FINITE GROUPS WITH REAL VALUED IRREDUCIBLE CHARACTERS OF PRIME DEGREE

SILVIO DOLFI, EMANUELE PACIFICI, AND LUCIA SANUS

Abstract. In this paper we describe the structure of finite groups whose real valued nonlinear irreducible characters have all prime degree. The more general situation in which the real valued irreducible characters of a finite group have all squarefree degree is also considered.

A classical problem in Character Theory is understanding to what extent the set $\mathrm{cd}(G)$ of (distinct) degrees of the irreducible characters of a finite group G determines the structure of G.

Here we consider the subset $\mathrm{cd}_{\mathrm{rv}}(G)$ of $\mathrm{cd}(G)$ whose elements are the degrees of the *real valued* irreducible characters of G. In particular, we consider the case when $\mathrm{cd}_{\mathrm{rv}}(G) \setminus \{1\}$ consists of prime numbers.

Theorem A. *Let G be a finite group. If every real valued nonlinear irreducible character of G has prime degree, then G is solvable.*

It may be worth mentioning that our proof of Theorem A depends indirectly on the Classification of Finite Simple Groups, as it involves an application of Theorem 4.2 in [4]. Instead, with a direct use of the Classification, we can prove that a finite group whose real valued irreducible characters have all squarefree degree is either solvable or an extension of a solvable group by the alternating group A_7 (Theorem 3.1).

We also prove the following.

Theorem B. *Let G be a finite group. If every real valued nonlinear irreducible character of G has prime degree, then $\mathrm{cd}_{\mathrm{rv}}(G)$ is contained in a set of the kind $\{1, 2, p\}$, where p is an odd prime.*

In the cases when $\mathrm{cd}_{\mathrm{rv}}(G) = \{1, p\}$ and $\mathrm{cd}_{\mathrm{rv}}(G) = \{1, 2\}$, we obtain a complete description of the structure of G. This is provided by Theorem 4.4 and Theorem 4.6.

The prime 2 plays a prominent role in the context of real valued characters. Recall, for instance, that by a classical result of Burnside only groups of even order may have real valued irreducible characters other than the principal character.

In this spirit, we determine upper bounds for the 2-length and the $2'$-length of finite groups with real valued irreducible characters of prime degree.

2000 *Mathematics Subject Classification.* 20C15.

The first and second author are partially supported by the MIUR project "Teoria dei gruppi e applicazioni". The third author is partially supported by the Ministerio de Educación y Ciencia proyecto MTM2007-61161.

Theorem C. *Let G be a finite group. If every real valued nonlinear irreducible character of G has prime degree, then $l_2(G) \leq 2$ and $l_{2'}(G) \leq 1$.*

The bounds in Theorem C are attained, as $\mathrm{cd}_{\mathrm{rv}}(S_4) = \{1, 2, 3\}$, and $l_2(S_4) = 2$, $l_{2'}(S_4) = 1$.

The notation is standard. Throughout the whole paper, every abstract group is tacitly assumed to be finite.

1. Preliminary results

Let G be a group, let $\mathrm{Irr}_{\mathrm{rv}}(G)$ denote the set of real valued irreducible characters of G and, as already mentioned, $\mathrm{cd}_{\mathrm{rv}}(G)$ the set of distinct degrees of the characters in $\mathrm{Irr}_{\mathrm{rv}}(G)$.

We recall that $\mathrm{Irr}_{\mathrm{rv}}(G) = \{1_G\}$ if $|G|$ is odd. Also, if N is a normal subgroup of G, we clearly have $\mathrm{cd}_{\mathrm{rv}}(G/N) \subseteq \mathrm{cd}_{\mathrm{rv}}(G)$.

We shall also make use of the following theorems, which relate the arithmetical structure of $\mathrm{cd}_{\mathrm{rv}}(G)$ and the group structure of G. The first one is Theorem 4.2 in [4].

Theorem 1.1. *Let G be a group, and T a Sylow 2-subgroup of G. All the elements in $\mathrm{cd}_{\mathrm{rv}}(G)$ are odd numbers if and only if T is normal in G and $\mathrm{cd}_{\mathrm{rv}}(T) = \{1\}$.*

A 2-group T such that $\mathrm{cd}_{\mathrm{rv}}(T) = \{1\}$ will be called a 2*-group of Chillag-Mann type*, as this class of groups was studied in [2]. It is easily seen that T is of this type if and only if the kernel of every real valued character of T contains the Frattini subgroup $\Phi(T)$. In [2], it is proved that this happens if and only if every element of $\Phi(T)$ has the same number of square roots in T.

The next two results appear in [13].

Theorem 1.2 ([13, Theorem A])**.** *Let G be a group. All the elements in $\mathrm{cd}_{\mathrm{rv}}(G)\backslash\{1\}$ are even numbers if and only if G has a normal 2-complement.*

Theorem 1.3 ([13, Theorem C(b)])**.** *Let G be a group, and T a Sylow 2-subgroup of G. All the elements in $\mathrm{cd}_{\mathrm{rv}}(G)$ are powers of 2 if and only if G has a normal 2-complement K and T centralizes K'.*

For our purposes, we shall also need results relating the real valued characters of a group to those of its normal subgroups, in order to apply Clifford Theory. We start with a general observation. Let N be a normal subgroup of G, $\chi \in \mathrm{Irr}_{\mathrm{rv}}(G)$, and let $\Theta = \{\theta_1, \theta_2, \ldots, \theta_t\}$ be the set of irreducible constituents of χ_N. As χ is real valued, the map $\gamma : \theta_i \mapsto \overline{\theta_i}$ is a permutation of Θ. Now, if $t = |\Theta|$ is odd, then γ fixes at least one θ_i, which is therefore real valued. Since the characters in Θ form an orbit under the natural action of G on $\mathrm{Irr}(N)$, it follows that $\theta_i \in \mathrm{Irr}_{\mathrm{rv}}(N)$ for all i in $\{1, ..., t\}$.

From this one gets the following known results.

Lemma 1.4. *Let G be a group, and N a normal subgroup of G such that $|N|$ is odd. If N centralizes a Sylow 2-subgroup of G, then $N \leq \mathrm{Ker}(\chi)$ for every $\chi \in \mathrm{Irr}_{\mathrm{rv}}(G)$.*

Proof. Let $\chi \in \mathrm{Irr}_{\mathrm{rv}}(G)$, and write

$$\chi_N = e\sum_{i=1}^{t}\theta_i\,.$$

Denoting by $I_G(\theta_1)$ the inertia subgroup of θ_1 in G, we have that $t = |G : I_G(\theta_1)|$ is odd, because N is centralized by a Sylow 2-subgroup of G. Thus, θ_i lies in $\mathrm{Irr}_{\mathrm{rv}}(N) = \{1_N\}$, and hence $N \leq \mathrm{Ker}(\chi)$. ∎

Lemma 1.5. *Let G be a group, N a normal subgroup of G, and $\chi \in \mathrm{Irr}_{\mathrm{rv}}(G)$. If $|G : N|$ is odd, then every irreducible constituent of χ_N is real valued.*

Proof. Just recall that the number of distinct irreducible constituents of χ_N is a divisor of $|G : N|$. ∎

Now some tools for building up real valued characters of a group. The following lemma is essentially Lemma 2.2(b) in [11].

