Matemática
Contemporânea

Vol. 49, 263–307
http://doi.org/10.21711/231766362022/rmc4911

Codazzi surfaces in 4-manifolds

Giulio Colombo [1], **Gary R. Jensen** [2] and **Marco Rigoli** [3]

[1]Dipartimento di Matematica, Università degli Studi di Milano, 20133 Milano, Italy

[2]Department of Mathematics, Washington University, One Brookings Drive, St. Louis, MO 63130, USA

[3]Dipartimento di Matematica, Università degli Studi di Milano, 20133 Milano, Italy

Dedicated to Professor Renato Tribuzy
on the occasion of his 75th birthday

Abstract. We study the geometry of Codazzi surfaces immersed in 4-manifolds with mean curvature vector satisfying a differential inequality that generalizes the condition of having parallel mean curvature. In this way, we extend some rigidity results obtained in the past by several authors. By similar techniques, we also study the geometry of smooth maps between Riemann surfaces whose tension field is suitably controlled by the energy density.

Keywords: Codazzi surfaces, harmonic maps, Gauss sections, twistor sections, Hopf differentials.

2020 Mathematics Subject Classification: primary 53C42, 53C43; secondary 53C55, 30F30.

E-mail: giulio.colombo@unimi.it
E-mail: gary@wustl.edu
E-mail: marco.rigoli55@gmail.com

1 Introduction

In recent years the richness of the geometry of surfaces in 4-dimensional manifolds has attracted the attention of many authors. In particular, starting from the pioneering work of Calabi, [9], and Chern, [12], the research has been focusing on minimal surfaces and culminated with the result of Bryant, [8], on the possibility of conformally and minimally immerse compact surfaces in $\mathbb{S}^4$, and with the fundamental achievements of Eschenburg, Guadalupe and Tribuzy, [19], on the geometry of minimal surfaces in $\mathbb{CP}^2$. Later, Eells and Salamon, [14], introduced twistorial methods to study the case of an arbitrary 4-dimensional target N and, making a clear distinction between harmonicity and conformality of the map $f : M \to N$, they were able to reinterpret, extend and complete in a well geometrically organized picture, practically, all of the previous research. (For a complete survey on the subject we refer to [13].)

The harmonicity condition, or at least the fact that the surface had parallel mean curvature vector, played a fundamental role in a number of results and in particular on the existence of Abelian differentials intimately related to geometrical properties. Different authors have been developing a number of techniques to better understand and simplify to its roots the basic phenomenon (see [26, 27, 38, 21, 18, 1, 2, 3]). Thus Eschenburg and Tribuzy, [20], generalized a result of Chern, [11], and showed, with a number of applications, that the harmonicity assumption, expressed by the vanishing of the tension field $\tau(f)$, can be relaxed to appropriate, geometrically significant, inequalities.

One of the aims of this paper is to give a number of further applications of the Eschenburg-Tribuzy technique to obtain topological-type restrictions on the geometry of the surface and to identify a class of them, that we call Codazzi surfaces, whose behavior parallels that of the generic surface in a 4-space form. The notion of Codazzi surface is intimately related to the Gauss lift γ_f associated to the immersion and more precisely to the Ruh-Vilms property as introduced and studied in [40] (see also [31]). We say that a class of isometric immersions $f : M \to N$ has the Ruh-Vilms property if, for every immersion f in such class, γ_f is harmonic if and only if f has parallel mean curvature. Ruh and Vilms [37] first observed that this equivalence holds for the class of immersions in Euclidean spaces $N = \mathbb{R}^n$. In this way, for instance, we produce harmonic maps starting from weaker properties of f.

The following results are a sample of those that we obtain in the paper. For terminology and notation the reader is referred to section 2.

Theorem 1.1. *Let $f : M \to N$ be a smooth map between Riemann surfaces satisfying $|\tau(f)| \leq \gamma \min\{e'(f), e''(f)\}$ on M. Then f is weakly conformal if and only if is either holomorphic of anti-holomorphic.*

As a consequence of Theorem 3.11 in section 3 we generalize a result, basically already in [14] and further improved by Salamon (see [13]), to the following

Theorem 1.2. *Let $f : M \to \mathbb{S}^4$ be an isometrically immersed compact surface, isotropic with negative spin and such that*

$$|\nabla \mathbf{H}^{(1,0)}| \leq \gamma(|\mathbf{H}|^2 - K + 1) = \gamma K^{\perp} .$$

Then f is totally umbilical if and only if $\chi(TM^{\perp}) = 0$. Otherwise

$$\chi(TM^{\perp}) = 2\chi(M) + m$$

where m is the total number of umbilical points counted with multiplicities.

Here and in the sequel $\chi(\ )$ denotes the Euler characteristic. Theorem 4.6 of section 4 immediately gives

Theorem 1.3. *Let $f : M \to \mathbb{CP}^2$ be an isometrically immersed surface with constant Kähler angle and neither holomorphic nor anti-holomorphic. Then there exists a metric on the total space $G_2(T\mathbb{CP}^2)$ of the Grassmann bundle of 2-planes over $\mathbb{CP}^2$ such that the following properties are equivalent:*

i) *f is a Codazzi surface*

ii) *f is totally real (or, in alternative terminology, Lagrangian)*

iii) *The Gauss lift $\gamma_f : M \to G_2(T\mathbb{CP}^2)$ is harmonic if and only if f has parallel mean curvature vector.*

The paper is divided in three sections as follows. In section 2 we recall some basic formulas and results to be used in the sequel. Section 3 is devoted to finding topological restrictions on the geometry of the surface once appropriate differential inequalities are satisfied. Finally, section 4 deals with the case of parallel mean curvature vector.

The Einstein summation convention over repeated indexes is in force unless otherwise specified.

2 Generalities

In this paper we will adopt both the complex variable notation and the moving frame technique. Let (M, g) be a connected Riemann surface with a specified metric g in its conformal class. If $z : U \subseteq M \to \mathbb{C}$ is a local holomorphic coordinate, then the real and imaginary part of $z = x + iy$ yield a pair of real coordinates $(x, y) : U \to \mathbb{R}^2$ in which respect the metric g and the volume form $\mathrm{d}A$ of M are given by

$$g = \lambda^2((\mathrm{d}x)^2 + (\mathrm{d}y)^2)\,, \qquad \mathrm{d}A = \lambda^2\,\mathrm{d}x \wedge \mathrm{d}y\,. \tag{2.1}$$

for a smooth positive function λ. Hereafter, for any 1-forms ω, ψ and for any integer $n \geq 1$ we adopt the notations

$$\omega^n = \underbrace{\omega \otimes \cdots \otimes \omega}_{n \text{ times}}, \qquad \omega\psi = \frac{1}{2}(\omega \otimes \psi + \psi \otimes \omega)\,, \qquad \omega \wedge \psi = \omega \otimes \psi - \psi \otimes \omega\,.$$

Conversely, if (M, g) is an oriented Riemannian manifold of dimension 2 then for every $p \in M$ it is possible to find a chart $(x, y) : U \subseteq M \to \mathbb{R}^2$ in a neighbourhood U of p such that (2.1) holds for some smooth positive function λ. Such (x, y) are called isothermal coordinates and λ is called the corresponding conformal factor. If $(x', y') : U' \to \mathbb{R}^2$ is another pair of isothermal coordinates with corresponding conformal factor λ' defined in a neighbourhood U' of p, then the functions $x' = x'(x, y)$ and $y' = y'(x, y)$ satisfy Cauchy-Riemann equations

$$\frac{\partial x'}{\partial x} = \frac{\partial y'}{\partial y}\,, \qquad \frac{\partial x'}{\partial y} = -\frac{\partial y'}{\partial x}$$

and therefore the transition function $z' = z'(z)$ between the complex charts $z = x + iy : U \to \mathbb{C}$ and $z' = x' + iy' : U' \to \mathbb{C}$ is holomorphic. In other words, the existence of an atlas for (M, g) given by a collection of isothermal coordinate charts endows M with a structure of Riemann surface, independent of the chosen atlas.

More generally, for every $p \in M$ it is possible to find a couple $\{\theta^1, \theta^2\}$ of 1-forms defined in a neighbourhood $U \subseteq M$ of p such that

$$g = (\theta^1)^2 + (\theta^2)^2\,, \qquad \mathrm{d}A = \theta^1 \wedge \theta^2\,.$$

The ordered pair $\{\theta^1, \theta^2\}$ is called a local oriented orthonormal coframe for M. For any choice of local isothermal coordinates $(x, y) : U \to \mathbb{R}^2$ with corresponding conformal factor λ, the ordered pair $(\lambda\,\mathrm{d}x, \lambda\,\mathrm{d}y)$ is a local oriented orthonormal coframe that we say to be induced by (x, y). Not every local oriented orthonormal coframe arises in this way. Indeed, if $(x, y) : U \to \mathbb{R}^2$ are local isothermal coordinates then for every smooth function $\beta : U \to \mathbb{R}$ the pair $\{\theta^1, \theta^2\}$ given by

$$\begin{aligned}\theta^1 &= \lambda \cos\beta\,\mathrm{d}x + \lambda \sin\beta\,\mathrm{d}y\\ \theta^2 &= -\lambda \sin\beta\,\mathrm{d}x + \lambda\cos\beta\,\mathrm{d}y\end{aligned}$$

is a local oriented orthonormal coframe. Conversely, any local oriented orthonormal coframe $\{\theta^1, \theta^2\}$ on a contractible neighbourhood U of a point $p \in M$ can be expressed in this way for a unique (up to additive constants) univalent smooth function $\beta : U \to \mathbb{R}$, but it is induced by local isothermal coordinates $(x', y') : U \to \mathbb{R}^2$ if and only if $\Delta\beta = 0$.

Given a local holomorphic coordinate $z = x + iy : U \to \mathbb{C}$ we set as usual

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

and

$$\mathrm{d}z = \mathrm{d}x + i\mathrm{d}y\,, \qquad \mathrm{d}\bar z = \mathrm{d}x - i\mathrm{d}y\,.$$

The pairs $\{\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar z}\}$ and $\{\mathrm{d}z, \mathrm{d}\bar z\}$ are local $\mathbb{C}$-frames (that is, pairs of $\mathbb{C}$-linearly independent local sections) for the complexified tangent and cotangent bundles $TM_{\mathbb{C}} = TM \otimes \mathbb{C}$ and $T^*M_{\mathbb{C}} = T^*M \otimes \mathbb{C}$, respectively. The Levi-Civita connection ∇ of M extends complex linearly to both bundles. In particular, we have

$$\nabla\frac{\partial}{\partial z} = \frac{\partial \log \lambda^2}{\partial z}\,\mathrm{d}z \otimes \frac{\partial}{\partial z}\,, \qquad \nabla\frac{\partial}{\partial \bar z} = \frac{\partial \log \lambda^2}{\partial \bar z}\,\mathrm{d}\bar z \otimes \frac{\partial}{\partial \bar z} \tag{2.2}$$

and

$$\nabla \mathrm{d}z = -\frac{\partial \log \lambda^2}{\partial z}\,\mathrm{d}z \otimes \mathrm{d}z\,, \qquad \nabla \mathrm{d}\bar z = -\frac{\partial \log \lambda^2}{\partial \bar z}\,\mathrm{d}\bar z \otimes \mathrm{d}\bar z\,. \tag{2.3}$$

The complex cotangent bundle splits as the direct sum $T^*M^{(1,0)} \oplus T^*M^{(0,1)}$ of the subbundles of 1-forms of type $(1,0)$ and $(0,1)$, respectively. If z is a local holomorphic coordinate, then sections s of $T^*M^{(1,0)}$ (respectively, $T^*M^{(0,1)}$) are the differential forms that admit a local expression

$$s = s_U \mathrm{d}z \qquad (\text{resp., } s = s_U \mathrm{d}\bar z)$$

with s_U any complex valued function. If instead we are working with a local oriented orthonormal coframe $\{\theta^1, \theta^2\}$ on M, then the forms of type $(1,0)$ are the complex multiples of

$$\varphi = \theta^1 + i\theta^2$$

and the forms of type $(0,1)$ are the multiples of $\bar\varphi = \theta^1 - i\theta^2$. Denoting by $\{e_1, e_2\}$ the local frame dual to $\{\theta^1, \theta^2\}$, we can associate to it the Levi-Civita connection forms $\{\theta^i_j\}_{1\le i,j\le 2}$, that is, a collection of 1-forms such that

$$\nabla e_j = \theta^i_j \otimes e_i\,.$$

Orthonormality of the frame implies that $\theta^1_1 = \theta^2_2 = 0$ while $\theta^1_2 = -\theta^2_1$ and we have

$$\nabla\varphi = i\theta^1_2 \otimes \varphi\,, \qquad \nabla\bar\varphi = -i\theta^1_2 \otimes \bar\varphi = i\theta^2_1 \otimes \bar\varphi\,.$$

For any pair of non-negative integers p, q we let

$$T^*M^{(p,q)} = (T^*M^{(1,0)})^{\otimes p} \otimes (T^*M^{(0,1)})^{\otimes q}$$

be the bundle of $(p+q)$-covariant tensor fields of type (p, q). With the notation introduced by Calabi, [9], we have a splitting $\nabla = \nabla' + \nabla''$ of the Levi-Civita connection induced on $T^*M^{(p,q)}$ as a sum of two connections

$$\nabla' : T^*M^{(p,q)} \to T^*M^{(p+1,q)}\,, \qquad \nabla'' : T^*M^{(p,q)} \to T^*M^{(p,q+1)}\,.$$

In particular, we have $\nabla'\mathrm{d}\bar z = 0$ and $\nabla''\mathrm{d}z = 0$.

