map $\mathcal{V} \rightarrow h_S \times h_\perp^0$, $z \mapsto \mathcal{E}(z) = (\mathcal{E}_S(z), 0)$ is real analytic. Furthermore, for any $z \in \mathcal{V}$, $\widehat{z}_1, \dots, \widehat{z}_l \in h_0^0$, $l \geq 1$, one has*

$$\begin{aligned} \|\mathcal{E}_S(z)\| &\lesssim \|z_\perp\|_0^2, \quad \|d\mathcal{E}_S(z)[\widehat{z}_1]\| \lesssim \|z_\perp\|_0 \|\widehat{z}_1\|_0, \\ \|d^l \mathcal{E}_S(z)[\widehat{z}_1, \dots, \widehat{z}_l]\| &\lesssim_l \prod_{j=1}^l \|\widehat{z}_j\|_0, \quad l \geq 2. \end{aligned}$$

6 In view of the formulas of the components of $X(\tau, z)$ in (3.17), one then infers from Lemma 2.2 and
7 Lemma 3.1 the following results.

Lemma 3.2. *For any $s \geq 0$, the non-autonomous vector field*

$$X : [0, 1] \times (\mathcal{V} \cap h_0^s) \rightarrow h_0^s$$

is real analytic and the following estimates hold: for any $z \in \mathcal{V} \cap h_0^s$, $0 \leq \tau \leq 1$, $\widehat{z} \in h_0^s$,

$$\|X(\tau, z)\|_s \lesssim_s \|z_\perp\|_s \|z_\perp\|_0, \quad \|dX(\tau, z)[\widehat{z}]\|_s \lesssim_s \|z_\perp\|_s \|\widehat{z}\|_0 + \|z_\perp\|_0 \|\widehat{z}\|_s.$$

If in addition $\widehat{z}_1, \dots, \widehat{z}_l \in h_0^s$, $l \geq 2$, then

$$\|d^l X(\tau, z)[\widehat{z}_1, \dots, \widehat{z}_l]\|_s \lesssim_{s,l} \sum_{j=1}^l \|\widehat{z}_j\|_s \prod_{i \neq j} \|\widehat{z}_i\|_0 + \|z_\perp\|_s \prod_{j=1}^l \|\widehat{z}_j\|_0.$$

8 By a standard contraction argument, there exists an open neighborhood $\mathcal{V}'_S \subset \mathcal{V}_S$ of $\mathcal{K} \subset h_S$ and a ball
9 $\mathcal{V}'_\perp \subset \mathcal{V}_\perp$, centered at 0, so that for any $\tau, \tau_0 \in [0, 1]$, the flow map $\Psi_X^{\tau_0, \tau}$ of the non-autonomous differential
10 equation $\partial_\tau z = X(\tau, z)$ is well defined on $\mathcal{V}' := \mathcal{V}'_S \times \mathcal{V}'_\perp$ and

$$\Psi_X^{\tau_0, \tau} : \mathcal{V}' \rightarrow \mathcal{V} \quad (3.18)$$

11 is real analytic. Arguing as in the proof of [11, Lemma 4.4] one obtains the following estimates for $\Psi_X^{\tau_0, \tau} - \text{id}$.

Lemma 3.3. *Shrinking the ball $\mathcal{V}'_\perp \subset h_\perp^0$ in $\mathcal{V}' = \mathcal{V}'_S \times \mathcal{V}'_\perp$, if needed, it follows that for any $s \geq 0$, $\tau_0, \tau \in [0, 1]$, the map $\Psi_X^{\tau_0, \tau} - \text{id} : \mathcal{V}' \cap h_0^s \rightarrow h_0^s$ is real analytic and for any $z \in \mathcal{V}' \cap h_0^s$, $0 \leq \tau_0, \tau \leq 1$, $\widehat{z} \in h_0^s$,*

$$\|\Psi_X^{\tau_0, \tau}(z) - z\|_s \lesssim_s \|z_\perp\|_s \|z_\perp\|_0, \quad \|(d\Psi_X^{\tau_0, \tau}(z) - \text{Id})[\widehat{z}]\|_s \lesssim_s \|z_\perp\|_s \|\widehat{z}\|_0 + \|z_\perp\|_0 \|\widehat{z}\|_s.$$

If in addition $\widehat{z}_1, \dots, \widehat{z}_l \in h_0^s$, $l \geq 2$, then

$$\|d^l \Psi_X^{\tau_0, \tau}(z)[\widehat{z}_1, \dots, \widehat{z}_l]\|_s \lesssim_s \sum_{j=1}^l \|\widehat{z}_j\|_s \prod_{i \neq j} \|\widehat{z}_i\|_0 + \|z_\perp\|_s \prod_{j=1}^l \|\widehat{z}_j\|_0.$$

The main goal of this section is to derive expansions for the flow maps $\Psi_X^{\tau_0, \tau}(z)$. To this end we first derive such expansions for the vector field $X(\tau, z)$. In view of the formulas for the components of $X(\tau, z)$, one infers from Lemma 2.2 the following

Lemma 3.4. *For any $N \geq 1$, $0 \leq \tau \leq 1$, and $z \in \mathcal{V}$, $X(\tau, z) = -\mathcal{L}_\tau(z)^{-1}[J\mathcal{E}(z)]$ has an expansion of the form $(0, \mathcal{OP}_N(\tau, z; X)) + \mathcal{R}_N(\tau, z; X)$ where*

$$\mathcal{OP}_N(\tau, z; X) = \mathcal{F}_{N_S}^+ \circ \sum_{k=0}^N a_k^+(\tau, z; X) D^{-k} [(\mathcal{F}_{N_S}^+)^{-1} z_\perp] + \mathcal{F}_{N_S}^- \circ \sum_{k=0}^N a_k^-(\tau, z; X) (-D)^{-k} [(\mathcal{F}_{N_S}^-)^{-1} z_\perp] \quad (3.19)$$

and for any $s \geq 0$ and $k \geq 0$, the maps

$$[0, 1] \times \mathcal{V} \rightarrow H_{\mathbb{C}}^s, \quad (\tau, z) \mapsto a_k^\pm(\tau, z; X), \quad [0, 1] \times (\mathcal{V} \cap h_0^s) \rightarrow h_0^{s+N+1}, \quad (\tau, z) \mapsto \mathcal{R}_N(\tau, z; X),$$

are real analytic and $a_k^-(\tau, z; X) = \overline{a_k^+(\tau, z; X)}$. The coefficients $a_0^\pm(\tau, z; X)$ take values in $i\mathbb{R}$. Writing $q = \Psi^{bo}(z_S, 0)$, they are given by

$$a_0^+(\tau, z; X) = -i\tau \partial_x^{-1} (d_S q[X_S(\tau, z)]), \quad a_0^-(\tau, z; X) = -a_0^+(\tau, z; X). \quad (3.20)$$

Furthermore, for any $k \geq 0$, $0 \leq \tau \leq 1$, $z \in \mathcal{V}$, $\widehat{z} \in h_0^0$,

$$\|a_k^\pm(\tau, z; X)\|_s \lesssim_{s,k} \|z_\perp\|_0^2, \quad \|da_k^\pm(\tau, z; X)[\widehat{z}]\|_s \lesssim_{s,k} \|z_\perp\|_0 \|\widehat{z}\|_0.$$

If in addition $\widehat{z}_1, \dots, \widehat{z}_l \in h_0^0$, $l \geq 2$, then

$$\|d^l a_k^\pm(\tau, z; X)[\widehat{z}_1, \dots, \widehat{z}_l]\|_s \lesssim_{s,k,l} \prod_{j=1}^l \|\widehat{z}_j\|_0.$$

For any $z \in \mathcal{V} \cap h_0^s$, $0 \leq \tau \leq 1$, $\widehat{z} \in h_0^s$, the remainder term $\mathcal{R}_N(\tau, z; X)$ satisfies

$$\|\mathcal{R}_N(\tau, z; X)\|_{s+N+1} \lesssim_{s,N} \|z_\perp\|_0^2 + \|z_\perp\|_0^2 \|z_\perp\|_s \lesssim_{s,N} \|z_\perp\|_0 \|z_\perp\|_s,$$

$$\|d\mathcal{R}_N(\tau, z; X)[\widehat{z}]\|_{s+N+1} \lesssim_{s,N} \|z_\perp\|_0^2 \|\widehat{z}\|_s + \|z_\perp\|_0 \|\widehat{z}\|_0 (1 + \|z_\perp\|_s) \lesssim_{s,N} \|z_\perp\|_0 (\|\widehat{z}\|_s + \|z_\perp\|_s \|\widehat{z}\|_0).$$

If in addition $\widehat{z}_1, \dots, \widehat{z}_l \in h_0^s$, $l \geq 2$, then

$$\|d^l \mathcal{R}_N(\tau, z; X)[\widehat{z}_1, \dots, \widehat{z}_l]\|_{s+N+1} \lesssim_{s,N,l} \sum_{j=1}^l \|\widehat{z}_j\|_s \prod_{i \neq j} \|\widehat{z}_i\|_0 + \|z_\perp\|_s \prod_{j=1}^l \|\widehat{z}_j\|_0.$$

Proof. In a first step, we compute the expansion of $X_\perp(\tau, z) = -\tau J_\perp \mathcal{L}_\perp^S(z)[X_S(\tau, z)]$ (cf. (3.17)). By Lemma 2.2(ii), one has $\mathcal{L}_\perp^S(z)[\widehat{z}_S] = \mathcal{OP}_{N+1}(z_S; \mathcal{L}_\perp^S)[\widehat{z}_S] + \mathcal{R}_{N+1}(z; \mathcal{L}_\perp^S)[\widehat{z}_S]$ where for any $\widehat{z}_S \in h_S$, $\mathcal{OP}_{N+1}(z_S; \mathcal{L}_\perp^S)[\widehat{z}_S]$ is the pseudo-differential operator (cf. (2.37))

$$\mathcal{F}_{N_S}^+ \circ \sum_{k=1}^{N+1} \mathcal{A}_k^+(z_S; \mathcal{L}_\perp^S)[\widehat{z}_S] \cdot D^{-k} [(\mathcal{F}_{N_S}^+)^{-1} z_\perp] + \mathcal{F}_{N_S}^- \circ \sum_{k=1}^{N+1} \mathcal{A}_k^-(z_S; \mathcal{L}_\perp^S)[\widehat{z}_S] \cdot (-D)^{-k} [(\mathcal{F}_{N_S}^-)^{-1} z_\perp]. \quad (3.21)$$

Since by the definition (2.29) of J , one has $J_\perp \mathcal{F}_{N_S}^+ = \mathcal{F}_{N_S}^+ \partial_x$ and $\partial_x = iD$, it follows that

$$\begin{aligned} -\tau J_\perp \mathcal{OP}_{N+1}(z_S; \mathcal{L}_\perp^S)[X_S(\tau, z)] &= -i\tau \mathcal{F}_{N_S}^+ \circ D \sum_{k=1}^{N+1} \mathcal{A}_k^+(z_S; \mathcal{L}_\perp^S)[X_S(\tau, z)] \cdot D^{-k} [(\mathcal{F}_{N_S}^+)^{-1} z_\perp] \\ &\quad + i\tau \mathcal{F}_{N_S}^- \circ (-D) \sum_{k=1}^{N+1} \mathcal{A}_k^-(z_S; \mathcal{L}_\perp^S)[X_S(\tau, z)] \cdot (-D)^{-k} [(\mathcal{F}_{N_S}^-)^{-1} z_\perp] \end{aligned}$$

1 and hence by the chain rule,

$$\begin{aligned}
D \sum_{k=1}^{N+1} \mathcal{A}_k^+(z_S; \mathcal{L}_\perp^S) [X_S(\tau, z)] \cdot D^{-k} [(\mathcal{F}_{N_S}^+)^{-1} z_\perp] &= \sum_{k=0}^N \mathcal{A}_{k+1}^+(z_S; \mathcal{L}_\perp^S) [X_S(\tau, z)] \cdot D^{-k} [(\mathcal{F}_{N_S}^+)^{-1} z_\perp] \\
&- i \sum_{k=1}^{N+1} \partial_x (\mathcal{A}_k^+(z_S; \mathcal{L}_\perp^S) [X_S(\tau, z)]) \cdot D^{-k} [(\mathcal{F}_{N_S}^+)^{-1} z_\perp].
\end{aligned} \tag{3.22}$$

It then follows that $X(\tau, z)$ has an expansion of the form $(0, \mathcal{OP}_N(\tau, z; X)) + \mathcal{R}_N(\tau, z; X)$ with $\mathcal{OP}_N(\tau, z; X)$ given by

$$\mathcal{F}_{N_S}^+ \circ \sum_{k=0}^N a_k^+(\tau, z; X) D^{-k} [(\mathcal{F}_{N_S}^+)^{-1} z_\perp] + \mathcal{F}_{N_S}^- \circ \sum_{k=0}^N a_k^-(\tau, z; X) (-D)^{-k} [(\mathcal{F}_{N_S}^-)^{-1} z_\perp],$$

where the coefficients $a_k^\pm(\tau, z; X)$ can be read off from (3.22). In view of Lemma 2.2, the formula (3.17) for $X(\tau, z)$, and Lemma 3.1, the coefficients $a_k^\pm(\tau, z; X)$ and the remainder term $\mathcal{R}_N(\tau, z; X)$ have the stated properties. Finally by (3.22), the coefficients $a_0^\pm(\tau, z; X)$ are given by $\mp i\tau \mathcal{A}_1^+(z_S; \mathcal{L}_\perp^S) [X_S(\tau, z)]$ and hence by Remark 2.4(i),

$$a_0^+(\tau, z; X) = -i\tau \partial_x^{-1} (d_S q [X_S(\tau, z)]), \quad a_0^-(\tau, z; X) = \overline{a_0^+(\tau, z; X)} = i\tau \partial_x^{-1} (d_S q [X_S(\tau, z)])$$

2 where for the latter identity we used that $\partial_x^{-1} (d_S q [X_S(\tau, z)])$ is real valued. \square

3 After these preliminary considerations we can now state the main result of this section, saying that for
4 any $\tau_0, \tau \in [0, 1]$, the flow map $\Psi_X^{\tau_0, \tau}$, defined on \mathcal{V}' and taking values in \mathcal{V} , admits an expansion, referred to
5 as parametrix for the solution of the initial value problem of $\partial_\tau z(\tau) = X(\tau, z(\tau))$.

6 **Theorem 3.1.** (i) For any $\tau_0, \tau \in [0, 1]$, $N \in \mathbb{N}$, and $z = (z_S, z_\perp) \in \mathcal{V}'$,

$$\Psi_X^{\tau_0, \tau}(z) = z + (0, \mathcal{OP}_N(z; \Psi_X^{\tau_0, \tau})) + \mathcal{R}_N(z; \Psi_X^{\tau_0, \tau}), \tag{3.23}$$

7 where $\mathcal{OP}_N(z; \Psi_X^{\tau_0, \tau})$ is the pseudo-differential operator

$$\mathcal{F}_{N_S}^+ \circ \sum_{k=0}^N a_k^+(z; \Psi_X^{\tau_0, \tau}) D^{-k} [(\mathcal{F}_{N_S}^+)^{-1} z_\perp] + \mathcal{F}_{N_S}^- \circ \sum_{k=0}^N a_k^-(z; \Psi_X^{\tau_0, \tau}) (-D)^{-k} [(\mathcal{F}_{N_S}^-)^{-1} z_\perp] \tag{3.24}$$

and for any $\tau_0, \tau \in [0, 1]$, $0 \leq k \leq N$, and $s \geq 0$, the maps

$$\mathcal{V}' \rightarrow H_{\mathbb{C}}^s, z \mapsto a_k^\pm(z; \Psi_X^{\tau_0, \tau}), \quad \mathcal{V}' \cap h_0^s \rightarrow h_0^{s+N+1}, z \mapsto \mathcal{R}_N(z; \Psi_X^{\tau_0, \tau}),$$

are real analytic and $a_k^-(z; \Psi_X^{\tau_0, \tau}) = \overline{a_k^+(z; \Psi_X^{\tau_0, \tau})}$. Furthermore, for any $k \geq 0$, $z \in \mathcal{V}'$, $\widehat{z} \in h_0^0$,

$$\|a_k^\pm(z; \Psi_X^{\tau_0, \tau})\|_s \lesssim_{s,k} \|z_\perp\|_0^2, \quad \|da_k^\pm(z; \Psi_X^{\tau_0, \tau})[\widehat{z}]\|_s \lesssim_{s,k} \|z_\perp\|_0 \|\widehat{z}\|_0.$$

If in addition $\widehat{z}_1, \dots, \widehat{z}_l \in h_0^0$, $l \geq 2$, then

$$\|d^l a_k(z; \Psi_X^{\tau_0, \tau})[\widehat{z}_1, \dots, \widehat{z}_l]\|_s \lesssim_{s,k,l} \prod_{j=1}^l \|\widehat{z}_j\|_0.$$

The remainder term satisfies the following estimates: for any $z \in \mathcal{V}' \cap h_0^s$ and $\widehat{z} \in h_0^s$,

$$\|\mathcal{R}_N(z; \Psi_X^{\tau_0, \tau})\|_{s+N+1} \lesssim_{s,N} \|z_\perp\|_s \|z_\perp\|_0, \quad \|d\mathcal{R}_N(z; \Psi_X^{\tau_0, \tau})[\widehat{z}]\|_{s+N+1} \lesssim_{s,N} \|z_\perp\|_s \|\widehat{z}\|_0 + \|z_\perp\|_0 \|\widehat{z}\|_s,$$

and for any $\widehat{z}_1, \dots, \widehat{z}_l \in h_0^s$, $l \geq 2$,

$$\|d^l \mathcal{R}_N(z; \Psi_X^{\tau_0, \tau})[\widehat{z}_1, \dots, \widehat{z}_l]\|_{s+N+1} \lesssim_{s,N,l} \sum_{j=1}^l \|\widehat{z}_j\|_s \prod_{i \neq j} \|\widehat{z}_i\|_0 + \|z_\perp\|_s \prod_{j=1}^l \|\widehat{z}_j\|_0.$$

(ii) In particular, the statements of item (i) hold for $\Psi_C := \Psi_X^{0,1} : \mathcal{V}' \rightarrow \Psi_X^{0,1}(\mathcal{V}')$, referred to as symplectic corrector, and $\Psi_X^{1,0} : \mathcal{V}' \rightarrow \Psi_X^{1,0}(\mathcal{V}')$, which by a slight abuse of terminology with respect to its domain of definition, we refer to as the inverse of Ψ_C and denote by Ψ_C^{-1} . The expansion of the map Ψ_C is then written as ($z \in \mathcal{V}'$)

$$\Psi_C(z) = z + (0, \mathcal{OP}_N(z; \Psi_C)) + \mathcal{R}_N(z; \Psi_C),$$

1 where $\mathcal{OP}_N(z; \Psi_C)$ is the pseudo-differential operator

$$\mathcal{F}_{N_S}^+ \circ \sum_{k=0}^N a_k^+(z; \Psi_C) D^{-k} [(\mathcal{F}_{N_S}^+)^{-1} z_\perp] + \mathcal{F}_{N_S}^- \circ \sum_{k=0}^N a_k^-(z; \Psi_C) (-D)^{-k} [(\mathcal{F}_{N_S}^-)^{-1} z_\perp] \quad (3.25)$$

with

$$a_k^\pm(z; \Psi_C) := a_k^\pm(z; \Psi_X^{0,1}), \quad \mathcal{R}_N(z; \Psi_C) := \mathcal{R}_N(z; \Psi_X^{0,1}).$$

Similarly, the expansion for the inverse $\Psi_C^{-1}(z)$, $z \in \mathcal{V}'$, is of the form

$$\Psi_C(z)^{-1} = z + (0, \mathcal{OP}_N(z; \Psi_C^{-1})) + \mathcal{R}_N(z; \Psi_C^{-1})$$

where $\mathcal{OP}_N(z; \Psi_C^{-1})$ is given by

$$\mathcal{F}_{N_S}^+ \circ \sum_{k=0}^N a_k^+(z; \Psi_C^{-1}) D^{-k} [(\mathcal{F}_{N_S}^+)^{-1} z_\perp] + \mathcal{F}_{N_S}^- \circ \sum_{k=0}^N a_k^-(z; \Psi_C^{-1}) (-D)^{-k} [(\mathcal{F}_{N_S}^-)^{-1} z_\perp]$$

with

$$a_k^\pm(z; \Psi_C^{-1}) := a_k^\pm(z; \Psi_X^{1,0}), \quad \mathcal{R}_N(z; \Psi_C^{-1}) := \mathcal{R}_N(z; \Psi_X^{1,0}).$$

2 As a consequence, $a_k^-(z; \Psi_C) = \overline{a_k^+(z; \Psi_C)}$ and $a_k^-(z; \Psi_C^{-1}) = \overline{a_k^+(z; \Psi_C^{-1})}$.

3 *Proof.* Clearly, item (ii) is a direct consequence of (i). Since the proof of item (i) is quite lengthy, we divide
4 it up into several steps. First note that the flow map $\Psi^{\tau_0, \tau} \equiv \Psi_X^{\tau_0, \tau}$ is a bounded nonlinear operator acting
5 on $\mathcal{V}' \cap h_0^s$, $s \geq 0$, which satisfies the differential equation

$$\partial_\tau \Psi^{\tau_0, \tau}(z) = Y_{\tau_0}(\tau, z), \quad Y_{\tau_0}(\tau, z) := X(\tau, \Psi^{\tau_0, \tau}(z)). \quad (3.26)$$

Using the latter equation, the coefficients $a_k^\pm(z; \Psi^{\tau_0, \tau})$, $k \geq 0$, of the parametrix (3.23) are determined inductively. In a first step, we want to obtain an expansion of

$$Y_{\tau_0}(\tau, z) = -\mathcal{L}_\tau(z)^{-1} [J\mathcal{E}(\Psi^{\tau_0, \tau}(z))], \quad 0 \leq \tau_0, \tau \leq 1, \quad z \in \mathcal{V}',$$

6 of the form $Y_{\tau_0}(\tau, z) = (0, \mathcal{OP}_N(\tau, z; Y_{\tau_0})) + \mathcal{R}_N(\tau, z; Y_{\tau_0})$ where $\mathcal{OP}_N(\tau, z; Y_{\tau_0})$ is a pseudo-differential
7 operator and $\mathcal{R}_N(\tau, z; Y_{\tau_0})$ a remainder term with the usual properties. By (3.19), one has for any $0 \leq \tau_0$,
8 $\tau \leq 1$, $z \in \mathcal{V}'$,

$$Y_{\tau_0}(\tau, z) = (0, \mathcal{OP}_N(\tau, \Psi^{\tau_0, \tau}(z); X)) + \mathcal{R}_N(\tau, \Psi^{\tau_0, \tau}(z); X), \quad (3.27)$$

9

$$\begin{aligned} \mathcal{OP}_N(\tau, \Psi^{\tau_0, \tau}(z); X) &= \mathcal{F}_{N_S}^+ \circ \sum_{k=0}^N a_k^+(\tau, z; \tau_0) D^{-k} [(\mathcal{F}_{N_S}^+)^{-1} \Psi^{\tau_0, \tau}(z)_\perp] \\ &+ \mathcal{F}_{N_S}^- \circ \sum_{k=0}^N a_k^-(\tau, z; \tau_0) (-D)^{-k} [(\mathcal{F}_{N_S}^-)^{-1} \Psi^{\tau_0, \tau}(z)_\perp]. \end{aligned} \quad (3.28)$$

10 where, to simplify notation, $a_k^\pm(\tau, z; \tau_0)$ is defined as

$$a_k^\pm(\tau, z; \tau_0) := a_k^\pm(\tau, \Psi^{\tau_0, \tau}(z); X). \quad (3.29)$$

11 In a first step, we argue formally and want to derive an expansion for $Y_{\tau_0}(\tau, z)$ in terms of the (not yet
12 determined) expansion (3.23)-(3.24) of $\Psi_X^{\tau_0, \tau}(z)$. We begin with expanding the term $D^{-k} [(\mathcal{F}_{N_S}^+)^{-1} \Psi^{\tau_0, \tau}(z)_\perp]$
13 in (3.28).

Expansion of $D^{-k}[(\mathcal{F}_{N_S}^+)^{-1}\Psi^{\tau_0,\tau}(z)_\perp]$, $0 \leq k \leq N$: Substituting the expansion (3.23)-(3.24) into the expression $D^{-k}[(\mathcal{F}_{N_S}^+)^{-1}\Psi^{\tau_0,\tau}(z)_\perp]$ one obtains

$$\begin{aligned} D^{-k}[(\mathcal{F}_{N_S}^+)^{-1}\Psi^{\tau_0,\tau}(z)_\perp] &= D^{-k} \left([(\mathcal{F}_{N_S}^+)^{-1}z_\perp] + \sum_{j=0}^N a_j^+(z; \Psi^{\tau_0,\tau}) D^{-j}[(\mathcal{F}_{N_S}^+)^{-1}z_\perp] + (\mathcal{F}_{N_S}^+)^{-1}\mathcal{R}_N(z; \Psi^{\tau_0,\tau})_\perp \right) \\ &= D^{-k} \left([(\mathcal{F}_{N_S}^+)^{-1}z_\perp] + \sum_{j=0}^{N-k} a_j^+(z; \Psi^{\tau_0,\tau}) D^{-j}[(\mathcal{F}_{N_S}^+)^{-1}z_\perp] \right) + \mathcal{R}_{N,k}^{(+,1)}(\tau, z; \tau_0) \end{aligned} \quad (3.30)$$

1 where

$$\mathcal{R}_{N,k}^{(+,1)}(\tau, z; \tau_0) := D^{-k} \circ \sum_{j=N-k+1}^N a_j^+(z; \Psi^{\tau_0,\tau}) D^{-j}[(\mathcal{F}_{N_S}^+)^{-1}z_\perp] + D^{-k}[(\mathcal{F}_{N_S}^+)^{-1}\mathcal{R}_N(z; \Psi^{\tau_0,\tau})_\perp]. \quad (3.31)$$

Using Lemma D.2(i) and the notation established there, one has for any $0 \leq j \leq N-k$,

$$\begin{aligned} D^{-k} \circ (a_j^+(z; \Psi^{\tau_0,\tau}) D^{-j}[(\mathcal{F}_{N_S}^+)^{-1}z_\perp]) &= \sum_{i=0}^{N-k-j} C_i(k, j) \cdot D^i a_j^+(z; \Psi^{\tau_0,\tau}) \cdot D^{-k-j-i}[(\mathcal{F}_{N_S}^+)^{-1}z_\perp] \\ &\quad + \mathcal{R}_{N,k,j}^{(+,2)}(\tau, z; \tau_0) \end{aligned}$$

2 where

$$\mathcal{R}_{N,k,j}^{(+,2)}(\tau, z; \tau_0) := \mathcal{R}_{N,k,j}^{\psi do}(a_j^+(z; \Psi^{\tau_0,\tau}))[(\mathcal{F}_{N_S}^+)^{-1}z_\perp]. \quad (3.32)$$

3 By Lemma D.2, for any $z \in \mathcal{V}' \cap h_0^s$, $s \geq 0$, $0 \leq \tau, \tau_0 \leq 1$, $0 \leq j \leq N$,

$$\|\mathcal{R}_{N,k,j}^{(+,2)}(\tau, z; \tau_0)\|_{s+N+1} \lesssim_{s,N} \max_{0 \leq i \leq N} \|a_i^+(z; \Psi^{\tau_0,\tau})\|_{s+2N} \|z_\perp\|_s. \quad (3.33)$$

Combining the above expansions, the formula (3.30) for $D^{-k}[(\mathcal{F}_{N_S}^+)^{-1}\Psi^{\tau_0,\tau}(z)_\perp]$ then reads

$$D^{-k}[(\mathcal{F}_{N_S}^+)^{-1}z_\perp] + \sum_{j=0}^{N-k} \sum_{i=0}^{N-k-j} C_i(k, j) \cdot D^i a_j^+(z; \Psi^{\tau_0,\tau}) \cdot D^{-k-j-i}[(\mathcal{F}_{N_S}^+)^{-1}z_\perp] + \mathcal{R}_{N,k}^{(+,3)}(\tau, z; \tau_0)$$

4 where

$$\mathcal{R}_{N,k}^{(+,3)}(\tau, z; \tau_0) := \mathcal{R}_{N,k}^{(+,1)}(\tau, z; \tau_0) + \sum_{j=0}^{N-k} \mathcal{R}_{N,k,j}^{(+,2)}(\tau, z; \tau_0). \quad (3.34)$$

Changing in the double sum $\sum_{j=0}^{N-k} \sum_{i=0}^{N-k-j}$ the index i of summation to $n := i+j$ and then interchanging the order of summation, one obtains

$$\sum_{j=0}^{N-k} \sum_{i=0}^{N-k-j} C_i(k, j) \cdot D^i a_j^+ \cdot D^{-k-j-i} = \sum_{n=0}^{N-k} \sum_{j=0}^n C_{n-j}(k, j) \cdot D^{n-j} a_j^+ \cdot D^{-k-n},$$

5 implying that $D^{-k}[(\mathcal{F}_{N_S}^+)^{-1}\Psi^{\tau_0,\tau}(z)_\perp]$ equals

$$D^{-k}[(\mathcal{F}_{N_S}^+)^{-1}z_\perp] + \sum_{n=0}^{N-k} \left(\sum_{j=0}^n C_{n-j}(k, j) \cdot D^{n-j} a_j^+(z; \Psi^{\tau_0,\tau}) \right) \cdot D^{-k-n}[(\mathcal{F}_{N_S}^+)^{-1}z_\perp] + \mathcal{R}_{N,k}^{(+,3)}(\tau, z; \tau_0). \quad (3.35)$$

Expansion of $\sum_{k=0}^N a_k^+(\tau, z; \tau_0) D^{-k}[(\mathcal{F}_{N_S}^+)^{-1}\Psi^{\tau_0,\tau}(z)_\perp]$: Substituting for any $0 \leq k \leq N$ the expansion

(3.35) into $\sum_{k=0}^N a_k^+(\tau, z; \tau_0) D^{-k}[(\mathcal{F}_{N_S}^+)^{-1} \Psi^{\tau_0, \tau}(z)_\perp]$, one gets

$$\begin{aligned} & \sum_{k=0}^N a_k^+(\tau, z; \tau_0) D^{-k}[(\mathcal{F}_{N_S}^+)^{-1} \Psi^{\tau_0, \tau}(z)_\perp] = \sum_{k=0}^N a_k^+(\tau, z; \tau_0) D^{-k}[(\mathcal{F}_{N_S}^+)^{-1} z_\perp] \\ & + \sum_{k=0}^N \sum_{n=0}^{N-k} \sum_{j=0}^n C_{n-j}(k, j) a_k^+(\tau, z; \tau_0) \cdot D^{n-j} a_j^+(z; \Psi^{\tau_0, \tau}) \cdot D^{-k-n}[(\mathcal{F}_{N_S}^+)^{-1} z_\perp] \\ & + \sum_{k=0}^N a_k^+(\tau, z; \tau_0) \mathcal{R}_{N,k}^{(+,3)}(\tau, z; \tau_0). \end{aligned} \quad (3.36)$$

Changing the index of summation n to $l := k + n$ and then interchanging the sum with respect to k and l and subsequently with respect to k and j , the triple sum in (3.36) becomes

$$\begin{aligned} & \sum_{k=0}^N \sum_{l=k}^N \sum_{j=0}^{l-k} C_{l-k-j}(k, j) a_k^+(\tau, z; \tau_0) \cdot D^{l-k-j} a_j^+(z; \Psi^{\tau_0, \tau}) \cdot D^{-l}[(\mathcal{F}_{N_S}^+)^{-1} z_\perp] \\ & = \sum_{l=0}^N \left(\sum_{k=0}^l \sum_{j=0}^{l-k} C_{l-k-j}(k, j) a_k^+(\tau, z; \tau_0) \cdot D^{l-k-j} a_j^+(z; \Psi^{\tau_0, \tau}) \right) D^{-l}[(\mathcal{F}_{N_S}^+)^{-1} z_\perp] \\ & = \sum_{l=0}^N \left(\sum_{j=0}^l \sum_{k=0}^{l-j} C_{l-k-j}(k, j) a_k^+(\tau, z; \tau_0) \cdot D^{l-k-j} a_j^+(z; \Psi^{\tau_0, \tau}) \right) D^{-l}[(\mathcal{F}_{N_S}^+)^{-1} z_\perp]. \end{aligned} \quad (3.37)$$

Combining (3.36) - (3.37) and writing n for k and k for l in (3.37), one infers that

$$\begin{aligned} & \sum_{k=0}^N a_k^+(\tau, z; \tau_0) D^{-k}[(\mathcal{F}_{N_S}^+)^{-1} \Psi^{\tau_0, \tau}(z)_\perp] = \sum_{k=0}^N a_k^+(\tau, z; \tau_0) D^{-k}[(\mathcal{F}_{N_S}^+)^{-1} z_\perp] \\ & + \sum_{k=0}^N \left(\sum_{j=0}^k \sum_{n=0}^{k-j} C_{k-n-j}(n, j) a_n^+(\tau, z; \tau_0) \cdot D^{k-n-j} a_j^+(z; \Psi^{\tau_0, \tau}) \right) \cdot D^{-k}[(\mathcal{F}_{N_S}^+)^{-1} z_\perp] \\ & + \sum_{k=0}^N a_k^+(\tau, z; \tau_0) \mathcal{R}_{N,k}^{(+,3)}(\tau, z; \tau_0), \end{aligned} \quad (3.38)$$

1 with $\mathcal{R}_{N,k}^{(+,3)}(\tau, z; \tau_0)$ given by (3.34).

