Overconvergent Eichler-Shimura isomorphisms

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1 Introduction

We start by fixing a prime integer $p > 2$, a complete discrete valuation field K of characteristic 0, ring of integers $\mathcal{O}_K$ and residue field $\mathbb{F}$, a perfect field of characteristic p.

Let us first recall the classical Eichler-Shimura isomorphism. We fix $N \geq 3$ an integer not divisible by p and let $\Gamma := \Gamma_1(N) \cap \Gamma_0(p) \subset \mathbf{SL}_2(\mathbb{Z})$. Let us denote by $X := X(N, p)$ the modular curve over $\mathrm{Spec}(\mathbb{Z}[1/(Np)])$ which classifies generalized elliptic curves with Γ-level structure, $E \to X$ the universal semi-abelian scheme and $\omega := \omega_{E/X} = e^*(\Omega^1_{E/X})$ the invertible sheaf on X of invariant 1-differentials, where $e : X \to E$ denotes the zero-section. With these notations we have

Theorem 1.1 (see [D]). *For every $k \in \mathbb{Z}$, $k \geq 0$ we have a natural isomorphism compatible with the action of Hecke operators*

$$\mathrm{H}^1\big(\Gamma, V_{k,\mathbb{C}}\big) \cong \mathrm{H}^0\big(X_{\mathbb{C}}, \omega^{k+2}\big) \oplus \overline{\mathrm{H}^0\big(X_{\mathbb{C}}, \omega^k \otimes \Omega^1_{X/\mathbb{C}}\big)},$$

where $V_{k,\mathbb{C}}$ is the natural Γ-representation $V_{k,\mathbb{C}} := \mathrm{Sym}^k(\mathbb{C}^2)$ and the overline on the second term on the right means "complex conjugation".

The elements of $\mathrm{H}^1\big(\Gamma, V_{k,\mathbb{C}}\big)$ are called (classical) weight k *modular symbols* while the elements appearing on the right hand side of the Eichler-Shimura isomorphism are (classical) *modular, respectively cusp forms* of weight $k + 2$.

There is a more arithmetic version of the above theorem, which we will also call a classical Eichler-Shimura isomorphism. Namely let us consider now the modular curve X over the p-adic field K and for $k \geq 0$ an integer, we let $V_k := \mathrm{Sym}^k(\mathbb{Q}_p^2)$ with its natural action of Γ. Then $\mathrm{H}^1\big(\Gamma, V_k(1)\big)$ can be seen as an étale cohomology group over $Y_{\overline{K}} := (X - \{\mathrm{cusps}\})_{\overline{K}}$ (see section §5 for more details), this $\mathbb{Q}_p$-vector space is endowed both with a natural action of the Galois group $G_K := \mathrm{Gal}(\overline{K}/K)$ and a commuting action of the Hecke operators.

Theorem 1.2 ([F1]). *We have a natural, G_K and Hecke equivariant isomorphism*

$$\mathfrak{F}_k : \mathrm{H}^1\big(\Gamma, V_k(1)\big) \otimes_K \mathbb{C}_p \cong \Big(\mathrm{H}^0\big(X, \omega^{k+2}\big) \otimes_K \mathbb{C}_p\Big) \oplus \Big(\mathrm{H}^1\big(X, \omega^{-k}\big) \otimes_K \mathbb{C}_p(k+1)\Big),$$

where $\mathbb{C}_p$ is the p-adic completion of $\overline{K}$ and the symbols (1) on the left hand side, respectively $(k+1)$ on the right hand side of the isomorphism denote Tate twists.

In this article we are mainly concerned with the p-adic variation of modular forms and modular symbols, and in fact with the relationship between these two variations.

Let us recall that both modular forms and modular symbols have very interesting p-adic properties, more precisely for (classical) weights k, k' congruent modulo $(p-1)p^{r-1}$, $r \geq 1$ an integer, we have natural congruences modulo p^r between certain modular symbols of weights k, k' as well as between certain modular forms of weight $k+2$, $k'+2$. This suggested (in the late 1980's) that they could be p-adically interpolated, i.e., could be organized in p-adic analytic families of modular symbols, respectively modular forms (at least the finite slope ones).

In the present article we would like to interpolate p-adically Faltings' isomorphisms $\mathfrak{F}_k$, i.e., construct p-adic families of such isomorphisms connecting the p-adic families of modular symbols to p-adic families of modular forms (with $\mathbb{C}_p$-coefficients).

Let us first point out that there is no obvious indication in [F1] that such p-adic families should exist. Such an indication would be if considering the natural integral structures of the vector spaces on both left hand and right hand sides of theorem 1.2, the integral versions $\mathfrak{F}_k^o$, $\mathfrak{F}_{k'}^o$ of $\mathfrak{F}_k$, $\mathfrak{F}_{k'}$ respectively, defined in loc. cit. would be compatible with the congruences.