The same construction generalizes to linear connections on complex vector bundles as follows. For any complex vector bundle E and for any pair of non-negative integers p, q we can consider the complex bundle $T^*M^{(p,q)} \otimes E$ of E-valued tensor fields of type (p, q). Setting $k = \operatorname{rank} E$, if $\{E_1, \ldots, E_k\}$ is a local frame for E defined on an open set $U \subseteq M$ and $z : U \to \mathbb{C}$ is a holomorphic coordinate then the sections s of $T^*M^{(p,q)} \otimes E$ are the tensor fields that admit a local expression

$$s = \sum_{a=1}^{k} s^a_U (\mathrm{d}z)^p \otimes (\mathrm{d}\bar z)^q \otimes E_a \tag{2.4}$$

with s^a_U, $a = 1, \ldots, k$, complex valued functions on U. Equivalently, if $\{\theta^1, \theta^2\}$ is local oriented orthonormal coframe on M then the sections of $T^*M^{(p,q)} \otimes E$ are locally expressed as

$$s = \sum_{a=1}^{k} \tilde s^a_U \varphi^p \otimes \bar\varphi^q \otimes E_a$$

with $\tilde s^a_U : U \to \mathbb{C}$ for $a = 1, \ldots, k$. The bundle $T^*M \otimes E$ of E-valued 1-forms splits as the direct sum

$$T^*M \otimes E = (T^*M^{(1,0)} \otimes E) \oplus (T^*M^{(0,1)} \otimes E)\,. \tag{2.5}$$

If ${}^E\nabla$ is a linear connection on E, then its canonical extension to $T^*M^{(p,q)} \otimes E$ that we still denote by ${}^E\nabla$ can be splitted as the direct sum of two connections

$$
\begin{aligned}
{}^E\nabla' &: T^*M^{(p,q)} \otimes E \to T^*M^{(p+1,q)} \otimes E\,,\\
{}^E\nabla'' &: T^*M^{(p,q)} \otimes E \to T^*M^{(p,q+1)} \otimes E\,.
\end{aligned}
$$

We briefly describe locally their action on a section s of $T^*M^{(p,q)} \otimes E$. To any local frame $\{E_1, \dots, E_k\}$ for E defined on an open set $U \subseteq M$ we can associate a collection of 1-forms $\{{}^E\Theta^a_b\}_{1 \le a,b \le k}$, that we call connection forms, so that

$$
{}^E\nabla E_b = {}^E\Theta^a_b \otimes E_a\,.
$$

If $z : U \to \mathbb{C}$ is a holomorphic coordinate then locally we can express s as in (2.4) and split ${}^E\Theta^a_b$ into its $(1,0)$ and $(0,1)$ parts as

$$
{}^E\Theta^a_b = {}^E\Gamma^a_{bz}\,\mathrm{d}z + {}^E\Gamma^a_{b\bar z}\,\mathrm{d}\bar z\,.
$$

Then

$$
{}^E\nabla' s = \sum_{a=1}^{k} \left(\frac{\partial s^a_U}{\partial z} - p\frac{\partial \log \lambda^2}{\partial z} s^a_U + {}^E\Gamma^a_{bz} s^b_U \right) (\mathrm{d}z)^{p+1} \otimes (\mathrm{d}\bar z)^q \otimes e_a \tag{2.6}
$$

$$
{}^E\nabla'' s = \sum_{a=1}^{k} \left(\frac{\partial s^a_U}{\partial \bar z} - q\frac{\partial \log \lambda^2}{\partial \bar z} s^a_U + {}^E\Gamma^a_{b\bar z} s^b_U \right) (\mathrm{d}z)^{p} \otimes (\mathrm{d}\bar z)^{q+1} \otimes e_a \tag{2.7}
$$

and

$$
{}^E\nabla s = {}^E\nabla' s + {}^E\nabla'' s\,.
$$

We also remark that if $\{\theta^1, \theta^2\}$ is a local oriented orthonormal coframe on U and s is given as

$$
s = \sum_{a=1}^{k} \tilde s^a_U \varphi^p \otimes \bar\varphi^q \otimes E_a
$$

then we have

$$
\nabla s = \left(\mathrm{d}\tilde s^a_U + i(p-q)\tilde s^a_U \theta^1_2 + \tilde s^b_U\, {}^E\Theta^a_b \right) \otimes \varphi^p \otimes \bar\varphi^q \otimes E_a\,.
$$

Let (M, g) be a connected Riemann surface with a specified metric g in its conformal class and let $(N, \langle\ ,\ \rangle)$ be an oriented Riemannian manifold that we tacitely assume to be 4-dimensional unless otherwise stated. For a given smooth map $f : M \to N$ we consider the differential $\mathrm{d}f$ as a section of the bundle $T^*M \otimes f^{-1}TN$. Thus indicating its Hilbert-Schmidt norm with $|\mathrm{d}f|$, the energy of f, on the compact domain $D \subset M$, is given by

$$E_D(f) = \frac{1}{2}\int_D |\mathrm{d}f|^2 \mathrm{d}A \tag{2.8}$$

where $\mathrm{d}A$ is the area element of the metric g on M. The function $e(f) = \frac{1}{2}|\mathrm{d}f|^2$ is referred to as the energy density of the map. Critical points of the energy functional (2.8) are called harmonic maps and are characterized by the vanishing of the tension field $\tau(f)$, a section of $f^{-1}TN$, defined by

$$\tau(f) = \mathrm{Tr}_g \nabla \mathrm{d}f \tag{2.9}$$

where with ∇ we indicate the canonical connections on TM, $f^{-1}TN$ and all related and pertinent bundles in the subsequent discussion.

In case N is Kählerian, that is, it carries an orthogonal $(1,1)$-tensor field J with $\nabla J = 0$ and $J^2 = -\,\mathrm{id}$, we can split the complexified tangent bundle $TN \otimes \mathbb{C}$ as the direct sum

$$TN \otimes \mathbb{C} = TN^{(1,0)} \oplus TN^{(0,1)} \tag{2.10}$$

of the complex subbundles of vector fields of type $(1,0)$ and $(0,1)$, respectively. They are defined as the eigendistributions, corresponding to eigenvalues i and $-i$, of the $\mathbb{C}$-linear extension $J^{\mathbb{C}} : TN \otimes \mathbb{C} \to TN \otimes \mathbb{C}$ of J. $TN \otimes \mathbb{C}$ is naturally isomorphic to the complex tangent bundle of the underlying complex manifold N, with $TN^{(1,0)}$ and $TN^{(0,1)}$ corresponding to the holomorphic and antiholomorphic tangent bundles. If $\{w^a\}_{a=1}^n = \{u^a + iv^a\} : V \subseteq N \to \mathbb{C}^n$ is a holomorphic chart, with $2n$ the dimension of N as a real manifold, and we set

$$\frac{\partial}{\partial w^a} = \frac{1}{2}\left(\frac{\partial}{\partial u^a} - i\frac{\partial}{\partial v^a}\right), \qquad \frac{\partial}{\partial \bar{w}^a} = \frac{1}{2}\left(\frac{\partial}{\partial u^a} + i\frac{\partial}{\partial v^a}\right)$$

then $\{\frac{\partial}{\partial w^a}\}$ and $\{\frac{\partial}{\partial \bar{w}^a}\}$ are local $\mathbb{C}$-frames for $TN^{(1,0)}$ and $TN^{(0,1)}$, respectively. The complexified cotangent bundle also splits as the direct sum

$$T^*N \otimes \mathbb{C} = T^*N^{(1,0)} \oplus T^*N^{(0,1)}$$

of the subbundles of holomorphic and antiholomorphic 1-forms, for which local frames are given by the collections $\{\mathrm{d}w^a\}$ and $\{\mathrm{d}\bar{w}^a\}$, where

$$\mathrm{d}w^a = \mathrm{d}u^a + i\mathrm{d}v^a\,, \qquad \mathrm{d}\bar{w}^a = \mathrm{d}u^a - i\mathrm{d}v^a\,.$$

We still denote as $\langle\,,\,\rangle$ the $\mathbb{C}$-bilinear extension to $(TN \otimes \mathbb{C}) \times (TN \otimes \mathbb{C})$ of the original (real) Riemannian metric on TN. An operation of complex conjugation can be defined on the bundle $TN \otimes \mathbb{C}$ as the involutive transformation

$$\overline{(\cdot)} : TN \otimes \mathbb{C} \to TN \otimes \mathbb{C}$$

whose action on decomposable tensors $\lambda v \equiv v \otimes \lambda \in TN \otimes \mathbb{C}$ is given by $\overline{\lambda v} = \bar{\lambda} v$. We have $\overline{TN^{(1,0)}} = TN^{(0,1)}$. In particular, for any local holomorphic chart it holds $\overline{\frac{\partial}{\partial w^a}} = \frac{\partial}{\partial \bar{w}^a}$ for $a = 1, \ldots, n$. With this notion of complex conjugation, the sesquilinear form $h : (TN\otimes\mathbb{C})\times(TN\otimes\mathbb{C}) \to \mathbb{C}$ given by

$$h(X,Y) = \langle X, \overline{Y}\rangle \tag{2.11}$$

is a Hermitian product. For any $X \in TN \otimes \mathbb{C}$ we set $|X| = |X|_h = \sqrt{h(X,X)}$. We remark that the subbundles $TN^{(1,0)}$ and $TN^{(0,1)}$ are orthogonal to each other with respect to h.