Expansion of $\sum_{k=0}^N a_k^-(\tau, z; \tau_0) (-D)^{-k}[(\mathcal{F}_{N_S}^-)^{-1} \Psi^{\tau_0, \tau}(z)_\perp]$: Analogous to the case '+', one obtains

$$\begin{aligned} & \sum_{k=0}^N a_k^-(\tau, z; \tau_0) (-D)^{-k}[(\mathcal{F}_{N_S}^-)^{-1} \Psi^{\tau_0, \tau}(z)_\perp] = \sum_{k=0}^N a_k^-(\tau, z; \tau_0) (-D)^{-k}[(\mathcal{F}_{N_S}^-)^{-1} z_\perp] \\ & + \sum_{k=0}^N \left(\sum_{j=0}^k \sum_{n=0}^{k-j} C_{k-n-j}(n, j) a_n^-(\tau, z; \tau_0) \cdot (-D)^{k-n-j} a_j^-(z; \Psi^{\tau_0, \tau}) \right) \cdot (-D)^{-k}[(\mathcal{F}_{N_S}^-)^{-1} z_\perp] \\ & + \sum_{k=0}^N a_k^-(\tau, z; \tau_0) \mathcal{R}_{N,k}^{(-,3)}(\tau, z; \tau_0), \end{aligned} \quad (3.39)$$

2 where for any $0 \leq k \leq N$, $\mathcal{R}_{N,k}^{(-,3)}(\tau, z; \tau_0)$ is defined analogous to $\mathcal{R}_{N,k}^{(+,3)}(\tau, z; \tau_0)$,

$$\mathcal{R}_{N,k}^{(-,3)}(\tau, z; \tau_0) := \mathcal{R}_{N,k}^{(-,1)}(\tau, z; \tau_0) + \sum_{j=0}^{N-k} \mathcal{R}_{N,k,j}^{(-,2)}(\tau, z; \tau_0), \quad (3.40)$$

the remainder $\mathcal{R}_{N,k}^{(-,1)}(\tau, z; \tau_0)$ analogous to $\mathcal{R}_{N,k}^{(+,1)}(\tau, z; \tau_0)$,

$$\begin{aligned} \mathcal{R}_{N,k}^{(-,1)}(\tau, z; \tau_0) := & (-D)^{-k} \circ \sum_{j=N-k+1}^N a_j^-(z; \Psi^{\tau_0, \tau}) (-D)^{-j} [(\mathcal{F}_{N_S}^-)^{-1} z_\perp] \\ & + (-D)^{-k} [(\mathcal{F}_{N_S}^-)^{-1} \mathcal{R}_N(z; \Psi^{\tau_0, \tau})_\perp]. \end{aligned} \quad (3.41)$$

1 and $\mathcal{R}_{N,k,j}^{(-,2)}(\tau, z; \tau_0)$ analogous to $\mathcal{R}_{N,k,j}^{(+,2)}(\tau, z; \tau_0)$,

$$\mathcal{R}_{N,k,j}^{(-,2)}(\tau, z; \tau_0) := \mathcal{R}_{N,k,j}^{\psi do}(a_j^-(z; \Psi^{\tau_0, \tau})) [(\mathcal{F}_{N_S}^-)^{-1} z_\perp]. \quad (3.42)$$

2 Furthermore, for any $z \in \mathcal{V}' \cap h_0^s$, $s \geq 0$, $0 \leq \tau, \tau_0 \leq 1$, $0 \leq j \leq N$, one has (cf. (3.33))

$$\|\mathcal{R}_{N,k,j}^{(-,2)}(\tau, z; \tau_0)\|_{s+N+1} \lesssim_{s,N} \max_{0 \leq i \leq N} \|a_i^-(z; \Psi^{\tau_0, \tau})\|_{s+2N} \|z_\perp\|_s. \quad (3.43)$$

3 *Expansion of $Y_{\tau_0}(\tau, z)$:* From (3.28) and (3.38) - (3.39) one infers that $Y_{\tau_0}(\tau, z) = X(\tau, \Psi^{\tau_0, \tau}(z))$ has an
4 expansion of the form

$$Y_{\tau_0}(\tau, z) = (0, \mathcal{OP}_N(\tau, z; Y_{\tau_0})) + \mathcal{R}_N(\tau, z; Y_{\tau_0}) \quad (3.44)$$

with $\mathcal{OP}_N(\tau, z; Y_{\tau_0})$ and $\mathcal{R}_N(\tau, z; Y_{\tau_0})$ given as follows: using that $C_0(k, j) = 1$ (cf. Lemma D.2(i)) one concludes that the pseudo-differential operator $\mathcal{OP}_N(\tau, z; Y_{\tau_0})$ reads

$$\begin{aligned} & \mathcal{F}_{N_S}^+ \circ (a_0^+(\tau, z; \tau_0)(1 + a_0^+(z; \Psi^{\tau_0, \tau}))) \cdot [(\mathcal{F}_{N_S}^+)^{-1} z_\perp] \\ & + \mathcal{F}_{N_S}^+ \circ \sum_{k=1}^N (a_0^+(\tau, z; \tau_0) a_k^+(z; \Psi^{\tau_0, \tau}) + b_k^+(\tau, z; \tau_0)) \cdot D^{-k} [(\mathcal{F}_{N_S}^+)^{-1} z_\perp] \\ & + \mathcal{F}_{N_S}^- \circ (a_0^-(\tau, z; \tau_0)(1 + a_0^-(z; \Psi^{\tau_0, \tau}))) \cdot [(\mathcal{F}_{N_S}^-)^{-1} z_\perp] \\ & + \mathcal{F}_{N_S}^- \circ \sum_{k=1}^N (a_0^-(\tau, z; \tau_0) a_k^-(z; \Psi^{\tau_0, \tau}) + b_k^-(\tau, z; \tau_0)) \cdot (-D)^{-k} [(\mathcal{F}_{N_S}^-)^{-1} z_\perp] \end{aligned} \quad (3.45)$$

5 where for any $k \geq 1$,

$$b_k^\pm(\tau, z; \tau_0) := a_k^\pm(\tau, z; \tau_0) + \sum_{j=0}^{k-1} \sum_{n=0}^{k-j} C_{k-n-j}(n, j) a_n^\pm(\tau, z; \tau_0) (\pm D)^{k-n-j} a_j^\pm(z; \Psi^{\tau_0, \tau}) \quad (3.46)$$

and the remainder term $\mathcal{R}_N(\tau, z; Y_{\tau_0})$ is defined as

$$\begin{aligned} & (0, \mathcal{F}_{N_S}^+ \circ \sum_{k=0}^N a_k^+(\tau, z; \tau_0) \mathcal{R}_{N,k}^{(+,3)}(\tau, z; \tau_0) + \mathcal{F}_{N_S}^- \circ \sum_{k=0}^N a_k^-(\tau, z; \tau_0) \mathcal{R}_{N,k}^{(-,3)}(\tau, z; \tau_0)) \\ & + \mathcal{R}_N(\tau, \Psi^{\tau_0, \tau}(z); X). \end{aligned} \quad (3.47)$$

6 *Definition and estimates of $a_k^\pm(z; \Psi^{\tau_0, \tau})$, $k \geq 0$:* The fact that the coefficients $b_k^\pm(\tau, z; \tau_0)$, $k \geq 1$, depend on
7 the unknown coefficients $a_j^\pm(z; \Psi^{\tau_0, \tau})$ with $0 \leq j \leq k-1$, but not on $a_j^\pm(z; \Psi^{\tau_0, \tau})$ with $j \geq k$, allows to define
8 $a_k^\pm(z; \Psi^{\tau_0, \tau})$ recursively by using equation (3.26). Clearly, the coefficients $a_k^\pm(z; \Psi^{\tau_0, \tau})$ can be defined in the
9 same way. Our candidates for $a_k^\pm(z; \Psi^{\tau_0, \tau})$ are thus obtained by solving $\partial_\tau \mathcal{OP}_N(z; \Psi_X^{\tau_0, \tau}) = \mathcal{OP}_N(\tau, z; Y_{\tau_0})$,
10 or in more detail,

$$\partial_\tau a_0^\pm(z; \Psi^{\tau_0, \tau}) = a_0^\pm(\tau, z; \tau_0) \cdot (1 + a_0^\pm(z; \Psi^{\tau_0, \tau})), \quad a_0^\pm(z; \Psi^{\tau_0, \tau_0}) = 0, \quad (3.48)$$

11 and for any $k \geq 1$

$$\partial_\tau a_k^\pm(z; \Psi^{\tau_0, \tau}) = a_0^\pm(\tau, z; \tau_0) a_k^\pm(z; \Psi^{\tau_0, \tau}) + b_k^\pm(\tau, z; \tau_0), \quad a_k^\pm(z; \Psi^{\tau_0, \tau_0}) = 0. \quad (3.49)$$

The solution of (3.48) then leads to the definition

$$a_0^\pm(z; \Psi^{\tau_0, \tau}) := e^{\alpha^\pm(\tau, z; \tau_0)} - 1, \quad \alpha^\pm(\tau, z; \tau_0) := \int_{\tau_0}^{\tau} a_0^\pm(t, z; \tau_0) dt \quad (3.50)$$

and, recursively, for any $k \geq 1$, the one of (3.49) to

$$a_k^\pm(z; \Psi^{\tau_0, \tau}) := e^{\alpha^\pm(\tau, z; \tau_0)} \int_{\tau_0}^{\tau} e^{-\alpha^\pm(t, z; \tau_0)} b_k^\pm(t, z; \tau_0) dt. \quad (3.51)$$

Going through the arguments above, one verifies that for any $k \geq 0$, $a_k^-(z; \Psi^{\tau_0, \tau}) = \overline{a_k^+(z; \Psi^{\tau_0, \tau})}$. To prove the claimed estimates for $a_k^\pm(z; \Psi^{\tau_0, \tau})$, $k \geq 0$, we first estimate $a_k^\pm(\tau, z; \tau_0)$. Recall that by (3.29), $a_k^\pm(\tau, z; \tau_0) = a_k^\pm(\tau, \Psi^{\tau_0, \tau}(z); X)$. By Lemma 3.4 and Lemma 3.3 one has for any $0 \leq \tau_0, \tau \leq 1$, $z \in \mathcal{V}'$, and $s \geq 0$,

$$\|a_k^\pm(\tau, z; \tau_0)\|_s \lesssim_{s,k} \|\pi_\perp \Psi^{\tau_0, \tau}(z)\|_0^2 \lesssim_{s,k} \|z_\perp\|_0^2. \quad (3.52)$$

It then follows from the definition (3.50) of $a_0^\pm(z; \Psi^{\tau_0, \tau})$ that for any $s \geq 0$,

$$\|a_0^\pm(z; \Psi^{\tau_0, \tau})\|_s \lesssim_s \|z_\perp\|_0^2, \quad \forall 0 \leq \tau_0, \tau \leq 1, \forall z \in \mathcal{V}'. \quad (3.53)$$

To prove corresponding estimates for $a_k^\pm(z; \Psi^{\tau_0, \tau})$ with $1 \leq k \leq N$, we argue by induction. Assume that for any $0 \leq j \leq k-1$ and $s \geq 0$,

$$\|a_j^\pm(z; \Psi^{\tau_0, \tau})\|_s \lesssim_{s,j} \|z_\perp\|_0^2, \quad \forall 0 \leq \tau_0, \tau \leq 1, \forall z \in \mathcal{V}'. \quad (3.54)$$

By the definition (3.51) of $a_k^\pm(z; \Psi^{\tau_0, \tau})$, the definition (3.46) of $b_k^\pm(\tau, z; \tau_0)$, and by the induction hypothesis, Lemma 3.3 - Lemma 3.4, and the interpolation Lemma D.4, one then concludes that the estimate (3.54) is also satisfied for $j = k$. Using the analyticity properties established for $a_k^\pm(\tau, z; \tau_0)$ and $\Psi^{\tau_0, \tau}(z)$, one verifies the ones stated for the coefficients $a_k^\pm(z; \Psi^{\tau_0, \tau})$.

Estimates of the derivatives of $a_k^\pm(z; \Psi^{\tau_0, \tau})$: By Lemma 3.4, Lemma 3.3, and the chain rule one has for any $0 \leq k \leq N$, $0 \leq \tau_0, \tau \leq 1$, $z \in \mathcal{V}'$, $\widehat{z} \in h_0^0$, $s \geq 0$,

$$\|da_k^\pm(\tau, z; \tau_0)[\widehat{z}]\|_s \lesssim_{s,k} \|\pi_\perp \Psi^{\tau_0, \tau}(z)\|_0 \|d\Psi^{\tau_0, \tau}(z)[\widehat{z}]\|_0 \lesssim_s \|z_\perp\|_0 \|\widehat{z}\|_0. \quad (3.55)$$

If in addition, $\widehat{z}_1, \dots, \widehat{z}_l \in h_0^0$, $l \geq 2$,

$$\|d^l a_k^\pm(\tau, z; \tau_0)[\widehat{z}_1, \dots, \widehat{z}_l]\|_s \lesssim_s \prod_{j=1}^l \|\widehat{z}_j\|_0. \quad (3.56)$$

By the definition of $a_0^\pm(z; \Psi^{\tau_0, \tau})$, and (3.53), (3.55) - (3.56) it then follows that the claimed estimate for $\|d^l a_0^\pm(z; \Psi^{\tau_0, \tau})[\widehat{z}_1, \dots, \widehat{z}_l]\|_s$ hold for any $l \geq 1$. To prove corresponding estimates for the derivatives of $a_k^\pm(z; \Psi^{\tau_0, \tau})$ with $1 \leq k \leq N$, we again argue by induction. Assume that for any $0 \leq j \leq k-1$ and $s \geq 0$,

$$\|da_j^\pm(z; \Psi^{\tau_0, \tau})[\widehat{z}]\|_s \lesssim_{s,j} \|z_\perp\|_0 \|\widehat{z}\|_0, \quad \forall 0 \leq \tau_0, \tau \leq 1, \forall z \in \mathcal{V}', \widehat{z} \in h_0^0. \quad (3.57)$$

By the definition (3.51) of $a_k^\pm(z; \Psi^{\tau_0, \tau})$ and the estimates (3.52) - (3.55) it then follows that (3.57) also holds for $j = k$. The estimates for $\|d^l a_k^\pm(z; \Psi^{\tau_0, \tau})[\widehat{z}_1, \dots, \widehat{z}_l]\|_s$ with $l \geq 2$ are derived in a similar fashion.

Definition and estimate of $\mathcal{R}_N(z; \Psi^{\tau_0, \tau})$: The remainder term $\mathcal{R}_N(z; \Psi^{\tau_0, \tau})$ is defined so that the identity (3.23) holds,

$$\mathcal{R}_N(z; \Psi_X^{\tau_0, \tau}) := \Psi_X^{\tau_0, \tau}(z) - z - (0, \mathcal{OP}_N(z; \Psi_X^{\tau_0, \tau})).$$

By (3.26) and the expansion (3.44) of $Y_{\tau_0}(\tau, z)$, the remainder term $\mathcal{R}_N(z; \Psi^{\tau_0, \tau})$ satisfies

$$\partial_\tau \mathcal{R}_N(z; \Psi_X^{\tau_0, \tau}) = \mathcal{R}_N(\tau, z; Y_{\tau_0}), \quad \mathcal{R}_N(z; \Psi_X^{\tau_0, \tau_0}) = 0,$$

1 and hence

$$\mathcal{R}_N(z; \Psi_X^{\tau_0, \tau}) = \int_{\tau_0}^{\tau} \mathcal{R}_N(t, z; Y_{\tau_0}) dt. \quad (3.58)$$

We recall that the remainder term $\mathcal{R}_N(t, z; Y_{\tau_0})$ is given by (3.47),

$$(0, \mathcal{F}_{N_S}^+ \circ \sum_{k=0}^N a_k^+(t, z; \tau_0) \mathcal{R}_{N,k}^{(+,3)}(t, z; \tau_0) + \mathcal{F}_{N_S}^- \circ \sum_{k=0}^N a_k^-(t, z; \tau_0) \mathcal{R}_{N,k}^{(-,3)}(t, z; \tau_0)) + \mathcal{R}_N(\tau, \Psi^{\tau_0, t}(z); X),$$

where $\mathcal{R}_{N,k}^{(+,3)}(t, z; \tau_0)$ is given in (3.34) and $\mathcal{R}_{N,k}^{(-,3)}(t, z; \tau_0)$ in (3.40). We estimate the two components $\pi_S \int_{\tau_0}^{\tau} \mathcal{R}_N(t, z; Y_{\tau_0}) dt$ and $\pi_{\perp} \int_{\tau_0}^{\tau} \mathcal{R}_N(t, z; Y_{\tau_0}) dt$ of $\int_{\tau_0}^{\tau} \mathcal{R}_N(t, z; Y_{\tau_0}) dt$ separately. By (3.58) and the formula (3.47) for $\mathcal{R}_N(\tau, z; Y_{\tau_0})$, recalled above,

$$\pi_S \int_{\tau_0}^{\tau} \mathcal{R}_N(t, z; Y_{\tau_0}) dt = \int_{\tau_0}^{\tau} \pi_S \mathcal{R}_N(t, \Psi^{\tau_0, t}(z); X) dt$$

and

$$\begin{aligned} \pi_{\perp} \int_{\tau_0}^{\tau} \mathcal{R}_N(t, z; Y_{\tau_0}) dt &= \int_{\tau_0}^{\tau} \pi_{\perp} \mathcal{R}_N(t, \Psi^{\tau_0, t}(z); X) dt \\ &+ \mathcal{F}_{N_S}^+ \circ \sum_{k=0}^N \int_{\tau_0}^{\tau} a_k^+(t, z; \tau_0) \mathcal{R}_{N,k}^{(+,3)}(t, z; \tau_0) dt + \mathcal{F}_{N_S}^- \circ \sum_{k=0}^N \int_{\tau_0}^{\tau} a_k^-(t, z; \tau_0) \mathcal{R}_{N,k}^{(-,3)}(t, z; \tau_0) dt. \end{aligned}$$

2 By Lemma 3.4 and Lemma 3.3, for any $z \in \mathcal{V}'$, $0 \leq \tau_0, t \leq 1$,

$$\|\pi_S \mathcal{R}_N(t, \Psi^{\tau_0, t}(z); X)\| \lesssim_N \|\mathcal{R}_N(t, \Psi^{\tau_0, t}(z); X)\|_0 \lesssim_N \|z_{\perp}\|_0^2, \quad (3.59)$$

implying that

$$\left\| \int_{\tau_0}^{\tau} \pi_S \mathcal{R}_N(t, \Psi^{\tau_0, t}(z); X) dt \right\| \lesssim_{s, N} \|z_{\perp}\|_0^2.$$

The component $\pi_{\perp} \int_{\tau_0}^{\tau} \mathcal{R}_N(t, z; Y_{\tau_0}) dt$ is estimated by Gronwall's lemma. First we have to determine the terms in $\mathcal{R}_{N,k}^{(\pm,3)}(t, z; \tau_0)$ which contain $\mathcal{R}_N(t, \Psi^{\tau_0, t}(z); X)$. By the definition of $\mathcal{R}_{N,k}^{(\pm,3)}(t, z; \tau_0)$ (cf. (3.34), (3.40)),

$$\int_{\tau_0}^{\tau} a_k^{\pm}(t, z; \tau_0) \mathcal{R}_{N,k}^{(\pm,3)}(t, z; \tau_0) dt = \int_{\tau_0}^{\tau} a_k^{\pm}(t, z; \tau_0) \mathcal{R}_{N,k}^{(\pm,1)}(t, z; \tau_0) dt + \sum_{j=0}^{N-k} \int_{\tau_0}^{\tau} a_k^{\pm}(t, z; \tau_0) \mathcal{R}_{N,k,j}^{(\pm,2)}(t, z; \tau_0) dt.$$

Note that the terms $\mathcal{R}_{N,k,j}^{(\pm,2)}(t, z; \tau_0)$ do not involve $\mathcal{R}_N(t, \Psi^{\tau_0, t}(z); X)$ (cf. (3.32), (3.42)), whereas in contrast, $\mathcal{R}_{N,k}^{(\pm,1)}(t, z; \tau_0)$ do. Indeed, by the definitions (3.31) and (3.41), $\int_{\tau_0}^{\tau} a_k^{\pm}(t, z; \tau_0) \mathcal{R}_{N,k}^{(\pm,1)}(t, z; \tau_0) dt$ equals

$$\begin{aligned} &\int_{\tau_0}^{\tau} a_k^{\pm}(t, z; \tau_0) \cdot (\pm D)^{-k} \left(\sum_{j=N-k+1}^N a_j^{\pm}(z; \Psi^{\tau_0, t}) \cdot (\pm D)^{-j} [(\mathcal{F}_{N_S}^{\pm})^{-1} z_{\perp}] \right) dt \\ &+ \int_{\tau_0}^{\tau} a_k^{\pm}(t, z; \tau_0) \cdot (\pm D)^{-k} [(\mathcal{F}_{N_S}^{\pm})^{-1} \pi_{\perp} \mathcal{R}_N(z; \Psi^{\tau_0, t})] dt. \end{aligned}$$

One then infers from (3.58) that $\pi_{\perp} \mathcal{R}_N(z; \Psi^{\tau_0, \tau})$ satisfies the integral equation

$$\begin{aligned} \pi_{\perp} \mathcal{R}_N(z; \Psi^{\tau_0, \tau}) &= B_N(\tau, z; \tau_0) + \mathcal{F}_{N_S}^+ \circ \int_{\tau_0}^{\tau} \sum_{k=0}^N a_k^+(t, z; \tau_0) \cdot D^{-k} [(\mathcal{F}_{N_S}^+)^{-1} \pi_{\perp} \mathcal{R}_N(z; \Psi^{\tau_0, t})] dt \\ &+ \mathcal{F}_{N_S}^- \circ \int_{\tau_0}^{\tau} \sum_{k=0}^N a_k^-(t, z; \tau_0) \cdot (-D)^{-k} [(\mathcal{F}_{N_S}^-)^{-1} \pi_{\perp} \mathcal{R}_N(z; \Psi^{\tau_0, t})] dt, \end{aligned} \quad (3.60)$$

where

$$\begin{aligned}
B_N(\tau, z; \tau_0) &:= \int_{\tau_0}^{\tau} \pi_{\perp} \mathcal{R}_N(t, \Psi^{\tau_0, t}(z); X) dt + \mathcal{F}_{N_S}^+ \circ \sum_{k=0}^N \sum_{j=0}^{N-k} \int_{\tau_0}^{\tau} a_k^+(t, z; \tau_0) \mathcal{R}_{N, k, j}^{(+, 2)}(t, z; \tau_0) dt \\
&\quad + \mathcal{F}_{N_S}^+ \circ \sum_{k=0}^N \sum_{j=N-k+1}^N \int_{\tau_0}^{\tau} a_k^+(t, z; \tau_0) \cdot D^{-k} (a_j^+(z; \Psi^{\tau_0, t}) \cdot D^{-j} [(\mathcal{F}_{N_S}^+)^{-1} z_{\perp}]) dt \\
&\quad + \mathcal{F}_{N_S}^- \circ \sum_{k=0}^N \sum_{j=0}^{N-k} \int_{\tau_0}^{\tau} a_k^-(t, z; \tau_0) \mathcal{R}_{N, k, j}^{(-, 2)}(t, z; \tau_0) dt \\
&\quad + \mathcal{F}_{N_S}^- \circ \sum_{k=0}^N \sum_{j=N-k+1}^N \int_{\tau_0}^{\tau} a_k^-(t, z; \tau_0) \cdot (-D)^{-k} (a_j^-(z; \Psi^{\tau_0, t}) \cdot (-D)^{-j} [(\mathcal{F}_{N_S}^-)^{-1} z_{\perp}]) dt.
\end{aligned} \tag{3.61}$$

By the estimates of $\mathcal{R}_{N, k, j}^{(\pm, 2)}(t, z; \tau_0)$ in (3.33), (3.43), the ones of $a_k^{\pm}(\tau, z; X)$ and $\mathcal{R}_N(\tau, z; X)$ in Lemma 3.4, the estimates of $\Psi^{\tau_0, t}(z)$ in Lemma 3.3, and the ones of $a_k^{\pm}(z; \Psi^{\tau_0, \tau})$ in (3.54), and using the interpolation Lemma D.4, one obtains for any $s \geq 0$,

$$\|B_N(\tau, z; \tau_0)\|_{s+N+1} \lesssim_{s, N} \|z_{\perp}\|_s \|z_{\perp}\|_0, \quad \forall z \in \mathcal{V}' \cap h_0^s, \quad \forall 0 \leq \tau_0, \tau \leq 1.$$

Note that $\sum_{k=0}^N a_k^{\pm}(t, z; \tau_0)(\pm D)^{-k}$ is a pseudo-differential operator of order 0 where the coefficients $a_k^{\pm}(t, z; \tau_0)$ satisfy $\|a_k^{\pm}(t, z; \tau_0)\|_s \lesssim_{s, k} \|z_{\perp}\|_0^2$ (cf. (3.52)). Hence for any $z \in \mathcal{V}' \cap h_0^s$, $0 \leq \tau_0, \tau \leq 1$,

$$\left\| \sum_{k=0}^N a_k^{\pm}(t, z; \tau_0)(\pm D)^{-k} [(\mathcal{F}_{N_S}^{\pm})^{-1} \pi_{\perp} \mathcal{R}_N(z; \Psi^{\tau_0, t})] \right\|_{s+N+1} \lesssim_{s, N} \|z_{\perp}\|_0^2 \|(\mathcal{F}_{N_S}^{\pm})^{-1} \pi_{\perp} \mathcal{R}_N(z; \Psi^{\tau_0, t})\|_{s+N+1}.$$

- 1 By Gronwall's Lemma and since \mathcal{V}'_{\perp} is a ball of sufficiently small radius, the integral equation (3.60) yields
2 that for any $s \geq 0$,

$$\|\pi_{\perp} \mathcal{R}_N(z; \Psi^{\tau_0, \tau})\|_{s+N+1} \lesssim_{s, N} \|z_{\perp}\|_s \|z_{\perp}\|_0, \quad \forall z \in \mathcal{V}' \cap h_0^s, \quad \forall 0 \leq \tau_0, \tau \leq 1, \tag{3.62}$$

- 3 which together with (3.59) proves the claimed estimates of $\mathcal{R}_N(z; \Psi^{\tau_0, \tau})$. The stated analyticity property of
4 $\mathcal{R}_N(z; \Psi_X^{\tau_0, \tau})$ then follows from the already established analyticity properties of $\Psi_X^{\tau_0, \tau}(z)$, $a_k^{\pm}(\tau, z; \tau_0)$, and
5 $a_k^{\pm}(z; \Psi^{\tau_0, \tau})$ (cf. e.g. [10, Theorem A.5]).