But let us recall from [F1] that in fact $\mathfrak{F}_k^o$ and $\mathfrak{F}_{k'}^o$ are only defined up to p-power torsion of degree linear in k, respectively k' kernel and cokernel, so the question of the compatibility with congruences doesn't even make sense. One of the reasons the p-power torsion appears in the above mentioned definition is the fact that the Hodge-Tate sequence associated to a family of elliptic curves E/R (see lemma 4.2) is in general not an exact sequence, it is only exact up to p-power torsion. We have noticed in [AIS] (see also lemma 4.3 of this article) that if the family of elliptic curves E/R has a canonical subgroup over $R[1/p]$ and all the sections of that canonical subgroup are defined over $R[1/p]$ then the Hodge-Tate sequence associated to E/R can be "corrected", i.e., it can be functorially modified to become an exact sequence.
Using these corrected Hodge-Tate sequences, after restriction to a strict neighborhood of the infinity component of the ordinary locus in the modular curve X one can find **new integral structures** of all the objects appearing in theorem 1.2 which are now compatible with congruences and therefore can be interpolated. Moreover, as is the case with most objects obtained via p-adic interpolation, this new map is unique.

Let us now be more precise. First of all, the parameter space for the above mentioned p-adic families, denoted $\mathcal{W}$ and called *the weight space*, is the rigid analytic space associated to the complete noetherian semilocal algebra $\Lambda := \mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]$. We set $T_0 := \mathbb{Z}_p^\times \times \mathbb{Z}_p$, seen as a compact subset of $\mathbb{Z}_p^2$, endowed with a natural action of the compact group $\mathbb{Z}_p^\times$ and of the Iwahori subgroup of $\mathbf{GL}_2(\mathbb{Z}_p)$. If $k \in \mathcal{W}(K)$ is a weight, we denote by D_k the K-Banach space of analytic distributions on T_0, homogeneous of degree k for the action of $\mathbb{Z}_p^\times$. Then D_k is a Γ-representation. The same construction can be performed in a slightly more complicated situation: let $U \subset \mathcal{W}$ be a wide open disk defined over K, let Λ_U denote the $\mathcal{O}_K$-algebra of bounded rigid analytic functions on U, let $B_U := \Lambda_U \otimes_{\mathcal{O}_K} K$ and we denote by $k_U\colon \mathbb{Z}_p^\times \longrightarrow \Lambda_U^\times$ the associated universal character. We denote by D_U the B_U-Banach module of B_U-valued compact analytic distributions on T_0, homogeneous of degree k_U for the action of $\mathbb{Z}_p^\times$. Then D_U is also a Γ-representation and we denote by D_U^o the integral distributions, i.e., the ones with values in Λ_U. See section §3 for more details.

Of course if $k \in U(K)$ the two Γ-representations above are connected by a Γ-equivariant specialization map $D_U \to D_k$. We say that the classes in $\mathrm{H}^1\big(\Gamma, D_k(1)\big)$ are overconvergent modular symbols and the ones in $\mathrm{H}^1\big(\Gamma, D_U(1)\big)$ are p-adic families of overconvergent modular symbols.

We would like to point out that we have introduced a small modification to the usual way p-adic families of modular symbols are defined, namely we have used a wide open disk instead of an affinoid as parameter space for the weights of our family. As a result the (integral) family of modular symbols $\mathrm{H}^1\big(\Gamma, D_U^o(1)\big)$ is a Λ_U-module. Without wishing to be pedantic, we'd like to stress that this small modification is essential for the following interpretation: the integral distribution module D_U^o has a natural filtration $\{\mathrm{Fil}^i(D_U^o)\}_{i\geq 0}$ which is Γ-invariant and whose graded quotients are artinian $\mathcal{O}_K$-modules (see section §3 for more details). This allows one to identify naturally the p-adic family of modular symbols over U with the continuous cohomology group

$$\mathrm{H}^1_{\mathrm{cont}}\Big(\Gamma, \big(D^o_U/\mathrm{Fil}^i(D^o_U)(1)\big)_{i\geq 0}\otimes K\Big),$$

which then can be identified by GAGA with the étale cohomology group on $Y_{\overline{\mathbb{Q}}}$ of the associated ind-continuous étale sheaf.

Due to this identification $\mathrm{H}^1\big(\Gamma, D_U(1)\big)$ has a natural $G_{\mathbb{Q}}$-action and at the same time the completely continuous action of U_p allows finite slope decompositions (to the expense of maybe shrinking U).

Following-up a remark of the referee of this article, if V is an affinoid sub-domain of $\mathcal{W}$ then using results in the literature (for example the fact that all the reduced eigencurves for $\mathbf{GL}_{2/\mathbb{Q}}$ known to man are isomorphic) one can see that there is a natural $G_{\mathbb{Q}}$-action on $\mathrm{H}^1\big(\Gamma, D_V\big)$ but, unless $V=\{k\}$, we do not know how to show that this Galois representation is the one associated to an étale cohomology group on $Y_{\overline{\mathbb{Q}}}$ and we suspect that this is not always the case. Moreover, the above mentioned isomorphism of all eigevarieties is not known in the higher dimensional case while our method seems to work at least for all PEL Shimura varieties.

We now continue and introduce the other main actors in this drama, namely the overconvergent and p-adic families of modular forms. For each $w\in\mathbb{Q}$ such that $0<w<p/(p+1)$ we denote by $X(w)$ the strict neighborhood of the component containing the cusp ∞ of the ordinary locus of width p^w in the rigid analytic curve $(X_{/K})^{\mathrm{an}}$ (see section §2 for more details). For every $k\in\mathcal{W}(K)$, in [AIS] we have shown that there exist a w as above and an invertible, modular sheaf $\omega_w^{\dagger,k}$ on $X(w)$ such that if $k\in\mathbb{Z}$ then $\omega_w^{\dagger,k}\cong\omega^k|_{X(w)}$. We call the elements of $\mathrm{H}^0\big(X(w),\omega_w^{\dagger,k}\big)$ *overconvergent modular forms of weight* k (and radius of overconvergence w). In [AIS] it is shown that after taking the limit for $w\to 0$ we obtain precisely the Hecke module of overconvergent modular forms of weight k introduced by Robert Coleman [Co]. Similarly, if $U\subset\mathcal{W}$ is a wide open disk and k_U its universal weight, there is a w and a modular sheaf of B_U-Banach modules $\omega_w^{\dagger,k_U}$ such that the elements of $\mathrm{H}^0\big(X(w),\omega_w^{\dagger,k_U}\big)$ are p-adic families of overconvergent modular forms over U.