In this setting, the $\mathbb{C}$-linear extension

$$\mathrm{d}^{\mathbb{C}} f : TM \otimes \mathbb{C} \to TN \otimes \mathbb{C}$$

of $\mathrm{d}f : TM \to TN$ can be splitted as the direct sum of four maps (we stick to the notation in [17, pages 221-222])

$$\begin{array}{ll} \partial f : TM^{(1,0)} \to TN^{(1,0)} & \qquad \bar{\partial} f : TM^{(0,1)} \to TN^{(1,0)} \\ \partial \bar{f} : TM^{(1,0)} \to TN^{(0,1)} & \qquad \bar{\partial} \bar{f} : TM^{(0,1)} \to TN^{(0,1)}\,. \end{array}$$

The map f is holomorphic if and only if $\bar\partial f = 0$, and antiholomorphic if and only if $\partial f = 0$. The third and fourth maps in the above decomposition are complex conjugate to the first and second,

$$\partial \bar f = \overline{\bar\partial f}\,, \qquad \bar\partial \bar f = \overline{\partial f}$$

in the following sense: if z is a local holomorphic coordinate on M, then

$$\partial \bar f(\partial_z) = \overline{\bar\partial f(\partial_{\bar z})}\,, \qquad \bar\partial \bar f(\partial_{\bar z}) = \overline{\partial f(\partial_z)}\,, \tag{2.12}$$

where we are writing for short $\partial_z = \frac{\partial}{\partial z}$ and $\partial_{\bar z} = \frac{\partial}{\partial \bar z}$. Thus, the energy density of f decomposes as the sum $e(f) = e'(f) + e''(f)$ of

$$e'(f) = |\partial f|^2 \equiv |\bar\partial \bar f|^2\,, \qquad e''(f) = |\bar\partial f|^2 \equiv |\partial \bar f|^2\,, \tag{2.13}$$

where $|\cdot|$ denotes the Hilbert-Schmidt norm. Explicitely, if we write the metric of M as $g = \lambda^2\,\mathrm{d}z\,\mathrm{d}\bar z \equiv \frac{\lambda^2}{2}(\mathrm{d}z\otimes\mathrm{d}\bar z+\mathrm{d}\bar z\otimes\mathrm{d}z)$ for a suitable conformal factor $\lambda > 0$ then $|\partial_z| = |\partial_{\bar z}| = \frac{\lambda}{\sqrt{2}}$ and

$$\begin{aligned} e'(f) &= \frac{2}{\lambda^2}\langle \partial f(\partial_z), \overline{\partial f(\partial_z)}\rangle \equiv \frac{2}{\lambda^2}\langle \bar\partial \bar f(\partial_{\bar z}), \overline{\bar\partial \bar f(\partial_{\bar z})}\rangle\,, \\ e''(f) &= \frac{2}{\lambda^2}\langle \bar\partial f(\partial_{\bar z}), \overline{\bar\partial f(\partial_{\bar z})}\rangle \equiv \frac{2}{\lambda^2}\langle \partial \bar f(\partial_z), \overline{\partial \bar f(\partial_z)}\rangle\,. \end{aligned} \tag{2.14}$$

By also defining

$$\rho = \frac{2}{\lambda^2}\langle \partial f(\partial_z), \overline{\bar\partial f(\partial_{\bar z})}\rangle \tag{2.15}$$

we have

$$f^*\langle\,,\,\rangle = e(f)g + \rho\lambda^2(\mathrm{d}z)^2 + \bar\rho\lambda^2(\mathrm{d}\bar z)^2 \tag{2.16}$$

where $f^*\langle\,,\,\rangle = \langle \mathrm{d}^{\mathbb{C}} f\,\cdot\,, \mathrm{d}^{\mathbb{C}} f\,\cdot\,\rangle$ denotes the pull-back via $\mathrm{d}^{\mathbb{C}}$ of the symmetric $\mathbb{C}$-bilinear extension to $TN \otimes \mathbb{C}$ of the Riemannian metric of N. (2.16) can be easily checked writing $\mathrm{d}^{\mathbb{C}} f = \partial f + \partial \bar f + \bar\partial f + \bar\partial \bar f$ and then using (2.12) and orthogonality of $TN^{(1,0)}$ and $TN^{(0,1)}$ with respect to h. From (2.16) we see that conformality of f, in case f is an immersion, is equivalent to the vanishing of ρ. If N is a Riemann surface then the latter happens if and only if at every point in M at least one among $e'(f)$ or

$e''(f)$ is zero. Indeed, in this case $\partial f(\partial_z)$ and $\bar\partial f(\partial_{\bar z})$ are always $\mathbb{C}$-linearly dependent since $TN^{(1,0)}$ has rank one, so it follows from (2.14)-(2.15) that

$$\rho\bar\rho = e'(f)e''(f)\,. \tag{2.17}$$

Setting $E = f^{-1}(TN \otimes \mathbb{C})$, the linear maps

$$\begin{aligned} \partial f + \partial \bar f &: TM^{(1,0)} \to TN \otimes \mathbb{C}\,,\\ \bar\partial f + \bar\partial \bar f &: TM^{(0,1)} \to TN \otimes \mathbb{C} \end{aligned}$$

can be regarded as sections of the complex bundles $T^*M^{(1,0)} \otimes E$ and $T^*M^{(0,1)} \otimes E$, respectively. If z is a (local) holomorphic coordinate on M then we can write

$$\partial f + \partial \bar f = \frac{\partial f}{\partial z} \otimes \mathrm{d}z\,, \qquad \bar\partial f + \bar\partial \bar f = \frac{\partial f}{\partial \bar z} \otimes \mathrm{d}\bar z$$

with $\frac{\partial f}{\partial z}$, $\frac{\partial f}{\partial \bar z}$ sections of E, so that

$$\mathrm{d}^{\mathbb{C}} f = \frac{\partial f}{\partial z} \otimes \mathrm{d}z + \frac{\partial f}{\partial \bar z} \otimes \mathrm{d}\bar z\,. \tag{2.18}$$

We also set

$$\frac{\partial f}{\partial z} \otimes \mathrm{d}z = \mathrm{d}f^{(1,0)}\,, \qquad \frac{\partial f}{\partial \bar z} \otimes \mathrm{d}\bar z = \mathrm{d}f^{(0,1)}$$

to highlight that (2.18) gives the decomposition of $\mathrm{d}^{\mathbb{C}} f$, as a section of $T^*M \otimes E$, according to the direct sum in (2.5). Notice that in this setting the tension field (2.9) can be expressed as

$$\tau(f) = \frac{4}{\lambda^2} \nabla_{\frac{\partial}{\partial z}} \frac{\partial f}{\partial \bar z} = \frac{4}{\lambda^2} \nabla_{\frac{\partial}{\partial \bar z}} \frac{\partial f}{\partial z}\,. \tag{2.19}$$

For $f : M \to N$ an isometric immersion with N Kählerian, another invariant, the Kähler angle $\alpha : M \to [0, \pi]$, plays an important role. α is invariantly defined by

$$f^*k = \cos\alpha \, \mathrm{d}A \tag{2.20}$$

where k is the Kähler form of N. Geometrically, given an orthonormal oriented basis e_1, e_2 of T_pM, $p \in M$, and indicating with the same letter

J the pull-back of the almost complex structure, α measures the angle between e_1 and Je_2. Observe that f is holomorphic or anti-holomorphic, for short $\pm$ holomorphic, respectively when $\cos\alpha = \pm 1$ everywhere on M, that is, $J(TM) = TM$. In case $\cos\alpha \equiv 0$ on M, that is, $J(TM) = TM^\perp$, the normal bundle of the immersion, f is called totally real or, with an alternative terminology, a Lagrangian surface.

In the study of the geometry of an isometric immersion $f : M \to N$ we systematically use the notion of isotropy with positive or negative spin, $\pm$ spin for short. Following [14], one way to introduce this concept is by considering the twistor space Z over N and the twistor lifts $\varphi_\pm$ canonically associated to f. On Z there exists a natural almost complex structure J_1 (integrable in favourable circumstances according to the Atiyah-Hitchin-Singer's theorem, [4]) with respect to which the twistor lifts $\varphi_\pm : M \to Z$ are holomorphic if and only if f is isotropic with $\pm$ spin, respectively. Here we avoid direct use of φ_+ and φ_- by equivalently formulating isotropy via the vanishing of appropriate contact invariants, $s_\pm$, on M that we define as follows (for the equivalence and further details the reader is referred to [30]).

Let $\mathbf{e} = \{e_a\}$, $1 \le a \le 4$, be a (local) Darboux frame along f, so that $\{e_1, e_2\}$ is a (local) oriented orthonormal basis of TM and $\{e_3, e_4\}$ of $TM^\perp$ considered oriented coherently with the orientations of M and N. From now on we fix the index ranges

$$1 \le a, b, c, \dots \le 4\,, \quad 1 \le i, j, k, \dots \le 2\,, \quad 3 \le \alpha, \beta, \gamma, \dots \le 4\,.$$

For $\{\theta^i\}$ orthonormal forms dual to $\{e_i\}$ on M, the second fundamental tensor, $\mathrm{II} = \nabla \mathrm{d}f$, expresses as

$$\mathrm{II} = h^\alpha_{ij}\, \theta^i \otimes \theta^j \otimes e_\alpha\,. \tag{2.21}$$

Since f is an isometry the mean curvature vector $\mathbf{H}$ is $\frac{1}{2}\tau(f)$ so that minimality and harmonicity of f coincide and the $(1,0)$ forms on M are exactly the multiples of $\varphi = \theta^1 + i\theta^2$. We consider now the complexified normal bundle $TM^\perp \otimes \mathbb{C}$ and split it into two complementary line bundles

$$TM^{\perp(1,0)} = \mathbb{C}\{e_3 - ie_4\}\,, \qquad TM^{\perp(0,1)} = \mathbb{C}\{e_3 + ie_4\} \tag{2.22}$$

which are globally well defined. Notice that $TM^{\perp(1,0)} \simeq TM^{\perp}$ with the correct orientation. Indicating with L^α the Hopf's transform of the symmetric matrix (h^α_{ij}), that is,

$$L^\alpha = \frac{1}{2}(h^\alpha_{11} - h^\alpha_{22}) - ih^\alpha_{12}\,,$$

we can split II according to form type as

$$\mathrm{II} = \frac{1}{2}L^\alpha\varphi^2 \otimes e_\alpha + g \otimes \mathbf{H} + \frac{1}{2}\overline{L^\alpha}\bar{\varphi}^2 \otimes e_\alpha \tag{2.23}$$

and we obtain two global sections Σ_+ and Σ_- respectively of $T^*M^{(1,0)} \otimes T^*M^{(1,0)} \otimes T^*M^{\perp(1,0)}$ and $T^*M^{(1,0)} \otimes T^*M^{(1,0)} \otimes T^*M^{\perp(0,1)}$ by setting

$$\Sigma_\pm = \frac{1}{\sqrt{2}}(L^3 \pm iL^4)\varphi^2 \otimes (e_3 \mp ie_4)\,. \tag{2.24}$$

Thus the $(2,0)$ part of II is given by

$$\mathrm{II}^{(2,0)} = \frac{1}{2}L^\alpha\varphi^2 \otimes e_\alpha = \frac{1}{2\sqrt{2}}(\Sigma_+ + \Sigma_-)$$

and the scalar quantities

$$s_\pm = \frac{1}{2\sqrt{2}}|\Sigma_\pm| \tag{2.25}$$

define two global contact invariants with smooth squares. Their geometrical significance is expressed by the following relations with curvatures obtained via Gauss and Ricci equations:

$$\begin{cases} s_+^2 + s_-^2 = |\mathbf{H}|^2 - K + R_{1212} \\ s_+^2 - s_-^2 = K^\perp - R_{1234}\,. \end{cases} \tag{2.26}$$

Here K, $K^\perp$ are respectively the Gaussian curvature and the normal curvature of $f : M \to N$, while R_{1212}, R_{1234} are the pull-backs of the correspondent components of the curvature tensor of N under the Darboux frame $\mathbf{e}$. (Thus in particular they are well defined contact invariants.) Setting

$$u = s_+^2 + s_-^2 = \sum_\alpha |L^\alpha|^2 \tag{2.27}$$

the zeros of u are precisely the umbilical points of f, as one can see by (2.23).