Estimates of the derivatives of $\mathcal{R}_N(z; \Psi^{\tau_0, \tau})$: The estimates of the derivatives of $\mathcal{R}_N(z; \Psi^{\tau_0, \tau})$ can be obtained in a similar way as the ones for $\mathcal{R}_N(z; \Psi^{\tau_0, \tau})$. Indeed, for any $s \geq 0$, $z \in \mathcal{V}' \cap h_0^s$, $0 \leq \tau_0, \tau \leq 1$, $\hat{z} \in h_0^s$, $d\mathcal{R}_N(z; \Psi^{\tau_0, \tau})[\hat{z}] = \int_{\tau_0}^{\tau} d\mathcal{R}_N(t, z; Y_{\tau_0})[\hat{z}] dt$ can be computed as

$$\begin{aligned}
&\left(0, \mathcal{F}_{N_S}^+ \circ \int_{\tau_0}^{\tau} \sum_{k=0}^N d(a_k^+(t, z; \tau_0) \mathcal{R}_{N, k}^{(+, 3)}(t, z; \tau_0))[\hat{z}] dt + \mathcal{F}_{N_S}^- \circ \int_{\tau_0}^{\tau} \sum_{k=0}^N d(a_k^-(t, z; \tau_0) \mathcal{R}_{N, k}^{(-, 3)}(t, z; \tau_0))[\hat{z}] dt \right) \\
&\quad + \int_{\tau_0}^{\tau} d(\mathcal{R}_N(t, \Psi^{\tau_0, t}(z); X))[\hat{z}] dt.
\end{aligned}$$

- 6 Again, we estimate $\pi_S(d\mathcal{R}_N(z; \Psi^{\tau_0, \tau})[\hat{z}]) = \int_{\tau_0}^{\tau} \pi_S(d(\mathcal{R}_N(t, z; Y_{\tau_0}))[\hat{z}]) dt$ and $\pi_{\perp}(d\mathcal{R}_N(z; \Psi^{\tau_0, \tau})[\hat{z}])$ sepa-
7 rately. By Lemma 3.4, Lemma 3.3, and the chain rule, one has

$$\left\| \int_{\tau_0}^{\tau} \pi_S(d(\mathcal{R}_N(t, \Psi^{\tau_0, t}(z); X))[\hat{z}]) dt \right\| \lesssim_N \|z_{\perp}\|_0 \|\hat{z}\|_0, \tag{3.63}$$

whereas by (3.60), $\pi_{\perp}(d\mathcal{R}_N(z; \Psi^{\tau_0, \tau})[\hat{z}])$ satisfies

$$\begin{aligned}
\pi_{\perp}(d\mathcal{R}_N(z; \Psi^{\tau_0, \tau})[\hat{z}]) &= B_N^{(1)}(\tau, z; \tau_0)[\hat{z}] + \mathcal{F}_{N_S}^+ \circ \int_{\tau_0}^{\tau} \sum_{k=0}^N a_k^+(t, z; \tau_0) \cdot D^{-k} [(\mathcal{F}_{N_S}^+)^{-1} \pi_{\perp} d\mathcal{R}_N(z; \Psi^{\tau_0, t})[\hat{z}]] dt \\
&\quad + \mathcal{F}_{N_S}^- \circ \int_{\tau_0}^{\tau} \sum_{k=0}^N a_k^-(t, z; \tau_0) \cdot (-D)^{-k} [(\mathcal{F}_{N_S}^-)^{-1} \pi_{\perp} d\mathcal{R}_N(z; \Psi^{\tau_0, t})[\hat{z}]] dt,
\end{aligned} \tag{3.64}$$

with $B_N^{(1)}(\tau, z; \tau_0)[\widehat{z}]$ given by

$$\begin{aligned} B_N^{(1)}(\tau, z; \tau_0)[\widehat{z}] = & dB_N(\tau, z; \tau_0)[\widehat{z}] + \mathcal{F}_{N_S}^+ \circ \int_{\tau_0}^{\tau} \sum_{k=0}^N da_k^+(t, z; \tau_0)[\widehat{z}] \cdot D^{-k}[(\mathcal{F}_{N_S}^+)^{-1} \pi_{\perp} \mathcal{R}_N(z; \Psi^{\tau_0, t})] dt \\ & + \mathcal{F}_{N_S}^- \circ \int_{\tau_0}^{\tau} \sum_{k=0}^N da_k^-(t, z; \tau_0)[\widehat{z}] \cdot (-D)^{-k}[(\mathcal{F}_{N_S}^-)^{-1} \pi_{\perp} \mathcal{R}_N(z; \Psi^{\tau_0, t})] dt. \end{aligned}$$

Since

$$\|B_N^{(1)}(\tau, z; \tau_0)[\widehat{z}]\|_{s+N+1} \lesssim_{s, N} \|z_{\perp}\|_s \|\widehat{z}\|_0 + \|z_{\perp}\|_0 \|\widehat{z}\|_s, \quad \forall z \in \mathcal{V}' \cap h_0^s, \widehat{z} \in h_0^s, 0 \leq \tau_0, \tau \leq 1,$$

1 we infer from (3.64) by Gronwall's lemma that

$$\|\pi_{\perp}(d\mathcal{R}_N(z; \Psi^{\tau_0, \tau})[\widehat{z}])\|_{s+N+1} \lesssim_{s, N} \|z_{\perp}\|_s \|\widehat{z}\|_0 + \|z_{\perp}\|_0 \|\widehat{z}\|_s, \quad \forall z \in \mathcal{V}' \cap h_0^s, \widehat{z} \in h_0^s, 0 \leq \tau_0, \tau \leq 1, \quad (3.65)$$

2 which together with (3.63) proves the claimed estimate for $d\mathcal{R}_N(z; \Psi^{\tau_0, \tau})[\widehat{z}]$. In a similar fashion, one derives
3 the estimates for $\|d^l \mathcal{R}_N(z; \Psi^{\tau_0, \tau})[\widehat{z}_1, \dots, \widehat{z}_l]\|_{s+N+1}$ with $\widehat{z}_1, \dots, \widehat{z}_l \in h_0^s, l \geq 2$. \square

4 It turns out that the flow maps $\Psi_X^{\tau_0, \tau}$ and hence the symplectic corrector Ψ_C and its inverse Ψ_C^{-1} preserve
5 the reversible structures, introduced in Section 1. To state the result in more detail, note that without loss
6 of generality, we may assume that the neighborhood $\mathcal{V}' = \mathcal{V}'_S \times \mathcal{V}'_{\perp}$ (cf Lemma 3.3) is invariant under the
7 map \mathcal{S}_{rev} .

Addendum to Theorem 3.1 (i) For any $0 \leq \tau_0, \tau \leq 1$, $\Psi_X^{\tau_0, \tau} \circ \mathcal{S}_{rev} = \mathcal{S}_{rev} \circ \Psi_X^{\tau_0, \tau}$ on \mathcal{V}' and for any
 $z \in \mathcal{V}'$, $x \in \mathbb{R}$, $N \in \mathbb{N}$, and $0 \leq k \leq N$, the coefficients $a_k^{\pm}(z; \Psi_X^{\tau_0, \tau})$ and the remainder term $\mathcal{R}_N(z; \Psi_X^{\tau_0, \tau})$ of
the expansion (3.24) of $\Psi_X^{\tau_0, \tau}(z)$ satisfy

$$\begin{aligned} a_k^+(\mathcal{S}_{rev} z; \Psi_X^{\tau_0, \tau})(x) &= a_k^-(z; \Psi_X^{\tau_0, \tau})(-x), \quad a_k^-(\mathcal{S}_{rev} z; \Psi_X^{\tau_0, \tau})(x) = a_k^+(z; \Psi_X^{\tau_0, \tau})(-x), \\ \mathcal{R}_N(\mathcal{S}_{rev} z; \Psi_X^{\tau_0, \tau}) &= \mathcal{S}_{rev}(\mathcal{R}_N(z; \Psi_X^{\tau_0, \tau})). \end{aligned}$$

8 (ii) As a consequence, Ψ_C and Ψ_C^{-1} are invariant under \mathcal{S}_{rev} on \mathcal{V}' ,

$$\Psi_C \circ \mathcal{S}_{rev} = \mathcal{S}_{rev} \circ \Psi_C, \quad \Psi_C^{-1} \circ \mathcal{S}_{rev} = \mathcal{S}_{rev} \circ \Psi_C^{-1} \quad (3.66)$$

and for any $z \in \mathcal{V}'$, $x \in \mathbb{R}$, $N \in \mathbb{N}$, and $0 \leq k \leq N$,

$$\begin{aligned} a_k^+(\mathcal{S}_{rev} z; \Psi_C)(x) &= a_k^-(z; \Psi_C)(-x), \quad a_k^-(\mathcal{S}_{rev} z; \Psi_C)(x) = a_k^+(z; \Psi_C)(-x), \\ \mathcal{R}_N(\mathcal{S}_{rev} z; \Psi_C) &= \mathcal{S}_{rev}(\mathcal{R}_N(z; \Psi_C)). \end{aligned}$$

Proof of Addendum to Theorem 3.1 Clearly, item (ii) is a direct consequence of item (i). By the
Addendum to Lemma 2.2, the operator $\mathcal{L}(z)$, introduced in (2.31), satisfies $\mathcal{L}(\mathcal{S}_{rev} z) \circ \mathcal{S}_{rev} = -\mathcal{S}_{rev} \circ \mathcal{L}(z)$
on \mathcal{V} . It implies that for any $z \in \mathcal{V}$, $\mathcal{E}(\mathcal{S}_{rev} z) = -\mathcal{S}_{rev} \mathcal{E}(z)$ where $\mathcal{E}(z)$ has been introduced in (3.6).
Altogether we then conclude that the vector field $X(\tau, z)$, introduced in (3.16), satisfies

$$X(\tau, \mathcal{S}_{rev} z) = \mathcal{S}_{rev} X(\tau, z), \quad \forall z \in \mathcal{V}, 0 \leq \tau \leq 1$$

and hence by the uniqueness of the initial value problem of $\partial_{\tau} z = X(\tau, z)$, the solution map satisfies

$$\Psi_X^{\tau_0, \tau}(\mathcal{S}_{rev} z) = \mathcal{S}_{rev} \Psi_X^{\tau_0, \tau}(z), \quad \forall z \in \mathcal{V}', 0 \leq \tau_0, \tau \leq 1.$$

9 The claimed identities for $a_k^{\pm}(z; \Psi_X^{\tau_0, \tau})$ and $\mathcal{R}_N(z; \Psi_X^{\tau_0, \tau})$ then follow from the expansion (3.23). \square

We finish this section with a discussion of two applications of Theorem 3.1. The first one concerns the
expansion of the transpose $(d\Psi_X^{0, \tau}(z))^{\top}$ of the differential $d\Psi_X^{0, \tau}(z)$ which will be used in Section 4 in the
proof of Lemma 4.10. Recall that for any $z \in \mathcal{V}'$, and $\widehat{z}, \widehat{w} \in h_0^0$,

$$\Lambda_{\tau}(z)[\widehat{z}, \widehat{w}] = \langle \widehat{z} | J^{-1} \mathcal{L}_{\tau}(z) [\widehat{w}] \rangle, \quad \mathcal{L}_{\tau}(z) = \text{Id} + \tau J \mathcal{L}(z), \quad 0 \leq \tau \leq 1,$$

and that the flow $\Psi_X^{0,\tau}$ satisfies $\partial_\tau((\Psi_X^{0,\tau})^*\Lambda_\tau) = 0$ and hence $(\Psi_X^{0,\tau})^*\Lambda_\tau = \Lambda_0$. By the definition of the pullback this means that for any $z \in V'$, $0 \leq \tau \leq 1$, and $\widehat{z}, \widehat{w} \in h_0^0$,

$$\langle d\Psi_X^{0,\tau}(z)[\widehat{z}] | J^{-1}\mathcal{L}_\tau(\Psi_X^{0,\tau}(z))[d\Psi_X^{0,\tau}(z)\widehat{w}] \rangle = \langle \widehat{z} | J^{-1}\widehat{w} \rangle$$

implying that

$$(d\Psi_X^{0,\tau}(z))^\top J^{-1}\mathcal{L}_\tau(\Psi_X^{0,\tau}(z))d\Psi_X^{0,\tau}(z) = J^{-1}.$$

1 Using that $(d\Psi_X^{0,\tau}(z))^{-1} = d\Psi_X^{\tau,0}(\Psi_X^{0,\tau}(z))$ one obtains the following formula for $(d\Psi_X^{0,\tau}(z))^\top$,

$$(d\Psi_X^{0,\tau}(z))^\top = J^{-1} \circ d\Psi_X^{\tau,0}(\Psi_X^{0,\tau}(z)) \circ (\mathcal{L}_\tau(\Psi_X^{0,\tau}(z)))^{-1} \circ J. \quad (3.67)$$

2 Note that $d\Psi_X^{\tau,0}(\Psi_X^{0,\tau}(z))$ and $(\mathcal{L}_\tau(\Psi_X^{0,\tau}(z)))^{-1}$ are bounded linear operators on h_0^0 , implying that $(d\Psi_X^{0,\tau}(z))^\top$ is one on h_0^0 , and that these operators and their derivatives depend continuously on $0 \leq \tau \leq 1$.

Corollary 3.1. *For any $0 \leq \tau \leq 1$, $z \in \mathcal{V}'$, the transpose $(d\Psi_X^{0,\tau}(z))^\top$ (with respect to the standard inner product) of the differential $d\Psi_X^{0,\tau}(z)$ is a bounded linear operator $(d\Psi_X^{0,\tau}(z))^\top : h_0^1 \rightarrow h_0^1$ and for any $N \in \mathbb{N}$, $(d\Psi_X^{0,\tau}(z))^\top$ admits an expansion of the form*

$$\text{Id} + \iota_\perp \circ \mathcal{OP}(z; (d\Psi_X^{0,\tau})^\top) + \mathcal{R}_N(z; (d\Psi_X^{0,\tau})^\top),$$

where for any $\widehat{z} \in h_0^1$, $\mathcal{OP}(z; (d\Psi_X^{0,\tau})^\top)[\widehat{z}]$ is given by

$$\begin{aligned} & \mathcal{F}_{N_S}^+ \circ \sum_{k=0}^N a_k^+(z; (d\Psi_X^{0,\tau})^\top) \cdot D^{-k}[(\mathcal{F}_{N_S}^+)^{-1}\widehat{z}_\perp] + \mathcal{F}_{N_S}^+ \circ \sum_{k=0}^N \mathcal{A}_k^+(z; (d\Psi_X^{0,\tau})^\top)[\widehat{z}] \cdot D^{-k}[(\mathcal{F}_{N_S}^+)^{-1}z_\perp] \\ & + \mathcal{F}_{N_S}^- \circ \sum_{k=0}^N a_k^-(z; (d\Psi_X^{0,\tau})^\top) \cdot (-D)^{-k}[(\mathcal{F}_{N_S}^-)^{-1}\widehat{z}_\perp] + \mathcal{F}_{N_S}^- \circ \sum_{k=0}^N \mathcal{A}_k^-(z; (d\Psi_X^{0,\tau})^\top)[\widehat{z}] \cdot (-D)^{-k}[(\mathcal{F}_{N_S}^-)^{-1}z_\perp] \end{aligned}$$

and for any integer $s \geq 0$, $k \geq 0$, and $N \geq 1$,

$$\begin{aligned} & a_k^\pm(\cdot; (d\Psi_X^{0,\tau})^\top) : \mathcal{V}' \rightarrow H_\mathbb{C}^s, \quad z \mapsto a_k^\pm(z; (d\Psi_X^{0,\tau})^\top), \\ & \mathcal{A}_k^\pm(\cdot; (d\Psi_X^{0,\tau})^\top) : \mathcal{V}' \rightarrow \mathcal{B}(h_0^1, H_\mathbb{C}^s), \quad z \mapsto \mathcal{A}_k^\pm(z; (d\Psi_X^{0,\tau})^\top), \\ & \mathcal{R}_N(\cdot; (d\Psi_X^{0,\tau})^\top) : \mathcal{V}' \cap h_0^s \rightarrow \mathcal{B}(h_0^{s+1}, h_0^{s+1+N+1}), \quad z \mapsto \mathcal{R}_N(z; (d\Psi_X^{0,\tau})^\top) \end{aligned}$$

are real analytic maps. Furthermore, for any $z \in \mathcal{V}'$, $k \geq 0$ and $\widehat{z} \in h_0^1$,

$$a_k^-(z; (d\Psi_X^{0,\tau})^\top) = \overline{a_k^+(z; (d\Psi_X^{0,\tau})^\top)}, \quad \mathcal{A}_k^-(z; (d\Psi_X^{0,\tau})^\top)[\widehat{z}] = \overline{\mathcal{A}_k^+(z; (d\Psi_X^{0,\tau})^\top)[\widehat{z}]}.$$

If in addition, $\widehat{z}_1, \dots, \widehat{z}_l \in h_0^0$, $l \geq 1$, then

$$\|a_k^\pm(z; (d\Psi_X^{0,\tau})^\top)\|_s \lesssim_{s,k} \|z_\perp\|_0^2, \quad \|da_k^\pm(z; (d\Psi_X^{0,\tau})^\top)[\widehat{z}_1]\|_s \lesssim_{s,k} \|z_\perp\|_0 \|\widehat{z}_1\|_0,$$

$$\|d^l a_k^\pm(z; (d\Psi_X^{0,\tau})^\top)[\widehat{z}_1, \dots, \widehat{z}_l]\|_s \lesssim_{s,k,l} \prod_{j=1}^l \|\widehat{z}_j\|_0,$$

and

$$\|\mathcal{A}_k^\pm(z; (d\Psi_X^{0,\tau})^\top)[\widehat{z}]\|_s \lesssim_{s,k} \|z_\perp\|_0 \|\widehat{z}\|_1, \quad \|d^l(\mathcal{A}_k^\pm(z; (d\Psi_X^{0,\tau})^\top)[\widehat{z}])[\widehat{z}_1, \dots, \widehat{z}_l]\|_s \lesssim_{s,k,l} \|\widehat{z}\|_1 \prod_{j=1}^l \|\widehat{z}_j\|_0.$$

The remainder $\mathcal{R}_N(z; (d\Psi_X^{0,\tau})^\top)$ satisfies for any $z \in \mathcal{V}' \cap h_0^s$, $\widehat{z} \in h_0^{s+1}$, and $\widehat{z}_1, \dots, \widehat{z}_l \in h_0^s$, $l \in \mathbb{N}$,

$$\begin{aligned} & \|\mathcal{R}_N(z; (d\Psi_X^{0,\tau})^\top)[\widehat{z}]\|_{s+1+N+1} \lesssim_{s,N} \|z_\perp\|_0 \|\widehat{z}\|_{s+1} + \|z_\perp\|_s \|\widehat{z}\|_1, \\ & \|d^l(\mathcal{R}_N(z; (d\Psi_X^{0,\tau})^\top)[\widehat{z}])[\widehat{z}_1, \dots, \widehat{z}_l]\|_{s+1+N+1} \\ & \lesssim_{s,N,l} \|\widehat{z}\|_{s+1} \prod_{j=1}^l \|\widehat{z}_j\|_0 + \|\widehat{z}\|_1 \sum_{j=1}^l \|\widehat{z}_j\|_s \prod_{i \neq j} \|\widehat{z}_i\|_0 + \|z_\perp\|_s \|\widehat{z}\|_1 \prod_{j=1}^l \|\widehat{z}_j\|_0. \end{aligned}$$

1 **Remark 3.1.** Corollary 3.1 holds in particular for $d\Psi_C(z)^\top = (d\Psi_X^{0,1}(z))^\top$.

Proof. The starting point is the formula (3.67) for $d\Psi_X^{0,\tau}(z)^\top$. Since

$$\|J\hat{z}\|_s \lesssim_s \|\hat{z}\|_{s+1}, \quad \|J^{-1}\hat{z}\|_{s+1} \lesssim_s \|\hat{z}\|_s,$$

it suffices to derive corresponding estimates for the operator $d\Psi_X^{\tau,0}(\Psi_X^{0,\tau}(z)) \circ \mathcal{L}_\tau(\Psi_X^{0,\tau}(z))^{-1}$. By Theorem 3.1, for any $w \in \mathcal{V}'$ and $\hat{w} \in h_0^0$, one has

$$d\Psi_X^{\tau,0}(w) = \hat{w} + (0, d(\mathcal{OP}_N(w; \Psi_X^{\tau,0}))[\hat{w}] + d\mathcal{R}_N(w; \Psi_X^{\tau,0})[\hat{w}])$$

where $d(\mathcal{OP}_N(w; \Psi_X^{\tau,0}))[\hat{w}]$ is given by

$$\begin{aligned} & \mathcal{F}_{N_S}^+ \circ \sum_{k=0}^N a_k^+(w; \Psi_X^{\tau,0}) \cdot D^{-k}[(\mathcal{F}_{N_S}^+)^{-1}\hat{w}_\perp] + \mathcal{F}_{N_S}^+ \circ \sum_{k=0}^N da_k^+(w; \Psi_X^{\tau,0})[\hat{w}] \cdot D^{-k}[(\mathcal{F}_{N_S}^+)^{-1}[w_\perp] \\ & + \mathcal{F}_{N_S}^- \circ \sum_{k=0}^N a_k^-(w; \Psi_X^{\tau,0}) \cdot (-D)^{-k}[(\mathcal{F}_{N_S}^-)^{-1}\hat{w}_\perp] + \mathcal{F}_{N_S}^- \circ \sum_{k=0}^N da_k^-(w; \Psi_X^{\tau,0})[\hat{w}] \cdot (-D)^{-k}[(\mathcal{F}_{N_S}^-)^{-1}[w_\perp]. \end{aligned}$$

These formulas are applied to $w = \Psi_X^{0,\tau}(z) = z + (0, \mathcal{OP}_N(z; \Psi_X^{0,\tau})) + \mathcal{R}_N(z; \Psi_X^{0,\tau})$ where

$$\mathcal{OP}_N(z; \Psi_X^{0,\tau}) = \mathcal{F}_{N_S}^+ \circ \sum_{k=0}^N a_k^+(z; \Psi_X^{0,\tau}) \cdot D^{-k}[(\mathcal{F}_{N_S}^+)^{-1}z_\perp] + \mathcal{F}_{N_S}^- \circ \sum_{k=0}^N a_k^-(z; \Psi_X^{0,\tau}) \cdot (-D)^{-k}[(\mathcal{F}_{N_S}^-)^{-1}z_\perp].$$

By (3.15), $\mathcal{L}_\tau(w)^{-1} = (\text{Id} + \tau J\mathcal{L}(w))^{-1}$ is of the form

$$(\text{Id} + \tau J\mathcal{L}(w))^{-1} = \begin{pmatrix} C_{11}^{-1} & -\tau C_{11}^{-1}B_{12} \\ -\tau J_\perp \mathcal{L}_\perp^S(w)C_{11}^{-1} & \text{Id}_\perp + \tau^2 J_\perp \mathcal{L}_\perp^S(w)C_{11}^{-1}B_{12} \end{pmatrix},$$

2 with C_{11} and B_{12} given by (3.13) and (3.12), respectively. We then obtain an expansion of $\mathcal{L}_\tau(w)^{-1}$
3 from the expansion of $\mathcal{L}_\perp^S(w)$, provided by Lemma 2.2(ii). The formulas are then again applied to $w =$
4 $\Psi_X^{0,\tau}(z) = z + (0, \mathcal{OP}_N(z; \Psi_X^{0,\tau})) + \mathcal{R}_N(z; \Psi_X^{0,\tau})$. Combining these expansions, one obtains an expansion
5 of $d\Psi_X^{\tau,0}(\Psi_X^{0,\tau}(z)) \circ \mathcal{L}_\tau(\Psi_X^{0,\tau}(z))^{-1}$ as stated. The claimed estimates follow from Lemma 2.2 and Theorem
6 3.1. \square

As a second application of Theorem 3.1, we compute the Taylor expansion of the symplectic corrector $\Psi_C(z_S, z_\perp)$ in z_\perp around 0. This expansion will be needed in the subsequent section to show that the BO Hamiltonian, when expressed in the new coordinates provided by the map $\Psi_L \circ \Psi_C$, is in Birkhoff normal form up to order three. Note that by Theorem 3.1, for any $z_S \in \mathcal{V}'_S$, $\hat{z}_\perp \in h_\perp^0$, $0 \leq k \leq N$,

$$\begin{aligned} a_k^\pm((z_S, 0); \Psi_C) &= 0, & d_\perp a_k^\pm((z_S, 0); \Psi_C)[\hat{z}_\perp] &= 0, \\ \mathcal{R}_N((z_S, 0); \Psi_C) &= 0, & d_\perp(\mathcal{R}_N((z_S, 0); \Psi_C))[\hat{z}_\perp] &= 0. \end{aligned}$$

7 Hence the Taylor expansion of $\mathcal{R}_N(z; \Psi_C)$ in z_\perp of order three around 0 reads

$$\mathcal{R}_N(z; \Psi_C) = \mathcal{R}_{N,2}(z; \Psi_C) + \mathcal{R}_{N,3}(z; \Psi_C), \quad \mathcal{R}_{N,2}(z; \Psi_C) := \frac{1}{2}d_\perp^2 \mathcal{R}_N((z_S, 0); \Psi_C)[z_\perp, z_\perp] \quad (3.68)$$

8 with the Taylor remainder term $\mathcal{R}_{N,3}(z; \Psi_C)$ given by

$$\mathcal{R}_{N,3}(z; \Psi_C) = \int_0^1 d_\perp^3 \mathcal{R}_N((z_S, tz_\perp); \Psi_C)[z_\perp, z_\perp, z_\perp] \frac{1}{2}(1-t)^2 dt \quad (3.69)$$

9 whereas for any $0 \leq k \leq N$, $\mathcal{F}_{N_S}^\pm(a_k^\pm(z; \Psi_C)(\pm D)^{-k}[(\mathcal{F}_{N_S}^\pm)^{-1}z_\perp])$ vanishes in z_\perp at 0 up to order two.

Furthermore, we need to compute the Taylor expansion of $d\Psi_C(z)^\top = (d\Psi_X^{0,1}(z))^\top$ in z_\perp around 0. According to Corollary 3.1, the term $\mathcal{R}_N(z; d\Psi_C^\top)$ in the expansion of $d\Psi_C(z)^\top$ satisfies $\mathcal{R}_N((z_S, 0); d\Psi_C^\top) = 0$ for any $(z_S, 0) \in \mathcal{V}'$ and hence for any $\hat{z} \in h_0^1$, the Taylor expansion of $\mathcal{R}_N(z; d\Psi_C^\top)[\hat{z}]$ of order 2 in z_\perp around 0 reads

$$\mathcal{R}_N(z; d\Psi_C^\top)[\hat{z}] = \mathcal{R}_{N,1}(z; d\Psi_C^\top)[\hat{z}] + \mathcal{R}_{N,2}(z; d\Psi_C^\top)[\hat{z}], \quad (3.70)$$

where $\mathcal{R}_{N,2}(z; d\Psi_C^\top)[\hat{z}]$ denotes the Taylor remainder term of order 2.