Here are the main result of this article.

Fix $U\subset\mathcal{W}^*$ a wide open disk defined over K of so called *accessible weights*, namely of weights k such that $|k(t)^{p-1}-1|<p^{-1/(p-1}$. Let $w\in\mathbb{Q}$, $0<w<p/(p+1)$ be such that $\omega_w^{\dagger,k_U+2}$ is defined over $X(w)$. We define a **geometric** $(B_U\hat{\otimes}\mathbb{C}_p)$-linear homomorphism

$$\Psi_U\colon \mathrm{H}^1\big(\Gamma, D_U\big)\hat{\otimes}_K\mathbb{C}_p(1)\longrightarrow \mathrm{H}^0\big(X(w),\omega_w^{\dagger,k_U+2}\big)\hat{\otimes}_K\mathbb{C}_p,$$

which is equivariant for the action of the Galois group $G_K=\mathrm{Gal}(\overline{K}/K)$ and the action of the Hecke operators T_ℓ, for ℓ not dividing pN and U_ℓ for ℓ dividing pN and most importantly it is compatible with **specializations**. In other words, Ψ_U interpolates (Faltings' isomorphisms $\mathfrak{F}_k$ composed with the projection on the H^0-component), for classical weights $k\in U$. It follows that Ψ_U is unique with this property, for a more precise statement see remark 2 of section §6.1.

Let now $h\geq 0$ be an integer. We suppose that U is such that both $\mathrm{H}^1\big(\Gamma, D_U\big)$ and $\mathrm{H}^0\big(X(w),\omega_w^{\dagger,k_U}\big)$ have slope $\leq h$ decompositions and that there is an integer $k_0>h-1$ such that $k_0\in U(K)$.

If N is a $B_U[U_p]$-module which has a slope $\le h$-decomposition we denote by $N^{(h)}$ the slope $\le h$ submodule of N. All these being said, Ψ_U induces a morphism on slope $\le h$ parts:

$$\Psi_U^{(h)}\colon \mathrm{H}^1\big(\Gamma, D_U\big)^{(h)}\hat{\otimes}_K\mathbb{C}_p(1) \longrightarrow \mathrm{H}^0\big(X(w), \omega_w^{\dagger,k_U}\big)^{(h)}\hat{\otimes}_K\mathbb{C}_p.$$

We prove the following:

Theorem 1.3. *There is a finite subset of weights $Z \subset U(\mathbb{C}_p)$ such that:*

a) For each $k \in U(K) - Z$ there exists a finite dimensional K-vector space $S_{k+2}^{(h)}$ on which the Hecke operators T_ℓ for $(\ell, Np) = 1$ and U_ℓ for ℓ dividing pN act, U_p acts with slope $\le h$ and such that we have natural, G_K and Hecke-equivariant isomorphisms

$$\mathrm{H}^1\big(\Gamma, D_k\big)^{(h)} \otimes_K \mathbb{C}_p(1) \cong \Big(\mathrm{H}^0\big(X(w), \omega_w^{\dagger,k+2}\big)^{(h)} \otimes_K \mathbb{C}_p\Big) \oplus \Big(S_{k+2}^{(h)} \otimes_K \mathbb{C}_p(k+1)\Big).$$

Here the projection of $\mathrm{H}^1\big(\Gamma, D_k\big)^{(h)} \otimes_K \mathbb{C}_p(1)$ onto $\mathrm{H}^0\big(X(w), \omega_w^{\dagger,k+2}\big)^{(h)} \otimes_K \mathbb{C}_p$ is determined by the geometric morphism $\Psi_U^{(h)}$ above.

Moreover, the characteristic polynomial of T_ℓ for ℓ not dividing Np and that of U_ℓ for ℓ dividing Np acting on $S_{k+2}^{(h)}$ is equal to the characteristic polynomial of T_ℓ, respectively U_ℓ acting on the space of overconvergent cusp forms of weight $k+2$ and slope less or equal to h, $\mathrm{H}^0\big(X(w), \omega_w^{\dagger,k} \otimes \Omega^1_{X(w)/K}\big)^{(h)}$.

b) We have a family version of a) above: for every wide open disk $V \subset U$ defined over K such that $V(\mathbb{C}_p) \cap Z = \phi$, there is a finite free B_V-module $S_V^{(h)}$ on which the Hecke operators T_ℓ (for ℓ not dividing pN) and U_ℓ (for ℓ dividing pN) act, U_p acts with slope less or equal to h, and we have a natural isomorphism G_K and Hecke equivariant

$$\mathrm{H}^1\big(\Gamma, D_V^{(h)}\big)\hat{\otimes}_K\mathbb{C}_p(1) \cong \Big(\mathrm{H}^0\big(X(w), \omega_w^{\dagger,k_V+2}\big)^{(h)}\hat{\otimes}_K\mathbb{C}_p\Big) \oplus \Big(S_V\hat{\otimes}_K\mathbb{C}_p(\chi_V^{\mathrm{univ}} \cdot \chi)\Big),$$

where $\chi_V^{\mathrm{univ}}\colon G_K \longrightarrow \Lambda_V^\times$ is the universal cyclotomic character of V. As at a), the first projection is determined by the geometric map $\Psi_U^{(h)}$.

c) If V is as at b) above let $k \in V(K)$ and let us denote by t_k a uniformizer of B_V at k. Then we have natural isomorphisms as Hecke modules

$$S_V^{(h)}/t_k S_V^{(h)} \cong S_{k+2}^{(h)},$$

where $S_{k+2}^{(h)}$ is the Hecke module appearing at a).