Lemma 1.6. *Let G be a group which acts by automorphisms on the group M. If $|G/C_G(M)|$ is even, then there exist $x \in G$ and $\mu \in \mathrm{Irr}(M)$ with $\mu \neq 1_M$, such that $\mu^x = \bar{\mu}$.*

Proof. Let $C_G(M)x$ be an involution in $G/C_G(M)$. We can certainly find an element m of M such that $m^x \neq m$. Setting $y := m^{-1}m^x$, we get $(y^x)^{-1} = y$ with $y \neq 1$. Consider now the permutation π on the elements of M defined by $z^\pi := (z^x)^{-1}$ (observe that π^2 is the identity map). This π induces a well defined permutation on the set of conjugacy classes of M, given by $(z^M)^\pi := (z^\pi)^M$ for every z in M. Moreover, for λ in $\mathrm{Irr}(M)$ and z in M, we get

$$\lambda^\pi(z) := \lambda(z^{\pi^{-1}}) = \lambda(z^\pi) = \lambda((z^x)^{-1}) = \bar{\lambda}(z^x) = \bar{\lambda}^{x^{-1}}(z).$$

Therefore, π induces the permutation $\lambda \mapsto \bar{\lambda}^{x^{-1}}$ on $\mathrm{Irr}(M)$.

We are now in a position to apply the Brauer's Permutation Lemma (see [9, 6.32]): since π fixes the conjugacy class of the nonidentity element y, it must fix also a nonprincipal μ in $\mathrm{Irr}(M)$. In other words, there exists a nonprincipal μ in $\mathrm{Irr}(M)$ such that $\mu = \bar{\mu}^{x^{-1}}$, hence $\mu^x = \bar{\mu}$, as desired. ∎

Lemma 1.7. *Let G be a group, H a subgroup of G, ψ a character of H, and $x \in N_G(H)$ such that $\psi^x = \bar{\psi}$. Then ψ^G is a real valued character of G.*

Proof. Since x normalizes H, by the definition of induced character we have $(\psi^x)^G = \psi^G$. Then, $\overline{\psi^G} = (\bar{\psi})^G = (\psi^x)^G = \psi^G$. ∎

Lemma 1.8. *Let G be a group, N a normal subgroup of G with $(|N|, |G : N|) = 1$, and $\theta \in \mathrm{Irr}(N)$. Assume that there exists an $x \in G$ such that $\theta^x = \bar{\theta}$. Then there exists a real valued $\chi \in \mathrm{Irr}(G)$ such that $|G : I_G(\theta)|$ divides $\chi(1)$.*

Proof. Let I be the inertia subgroup of θ in G. Observe that $I = I_G(\overline{\theta})$ and that $x \in N_G(I)$. Since $(|N|, |G : N|) = 1$, by Corollary 6.28 in [9] there exists a unique $\psi \in \mathrm{Irr}(I|\theta)$ such that the determinantal orders $o(\psi)$ and $o(\theta)$ coincide. Now, both ψ^x and $\overline{\psi}$ lie over $\theta^x = \overline{\theta}$. Since $o(\psi^x) = o(\theta^x) = o(\overline{\theta}) = o(\overline{\psi})$, we get

$\psi^x = \overline{\psi}$. By Clifford Theory, $\chi = \psi^G$ is irreducible, and $|G : I_G(\theta)|$ divides $\chi(1)$. By Lemma 1.7, χ is real valued. ■

With some extra assumptions concerning the prime 2, one can somewhat control the occurrence of real valued characters "lying over" real valued characters of normal subgroups.

Proposition 1.9 ([12, 2.1 and 2.2]). *Let N be a normal subgroup of G, and $\theta \in \mathrm{Irr}_{\mathrm{rv}}(N)$. If $|G : N|$ is odd, then θ allows a unique real valued extension to $I_G(\theta)$. Further, there exists a unique real valued character χ in $\mathrm{Irr}(G|\theta)$.*

Proposition 1.10 ([12, 2.3]). *Let N be a normal subgroup of G, and $\theta \in \mathrm{Irr}_{\mathrm{rv}}(N)$. Suppose that $\theta(1)$ is odd and that $o(\theta) = 1$. Then θ allows a real valued extension to $I_G(\theta)$, and there exists a real valued character χ in $\mathrm{Irr}(G|\theta)$.*

Next, two easy and well known results.

Lemma 1.11 ([6, V.8.9(d)]). *Let x be an automorphism of a group G. If x has order 2 and $C_G(x) = 1$, then G is abelian and $g^x = g^{-1}$ for every $g \in G$.*

Lemma 1.12. *Let M be a minimal normal subgroup of a group G. Then M has no irreducible character of degree 2.*

Proof. We can clearly assume that M is nonabelian, so that $M = S_1 \times S_2 \times \cdots S_n$, where $S_i \simeq S$ is a nonabelian simple group. An irreducible character of M is the "direct product" of irreducible characters of the S_i. So, if there is a $\chi \in \mathrm{Irr}(M)$ with $\chi(1) = 2$, there must be a $\theta \in \mathrm{Irr}(S)$ with $\theta(1) = 2$. As $\theta(1)$ divides $|S|$, there exists an involution $x \in S$. If Θ is a representation affording θ, then the eigenvalues of $\Theta(x)$ are either 1 or -1. But $\det(\Theta(x)) = 1$, as $\det(\Theta)$ is a homomorphism from the perfect group S to the abelian group $\mathbb{C}^*$, and this implies that the two eigenvalues of $\Theta(x)$ coincide. Hence $|\theta(x)| = \theta(1)$, and then $x \in Z(\theta) = 1$, a contradiction. ■

Finally, we point out the following elementary fact, that we are going to use in the proof of Theorem C.

Lemma 1.13. *Let G be a group, and assume $G = A \times B$, where $|A|$ is a prime q. Then the number of complements for A in G is a power of q.*

Proof. Let us denote by Ω the set of all complements for A in G. Observe that, if H is in Ω, then H contains G', and H/G' is a complement for the direct factor $G'A/G'$ (whose order is q) in G/G'. This yields that, denoting by $\bar{\Omega}$ the set of complements for $G'A/G'$ in G/G', the map $H \mapsto H/G'$ from Ω to the set of subgroups of G/G' has image in $\bar{\Omega}$. This map is injective, and it is easy to see that its image is the whole $\bar{\Omega}$, so that $|\Omega| = |\bar{\Omega}|$. In other words, we can assume that G is abelian.

Next, if H is in Ω, we clearly have $O_{q'}(G) \leq H$. Arguing as above with $O_{q'}(G)$ in place of G', we can assume that G is an abelian q-group.

As the last reduction, we now have that every H in Ω contains $\Phi(G)$. Therefore we can assume that G is an elementary abelian q-group, which can be viewed as an n-dimensional vector space over $\mathrm{GF}(q)$ for a suitable n in $\mathbb{N}$.