Corollary 3.2. (i) For any $z' \in \mathcal{V}'$ and any integer $N \geq 1$, the Taylor expansion of the symplectic corrector $\Psi_C(z_S, z_\perp)$ in z_\perp around 0 reads

$$\Psi_C(z) = (z_S, 0) + (0, z_\perp) + \mathcal{R}_{N,2}(z; \Psi_C) + \Psi_{C,3}(z),$$

where

$$\Psi_{C,3}(z) \equiv \Psi_{C,N,3}(z) := (0, \mathcal{OP}_N(z; \Psi_C)) + \mathcal{R}_{N,3}(z; \Psi_C) \quad (3.71)$$

and $\mathcal{OP}_N(z; \Psi_C)$ is given by (3.25). For any $s \geq 0$, the map $\mathcal{V}' \cap h_0^s \rightarrow h_0^{s+N+1}$, $z \mapsto \mathcal{R}_{N,2}(z; \Psi_C)$, is real analytic and the following estimates hold: for any $z \in \mathcal{V}' \cap h_0^s$, $\hat{z} \in h_0^s$,

$$\|\mathcal{R}_{N,2}(z; \Psi_C)\|_{s+N+1} \lesssim_{s,N} \|z_\perp\|_s \|z_\perp\|_0, \quad \|d\mathcal{R}_{N,2}(z; \Psi_C)[\hat{z}]\|_{s+N+1} \lesssim_{s,N} \|z_\perp\|_0 \|\hat{z}\|_s + \|z_\perp\|_s \|\hat{z}\|_0.$$

If in addition, $\hat{z}_1, \dots, \hat{z}_l \in h_0^s$, $l \geq 2$,

$$\|d^l \mathcal{R}_{N,2}(z; \Psi_C)[\hat{z}_1, \dots, \hat{z}_l]\|_{s+N+1} \lesssim_{s,N,l} \sum_{j=1}^l \|\hat{z}_j\|_s \prod_{i \neq j} \|\hat{z}_i\|_0 + \|z_\perp\|_s \prod_{j=1}^l \|\hat{z}_j\|_0.$$

Similarly, for any $s \geq 0$, the map $\mathcal{V}' \cap h_0^s \rightarrow h_0^{s+N+1}$, $z \mapsto \mathcal{R}_{N,3}(z; \Psi_C)$, is real analytic and the following estimates hold: for any $z \in \mathcal{V}' \cap h_0^s$, $\hat{z}_1, \hat{z}_2 \in h_0^s$,

$$\|\mathcal{R}_{N,3}(z; \Psi_C)\|_{s+N+1} \lesssim_{s,N} \|z_\perp\|_s \|z_\perp\|_0^2, \quad \|d\mathcal{R}_{N,3}(z; \Psi_C)[\hat{z}_1]\|_{s+N+1} \lesssim_{s,N} \|z_\perp\|_s \|z_\perp\|_0 \|\hat{z}_1\|_0 + \|z_\perp\|_0^2 \|\hat{z}_1\|_s,$$

$$\|d^2 \mathcal{R}_{N,3}(z; \Psi_C)[\hat{z}_1, \hat{z}_2]\|_{s+N+1} \lesssim_{s,N} \|z_\perp\|_0 (\|\hat{z}_1\|_s \|\hat{z}_2\|_0 + \|\hat{z}_1\|_0 \|\hat{z}_2\|_s) + \|z_\perp\|_s \|\hat{z}_1\|_0 \|\hat{z}_2\|_0.$$

If in addition, $\hat{z}_1, \dots, \hat{z}_l \in h_0^s$, $l \geq 3$,

$$\|d^l \mathcal{R}_{N,3}(z; \Psi_C)[\hat{z}_1, \dots, \hat{z}_l]\|_{s+N+1} \lesssim_{s,N,l} \sum_{j=1}^l \|\hat{z}_j\|_s \prod_{i \neq j} \|\hat{z}_i\|_0 + \|z_\perp\|_s \prod_{j=1}^l \|\hat{z}_j\|_0.$$

(ii) For any integer $N \geq 1$ and $\hat{z} \in h_0^1$, the Taylor expansion of $d\Psi_C(z)^\top[\hat{z}]$ with respect to the component z_\perp of $z = (z_S, z_\perp)$ around 0 can be computed as

$$d\Psi_C(z)^\top[\hat{z}] = \hat{z} + \Psi_{C,1}^\top(z)[\hat{z}] + \Psi_{C,2}^\top(z)[\hat{z}],$$

where $\Psi_{C,1}^\top(z) := \mathcal{R}_{N,1}(z; d\Psi_C^\top)$ (cf. (3.70)), and for any $\hat{z} \in h_0^0$,

$$\Psi_{C,2}^\top(z)[\hat{z}] := (0, \mathcal{OP}(z; d\Psi_C^\top)[\hat{z}]) + \mathcal{R}_{N,2}(z; d\Psi_C^\top)[\hat{z}]$$

with $\mathcal{R}_{N,2}(z; d\Psi_C^\top)$ given by (3.70) and $\mathcal{OP}(z; d\Psi_C^\top)[\hat{z}]$ by Corollary 3.1,

$$\begin{aligned} & \mathcal{F}_{N_S}^+ \circ \sum_{k=0}^N a_k^+(z; d\Psi_C^\top) \cdot D^{-k}[(\mathcal{F}_{N_S}^+)^{-1} \hat{z}_\perp] + \mathcal{F}_{N_S}^+ \circ \sum_{k=0}^N \mathcal{A}_k^+(z; d\Psi_C^\top)[\hat{z}] \cdot D^{-k}[(\mathcal{F}_{N_S}^+)^{-1} z_\perp] \\ & + \mathcal{F}_{N_S}^- \circ \sum_{k=0}^N a_k^-(z; d\Psi_C^\top) \cdot (-D)^{-k}[(\mathcal{F}_{N_S}^-)^{-1} \hat{z}_\perp] + \mathcal{F}_{N_S}^- \circ \sum_{k=0}^N \mathcal{A}_k^-(z; d\Psi_C^\top)[\hat{z}] \cdot (-D)^{-k}[(\mathcal{F}_{N_S}^-)^{-1} z_\perp]. \end{aligned}$$

For any $i = 1, 2$, $s \geq 0$,

$$\mathcal{R}_{N,i}(\cdot; d\Psi_C^\top) : \mathcal{V}' \cap h_0^s \rightarrow \mathcal{B}(h_0^{s+1}, h_0^{s+1+N+1}), \quad z \mapsto \mathcal{R}_{N,i}(z; d\Psi_C^\top),$$

is a real analytic map. Furthermore, for any $z \in \mathcal{V}' \cap h_0^s$, $\widehat{z} \in h_0^{s+1}$,

$$\|\mathcal{R}_{N,1}(z; d\Psi_C^\top)[\widehat{z}]\|_{s+1+N+1} \lesssim_{s,N} \|z_\perp\|_0 \|\widehat{z}\|_{s+1} + \|z_\perp\|_s \|\widehat{z}\|_1.$$

If in addition, $\widehat{z}_1, \dots, \widehat{z}_l \in h_0^s$, $l \in \mathbb{N}$,

$$\begin{aligned} & \|d^l(\mathcal{R}_{N,1}(z; d\Psi_C^\top)[\widehat{z}])[\widehat{z}_1, \dots, \widehat{z}_l]\|_{s+1+N+1} \\ & \lesssim_{s,N,l} \|\widehat{z}\|_{s+1} \prod_{j=1}^l \|\widehat{z}_j\|_0 + \|\widehat{z}\|_1 \sum_{j=1}^l \|\widehat{z}_j\|_s \prod_{i \neq j} \|\widehat{z}_i\|_0 + \|z_\perp\|_s \|\widehat{z}\|_1 \prod_{j=1}^l \|\widehat{z}_j\|_0. \end{aligned}$$

Similarly, for any $z \in \mathcal{V}' \cap h_0^s$, $\widehat{z} \in h_0^{s+1}$, $\widehat{z}_1 \in h_0^s$,

$$\|\mathcal{R}_{N,2}(z; d\Psi_C^\top)[\widehat{z}]\|_{s+1+N+1} \lesssim_{s,N} \|z_\perp\|_0^2 \|\widehat{z}\|_{s+1} + \|z_\perp\|_s \|z_\perp\|_0 \|\widehat{z}\|_1,$$

and

$$\|d(\mathcal{R}_{N,2}(z; d\Psi_C^\top)[\widehat{z}])[\widehat{z}_1]\|_{s+1+N+1} \lesssim_{s,N} \|z_\perp\|_0 \|\widehat{z}_1\|_0 \|\widehat{z}\|_{s+1} + \|z_\perp\|_0 \|\widehat{z}_1\|_s \|\widehat{z}\|_1 + \|z_\perp\|_s \|\widehat{z}_1\|_0 \|\widehat{z}\|_1.$$

If in addition, $\widehat{z}_2, \dots, \widehat{z}_l \in h_0^s$, $l \geq 2$, then

$$\begin{aligned} & \|d^l(\mathcal{R}_{N,2}(z; d\Psi_C^\top)[\widehat{z}])[\widehat{z}_1, \dots, \widehat{z}_l]\|_{s+1+N+1} \\ & \lesssim_{s,N,l} \|\widehat{z}\|_{s+1} \prod_{j=1}^l \|\widehat{z}_j\|_0 + \|\widehat{z}\|_1 \sum_{j=1}^l \|\widehat{z}_j\|_s \prod_{i \neq j} \|\widehat{z}_i\|_0 + \|\widehat{z}\|_1 \|z_\perp\|_s \prod_{j=1}^l \|\widehat{z}_j\|_0. \end{aligned}$$

- ¹ *Proof.* (i) The claimed properties of $\mathcal{R}_{N,2}(z; \Psi_C)$ follow directly from Theorem 3.1. In view of the formula
² (3.69) the same is true for the ones of $\mathcal{R}_{N,3}(z; \Psi_C)$. Item (ii) is a direct consequence of Corollary 3.1. \square

³ 4 The BO Hamiltonian in new coordinates

In this section we provide an expansion of the transformed BO Hamiltonian $\mathcal{H} = H^{bo} \circ \Psi$ where the map $\Psi = \Psi_L \circ \Psi_C$ is the composition of Ψ_L (cf. Section 2) with the symplectic corrector Ψ_C (cf. Section 3) and H^{bo} is the BO Hamiltonian, introduced in (1.3),

$$H^{bo}(q) = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{1}{2} (|\partial_x|^{\frac{1}{2}} q)^2 - \frac{1}{3} q^3 \right) dx.$$

First we need to make some preliminary considerations. Recall that for any finite subset $S_+ \subset \mathbb{N}$, the map Ψ_S , introduced in (1.16), establishes a one to one correspondance between \mathcal{M}_S^o and the set M_S^o of proper S -gap potentials where $S = S_+ \cup (-S_+)$. For any proper S -gap potential q , the corresponding BO actions

$$I = (I_S, I_\perp), \quad I_S = (I_j)_{j \in S_+}, \quad I_\perp = (I_j)_{j \in S_+^\perp},$$

- ⁴ defined in terms of the Birkhoff coordinates $\Phi^{bo}(q)$, satisfy $I_\perp = 0$ and $I_j > 0$ for any $j \in S_+$. Denote by
⁵ $\Omega_S(I_S)$ and $\Omega_\perp(I_S)$ the diagonal linear operators defined by

$$\Omega_S(I_S) := \text{diag}((\Omega_n(I_S))_{n \in S}) : h_S \rightarrow h_S, (z_n)_{n \in S} \mapsto (\Omega_n(I_S) z_n)_{n \in S}, \quad (4.1)$$

⁶

$$\Omega_\perp(I_S) := \text{diag}((\Omega_n(I_S))_{n \in S^\perp}) : h_\perp^1 \rightarrow h_\perp^0, (z_n)_{n \in S^\perp} \mapsto (\Omega_n(I_S) z_n)_{n \in S^\perp}, \quad (4.2)$$

1 where for any $n \geq 1$, $\Omega_n(I_S)$ is defined by

$$\Omega_n(I_S) := \frac{1}{n} \omega_n^{bo}((I_S, 0)), \quad \Omega_{-n}(I_S) := \Omega_n(I_S) \quad (4.3)$$

and $\omega_n^{bo}(I)$, $n \geq 1$, are the BO frequencies, viewed as a function of the actions (cf. (1.11)),

$$\omega_n^{bo}(I) = n^2 - 2 \sum_{k=1}^{\infty} \min\{n, k\} I_k, \quad \forall n \geq 1.$$

2 **Lemma 4.1.** *For any finite gap potential $q \in M_S$, $n \neq 1$, one has*

$$\Omega_n(I_S) = |n| - \frac{2}{|n|} \sum_{k \in S_+} \min\{|n|, k\} I_k. \quad (4.4)$$

3 *In particular,*

$$\Omega_n(I_S) = |n| - (2 \sum_{k \in S_+} k I_k) \frac{1}{|n|}, \quad \forall |n| \geq N_S + 1. \quad (4.5)$$

4 For what follows, we need to consider the linearized Benjamin-Ono equation. Let $\hat{u}(t) = \partial_\varepsilon|_{\varepsilon=0} u_\varepsilon(t)$
 5 where $u_\varepsilon(t)$ is a one parameter family of solutions of (1.1), corresponding to a one parameter family of initial
 6 data $u_\varepsilon(0)$. Then $\hat{u}(t)$ satisfies the linearized BO equation,

$$\partial_t \hat{u}(t) = \partial_x d\nabla H^{bo}(u_0(t))[\hat{u}(t)]. \quad (4.6)$$

Furthermore, $z_\varepsilon(t) = \Phi^{bo}(u_\varepsilon(t))$ solves

$$\partial_t z_\varepsilon(t) = J\nabla \mathcal{H}^{bo}(z_\varepsilon(t)) = J\Omega(I_\varepsilon)z_\varepsilon(t), \quad I_\varepsilon = (I_{\varepsilon, n})_{n \geq 1} = \left(\frac{1}{n} z_{\varepsilon, n}(0) \cdot z_{\varepsilon, -n}(0)\right)_{n \geq 1}.$$

7 Hence

$$\hat{z}(t) := \partial_\varepsilon|_{\varepsilon=0} z_\varepsilon(t) = d\Phi^{bo}(u_0(t))[\hat{u}(t)] \quad (4.7)$$

8 satisfies the linear equation

$$\partial_t \hat{z}(t) = J\Omega(I_0)[\hat{z}(t)] + J\partial_\varepsilon|_{\varepsilon=0} \Omega(I_\varepsilon)[z_0(t)], \quad (4.8)$$

9 where for any $n \geq 1$,

$$\partial_\varepsilon|_{\varepsilon=0} \Omega_n(I_\varepsilon) = \sum_{k \geq 1} \partial_{I_k} \Omega_n(I_0) \frac{1}{k} (\partial_\varepsilon|_{\varepsilon=0} z_{\varepsilon, k}(0) \cdot z_{0, -k}(0) + z_{0, k}(0) \cdot \partial_\varepsilon|_{\varepsilon=0} z_{\varepsilon, -k}(0)). \quad (4.9)$$

Now assume that $t \mapsto q(t)$ is a solution of the BO equation (1.1) in M_S^0 with $z(t) := \Phi^{bo}(q(t)) \in \mathcal{V}$, $t \in \mathbb{R}$. Note that $z(t)$ is of the form $(z_S(t), 0)$, and the actions $I = (I_S, 0)$ of $q(t)$ satisfy $I_S = (\frac{1}{n} z_n(0) \cdot z_{-n}(0))_{n \in S_+}$ since $I = (I_n)_{n \geq 1}$ are independent of t . We need to investigate $\partial_x d\nabla H^{bo}(q(t))[\hat{q}(t)]$ where $\hat{q}(t)$ solves $\partial_t \hat{q}(t) = \partial_x d\nabla H^{bo}(q(t))[\hat{q}(t)]$ with $\hat{z}(0) = d\Phi^{bo}(q(0))[\hat{q}(0)]$ in h_0^2 and $\hat{z}_S(0) = 0$. One has

$$\hat{q}(0) = \Psi_1(z_S(0))[\hat{z}_\perp(0)] (= d\Psi^{bo}(z_S(0), 0)[0, \hat{z}_\perp(0)]), \quad \hat{z}_\perp(0) \in h_\perp^2.$$

By (4.8) - (4.9) it then follows that $\hat{z}_S(t) = 0$ for any $t \in \mathbb{R}$, hence $\hat{q}(t) = \Psi_1(z_S(t))[\hat{z}_\perp(t)]$, and that

$$\partial_t \hat{z}_\perp(t) = J_\perp \Omega_\perp(I_S)[\hat{z}_\perp(t)],$$

10 or more explicitly, for any $n \in S^\perp$,

$$\partial_t \hat{z}_n(t) = in \Omega_n(I_S) \hat{z}_n(t). \quad (4.10)$$

By differentiating $\hat{q}(t) = \Psi_1(z_S(t))[\hat{z}_\perp(t)]$ with respect to t , one gets

$$\begin{aligned} \partial_t \hat{q}(t) &= \Psi_1(z_S(t))[\partial_t \hat{z}_\perp(t)] + d_S(\Psi_1(z_S(t))[\hat{z}_\perp(t)])[\partial_t z_S(t)] \\ &= \Psi_1(z_S(t)) [J_\perp \Omega_\perp(I_S)[\hat{z}_\perp(t)]] + d_S(\Psi_1(z_S(t))[\hat{z}_\perp(t)])[\partial_t z_S(t)]. \end{aligned} \quad (4.11)$$

Comparing (4.6) and (4.11) and using that $\partial_t z_S(t) = J_S \Omega_S(I_S)[z_S(t)]$, one gets

$$\begin{aligned} \partial_x d\nabla H^{bo}(q(t))[\Psi_1(z_S(t))[\widehat{z}_\perp(t)]] &= \Psi_1(z_S(t))[J_\perp \Omega_\perp(I_S)[\widehat{z}_\perp(t)]] \\ &+ d_S(\Psi_1(z_S(t))[\widehat{z}_\perp(t)])[J_S \Omega_S(I_S)[z_S(t)]] . \end{aligned} \quad (4.12)$$

Applying $\Psi_1(z_S(t))^{-1}$ to both sides of (4.12), then yields

$$\begin{aligned} \Psi_1(z_S(t))^{-1} \partial_x d\nabla H^{bo}(q(t))[\Psi_1(z_S(t))[\widehat{z}_\perp(t)]] &= J_\perp \Omega_\perp(I_S)[\widehat{z}_\perp(t)] \\ &+ \Psi_1(z_S(t))^{-1} d_S(\Psi_1(z_S(t))[\widehat{z}_\perp(t)])[J_S \Omega_S(I_S)[z_S(t)]] . \end{aligned} \quad (4.13)$$

Since $\Psi_1(z_S)$ is symplectic one has $\Psi_1(z_S)^\top \partial_x^{-1} \Psi_1(z_S) = J_\perp^{-1}$ or $\Psi_1(z_S)^{-1} \partial_x = J_\perp \Psi_1(z_S)^\top$, implying that

$$\begin{aligned} J_\perp \Psi_1(z_S(t))^\top d\nabla H^{bo}(q(t))[\Psi_1(z_S(t))[\widehat{z}_\perp(t)]] &= J_\perp \Omega_\perp(I_S)[\widehat{z}_\perp(t)] \\ &+ \Psi_1(z_S(t))^{-1} d_S(\Psi_1(z_S(t))[\widehat{z}_\perp(t)])[J_S \Omega_S(I_S)[z_S(t)]] \end{aligned} \quad (4.14)$$

and hence for any $z_S \in \mathcal{V}_S$, $I_S = (\frac{1}{n} z_n z_{-n})_{n \in S_+}$, $q = \Psi^{bo}(z_S, 0)$, and $\widehat{z}_\perp \in h_\perp^2$,

$$\Psi_1(z_S)^\top d\nabla H^{bo}(q)[\Psi_1(z_S)[\widehat{z}_\perp]] = \Omega_\perp(I_S)[\widehat{z}_\perp] + \mathcal{G}(z_S)[\widehat{z}_\perp] \quad (4.15)$$

1 where $\mathcal{G}(z_S) : h_\perp^0 \rightarrow h_\perp^0$ is given by

$$\mathcal{G}(z_S)[\widehat{z}_\perp] := J_\perp^{-1} \Psi_1(z_S)^{-1} d_S(\Psi_1(z_S)[\widehat{z}_\perp])[J_S \Omega_S(I_S)[z_S]] . \quad (4.16)$$

2 In the next lemma we record an expansion for the operator $\mathcal{G}(z_S)$.

Lemma 4.2. *For any integer $N \geq 1$, the operator $\mathcal{G}(z_S) : h_\perp^0 \rightarrow h_\perp^0$ admits an expansion of the form $\mathcal{OP}(z_S; \mathcal{G}) + \mathcal{R}_N(z_S; \mathcal{G})$,*

$$\mathcal{OP}(z_S; \mathcal{G}) = \mathcal{F}_{N_S}^+ \circ \sum_{k=0}^N a_k^+(z_S; \mathcal{G}) D^{-k} \circ (\mathcal{F}_{N_S}^+)^{-1} + \mathcal{F}_{N_S}^- \circ \sum_{k=0}^N a_k^-(z_S; \mathcal{G}) (-D)^{-k} \circ (\mathcal{F}_{N_S}^-)^{-1} ,$$

where for any $0 \leq k \leq N$, $s \geq 0$, the maps

$$\mathcal{V}_S \rightarrow H_{\mathbb{C}}^s, \quad z_S \mapsto a_k^\pm(z_S; \mathcal{G}), \quad \mathcal{V}_S \mapsto \mathcal{B}(h_\perp^s, h_\perp^{s+N+1}), \quad z_S \mapsto \mathcal{R}_N(z_S; \mathcal{G}),$$

3 are real analytic and $a_k^-(z_S; \mathcal{G}) = \overline{a_k^+(z_S; \mathcal{G})}$.

4 *Proof.* In view of the definition (4.16) of \mathcal{G} , the lemma follows from Corollary 2.2 and Lemma D.2. \square

5 After this preliminary discussion, we can now study the transformed Hamiltonian $H^{bo} \circ \Psi$ where $\Psi =$
6 $\Psi_L \circ \Psi_C$. We split the analysis into two parts. First we expand $\mathcal{H}^{(1)} := H^{bo} \circ \Psi_L$ and then we analyze
7 $\mathcal{H}^{(2)} = \mathcal{H}^{(1)} \circ \Psi_C$.

8 **Expansion of $\mathcal{H}^{(1)} := H^{bo} \circ \Psi_L$**

9 To expand $H^{bo} \circ \Psi_L$, it is useful to write $H^{bo}(u)$ as $H^{bo}(u) = H_2^{bo}(u) + H_3^{bo}(u)$ where

$$H_2^{bo}(u) := \frac{1}{2} \langle |\partial_x| u, u \rangle, \quad H_3^{bo}(u) := \frac{1}{2\pi} \int_0^{2\pi} -\frac{1}{3} u^3 dx . \quad (4.17)$$

10 The L^2 -gradient ∇H^{bo} of H^{bo} and its derivative are then given by

$$\nabla H^{bo}(u) = |\partial_x| u - u^2, \quad d\nabla H^{bo}(u) = |\partial_x| - 2u . \quad (4.18)$$

Let $z_S \in \mathcal{V}_S$ and $q = \Psi^{bo}(z_S, 0)$ be given. The Taylor expansion of $H^{bo}(q + v)$ around q in direction $v = \Psi_1(z_S)[z_\perp]$ with $z_\perp \in \mathcal{V}_\perp \cap h_\perp^1$ reads

$$H^{bo}(q + v) = H^{bo}(q) + \langle \nabla H^{bo}(q), v \rangle + \frac{1}{2} \langle d\nabla H^{bo}(q)[v], v \rangle + \frac{1}{2\pi} \int_0^{2\pi} -\frac{1}{3} v^3 dx . \quad (4.19)$$

Since $v = d\Psi^{bo}(z_S, 0)[0, z_\perp]$ one has $\langle \nabla H^{bo}(q), v \rangle = \partial_\varepsilon|_{\varepsilon=0} H^{bo}(\Psi^{bo}(z_S, \varepsilon z_\perp))$. Recall that $\mathcal{H}^{bo} = H^{bo} \circ \Psi^{bo}$ is a function of the actions $I = (I_n)_{n \geq 1}$ alone and that $I_n = \frac{1}{n} z_n \cdot z_{-n}$, $n \geq 1$. It implies that

$$\partial_\varepsilon|_{\varepsilon=0} H^{bo}(\Psi^{bo}(z_S, \varepsilon z_\perp)) = \sum_{n \in S_+^\perp} \omega_n(I_S, 0) \partial_\varepsilon|_{\varepsilon=0} \varepsilon^2 I_n = 0$$

and hence $\langle \nabla H^{bo}(q), v \rangle = 0$. Since $\Psi_L(z) = q + \Psi_1(z_S)[z_\perp]$, the Hamiltonian $\mathcal{H}^{(1)}(z) = H^{bo}(\Psi_L(z))$ can be computed as

$$\mathcal{H}^{(1)}(z) = H^{bo}(q) + \frac{1}{2} \langle d\nabla H^{bo}(q)[\Psi_1(z_S)z_\perp], \Psi_1(z_S)[z_\perp] \rangle + \frac{1}{2\pi} \int_0^{2\pi} -\frac{1}{3} (\Psi_1(z_S)[z_\perp])^3 dx.$$

By formula (4.15),

$$\langle d\nabla H^{bo}(q)[\Psi_1(z_S)[z_\perp]], \Psi_1(z_S)[z_\perp] \rangle = \langle \Omega_\perp(I_S)[z_\perp], z_\perp \rangle + \langle \mathcal{G}(z_S)[z_\perp], z_\perp \rangle.$$

Since $\Psi_1(z_S)^\top \circ d\nabla H^{bo}(q) \circ \Psi_1(z_S)$ and $\Omega_\perp(I_S)$ are symmetric, so is the operator $\mathcal{G}(z_S)$. In summary,

$$\mathcal{H}^{(1)}(z) = \mathcal{H}_S^{bo}(z) + \frac{1}{2} \langle \Omega_\perp(I_S)[z_\perp], z_\perp \rangle + \mathcal{P}_2^{(1)}(z) + \mathcal{P}_3^{(1)}(z) \quad (4.20)$$

where for any $z = (z_S, z_\perp) \in \mathcal{V} \cap h_0^1$,

$$\mathcal{H}_S^{bo}(z) := H^{bo}(\Psi^{bo}(z_S, 0)), \quad \mathcal{P}_2^{(1)}(z) := \frac{1}{2} \langle \mathcal{G}(z_S)[z_\perp], z_\perp \rangle, \quad (4.21)$$

$$\mathcal{P}_3^{(1)}(z) := \frac{1}{2\pi} \int_0^{2\pi} -\frac{1}{3} (\Psi_1(z_S)[z_\perp])^3 dx.$$

Here, the superscript (1) in $\mathcal{P}_2^{(1)}(z)$ and $\mathcal{P}_3^{(1)}(z)$ refers to the Hamiltonian $\mathcal{H}^{(1)}$ whereas the subscripts in these functionals refer to the fact that $\mathcal{P}_2^{(1)}(z)$ is quadratic in z_\perp and $\mathcal{P}_3^{(1)}(z)$ is at least of order three in z_\perp . Furthermore, note that $\mathcal{H}_S^{bo}(z) = \mathcal{H}_S^{bo}(\Pi_S z)$ where we recall that $\Pi_S : h_S \times h_\perp^0 \rightarrow h_S \times h_\perp^0$ denotes the projection, given by $(\widehat{z}_S, \widehat{z}_\perp) \mapsto (\widehat{z}_S, 0)$ (cf. (3.8)).

Recall from (D.3) that for any $a \in H^1$, the paraproduct $T_a u$ of the function a with $v \in L^2$ with respect to the cut-off function χ is defined as $(T_a v)(x) = \sum_{k, n \in \mathbb{Z}} \chi(k, n) a_k v_n e^{i2\pi(k+n)x}$ with v_n , $n \in \mathbb{Z}$, denoting the Fourier coefficients of v and a_k , $k \in \mathbb{Z}$, the ones of a .