Let us remark that the set of "bad weights" denoted Z in theorem 1.3 contains at least the classical weights (k, χ) in U such that $0 \le k \le h-1$.

The theorem above has as immediate consequence the following geometric interpretation of the global Galois representations attached to generic overconvergent cuspidal eigenforms of finite slope. Let U, h, Z be as in theorem 1.3 and $k \in U(K) - Z$. Let f be an overconvergent cuspidal eigenform of weight k and slope $\le h$, in other words f is an eigenform for all the Hecke operators T_ℓ for ℓ not dividing Np and for U_ℓ for ℓ dividing Np. We denote by K_f the finite extension of K generated by all the Hecke eigenvalues of f.

Theorem 1.4. *The $G_{\mathbb{Q}}$-representation $\mathrm{H}^1\big(\Gamma, D_k(1)\big)_f^{(h)}$ is a two dimensional K_f-vector space and it is isomorphic to the p-adic $G_{\mathbb{Q}}$-representation attached to f by the theory of pseudo-representations.*

Remark 1.5. As we remarked above, the result of theorem 1.4 could be also deduced by the fact that the reduced eigencurve defined using modular symbols and the one defined by the completed cohomology of a certain tower of modular curves are isomorphic. The interest in giving a new proof consists in the fact that that in higher dimensions our proof seems to work while the eigenvariety defined using the completed cohomology of towers of Shimura varieties is not yet known to have the expected dimension.

The main difficulty in proving these theorems is the definition of the geometric map $\Psi_U^{(h)}$ having all the required properties. We see it as a map comparing a p-adic étale cohomology group, $\mathrm{H}^1\big(\Gamma, D_U(1)\big)$ with a differential object, namely $\mathrm{H}^0\big(X(w), \omega_w^{\dagger, k_U+2}\big)$. We obtain it as a Hodge-Tate comparison map except that on the one hand the étale cohomology group is global (on X) while the differential object only lives on the affinoid $X(w)$. Moreover, to make things worse the étale sheaf associated to the Γ-representation D_U is **not** a Hodge-Tate sheaf, i.e., it does not define (locally on X) Hodge-Tate representations in the sense of [Hy]. Its cohomology is not a Hodge-Tate G_K-representation!

Let us explain the main new ideas in this article. We denote by $\mathfrak{X}(N,p)$ and $\mathfrak{X}(w)$ the log Faltings' sites associated to certain log formal models of X and respectively $X(w)$. There is a continuous functor $\nu\colon \mathfrak{X}(N,p) \longrightarrow \mathfrak{X}(w)$ which allows to move sheaves from one site to another.

Let $\mathcal{D}_U$ denote the ind-continuous étale sheaf associated to D_U, it can be seen as sheaf on $\mathfrak{X}(N,p)$, then $\nu^*(\mathcal{D}_U)$ is a sheaf on $\mathfrak{X}(w)$. At this point something remarkable happens, namely using the Hodge-Tate sequence one can construct a natural $\widehat{\mathcal{O}}_{\mathfrak{X}(w)}[1/p]$-linear morphism of shaves on $\mathfrak{X}(w)$:

$$\delta_k^\vee(w)\colon \nu^*(\mathcal{D}_U)\hat{\otimes}\widehat{\mathcal{O}}_{\mathfrak{X}(w)} \longrightarrow \omega_w^{\dagger, k_U+2}\hat{\otimes}\widehat{\mathcal{O}}_{\mathfrak{X}(w)}.$$

This fact allows us to define the map $\Psi_U^{(h)}$ as the composition

$$\mathrm{H}^1\big(\Gamma, D_U\big)\hat{\otimes}_K\mathbb{C}_p(1) \longrightarrow \mathrm{H}^1\big(\mathfrak{X}(N,p), \mathcal{D}_U\hat{\otimes}\widehat{\mathcal{O}}_{\mathfrak{X}(N,p)}(1)\big) \longrightarrow \mathrm{H}^1\big(\mathfrak{X}(w), \nu^*(\mathcal{D}_U)\hat{\otimes}\widehat{\mathcal{O}}_{\mathfrak{X}(w)}(1)\big) \longrightarrow$$

$$\longrightarrow \mathrm{H}^1\big(\mathfrak{X}(w), \omega_w^{\dagger, k_U}\hat{\otimes}\widehat{\mathcal{O}}_{\mathfrak{X}(w)}(1)\big) \longrightarrow \mathrm{H}^0\big(X(w), \omega_w^{\dagger, k_U+2}\big)\hat{\otimes}\mathbb{C}_p.$$

The theory developed in [AIS] plays a crucial role in the definition of the maps $\delta_k^\vee(w)$. It was in fact the search for such maps which lead us to discover the modular sheaves $\omega_w^{\dagger, k}$.