Proof of (2.26). Gauss and Ricci equations read as

$$K = R_{1212} + h^\alpha_{11}h^\alpha_{22} - h^\alpha_{12}h^\alpha_{12}\,, \qquad K^\perp = R_{1234} + h^3_{1k}h^4_{2k} - h^3_{2k}h^4_{1k}$$

hence (2.26) is equivalent to

$$\begin{cases} s_+^2 + s_-^2 = |\mathbf{H}|^2 - h^\alpha_{11}h^\alpha_{22} + h^\alpha_{12}h^\alpha_{12} \\ s_+^2 - s_-^2 = h^3_{1k}h^4_{2k} - h^3_{2k}h^4_{1k} \end{cases}$$

and a direct computation yields (note that $|\varphi| = |e_3 \pm ie_4| = \sqrt{2}$)

$$\begin{aligned} s_+^2 = \frac{1}{2}|L^3 + iL^4|^2 &= \frac{1}{2}\left[\left(\frac{1}{2}(h^3_{11} - h^3_{22}) + h^4_{12}\right)^2 + \left(\frac{1}{2}(h^4_{11} - h^4_{22}) - h^3_{12}\right)^2\right] \\ &= \frac{1}{2}\left(|\mathbf{H}|^2 - h^\alpha_{12}h^\alpha_{12} + h^\alpha_{11}h^\alpha_{22} + h^3_{1k}h^4_{2k} - h^3_{2k}h^4_{1k}\right)\,, \\ s_-^2 = \frac{1}{2}|L^3 - iL^4|^2 &= \frac{1}{2}\left[\left(\frac{1}{2}(h^3_{11} - h^3_{22}) - h^4_{12}\right)^2 + \left(\frac{1}{2}(h^4_{11} - h^4_{22}) + h^3_{12}\right)^2\right] \\ &= \frac{1}{2}\left(|\mathbf{H}|^2 - h^\alpha_{12}h^\alpha_{12} + h^\alpha_{11}h^\alpha_{22} - h^3_{1k}h^4_{2k} + h^3_{2k}h^4_{1k}\right)\,. \end{aligned} \tag{2.28}$$

□

Definition 2.1. At a point p, f is isotropic with positive (respectively, negative) spin if $s_+(p) = 0$ (respectively $s_-(p) = 0$). It is isotropic with positive (negative) spin if it has the respective property at every point of M. We will simply say that f is isotropic when at least one of the two possibilities occurs.

Remark 2.2. This definition follows that of Bryant, [8], for minimal surfaces in $\mathbb{S}^4$, Calabi, [9], called isotropic minimal surfaces in $\mathbb{S}^4$ pseudo-holomorphic curves. Our notion of isotropy corresponds to real isotropy of Eells and Wood, [17], and Chern, [12]. Observe that in our setting an isotropic f need not to be minimal.

Even if some of our results can be extended to more general situations, we will focus our attention on a special class of surfaces that we call Codazzi surfaces.

Definition 2.3. Let $f : M \to N$ be an isometric immersion of a Riemann surface (M, g) into a Riemannian manifold $(N, \langle\, ,\, \rangle)$. We say that M is a Codazzi surface if its second fundamental form II satisfies the Codazzi equation

$$(\nabla_X \mathrm{II})(Y, Z) = (\nabla_Y \mathrm{II})(X, Z) \tag{2.29}$$

for all vector fields $X, Y, Z \in \mathfrak{X}(M)$.

The terminology comes from the characterizing property of having a symmetric covariant derivative of the second fundamental tensor. In other words, according to standard terminology, see for instance [6], II is a Codazzi tensor. Due to the fact that M is 2-dimensional, this latter property is expressed by the vanishing of the tensor

$$\mathrm{Ric}^{\perp}(f) = R^{\alpha}_{kik}\theta^i \otimes e_\alpha = R^{\alpha}_{i}\theta^i \otimes e_\alpha \tag{2.30}$$

which first appeared in the study of the Gauss lift γ_f into $G_2(TN)$, the Grassmann bundle of oriented 2-planes over N, of an isometric immersion f in the work of C. M. Wood, [40], and [31]. Indeed we will show that isotropy and the Codazzi property are strictly related to the Ruh-Vilms property that we state as

$$\gamma_f \text{ is harmonic if and only if } f \text{ has parallel mean curvature.} \tag{RV}$$

It is perhaps well known that for a Codazzi surface conditions $\nabla \mathbf{H} = 0$ and $\nabla'' \mathrm{II}^{(2,0)} = 0$ are equivalent. This is true considering ambient manifolds N of any dimension ≥ 3, and in case $\dim N = 3$ it amounts to saying that a Codazzi surface $f : M \to N$ has constant mean curvature if and only if the Hopf differential $L^3\varphi^2$ is holomorphic.

We provide a short proof of the above statement by moving frame technique along the lines of the original argument by Hopf, [26, 27], for $N = \mathbb{R}^3$. The coefficients of the covariant derivative of II

$$\nabla \mathrm{II} = h^{\alpha}_{ij,k}\, \theta^k \otimes \theta^i \otimes \theta^j \otimes e_\alpha$$

are given by

$$h^{\alpha}_{ij,k}\, \theta^k = \mathrm{d}h^{\alpha}_{ij} - h^{\alpha}_{kj}\, \theta^k_i - h^{\alpha}_{ik}\, \theta^k_j + h^{\beta}_{ij}\, \theta^{\alpha}_{\beta}$$

and they satisfy Codazzi equations

$$h^{\alpha}_{ij,k} - h^{\alpha}_{ik,j} = -R^{\alpha}{}_{ijk} \,.$$

In particular

$$h^{\alpha}_{11,2} - h^{\alpha}_{12,1} = R^{\alpha}_{2} \,, \qquad h^{\alpha}_{22,1} - h^{\alpha}_{21,2} = R^{\alpha}_{1} \,. \tag{2.31}$$

We write the covariant derivative of $\mathbf{H}$ as $\nabla\mathbf{H} = H^{\alpha}_{k}\theta^{k} \otimes e_{\alpha}$ and we have

$$H^{\alpha}_{k} = \frac{1}{2}(h^{\alpha}_{11,k} + h^{\alpha}_{22,k}) \,. \tag{2.32}$$

Lastly, the covariant derivative of the tensor field $\mathrm{II}^{(2,0)} = \frac{1}{2}L^{\alpha}\varphi^{2}e_{\alpha}$ is given by

$$2\nabla\mathrm{II}^{(2,0)} = (\mathrm{d}L^{\alpha} + L^{\beta}\theta^{\alpha}_{\beta} + 2iL^{\alpha}\theta^{1}_{2}) \otimes \varphi^{2} \otimes e_{\alpha}$$

and it is easy to check that

$$\mathrm{d}L^{\alpha} + L^{\beta}\theta^{\alpha}_{\beta} + 2iL^{\alpha}\theta^{1}_{2} = \frac{1}{2}(h^{\alpha}_{11,k} - h^{\alpha}_{22,k})\theta^{k} - ih^{\alpha}_{12,k}\theta^{k} \,.$$

Using $\theta^{1} = \frac{1}{2}(\varphi + \bar{\varphi})$ and $\theta^{2} = -\frac{i}{2}(\varphi - \bar{\varphi})$ we infer that

$$2\nabla'\mathrm{II}^{(2,0)} = \left(\frac{1}{4}h^{\alpha}_{11,1} - \frac{1}{4}h^{\alpha}_{22,1} - \frac{i}{2}h^{\alpha}_{12,1} - \frac{i}{4}h^{\alpha}_{11,2} + \frac{i}{4}h^{\alpha}_{22,2} - \frac{1}{2}h^{\alpha}_{12,2}\right)\varphi^{3}\otimes e_{\alpha}$$

$$2\nabla''\mathrm{II}^{(2,0)} = \left(\frac{1}{4}h^{\alpha}_{11,1} - \frac{1}{4}h^{\alpha}_{22,1} + \frac{i}{4}h^{\alpha}_{11,2} - \frac{i}{4}h^{\alpha}_{22,2} - \frac{i}{2}h^{\alpha}_{12,1} + \frac{1}{2}h^{\alpha}_{12,2}\right)\varphi^{2}\otimes\bar{\varphi}\otimes e_{\alpha}$$

and after a little manipulation using (2.31) and (2.32) we get

$$2\nabla''\mathrm{II}^{(2,0)} = \left(\frac{1}{2}(H^{\alpha}_{1} - R^{\alpha}_{1}) - \frac{i}{2}(H^{\alpha}_{2} - R^{\alpha}_{2})\right)\varphi^{2} \otimes \bar{\varphi} \otimes e_{\alpha} \,.$$

In local notation we can restate this as

$$(\mathrm{d}L^{\alpha} + L^{\beta}\theta^{\alpha}_{\beta} + 2iL^{\alpha}\theta^{1}_{2}) \wedge \varphi + \nabla H^{\alpha} \wedge \bar{\varphi} = \frac{1}{2}(R^{\alpha}_{1} - iR^{\alpha}_{2})\,\varphi \wedge \bar{\varphi} \,. \tag{2.33}$$

If f is Codazzi, that is, $R^{\alpha}_{i} = 0$, then we have $\nabla''\mathrm{II}^{(2,0)} = 0$ if and only if $\nabla'\mathbf{H} = 0$, that in turn amounts to $\nabla\mathbf{H} = 0$ since $\mathbf{H}$ is a real section of the normal bundle. Note that if all of these conditions are satisfied, namely $R^\alpha_i = H^\alpha_i = 0$, then

$$h^\alpha_{12,1} = h^\alpha_{11,2} = -h^\alpha_{22,2} \qquad \text{and} \qquad h^\alpha_{12,2} = h^\alpha_{22,1} = -h^\alpha_{11,1}$$

hence from the above expressions for $\nabla' \mathrm{II}^{(2,0)}$

$$\mathrm{d}L^\alpha + L^\beta \theta^\alpha_\beta + 2iL^\alpha \theta^1_2 = (h^\alpha_{11,1} + ih^\alpha_{22,2})\varphi \,. \tag{2.34}$$

From (2.33) it is also apparent that $\nabla''\Sigma_+ = 0$ (respectively, $\nabla''\Sigma_- = 0$) if and only if $\nabla'\mathbf{H}^{(1,0)} = 0$ (resp., $\nabla'\mathbf{H}^{(0,1)} = 0$).

It is clear that the Codazzi property generally depends on f, even so it is automatically satisfied when N has constant sectional curvature. A second large class of Codazzi surfaces is given by those for which $\nabla \mathrm{II} = 0$. For N the Euclidean space these latter have been classified by Ferus, [23]. We give here a further example that will be repeatedly used in the sequel.

Let N be Kähler and with constant holomorphic sectional curvature c. The Riemann curvature tensor of N can be expressed in the form

$$\begin{aligned}\langle R(X,Y)Z,W\rangle = \frac{c}{4}(&\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle \\ &+ \langle X,JW\rangle\langle Y,JZ\rangle - \langle X,JZ\rangle\langle Y,JW\rangle \\ &- 2\langle X,JY\rangle\langle Z,JW\rangle)\end{aligned} \tag{2.35}$$

for X, Y, Z, W vector fields on N. Indicate with $(\ ,\ )$ the Hermitian inner product on N defined by

$$(X,Y) = \langle X,Y\rangle + i\langle X,JY\rangle \,. \tag{2.36}$$

For $\mathbf{e}$ a Darboux frame along the isometry $f : M \to N$ we let $\{E_i\}$ be the special unitary frame defined in [19], page 590, and related to $\mathbf{e}$ via the equations

$$\begin{cases} e_1 = uE_1 + \bar{v}E_2 \,, & e_2 = i(uE_1 - \bar{v}E_2) \\ e_3 = -\bar{v}E_1 + uE_2 \,, & e_4 = i(\bar{v}E_1 + uE_2) \end{cases} \tag{2.37}$$

where u, v are smooth (locally defined) functions of M such that

$$|u| = \cos\frac{\alpha}{2}\,, \qquad |v| = \sin\frac{\alpha}{2}$$

α the Kähler angle defined in (2.20). Set $u = e^{i\theta}\cos\frac{\alpha}{2}$, $v = e^{i\psi}\sin\frac{\alpha}{2}$.

Using (2.35), (2.36), (2.37) it is immediate to obtain

$$\begin{aligned} R^3{}_{121} &= \frac{3}{8}c\sin(2\alpha)\cos(\theta+\psi)\,, \qquad & R^3{}_{212} &= -\frac{3}{8}c\sin(2\alpha)\sin(\theta+\psi) \\ R^4{}_{212} &= \frac{3}{8}c\sin(2\alpha)\sin(\theta+\psi)\,, \qquad & R^4{}_{212} &= \frac{3}{8}c\sin(2\alpha)\cos(\theta+\psi)\,. \end{aligned}$$

We therefore verify that

$$f \text{ is Codazzi if and only if either } f \text{ is } \pm \text{ holomorphic or } f \text{ is totally real.} \tag{2.38}$$

Remark 2.4. This fact is also known to A. Elghanmi.