Lemma 4.3. *For any integer $N \geq 1$, there exists an integer $\sigma_N \geq N$ (loss of regularity) so that on $\mathcal{V} \cap h_0^{\sigma_N}$, the L^2 -gradient $\nabla \mathcal{P}_3^{(1)}$ of $\mathcal{P}_3^{(1)}$ admits the asymptotic expansion of the form $(0, \mathcal{OP}(z; \nabla \mathcal{P}_3^{(1)})) + \mathcal{R}_N(z; \nabla \mathcal{P}_3^{(1)})$, where $\mathcal{OP}(z; \nabla \mathcal{P}_3^{(1)})$ is the para-differential operator*

$$\mathcal{OP}(z; \nabla \mathcal{P}_3^{(1)}) = \mathcal{F}_{N_S}^+ \circ \sum_{k=0}^N T_{a_k^+(z; \nabla \mathcal{P}_3^{(1)})} D^{-k} [(\mathcal{F}_{N_S}^+)^{-1} z_\perp] + \mathcal{F}_{N_S}^- \circ \sum_{k=0}^N T_{a_k^-(z; \nabla \mathcal{P}_3^{(1)})} (-D)^{-k} [(\mathcal{F}_{N_S}^-)^{-1} z_\perp] \quad (4.22)$$

and where for any $s \geq 0$, $0 \leq k \leq N$, the maps

$$\mathcal{V} \cap h_0^{s+\sigma_N} \rightarrow H_{\mathbb{C}}^s, z \mapsto a_k^\pm(z; \nabla \mathcal{P}_3^{(1)}), \quad \mathcal{V} \cap h_0^{s \vee \sigma_N} \rightarrow h_0^{s+N+1}, z \mapsto \mathcal{R}_N(z; \nabla \mathcal{P}_3^{(1)})$$

are real analytic and $a_k^+(z; \nabla \mathcal{P}_3^{(1)}) = \overline{a_k^-(z; \nabla \mathcal{P}_3^{(1)})}$. Furthermore, for any $z \in \mathcal{V} \cap h_0^{s+\sigma_N}$ with $\|z\|_{\sigma_N} \leq 1$, $\|a_k^\pm(z; \nabla \mathcal{P}_3^{(1)})\|_s \lesssim_{s,N} \|z_\perp\|_{s+\sigma_N}$. If in addition $\widehat{z}_1, \dots, \widehat{z}_l \in h_0^{s+\sigma_N}$, $l \geq 1$, then

$$\|d^l a_k^\pm(z; \nabla \mathcal{P}_3^{(1)})[\widehat{z}_1, \dots, \widehat{z}_l]\|_s \lesssim_{s,k,l} \sum_{j=1}^l \|\widehat{z}_j\|_{s+\sigma_N} \prod_{i \neq j} \|\widehat{z}_i\|_{\sigma_N} + \|z_\perp\|_{s+\sigma_N} \prod_{j=1}^l \|\widehat{z}_j\|_{\sigma_N}. \quad (4.23)$$

Similarly, for any $z \in \mathcal{V} \cap h_0^{s \vee \sigma_N}$ with $\|z\|_{\sigma_N} \leq 1$, $\widehat{z} \in h_0^{s \vee \sigma_N}$, $\|\mathcal{R}_N(z; \nabla \mathcal{P}_3^{(1)})\|_{s+N+1} \lesssim_{s,N} \|z_\perp\|_{s \vee \sigma_N} \|z_\perp\|_{\sigma_N}$ and

$$\|d\mathcal{R}_N(z; \nabla \mathcal{P}_3^{(1)})[\widehat{z}]\|_{s+N+1} \lesssim_{s,N} \|z_\perp\|_{s \vee \sigma_N} \|\widehat{z}\|_{\sigma_N} + \|z_\perp\|_{\sigma_N} \|\widehat{z}\|_{s \vee \sigma_N}. \quad (4.24)$$

1 If in addition $\widehat{z}_1, \dots, \widehat{z}_l \in h_0^{s \vee \sigma_N}$, $l \geq 2$, then

$$\|d^l \mathcal{R}_N(z; \nabla \mathcal{P}_3^{(1)})[\widehat{z}_1, \dots, \widehat{z}_l]\|_{s+N+1} \lesssim_{s,N,l} \sum_{j=1}^l \|\widehat{z}_j\|_{s \vee \sigma_N} \prod_{i \neq j} \|\widehat{z}_i\|_{\sigma_N} + \|z_\perp\|_{s \vee \sigma_N} \prod_{j=1}^l \|\widehat{z}_j\|_{\sigma_N}. \quad (4.25)$$

Proof. By a straightforward calculation, one has $\nabla_\perp \mathcal{P}_3^{(1)}(z) = -\Psi_1(z_S)^\top (\Psi_1(z_S)[z_\perp])^2$. By the Bony decomposition given in Lemma D.1(ii),

$$(\Psi_1(z_S)[z_\perp])^2 = 2T_{\Psi_1(z_S)[z_\perp]} \Psi_1(z_S)[z_\perp] + \mathcal{R}^{(B)}(\Psi_1(z_S)[z_\perp], \Psi_1(z_S)[z_\perp]).$$

2 The expansion (4.22) and the stated estimates follow from Corollary 2.2, Corollary 2.4, and Lemmata D.1,
3 D.2, and D.4. \square

4 **Expansion of $\mathcal{H}^{(2)} := \mathcal{H}^{(1)} \circ \Psi_C$**

5 To compute the expansion of $\mathcal{H}^{(2)}(z) = \mathcal{H}^{(1)}(\Psi_C(z))$ on $\mathcal{V}' \cap h_0^1$, we study the composition of each of the
6 terms in (4.20) with the symplectic corrector Ψ_C separately. Recall that Ψ_C is defined on \mathcal{V}' and takes
7 values in \mathcal{V} .

Term \mathcal{H}_S^{bo} . By Corollary 3.2, $\Psi_C(z)$ has a Taylor expansion in z_\perp around 0 of the form

$$\Psi_C(z) = (z_S, 0) + (0, z_\perp) + \tilde{\Psi}_C(z), \quad \tilde{\Psi}_C(z) := \mathcal{R}_{N,2}(z; \Psi_C) + \Psi_{C,3}(z), \quad \Psi_{C,3}(z) \equiv \Psi_{C,N,3}(z),$$

where $\mathcal{R}_{N,2}(z; \Psi_C)$ is the term of order two, given by $\mathcal{R}_{N,2}(z; \Psi_C) = \frac{1}{2} d_\perp^2 \mathcal{R}_N((z_S, 0); \Psi_C)[z_\perp, z_\perp]$ (cf. (3.68)),
and $\Psi_{C,3}(z)$ is given by (3.71)

$$\Psi_{C,3}(z) = (0, \mathcal{OP}_N(z; \Psi_C)) + \mathcal{R}_{N,3}(z; \Psi_C)$$

with $\mathcal{R}_{N,3}(z; \Psi_C)$ denoting the Taylor remainder term (3.69). Since $\mathcal{H}_S^{bo}(z) = \mathcal{H}_S^{bo}(\Pi_S z)$ (cf. (4.21)), the
Taylor expansion of $\mathcal{H}_S^{bo}(\Psi_C(z)) = \mathcal{H}_S^{bo}(z + \tilde{\Psi}_C(z))$ reads

$$\mathcal{H}_S^{bo}(\Psi_C(z)) = \mathcal{H}_S^{bo}(z) + \langle \nabla_S \mathcal{H}_S^{bo}(z), \pi_S \mathcal{R}_{N,2}(z; \Psi_C) \rangle + \mathcal{P}_3^{(2a)}(z), \quad (4.26)$$

where $\mathcal{P}_3^{(2a)}(z)$ is the Taylor remainder term of order three, given by

$$\langle \nabla_S \mathcal{H}_S^{bo}(z), \pi_S \Psi_{C,3}(z) \rangle + \int_0^1 (1-\tau) \langle d_S \nabla_S \mathcal{H}_S^{bo}(z + \tau \tilde{\Psi}_C(z))[\pi_S \tilde{\Psi}_C(z)], \pi_S \tilde{\Psi}_C(z) \rangle d\tau$$

and $\pi_S : h_S \times h_\perp^0 \rightarrow h_S$ denotes the map given by $z = (z_S, z_\perp) \mapsto z_S$ (cf. (3.10)). Since $\pi_S \Psi_{C,3}(z) =$
 $\pi_S \mathcal{R}_{N,3}(z; \Psi_C)$ and $\pi_S \tilde{\Psi}_C(z) = \pi_S \mathcal{R}_N(z; \Psi_C) = \pi_S (\mathcal{R}_{N,2}(z; \Psi_C) + \mathcal{R}_{N,3}(z; \Psi_C))$ (cf. (3.68)), one has

$$\begin{aligned} \mathcal{P}_3^{(2a)}(z) &= \langle \nabla_S \mathcal{H}_S^{bo}(z), \pi_S \mathcal{R}_{N,3}(z; \Psi_C) \rangle \\ &+ \int_0^1 (1-\tau) \langle d_S \nabla_S \mathcal{H}_S^{bo}(z + \tau \tilde{\Psi}_C(z))[\pi_S \mathcal{R}_N(z; \Psi_C)], \pi_S \mathcal{R}_N(z; \Psi_C) \rangle d\tau. \end{aligned} \quad (4.27)$$

8 In the next lemma we show that $\nabla \mathcal{P}_3^{(2a)}(z)$ is in h_0^{s+N+1} for any $z \in \mathcal{V}' \cap h_0^s$.

Lemma 4.4. *The Hamiltonian $\mathcal{P}_3^{(2a)} : \mathcal{V}' \rightarrow \mathbb{R}$ is real analytic and for any integers $s \geq 0$, $N \geq 1$, the map
 $\mathcal{V}' \cap h_0^s \rightarrow h_0^{s+N+1}$, $z \mapsto \nabla \mathcal{P}_3^{(2a)}(z)$ is real analytic. Furthermore, for any $z \in \mathcal{V}' \cap h_0^s$, and $\widehat{z} \in h_0^s$,*

$$\|\nabla \mathcal{P}_3^{(2a)}(z)\|_{s+N+1} \lesssim_{s,N} \|z_\perp\|_s \|z_\perp\|_0, \quad \|d \nabla \mathcal{P}_3^{(2a)}(z)[\widehat{z}]\|_{s+N+1} \lesssim_{s,N} \|z_\perp\|_s \|\widehat{z}\|_0 + \|z_\perp\|_0 \|\widehat{z}\|_s.$$

If in addition $\widehat{z}_1, \dots, \widehat{z}_l \in h_0^s$, $l \geq 2$,

$$\|d^l \nabla \mathcal{P}_3^{(2a)}(z)[\widehat{z}_1, \dots, \widehat{z}_l]\|_{s+N+1} \lesssim_{s,N,l} \sum_{j=1}^l \|\widehat{z}_j\|_s \prod_{i \neq j} \|\widehat{z}_i\|_0 + \|z_\perp\|_s \prod_{j=1}^l \|\widehat{z}_j\|_0.$$

Proof. We begin by analyzing the first term $\langle \nabla_S \mathcal{H}_S^{bo}(z) | \pi_S R_{N,3}(z; \Psi_C) \rangle$ on the right hand side of (4.27). It is given by the finite sum $\sum_{n \in S} h_n(z)$ where

$$h_n(z) := (\nabla \mathcal{H}_S^{bo}(z))_n \cdot (R_{N,3}(z))_{-n} = \partial_{z_{-n}} \mathcal{H}_S^{bo}(z) \cdot \langle \mathcal{R}_{N,3}(z; \Psi_C), e_n \rangle, \quad \forall n \in S,$$

and $(e_n)_{n \in S}$ denotes the standard basis of h_S . The derivative of h_n in direction $\hat{z} \in h_0^0$ then reads

$$\begin{aligned} \langle \nabla h_n(z), \hat{z} \rangle &= \langle \nabla \partial_{z_{-n}} \mathcal{H}_S^{bo}(z), \hat{z} \rangle \cdot \langle \mathcal{R}_{N,3}(z; \Psi_C), e_n \rangle + \partial_{z_{-n}} \mathcal{H}_S^{bo}(z) \cdot \langle d\mathcal{R}_{N,3}(z; \Psi_C)[\hat{z}], e_n \rangle \\ &= \langle \nabla \partial_{z_{-n}} \mathcal{H}_S^{bo}(z), \hat{z} \rangle \cdot \langle \mathcal{R}_{N,3}(z; \Psi_C), e_n \rangle + \partial_{z_{-n}} \mathcal{H}_S^{bo}(z) \cdot \langle (d\mathcal{R}_{N,3}(z; \Psi_C))^\top [e_n], \hat{z} \rangle, \end{aligned}$$

implying that

$$\nabla h_n(z) = \langle \mathcal{R}_{N,3}(z; \Psi_C), e_n \rangle \cdot \nabla \partial_{z_{-n}} \mathcal{H}_S^{bo}(z) + \partial_{z_{-n}} \mathcal{H}_S^{bo}(z) \cdot (d\mathcal{R}_{N,3}(z; \Psi_C))^\top [e_n].$$

- 1 By Corollary 3.2, for any $s \geq 0$, $\mathcal{V}' \cap h_0^s \rightarrow h_0^{s+N+1}$, $z \mapsto \nabla h_n(z)$ is real analytic and satisfies the estimates
2 $\|\nabla h_n(z)\|_{s+N+1} \lesssim_{s,N} \|z_\perp\|_s \|z_\perp\|_0$. The estimates for the higher order derivatives of h_n , $n \in S$, are obtained
3 by differentiating the expression for $\nabla h_n(z)$ and using the estimates of Corollary 3.2.

In order to analyze the second term on the right hand side of (4.27) it suffices to study the functions $h_{n,k}(z; \tau)$, $n, k \in S$, given by

$$h_{n,k}(z; \tau) := \partial_{z_{-n}} \partial_{z_{-k}} \mathcal{H}_S^{bo}(z + \tau \tilde{\Psi}_C(z)) \cdot \langle \mathcal{R}_N(z; \Psi_C), e_n \rangle \cdot \langle \mathcal{R}_N(z; \Psi_C), e_k \rangle$$

where $0 \leq \tau \leq 1$. Clearly, $h_{n,k}(z; \tau)$ depends continuously on τ as do all the derivatives with respect to the variable z . Since $\mathcal{H}_S^{bo}(z + \tau \tilde{\Psi}_C(z))$ only depends on $\pi_S(z + \tau \tilde{\Psi}_C(z))$ one sees that

$$\begin{aligned} \langle \nabla h_{n,k}(z; \tau), \hat{z} \rangle &= \langle \nabla_S (\partial_{z_{-n}} \partial_{z_{-k}} \mathcal{H}_S^{bo}(z + \tau \tilde{\Psi}_C(z))), \pi_S(\text{Id} + \tau d\tilde{\Psi}_C(z))[\hat{z}] \rangle \cdot \langle \mathcal{R}_N(z; \Psi_C), e_n \rangle \cdot \langle \mathcal{R}_N(z; \Psi_C), e_k \rangle \\ &\quad + \partial_{z_{-n}} \partial_{z_{-k}} \mathcal{H}_S^{bo}(z + \tau \tilde{\Psi}_C(z)) \cdot \langle (d\mathcal{R}_N(z; \Psi_C))^\top [e_n], \hat{z} \rangle \cdot \langle \mathcal{R}_N(z; \Psi_C), e_k \rangle \\ &\quad + \partial_{z_{-n}} \partial_{z_{-k}} \mathcal{H}_S^{bo}(z + \tau \tilde{\Psi}_C(z)) \cdot \langle \mathcal{R}_N(z; \Psi_C), e_n \rangle \cdot \langle (d\mathcal{R}_N(z; \Psi_C))^\top [e_k], \hat{z} \rangle, \end{aligned}$$

implying that

$$\begin{aligned} \nabla h_{n,k}(z; \tau) &= (\text{Id} + \tau d\tilde{\Psi}_C(z))^\top [\Pi_S \nabla_S (\partial_{z_{-n}} \partial_{z_{-k}} \mathcal{H}_S^{bo}(z + \tau \tilde{\Psi}_C(z)))] \cdot \langle \mathcal{R}_N(z; \Psi_C), e_n \rangle \cdot \langle \mathcal{R}_N(z; \Psi_C), e_k \rangle \\ &\quad + \partial_{z_{-n}} \partial_{z_{-k}} \mathcal{H}_S^{bo}(z + \tau \tilde{\Psi}_C(z)) \cdot \langle \mathcal{R}_N(z; \Psi_C), e_k \rangle \cdot (d\mathcal{R}_N(z; \Psi_C))^\top [e_n] \\ &\quad + \partial_{z_{-n}} \partial_{z_{-k}} \mathcal{H}_S^{bo}(z + \tau \tilde{\Psi}_C(z)) \cdot \langle \mathcal{R}_N(z; \Psi_C), e_n \rangle \cdot (d\mathcal{R}_N(z; \Psi_C))^\top [e_k]. \end{aligned}$$

- 4 By Corollary 3.2, for any $s \geq 0$, the map $\mathcal{V}' \cap h_0^s \rightarrow h_0^{s+N+1}$, $z \mapsto \nabla h_{n,k}(z; y)$ is real analytic and satisfies the
5 estimate $\|\nabla h_{n,k}(z; y)\|_{s+N+1} \lesssim_{s,N} \|z_\perp\|_s \|z_\perp\|_0^2$. The estimates for the higher order derivatives are obtained
6 by differentiating $\nabla h_{n,k}$ and applying again Corollary 3.2. \square

7 In a next step we analyze $\mathcal{H}_\Omega(\Psi_C(z))$ where for any $z \in h_0^1$,

$$\mathcal{H}_\Omega(z) := \frac{1}{2} \langle \Omega_\perp(I_S)[z_\perp], z_\perp \rangle, \quad (4.28)$$

8 and according to (4.2),

$$\Omega_\perp(I_S) = |D_\perp| + \Omega_\perp^{(0)}(I_S), \quad |D_\perp| := \text{diag}_{n \in S^\perp}(|n|), \quad (4.29)$$

9 where by (4.2) - (4.4),

$$\Omega_\perp^{(0)}(I_S) := \text{diag}_{n \in S^\perp}(\Omega_n(I_S) - |n|) = \text{diag}_{n \in S^\perp}(-\frac{2}{|n|} \sum_{k \in S_+} \min\{|n|, k\} I_k). \quad (4.30)$$

10 Term $\mathcal{H}_\Omega(z)$. First, for further reference, we rewrite $\Omega_\perp^{(0)}(I_S)$ in the form of an expansion as follows.

1 **Lemma 4.5.** *The operator $\Omega_{\perp}^{(0)}(I_S)$ can be written in the form*

$$\mathcal{F}_{N_S}^+ \circ (a_1^+(I_S; \Omega_{\perp}^{(0)}) D^{-1}) (\mathcal{F}_{N_S}^+)^{-1} + \mathcal{F}_{N_S}^- \circ (a_1^-(I_S; \Omega_{\perp}^{(0)}) (-D)^{-1}) (\mathcal{F}_{N_S}^-)^{-1} + \mathcal{R}_N(I_S; \Omega_{\perp}^{(0)}), \quad (4.31)$$

where

$$a_1^+(I_S; \Omega_{\perp}^{(0)}) := -2 \sum_{k \in S_+} k I_k, \quad a_1^-(I_S; \Omega_{\perp}^{(0)}) := a_1^+(I_S; \Omega_{\perp}^{(0)}),$$

and $\mathcal{R}_N(I_S; \Omega_{\perp}^{(0)})$ is defined by the identity (4.31). For any $s \geq 0$,

$$\mathbb{R}_{>0}^{S_+} \rightarrow \mathbb{R}, \quad I_S \mapsto a_1^+(I_S; \Omega_{\perp}^{(0)}), \quad \mathbb{R}_{>0}^{S_+} \rightarrow \mathcal{B}(h_{\perp}^s, h_{\perp}^{s+N+1}), \quad I_S \mapsto \mathcal{R}_N(I_S; \Omega_{\perp}^{(0)}),$$

2 are real analytic.

To analyze $\mathcal{H}_{\Omega}(\Psi_C(z))$, we write the quadratic form $2\mathcal{H}_{\Omega}(\Psi_C(z)) = \langle \Omega_{\perp}(I_S)[z_{\perp}], z_{\perp} \rangle$, $z_{\perp} \in h_{\perp}^1$, as a sum

$$\langle \Omega_{\perp}(I_S)[z_{\perp}], z_{\perp} \rangle = \langle |D_{\perp}|[z_{\perp}], z_{\perp} \rangle + \langle \Omega_{\perp}^{(0)}(I_S)[z_{\perp}], z_{\perp} \rangle$$

and consider $\langle |D_{\perp}|[z_{\perp}], z_{\perp} \rangle$ and $\langle \Omega_{\perp}^{(0)}(I_S)[z_{\perp}], z_{\perp} \rangle$ separately. Substituting $\pi_{\perp} \Psi_C(z) = z_{\perp} + \pi_{\perp} \tilde{\Psi}_C(z)$ for z_{\perp} in $\langle |D_{\perp}|[z_{\perp}], z_{\perp} \rangle$, one gets

$$\begin{aligned} \langle |D_{\perp}|[z_{\perp} + \pi_{\perp} \tilde{\Psi}_C(z)], z_{\perp} + \pi_{\perp} \tilde{\Psi}_C(z) \rangle &= \langle |D_{\perp}|[z_{\perp}], z_{\perp} \rangle + \langle |D_{\perp}|[z_{\perp}], \pi_{\perp} \tilde{\Psi}_C(z) \rangle \\ &+ \langle |D_{\perp}|[\pi_{\perp} \tilde{\Psi}_C(z)], z_{\perp} \rangle + \langle |D_{\perp}|[\pi_{\perp} \tilde{\Psi}_C(z)], \pi_{\perp} \tilde{\Psi}_C(z) \rangle, \end{aligned} \quad (4.32)$$

where the map π_{\perp} is defined in (3.10). With a view towards the expansion of $H^{bo} \circ \Psi$, stated in Theorem 1.1, we treat the difference

$$\frac{1}{2} \langle |D_{\perp}|[z_{\perp} + \pi_{\perp} \tilde{\Psi}_C(z)], z_{\perp} + \pi_{\perp} \tilde{\Psi}_C(z) \rangle - \frac{1}{2} \langle |D_{\perp}|[z_{\perp}], z_{\perp} \rangle$$

as part of the error term $\mathcal{P}_3(z)$. It needs special attention since the Hamiltonian vector fields, associated to the functionals

$$\langle |D_{\perp}|[z_{\perp}], \pi_{\perp} \tilde{\Psi}_C(z) \rangle, \quad \langle |D_{\perp}|[\pi_{\perp} \tilde{\Psi}_C(z)], z_{\perp} \rangle,$$

3 could be unbounded. We write

$$\mathcal{H}_{\Omega}(\Psi_C(z)) = \mathcal{H}_{\Omega}(z) + \mathcal{P}_3^{(2b)}(z), \quad \mathcal{P}_3^{(2b)}(z) := \mathcal{H}_{\Omega}(\Psi_C(z)) - \mathcal{H}_{\Omega}(z), \quad (4.33)$$

with $\mathcal{H}_{\Omega}(z)$ given by (4.28). Note that by the mean value theorem,

$$\mathcal{P}_3^{(2b)}(z) = \int_0^1 \mathcal{P}_{\Omega}(\tau, \Psi_X^{0,\tau}(z)) d\tau, \quad (4.34)$$

4 where for $\tau \in [0, 1]$, $z \in \mathcal{V}$,

$$\mathcal{P}_{\Omega}(\tau, z) := \langle \nabla \mathcal{H}_{\Omega}(z), X(\tau, z) \rangle. \quad (4.35)$$

In a first step we analyze $\mathcal{P}_{\Omega}(\tau, z)$. One has

$$\langle \nabla \mathcal{H}_{\Omega}(z), X(\tau, z) \rangle = \langle \nabla_S \mathcal{H}_{\Omega}(z), \pi_S X(\tau, z) \rangle + \frac{1}{2} \langle \Omega_{\perp}(I_S) z_{\perp}, \pi_{\perp} X(\tau, z) \rangle. \quad (4.36)$$

Since $\mathcal{H}_{\Omega} = \frac{1}{2} \langle \Omega_{\perp}(I_S)[z_{\perp}], z_{\perp} \rangle$ and $\Omega_{\perp}(I_S) = |D_{\perp}| + \Omega_{\perp}^{(0)}(I_S)$ one has

$$\begin{aligned} \langle \nabla_S \mathcal{H}_{\Omega}(z), \pi_S X(\tau, z) \rangle &= \sum_{j \in S} \partial_{z-j} \mathcal{H}_{\Omega}(z) \cdot \langle X(\tau, z), e_j \rangle \\ &= \frac{1}{2} \sum_{j \in S} \langle \partial_{z-j} \Omega_{\perp}^{(0)}(I_S)[z_{\perp}], z_{\perp} \rangle \langle X(\tau, z), e_j \rangle. \end{aligned} \quad (4.37)$$

Concerning the term $\frac{1}{2}\langle \Omega_\perp(I_S)z_\perp, \pi_\perp X(\tau, z) \rangle$ in (4.36), recall that (cf. (3.16), (3.4))

$$X(\tau, z) = -\mathcal{L}_\tau(z)^{-1}[J\mathcal{E}(z)], \quad \mathcal{L}_\tau(z) = \text{Id} + \tau J\mathcal{L}(z),$$

and hence

$$X(\tau, z) = -J\mathcal{E}(z) + \tau J\mathcal{L}(z)[X(\tau, z)]. \quad (4.38)$$

Since $\mathcal{E}(z) = (\mathcal{E}_S(z), 0)$ and $J^\top = -J$, one gets

$$\begin{aligned} \langle \Omega_\perp(I_S)z_\perp, \pi_\perp X(\tau, z) \rangle &= \langle \Omega_\perp(I_S)z_\perp, \pi_\perp \tau J\mathcal{L}(z)X(\tau, z) \rangle \\ &= -\tau \langle J_\perp \Omega_\perp(I_S)z_\perp, \pi_\perp \mathcal{L}(z)X(\tau, z) \rangle. \end{aligned} \quad (4.39)$$

By (2.33) the component $\mathcal{L}_\perp^\perp(z)$ of $\mathcal{L}(z)$ vanishes, implying that $\pi_\perp \mathcal{L}(z)X(\tau, z) = \mathcal{L}_\perp^S(z)\pi_S X(\tau, z)$. Substituting the latter expression into (4.39) and using that by (2.36), $\mathcal{L}_\perp^S(z)^\top = -\mathcal{L}_\perp^\perp(z)$, one concludes that

$$\begin{aligned} \langle \Omega_\perp(I_S)z_\perp, \pi_\perp X(\tau, z) \rangle &= -\tau \langle J_\perp \Omega_\perp(I_S)z_\perp, \mathcal{L}_\perp^S(z)\pi_S X(\tau, z) \rangle \\ &= \tau \langle \mathcal{L}_\perp^\perp(z)J_\perp \Omega_\perp(I_S)z_\perp, \pi_S X(\tau, z) \rangle. \end{aligned} \quad (4.40)$$

Furthermore, by (2.35),

$$\mathcal{L}_\perp^\perp(z)[J_\perp \Omega_\perp(I_S)z_\perp] = \left(\langle \partial_x^{-1} \Psi_1(z_S)[J_\perp \Omega_\perp(I_S)z_\perp] | \partial_{z_n} \Psi_1(z_S)[z_\perp] \rangle \right)_{n \in S}. \quad (4.41)$$

Since $\Omega_\perp(I_S) = |D_\perp| + \Omega_\perp^{(0)}(I_S)$ and

$$J_\perp |D_\perp| = iD_\perp |D_\perp|, \quad D_\perp := \text{diag}_{n \in S^\perp}(n),$$

we need to analyze $\Psi_1(z_S)iD_\perp |D_\perp|$. By Remark 2.1(i), $\Psi_1(z_S)iD_\perp |D_\perp|$ is a bounded linear operator $h_\perp^s \rightarrow H_0^{s-2}$ for any $s \geq 0$.

Lemma 4.6. *For any integer $N \geq 0$ and any $z_S \in \mathcal{V}_S$, the operator*

$$\mathcal{T}(z_S) := \Psi_1(z_S)iD_\perp |D_\perp| - \partial_x |\partial_x \Psi_1(z_S)|$$

admits an expansion of the form $\mathcal{OP}(z_S; \mathcal{T}) + \mathcal{R}_N(z_S; \mathcal{T})$ where

$$\mathcal{OP}(z_S; \mathcal{T}) = \sum_{k=-1}^N a_k^+(z_S; \mathcal{T}) D^{-k} (\mathcal{F}_{N_S}^+)^{-1} + \sum_{k=-1}^N a_k^-(z_S; \mathcal{T}) (-D)^{-k} (\mathcal{F}_{N_S}^-)^{-1}$$

and for any $s \geq 0$, $-1 \leq k \leq N$, the maps

$$\mathcal{V}_S \rightarrow H_{\mathbb{C}}^s, z_S \mapsto a_k^\pm(z_S; \mathcal{T}), \quad \mathcal{V}_S \rightarrow \mathcal{B}(h_\perp^s, H^{s+N+1}), z_S \mapsto \mathcal{R}_N(z_S; \mathcal{T}),$$

are real analytic and $a_k^-(z_S; \mathcal{T}) = \overline{a_k^+(z_S; \mathcal{T})}$. A similar statement holds for the transpose $\mathcal{T}(z_S)^\top$ of $\mathcal{T}(z_S)$.

Proof. Since by Corollary 2.2, $\Psi_1(z_S)$ has an expansion of order zero and $D_\perp |D_\perp|$ is a diagonal operator of order two, the operator $\mathcal{T}(z_S)$, being of commutator type, has an expansion of order one. The claimed statements then follow from Corollary 2.2 (expansion of $\Psi_1(z_S)$) and Corollary 2.4 (expansion of $\Psi_1(z_S)^\top$). \square

Using that $J_\perp \Omega_\perp(I_S) = iD_\perp |D_\perp| + J_\perp \Omega_\perp^{(0)}(I_S)$, the operator $\partial_x^{-1} \Psi_1(z_S) J_\perp \Omega_\perp(I_S)$, appearing in formula (4.41), reads

$$\partial_x^{-1} \Psi_1(z_S) J_\perp \Omega_\perp(I_S) = \partial_x^{-1} \Psi_1(z_S) iD_\perp |D_\perp| + \partial_x^{-1} \Psi_1(z_S) J_\perp \Omega_\perp^{(0)}(I_S). \quad (4.42)$$

By the definition of $\mathcal{T}(z_S)$ one then gets

$$\partial_x^{-1} \Psi_1(z_S) iD_\perp |D_\perp| = i|\partial_x \Psi_1(z_S)| + \partial_x^{-1} \mathcal{T}(z_S).$$

By (4.41) it then follows that for any $n \in S$,

$$(\mathcal{L}_S^\perp(z)[J_\perp \Omega_\perp(I_S)z_\perp])_n = i\langle |\partial_x| \Psi_1(z_S)[z_\perp] | \partial_{z_n} \Psi_1(z_S)[z_\perp] \rangle + \langle \mathcal{T}_{1,n}(z_S)[z_\perp] | z_\perp \rangle, \quad (4.43)$$

where for any $z_S \in \mathcal{V}_S$ and $n \in S$, the operator $\mathcal{T}_{1,n}(z_S)$ is given by (cf. (4.42))

$$\mathcal{T}_{1,n}(z_S) := (\partial_{z_n} \Psi_1(z_S))^\top \partial_x^{-1} \mathcal{T}(z_S) + (\partial_{z_n} \Psi_1(z_S))^\top \partial_x^{-1} \Psi_1(z_S) J_\perp \Omega_\perp^{(0)}(I_S). \quad (4.44)$$

Since $(\partial_{z_n} \Psi_1(z_S))^\top$ (Corollary 2.4) and $\partial_x^{-1} \mathcal{T}(z_S)$ (Lemma 4.6) have both an expansion of order zero, one infers that $\mathcal{T}_{1,n}(z_S)$ has also such an expansion. More precisely, the following result holds.