Our finding was that the theory of the canonical subgroup can be used in order to provide a new integral structure on the sheaf of invariant differentials of the universal generalized elliptic curve making the Hodge-Tate sequence exact integrally (namely without inverting p!). With this accomplished, a definition à la Katz provides the sought for sheaves $\omega_w^{\dagger, k}$ for any k and the maps $\delta_k^\vee(w)$. We believe that this application to the problem of making the Hodge-Tate sequence integrally exact constitutes the essence of the theory of the canonical subgroup. This intimate relation with the existence of the canonical subgroup should justify the fact that $\delta_k^\vee(w)$ can be defined **only** over $\mathfrak{X}(w)$ and not over the whole $\mathfrak{X}(N,p)$.

We believe that the ideas and techniques presented here could be applied without much change in other settings in order to prove overconvergent Eichler-Shimura isomorphisms (for example for Shimura curves, for Hilbert modular varieties etc.). We realized that a very convenient concept to use in order to define Hecke operators on Faltings' cohomology groups is that of localized (or induced) topos. We made a careful study of various localized logarithmic Faltings' sites and showed that trace maps can be defined (see section §2.4). This allows us to work in situations where we do not have explicit descriptions of good integral models of the curves involved (for example $X_1(Np^r)$ for $r > 1$) and would allow extensions of these results to higher dimensional Shimura varieties.

Let us finally remark that it would be possible to give geometric interpretations both of the Hecke modules $S_V^{(h)}$ and of the maps $\mathrm{H}^1\big(\Gamma, D_V\big)^{(h)}\hat{\otimes}\mathbb{C}_p(1) \longrightarrow S_V^{(h)}\hat{\otimes}\mathbb{C}_p(\chi_V^{\mathrm{univ}} \cdot \chi)$ appearing in theorem 1.3 and we propose to write these in a future article.

Notations In what follows we will denote by calligraphic letters $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \ldots$ log formal schemes over $\mathcal{O}_K$ and by $\underline{\mathcal{X}}$, $\underline{\mathcal{Y}}$, $\underline{\mathcal{Z}}$ respectively the formal schemes underlying $\mathcal{X}$ respectively $\mathcal{Y}$, respectively $\mathcal{Z}$. We will denote by X, Y, Z, ... respectively the log rigid analytic generic fibers of $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$, ... and by $\underline{X}$, $\underline{Y}$, $\underline{Z}$, ... respectively the underlying rigid spaces.

2 Faltings' topoi

2.1 The geometric set-up.

Let $p > 2$ be a prime integer, K a complete discrete valuation field of characteristic 0 and perfect residue field $\mathbb{F}$ of characteristic p and $N \geq 3$ a positive integer not divisible by p. We fix once for all an algebraic closure $\overline{K}$ of K and an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{K}$, where $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. We denote by $\mathbb{C}_p$ the completion of $\overline{K}$ and by G_K the Galois group of $\overline{K}$ over K. We denote by v the valuation on $\mathbb{C}_p$, normalized such that $v(p) = 1$.

Now we'd like to recall the basic geometric set-up from [AIS]. Let $w \in \mathbb{Q}$ be such that $0 \leq w \leq p/(p+1)$ and let us suppose that there is an element (which will be denoted p^v) in K whose valuation is $v := w/(p-1)$. We fix an integer $r \geq 1$ and we suppose that $w < 2/(p^r - 1)$ if $p > 3$ and $w < 1/3^r$ if $p = 3$. The assumptions on p being odd and on the bounds for w will be fundamental in the sequel (see §4.1 for example) and are related to the existence of the canonical subgroup.

We consider the following tower of rigid analytic modular curves over K (in this section there are no log structures):

$$X_1(Np^r) \longrightarrow X(N, p^r) \longrightarrow X_1(N),$$

where $X_1(Np^r)$, respectively $X_1(N)$ classify generalized elliptic curves with $\Gamma_1(Np^r)$ respectively $\Gamma_1(N)$)-level structure, while $X(N, p^r)$ classifies generalized elliptic curves with $\Gamma_1(N) \cap \Gamma_0(p^r)$-level structure. The morphism $X(N, p^r) \longrightarrow X_1(N)$ is the one which forgets the $\Gamma_0(p^r)$-level structure.

We denote by Ha a lift of the Hasse invariant (for example $\mathrm{Ha} = E_{p-1}$, the normalized Eisenstein series of level 1 and weight $p-1$, if $p > 3$) which we view as a modular form on $X_1(N)$. We define the rigid analytic space

$$X(w) := \{x \in X_1(N) \quad | \quad |\mathrm{Ha}(x)| \geq p^{-w}\} \subset X_1(N),$$

and remark that the morphism $X(N,p^r) \longrightarrow X_1(N)$ has a canonical section over $X(w)$ whose image we also denote by $X(w)$). We define $X(p^r)(w) := X_1(Np^r) \times_{X_1(N,p^r)} X(w)$ and view $X(w)$ (respectively $X(p^r)(w)$) as a connected affinoid subdomain of $X_1(N)$ and via the above mentioned section of $X(N,p^r)$ (respectively of $X_1(Np^r)$).