Finally, $|\Omega|$ is the number of hyperplanes of G not containing A. This is given by the total number of hyperplanes of G minus the total number of hyperplanes of the quotient space G/A (which has dimension $n-1$). It is now clear that we get $|\Omega| = q^{n-1}$, as claimed. ∎

Occasionally we shall also make use, with no reference, of some well known results concerning coprime actions, and of the Odd Order Theorem by Feit and Thompson: if a finite group has odd order, then it is solvable.

2. Proof of Theorem A

We now restate and prove Theorem A.

Theorem 2.1. *Let G be a group, and assume that every element in $\mathrm{cd}_{\mathrm{rv}}(G) \setminus \{1\}$ is a prime number. Then G is solvable.*

Proof. We argue by induction on $|G|$. Let M be a minimal normal subgroup of G. Since the assumption is inherited by factor groups, by induction G/M is solvable. It is hence enough to show that M is solvable.

If M is not solvable then, by Theorem 1.2, there exists a real valued nonlinear $\theta \in \mathrm{Irr}(M)$ of odd degree. Also, $o(\theta) = 1$ and, by Proposition 1.10, there exists a real valued $\chi \in \mathrm{Irr}(G|\theta)$. Since $1 \neq \theta(1) | \chi(1)$ and $\chi(1)$ is prime, we have $\chi_M = \theta$. By Gallagher's Theorem ([9, 6.17]), it follows that $\mathrm{cd}_{\mathrm{rv}}(G/M) = \{1\}$.

If every real valued irreducible character of G has odd degree, then, by Theorem 1.1, G is solvable. So, there exists a real valued $\chi \in \mathrm{Irr}(G)$ such that $\chi(1) = 2$. By Lemma 1.12, χ_M is the sum of two linear characters and, as 1_M is the only linear character of M, it follows that $M \leq \mathrm{Ker}(\chi)$. Hence $2 \in \mathrm{cd}_{\mathrm{rv}}(G/M)$, a contradiction.

Therefore M is solvable, and the proof is complete. ∎

3. Squarefree degrees

We now consider, as a natural generalization, the groups with real valued irreducible characters of squarefree degree. Here we can not hope for solvability, as

$$\mathrm{cd}_{\mathrm{rv}}(A_7) = \{1, 6, 14, 15, 21, 35\}.$$

Anyway, using the Classification of Finite Simple Groups, we can prove the following.

Theorem 3.1. *Let G be a group, and assume that every element in $\mathrm{cd}_{\mathrm{rv}}(G)$ is a squarefree number. Then either G is solvable, or there exists a solvable normal subgroup R of G such that $G/R \simeq A_7$.*

For proving Theorem 3.1 we need the next preliminary result.

Lemma 3.2. *Let G be a group, and assume that every element in* $\mathrm{cd}_{\mathrm{rv}}(G)$ *is a squarefree number. Then every nonabelian chief factor of G is isomorphic to A_7.*

Proof. Let M/N be a nonabelian chief factor of G. We first show that M/N is a simple group.

As M/N is a minimal normal subgroup of G/N, there exists a nonabelian simple group S such that M/N is isomorphic to the direct product of n copies of S, where n is a suitable positive integer. By Theorem 1.2, there exists a σ in $\mathrm{Irr}_{\mathrm{rv}}(S)$ such that $\sigma(1)$ is an odd number greater than 1. Let θ be the irreducible character of M/N defined as the product of n copies of σ. We have that $\theta(1)$ is odd, $o(\theta)$ is 1, and of course θ is real valued. Therefore, we are in a position to apply Proposition 1.10, concluding that there exists a real valued χ in $\mathrm{Irr}((G/N)|\theta)$. Now, χ (which we regard as a character of G, by inflation) has degree divisible by $\sigma(1)^n$. This yields $n = 1$, and hence that $M/N \simeq S$ is a simple group.

Also, the previous paragraph shows that every odd number in $\mathrm{cd}_{\mathrm{rv}}(M/N)$ is squarefree.

Let C be the subgroup of G containing N and such that $C/N = C_{G/N}(M/N)$. The group G/C acts faithfully by conjugation on CM/C, which is isomorphic to S, and so G/C is an almost-simple group with socle CM/C. In what follows, we analyze the possible isomorphism type of S.

A direct check of [3] shows that the Tits group and every sporadic simple group, except J_1, have a real valued irreducible character whose degree is odd but not squarefree. Anyway, the group J_1 has a real valued irreducible character whose degree is not squarefree, and since $\mathrm{Out}(J_1)$ is trivial, $S \simeq J_1$ would imply $G/C \simeq J_1$, a contradiction.

Next, assume that S is isomorphic to an alternating group A_n, with $n \geq 5$ and $n \neq 7$. As $\mathrm{cd}_{\mathrm{rv}}(A_6)$ contains 9, we can also assume $n \neq 6$, so that G/C is isomorphic either to A_n or to S_n. As explained in [7, proof of Lemma 2.1], for every $n \geq 5$, $n \neq 7$, the group S_n has a (real valued) irreducible character whose degree is not squarefree and which restricts irreducibly to A_n, again a contradiction.

Finally, if S is isomorphic to a simple group of Lie type, except the Tits group, then we can consider the Steinberg character of S. This irreducible character allows a real valued extension to $\mathrm{Aut}(S)$ (see [15, Remark]), and from [3, Table 6 on page xvi] we see that its degree is not squarefree unless S is isomorphic to $\mathrm{PSL}(2,p)$ where p is an odd prime (greater than 3). But then G/C is isomorphic either to $\mathrm{PSL}(2,p)$ or to $\mathrm{PGL}(2,p)$, so that $\mathrm{cd}_{\mathrm{rv}}(G/C)$ contains $p+1$ and $p-1$ (see [5, Theorem 38.1] and [16, Table III]), except for the case $G/C \simeq \mathrm{PSL}(2,5) \simeq A_5$ which has been already considered. Of course, one among $p+1$ and $p-1$ is divisible by 4, against our assumptions.

We conclude that the only possibility is $S \simeq A_7$, as desired. ∎

Proof of Theorem 3.1. We argue by induction on $|G|$. Assume first that there exists a solvable minimal normal subgroup M of G. Since the hypothesis in the statement is inherited by G/M, either G/M is solvable or there exists a normal subgroup R of G, containing M, such that $(G/M)/(R/M) \simeq A_7$ and R/M is solvable.

In the former case we get that G is solvable, whereas in the latter case we obviously have that R is a solvable normal subgroup of G and $G/R \simeq (G/M)/(R/M) \simeq A_7$.

In view of the previous discussion, we can assume that every minimal normal subgroup of G is nonsolvable, and thus, by Lemma 3.2, isomorphic to A_7. If G has two distinct minimal normal subgroups U_1 and U_2, then the normal subgroup $U_1 \times U_2 \simeq A_7 \times A_7$ has a real valued irreducible character θ whose degree is odd but not squarefree, and $o(\theta) = 1$. By Proposition 1.10, there exists a real valued character χ in $\mathrm{Irr}(G|\theta)$ and $\chi(1)$ is divisible by $\theta(1)$, a contradiction.