Lemma 4.7. *For any $n \in S$ and $N \in \mathbb{N}$, the operator $\mathcal{T}_{1,n}(z_S)$, defined by (4.44) for $z_S \in \mathcal{V}_S$, admits an expansion of the form $\mathcal{OP}(z_S; \mathcal{T}_{1,n}) + \mathcal{R}_N(z_S; \mathcal{T}_{1,n})$ where*

$$\mathcal{OP}(z_S; \mathcal{T}_{1,n}) = \mathcal{F}_{N_S}^+ \circ \sum_{k=0}^N a_k^+(z_S; \mathcal{T}_{1,n}) D^{-k} (\mathcal{F}_{N_S}^+)^{-1} + \mathcal{F}_{N_S}^- \circ \sum_{k=0}^N a_k^-(z_S; \mathcal{T}_{1,n}) (-D)^{-k} (\mathcal{F}_{N_S}^-)^{-1}$$

and for any $s \geq 0$, $0 \leq k \leq N$, the maps

$$\mathcal{V}_S \rightarrow H_{\mathbb{C}}^s, z_S \mapsto a_k^\pm(z_S; \mathcal{T}_{1,n}), \quad \mathcal{V}_S \rightarrow \mathcal{B}(h_\perp^s, h_\perp^{s+N+1}), z_S \mapsto \mathcal{R}_N(z_S; \mathcal{T}_{1,n}),$$

are real analytic and $a_k^-(z_S; \mathcal{T}_{1,n}) = \overline{a_k^+(z_S; \mathcal{T}_{1,n})}$.

Proof. The claimed statements follow from Corollary 2.4 (expansion of $\Psi_1(z_S)^\top$), Lemma 4.6 (expansion of $\mathcal{T}(z_S)$), Corollary 2.2 (expansion of $\Psi_1(z_S)$) Lemma 4.5 (expansion of $\Omega_\perp^{(0)}(I_S)$), and Lemma D.2. \square

We now turn our attention to the term $i\langle |\partial_x| \Psi_1(z_S)[z_\perp] | \partial_{z_n} \Psi_1(z_S)[z_\perp] \rangle$ in (4.43). By (4.17)

$$d\nabla H^{bo}(q) = |\partial_x| + d\nabla H_3^{bo}(q) = |\partial_x| - 2q, \quad H_3^{bo}(q) := \frac{1}{2\pi} \int_0^{2\pi} -\frac{1}{3} q^3 dx,$$

and hence $|\partial_x| = d\nabla H^{bo}(q) + 2q$. Since $\Psi_1(z_S)[z_\perp]$ is real valued, one then infers that for any $n \in S$

$$\begin{aligned} \langle |\partial_x| \Psi_1(z_S)[z_\perp] | \partial_{z_n} \Psi_1(z_S)[z_\perp] \rangle &= \partial_{z_{-n}} \frac{1}{2} \langle |\partial_x| \Psi_1(z_S)[z_\perp] | \Psi_1(z_S)[z_\perp] \rangle \\ &= \frac{1}{2} \partial_{z_{-n}} \langle d\nabla H^{bo}(q) [\Psi_1(z_S)[z_\perp]] | \Psi_1(z_S)[z_\perp] \rangle + \frac{1}{2} \partial_{z_{-n}} \langle 2q \Psi_1(z_S)[z_\perp] | \Psi_1(z_S)[z_\perp] \rangle. \end{aligned} \quad (4.45)$$

Since by (4.15),

$$\Psi_1(z_S)^\top d\nabla H^{bo}(q) \Psi_1(z_S) = \Omega_\perp(I_S) + \mathcal{G}(z_S)$$

we conclude that $i\langle |\partial_x| \Psi_1(z_S)[z_\perp] | \partial_{z_n} \Psi_1(z_S)[z_\perp] \rangle$ equals

$$\frac{i}{2} \langle \partial_{z_{-n}} (\Omega_\perp(I_S) + \partial_{z_{-n}} \mathcal{G}(z_S)) [z_\perp], z_\perp \rangle + \frac{i}{2} \langle \partial_{z_{-n}} (2\Psi_1(z_S)^\top q \Psi_1(z_S)) [z_\perp], z_\perp \rangle. \quad (4.46)$$

Using again that for any $n \in S$, $\partial_{z_{-n}} \Omega_\perp(I_S) = \partial_{z_{-n}} \Omega_\perp^{(0)}(I_S)$ (cf. (4.2)), one thus obtains

$$i\langle |\partial_x| \Psi_1(z_S)[z_\perp] | \partial_{z_n} \Psi_1(z_S)[z_\perp] \rangle = \langle \mathcal{T}_{2,n}(z_S)[z_\perp], z_\perp \rangle$$

where

$$\mathcal{T}_{2,n}(z_S) := \frac{i}{2} \partial_{z_{-n}} (\Omega_\perp^{(0)}(I_S) + \mathcal{G}(z_S) + 2\Psi_1(z_S)^\top q \Psi_1(z_S)). \quad (4.47)$$

Lemma 4.8. For any $n \in S$ and any integer $N \geq 0$, the operator $\mathcal{T}_{2,n}(z_S) : h_{\perp}^0 \rightarrow h_{\perp}^0$, defined by (4.47) for $z_S \in \mathcal{V}_S$, admits an expansion of the form $\mathcal{OP}(z_S; \mathcal{T}_{2,n}) + \mathcal{R}_N(z_S; \mathcal{T}_{2,n})$ where

$$\mathcal{OP}(z_S; \mathcal{T}_{2,n}) = \mathcal{F}_{N_S}^+ \circ \sum_{k=0}^N a_k^+(z_S; \mathcal{T}_{2,n}) D^{-k} (\mathcal{F}_{N_S}^+)^{-1} + \mathcal{F}_{N_S}^- \circ \sum_{k=0}^N a_k^-(z_S; \mathcal{T}_{2,n}) (-D)^{-k} (\mathcal{F}_{N_S}^-)^{-1}$$

and where for any $s \geq 0$, $0 \leq k \leq N$, the maps

$$\mathcal{V}_S \rightarrow H_{\mathbb{C}}^s, z_S \mapsto a_k^{\pm}(z_S; \mathcal{T}_{2,n}), \quad \mathcal{V}_S \rightarrow \mathcal{B}(h_{\perp}^s, h_{\perp}^{s+N+1}), z_S \mapsto \mathcal{R}_N(z_S; \mathcal{T}_{2,n}),$$

are real analytic and $a_k^-(z_S; \mathcal{T}_{2,n}) = \overline{a_k^+(z_S; \mathcal{T}_{2,n})}$. A similar statement holds for the transpose $\mathcal{T}_{2,n}(z_S)^{\top}$ of the operator $\mathcal{T}_{2,n}(z_S)$.

Proof. The claimed results follow from Lemmata 2.4, 4.2, 4.5, and Lemma D.2. \square

By (4.35) - (4.36), (4.40) - (4.43), and (4.47) the Hamiltonian $\mathcal{P}_{\Omega}(\tau, z)$, defined in (4.35), is given by

$$\mathcal{P}_{\Omega}(\tau, z) = \langle \nabla_S \mathcal{H}_{\Omega}(z), \pi_S X(\tau, z) \rangle + \frac{\tau}{2} \sum_{j \in S} \langle (\mathcal{T}_{1,j}(z_S) + \mathcal{T}_{2,j}(z_S)) [z_{\perp}], z_{\perp} \rangle \cdot \langle X(\tau, z), e_j \rangle.$$

Hence by (4.37),

$$\mathcal{P}_{\Omega}(\tau, z) = \frac{1}{2} \sum_{j \in S} \langle \mathcal{T}_{3,j}(\tau, z_S) [z_{\perp}], z_{\perp} \rangle \cdot \langle X(\tau, z), e_j \rangle \quad (4.48)$$

where for any $j \in S$, $z_S \in \mathcal{V}_S$, and $0 \leq \tau \leq 1$, the operator $\mathcal{T}_{3,j}(\tau, z_S) : h_{\perp}^0 \rightarrow h_{\perp}^0$ is defined by

$$\mathcal{T}_{3,j}(\tau, z_S) := \partial_{z_{-j}} \Omega_{\perp}^{(0)}(I_S) + \tau \mathcal{T}_{1,j}(z_S) + \tau \mathcal{T}_{2,j}(z_S). \quad (4.49)$$

The Hamiltonian $\mathcal{P}_{\Omega}(\tau, z)$ has the following properties.

Lemma 4.9. For any $0 \leq \tau \leq 1$ and any integer $N \geq 0$, the Hamiltonian $\mathcal{P}_{\Omega}(\tau, \cdot) : \mathcal{V} \rightarrow \mathbb{R}$ is real analytic and $\nabla \mathcal{P}_{\Omega}(\tau, z)$ admits the expansion of the form $(0, \mathcal{OP}(\tau, z; \nabla \mathcal{P}_{\Omega})) + \mathcal{R}_N(\tau, z; \nabla \mathcal{P}_{\Omega})$ where

$$\mathcal{OP}(\tau, z; \nabla \mathcal{P}_{\Omega}) = \mathcal{F}_{N_S}^+ \circ \sum_{k=0}^N a_k^+(\tau, z; \nabla \mathcal{P}_{\Omega}) D^{-k} [(\mathcal{F}_{N_S}^+)^{-1} z_{\perp}] + \mathcal{F}_{N_S}^- \circ \sum_{k=0}^N a_k^-(\tau, z; \nabla \mathcal{P}_{\Omega}) (-D)^{-k} [(\mathcal{F}_{N_S}^-)^{-1} z_{\perp}]$$

and where for any $s \geq 0$, $0 \leq k \leq N$, the maps

$$\mathcal{V} \rightarrow H_{\mathbb{C}}^s, z \mapsto a_k^{\pm}(\tau, z; \nabla \mathcal{P}_{\Omega}), \quad \mathcal{V} \cap h_0^s \rightarrow h_0^{s+N+1}, z \mapsto \mathcal{R}_N(\tau, z; \nabla \mathcal{P}_{\Omega}),$$

are real analytic and $a_k^-(\tau, z; \nabla \mathcal{P}_{\Omega}) = \overline{a_k^+(\tau, z; \nabla \mathcal{P}_{\Omega})}$. Furthermore, for any $0 \leq \tau \leq 1$, $z \in \mathcal{V}$, $\widehat{z} \in h_0^0$,

$$\|a_k^{\pm}(\tau, z; \nabla \mathcal{P}_{\Omega})\|_s \lesssim_s \|z_{\perp}\|_0^2, \quad \|da_k^{\pm}(\tau, z; \nabla \mathcal{P}_{\Omega})[\widehat{z}]\|_s \lesssim_s \|z_{\perp}\|_0 \|\widehat{z}\|_0.$$

If in addition, $\widehat{z}_1, \dots, \widehat{z}_l \in h_0^0$, $l \geq 2$, then

$$\|d^l a_k^{\pm}(\tau, z; \nabla \mathcal{P}_{\Omega})[\widehat{z}_1, \dots, \widehat{z}_l]\|_s \lesssim_{s,l} \prod_{j=1}^l \|\widehat{z}_j\|_0.$$

Similarly, for any $0 \leq \tau \leq 1$, $z \in \mathcal{V} \cap h_0^s$, $\widehat{z}_1, \widehat{z}_2 \in h_0^s$,

$$\|\mathcal{R}_N(\tau, z; \nabla \mathcal{P}_{\Omega})\|_{s+N+1} \lesssim_{s,N} \|z_{\perp}\|_s \|z_{\perp}\|_0^2,$$

$$\|d\mathcal{R}_N(\tau, z; \nabla \mathcal{P}_{\Omega})[\widehat{z}_1]\|_{s+N+1} \lesssim_{s,N} \|z_{\perp}\|_0^2 \|\widehat{z}_1\|_s + \|z_{\perp}\|_s \|z_{\perp}\|_0 \|\widehat{z}_1\|_0,$$

$$\|d^2 \mathcal{R}_N(\tau, z; \nabla \mathcal{P}_{\Omega})[\widehat{z}_1, \widehat{z}_2]\|_{s+N+1} \lesssim_{s,N} \|z_{\perp}\|_0 (\|\widehat{z}_1\|_s \|\widehat{z}_2\|_0 + \|\widehat{z}_1\|_0 \|\widehat{z}_2\|_s) + \|z_{\perp}\|_s \|\widehat{z}_1\|_0 \|\widehat{z}_2\|_0,$$

and if in addition $\widehat{z}_1, \dots, \widehat{z}_l \in h_0^s$, $l \geq 3$, then

$$\|d^l \mathcal{R}_N(\tau, z; \nabla \mathcal{P}_{\Omega})[\widehat{z}_1, \dots, \widehat{z}_l]\|_{s+N+1} \lesssim_{s,N,l} \sum_{j=1}^l \|\widehat{z}_j\|_s \prod_{i \neq j} \|\widehat{z}_i\|_0 + \|z_{\perp}\|_s \prod_{j=1}^l \|\widehat{z}_j\|_0.$$

Proof. One has $\nabla_S \mathcal{P}_\Omega(\tau, z) = (\partial_{z_{-n}} \mathcal{P}_\Omega(\tau, z))_{n \in S}$ with

$$\begin{aligned} \partial_{z_{-n}} \mathcal{P}_\Omega(\tau, z) &= \frac{1}{2} \sum_{j \in S} \langle \partial_{z_{-n}} \mathcal{T}_{3,j}(\tau, z_S)[z_\perp], z_\perp \rangle \cdot \langle X(\tau, z), e_j \rangle \\ &\quad + \frac{1}{2} \sum_{j \in S} \langle \mathcal{T}_{3,j}(\tau, z_S)[z_\perp], z_\perp \rangle \cdot \langle \partial_{z_{-n}} X(\tau, z), e_j \rangle, \end{aligned}$$

whereas $\nabla_\perp \mathcal{P}_\Omega(\tau, z)$ can be computed to be

$$\nabla_\perp \mathcal{P}_\Omega(\tau, z) = \sum_{j \in S} \langle X(\tau, z), e_j \rangle \mathcal{T}_{3,j}(\tau, z_S)[z_\perp] + \frac{1}{2} \sum_{j \in S} \langle \mathcal{T}_{3,j}(\tau, z_S)[z_\perp], z_\perp \rangle (d_\perp X(\tau, z))^\top [e_j].$$

1 The claimed statements then follow by Lemmata 3.4, 4.5, 4.7, 4.8. \square

2 We are now ready to analyze the gradient of the Hamiltonian $\mathcal{P}_3^{(2b)}(z) := \int_0^1 \mathcal{P}_\Omega(\tau, \Psi_X^{0,\tau}(z)) d\tau$ (cf. (4.34)).

Lemma 4.10. *The Hamiltonian $\mathcal{P}_3^{(2b)} : \mathcal{V}' \rightarrow \mathbb{R}$ is real analytic and for any integer $N \geq 0$, its gradient $\nabla \mathcal{P}_3^{(2b)}(z)$ admits the expansion of the form $(0, \mathcal{OP}(z; \nabla \mathcal{P}_3^{(2b)})) + \mathcal{R}_N(z; \nabla \mathcal{P}_3^{(2b)})$ where*

$$\mathcal{OP}(z; \nabla \mathcal{P}_3^{(2b)}) = \mathcal{F}_{N_S}^+ \circ \sum_{k=0}^N a_k^+(z; \nabla \mathcal{P}_3^{(2b)}) D^{-k} [(\mathcal{F}_{N_S}^+)^{-1} z_\perp] + \mathcal{F}_{N_S}^- \circ \sum_{k=0}^N a_k^-(z; \nabla \mathcal{P}_3^{(2b)}) (-D)^{-k} [(\mathcal{F}_{N_S}^-)^{-1} z_\perp]$$

and where for any $s \geq 0$, $0 \leq k \leq N$, the maps

$$\mathcal{V}' \rightarrow H_{\mathbb{C}}^s, z \mapsto a_k^\pm(z; \nabla \mathcal{P}_3^{(2b)}), \quad \mathcal{V}' \cap h_0^s \rightarrow h_0^{s+N+1}, z \mapsto \mathcal{R}_N(z; \nabla \mathcal{P}_3^{(2b)}),$$

are real analytic and $a_k^-(z; \nabla \mathcal{P}_3^{(2b)}) = \overline{a_k^+(z; \nabla \mathcal{P}_3^{(2b)})}$. Furthermore, the following estimates hold: for any $z \in \mathcal{V}'$, $\widehat{z} \in h_0^0$,

$$\|a_k^\pm(z; \nabla \mathcal{P}_3^{(2b)})\|_s \lesssim_s \|z_\perp\|_0^2, \quad \|da_k^\pm(z; \nabla \mathcal{P}_3^{(2b)})[\widehat{z}]\|_s \lesssim_s \|z_\perp\|_0 \|\widehat{z}\|_0,$$

3 and if in addition $\widehat{z}_1, \dots, \widehat{z}_l \in h_0^0$, $l \geq 2$, then $\|d^l a_k^\pm(z; \nabla \mathcal{P}_3^{(2b)})[\widehat{z}_1, \dots, \widehat{z}_l]\|_s \lesssim_{s,l} \prod_{j=1}^l \|\widehat{z}_j\|_0$. Similarly, for any $z \in \mathcal{V}' \cap h_0^s$, $\widehat{z}_1, \widehat{z}_2 \in h_0^s$, one has

$$\|\mathcal{R}_N(z; \nabla \mathcal{P}_3^{(2b)})\|_{s+N+1} \lesssim_{s,N} \|z_\perp\|_s \|z_\perp\|_0^2,$$

$$\|d\mathcal{R}_N(z; \nabla \mathcal{P}_3^{(2b)})[\widehat{z}_1]\|_{s+N+1} \lesssim_{s,N} \|z_\perp\|_0^2 \|\widehat{z}_1\|_s + \|z_\perp\|_s \|z_\perp\|_0 \|\widehat{z}_1\|_0,$$

$$\|d^2 \mathcal{R}_N(z; \nabla \mathcal{P}_3^{(2b)})[\widehat{z}_1, \widehat{z}_2]\|_{s+N+1} \lesssim_{s,N} \|z_\perp\|_0 (\|\widehat{z}_1\|_s \|\widehat{z}_2\|_0 + \|\widehat{z}_1\|_0 \|\widehat{z}_2\|_s) + \|z_\perp\|_s \|\widehat{z}_1\|_0 \|\widehat{z}_2\|_0,$$

and if in addition $\widehat{z}_1, \dots, \widehat{z}_l \in h_0^s$, $l \geq 2$, then

$$\|d^l \mathcal{R}_N(z; \nabla \mathcal{P}_3^{(2b)})[\widehat{z}_1, \dots, \widehat{z}_l]\|_{s+N+1} \lesssim_{s,N,l} \sum_{j=1}^l \|\widehat{z}_j\|_s \prod_{i \neq j} \|\widehat{z}_i\|_0 + \|z_\perp\|_s \prod_{j=1}^l \|\widehat{z}_j\|_0.$$

Proof. By a straightforward computation, one has for any $z \in \mathcal{V}'$,

$$\nabla \mathcal{P}_3^{(2b)}(z) = \int_0^1 (d\Psi_X^{0,\tau}(z))^\top \nabla \mathcal{P}_\Omega(\tau, \Psi_X^{0,\tau}(z)) d\tau.$$

4 The claimed statements then follow by applying Corollary 3.1 (expansion of $d\Psi_X^{0,\tau}(z)^\top$), Lemma 4.9 (expansion of $\nabla \mathcal{P}_\Omega(\tau, z)$), Theorem 3.1 (expansion of $\Psi_X^{0,\tau}(z)$), and Lemma D.2. \square

Terms $\mathcal{P}_2^{(1)}$ and $\mathcal{P}_3^{(1)}$. Recall that the Hamiltonians $\mathcal{P}_2^{(1)}$ and $\mathcal{P}_3^{(1)}$ were introduced in (4.21). We write

$$\begin{aligned}\mathcal{P}_2^{(1)}(\Psi_C(z)) + \mathcal{P}_3^{(1)}(\Psi_C(z)) &= \mathcal{P}_2^{(1)}(z) + \mathcal{P}_3^{(2c)}(z), \\ \mathcal{P}_3^{(2c)}(z) &:= \mathcal{P}_2^{(1)}(\Psi_C(z)) - \mathcal{P}_2^{(1)}(z) + \mathcal{P}_3^{(1)}(\Psi_C(z)),\end{aligned}\tag{4.50}$$

where by the mean value theorem

$$\mathcal{P}_2^{(1)}(\Psi_C(z)) - \mathcal{P}_2^{(1)}(z) = \int_0^1 \langle \nabla \mathcal{P}_2^{(1)}(z + y(\Psi_C(z) - z)), \Psi_C(z) - z \rangle dy.$$

1 The Hamiltonian $\mathcal{P}_3^{(2c)}(z)$ has the following properties.

Lemma 4.11. *The Hamiltonian $\mathcal{P}_3^{(2c)} : \mathcal{V}' \cap h_0^1 \rightarrow \mathbb{R}$ is real analytic and for any integer $N \geq 0$ its gradient $\nabla \mathcal{P}_3^{(2c)}(z)$ admits the expansion of the form $(0, \mathcal{OP}(z; \nabla \mathcal{P}_3^{(2c)})) + \mathcal{R}_N(z; \nabla \mathcal{P}_3^{(2c)})$ where*

$$\mathcal{OP}(z; \nabla \mathcal{P}_3^{(2c)}) = \mathcal{F}_{N_S}^+ \circ \sum_{k=0}^N T_{a_k^+(z; \nabla \mathcal{P}_3^{(2c)})} D^{-k} [(\mathcal{F}_{N_S}^+)^{-1} z_\perp] + \mathcal{F}_{N_S}^- \circ \sum_{k=0}^N T_{a_k^-(z; \nabla \mathcal{P}_3^{(2c)})} (-D)^{-k} [(\mathcal{F}_{N_S}^-)^{-1} z_\perp]$$

with the property that there exists an integer $\sigma_N \geq N$ (loss of regularity) such that for any $s \geq 0$, $0 \leq k \leq N$, the maps

$$\mathcal{V}' \cap h^{s+\sigma_N} \rightarrow H_{\mathbb{C}}^s, z \mapsto a_k^\pm(z; \nabla \mathcal{P}_3^{(2c)}), \quad \mathcal{V}' \cap h_0^{s \vee \sigma_N} \rightarrow h_0^{s+N+1}, z \mapsto \mathcal{R}_N(z; \nabla \mathcal{P}_3^{(2c)})$$

are real analytic and $a_k^-(z; \nabla \mathcal{P}_3^{(2c)}) = \overline{a_k^+(z; \nabla \mathcal{P}_3^{(2c)})}$. Furthermore, for any $s \geq 0$, $z \in \mathcal{V}' \cap h_0^{s+\sigma_N}$ with $\|z\|_{\sigma_N} \leq 1$, $\widehat{z}_1, \dots, \widehat{z}_l \in h_0^{s+\sigma_N}$, $l \geq 1$,

$$\begin{aligned}\|a_k^\pm(z; \nabla \mathcal{P}_3^{(2c)})\|_s &\lesssim_{s,N} \|z_\perp\|_{s+\sigma_N}, \\ \|d^l a_k^\pm(z; \nabla \mathcal{P}_3^{(2c)})[\widehat{z}_1, \dots, \widehat{z}_l]\|_s &\lesssim_{s,N,l} \sum_{j=1}^l \|\widehat{z}_j\|_{s+\sigma_N} \prod_{i \neq j} \|\widehat{z}_i\|_{\sigma_N} + \|z_\perp\|_{s+\sigma_N} \prod_{j=1}^l \|\widehat{z}_j\|_{\sigma_N}.\end{aligned}$$

Similarly, for any $s \geq 0$, $z \in \mathcal{V}' \cap h_0^{s \vee \sigma_N}$ with $\|z\|_{\sigma_N} \leq 1$, $\widehat{z} \in h_0^{s \vee \sigma_N}$,

$$\begin{aligned}\|\mathcal{R}_N(z; \nabla \mathcal{P}_3^{(2c)})\|_{s+N+1} &\lesssim_{s,N} \|z_\perp\|_{s \vee \sigma_N} \|z_\perp\|_{\sigma_N}, \\ \|d\mathcal{R}_N(z; \nabla \mathcal{P}_3^{(2c)})[\widehat{z}]\|_{s+N+1} &\lesssim_{s,N} \|z_\perp\|_{\sigma_N} \|\widehat{z}\|_{s \vee \sigma_N} + \|z_\perp\|_{s \vee \sigma_N} \|\widehat{z}\|_{\sigma_N},\end{aligned}$$

and if in addition $\widehat{z}_1, \dots, \widehat{z}_l \in h_0^{s \vee \sigma_N}$, $l \geq 2$, then

$$\|d^l \mathcal{R}_N(z; \nabla \mathcal{P}_3^{(2c)})[\widehat{z}_1, \dots, \widehat{z}_l]\|_{s+N+1} \lesssim_{s,N,l} \sum_{j=1}^l \|\widehat{z}_j\|_{s \vee \sigma_N} \prod_{i \neq j} \|\widehat{z}_i\|_{\sigma_N} + \|z_\perp\|_{s \vee \sigma_N} \prod_{j=1}^l \|\widehat{z}_j\|_{\sigma_N}.$$

2 *Proof.* The lemma follows by differentiating the Hamiltonian $\mathcal{P}_3^{(2c)}$, defined in (4.50) and then applying
3 Corollary 3.2, Lemmata 4.2, 4.3 and using Lemmata D.1, D.2. \square

Altogether we have found the following expansion of $\mathcal{H}^{(2)}(z) = \mathcal{H}^{(1)}(\Psi_C(z))$: By (4.20) (expansion of $\mathcal{H}^{(1)}$) and by the definition $\mathcal{H}_\Omega(z) = \frac{1}{2} \langle \Omega_\perp(I_S)[z_\perp], [z_\perp] \rangle$ (cf. (4.28)), one has for any $z \in \mathcal{V}'$

$$\begin{aligned}\mathcal{H}^{(2)}(z) &= \mathcal{H}_S^{bo}(\Psi_C(z)) + \mathcal{H}_\Omega(\Psi_C(z)) + \mathcal{P}_2^{(1)}(\Psi_C(z)) + \mathcal{P}_3^{(1)}(\Psi_C(z)) \\ &= \mathcal{H}_S^{bo}(z) + \mathcal{H}_\Omega(z) + \mathcal{P}_2^{(1)}(z) + (\mathcal{H}_S^{bo}(\Psi_C(z)) - \mathcal{H}_S^{bo}(z)) + (\mathcal{H}_\Omega(\Psi_C(z)) - \mathcal{H}_\Omega(z)) + \mathcal{P}_3^{(2c)}(z),\end{aligned}$$

where $\mathcal{P}_3^{(2c)}(z) = \mathcal{P}_2^{(1)}(\Psi_C(z)) - \mathcal{P}_2^{(1)}(z) + \mathcal{P}_3^{(1)}(\Psi_C(z))$ (cf. (4.50)). Using $H^{bo}(\Psi^{bo}(z_S, 0)) = \mathcal{H}^{bo}(I_S)$ (cf. (4.21)) and the identity (cf. (4.26)),

$$\mathcal{H}_S^{bo}(\Psi_C(z)) - \mathcal{H}_S^{bo}(z) = \langle \nabla_S \mathcal{H}_S^{bo}(z), \pi_S \mathcal{R}_{N,2}(z; \Psi_C) \rangle + \mathcal{P}_3^{(2a)}(z),$$

as well as the definition $\mathcal{P}_3^{(2b)}(z) = \mathcal{H}_\Omega(\Psi_C(z)) - \mathcal{H}_\Omega(z)$ (cf. (4.33)) it follows that for $z = (z_S, z_\perp) \in \mathcal{V}'$, the Hamiltonian $\mathcal{H}^{(2)}(z)$ is given by

$$\mathcal{H}^{(2)}(z) = \mathcal{H}^{bo}(I_S) + \frac{1}{2} \langle \Omega_\perp(I_S)[z_\perp], z_\perp \rangle + \mathcal{P}_2^{(2)}(z) + \mathcal{P}_3^{(2)}(z), \quad (4.51)$$

where

$$\mathcal{P}_2^{(2)}(z) := \langle \nabla_S \mathcal{H}_S^{bo}(z), \pi_S \mathcal{R}_{N,2}(z; \Psi_C) \rangle + \mathcal{P}_2^{(1)}(z), \quad \mathcal{P}_3^{(2)}(z) := \mathcal{P}_3^{(2a)}(z) + \mathcal{P}_3^{(2b)}(z) + \mathcal{P}_3^{(2c)}(z). \quad (4.52)$$

We recall that by (3.68) and (4.21),

$$\mathcal{R}_{N,2}(z; \Psi_C) = \frac{1}{2} d_\perp^2 \mathcal{R}_N((z_S, 0); \Psi_C)[z_\perp, z_\perp], \quad \mathcal{P}_2^{(1)}(z) = \frac{1}{2} \langle \mathcal{G}(z_S)[z_\perp], z_\perp \rangle. \quad (4.53)$$

Note that $\mathcal{P}_2^{(2)}(z)$ is quadratic with respect to z_\perp , whereas $\mathcal{P}_3^{(2)}$ is a remainder term of order three in z_\perp . Being quadratic with respect to z_\perp , $\mathcal{P}_2^{(2)}$ can be written as

$$\mathcal{P}_2^{(2)}(z) = \frac{1}{2} \langle d_\perp \nabla_\perp \mathcal{P}_2^{(2)}(z_S, 0)[z_\perp], z_\perp \rangle. \quad (4.54)$$

The following lemma is the analogue of a corresponding result for the KdV equation, due to Kuksin [18].