We denote by $\mathcal{X}_1(N)$, $\mathcal{X}(N,p^r)$ and $\mathcal{X}_1(Np^r)$ the p-adic formal schemes over $\mathcal{O}_K$ obtained by completing the proper schemes over $\mathcal{O}_K$ classifying generalized elliptic curves with $\Gamma_1(N)$, respectively $\Gamma_1(N)\cap\Gamma_0(p^r)$, respectively $\Gamma_1(Np^r)$-level structures along, respectively, their special fibers. Let $\mathcal{X}(w)$ denote the open formal sub-scheme of the formal blow-up of $\mathcal{X}_1(N)$ defined by the ideal sheaf of $\mathcal{O}_{\mathcal{X}_1(N)}$ generated by the sections p^w and $\mathrm{Ha}(\mathcal{E}/\mathcal{X}_1(N),\omega)$ which is the complement of the section at ∞ of the exceptional divisor of the blowing-up. Here $\mathcal{E} \longrightarrow \mathcal{X}_1(N)$ is the universal generalized elliptic curve and ω is a global invariant 1-differential form of $\mathcal{E}$ over $\mathcal{X}_1(N)$. Finally we let $\mathcal{X}(p^r)(w)$ denote the normalization of $\mathcal{X}(w)$ in $X_1(Np)(w)$ (see §3 of [AIS] for more details).

Let us remark that we have constructed a natural commutative diagram of formal schemes, rigid analytic spaces and morphisms which is our basic geometric setup:

$$\begin{array}{ccccc} \mathcal{X}(p^r)(w) & \longrightarrow & \mathcal{X}(w) & = & \mathcal{X}(w) \\ u\uparrow & & u\uparrow & & u\uparrow \\ X(p^r)(w) & \longrightarrow & X(w) & = & X(w) \\ \cap & & \cap & & \cap \\ X_1(Np^r) & \longrightarrow & X(N,p^r) & \longrightarrow & X_1(N) \end{array}$$

In the above diagram u denotes the various specialization (or reduction) morphisms.

Finally, we have the following basic commutative diagram of formal schemes and rigid spaces:

$$(*) \quad \begin{array}{ccc} \mathcal{X}(w) & \stackrel{\nu}{\longrightarrow} & \mathcal{X}(N,p) \\ u\uparrow & & u\uparrow \\ X(w) & \subset & X(N,p) \end{array}$$

2.2 Log Structures

In this section we will describe log structures on the formal schemes and rigid spaces appearing in the commutative diagram $(*)$ in section §2.1.

Let us now fix N, r and w as above and denote by $\underline{\mathcal{X}}(w), \underline{\mathcal{X}}(N,p)$ the formal schemes denoted $\mathcal{X}(w), \mathcal{X}(N,p)$ in section 2.1. We denote by π a fixed uniformizer of K. By its definition, if $\mathcal{U} = \mathrm{Spf}(R_U) \hookrightarrow \underline{\mathcal{X}}(w)$ is an affine open then $\mathcal{U}$ is either smooth over $\mathcal{O}_K$ or if $\mathcal{U}$ contains a supersingular point, then there is $a \in \mathbb{N}$, which depends only on K and w, and a formally étale morphism $\mathcal{U} \longrightarrow \mathrm{Spf}(R')$, where $R' := \mathcal{O}_K\{X,Y\}/(XY-\pi^a)$.

Let us consider on $\underline{S} := \mathrm{Spf}(\mathcal{O}_K)$ the log structure M given by the closed point and let us denote by $S := (\underline{S}, M)$ the associated log formal scheme. Let us recall that it has a local chart given by $\mathbb{N} \longrightarrow \mathcal{O}_K$ sending $1 \to \pi$.

There exists a fine and saturated log structure $N_{\mathcal{X}}$ on $\underline{\mathcal{X}}(w)$, with a morphism of log formal schemes $f\colon \mathcal{X}(w) := (\underline{\mathcal{X}}(w), N) \longrightarrow S = (\underline{S}, M)$ which can be described locally as follows. Let $\mathcal{U} = \mathrm{Spf}(R)$ be an open affine of $\underline{\mathcal{X}}$ as above, then

i) if $\mathcal{U}$ is smooth over S let us denote by $\Sigma_{\mathcal{U}}$ the divisor of cusps of $\mathcal{U}$. Then $N|_{\mathcal{U}}$ is the log structure associated to the divisor $\Sigma_{\mathcal{U}}$. It has a local chart of the form $\mathbb{N} \longrightarrow R$ sending 1 to a uniformizer at all the cusps.

ii) if $\mathcal{U}$ is not smooth over $\underline{S}$ we'll suppose that it does not contain cusps, let R' be as above and let us denote $\Psi_R\colon R' \longrightarrow R_{\mathcal{U}}$ the étale morphism of $\mathcal{O}_K$-algebras defined above.

Let us consider the following commutative diagram of monoids and morphisms of monoids (see [AI], §2.1).

$$\begin{array}{ccc} \mathbb{N}^2 & \xrightarrow{\psi_R} & R' \\ \Delta\uparrow & & \uparrow \\ \mathbb{N} & \xrightarrow{\psi_a} & \mathcal{O}_K \end{array}$$

where $\psi_R(m,n) = X^m Y^n$, $\psi_a(n) = \pi^{an}$ and $\Delta(n) = (n,n)$ for all $n, m \in \mathbb{N}$.