The conclusion is that G has a unique minimal normal subgroup $U \simeq A_7$, so that either $G = U$, or $G \simeq S_7$. Since S_7 has a real valued irreducible character whose degree is not squarefree, the only possibility is $G = U$, and the proof is complete. ■

We note here that Theorem 2.1 can be also deduced from Theorem 3.1, since A_7 has real valued irreducible characters whose degree is not a prime.

We also remark that, by Theorem 2.8 in [7], if G is a nonsolvable group such that every element in $\mathrm{cd}(G)$ is a squarefree number, then there exists a solvable normal subgroup R of G such that $G \simeq R \times A_7$. It may be worth stressing that we can not pursue such a strong conclusion under the weaker assumptions of Theorem 3.1. In fact the group $3.A_7$, whose character table appears in [3, page 10], fulfills the assumptions of Theorem 3.1, but it is a *non-split* extension of a normal subgroup of order 3 by A_7.

Finally, we observe that there is no upper bound for $|\,\mathrm{cd}_{\mathrm{rv}}(G)|$, when G varies in the class of finite groups with real valued irreducible characters of squarefree degree. We denote by F_p the semidirect product of the additive group of the field $\mathbb{K} = \mathrm{GF}(2^k)$, for some $k \in \mathbb{N}$, by the subgroup of prime order p, p a suitable prime, of the multiplicative group $\mathbb{K}^{\times}$. Then the nonlinear irreducible characters of F_p are real valued and $\mathrm{cd}_{\mathrm{rv}}(F_p) = \mathrm{cd}(F_p) = \{1, p\}$. By using Zsigmondy's Theorem, for every positive integer n we can produce groups $F_{p_1}, F_{p_2}, \ldots, F_{p_n}$, for distinct primes $p_1, p_2, \ldots, p_n$. Consider

$$G = F_{p_1} \times F_{p_2} \times \cdots \times F_{p_n}.$$

Then

$$\mathrm{cd}_{\mathrm{rv}}(G) = \{\prod_{i \in I} p_i \,|\, I \subseteq \{1, 2, \ldots, n\}\}$$

where the product on $I = \emptyset$ is meant to be 1.

Hence the degrees of the real valued irreducible characters of G are all squarefree, and $|\,\mathrm{cd}_{\mathrm{rv}}(G)| = 2^n$. Also, there are n distinct prime divisors of the degrees in $\mathrm{cd}_{\mathrm{rv}}(G)$.

4. Degree patterns.

In this section we shall prove Theorem B, which we state again.

Theorem 4.1. *Let G be a group, and assume that every element in $\mathrm{cd}_{\mathrm{rv}}(G) \setminus \{1\}$ is a prime number. Then $\mathrm{cd}_{\mathrm{rv}}(G)$ is contained in a set of the kind $\{1, 2, p\}$, where p is an odd prime.*

The proof of Theorem 4.1 will be postponed after the proof of Theorem 4.4. We first analyze the structure of groups whose real valued nonlinear irreducible characters have odd prime degree. In Theorem 4.4 we shall see that in this case the nonlinear characters in $\mathrm{Irr}_{\mathrm{rv}}(G)$ are forced to have all the same degree.

Let us introduce some notation. We shall denote by $\Gamma(K)$ the *semilinear group* on $K = \mathrm{GF}(r^n)$, where r is a prime:

$$\Gamma(K) := \{x \mapsto ax^{\sigma} \ : \ a \in K \setminus \{0\}, \sigma \in \mathrm{Gal}(K)\}.$$

(Here $\mathrm{Gal}(K)$ denotes the Galois group of K over its prime subfield). The subgroup of $\Gamma(K)$ consisting of all the maps of the kind $x \mapsto ax$ with $a \in K \setminus \{0\}$ will be denoted by $\Gamma_0(K)$.

Example 4.2. Let G be the semidirect product of $K = \mathrm{GF}(r^n)$ by the subgroup $H = \langle \alpha, \beta \rangle$ of $\Gamma(K)$, where $\alpha : x \mapsto ax$, $\beta : x \mapsto x^{\sigma}$ are such that a has prime order q in $K \setminus \{0\}$, and σ has prime order p in $\mathrm{Gal}(K)$; also, assume that p, q, r satisfy the relation

$$\frac{r^n - 1}{r^{n/p} - 1} = q \ . \qquad (1)$$

Observe that q is coprime with $r^{n/p} - 1$, because $((s^p - 1)/(s - 1), s - 1)$ divides p for every integer $s > 1$; but, as an easy consequence of (1), we have $q \neq p$. As $|C_{\Gamma_0(K)}(\beta)| = |C_{K \setminus \{0\}}(\beta)| = r^{n/p} - 1$, it follows that H is a nonabelian group of order pq.

Further, since no nontrivial element of K can be centralized by two distinct Sylow p-subgroups of H, by (1) it follows that $\{C_K(P) \setminus \{0\} \ : \ P \in \mathrm{Syl}_p(H)\}$ is a partition of $K \setminus \{0\}$. So, again by (1), it follows that $|H : C_H(x)| = q$ for all nontrivial $x \in K$.

Now, K has a structure of $\mathrm{GF}(r)$-vector space, and we can view it as a $\mathrm{GF}(r)[H]$-module. If L is a submodule of it, we get that $\{C_L(P) \setminus \{0\} \ : \ P \in \mathrm{Syl}_p(H)\}$ is a partition of $L \setminus \{0\}$, and hence $q = (r^m - 1)/(r^h - 1)$ where $r^m = |L|$ and $r^h = |C_L(P)|$ for any $P \in \mathrm{Syl}_p(H)$. Then the uniqueness of the representation of q in base r implies that $h = n/p$ and that $m = n$. Hence, K is an irreducible $\mathrm{GF}(r)[H]$-module.

Thus, K is minimal normal in G, and the Frobenius group H of order pq acts on K in such a way that $|H : C_H(x)| = q$ for every nontrivial x in K.

We are going to apply the following lemma in the proof of Theorem 4.4. Anyway, we state it in greater generality than needed there, since the proof is pretty much the same.

Lemma 4.3. *Let H be a solvable group of automorphisms of a group K. Assume that H and K have coprime orders and that, for every nontrivial $x \in K$, $|H : C_H(x)|$ is a prime. Then one of the following occurs.*

(a) *H has prime order, K is nilpotent and KH is a Frobenius group with kernel K and complement H; or*

(b) *H is a nonabelian group of order pq, where p, q are distinct primes, K is an elementary abelian r-group for a suitable prime r, and KH is isomorphic to one of the groups described in Example 4.2.*

Proof. If H has prime order, then it acts fixed-point freely on K. So, KH is a Frobenius group with kernel K and complement H, and by [6, V.8.14] K is nilpotent.

We hence assume that $|H|$ is not a prime. As $(|H|, |K|) = 1$, by Lemma 2.6.2 of [8] there exists an abelian group of squarefree exponent A such that $H \leq \mathrm{Aut}(A)$ and A and K are isomorphic as H-sets. In particular, $|K| = |A|$ and $|H : C_H(a)|$ is a prime for every nontrivial $a \in A$.