Lemma 4.12. *The Hamiltonian $\mathcal{P}_2^{(2)}$ vanishes on \mathcal{V}' .*

Proof. In view of (4.54), it suffices to prove that for any $z_S \in \mathcal{V}'_S$, the operator $d_\perp \nabla_\perp \mathcal{P}_2^{(2)}(z_S, 0)$ vanishes. We establish that $d_\perp \nabla_\perp \mathcal{P}_2^{(2)}(z_S, 0) = 0$ by studying the linearization of $\partial_t w = J \nabla \mathcal{H}^{(2)}(w)$ along an arbitrary solution $w(t)$ of the form $w(t) = (w_S(t), 0)$. First we need to make some preliminary considerations. Let $t \mapsto q(t) \in M_S^0$ be a solution of the BO equation $\partial_t q = \partial_x \nabla H^{bo}(q)$ and denote by $t \mapsto z(t) := (z_S(t), 0)$ the corresponding solution in Birkhoff coordinates, defined by $q(t) = \Psi^{bo}(z(t))$. It satisfies $\partial_t z(t) = J \Omega(I_S)[z(t)]$ (cf. (1.7)). Furthermore, let $\hat{q}(t)$ be the solution of the equation, obtained by linearizing the BO equation along $q(t)$,

$$\partial_t \hat{q}(t) = \partial_x d \nabla H^{bo}(q(t))[\hat{q}(t)],$$

with initial data $\hat{q}^0 := d \Psi^{bo}(z_S(0), 0)[0, \hat{z}_\perp^0]$ and $\hat{z}_\perp^0 \in h_\perp^2$. Similarly, denote by $\hat{z}(t)$ the solution of the equation, obtained by linearizing $\partial_t z = J \Omega(I_S)[z]$ along the solution $z(t)$ with initial data $\hat{z}^0 = (0, \hat{z}_\perp^0)$, $\partial_t \hat{z}(t) = J d \nabla \mathcal{H}^{bo}(z(t))[\hat{z}(t)]$. Since $\partial_t z(t) = J \Omega(I_S)[z(t)]$ one concludes that (cf. (4.10))

$$\hat{z}(t) = (0, \hat{z}_\perp(t)), \quad \partial_t \hat{z}_\perp(t) = J_\perp \Omega_\perp(I_S)[\hat{z}_\perp(t)]. \quad (4.55)$$

Since Ψ^{bo} is symplectic and $\mathcal{H}^{bo} = H^{bo} \circ \Psi^{bo}$, one has $\hat{q}(t) = d \Psi^{bo}(z_S(t), 0)[\hat{z}(t)]$. Recall that for any $z_S \in \mathcal{V}'_S$, $\Psi_L(z_S, 0) = \Psi^{bo}(z_S, 0)$ (cf. definition (2.3) of Ψ_L) and $\Psi_C(z_S, 0) = (z_S, 0)$ (cf. Corollary 3.2), implying that $\Psi(z_S, 0) = \Psi^{bo}(z_S, 0)$ and hence $q(t) = \Psi(z_S(t), 0)$ for any t . Since $\Psi : \mathcal{V}' \rightarrow H_0^0$ is symplectic and $\mathcal{H}^{(2)} = H^{bo} \circ \Psi$, one sees that $z(t) = (z_S(t), 0)$ is also a solution of the equation $\partial_t w = J \nabla \mathcal{H}^{(2)}(w)$.

With these preliminary considerations made, we are ready to prove that $d_\perp \nabla_\perp \mathcal{P}_2^{(2)}(z_S, 0)$ vanishes. To this end consider the solution $\hat{w}(t)$ of the equation obtained by linearizing $\partial_t w = J \nabla \mathcal{H}^{(2)}(w)$ along the solution $z(t) = (z_S(t), 0)$ with initial data $\hat{w}^0 = (0, \hat{z}_\perp^0)$. Again using that the map Ψ is symplectic and $\mathcal{H}^{(2)} = H^{bo} \circ \Psi$, it follows that $d \Psi(z(t))[\hat{w}(t)]$ solves the linearized BO equation. Since $d \Psi(z(0)) = d \Psi^{bo}(z(0))$ and $\hat{w}^0 = \hat{z}^0$, one then concludes from the uniqueness of the initial value problem that $d \Psi(z(t))[\hat{w}(t)] = d \Psi^{bo}(z(t))[\hat{z}(t)]$ and hence $\hat{w}(t) = \hat{z}(t)$ for any t . It means that $\hat{z}(t)$ satisfies also the linear equation

$$\partial_t \hat{z}(t) = J d \nabla \mathcal{H}^{(2)}(z(t))[\hat{z}(t)].$$

In view of the expansion (4.51) of $\mathcal{H}^{(2)}$ one then infers that

$$\partial_t \widehat{z}_\perp(t) = J_\perp \Omega_\perp(I_S)[\widehat{z}_\perp(t)] + J_\perp d_\perp \nabla_\perp \mathcal{P}_2^{(2)}(z_S(t), 0)[\widehat{z}_\perp(t)].$$

1 Comparing the latter identity with (4.55) one concludes that in particular, $d_\perp \nabla_\perp \mathcal{P}_2^{(2)}(z_S(0), 0) = 0$. Since
 2 the initial data $z_S(0) \in \mathcal{V}'_S$ can be chosen arbitrarily, we thus have $d_\perp \nabla_\perp \mathcal{P}_2^{(2)}(z_S, 0) = 0$ for any $z_S \in \mathcal{V}'_S$ as
 3 claimed. \square

4 In summary, we have proved the following results on the Hamiltonian $\mathcal{H}^{(2)} = H^{bo} \circ \Psi$.

5 **Theorem 4.1.** *The Hamiltonian $\mathcal{H}^{(2)} : \mathcal{V}' \cap h_0^1 \rightarrow \mathbb{R}$ has an expansion of the form*

$$\mathcal{H}^{(2)}(z) = H^{bo}(q) + \frac{1}{2} \langle \Omega_\perp(I_S)[z_\perp], z_\perp \rangle + \mathcal{P}_3^{(2)}(z) \quad (4.56)$$

where $\Omega_\perp(I_S)$ is given by (4.2) and the remainder term $\mathcal{P}_3^{(2)}$, defined by (4.52), satisfies the following:
 $\mathcal{P}_3^{(2)} : \mathcal{V}' \cap h_0^1 \rightarrow \mathbb{R}$ is real analytic and for any integer $N \geq 1$, its gradient $\nabla \mathcal{P}_3^{(2)}(z)$ admits the asymptotic
 expansion $(0, \mathcal{OP}(z; \nabla \mathcal{P}_3^{(2)})) + \mathcal{R}_N(z; \nabla \mathcal{P}_3^{(2)})$ where

$$\mathcal{OP}(z; \nabla \mathcal{P}_3^{(2)}) = \mathcal{F}_{N_S}^+ \circ \sum_{k=0}^N T_{a_k^+(z; \nabla \mathcal{P}_3^{(2)})} D^{-k} [(\mathcal{F}_{N_S}^+)^{-1} z_\perp] + \mathcal{F}_{N_S}^- \circ \sum_{k=0}^N T_{a_k^-(z; \nabla \mathcal{P}_3^{(2)})} (-D)^{-k} [(\mathcal{F}_{N_S}^-)^{-1} z_\perp]$$

with the property that there exists an integer $\sigma_N \geq N$ (loss of regularity) so that for any $s \geq 0$,
 $0 \leq k \leq N$, the maps

$$\mathcal{V}' \cap h_0^{s+\sigma_N} \rightarrow H_{\mathbb{C}}^s, z \mapsto a_k^\pm(z; \nabla \mathcal{P}_3^{(2)}), \quad \mathcal{V}' \cap h_0^{s \vee \sigma_N} \rightarrow h_0^{s+N+1}, z \mapsto \mathcal{R}_N(z; \nabla \mathcal{P}_3^{(2)})$$

are real analytic and $a_k^-(z; \nabla \mathcal{P}_3^{(2)}) = \overline{a_k^+(z; \nabla \mathcal{P}_3^{(2)})}$. Furthermore, they satisfy the following estimates: for
 any $s \geq 0$, $z \in \mathcal{V}' \cap h_0^{s+\sigma_N}$ with $\|z\|_{\sigma_N} \leq 1$, and $\widehat{z}_1, \dots, \widehat{z}_l \in h_0^{s+\sigma_N}$, $l \geq 1$,

$$\begin{aligned} \|a_k^\pm(z; \nabla \mathcal{P}_3^{(2)})\|_s &\lesssim_{s,N} \|z_\perp\|_{s+\sigma_N}, \\ \|d^l a_k^\pm(z; \nabla \mathcal{P}_3^{(2)})[\widehat{z}_1, \dots, \widehat{z}_l]\|_s &\lesssim_{s,N,l} \sum_{j=1}^l \|\widehat{z}_j\|_{s+\sigma_N} \prod_{i \neq j} \|\widehat{z}_i\|_{\sigma_N} + \|z_\perp\|_{s+\sigma_N} \prod_{j=1}^l \|\widehat{z}_j\|_{\sigma_N}. \end{aligned}$$

Similarly, for any $s \geq 1$ and $z \in \mathcal{V}' \cap h_0^{s \vee \sigma_N}$ with $\|z_\perp\|_{\sigma_N} \leq 1$, $\widehat{z} \in h_0^{s \vee \sigma_N}$,

$$\begin{aligned} \|\mathcal{R}_N(z; \nabla \mathcal{P}_3^{(2)})\|_{s+N+1} &\lesssim_{s,N} \|z_\perp\|_{s \vee \sigma_N} \|z_\perp\|_{\sigma_N}, \\ \|d \mathcal{R}_N(z; \nabla \mathcal{P}_3^{(2)})[\widehat{z}]\|_{s+N+1} &\lesssim_{s,N} \|z_\perp\|_{\sigma_N} \|\widehat{z}\|_{s \vee \sigma_N} + \|z_\perp\|_{s \vee \sigma_N} \|\widehat{z}\|_{\sigma_N}. \end{aligned}$$

If in addition $\widehat{z}_1, \dots, \widehat{z}_l \in h_0^{s \vee \sigma_N}$, $l \geq 2$, then

$$\|d^l \mathcal{R}_N(z; \nabla \mathcal{P}_3^{(2)})[\widehat{z}_1, \dots, \widehat{z}_l]\|_{s+N+1} \lesssim_{s,N,l} \sum_{j=1}^l \|\widehat{z}_j\|_{s \vee \sigma_N} \prod_{i \neq j} \|\widehat{z}_i\|_{\sigma_N} + \|z_\perp\|_{s \vee \sigma_N} \prod_{j=1}^l \|\widehat{z}_j\|_{\sigma_N}.$$

6 *Proof.* The identity (4.56) follows from formula (4.51) and Lemma 4.12. The claimed asymptotic expansion
 7 of $\nabla \mathcal{P}_3^{(2)}$ and its properties follow from Lemmata 4.4, 4.10, 4.11 and Lemma D.1. \square

8 5 Summary of the proofs of Theorem 1.1 and Theorem 1.2

In this section we summarize the proofs of Theorem 1.1, of its addendum, and of Theorem 1.2. First recall
 that in view of the envisioned applications, these theorems are formulated in terms of action angle coordinates

on the submanifold M_S^0 of proper S -gap potentials. Denote by Ξ the map relating action angle variables and complex Birkhoff coordinates,

$$\Xi : \mathbb{T}^{S_+} \times \mathbb{R}_{>0}^{S_+} \times h_{\perp}^0 \rightarrow h_S \times h_{\perp}^0, (\theta_S, I_S, z_{\perp}) \mapsto (z_S(\theta_S, I_S), z_{\perp}),$$

where

$$z_S(\theta_S, I_S) = (z_n(\theta_S, I_S))_{n \in S}, \quad z_{\pm n} = \sqrt{2\pi n I_n} e^{\mp i \theta_n}, \quad \forall n \in S_+.$$

Clearly, Ξ is symplectic and, for any $s \geq 0$, the map $\Xi : \mathbb{T}^{S_+} \times \mathbb{R}_{>0}^{S_+} \times h_{\perp}^s \rightarrow h_S \times h_{\perp}^s$, is real analytic. Furthermore, in view of the definition (1.24), the map Ξ preserves the reversible structure. Hence the claimed results for the map $\Psi_L \circ \Psi_C \circ \Xi$ follow from the corresponding ones for the map $\Psi_L \circ \Psi_C$. In what follows we summarize the proofs of the results for $\Psi_L \circ \Psi_C$ corresponding to the ones claimed for $\Psi_L \circ \Psi_C \circ \Xi$.

Proof of Theorem 1.1. For notational convenience, we denote the composition $\Psi_L \circ \Psi_C$ by Ψ (cf. the discussion in the paragraph above with regard to the map Ψ of Theorem 1.1). By (3.18), Ψ is defined on the neighborhood $\mathcal{V}' = \mathcal{V}'_S \times \mathcal{V}'_{\perp}$ where \mathcal{V}'_S is a bounded neighborhood of any given compact subset $\mathcal{K} \subset h_S$ and \mathcal{V}'_{\perp} is a ball in h_{\perp}^0 of radius smaller than 1, centered at 0. The expansion of Ψ , corresponding to the one of **(AE1)**, follows from the expansion for the map Ψ_L , provided by Corollary 2.2, and the one for the map Ψ_C , provided by Theorem 3.1(ii).

The expansion of the transpose $d\Psi(z)^{\top}$ of the derivative $d\Psi(z)$, corresponding to the one of **(AE2)**, follows from the fact that $\Psi : \mathcal{V}' \rightarrow L_0^2$ is symplectic, meaning that for any $z \in \mathcal{V}'$, the operator $d\Psi(z)^{\top} : H_0^1 \rightarrow h_0^1$ satisfies $d\Psi(z)^{\top} = J^{-1}(d\Psi(z))^{-1} \partial_x$. The expansion of $\Psi(z)$ in **(AE1)** then leads to an expansion of $d\Psi(z)$ and in turn of $(d\Psi(z))^{-1}$ and hence of $d\Psi(z)^{\top}$. The reality conditions of the coefficients in the various expansions follow from the way they are constructed.

The expansion of the Hamiltonian $\mathcal{H}^{(2)} = H^{bo} \circ \Psi$ and of the remainder term $\mathcal{P}_3^{(2)}$ in the Taylor expansion (4.56), corresponding to the one in **(AE3)**, are provided in Theorem 4.1. By arguing as in Section 4, one sees that $\mathcal{H}^{(mo,2)}(z) := H^{mo} \circ \Psi(z)$ has a Taylor expansion of the form

$$\mathcal{H}^{(mo,2)}(z) = \frac{1}{2} \langle z_S, z_S \rangle + \frac{1}{2} \langle z_{\perp}, z_{\perp} \rangle + \mathcal{P}_3^{(mo,2)}(z)$$

As in the case of the Benjamin-Ono Hamiltonian, one obtains expansions of the Hamiltonian $\mathcal{H}^{(mo,2)}$ and of the remainder term $\mathcal{P}_3^{(mo,2)}$, corresponding to the ones stated in **(AE3)**. Actually, in this case the computations considerably simplify. \square

Proof of Addendum to Theorem 1.1. Clearly, the Fourier transform \mathcal{F} and its inverse preserve the reversible structure and by Proposition C.1, so do the Birkhoff map Φ^{bo} and its inverse Ψ^{bo} . Furthermore, by item (ii) of the Addendum to Theorem 2.1, and by the Addendum to Theorem 3.1, also the maps Ψ_L and Ψ_C and hence $\Psi_L \circ \Psi_C$ preserve the reversible structure, as do the coefficients and the remainder terms in their expansions as well as the transpose of their derivatives.

Clearly, the BO Hamiltonian H^{bo} and H^{mo} are reversible and therefore so are $\mathcal{H}^{(2)} = H^{bo} \circ \Psi$ and $\mathcal{H}^{(mo,2)} = H^{mo} \circ \Psi$. By (4.56) one then concludes that also the remainders $\mathcal{P}_3^{(2)}$ and $\mathcal{P}_3^{(mo,2)}$ are reversible. \square

Proof of Theorem 1.2. The estimates of the coefficients and the remainder in the expansion of $\Psi = \Psi_L \circ \Psi_C$, corresponding to the ones of **(Est1)**, follow from the estimates of the coefficients and the remainder in the expansion of the map Ψ_L , provided by Corollary 2.2, and the ones of the coefficients and the remainder in the expansion of the map Ψ_C , provided by Theorem 3.1.

The estimates of the coefficients and the remainder in the expansion of $d\Psi(z)^{\top}$, corresponding to the one of **(Est2)**, follow from the fact that $\Psi : \mathcal{V}' \rightarrow L_0^2$ is symplectic, meaning that for any $z \in \mathcal{V}'$, $d\Psi(z)^{\top} : H_0^1 \rightarrow h_0^1$ satisfies $d\Psi(z)^{\top} = J^{-1}(d\Psi(z))^{-1} \partial_x$ and the estimates **(Est1)** of the coefficients and the remainder in the expansion of $\Psi(z)$ which lead to corresponding estimates of the coefficients and the remainder in the expansion of $d\Psi(z)$ and in turn of $(d\Psi(z))^{-1}$.

The estimates of the remainder term $\mathcal{P}_3^{(2)}$ in the expansion of the Hamiltonian $\mathcal{H}^{(2)} = H^{bo} \circ \Psi$, corresponding to **(Est3)**, are provided by Theorem 4.1. The ones for the remainder term $\mathcal{P}_3^{(mo,2)}$ are derived in the same way. \square

A Spectral theory of the Lax operator L_u

In this appendix we review the spectral theory of the Lax operator of the Benjamin-Ono equation (1.1),

$$L_u f := -i\partial_x f - \Pi[uf], \quad f \in \text{dom}(L_u) := H_+^1, \quad H_+^1 := H^1 \cap H_+^0, \quad (\text{A.1})$$

with potential $u \in L^2 \equiv H^0$, where $L_+^2 \equiv H_+^0 := \{f \in L^2 : \langle f | e^{inx} \rangle = 0 \ \forall n < 0\}$ denotes the Hardy space of \mathbb{T} and $\Pi : L^2 \rightarrow L_+^2, f \mapsto \sum_{n \geq 0} \hat{f}(n) e^{inx}$ the Szegő projector. The operator L_u appears in the Lax pair formulation of (1.1), $\partial_t L_u = B_u L_u - L_u B_u$, where B_u is a certain skew adjoint pseudo-differential operator – see [3, Remark 2.1, Appendix A]. For the convenience of the reader, we recall some of the notations, introduced in the main body of the paper.

For any $u \in L^2$, the Lax operator L_u is the pseudo-differential operator of order one, acting on L_+^2 , which is self-adjoint, bounded from below, and has compact resolvent (cf. [3], [5]). Hence its spectrum $\text{spec}(L_u)$ consists of an unbounded sequence of real eigenvalues, each of finite multiplicity, which can be listed in increasing order,

$$\lambda_0(u) \leq \lambda_1(u) \leq \lambda_2(u) \leq \dots$$

In [3], [4], the following results are proved.

Theorem A.1. (i) For any $u \in L^2$, the eigenvalues $\lambda_n(u)$, $n \geq 0$, of L_u are separated by at least one,

$$\gamma_n(u) := \lambda_n(u) - \lambda_{n-1}(u) - 1 \geq 0, \quad \forall n \geq 1.$$

In particular, $\text{spec}(L_u)$ is simple. Furthermore, for any $n \geq 0$,

$$\lambda_n(u) = n - \sum_{k \geq n+1} \gamma_k(u),$$

and the following trace formulas hold,

$$\langle u | 1 \rangle = -\lambda_0(u) - \sum_{n \geq 1} \gamma_n(u), \quad \|u - \langle u | 1 \rangle\|^2 = 2 \sum_{n \geq 1} n \gamma_n(u).$$

(ii) Conversely, for any sequence $(r_n)_{n \geq 1}$ of nonnegative numbers, satisfying $\sum_{n \geq 1} n r_n < \infty$, there exists $u \in L^2$ with $\langle u | 1 \rangle = 0$ so that $\gamma_n(u) = r_n$ for any $n \geq 1$. It means that such sequences are a moduli space for the spectra of the Lax operators (A.1).

(iii) For any potential u in L^2 and any $s \geq 0$, $\sum_{n \geq 1} n^{1+2s} \gamma_n(u) < \infty$ if and only if u is in H^s .

(iv) For any $n \geq 0$, $\lambda_n : L^2 \rightarrow \mathbb{R}$ is real analytic. Hence for any $n \geq 1$, $\gamma_n : L^2 \rightarrow \mathbb{R}$ is real analytic as well.

(v) For any $u \in L^2$, u is constant if and only if $\gamma_n(u) = 0$ for any $n \geq 1$.

Remark A.1. (i) It has been shown in [3, Appendix C] that the spectrum of L_u , when considered as an operator on the Hardy space of the real line, consists of the union of bands of length one, $\cup_{n \geq 0} [\lambda_n, \lambda_n + 1]$. These bands are separated by the gaps $(\lambda_{n-1} + 1, \lambda_n)$, $n \geq 1$. For any $n \geq 1$, $\gamma_n(u)$ is the length of the gap $(\lambda_{n-1} + 1, \lambda_n)$ and is referred to as the n th gap length of the spectrum of L_u . (ii) In [4], Theorem A.1 has been extended to L_u with potentials $u \in H^s$, $-1/2 < s < 0$. For any $u \in H^s$, $\sum_{n \geq 1} n^{1+2s} \gamma_n(u) < \infty$.

For any $u \in L^2$, let $h_n(\cdot, u)$, $n \geq 0$, be an orthonormal basis of eigenfunctions of L_u . Note that for any $n \geq 0$, $h_n(\cdot, u)$ is in H_+^1 and $\|h_n(\cdot, u)\| = 1$. Since $\lambda_n(u)$ is simple, $h_n(\cdot, u)$ is then uniquely determined up to a phase factor. By [3, Lemma 2.5, Lemma 2.7],

$$\langle 1 | h_0(\cdot, u) \rangle \neq 0, \quad \langle h_n(\cdot, u) | e^{ix} h_{n-1}(\cdot, u) \rangle \neq 0, \quad \forall n \geq 1.$$

Definition A.1. For any $u \in L^2$, we denote by $f_n(\cdot, u)$, $n \geq 0$, the orthonormal basis of L_+^2 of eigenfunctions of L_u , corresponding to the eigenvalues $(\lambda_n(u))_{n \geq 0}$, uniquely determined by the normalization conditions

$$\langle 1 | f_0(\cdot, u) \rangle > 0, \quad \langle f_n(\cdot, u) | e^{ix} f_{n-1}(\cdot, u) \rangle > 0, \quad \forall n \geq 1. \quad (\text{A.2})$$

By [3, Corollary 3.4] one concludes that for any $u \in L^2$ and $n \geq 1$, $\gamma_n(u) = 0$ if and only if $\langle 1|f_n(\cdot, u) \rangle = 0$.
More generally, the following holds.

Lemma A.1. (i) For any $u \in L^2$ and any $n \geq 1$,

$$|\langle 1|f_n(\cdot, u) \rangle|^2 = \gamma_n(u) \kappa_n(u), \quad \kappa_n(u) := \frac{1}{\lambda_n(u) - \lambda_0(u)} \prod_{p \neq n} \left(1 - \frac{\gamma_p(u)}{\lambda_p(u) - \lambda_n(u)}\right), \quad (\text{A.3})$$

where the infinite product in (A.3) is absolutely convergent. Furthermore, for any $n \geq 1$ and $u \in L^2$, $\gamma_n(u) = 0$ if and only if $f_n(\cdot, u) = e^{ix} f_{n-1}(\cdot, u)$.

(ii) For any $u \in L^2$ and $n \geq 1$, $\kappa_n(u)$, defined in (A.3), satisfies $\kappa_n(u) > 0$ and there exists a constant $C \geq 1$ so that

$$\sup_{n \geq 1} n \kappa_n(u) \leq C, \quad \sup_{n \geq 1} \frac{1}{n \kappa_n(u)} \leq C.$$

In particular, $\sqrt{\kappa_n(u)}$, $n \geq 1$, is well-defined and $\sqrt{\kappa_n(u)} > 0$, where $\sqrt{\cdot} \equiv \sqrt[4]{\cdot}$ denotes the principal branch of the square root.

(iii) For any $n \geq 1$, $f_n : L^2 \rightarrow H_+^1$ and $\kappa_n : L^2 \rightarrow \mathbb{R}$ are real analytic.

Next, we compare the spectrum of L_{u_*} with the one of L_u , where for any $u \in L^2$,

$$u_*(x) := u(-x). \quad (\text{A.4})$$

By [3, Proposition 4.5], eigenvalues and eigenfunctions of L_{u_*} and L_u are related as follows.

Lemma A.2. (i) For any $u \in L^2$ and $n \geq 0$,

$$\lambda_n(u_*) = \lambda_n(u), \quad f_n(x, u_*) = \overline{f_n(-x, u)}. \quad (\text{A.5})$$

(ii) As a consequence,

$$\gamma_n(u_*) = \gamma_n(u), \quad \kappa_n(u_*) = \kappa_n(u), \quad \forall n \geq 1, \quad u \in L^2, \quad (\text{A.6})$$

and (when combined with Lemma A.1(iii)) $f_n : L^2 \rightarrow H_+^1$, $u \mapsto \overline{f_n(-\cdot, u)} = f_n(\cdot, u_*)$ as well as

$$L^2 \rightarrow \mathbb{C}, \quad u \mapsto \langle 1|f_n(\cdot, u) \rangle = \frac{1}{2\pi} \int_0^{2\pi} f_n(x, u_*) dx, \quad L^2 \rightarrow \mathbb{C}, \quad u \mapsto \overline{\langle 1|f_n(\cdot, u) \rangle} = \frac{1}{2\pi} \int_0^{2\pi} f_n(x, u) dx, \quad (\text{A.7})$$

are real analytic.

We finish this appendix with a discussion of finite gap potentials. Let

$$L_0^2 := \{u \in L^2 : \frac{1}{2\pi} \int_0^{2\pi} u dx = 0\}.$$

Definition A.2. An element u in L_0^2 is said to be a finite gap potential if the set $S_u := \{n \in \mathbb{N} : \gamma_n(u) > 0\}$ is finite. For any given finite subset $S_+ \subset \mathbb{N}$, u is said to be a S -gap potential if $S_u = S_+$ where S is defined as $S := S_+ \cup (-S_+)$. The set of S -gap potentials in L_0^2 is denoted by M_S and the subset $M_S^g := \{q \in M_S : \gamma_n(u) > 0 \forall n \in S_+\}$ is referred to as the set of proper S -gap potentials.

For any integer $N \geq 1$, let

$$\mathcal{U}_N := \{u \in L_0^2 : \gamma_N(u) > 0, \gamma_n(u) = 0 \forall n > N\}.$$

Note that the set of finite gap potentials in L_0^2 is given by the disjoint union $\cup_{N \geq 1} \mathcal{U}_N \cup \{0\}$ and that by Theorem A.1, $\mathcal{U}_N \subset \cap_{s \geq 0} H_0^s$ for any $N \geq 1$. Furthermore, by [3, Section 7], any $u \in \mathcal{U}_N$ is of the form

$$u(x) = \sum_{j=1}^N \left(\frac{1 - r_j^2}{1 - 2r_j \cos(x + \alpha_j) + r_j^2} - 1 \right), \quad 0 < r_j < 1, \quad 0 \leq \alpha_j < 2\pi, \quad \forall 1 \leq j \leq N. \quad (\text{A.8})$$

In [3, Section 7] (cf. also [7, Appendix A]) the following is proved.

Lemma A.3. (i) For any $s > -1/2$, the disjoint union $\cup_{N \geq 1} \mathcal{U}_N \cup \{0\}$ is dense in H_0^s .
(ii) For any $u \in \mathcal{U}_N$, $N \geq 1$, one has for any $n \geq N$,

$$\lambda_n(u) = n, \quad f_n(x, u) = e^{inx} g_\infty(x, u), \quad g_\infty(x, u) := e^{i\partial_x^{-1}u(x)}.$$

(iii) For $u = 0$, one has $\lambda_0(0) = 0$, $f_0(x, 0) = 1$, $g_\infty(x, 0) = 1$, and for any $n \geq 1$,

$$\lambda_n(0) = n, \quad f_n(x, 0) = e^{inx}, \quad n\kappa_n(0) = 1.$$

Furthermore, $\nabla \langle 1 | f_n \rangle(x, 0) = -\frac{1}{n} e^{-inx}$ for any $n \geq 1$ (cf. [3, Remark 5.5]).

B Birkhoff map

In this appendix we review the definition and properties of the Birkhoff coordinates, needed in this paper. We refer to [3] - [9] for proofs of the results stated and for more details in these matters. In [6], most of the results obtained on the Birkhoff map Φ are summarized. We point out that in the main body of the paper, we exclusively use a rescaled version Φ^{bo} of the Birkhoff map Φ , which we also refer to as Birkhoff map. For the convenience of the reader, we recall some of the notations, introduced in the main body of the paper.