Then the above diagram induces a natural isomorphism of $\mathcal{O}_K$-algebras

$$R' \cong \mathcal{O}_K\{\mathbb{N}^2\} \otimes_{\mathcal{O}_K\{\mathbb{N}\}} \mathcal{O}_K.$$

Let P denote the amalgamated sum (or co-fibered product), $P := \mathbb{N}^2 \oplus_{\mathbb{N}} \mathbb{N}$ associated to the diagram of monoids

$$\begin{array}{ccc} & & \mathbb{N} \\ & & \uparrow \\ \mathbb{N}^2 & \xleftarrow{\Delta} & \mathbb{N} \end{array}$$

where the vertical morphism sends $n \to an$. By functoriality we obtain a canonical morphism of monoids $P \longrightarrow R' \xrightarrow{\Psi_R} R_{\mathcal{U}}$ which defines a local chart of $\mathcal{X}$, i.e., $N_{\mathcal{X}}|_{\mathcal{U}}$ is the log structure associated to the pre log structure $P \longrightarrow R_{\mathcal{U}}$. Let us consider the natural diagram of monoids which defines P as the amalgamated sum $\mathbb{N}^2 \oplus_{\mathbb{N}} \mathbb{N}$

$$\begin{array}{ccc} P & \xleftarrow{h} & \mathbb{N} \\ \uparrow & & \uparrow \\ \mathbb{N}^2 & \xleftarrow{\Delta} & \mathbb{N} \end{array}$$

The morphism $h\colon \mathbb{N} \longrightarrow P$ defined by the above diagram is a local chart of the morphism $f\colon \mathcal{X} \longrightarrow (S, M)$.

Lemma 2.1. *The morphism $f\colon \mathcal{X}(w) \longrightarrow S$ is log smooth.*

Proof. Given the local description of $\mathcal{X}(w) = (\underline{\mathcal{X}}(w), N_{\mathcal{X}})$, $S = (\underline{S}, M)$ and f in terms of charts, it is enough to consider the case ii) above, i.e., we have a local chart $P \longrightarrow R' \xrightarrow{\Psi_R} R_{\mathcal{U}}$ and Ψ_R is étale. By the description in [Ka1] §5 of log smooth morphisms, it is enough the show that the morphism h is injective and that the order of torsion of the group $P^{\mathrm{gp}}/h(\mathbb{N}^{\mathrm{gp}})$ is invertible in $R_{\mathcal{U}}$. For this it would be useful to have an explicit description of P as amalgamated sum of monoids (see also [AI] §2.1).

Let us define the sequence of monoids

$$\frac{1}{a}\Delta(\mathbb{N}) + N^2 \subset \frac{1}{a}\mathbb{N}^2 \subset \mathbb{Q}^2$$

as: $\frac{1}{a}\mathbb{N}^2$ is the (additive) submonoid of $\mathbb{Q}^2$ of pairs of rational numbers $\big(\frac{n}{a}, \frac{m}{a}\big)$ with $n, m \in \mathbb{N}$ and $\frac{1}{a}\Delta(\mathbb{N}) + \mathbb{N}^2$ is its submonoid of pairs of rational numbers of the form $\big(\frac{n}{a} + \alpha, \frac{n}{a} + \beta\big)$, where $n, \alpha, \beta \in \mathbb{N}$.

We have natural morphisms of monoids $\mathbb{N}^2 \longrightarrow \frac{1}{a}\Delta(\mathbb{N}) + \mathbb{N}^2$ sending $(\alpha, \beta) \to (\alpha, \beta)$ and $h' \colon \mathbb{N} \longrightarrow \frac{1}{a}\Delta(\mathbb{N}) + \mathbb{N}^2$ given by $n \to \big(\frac{n}{a}, \frac{n}{a}\big)$ such that the diagram is commutative

$$\begin{array}{ccc} \frac{1}{a}\Delta + \mathbb{N}^2 & \stackrel{h'}{\longleftarrow} & \mathbb{N} \\ \uparrow & & \uparrow \\ \mathbb{N}^2 & \stackrel{\Delta}{\longleftarrow} & \mathbb{N} \end{array}$$

It is then easy to see that $P = \mathbb{N}^2 \oplus_{\mathbb{N}} \mathbb{N} \cong \frac{1}{a}\Delta(\mathbb{N}) + \mathbb{N}^2$ by verifying that the latter monoid satisfies the universal properties of the co-fibered product. It follows that the above chart on R' is explicitly given by $\frac{1}{a}\Delta(\mathbb{N}) + \mathbb{N}^2 \longrightarrow R'$ where $\big(\frac{n}{a} + \alpha, \frac{n}{a} + \beta\big) \longrightarrow X^\alpha Y^\beta \pi^n$. Of course one has to first verify that the association is well defined, which it is. One sees immediately that the monoid P is fine and saturated (as claimed at the beginning of this section) and moreover that the morphism $h \colon \mathbb{N} \longrightarrow P$ is under the identifications between P and $\frac{1}{a}\Delta(\mathbb{N}) + \mathbb{N}^2$ equal h', therefore it is injective and moreover it follows that the quotient group $P^{\rm gp}/h(\mathbb{Z})$ is torsion free. This proves the lemma. □