By Maschke's Theorem, A is a completely reducible H-module, possibly over fields of different characteristic.

Assume that there exists a nontrivial decomposition $A = B \oplus C$ of the H-module A. Then, for all $b \in B$ and $c \in C$, $C_H(bc) = C_H(b) \cap C_H(c)$. Our hypothesis implies that, for all nontrivial $b \in B$ and $c \in C$, we get $C_H(b) = C_H(c) = C_H(bc)$, whence every nontrivial element of A has the same centralizer in H. It easily follows that $|H|$ is a prime, against our assumption.

Therefore A is an irreducible H-module and, in particular, $|A| = |K| = r^n$ where r is a prime and n is a positive integer.

Let N be a nontrivial normal subgroup of H. Since $C_A(N)$ is a proper submodule of A, we have that $C_A(N)$ is trivial. As a consequence, for every nontrivial x in A, N does not lie in $C_H(x)$, and the maximality of $C_H(x)$ in H yields $H = C_H(x)N$.

As the next step, we claim that N acts irreducibly on A. In fact, let B be a nontrivial N-submodule of A, and consider a nontrivial $x \in B$ and an element h in H. We can write h as a product cn, where c is in $C_H(x)$ and n is in N. Now we get $x^h = x^{cn} = x^n \in B$, thus B is H-invariant. We conclude that $B = A$, and our claim is proved.

Observe that every nontrivial abelian normal subgroup of H must be cyclic, as it acts faithfully and irreducibly on A ([6, II.3.10]).

Consider now a minimal normal subgroup M of H. Since H is solvable, M is an elementary abelian q-group, where q is a suitable prime number, so M has order q. For every nontrivial x in A we get $H = MC_H(x)$, and this forces $|H : C_H(x)|$ to be q. Also, we clearly have $O_t(H) = 1$ for every prime $t \neq q$, so that $F := F(H)$ is a q-group. Finally, if we assume $\Phi(H) \neq 1$, we get $H = \Phi(H)C_H(x)$ (whence $H = C_H(x)$) for every nontrivial x in A, a contradiction. The conclusion is that F is an elementary abelian q-group, whence its order is indeed q.

Since F is abelian and it acts irreducibly on A, by Theorem 2.1 of [10] we can assume that $A = \mathrm{GF}(r^n)$, $H \leq \Gamma(A)$ and that $F \leq \Gamma_0(A)$. Further, by [6, II.3.10], $n = \dim_{\mathrm{GF}(r)}(A)$ is the order of r modulo q.

Write $L = C_H(x)$ for a nontrivial $x \in A$. Observe that $F \cap L = 1$, and hence L is a complement for F in A. So, L acts fixed-point freely (by conjugation) on

$F = C_H(F)$, and $H = FL$ is a Frobenius group with kernel F and complement L.

As $\Gamma(A)$ acts transitively on $A \setminus \{0\}$, up to conjugation in $\Gamma(A)$ we can assume that L is a subgroup of the stabilizer of 1 in $\Gamma(A)$, i.e. $L \leq \{x \mapsto x^\sigma | \sigma \in \mathrm{Gal}(A)\}$.

Assume now that L has a nontrivial proper subgroup U. Hence we have $|C_A(U)| = r^{n/|U|} > |C_A(L)| = r^{n/|L|}$ and there exists an element $y \in C_A(U)$ such that $y \notin C_A(L)$. Now, $C_H(y) \cap L \geq U \neq 1$ and $|C_H(y)| = |L|$. Since H is a Frobenius group, this yields $C_H(y) = L$, a contradiction. We conclude that L has no nontrivial proper subgroups, so its order is a prime number p. Hence, H is a Frobenius group of order pq.

Observe now that

$$q = \frac{r^n - 1}{r^{n/p} - 1},$$

as every nontrivial element of A is fixed by exactly one conjugate of L in H.

As K is an r-group and r does not divide $|H|$, we have that H acts faithfully on $\overline{K} = K/\Phi(K)$. Again, $C_{\overline{K}}(H) = \Phi(K)C_K(H)/\Phi(K)$ is trivial, and hence $|H : C_H(\overline{x})| = q$ for all nontrivial $\overline{x} \in \overline{K}$. As above, we see that F acts irreducibly on $\overline{K}$ and hence, by [6, II.3.10], we get that $\dim_{\mathrm{GF}(r)}(\overline{K})$ is the order of r modulo $q = |F|$. It follows that $\Phi(K) = 1$, and hence that K is an elementary abelian r-group. Working with K in place of A, we can thus identify K with $\mathrm{GF}(r^n)$ and H with a subgroup of order pq of $\Gamma(K)$. Then F is the unique subgroup of order q of $\Gamma_0(K)$, and $L \in \mathrm{Syl}_p(H)$ is conjugate in $\Gamma(K)$ to the subgroup $\langle \beta : x \mapsto x^\sigma \rangle$, where $\sigma \in \mathrm{Gal}(K)$ has order p. Therefore, $G = KH$ is isomorphic to one of the groups described in Example 4.2. ∎

We are now ready to determine the structure of the groups with real valued nonlinear irreducible characters of odd prime degree.

Theorem 4.4. *Let G be a group, T a Sylow 2-subgroup of G, and $U = O_{2'}(G)$. Then the following conditions are equivalent.*

(a) *Every element in $\mathrm{cd}_{\mathrm{rv}}(G) \setminus \{1\}$ is an odd prime.*
(b) *There exists an odd prime q such that $\mathrm{cd}_{\mathrm{rv}}(G) = \{1, q\}$.*
(c) *T is normal in G, T is a 2-group of Chillag-Mann type, and either $|G : TU|$ is a prime or, writing $\overline{G} = G/\Phi(T)U$, we have*

$$\overline{G} = Z(\overline{G}) \times G_0$$

where G_0 is isomorphic to one of the groups described in Example 4.2 (with $r = 2$ and p odd).

Remark 4.5. If the group G satisfies the equivalent conditions in the statement of Theorem 4.4, then (using the notation of that statement) the subgroup $\Phi(T)U$ is in the kernel of every real valued character. This is true for U by Lemma 1.4, since U centralizes the normal Sylow 2-subgroup T of G. Further, for any $\chi \in \mathrm{Irr}_{\mathrm{rv}}(G)$, by Lemma 1.5 the irreducible constituents of χ_T are real valued and hence, as T is of Chillag-Mann type, their kernels contain $\Phi(T)$. It follows that $\Phi(T) \leq \mathrm{Ker}(\chi)$.

Proof of Theorem 4.4. Let us start by proving that (a) implies (c). The fact that G has a normal Sylow 2-subgroup T of Chillag-Mann type is ensured by Theorem 1.1. We shall denote by L a complement for T in G.

Let θ be any irreducible character of T whose kernel contains $\Phi(T)$. Then we know that θ is real valued. Since T has odd index in G, by Proposition 1.9 there exists a real valued character χ in $\mathrm{Irr}(G|\theta)$. Now, $\chi(1)$ is divisible by $|G : I_G(\theta)| = |L : I_L(\theta)|$, so that $|L : I_L(\theta)|$ is either a prime or 1. Now, $V = T/\Phi(T)$ and $\mathrm{Irr}(V)$ are isomorphic L-sets because $(|L|, |V|) = 1$ ([9, 13.24]), and hence $|L : C_L(v)|$ is either a prime or 1 for every $v \in V$.