A final essential ingredient in our investigation is given by a substantial generalization due to Eschenburg and Tribuzy [20] of a result dating back to Bers [5] and Chern [10, 11]. By their method we have been able to generalize part of our results to the present form and feel that more can be obtained. We report here the essential point to our subsequent analysis. Let $E \to M$ be a complex vector bundle. A smooth section s of E is called of holomorphic type if near any zero p of s we have

$$s(z) = z^k s_0(z) \tag{2.39}$$

for some positive integer k, some continuous section s_0 with $s_0(p) \neq 0$ and any holomorphic coordinate z centered in p. Observe that when E is a line bundle, s is not the zero section and M is compact, index(s), that is, the sum of the finitely many zeros of s counted with multiplicities, is the first Chern number, $c(E) = \chi(E)$, of E. In particular for line bundles we will use the properties

$$c(E_1 \otimes E_2) = c(E_1) + c(E_2)\,, \qquad c(E^*) = -c(E)\,, \qquad c(\psi^{-1}E) = \deg(\psi)c(E)$$

with $\deg(\psi)$ indicating the degree of the map ψ.

Definition 2.5. A differentiable section s of a complex vector bundle $E \to M$ is said to satisfy a Cauchy-Riemann inequality on an open subset $U \subseteq M$ if for some $p > 2$ there exists $\gamma \in L^p_{\mathrm{loc}}(U)$ such that

$$|\nabla'' s| \le \gamma |s| \qquad \text{on } U\,.$$

If $z : U \subseteq M \to \mathbb{C}$ is a local holomorphic coordinate and $\{E_1, \dots, E_k\}$, $k = \operatorname{rank} E$, is a C^1 local frame for $E \to M$ defined on u, then by (2.7) we see that a section s of E expressed in local notation as

$$s = \sum_{a=1}^{k} s^a_U E_a$$

satisfies a Cauchy-Riemann inequality on U if and only if the function

$$s_U = (s^1_U, \dots, s^k_U) : U \to \mathbb{C}^k$$

satisfies

$$\left| \frac{\partial s_U}{\partial \bar z} \right| \le \tilde\gamma |s_U| \qquad \text{on } U \tag{2.40}$$

for some $\tilde\gamma \in L^p_{\mathrm{loc}}(U)$, $p > 2$. In case $k = 1$, $U \subseteq \mathbb{C}$ and $\tilde\gamma \in L^\infty(U)$, Bers [5, Section 1.6, final Remark] showed that functions satisfying (2.40) must either vanish or have isolated zeros of finite orders in U, and he called them approximately analytic. His proof was later generalized by Chern [11] to encompass the case of $\mathbb{C}^k$-valued functions, then the result was sharpened by Eschenburg and Tribuzy [20] who proved the following

Proposition 2.6 ([20])**.** *If a smooth section s of a complex vector bundle $E \to M$ satisfies a Cauchy-Riemann inequality on M, it is either identically zero or of holomorphic type.*

As remarked in [20] condition (2.40) does not guarantee any higher order regularity of s_0 appearing in (2.39), beside continuity. Anyway we underline the fact that the nature of Proposition 2.6 is of a very local character.

3 Topological restrictions

Lemma 3.1. *Let $f : M \to N$ be a smooth map and N a $2n$-dimensional Kähler manifold. Assume that $|\tau(f)|^2 \leq \gamma e'(f)$. Then $\mathrm{d}f^{(1,0)}$ is of holomorphic type. In particular either f is antiholomorphic or the zeroes of $e'(f)$ are isolated.*

Proof. Let z be a (local) holomorphic coordinate so that $g = \lambda^2 \,\mathrm{d}z\,\mathrm{d}\bar{z}$, $\lambda > 0$. From (2.3) we have

$$\nabla_{\frac{\partial}{\partial \bar{z}}} \mathrm{d}z = 0\,, \tag{3.1}$$

then using (2.18) and (2.19) we compute

$$\nabla_{\frac{\partial}{\partial \bar{z}}} \mathrm{d}f^{(1,0)} = \left(\nabla_{\frac{\partial}{\partial \bar{z}}} \frac{\partial f}{\partial z}\right) \otimes \mathrm{d}z + \frac{\partial f}{\partial z} \otimes \nabla_{\frac{\partial}{\partial \bar{z}}} \mathrm{d}z = \frac{\lambda^2}{4} \tau(f) \otimes \mathrm{d}z\,.$$

Since from (2.13)

$$|\mathrm{d}f^{(1,0)}|^2 = 2e'(f)$$

under the assumption of the lemma we conclude

$$\left|\nabla_{\frac{\partial}{\partial \bar{z}}} \mathrm{d}f^{(1,0)}\right| \leq \gamma |\mathrm{d}f^{(1,0)}|$$

and Proposition 2.6 completes the proof. $\square$

Clearly an analogous result holds when considering $\mathrm{d}f^{(0,1)}$. In particular

Proposition 3.2. *Let $f : M \to N$ be a smooth map and N a $2n$-dimensional Kähler manifold. Assume that*

$$|\tau(f)| \leq \gamma \min\{e'(f), e''(f)\}\,. \tag{3.2}$$

Then either f is constant or the zeroes of the energy density $e(f)$ are isolated.

Observe that in the assumptions of Proposition 3.2 from (2.16) we deduce that f is weakly conformal, in fact a branched immersion in the sense of [24], if and only if $\rho \equiv 0$. In case N is a Riemann surface we already observed in (2.17) that

$$\rho\bar{\rho} = e'(f)e''(f)\,.$$

As a consequence of Lemma 3.1, assuming (3.2) we have that $e'(f)e''(f) \equiv 0$ on M if and only if $e'(f) \equiv 0$ or $e''(f) \equiv 0$. Thus, we have

Theorem 3.3. *Let $f : M \to N$ be a smooth map between Riemann surfaces satisfying* (3.2)*. Then f is weakly conformal if and only if f is $\pm$ holomorphic.*

The next theorem generalizes a result of Eells and Wood, [16].

Theorem 3.4. *Let $f : M \to N$ be a smooth map between compact Riemann surfaces satisfying* (3.2)*. If $\chi(M) + |\deg(f)\chi(N)| > 0$ then f is $\pm$ holomorphic.*

Here and in the sequel $\chi(\ )$ denotes the Euler characteristic. The proof parallels that of Eells and Wood's theorem, that we restate here for later convenience.

Theorem 3.5 ([16])**.** *Let $f : M \to N$ be a harmonic map between compact Riemann surfaces. If $\chi(M) + |\deg(f)\chi(N)| > 0$ then f is $\pm$ holomorphic.*

Proof of Theorem 3.4. Assume that f is not $\pm$ holomorphic. Since N is a Riemann surface ∂f and $\partial \bar{f}$ are sections of the line bundles $T^*M^{(1,0)} \otimes f^{-1}TN^{(1,0)}$ and $T^*M^{(1,0)} \otimes f^{-1}TN^{(0,1)}$, respectively. To fix ideas let us consider ∂f. Then, from Lemma 3.1 and compactness of M, ∂f is of holomorphic type and has only finitely many zeroes in M, therefore we get

$$\begin{aligned}0 \le \operatorname{index}(\partial f) &= c(T^*M^{(1,0)} \otimes f^{-1}TN^{(1,0)}) = c(T^*M^{(1,0)}) + c(f^{-1}TN^{(1,0)})\\ &= c(T^*M) + \deg(f)c(TN) = -\chi(M) + \deg(f)\chi(N)\end{aligned}$$

that is,

$$\text{index}(\partial f) = -\chi(M) + \deg(f)\chi(N) \geq 0 \tag{3.3}$$

and analogously

$$\text{index}(\partial \bar{f}) = -\chi(M) - \deg(f)\chi(N) \geq 0 \tag{3.4}$$

so that

$$|\deg(f)\chi(N)| \leq -\chi(M) \tag{3.5}$$

contradicting the assumptions of the theorem. □

Corollary 3.6. *Let $f : M \to N$ be a smooth map between compact Riemann surfaces satisfying (3.2). Let $g(M)$ and $g(N)$ denote the genera of M and N, respectively.*

i) If $g(M) = 0$ then f is $\pm$ holomorphic.

ii) If $g(N) = 0$ and $|\deg(f)| \geq g(M)$ then f is $\pm$ holomorphic.

iii) If $g(N) = 1$ then either $g(M) \geq 1$ or f is constant.

Proof. By Theorem 3.4, under the assumptions of the corollary f is $\pm$ holomorphic whenever condition

$$\chi(M) + |\deg(f)\chi(N)| > 0 \tag{3.6}$$

is satisfied. Then i) and ii) directly follow from Theorem 3.4 together with relations $\chi(M) = 2 - 2g(M)$ and $\chi(N) = 2 - 2g(N)$.

To prove iii) let us assume $g(N) = 1$, $g(M) = 0$, so that (3.6) is satisfied and f is $\pm$ holomorphic, and suppose by contradiction that f is not constant. To fix ideas, assume that f is holomorphic. Then ∂f is a nonzero section of $T^*M^{(1,0)} \otimes f^{-1}TN^{(1,0)}$ of holomorphic type and by (3.3) it must be $\chi(M) \leq 0$, contradiction. If instead we assume that f is anti-holomorphic then we reach the same contradiction by considering (3.4). □

Remark 3.7. If $f : M \to N$ is a smooth non-constant map between compact Riemann surfaces and $g(N) \geq 2$ then condition $\chi(M) + |\deg(f)\chi(N)| > 0$ is never satisfied, since a theorem of Kneser, [33], see also [16, p. 264], ensures that in this setting it must be

$$\chi(M) \leq |\deg(f)|\chi(N) = -|\deg(f)\chi(N)|\,.$$

In fact, Kneser's theorem is of purely topological nature and holds for not null-homotopic continuous maps $f : M \to N$ between oriented topological surfaces when $\chi(N) < 0$. Nevertheless, in [16] the authors proved that Kneser's theorem can be deduced from Theorem 3.5 by a contradiction argument (suppose that $\chi(N) < 0$ and that there exists a continuous, not null-homotopic map $f' : M \to N$ such that $\chi(M) + |\deg(f')\chi(N)| > 0$; endow M and N with smooth metrics so that N has negative curvature, then consider a harmonic map $f : M \to N$ homotopic to f', which exists by [15]; then f is non-constant with $\deg(f) = \deg(f')$, thus $\chi(M) + |\deg(f)\chi(N)| > 0$ and f is $\pm$ holomorphic, but then either (3.3) or (3.4) imply $\chi(M) \leq -|\deg(f)\chi(N)|$, contradiction).

We now consider an isometric immersion $f : M \to N$ and a Darboux frame $\{e_\alpha\}$ along f. Given the mean curvature vector field $\mathbf{H} = H^\alpha e_\alpha$ according to the splitting (2.22) of $TM^\perp \otimes \mathbb{C}$ we can consider its $(1,0)$ part given by

$$\mathbf{H}^{(1,0)} = \frac{1}{2}(H^3 + iH^4)(e_3 - ie_4)\,.$$

To simplify notation we set

$$b = H^3 + iH^4 \tag{3.7}$$

and letting $\theta^\alpha_\beta = \langle \nabla e_\beta, e_\alpha \rangle$, with ∇ the covariant derivative in $TM^\perp$, we have

$$\mathrm{d}H^\alpha + H^\beta \theta^\alpha_\beta = \nabla H^\alpha = H^\alpha_i \theta^i\,. \tag{3.8}$$

Theorem 3.8. *Let $f : M \to N$ be an isometrically immersed surface satisfying*

$$|\nabla \mathbf{H}^{(1,0)}| \leq \gamma |\mathbf{H}^{(1,0)}|\,, \qquad \mathbf{H} \not\equiv 0\,.$$

If $\gamma = 0$ *a. e. then the normal bundle is flat. Otherwise if* $\gamma \neq 0$ *a. e.* $\mathbf{H}^{(1,0)}$ *is of holomorphic type and, for* M *compact,* $\operatorname{index}(\mathbf{H}^{(1,0)}) = \chi(TM^{\perp})$*.*

Proof. From (3.8) we have

$$\frac{\partial b}{\partial \bar{z}} = i\theta_4^3 \left(\frac{\partial}{\partial \bar{z}}\right) b + \nabla_{\frac{\partial}{\partial \bar{z}}} b \,. \tag{3.9}$$

It is not hard to verify that the assumption $|\nabla \mathbf{H}^{(1,0)}| \leq \gamma |\mathbf{H}^{(1,0)}|$ implies $|\nabla_{\frac{\partial}{\partial \bar{z}}} b| \leq \tilde{\gamma}|b|$, where generally $\tilde{\gamma} \neq \gamma$ a. e. If $\gamma = 0$ a. e. then the normal bundle has a nonzero parallel section so that it is flat. Otherwise for $\gamma \neq 0$ a. e. applying Proposition 2.6 we obtain that $\mathbf{H}^{(1,0)}$ is of holomorphic type. Since it is a section of $TM^{\perp(1,0)}$ and $\mathbf{H} \not\equiv 0$, compactness of M implies $\operatorname{index}(\mathbf{H}^{(1,0)}) = \chi(TM^{\perp})$. □

Corollary 3.9. *Let* $f : M \to N$ *be an isometrically immersed compact surface satisfying* $|\nabla \mathbf{H}^{(1,0)}| \leq \gamma |\mathbf{H}^{(1,0)}|$ *and* $\chi(TM^{\perp}) < 0$*. Then* f *is minimal.*

Remark 3.10. An analogous version of Theorem 3.8 and Corollary 3.9 can be given by considering $\mathbf{H}^{(0,1)}$ instead of $\mathbf{H}^{(1,0)}$.