The Benjamin-Ono equation (1.1) is known to be globally in time wellposed on H^s for any $s > -1/2$. See [4] and references therein. For our purposes, it suffices to consider (1.1) on H^s for $s \geq 0$. Furthermore, since $\langle u | 1 \rangle = \frac{1}{2\pi} \int_0^{2\pi} u dx$ is a prime integral of (1.1), we may assume that $\langle u | 1 \rangle = 0$. Indeed, for any solution $u(t, x)$ of (1.1) in H^s , $s > -1/2$, and any $c \in \mathbb{R}$, $c + u(t, x - 2ct)$ is again a solution of (1.1) in H^s . Denote by $S(t)$, $t \in \mathbb{R}$, the flow map of (1.1) on L_0^2 , meaning that $S(t)u$ denotes the solution of (1.1) with initial value $S(0)u = u \in L_0^2$, where

$$L_0^2 := \{u \in L^2 : \langle u | 1 \rangle = 0\}, \quad H_0^s := H^s \cap L_0^2, \quad \forall s \geq 0.$$

By Lemma A.1, for any $n \geq 1$, the map

$$L_0^2 \rightarrow \mathbb{C}, \quad u \mapsto \zeta_n(u) := \frac{\langle 1 | f_n(\cdot, u) \rangle}{\sqrt{\kappa_n(u)}} \quad (\text{B.1})$$

is well-defined and satisfies $|\zeta_n(u)|^2 = \gamma_n(u)$. Hence by Theorem A.1(iii) one has for any $u \in L_0^2$ and $s \geq 0$,

$$u \in H_0^s \quad \text{if and only if} \quad \zeta(u) := (\zeta_n(u))_{n \geq 1} \in h_+^{s+\frac{1}{2}}.$$

For any C^1 -functionals $F, G : L_0^2 \rightarrow \mathbb{C}$ with sufficiently regular L_0^2 -gradient $\nabla F, \nabla G$, we denote by $\{F, G\}$ the Poisson bracket, due to Gardner and Faddeev&Zakharov,

$$\{F, G\}(u) := \langle \partial_x \nabla F, \nabla G \rangle(u) = \frac{1}{2\pi} \int_0^{2\pi} \partial_x(\nabla F)(u) \cdot \nabla G(u) dx. \quad (\text{B.2})$$

For any $s \in \mathbb{R}$, let

$$h_+^s := \{w = (w_n)_{n \geq 1} \subset \mathbb{C} : \|w\|_s < \infty\}, \quad \|w\|_s := \left(\sum_{n \geq 1} n^{2s} |w_n|^2 \right)^{\frac{1}{2}}.$$

Theorem B.1. *The map*

$$\Phi : \bigsqcup_{s \geq 0} H_0^s \rightarrow \bigsqcup_{s \geq 0} h_+^{s+\frac{1}{2}}, \quad u \mapsto \zeta(u) = (\zeta_n(u))_{n \geq 1},$$

has the following properties:

(NF1) For any $s \geq 0$, $\Phi : H_{r,0}^s \rightarrow h_+^{s+1/2}$ is a real analytic diffeomorphism and Φ and its inverse Φ^{-1} map bounded subsets to bounded ones.

(NF2) The Poisson brackets between the functionals $\zeta_n(u)$, $\overline{\zeta_n(u)}$, $n \geq 1$, are well-defined and one has (cf. [3, Corollary 7.3]),

$$\{\zeta_n, \overline{\zeta_k}\}(u) = -i\delta_{n,k}, \quad \{\zeta_n, \zeta_k\}(u) = 0, \quad \forall n, k \geq 1.$$

In particular,

$$\{\gamma_n, \gamma_k\}(u) = \{|\zeta_n|^2, |\zeta_k|^2\}(u) = 0, \quad \forall n, k \geq 1.$$

(NF3) For any $u \in L_0^2$, $n \geq 1$, and $t \in \mathbb{R}$,

$$\zeta_n(\mathcal{S}(t)u) = e^{it\omega_n^{bo}} \zeta_n(u), \quad \omega_n^{bo} \equiv \omega_n^{bo}(u) := n^2 - 2 \sum_{k=1}^n k \gamma_k(u) - 2n \sum_{k>n} \gamma_k(u). \quad (\text{B.3})$$

For any $n \geq 1$, $|\zeta_n(\mathcal{S}(t)u)|^2$ is independent of t and $\zeta(\mathcal{S}(t)u) = (\zeta_n(\mathcal{S}(t)u))_{n \geq 1}$ evolves on the torus

$$\text{Tor}(\Phi(u)) := \{w = (w_n)_{n \geq 1} \in h_+^{1/2} : |w_n|^2 = \gamma_n(u) \ \forall n \geq 1\}. \quad (\text{B.4})$$

(NF4) The differential of Φ at 0 is given by (cf. Lemma A.3(iii))

$$d_0\Phi : L_0^2 \rightarrow h_+^{\frac{1}{2}}, \quad u \mapsto \left(-\frac{u_n}{\sqrt{n}}\right)_{n \geq 1}.$$

The map Φ is referred to as the Birkhoff map for the Benjamin-Ono equation and $\zeta_n(u)$, $n \geq 1$, as the Birkhoff coordinates of u . Furthermore, $\gamma_n(u)$, $n \geq 1$, are referred to as action variables.

Remark B.1. (i) For any $-1/2 < s < 0$, Φ can be extended as a real analytic map, $H_0^s \rightarrow h_+^{s+\frac{1}{2}}$, but it does not extend to $H_0^{-\frac{1}{2}}$ – see [4], [7], [8].

(ii) For any $u \in L_0^2$, $u_*(x) = u(-x)$ is also in L_0^2 . It then follows from (A.5)–(A.6) that

$$\zeta_n(u_*) = \overline{\zeta_n(u)}, \quad \forall n \geq 1, \quad u \in L_0^2. \quad (\text{B.5})$$

Corollary B.1. For any finite subset $S_+ \subset \mathbb{N}$ and $S := S_+ \cup (-S_+)$, the set of S -gap potentials and the one of proper S -gap potentials, introduced in Definition A.2, are given by

$$M_S = \Phi^{-1}\{(w_n)_{n \geq 1} : w_n = 0 \ \forall n \in \mathbb{N} \setminus S_+\}, \quad M_S^0 = \Phi^{-1}\{(w_n)_{n \geq 1} : w_n \neq 0 \text{ if and only if } n \in S_+\}.$$

M_S and M_S^0 are real analytic, symplectic submanifolds of L_0^2 of dimension S_+ , where L_0^2 is endowed with the symplectic form Λ_G , induced by the Poisson bracket (B.2),

$$\Lambda_G[u, v] = \langle u, \partial_x^{-1} v \rangle = \frac{1}{2\pi} \int_0^{2\pi} u(x) \partial_x^{-1} v(x) dx, \quad \forall u, v \in L_0^2.$$

We finish this appendix with a discussion of the isospectral set $\text{Iso}(u)$, containing $u \in L_0^2$, defined as

$$\text{Iso}(u) := \{v \in L_0^2 : \lambda_n(v) = \lambda_n(u) \ \forall n \geq 0\}.$$

Corollary B.2. For any $u \in L_0^2$,

$$\text{Iso}(u) = \Phi^{-1}(\text{Tor}(\Phi(u)))$$

where $\text{Tor}(\Phi(u))$ is given by (B.4). In particular, $\text{Iso}(u)$ can be viewed as a torus of dimension $|\{n \geq 1 : \gamma_n(u) > 0\}|$.

C Reversibility structure

In this appendix we prove that the Birkhoff map Φ^{bo} , defined in (1.8), and hence also its inverse Ψ^{bo} , preserve the reversible structure, defined by the maps

$$S_{rev} : L_0^2 \rightarrow L_0^2, \quad (S_{rev}q) := q_*, \quad q_*(x) = q(-x), \quad \mathcal{S}_{rev} : h_0^0 \rightarrow h_0^0, \quad (\mathcal{S}_{rev}w)_n := w_{-n}, \quad \forall n \neq 0. \quad (\text{C.1})$$

1 **Proposition C.1.** For any $q \in L_0^2$,

$$\Phi^{bo}(S_{rev}(q)) = S_{rev}(\Phi^{bo}(q)). \quad (C.2)$$

As a consequence, $S_{rev} \circ \Psi^{bo} = \Psi^{bo} \circ S_{rev}$ and by the chain rule, for any $q \in L_0^2(\mathbb{T})$ and $w \in h_0^0$,

$$(d_{S_{rev}(q)}\Phi^{bo}) \circ S_{rev} = S_{rev} \circ d_q\Phi^{bo}, \quad (d_{S_{rev}(w)}\Psi^{bo}) \circ S_{rev} = S_{rev} \circ d_w\Psi^{bo}.$$

2 *Proof.* By Remark B.1(ii) and the definition (1.6) of $(z_n(q))_{n \neq 0}$, one infers that $z_n(q_*) = z_{-n}(q)$ for any
3 $n \neq 0$ and $q \in L_0^2$. Identity (C.2) then follows from the definition (1.8) of Φ^{bo} . \square

4 D Properties of pseudodifferential and paradifferential calculus

5 In this appendix we record some well known facts about pseudodifferential and paradifferential calculus on
6 the torus \mathbb{T} . We refer to [20] for further details. For the convenience of the reader, we recall some of the
7 notations, introduced in the main body of the paper.

8 Let $\chi \in C^\infty(\mathbb{R}^2, \mathbb{R})$ be an admissible cut-off function. It means that χ is an even function and that there
9 exist $0 < \varepsilon_1 < \varepsilon_2 < 1$ so that for any $(\vartheta, \eta) \in \mathbb{R}^2$ and $\alpha, \beta \in \mathbb{Z}_{\geq 0}$,

$$\chi(\vartheta, \eta) = 1, \quad \forall |\vartheta| \leq \varepsilon_1 + \varepsilon_1|\eta|, \quad \chi(\vartheta, \eta) = 0, \quad \forall |\vartheta| \geq \varepsilon_2 + \varepsilon_2|\eta|, \quad (D.1)$$

10

$$|\partial_\vartheta^\alpha \partial_\eta^\beta \chi(\vartheta, \eta)| \leq C_{\alpha, \beta} (1 + |\eta|)^{-\alpha - \beta}. \quad (D.2)$$

11 For any $a \in H_{\mathbb{C}}^1$, the paraproduct $T_a u$ of the function a with $u \in L_{\mathbb{C}}^2$ (with respect to the cut-off function χ)
12 is defined as

$$(T_a u)(x) := \sum_{k, n \in \mathbb{Z}} \chi(k, n) a_k u_n e^{i(k+n)x}, \quad (D.3)$$

13 where u_n , $n \in \mathbb{Z}$, denote the Fourier coefficients of u , $u_n = \frac{1}{2\pi} \int_0^{2\pi} u(x) e^{-inx} dx$. Note that if a, u are
14 real valued, $T_a u$ is real valued as well since χ is real valued and even. Given any $s, s' \in \mathbb{Z}$, we denote
15 by $\mathcal{B}(H_{\mathbb{C}}^s, H_{\mathbb{C}}^{s'})$ the Banach space of all bounded linear operators $H_{\mathbb{C}}^s \rightarrow H_{\mathbb{C}}^{s'}$, endowed with the operator
16 norm $\|\cdot\|_{\mathcal{B}(H_{\mathbb{C}}^s, H_{\mathbb{C}}^{s'})}$. In case $s = s'$, we also write $\mathcal{B}(H_{\mathbb{C}}^s)$ instead of $\mathcal{B}(H_{\mathbb{C}}^s, H_{\mathbb{C}}^s)$. Given any linear operator
17 $A \in \mathcal{B}(H_{\mathbb{C}}^s, H_{\mathbb{C}}^{s'})$, we denote by A^\top the transpose of A with respect to the standard L^2 -inner product. It is
18 an element in $\mathcal{B}((H_{\mathbb{C}}^{s'})^*, (H_{\mathbb{C}}^s)^*)$ where $(H_{\mathbb{C}}^s)^*$ denotes the dual of $H_{\mathbb{C}}^s$.

19 **Lemma D.1.** (i) For any $s \in \mathbb{Z}_{\geq 0}$ and $a \in H_{\mathbb{C}}^1$, the linear operator $T_a : u \mapsto T_a u$ is in $\mathcal{B}(H_{\mathbb{C}}^s)$. Furthermore
20 the linear map $H_{\mathbb{C}}^1 \rightarrow \mathcal{B}(H_{\mathbb{C}}^s)$, $a \mapsto T_a$, is bounded, $\|T_a\|_{\mathcal{B}(H_{\mathbb{C}}^s)} \lesssim_s \|a\|_1$.
(ii) Let $a \in H_{\mathbb{C}}^{s_1}$, $b \in H_{\mathbb{C}}^{s_2}$ and $s_1, s_2 \in \mathbb{Z}_{\geq 1}$. Then

$$ab = T_a b + T_b a + \mathcal{R}^{(B)}(a, b),$$

where the bilinear map $\mathcal{R}^{(B)} : H_{\mathbb{C}}^{s_1} \times H_{\mathbb{C}}^{s_2} \rightarrow H_{\mathbb{C}}^{s_1+s_2-1}$, $(a, b) \mapsto \mathcal{R}^{(B)}(a, b)$, is continuous and satisfies the
estimate

$$\|\mathcal{R}^{(B)}(a, b)\|_{s_1+s_2-1} \lesssim_{s_1, s_2} \|a\|_{s_1} \|b\|_{s_2}.$$

(iii) Let $a \in H_{\mathbb{C}}^\rho$ with $\rho \in \mathbb{Z}_{\geq 2}$. Then for any $s \geq 0$, $T_a^\top - T_a \in \mathcal{B}(H_{\mathbb{C}}^s, H_{\mathbb{C}}^{s+\rho-1})$ and

$$\|T_a^\top - T_a\|_{\mathcal{B}(H_{\mathbb{C}}^s, H_{\mathbb{C}}^{s+\rho-1})} \lesssim_{s, \rho} \|a\|_\rho.$$

(iv) Let $a, b \in H_{\mathbb{C}}^\rho$ with $\rho \in \mathbb{Z}_{\geq 1}$. Then for any $s \geq 0$, $T_a \circ T_b - T_{ab} \in \mathcal{B}(H_{\mathbb{C}}^s, H_{\mathbb{C}}^{s+\rho-1})$ and

$$\|T_a \circ T_b - T_{ab}\|_{\mathcal{B}(H_{\mathbb{C}}^s, H_{\mathbb{C}}^{s+\rho-1})} \lesssim_{s, \rho} \|a\|_\rho \|b\|_\rho.$$

Recall that for any $k \geq 1$, $\partial_x^{-k} : H_{\mathbb{C}}^s \rightarrow H_{0, \mathbb{C}}^{s+k}$ is the bounded linear operator, defined by

$$\partial_x^{-k}[e^{inx}] = \frac{1}{(in)^k} e^{inx}, \quad \forall n \neq 0, \quad \partial_x^{-k}[1] = 0.$$

Lemma D.2. (i) Let $k, \ell \in \mathbb{Z}_{\geq 0}$ and $a \in C^\infty(\mathbb{T}, \mathbb{C})$. Then for any $s \in \mathbb{Z}_{\geq 0}$ and $N \in \mathbb{N}$ with $N \geq k + \ell$, the composition $\partial_x^{-k} \circ a \partial_x^{-\ell}$ is a bounded linear operator $H_{\mathbb{C}}^s \rightarrow H_{0, \mathbb{C}}^{s+k+\ell}$, admitting an expansion of the form

$$\partial_x^{-k} \circ a \partial_x^{-\ell} = \sum_{j=0}^{N-k-\ell} C_j(k, \ell) (\partial_x^j a) \partial_x^{-k-\ell-j} + \mathcal{R}_{N, k, \ell}^{\psi do}(a),$$

1 where $C_j(k, \ell)$, $0 \leq j \leq N - k - \ell$, are real constants with $C_0(k, \ell) = 1$, and the remainder $\mathcal{R}_{N, k, \ell}^{\psi do}(a)$ is a
2 bounded linear operator $H_{\mathbb{C}}^s \rightarrow H_{0, \mathbb{C}}^{s+N+1}$, satisfying the estimate

$$\|\mathcal{R}_{N, k, \ell}^{\psi do}(a)\|_{\mathcal{B}(H_{\mathbb{C}}^s, H_{0, \mathbb{C}}^{s+N+1})} \lesssim_{s, N} \|a\|_{s+2N}. \quad (\text{D.4})$$

In particular for any $N \geq 1$, $\partial_x^{-1} \circ a = \sum_{j=0}^{N-1} (-1)^j (\partial_x^j a) \partial_x^{-1-j} + \mathcal{R}_{N, 1, 0}^{\psi do}(a)$ where for any $h \in H^s$,

$$\mathcal{R}_{N, 1, 0}^{\psi do}(a)[h] = (-1)^N \partial_x^{-1} [(\partial_x^N a) \partial_x^{-N} h] + (\partial_x^{-1} a) \langle h | 1 \rangle.$$

(ii) Let $k, \ell \in \mathbb{Z}_{\geq 0}$ and $N \geq k + \ell$. There exists a constant $\sigma_N > N - k - \ell + 1$ so that for any $a \in H_{\mathbb{C}}^{\sigma_N}$ and any $s \in \mathbb{Z}_{\geq 0}$, the composition $\partial_x^{-k} \circ T_a \circ \partial_x^{-\ell}$ is a bounded linear operator $H_{\mathbb{C}}^s \rightarrow H_{\mathbb{C}}^{s+k+\ell}$ which admits an expansion of the form

$$\partial_x^{-k} \circ T_a \circ \partial_x^{-\ell} = \sum_{j=0}^{N-k-\ell} C_j(k, \ell) T_{\partial_x^j a} \partial_x^{-k-\ell-j} + \mathcal{R}_{N, k, \ell}^{(B)}(a),$$

3 where $C_j(k, \ell)$, $1 \leq j \leq N - k - \ell$, are the same constants as in item (i) and for any $s \geq 0$, the remainder
4 $\mathcal{R}_{N, k, \ell}^{(B)}(a)$ is a bounded linear operator $H_{\mathbb{C}}^s \rightarrow H_{\mathbb{C}}^{s+N+1}$, satisfying the estimate

$$\|\mathcal{R}_{N, n, k}^{(B)}(a)\|_{\mathcal{B}(H_{\mathbb{C}}^s, H_{\mathbb{C}}^{s+N+1})} \lesssim_{s, N} \|a\|_{\sigma_N}. \quad (\text{D.5})$$

The next results concern Hankel operators which are defined as follows. For any $s \geq 0$, let $H_{\pm}^s := H_{\mathbb{C}}^s \cap L_{\pm}^2$ where L_{\pm}^2 denote the Hardy spaces of \mathbb{T} , given by

$$L_{\pm}^2 \equiv H_{\pm}^0 := \{f \in L_{\mathbb{C}}^2 : \langle f | e^{inx} \rangle = 0 \ \forall \ \pm n < 0\}.$$

Furthermore, denote by

$$\Pi \equiv \Pi^+ : L_{\mathbb{C}}^2 \rightarrow L_+^2, f \mapsto \sum_{n \geq 0} \widehat{f}(n) e^{inx}, \quad \Pi^- : L_{\mathbb{C}}^2 \rightarrow L_-^2, f \mapsto \sum_{n \leq 0} \widehat{f}(n) e^{inx},$$

the Szegő projectors. For any $u \in H_{\mathbb{C}}^s$ with $s > 1/2$, the Hankel operators $H_u \equiv H_u^+$ and H_u^- are given by

$$H_u^+ : L_-^2 \rightarrow L_+^2, f \mapsto \Pi^+(uf), \quad H_u^- : L_+^2 \rightarrow L_-^2, f \mapsto \Pi^-(uf).$$

5 The following results follow from smoothing properties of Hankel operators, obtained in [9]. For the convenience of the reader we include the proof, given in [9].

7 **Lemma D.3.** For any $u \in H_{\mathbb{C}}^{s+\alpha}$ with $\alpha \geq 0$ and $f \in H_-^s$, the following holds:

8 (i) If $s > 1/2$, then $\|H_u^+ f\|_{s+\alpha} \lesssim_{s, \alpha} \|u\|_{s+\alpha} \|f\|_s$.

9 (ii) If $s = 1/2$ and $\varepsilon > 0$, then $\|H_u^+ f\|_{\frac{1}{2}+\alpha-\varepsilon} \lesssim_{\alpha} \|u\|_{\frac{1}{2}+\alpha} \|f\|_{\frac{1}{2}}$.

(iii) If $0 \leq s < 1/2$ and $\alpha \geq 1/2 - s$, then

$$\|H_u^+ f\|_{s+\beta} \lesssim_{s, \alpha} \|u\|_{s+\alpha} \|f\|_s, \quad \beta := \alpha - (1/2 - s).$$

10 Corresponding results hold for the operator H_u^- .

Proof. Since the estimates for H_u^- can be proved in the same way as the ones for H_u^+ , we only consider the latter ones. Let $u = \sum_{k \in \mathbb{Z}} \widehat{u}(k) e^{ikx} \in H_{\mathbb{C}}^{s+\alpha}$ and $f = \sum_{p \geq 0} \widehat{f}(-p) e^{-ipx} \in H_-^s$. Then with $n := -(k+p)$,

$$g(x) := \Pi \left(\sum_{k \in \mathbb{Z}, p \geq 0} \widehat{u}(-k) \widehat{f}(-p) e^{-(k+p)x} \right) = \sum_{n \geq 0} \widehat{g}(n) e^{inx}$$

where

$$\widehat{g}(n) := \sum_{p \geq 0} \widehat{u}(n+p) \widehat{f}(-p), \quad \forall n \geq 0.$$

By Cauchy Schwarz, one obtains

$$|\widehat{g}(n)|^2 \leq \|f\|_s^2 \sum_{p \geq 0} \frac{1}{\langle p \rangle^{2s}} |\widehat{u}(n+p)|^2$$

and thus

$$\|g\|_{s+\alpha}^2 = \sum_{n \geq 0} \langle n \rangle^{2(s+\alpha)} |\widehat{g}(n)|^2 \leq \|f\|_s^2 \sum_{\ell \geq 0} |\widehat{u}(\ell)|^2 \sum_{p+n=\ell} \frac{\langle n \rangle^{2(s+\alpha)}}{\langle p \rangle^{2s}}$$

(i) In the case $s > 1/2$, one has $2s > 1$ and hence

$$\sum_{p+n=\ell} \frac{\langle n \rangle^{2(s+\alpha)}}{\langle p \rangle^{2s}} \leq \langle \ell \rangle^{2(s+\alpha)} \sum_{0 \leq p \leq \ell} \frac{1}{\langle p \rangle^{2s}} \lesssim_s \langle \ell \rangle^{2(s+\alpha)} \left(1 + \frac{1}{\langle \ell \rangle^{2s-1}}\right),$$

yielding

$$\|g\|_{s+\alpha} \lesssim_s \|f\|_s \|u\|_{s+\alpha}.$$

(ii) In the case $s = 1/2$, one has $2s = 1$ and hence

$$\sum_{p+n=\ell} \frac{\langle n \rangle^{2(s+\alpha-\varepsilon)}}{\langle p \rangle^{2s}} \leq \langle \ell \rangle^{2(\frac{1}{2}+\alpha-\varepsilon)} \sum_{0 \leq p \leq \ell} \frac{1}{\langle p \rangle} \lesssim \langle \ell \rangle^{2(\frac{1}{2}+\alpha)} \langle \ell \rangle^{-2\varepsilon} \log \langle \ell \rangle,$$

yielding

$$\|g\|_{\frac{1}{2}+\alpha-\varepsilon} \lesssim \|f\|_{\frac{1}{2}} \|u\|_{\frac{1}{2}+\alpha}.$$

(iii) In the case $0 \leq s < 1/2$, one has $2s < 1$ and hence $1 - 2s > 0$. Therefore

$$\sum_{p+n=\ell} \frac{\langle n \rangle^{2(s+\beta)}}{\langle p \rangle^{2s}} \leq \langle \ell \rangle^{2(s+\beta)} \sum_{0 \leq p \leq \ell} \frac{1}{\langle p \rangle^{2s}} \lesssim_s \langle \ell \rangle^{2(s+\beta)} \langle \ell \rangle^{2(\frac{1}{2}-s)} = \langle \ell \rangle^{2(s+\alpha)},$$

1 where we used that by definition $\beta + 1/2 = s + \alpha$. It thus follows that $\|g\|_{s+\beta} \lesssim_s \|f\|_s \|u\|_{s+\alpha}$. □

Corollary D.1. Assume that $u \in \cap_{k \geq 0} H_{\mathbb{C}}^k$. Then H_u^+ and H_u^- are infinitely smoothing, i.e.,

$$H_u^+ \in \mathcal{B}(H_-^s, H_+^{s+N}), \quad H_u^- \in \mathcal{B}(H_+^s, H_-^{s+N}), \quad \forall s \geq 0, N \geq 1.$$

2 Finally, we record the following well known tame estimates of products of functions in Sobolev spaces.

Lemma D.4. For any $s \in \mathbb{Z}_{\geq 1}$,

$$\|uv\|_s \lesssim_s \|u\|_s \|v\|_1 + \|u\|_1 \|v\|_s, \quad \forall u, v \in H_{\mathbb{C}}^s.$$

References

- [1] T. BENJAMIN, *Internal waves of permanent form in fluids of great depth*, J. Fluid Mech. 29(1967), 559-592.
- [2] R. DAVIS, A. ACRIVOS, *Solitary internal waves in deep water*, J. Fluid Mech. 29(1967), 593-607.
- [3] P. GÉRARD, T. KAPPELER, *On the integrability of the Benjamin-Ono equation on the torus*, Comm. Pure Appl. Math. 74(2021), no. 8, 1685-1747.
- [4] P. GÉRARD, T. KAPPELER, P. TOPALOV, *Sharp well-posedness results for the Benjamin-Ono equation in $H^s(\mathbb{T}, \mathbb{R})$ and qualitative properties of its solutions*, to appear in Acta Math., arXiv:2004.04857.
- [5] P. GÉRARD, T. KAPPELER, P. TOPALOV, *On the spectrum of the Lax operator of the Benjamin-Ono equation on the torus*, J. Funct. Anal. 279(2020), no. 12, 108762, 75 pp.
- [6] P. GÉRARD, T. KAPPELER, P. TOPALOV, *On the Benjamin-Ono equation on \mathbb{T} and its periodic and quasiperiodic solutions*, to appear in J. of Spectr. Theory, arXiv:2103.0929.
- [7] P. GÉRARD, T. KAPPELER, P. TOPALOV, *On the analytic Birkhoff normal form of the Benjamin-Ono equation and applications*, Nonlinear Anal. 216(2022), paper no. 112687, 32 pp.
- [8] P. GÉRARD, T. KAPPELER, P. TOPALOV, *On the analyticity of the nonlinear Fourier transform of the Benjamin-Ono equation on \mathbb{T}* , arXiv:2109.08988.
- [9] P. GÉRARD, T. KAPPELER, P. TOPALOV, *On smoothing properties and Tao's gauge transform of the Benjamin-Ono equation on the torus*, arXiv:2109.00610.
- [10] B. GRÉBERT, T. KAPPELER, *The defocusing NLS equation and its normal form*, EMS Series of Lectures in Mathematics, European Mathematical Society (EMS), Zürich, 2014.
- [11] T. KAPPELER, R. MONTALTO, *Canonical coordinates with tame estimates for the defocusing NLS equation on the circle*, Int. Math. Res. Not. IMNR, 2018, issue 5, 6 March 2018, 1473-1531.
- [12] T. KAPPELER, R. MONTALTO, *Normal form coordinates for the KdV equation having expansions in terms of pseudodifferential operators*, Comm. in Math. Phys. 375(2020), no. 1, 833-913.
- [13] T. KAPPELER, R. MONTALTO, *On the stability of periodic multi-solitons of the KdV equation*, Comm. Math. Phys. 385(2021), no. 3, 1871-1956.
- [14] T. KAPPELER, J. PÖSCHEL, *KdV & KAM*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, 45. Springer-Verlag, Berlin, 2003.
- [15] T. KAPPELER, B. SCHAAD, P. TOPALOV, *Qualitative features of periodic solutions of KdV*, Comm. in PDEs 38(2013), no. 9, 1626-1673.
- [16] T. KAPPELER, B. SCHAAD, P. TOPALOV, *Semi-linearity of the nonlinear Fourier transform of the defocusing NLS equation*, Int. Math. Res. Notices, IMRN vol. 2016, no. 23, 7212-7229.
- [17] I. KRICHIEVER, *Perturbation theory in periodic problems for two-dimensional integrable systems*, Soviet Scientific Reviews C. Math. Phys. 9(1991), 1-103.
- [18] S. KUKSIN, *Analysis of Hamiltonian PDEs*, Oxford University Press, 2000.
- [19] S. KUKSIN, G. PERELMAN, *Vey theorem in infinite dimensions and its applications to KdV*, Disc. Cont. Dyn. Syst. Serie A, 27(2010), 1-24.
- [20] G. MÉTIVIER, *Para-differential calculus and applications to the Cauchy problem for nonlinear systems*, Publications of the Scuola Normale Superiore, Book 5, Edizioni della Normale, 2008.

¹ [21] J.-C. SAUT, *Benjamin-Ono and intermediate long wave equations: modeling, IST, and PDE*, Fields
² Institute Communications, 83, Springer, 2019.

³ T. Kappeler, Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, CH-8057 Zürich;
⁴ email: thomas.kappeler@math.uzh.ch

⁵
⁶ R. Montalto, Department of Mathematics, University of Milan, Via Saldini 50, 20133, Milano, Italy;
⁷ email: riccardo.montalto@unimi.it