Write $H = L/U$. Since $|L|$ and $|T|$ are coprime, we have $C_L(V) = C_L(T)$. Hence $C_L(V) = U$, and H acts faithfully on V. Further, $V = Z \times W$, where $Z = C_V(H)$ and $W = [V, H]$. Observe that $|H : C_H(w)|$ is a prime for every nontrivial $w \in W$. Hence we are in a position to apply Lemma 4.3, and we conclude that either $|H| = |G : TU|$ is a prime, or WH is isomorphic to one of the groups described in Example 4.2. Moreover, we get $\overline{G} \simeq VH$, so that $\overline{G} = Z(\overline{G}) \times G_0$, with $G_0 \simeq WH$.

We show next that (c) implies (b). Consider the quotient $\overline{G}$, and denote by $\overline{T}$ its Sylow 2-subgroup. We have that both $O_{2'}(\overline{G})$ and $\Phi(\overline{T})$ are trivial, and $\overline{G}$ satisfies our assumptions. Arguing by induction on the order of the group, and taking into account Remark 4.5, the claim follows if $\Phi(T)U$ is not trivial. Therefore, we can assume $\Phi(T) = U = 1$. Clearly, we can also assume that G has no nontrivial direct central factors. Thus, G is either a Frobenius group with Frobenius complement of prime order, or G is isomorphic to one of the groups of Example 4.2 (with $r = 2$). In any case, there exists an odd prime q such that $|G : C_G(x)| = q$ for every nontrivial $x \in T$. Let χ be a nonprincipal character in $\mathrm{Irr}_{\mathrm{rv}}(G)$, and let θ be an irreducible constituent of χ_T. Then θ is real valued by Lemma 1.5, and it is nonprincipal of degree 1.

By coprimality, $\mathrm{Irr}(T)$ and T are isomorphic G/T-sets. Hence $|G : I_G(\theta)| = q$, and $|I_G(\theta) : T|$ is either 1 or a prime. Then, by [9, 6.19], every $\psi \in \mathrm{Irr}(I_G(\theta)|\theta)$ extends θ and hence, by Clifford Correspondence, $\chi(1) = \theta(1)|G : I_G(\theta)| = q$.

We conclude that $\mathrm{cd}_{\mathrm{rv}}(G) = \{1, q\}$, as desired.

The fact that (b) implies (a) is straightforward. ∎

It might be worth pointing out that, while conditions (a) and (b) in Theorem 4.4 are equivalent for $\mathrm{cd}_{\mathrm{rv}}(G)$, the same conditions for $\mathrm{cd}(G)$ are not. For instance, the semidirect product $K\Gamma(K)$, where $K = \mathrm{GF}(2^3)$, is such that $\mathrm{cd}(G) = \{1, 3, 7\}$. As an explanation for this possibly surprising behavior, one can just recall that odd order groups have no real valued irreducible characters other than the principal character.

We add a further remark. Let G be as in Example 4.2. Then $\mathrm{cd}(G) = \{1, p, q\}$ by Lemma 2.3 of [14]. Assume also $r = 2$. If p is odd, then $\mathrm{cd}_{\mathrm{rv}}(G) = \{1, q\}$ (see (c) $\Rightarrow$ (b) of Theorem 4.4, in the special case $G = G_0$). But if $p = 2$, then $\mathrm{cd}_{\mathrm{rv}}(G) = \{1, 2, q\}$ by Theorem 1.1 and Theorem 1.2. Observe that in this case q

is a Fermat prime. As of this writing, it is not known exactly which odd primes q can occur in $\mathrm{cd}_{\mathrm{rv}}(G)$, for a group G, when this set is of the kind $\{1,2,q\}$.

Proof of Theorem 4.1. We consider the prime graph $\Gamma_{\mathrm{rv}}(G)$ on $\mathrm{cd}_{\mathrm{rv}}(G)$. The vertices of $\Gamma_{\mathrm{rv}}(G)$ are the primes dividing some degree in $\mathrm{cd}_{\mathrm{rv}}(G)$, and two primes p and q are connected in $\Gamma_{\mathrm{rv}}(G)$ if some degree in $\mathrm{cd}_{\mathrm{rv}}(G)$ is divisible by pq.

If the real valued nonlinear irreducible characters of G have all prime degree, then $\Gamma_{\mathrm{rv}}(G)$ has precisely $|\,\mathrm{cd}_{\mathrm{rv}}(G)|-1$ connected components. By Theorem A, G is solvable, and hence Theorem 5.1(ii) of [4] tells us that $\Gamma_{\mathrm{rv}}(G)$ has at most two connected components. It follows that $\mathrm{cd}_{\mathrm{rv}}(G)=\{1,p,q\}$, where p and q are primes (possibly, $p=q$). If p and q are distinct primes, then by Theorem 4.4 one of them must be 2, and we are done. ∎

Next, we shall derive some detailed structural information on a group G such that $\mathrm{cd}_{\mathrm{rv}}(G)=\{1,2\}$.

Theorem 4.6. *Let G be a group, and T a Sylow 2-subgroup of G. Then we have $\mathrm{cd}_{\mathrm{rv}}(G)=\{1,2\}$ if and only if G has a normal 2-complement K, $K'\leq C_K(T)$, and one of the following holds.*

(a) $G=T\times K$, *with* $\mathrm{cd}_{\mathrm{rv}}(T)=\{1,2\}$*; or*

(b) $\mathrm{cd}_{\mathrm{rv}}(T)\subseteq\{1,2\}$, $\mathrm{cd}_{\mathrm{rv}}(O_2(G))=\{1\}$ *and* $G/C_K(T)O_2(G)$ *is a Frobenius group with Frobenius complement of order* 2.

Proof. Let us assume $\mathrm{cd}_{\mathrm{rv}}(G)=\{1,2\}$. By Theorem 1.3, G has a normal 2-complement K and T centralizes K'. If T is normal in G, then we get (a). Therefore, we shall assume that T acts nontrivially (by conjugation) on K, and we shall prove (b).

Since T is isomorphic to G/K, we immediately get $\mathrm{cd}_{\mathrm{rv}}(T)\subseteq\{1,2\}$.

As for the claim that $G/C_K(T)O_2(G)$ is a Frobenius group with Frobenius complement of order 2, we argue by induction on the order of the group.

Set $N=C_K(T)O_2(G)$, and $\overline{G}=G/N$. Observe that $\mathrm{cd}_{\mathrm{rv}}(\overline{G})=\{1,2\}$. In fact, $\mathrm{cd}_{\mathrm{rv}}(\overline{G})\subseteq\mathrm{cd}_{\mathrm{rv}}(G)$; on the other hand, since $|\overline{G}|$ is even and $O_2(\overline{G})=1$, an application of Theorem 1.1 yields that $\overline{G}$ has nonlinear real valued irreducible characters. Also, denoting respectively by $\overline{K}$ and $\overline{T}$ the images of K and T under the natural homomorphism of G onto $\overline{G}$, by coprimality we get $C_{\overline{K}}(\overline{T})=1$. Now, if $N\neq 1$ we apply our inductive hypothesis, concluding that $\overline{G}=\overline{G}/C_{\overline{K}}(\overline{T})O_2(\overline{G})$ is a Frobenius group with Frobenius complement of order 2, as desired.