The next result generalizes Proposition 2.3 from [30] and a result of Salamon reported in [13, page 7.14].

Theorem 3.11. *Let* $f : M \to N$ *be an isometrically immersed, compact, Codazzi surface, isotropic with negative (respectively, positive) spin such that*

$$|\nabla \mathbf{H}^{(1,0)}| \leq \gamma s_{+} \qquad (\textit{resp.}\,, \; |\nabla \mathbf{H}^{(1,0)}| \leq \gamma s_{-}\,)$$

Then either f *is totally umbilical or*

$$\chi(TM^{\perp}) = 2\chi(M) + m \qquad (\textit{resp.}\,, \; \chi(TM^{\perp}) = -2\chi(M) - m) \tag{3.10}$$

where m *is the total number of umbilical points counted with multiplicities.*

Here the multiplicity of an umbilical point p is half of the order of zero at p of the function u defined in (2.27) for which, in the assumption of the Theorem, is meaningful to define the order of zero. This will clearly appear throughout the proof.

Proof. To simplify notation we set $S_+ = L^3 + iL^4$ where L^α is the Hopf's transform of section 2. Then Σ_+, section of $T^*M^{(1,0)} \otimes T^*M^{(1,0)} \otimes TM^{\perp(1,0)}$, is rewritten as $\Sigma_+ = \frac{1}{\sqrt{2}} S_+ \varphi^2 \otimes (e_3 - ie_4)$. As shown in Section 2 Codazzi equations can be restated as

$$(\mathrm{d}L^\alpha + 2iL^\alpha \theta^1_2 + L^\beta \theta^\alpha_\beta) \wedge \varphi + \nabla H^\alpha \wedge \bar{\varphi} = \frac{1}{2}(R^\alpha_1 + iR^\alpha_2)\varphi \wedge \bar{\varphi} \tag{3.11}$$

where $\theta^a_b = \langle \nabla e_b, e_a \rangle$ (notice that $\theta^1_2 = \omega$ previously defined) and R^α_i are the coefficients of the tensor $\mathrm{Ric}^\perp(f)$ defined in (2.30). From (3.11) and the fact that the surface is Codazzi we obtain

$$\frac{\partial S_+}{\partial \bar{z}} = \tilde{\gamma} S_+ + \nabla_{\frac{\partial}{\partial \bar{z}}} b \tag{3.12}$$

where b has been defined in (3.7) and $\tilde{\gamma}$ is some smooth (locally defined) function. (2.25) and the assumption $|\nabla \mathbf{H}^{(1,0)}| \le \gamma s_+$ imply from (3.12)

$$\left| \frac{\partial S_+}{\partial \bar{z}} \right| \le \gamma |S_+| \,.$$

Therefore applying Proposition 2.6, either the zeros of Σ_+ are isolated, and therefore in finite number since M is compact, or Σ_+ is identically zero. Since f is isotropic with negative spin, $u = s_+^2 = \frac{1}{8}|\Sigma_+|^2$ and therefore in the first case

$$m = \mathrm{index}(\Sigma_+) = -2\chi(M) + \chi(TM^\perp) \tag{3.13}$$

while in the second case f is totally umbilical. □

Remark 3.12. 1. In the assumption of the theorem from (2.26) we have

$$s_+^2 = |\mathbf{H}|^2 - K + R_{1212} = K^\perp - R_{1234}$$

2. In the case $N = \mathbb{S}^4$ the conclusion can be stated in the stronger form: "then f is totally umbilical if and only if $\chi(TM^\perp) = 0$. Otherwise (3.10) holds." To prove this last statement observe that according to the the above observation 1. and the Gauss-Bonnet-Chern theorem

$$\chi(TM^\perp) = \frac{1}{2\pi} \int_M u \, \mathrm{d}A \,.$$

More generally the same conclusion holds when f has negative spin, in case the contact invariant R_{1234} is non-positive. Analogously in the positive spin case. This proves Theorem 1.2 of the introduction.

We give a criterion for isotropy. In case N has constant sectional curvature ε and $\mathbf{H}$ is parallel, this has been given by [21], Theorem 5.2, where $f : M \to N(\varepsilon)$ is only assumed to be conformal. Indeed, next result can be extended to the conformal case too, but for simplicity we have confined ourselves to the isometric one.

Theorem 3.13. *Let $f : M \to N$ be an isometrically immersed, compact, Codazzi surface satisfying*

$$|\nabla \mathbf{H}| \leq \gamma \min\{s_+, s_-\} . \tag{3.14}$$

Then either f is isotropic or

$$2\chi(M) \leq -|\chi(TM^\perp)| .$$

Proof. In the assumption (3.14) both $\Sigma_\pm$ are of holomorphic type so that, if no one of them is identically zero, together with (3.13) we have

$$\mathrm{index}(\Sigma_-) = -2\chi(M) \leq -\chi(TM^\perp)$$

and the conclusion follows. □

We like to explicitely state the following result contained in Theorem 3.13.

Corollary 3.14. *Let $f : M \to N$ be an isometrically immersed, Codazzi surface satisfying (3.14). Then either the umbilical points are isolated or f has parallel mean curvature.*

Another consequence is given in the following:

Corollary 3.15. *Let $f : M \to N$ be an isometrically immersed Codazzi surface satisfying (3.14). Suppose that M is conformally equivalent to $\mathbb{CP}^1$. Then f has parallel mean curvature. Moreover, if $\chi(TM^\perp) \neq 0$ then f is necessarily minimal, while if $\chi(TM^\perp) = 0$ then f is totally umbilical.*

Proof. Since in this case $\chi(M) = 2$, from Theorem 3.13 we have that f is isotropic, therefore (3.14) implies $\nabla\mathbf{H} = 0$. If f is not minimal, $\mathbf{H}$ is a nonzero parallel section of $TM^\perp$ so that necessarily $\chi(TM^\perp) = 0$. To prove the last statement, assume that f is not totally umbilical and, without loss of generality, that $s_+ \not\equiv 0$. Then from (3.13) we have

$$\chi(TM^\perp) \geq 4$$

contradicting the assumptions of the corollary. □

The following result is of a similar nature.

Theorem 3.16. *Let $f : M \to N$ be an isometrically immersed, Codazzi surface satisfying*

$$|\nabla\mathbf{H}^{(1,0)}| \leq \gamma \min\{s_+, |\mathbf{H}|\}\,. \tag{3.15}$$

Then the globally defined form

$$\Lambda = bS_+\varphi^2 \tag{3.16}$$

is of holomorphic type. Furthermore if M is compact then either f is isotropic with positive spin or minimal or

$$2\chi(M) = -\mathrm{index}(\mathbf{H}^{(1,0)}) - \mathrm{index}(\Sigma_+)\,.$$

Remark 3.17. An analogous result holds when we consider the globally defined form $\Lambda' = \bar{b}S_-\varphi^2$.

Proof. First of all observe that under the assumption (3.15) both $\mathbf{H}^{(1,0)}$ and Σ_+ are of holomorphic type. Secondly (3.16) is globally well defined as one immediately checks for instance using (2.16), (2.17) of [30] (paying attention to the definition of $S_\pm$ there). From (3.9) and (3.12) we have

$$\frac{\partial}{\partial\bar{z}}(bS_+) = S_+\frac{\partial b}{\partial\bar{z}} + b\frac{\partial S_+}{\partial\bar{z}} = i\theta_4^3\left(\frac{\partial}{\partial\bar{z}}\right)bS_+ + \tilde{\gamma}bS_+ + (S_+ + b)\nabla_{\frac{\partial}{\partial\bar{z}}}b$$

so that, under the assumption (3.15) and using Proposition 2.6, Λ is of holomorphic type. Since Λ is a section of $T^*M^{(1,0)} \otimes T^*M^{(1,0)}$ the remaining part of the theorem follows. □

An interesting immediate consequence is the following

Corollary 3.18. *Let $f : M \to N$ be an isometrically immersed, compact, Codazzi surface satisfying $|\nabla \mathbf{H}| \leq \gamma \min\{s_+, s_-, |\mathbf{H}|\}$. If f is neither isotropic nor minimal then* $\mathrm{index}(\Sigma_-) = \mathrm{index}(\Sigma_+)$.

4 Parallel mean curvature

We relate the notion of Codazzi surface to the Ruh-Vilms property (RV) from section 3. We need briefly to describe the geometry of the Grassmann bundle $\pi : G_2(TN) \to N$ of orientend tangent 2-planes. For more details we refer to [31]. An element of $G_2(TN)$ is a pair (p, ζ), $p \in N$ and ζ a 2-dimensional oriented subspace of T_pN. The projection $\pi : (p, \zeta) \to p$ presents $G_2(TN)$ as a fibre bundle over N with standard fibre $SO(4)/SO(2) \times SO(2)$. Letting $O(N)$ be the bundle of orthonormal frames of N we define a map

$$\mu : O(N) \to G_2(TN) \tag{4.1}$$

by setting $\mu(p, e) \mapsto (p, \{e_1, e_2\})$, where $e = \{e_a\}$ is an orthonormal frame at $p \in N$ and $\{e_1, e_2\}$ is the oriented plane in T_pN spanned by e_1, e_2 with the orientation $e_1 \wedge e_2$. This induces the identification

$$G_2(TN) \simeq O(N)/SO(2) \times O(2)\,.$$

Let $\{\theta^a\}$, $\{\theta^a_b\}$ be the canonical forms on $O(N)$. For $t > 0$ we define the Riemannian metric h_t on $G_2(TN)$ as that metric characterized by the property

$$\mu^* h_t = P_t$$

where P_t is the $O(2) \times O(2)$ invariant symmetric bilinear form on $O(N)$ given by

$$P_t = \sum_{a,\alpha,i} (\theta^a)^2 + t^2 (\theta^\alpha_i)^2\,.$$

We will indicate with $\{\varepsilon_a\}$, $\{\varepsilon_{\alpha i}\}$ the frame on $G_2(TN)$ dual to the orthonormal coframe for the metric h_t determined by the pull-back under a local section of (4.1) of the forms θ^a, θ^α_i.

From now on $G_2(TN)$ will be considered with the (family of) Riemannian metric(s) h_t defined above.