We can hence assume $N=1$, and it will be enough to show that $|T|=2$. Note that, in this situation, K is abelian as $K'\leq C_K(T)=1$ and, since $O_2(G)=1$, T acts faithfully on the dual group $\hat{K}$ of the irreducible characters of K.

Let x be a central involution of T. By coprimality, $\hat{K}=A\times B$ where $A=[\hat{K},\langle x\rangle]$ and $B=C_{\hat{K}}(x)$ (observe that A and B are T-invariant). Then $C_A(x)=1$ and, by Lemma 1.11, $\alpha^x=\overline{\alpha}$ for every $\alpha\in A$. By Lemma 1.8, we get

$$|T:I_T(\alpha)|=|G:I_G(\alpha)|=2$$

for every nonprincipal $\alpha\in A$.

Let $\widetilde{T}$ denote the quotient $T/C_T(A)$. If $|\widetilde{T}| > 2$, then $\widetilde{T}$ has a subgroup $\widetilde{L}$ of order 4, which is clearly abelian and acts faithfully on A. As an immediate consequence of a theorem by Brodkey (see [1]), there exists an element α in A which lies in an $\widetilde{L}$-orbit of length 4, and this implies $|T : I_T(\alpha)| \geq 4$, a contradiction. The conclusion so far is $|\widetilde{T}| = 2$. As the next step, we show that $C_T(A)$ is trivial. If it is not, choose an involution y in $C_T(A)$ and a nonprincipal β in B such that $\beta^y = \bar{\beta}$ (which exists as y inverts the elements of $[B, y] \neq 1$). Again by Lemma 1.8, we get $|T : I_T(\beta)| = 2$. Consider now the character $\alpha\beta \in \mathrm{Irr}(K)$. We have

$$(\alpha\beta)^{xy} = \alpha^{yx}\beta^{xy} = \alpha^x\beta^y = \overline{\alpha\beta},$$

whence $|T : I_T(\alpha\beta)| = 2$. On the other hand, we get $I_T(\alpha\beta) = I_T(\alpha) \cap I_T(\beta)$, and $I_T(\alpha)I_T(\beta) = T$, so that $|T : I_T(\alpha\beta)| = 4$, a contradiction. We conclude that $|T| = 2$, and the claim follows.

It remains to show that $\mathrm{cd}_{\mathrm{rv}}(O_2(G)) = \{1\}$ (observe that $O_2(G)$ has index 2 in T). Suppose that there exists λ in $\mathrm{Irr}_{\mathrm{rv}}(O_2(G))$ which is not a linear character. This λ must be G-invariant, otherwise it is easy to see that $(1_K \times \lambda)^G \in \mathrm{Irr}_{\mathrm{rv}}(G)$ has degree greater than 2, a contradiction. Now, as T does not centralize K, we can find t in T and $\mu \neq 1_K$ in $\mathrm{Irr}(K)$ such that $\mu^t = \bar{\mu}$. We see that $\mu \times \lambda \in \mathrm{Irr}(KO_2(G))$ is such that $(\mu \times \lambda)^t = \overline{\mu \times \lambda}$, hence $(\mu \times \lambda)^G$ is in $\mathrm{Irr}_{\mathrm{rv}}(G)$, by Clifford Theory and Lemma 1.7. But $(\mu \times \lambda)^G(1) > 2$, the final contradiction.

We move now to the converse statement. It is clear that, if (a) holds, then $\mathrm{cd}_{\mathrm{rv}}(G) = \mathrm{cd}_{\mathrm{rv}}(T) = \{1, 2\}$. Therefore, we shall assume that G has a normal 2-complement K such that the Sylow 2-subgroup T of G centralizes K', together with (b). As $K' \leq C_K(T)$, by Lemma 1.4 every real valued character of G has K' in the kernel. Moreover, the quotient G/K' satisfies our assumptions. Therefore, arguing by induction on the order of the group, if $K' \neq 1$ the claim is proved. We can hence suppose $K' = 1$. Set $R = KO_2(G)$, so that $|G : R| = 2$. Consider $\chi \in \mathrm{Irr}_{\mathrm{rv}}(G)$. As $\mathrm{cd}_{\mathrm{rv}}(R) = \{1\}$ and χ_R has at most $|G : R| = 2$ irreducible constituents, we see that $\chi(1) \leq 2$ if the constituents of χ_R are real valued. We can hence assume $\chi_R = \varphi + \varphi^x$, where x is in T, $\varphi^x \neq \varphi \in \mathrm{Irr}(R)$, and φ, φ^x are not real valued. Write $\varphi = \alpha \times \beta$ for some $\alpha \in \mathrm{Irr}(K)$ and $\beta \in \mathrm{Irr}(O_2(G))$.

Now, $\bar{\varphi}$ is an irreducible constituent of $\overline{\chi_R} = \chi_R$. As $\bar{\varphi} \neq \varphi$, we have $\bar{\varphi} = \varphi^x$ and, in particular, $\beta^x = \bar{\beta}$. If β is real valued, we have $\beta(1) = 1$, whereas if it is not, by Lemma 1.7 (and Clifford Correspondence) we get $\beta^T \in \mathrm{Irr}_{\mathrm{rv}}(T)$. In any case we get $\beta(1) = 1$, so that $\chi(1) = 2\alpha(1)\beta(1) = 2$, and the proof is complete. ■

Example 4.7. We note that, in (b) of Theorem 4.6, the condition $\mathrm{cd}_{\mathrm{rv}}(T) \subseteq \{1, 2\}$ is in general not a consequence of the condition $\mathrm{cd}_{\mathrm{rv}}(O_2(G)) = \{1\}$ (whereas $\mathrm{cd}(O_2(G)) = \{1\}$ would imply $\mathrm{cd}(T) \subseteq \{1, 2\}$, as $O_2(G)$ would be an abelian normal subgroup of T having index 2 in T). In fact, consider the group

$$G = \langle a, b, c \mid a^8 = b^2 = c^2 = 1,\ b^a = bc,\ a^c = a^5 \rangle,$$

which has order 32. Then the subgroup $H = \langle a, c \rangle$ has index 2 in G and $\mathrm{cd}_{\mathrm{rv}}(H) = \{1\}$, but $\mathrm{cd}_{\mathrm{rv}}(G) = \{1, 2, 4\}$.

5. Proof of Theorem C

We state and prove Theorem C.

Theorem 5.1. *Let G be a group, and assume that every element in* $\mathrm{cd}_{\mathrm{rv}}(G)\setminus\{1\}$ *is a prime number. Then* $l_2(G)\leq 2$*, and* $l_{2'}(G)\leq 1$.