Given the isometric immersion $f : M \to N$ and a Darboux frame field $\mathbf{e} = \{e_a\}$ along f the Gauss lift $\gamma_f : M \to G_2(TN)$ is defined by

$$\gamma_f : q \mapsto (f(q), \{e_1(q), e_2(q)\}) .$$

The tension field of γ_f has been computed in the general case in [31] and it is given by

$$\tau(\gamma_f) = t^2 R^\alpha_{jik} h^\alpha_{ji} \varepsilon_k + (2H^\beta + t^2 R^\alpha_{ji\beta} h^\alpha_{ji})\varepsilon_\beta + t(2H^\alpha_k - R^\alpha_{iki})\varepsilon_{\alpha k} . \quad (4.2)$$

To simplify (4.2) we shall assume some curvature restrictions on N. Recall that in the 4-dimensional case at any point $p \in N$ the Weyl curvature operator W preserves the ± 1 eigenspaces of the Hodge operator $*$ on $\wedge_2 T_p N$ and therefore splits into the components

$$W = W^+ + W^- .$$

Consequently the oriented Riemannian manifold N is said to be self-dual (respectively, anti-selfdual) if at every point of N we have $W^- = 0$ (respectively $W^+ = 0$), [4]. Letting $\mathbf{e}$ be an oriented frame locally defined on N this can be conveniently expressed for our purposes by the vanishing of the matrix $C - \frac{s}{12}\mathbb{I}_3$, where $\mathbb{I}_3$ is the identity matrix of order 3, s is the scalar curvature of N and

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

is the symmetric matrix defined in Section 3 of [30], whose entries are given in terms of the coefficients of the Riemann tensor by the following expressions (see equation (3.13) of [30])

$$\begin{aligned}
C_{11} &= \frac{1}{2}(R_{1212} - 2R_{1234} + R_{3434}) \\
C_{22} &= \frac{1}{2}(R_{1313} + 2R_{1324} + R_{2424}) \\
C_{33} &= \frac{1}{2}(R_{1414} - 2R_{1423} + R_{2323}) \\
C_{12} = C_{21} &= \frac{1}{2}(R_{1312} - R_{1334} + R_{2412} - R_{2434}) \\
C_{13} = C_{31} &= \frac{1}{2}(R_{1412} - R_{1434} - R_{2312} + R_{2334}) \\
C_{23} = C_{32} &= \frac{1}{2}(R_{1413} + R_{1424} - R_{2313} - R_{2324})\,.
\end{aligned}$$

With this preparation we state

Theorem 4.1. *Let $f : M \to N$ be an isometrically immersed surface, isotropic with negative spin and such that the contact invariant R_{1234} is constant. Assume that N is self-dual Einstein with scalar curvature $s > 6R_{1234}$. Then for the choice $t^2 = \frac{12}{s-6R_{1234}}$ the following are equivalent:*

i) the surface is Codazzi

ii) (RV) *holds.*

Remark 4.2. 1. If R_{1234} is not constant or $s \leq 6R_{1234}$ in the further assumption that $f : M \to N$ is a Codazzi surface we have that γ_f is harmonic if and only if f is minimal.

2. An analogous result holds for f isotropic with positive spin and N anti-selfdual Einstein. Under this latter assumption there is a natural integrable almost complex structure J_+ on $G_2(TN)$, see for instance [30], [39]. Holomorphicity of γ_f with respect to J_+ is equivalent to the fact that f is totally umbilical. This perfectly resembles the case of the usual Gauss map for surfaces in $\mathbb{R}^4$.

3. For the equivalence, in the general case, of minimality of f and holomorphicity of γ_f with respect to an appropriate (never integrable) almost complex structure J_- on $G_2(TN)$ see [13], [14].

Proof. The proof, using a Darboux frame along f, is by inspection of formula (4.2). Under the assumptions $s_- = 0$, that is, $L^3 = iL^4$, self-duality, $C = \frac{s}{12}\mathbb{I}_3$, and the Einstein property of N, (4.2) reduces to

$$\tau(\gamma_f) = t^2 R^\alpha_{jik} h^\alpha_{ji} \varepsilon_k + t(2H^\alpha_k - R^\alpha_{iki})\varepsilon_{\alpha k} + \left[t^2\left(R_{1234} - \frac{s}{12}\right) + 2\right] H^\beta \varepsilon_\beta\,.$$

Since the vectors $\{\varepsilon_a, \varepsilon_{\alpha k}\}$ are linearly independent we obtain the desired conclusion. $\square$

When the curvature tensor of N and the isometric immersion $f : M \to N$ are particularly well related the assumption that f is isotropic with negative spin is not necessary. We will give an example with $N = \mathbb{CP}^2$. To begin with we analyse the notion of isotropy with negative spin in case N is Kähler. Assume that f is not $\pm$ holomorphic and that those points where $\mathrm{d}f$ is $\mathbb{C}$-linear are isolated, shortly that the complex tangent planes are isolated. Remark that for instance, for $|\mathbf{H}| \leq \gamma \sin\alpha$ this is guaranteed by Theorem 5 in [20]. Then, outside a discrete set of points Z, we can choose the local unitary frame $\{E_1, E_2\}$ of (2.37), whose existence does not depend on the fact that N has constant holomorphic sectional curvature, in such a way that $u = \cos\frac{\alpha}{2}$, $v = \sin\frac{\alpha}{2}$. Let $\omega_{ij} = (\nabla E_i, E_j)$, so that in particular ω_{11}, ω_{22} are purely imaginary, and let Ψ be the 1-form

$$\Psi = u\mathrm{d}v - v\mathrm{d}u - uv(\omega_{11} + \omega_{22})\,. \tag{4.3}$$

Using [19], see bottom of page 591, we can rewrite, in our notation,

$$\Psi = \frac{1}{2}\overline{S_-}\varphi + \frac{1}{2}\bar{b}\bar{\varphi}\,. \tag{4.4}$$

Since $2(u\mathrm{d}v - v\mathrm{d}u) = d\alpha$ is real (α being smooth outside Z), from (4.3) and (4.4) we have that the Kähler angle α is constant if and only if

$$S_- = -\bar{b}\,. \tag{4.5}$$

In particular in the minimal case we have

Proposition 4.3. *Let $f : M \to N$, N Kähler, be a minimal not $\pm$ holomorphic surface. Then the following are equivalent* *i) f is isotropic with negative spin*

ii) α is constant (different from 0, π).

Remark 4.4. 1. From [19], bottom of page 591, we have in our notation

$$\omega_{12} = \frac{1}{2}S_+\varphi + \frac{1}{2}b\bar{\varphi}$$

so that Lemma 3.2 of [19] can be completed to: "Let $f : M \to N$, N Kähler, be an isometrically immersed surface with isolated complex tangent planes. Then, with respect to a special unitary frame $\{E_1, E_2\}$, f is minimal if and only if ω_{12} is a $(1,0)$-form, while f is isotropic with positive spin if and only if ω_{12} is a $(0,1)$-form."

2. Assume that N has constant holomorphic sectional curvature $c \geq 0$, and let $f : M \to N$ be a compact minimal surface which is not $\pm$ holomorphic. Then f is totally real if and only if f is isotropic with negative spin. Indeed, analogously to what we did in section 2, we compute

$$R_{1212} = \frac{c}{4}(3\cos^2\alpha + 1)\,, \qquad R_{1234} = \frac{c}{4}(3\cos^2\alpha - 1) \tag{4.6}$$

so that from (2.26) we have

$$2s_-^2 = |\mathbf{H}|^2 - (K + K^\perp) + \frac{3}{2}c\cos^2\alpha\,.$$

Therefore f being isotropic with negative spin and minimal imply $K + K^\perp = \frac{3}{2}c\cos^2\alpha$. Since $c > 0$ we have $K + K^\perp \geq 0$. Thus, as M is compact and f is not $\pm$ holomorphic we conclude, from Theorem 6.2 of [19], that f is totally real. The converse is already contained in Proposition 4.3. As a consequence in Proposition 4.3, assuming that N has constant non-negative holomorphic sectional curvature, in either one of the cases i) or ii) compactness of M is equivalent to $\alpha \equiv \frac{\pi}{2}$.

We recall that, in case $f : M \to N$, N a Kähler manifold, f (for the sake of simplicity) an isometry, f not $\pm$ holomorphic, there is a notion of complex isotropy introduced by [17]. This (at least for f minimal, $N =$

$\mathbb{CP}^2$) is defined by the vanishing of the (holomorphic) form $(\nabla \mathrm{d}f, \mathrm{d}f)^{(3,0)}$. One checks that

$$|(\nabla \mathrm{d}f, \mathrm{d}f)^{(3,0)}|^2 = \sin^2\alpha \left(|\mathbf{H}|^2 + K^\perp - K + R_{1212} - R_{1234}\right) . \tag{4.7}$$

On the other hand from (2.26)

$$2s_+^2 = |\mathbf{H}|^2 + K^\perp - K + R_{1212} - R_{1234}$$

so that complex isotropy is equivalent to isotropy with positive spin.

Remark 4.5. Condition (4.7), with N of constant holomorphic sectional curvature, already appears in [20].

We now go back to our purpose of showing that in favourable circumstances the assumption f isotropic with negative spin is not necessary in Theorem 4.1. First of all let N have constant holomorphic sectional curvature c and $f : M \to N$ an isometry. Using the framing defined in (2.37) we compute

$$\begin{aligned} R^2{}_{323} &= \frac{c}{4} + 3c(\mathrm{Re}(uv))^2 = R^1{}_{414} \\ R^1{}_{314} &= 3c(\mathrm{Re}(uv))(\mathrm{Im}(uv)) = -R^1{}_{424} \end{aligned}$$

and therefore assuming α constant, $\alpha \neq 0$, $\alpha \neq \pi$, we can in fact consider a Darboux frame $\mathbf{e}$ along f for which

$$\begin{cases} R^2{}_{323} = \frac{c}{4} + 3c\sin^2\alpha = R^1{}_{414} \\ R^1{}_{314} = 0 = R^1{}_{424} \, . \end{cases} \tag{4.8}$$

For such a frame (4.5) holds, so that, together with (4.8) we simplify (4.2) to the following

$$\tau(\gamma_f) = t^2 R^\alpha_{jik} h^\alpha_{ji} \varepsilon_k + t(2H^\alpha_k - R^\alpha_{iki})\varepsilon_{\alpha k} + 2\left[1 - t^2\left(\frac{c}{4} + 3c\sin^2\alpha\right)\right] H^\beta \varepsilon_\beta \, . \tag{4.9}$$

Therefore, recalling characterization (2.38), (4.9) gives

Theorem 4.6. *Let N be a Kähler manifold with constant holomorphic sectional curvature c and let $f : M \to N$ be an isometrically immersed surface with constant Kähler angle and not $\pm$ holomorphic. Consider the Gauss lift $\gamma_f : M \to G_2(TN)$, if $ct^2 = \frac{1}{13}$, the following properties are equivalent:*

i) f is a Codazzi surface

ii) f is totally real

iii) (RV) *holds.*

Remark 4.7. 1. In the above assumptions, if $c \leq 0$ then i) and ii) are of course equivalent because of (2.38) but iii) is never satisfied.

2. In $\mathbb{CP}^2$ the implication ii) $\Rightarrow$ iii) is known to A. Elghanmi.

Proposition 4.8. *Let $f : M \to N$ be an isometrically immersed Codazzi surface with parallel mean curvature vector. Then $s_\pm$, $|\mathbf{H}|$ are either identically zero or their zero sets are discrete and outside them the following equations hold:*

$$\Delta \log s_\pm = 2K \mp K^\perp \tag{4.10}$$

$$\Delta \log |\mathbf{H}| = -K^\perp\,. \tag{4.11}$$

Remark 4.9. The proof of the proposition is identical to that given in [21] in case N has constant sectional curvature and therefore will not be repeated here. Anyway, (4.10) and (4.11) already appeared in the unpublished thesis of the third author.

We rewrite (4.10) in a second form more appropriate to our purpose.