Proof. If the degrees of the real valued nonlinear irreducible characters of G are either all odd or all even, then, by Theorem 1.1 and Theorem 1.2 respectively, G has either a normal Sylow 2-subgroup or a normal 2-complement, and we are done.

We hence assume $\mathrm{cd}_{\mathrm{rv}}(G)=\{1,2,p\}$, where p is an odd prime, and prove that $l_{2'}(G)\leq 1$. Clearly, this also implies $l_2(G)\leq 2$.

Let G be a counterexample of minimal order. Recall that, by Theorem 2.1, G is solvable. First, we claim that G has a unique minimal normal subgroup M, which is an elementary abelian q-group for a suitable odd prime q. In fact, if M, N are distinct minimal normal subgroups of G, then G embeds into the direct product $G/M\times G/N$. But the hypothesis in the statement is inherited by factor groups, so that (by induction) the $2'$-length of both G/M and G/N is at most 1, and the same holds for every subgroup of $G/M\times G/N$, including the isomorphic copy of G. We reached a contradiction, whence G has a unique minimal normal subgroup M. It is clear that M can not be a 2-group, otherwise the inductive hypothesis applied to G/M would yield $l_{2'}(G)\leq 1$.

Set $O/M:=O_2(G/M)$, and note that O/M can not be trivial, since otherwise the inductive hypothesis applied to G/M yields the existence of a normal 2-complement in G/M, a contradiction.

Observe now that the Frattini subgroup $\Phi:=\Phi(G)$ must be trivial. In fact, if $\Phi(G)\neq 1$, then $M\leq\Phi(G)$. We get that $\Phi O/\Phi\simeq O/(\Phi\cap O)$ is a 2-group. Hence, ΦO is a nilpotent group, so that O is nilpotent, whence O has a nontrivial Sylow 2-subgroup which is normal in G. This is a contradiction, as the $2'$-group M is the unique minimal normal subgroup of G.

Recall that, by [6, III.4.4], the condition $\Phi=1$ implies that every abelian normal subgroup of G has a complement in G. We shall take advantage of an application to M of this fact.

Using the above notation, we have that $O=M\rtimes S$, where S is a (nontrivial) Sylow 2-subgroup of O. S does not centralize M, because $O_2(G)=1$, and hence by Lemma 1.6 there exists $x\in S$ and a nonprincipal irreducible character μ of M such that $\mu^x=\bar{\mu}$.

Write $I=I_G(\mu)$. As M has a complement in G, it has a complement I_0 in I as well. Denoting by K the kernel of μ, it is easy to see that K is a normal subgroup of I. Moreover, since μ can be regarded as a faithful linear character of M/K, we get that M/K has prime order and it is a central subgroup of I/K. Thus, $I/K\simeq M/K\times KI_0/K$ and by Lemma 1.13 the number of complements of M/K in I/K is odd.

Observe now that both I and K are normalized by the 2-element x. Therefore, the conjugation by x induces a permutation on the set of complements for M/K

in I/K. As this set has an odd number of elements, we conclude that there exists a complement L/K for M/K in I/K which is normalized by x.

Let θ be the irreducible character of I arising as the inflation of $\mu \times 1_{L/K}$. Then θ is an extension of μ and $\theta^x = \bar{\theta}$.

Consider the character $\chi := \theta^G$ of G. By Clifford Correspondence, χ is irreducible, and it is real valued by Lemma 1.7. Since the 2-element x of G does not lie in I, $\chi(1) = |G : I|$ is forced to be 2.

We now observe that there exists $\psi \in \mathrm{Irr}_{\mathrm{rv}}(G/M)$ having degree p. Otherwise, by Theorem 1.2, G/M would have a normal 2-complement, whence we would get $l_{2'}(G) \leq 1$.

Since $|G : I| = 2$, I is normal in G and, by Corollary 6.19 of [9], ψ_I is irreducible. Recalling that θ is linear and that $\theta_M = \mu$, it follows that $\theta\psi_I \in \mathrm{Irr}(I|\mu)$ and hence, again by Clifford Correspondence, $(\theta\psi_I)^G$ is an irreducible character of G. But $(\theta\psi_I)^G = \theta^G\psi = \chi\psi$ is a real valued character of degree $2p$, against our assumptions. This contradiction completes the proof. ■

References

[1] J.S. Brodkey, *A note on finite groups with an abelian Sylow group*, Proc. Amer. Math. Soc. 14 (1963), 132–133.

[2] D. Chillag, A. Mann, *Nearly odd-order and nearly real finite groups*, Comm. Algebra 26 (1998), 2041–2064.

[3] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker, R.A. Wilson, *Atlas of finite groups*, Clarendon Press, Oxford, 1985.

[4] S. Dolfi, G. Navarro, P.H. Tiep, *Primes dividing the degrees of the real characters*, Math. Z. 259 (2008), 755–774.

[5] L. Dornhoff, *Groups representations theory, Part A*, Dekker, New York, 1971.

[6] B. Huppert, *Endliche Gruppen I*, Springer, Berlin, 1983.

[7] B. Huppert, O. Manz, *Degree-problems I. Squarefree character degrees*, Arch. Math. (Basel) 45 (1985), 125–132.

[8] B. Hartley, A. Turull, *On characters of coprime operator groups and the Glauberman character correspondence*, J. Reine Angew. Math. 451 (1994), 175–219.

[9] I.M. Isaacs, *Character theory of finite groups*, Dover, New York, 1976.

[10] O. Manz, T.R. Wolf, *Representations of solvable groups*, Cambridge University Press, Cambridge, 1993.

[11] G. Navarro, L. Sanus, P.H. Tiep, *Groups with two real Brauer characters*, J. Algebra 307 (2007), 891–898.

[12] G. Navarro, P.H. Tiep, *Rational irreducible characters and rational conjugacy classes in finite groups*, Trans. Amer. Math. Soc. 360 (5) (2008), 2443–2465.

[13] G. Navarro, L. Sanus, P.H. Tiep, *Real characters and degrees*, to appear in Israel J. Math..

[14] T. Noritzsch, *Groups having three complex irreducible character degrees*, J. Algebra 175 (1995), 767–798.

[15] P. Schmid, *Extending the Steinberg Representation*, J. Algebra 150 (1992), 254–256.

[16] R. Steinberg, *The representations of* $\mathrm{GL}(3,q)$, $\mathrm{GL}(4,q)$, $\mathrm{PGL}(3,q)$, *and* $\mathrm{PGL}(4,q)$, Canad. J. Math. 3 (1951), 225–235.

Silvio Dolfi, Dipartimento di Matematica U. Dini,
Università degli Studi di Firenze, viale Morgagni 67/a, 50134 Firenze, Italy.
E-mail address: dolfi@math.unifi.it

Emanuele Pacifici, Dipartimento di Matematica F. Enriques,
Università degli Studi di Milano, via Saldini 50, 20133 Milano, Italy.
E-mail address: pacifici@mat.unimi.it

Lucia Sanus, Departament d'Àlgebra, Facultat de Matemàtiques,
Universitat de València, 46100 Burjassot, València, Spain.
E-mail address: lucia.sanus@uv.es