Lemma 4.10. *Let $f : M \to N$ be an isometrically immersed Codazzi surface with parallel mean curvature vector. Then*

$$\Delta s_\pm^2 = 2s_\pm^2(2K \mp K^\perp) + 4|\nabla \mathrm{II}|^2 \pm 4A \tag{4.12}$$

where, indicating with h^α_{ijk} the coefficients of $\nabla\mathrm{II}$ with respect to a Darboux frame $\mathbf{e}$ along f,

$$A = h^3_{22,2}h^4_{11,1} - h^3_{11,1}h^4_{22,2}\,.$$

Proof. Put $\mathrm{d}L^\alpha + 2iL^\alpha\theta^1_2 + L^\beta\theta^\alpha_\beta = \nabla L^\alpha$. From (2.33) and the assumptions we deduce in (2.34) that $\nabla L^\alpha = L^\alpha_1\varphi$ for $L^\alpha_1 = h^\alpha_{11,1} + ih^\alpha_{22,2}$ and that

$$\nabla \mathrm{II} = h^\alpha_{ijk} e_\alpha \otimes \theta^i \otimes \theta^j \otimes \theta^k = \frac{1}{2} L^\alpha_1 e_\alpha \otimes \varphi^3 + \frac{1}{2}\overline{L^\alpha_1} e_\alpha \otimes \bar{\varphi}^3 .$$

Hence $|\nabla \mathrm{II}|^2 = \frac{1}{2}\sum_\alpha |L^\alpha_1|^2$. On the other hand

$$2\mathrm{d}s^2_+ = (L^3 + iL^4)\nabla(\bar{L}^3 - i\bar{L}^4) + (\bar{L}^3 - i\bar{L}^4)\nabla(L^3 + iL^4)$$

and therefore

$$|\mathrm{d}s^2_+|^2 = |L^3 + iL^4|^2 |\bar{L}^3 - i\bar{L}^4|^2 = 2s^2_+\left(2|\nabla \mathrm{II}|^2 + i(L^4_1\bar{L}^3_1 - L^3_1\bar{L}^4_1)\right) .$$

Then the first of (4.12) follows from (4.10) because $2A = i(L^4_1\bar{L}^3_1 - L^3_1\bar{L}^4_1)$ and

$$\Delta s^2_+ = s^2_+ \Delta \log s^2_+ + \frac{|\mathrm{d}s^2_+|^2}{s^2_+} .$$

The second formula of (4.12) is obtained in a similar way. □

To state next result it is worth to recall the following characterization of isotropy. Given the surface $f : M \to N$, fix a point $p \in M$ and in T_pM consider the parametrized unit circle

$$X(\sigma) = (\cos\sigma)e_1 + (\sin\sigma)e_2 , \qquad 0 \le \sigma \le 2\pi$$

where $\mathbf{e}$ is a Darboux coframe along f. The ellipse of curvature at p is defined to be the curve in $T_pM^\perp$ given parametrically by

$$Y(\sigma) = \mathrm{II}(X(\sigma), X(\sigma)) = \mathbf{H} + \frac{1}{2}(\cos 2\sigma)(V_{11} - V_{22}) + (\sin 2\sigma)V_{12}$$

where $V_{ij} = \mathrm{II}(e_i, e_j)$. Observe that

$$s_+ s_- = \left| \frac{1}{4}|V_{11} - V_{22}|^2 - |V_{12}|^2 + i\langle V_{11} - V_{22}, V_{12}\rangle \right|$$

and thus f is isotropic at p if and only if the ellipse of curvature at p is a circle (possibly of zero radius). Furthermore, the ellipse of curvature at p degenerates to a line segment (possibly of zero length) if and only if $V_{12} \wedge (V_{11} - V_{22}) = 0$ at p, which occurs if and only if $R_{1234} = K^\perp$ as immediately checked.

Theorem 4.11. *Let $f : M \to N$ be an isometrically immersed, complete Codazzi surface with parallel mean curvature vector and Gaussian curvature of constant sign. Consider the following two cases:*

1) $K \geq 0$, $\sup_M R_{1212} < +\infty$ and

- *if R_{1234} has constant sign, suppose either $R_{1234} \leq K^{\perp} \leq 0$ or $0 \leq K^{\perp} \leq R_{1234}$*
- *if the sign of R_{1234} varies, suppose either that the normal bundle is flat or that the ellipse of curvature is degenerate at each point*

2) $K \leq 0$, $|\mathbf{H}|^2 + \inf_M R_{1212} > 0$ and the ellipse of curvature is degenerate at each point.

Then in case 1) the surface is either totally umbilical or flat, while in case 2) the surface is flat.

Proof. Suppose 1) holds. Completeness of M and $K \geq 0$ imply, by a result of Huber, [28], that M is either compact or parabolic. $K \geq 0$ and $\sup_M R_{1212} < +\infty$ imply from (2.26) that $u = s_+^2 + s_-^2$ is bounded above. Each one of the further assumptions implies $(R_{1234} - K^{\perp})K^{\perp} \geq 0$. Using now (2.26) and (4.12) we obtain

$$\Delta u = 4uK + 2(R_{1234} - K^{\perp})K^{\perp} + 8|\nabla \mathrm{II}|^2 \tag{4.13}$$

so that u is a subharmonic function on M. By the maximum principle, u is constant. Hence, either $u \equiv 0$ and the surface is totally umbilical, or $u \neq 0$ and then $K \equiv 0$ from (4.13).

Suppose 2) holds. Since the ellipse of curvature is degenerate at each point we have $s_+ = s_-$, hence $u = 2s_+^2$. On the other hand by (4.10) we deduce

$$\Delta \log u = 4K\,. \tag{4.14}$$

Since $K \leq 0$ we have that $\log u$ is a superharmonic function, furthermore from (2.26), $K = R_{1212} + |\mathbf{H}|^2 - u$, and therefore from the assumptions we conclude that $\log u$ is bounded below. We define on M a metric $\tilde{g}$

conformally equivalent to the induced metric g by setting $\tilde{g} = \sqrt{u}g$. Completeness of g and the assumptions in 2) (by guaranteeing $\inf_M u > 0$) imply completeness of $\tilde{g}$. Moreover its Gaussian curvature $\tilde{K}$ is given by $\tilde{K} = -\frac{K}{\sqrt{u}}$ and is therefore non-negative. By the above result of Huber, $(M, \tilde{g})$ is either compact or parabolic. We deduce that u is constant and therefore from (4.14), $K \equiv 0$. □

Remark 4.12. 1. Recall, from (4.11) for instance, that if $f : M \to N$ is a Codazzi surface with parallel mean curvature vector but not minimal then $K^{\perp} \equiv 0$. Thus, in this case conditions 1) and 2) of Theorem 4.11 respectively become

1') $K \geq 0$, $\sup_M R_{1212} < +\infty$

2') $K \leq 0$, $|\mathbf{H}|^2 + \inf_M R_{1212} > 0$, $R_{1234} \equiv 0$.

2. Let N be of constant sectional curvature ε. Then 1) and 2) respectively become

1") $K \geq 0$, $K^{\perp} \equiv 0$

2") $K \leq 0$, $|\mathbf{H}|^2 + \varepsilon > 0$, $K^{\perp} \equiv 0$.

3. Thus, from points 1. and 2., Theorem 4.11 extends a result of Hoffman, [25], proved under additional assumptions $|\mathbf{H}| \neq 0$ and in case 2") $\varepsilon \geq 0$. The above proof is modeled on [25] and [32].

As a consequence of Theorem 4.11 we have

Corollary 4.13. *Let $f : M \to \mathbb{CP}^2$ be an isometrically immersed, complete, totally real, minimal surface with non-negative Gaussian curvature. Then the surface is either flat or totally geodesic.*

Proof. From (4.6) and Gauss equation

$$K = R_{1212} + \sum_{\alpha} \det(h^{\alpha}_{ij})$$

reality and minimality of f, we have $0 \leq K \leq 1$. But, from Theorem 3.4 of [19], in case f is minimal, f is totally real if and only if $K + K^{\perp} \equiv 0$. We therefore deduce $-1 \leq K^{\perp} \leq 0$. Being f totally real it is Codazzi and as a consequence we are in the assumptions of case 1) of Theorem 4.11, completing the proof. $\square$

Remark 4.14. In the assumptions of the corollary, if M is compact and not totally geodesic then, up to a rigid motion of $\mathbb{CP}^2$, the surface is a reparametrization of the Clifford torus

$$T = \{[z] \in \mathbb{CP}^2 : z_0\bar{z}_0 = z_1\bar{z}_1 = z_2\bar{z}_2\}$$

as defined in [19] or [34]. This follows from Corollary 3.9 in [19] which classifies compact, totally real minimal surfaces $f : M \to \mathbb{CP}^2$ with constant Gaussian curvature, extending previous results from [34] that characterize T as the unique compact totally real minimal flat surface in $\mathbb{CP}^2$.

Next result is a quantization property for the Gaussian curvature.

Theorem 4.15. *Let $f : M \to N$ be an isometrically immersed, compact, non totally umbilical, Codazzi surface with parallel mean curvature vector. If f is isotropic with $\pm$ spin and $K \geq \frac{1}{3}(|\mathbf{H}|^2 + R_{1212} \mp R_{1234})$ then $K \equiv \frac{1}{3}(|\mathbf{H}|^2 + R_{1212} \mp R_{1234})$.*

Proof. To fix ideas let f be isotropic with positive spin, that is, $s_+ \equiv 0$. Then from (2.26) and isotropy we have

$$K = R_{1212} - u + |\mathbf{H}|^2\,, \qquad K^{\perp} = R_{1234} - u\,, \qquad u = s_-^2\,.$$

Therefore using (4.10) we obtain

$$\begin{aligned} \frac{1}{2}\Delta \log u = 2K + K^{\perp} &= 2K + R_{1234} - u + R_{1212} + |\mathbf{H}|^2 - R_{1212} - |\mathbf{H}|^2 \\ &= 3K + R_{1234} - R_{1212} - |\mathbf{H}|^2 \\ &= 3\left(K - \frac{1}{3}(|\mathbf{H}|^2 + R_{1212} - R_{1234})\right). \end{aligned} \tag{4.15}$$

Since f is not totally umbilical, $\log u$ is a subharmonic function with singularities in a set of (isolated) points where it goes to $-\infty$. Thus it has a maximum in M and hence is constant by the maximum principle. The result then follows from (4.15). $\square$

Next result complements Corollary 4.13.

Corollary 4.16. *Let $f : M \to N$, N Kähler with constant holomorphic sectional curvature c, be an isometrically immersed, compact, totally real, Codazzi surface with parallel mean curvature vector. If f is isotropic with positive spin and $K \geq \frac{1}{3}(|\mathbf{H}|^2 + \frac{c}{2})$ then either f is totally umbilical, in which case $K \geq -\frac{3c}{8}\cos^2\alpha$, or $K \equiv \frac{1}{3}(|\mathbf{H}|^2 + \frac{c}{2})$.*

Proof. A direct application of Theorem 4.15 with (2.26) and (4.6). $\square$

Remark 4.17. 1. Other special cases of Theorem 4.15 are, for instance, in case N is self dual or anti-self dual Einstein. Indeed one respectively has $R_{1212} - R_{1234} = \frac{s}{12}$, $R_{1212} + R_{1234} = \frac{s}{12}$ for s the (constant) scalar curvature of N. In particular if N has constant sectional curvature ε then the conclusion of the theorem can be restated as

> If f is isotropic and $K \geq \frac{1}{3}(\varepsilon + |\mathbf{H}|^2)$ then either f is locally umbilical with $K \geq 0$ or $K \equiv \frac{1}{3}(\varepsilon + |\mathbf{H}|^2)$.

2. For $N = \mathbb{S}^4$ and $H = 0$ the above result was proved in [29] where it is extended to higher codimension. Another version of it in $\mathbb{CP}^n$, and always in the minimal case, appears in [7]. A further result in this direction for $f : M \to \mathbb{CP}^2$, f holomorphic and (as usual) $\mathbb{CP}^2$ with constant holomorphic sectional curvature 4 is in [36]. Precisely, M compact, $K \geq 2$ implies $K \equiv 2$. Nomizu and Smyth, [35], proved that the same conclusion holds if we assume the reverse inequality, that is, $K \leq 2$. While the first result is extendable to higher codimension, [36], (and even in complex Grassmannians, [41]) generally the second is not, [36].

3. If N has constant sectional curvature $\varepsilon > 0$, then for $f : M \to N$ an isometrically immersed, compact, isotropic and minimal surface $K \leq$

$\frac{\varepsilon}{3}$ implies $K \equiv \frac{\varepsilon}{3}$. In the general case, for instance assume f Codazzi, isotropic with positive spin, minimal, the inequality $K \leq \frac{1}{3}(|\mathbf{H}|^2 + R_{1212} - R_{1234})$ implies that $\log u$ is a superharmonic function and to apply the maximum principle we have to guarantee that $u > 0$ on M. On the other hand the same inequality implies, from (2.26), $u \geq \frac{1}{3}(2R_{1212} + R_{1234} - |\mathbf{H}|^2)$ and therefore the desired result holds whenever this latter quantity is stricly positive.

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