Recall that in the previous section we defined a morphism of formal schemes $\mathcal{X}(p^r)(w) \longrightarrow \mathcal{X}(w)$. We denote by $\underline{\mathcal{X}}^{(r)}(w) := \mathcal{X}(p^r)(w)$, we let also $N_r \longrightarrow \mathcal{O}_{\underline{\mathcal{X}}^{(r)}(w)}$ denote the inverse image log structure via the above morphism and denote by $\mathcal{X}^{(r)}(w) := \big(\underline{\mathcal{X}}^{(r)}(w), N_r\big)$ the associated log formal scheme. We also denote by $\underline{X}(w) := \mathcal{X}(w)^{\rm rig} = X(w), \underline{X}^{(r)}(w) := \mathcal{X}(p^r)(w)^{\rm rig} = X(p^r)(w)$ the rigid analytic generic fibers of the two formal schemes. We recall that $\underline{X}^{(r)}(w) \longrightarrow \underline{X}(w)$ is a finite, étale, Galois morphism with Galois group $G_r := (\mathbb{Z}/p^r\mathbb{Z})^\times$ and as $\underline{\mathcal{X}}^{(r)}(w)$ is the normalization of $\underline{\mathcal{X}}(w)$ in $\underline{X}^{(r)}(w)$, we have that G_r acts without fixed points on $\underline{\mathcal{X}}^{(r)}(w)$ and $\underline{\mathcal{X}}^{(r)}(w)/G_r \cong \underline{\mathcal{X}}(w)$. In what follows we denote by $X(w)$ and $X^{(r)}(w)$ the log rigid analytic generic fibers of the log formal schemes $\mathcal{X}(w)$ and respectively $\mathcal{X}^{(r)}(w)$. Their log structures are the horizontal ones defined by the divisors of cusps.

Moreover, as the formal scheme $\underline{\mathcal{X}}(N,p)$ is semistable, its special fiber is a divisor with normal crossings. We define the log structure on $\underline{\mathcal{X}}(N,p)$ to be the one associated to the divisor consisting in the union of the special fiber and the divisor of cusps and denote by $\mathcal{X}(N,p)$ the corresponding log formal scheme. Moreover we define on $\underline{X}(N,p)$ the log structure associated to the cusps of this modular curve and by $X(N,p)$ the corresponding log rigid space. Let us remark that the diagram $(*)$ of section §2.1, written there for formal schemes and rigid spaces in fact holds for log formal scheme and log rigid spaces and it is commutative.

Corollary 2.2. *The formal scheme $\underline{\mathcal{X}}(w)$ is flat over Spf($\mathcal{O}_K$), it is Cohen-Macaulay and so in particular normal. If $a = 1$ then $\underline{\mathcal{X}}(w)$ is a regular formal scheme.*

Proof. See [AI] §2.1.1 (3), where we show how to reduce to [Ka2] Thm. 4.1. □

In particular, if $\mathcal{U} = \mathrm{Spf}(R_{\mathcal{U}}) \hookrightarrow \underline{\mathcal{X}}$ is an affine open of $\underline{\mathcal{X}}$, then the $\mathcal{O}_K$-algebra $R_{\mathcal{U}}$ satisfies the assumptions (1), (2), (3) (FORM) and (4) of section §2.1 of [AI].

2.3 Faltings' topoi

Our main reference for the constructions in this section is [AI] section §1.2.

We will define Faltings' sites and topoi associated to the pairs of a log formal schemes and log rigid spaces: $\big(\mathcal{X}(w), X(w)\big)$ and respectively $\big(\mathcal{X}(N,p), X(N,p)\big)$ which will be denoted $\mathfrak{X}(w)$ and respectively $\mathfrak{X}(N,p)$.

We start by writing $\big(\mathcal{X}, X\big)$ for any one of the two pairs above. We'll define Faltings' site associated to this pair which we denote by $\mathfrak{X}$. Namely we first let $\mathcal{X}^{\rm ket}$ be the Kummer étale site of $\mathcal{X}$, which is the full sub-category of the category of log schemes $\mathcal{T}$, endowed with a Kummer log étale morphism $\mathcal{T} \longrightarrow \mathcal{X}$ (see [AI] §1.2 or [Il] section §2.1). We recall that the fiber product in this category is the fiber product of log formal schemes in the category of fine and saturated log formal schemes so in particular the underlying formal scheme of the fiber product is not necessarily the fiber product of the underlying formal schemes (see [Ka1]).

If $\mathcal{U}$ is an object in $\mathcal{X}^{\rm ket}$ then we denote by $\mathcal{U}_{\overline{K}}^{\rm fket}$ the finite Kummer étale site attached to $\mathcal{U}$ over $\overline{K}$ as defined in [AI] §1.2.2. An object in this site is a pair (W, L) where L is a finite extension of K contained in $\overline{K}$ and W is an object of the finite Kummer étale site of $\mathcal{U}_L$ which we denote by $\mathcal{U}_L^{\rm fket}$. Given two objects (W', L') and (W, L) of $\mathcal{U}_{\overline{K}}^{\rm fket}$, we define the morphisms in the category as

$$\mathrm{Hom}_{\mathcal{U}_{\overline{K}}^{\rm fket}}\big((W', L'), (W, L)\big) := \lim_{\rightarrow} \mathrm{Hom}_{L''}\big(W' \times_{L'} L'', W \times_L L''\big)$$

where the limit is over the finite extensions L'' of K contained in $\overline{K}$ which contain both L and L'.

Now, to define $\mathfrak{X}$ we denote by $E_{\mathcal{X}_{\overline{K}}}$ the category such that

i) the objects are pairs $(\mathcal{U}, W)$ such that $\mathcal{U} \in \mathcal{X}^{\rm ket}$ and $W \in \mathcal{U}_{\overline{K}}^{\rm fket}$

ii) a morphism $(\mathcal{U}', W') \longrightarrow (\mathcal{U}, W)$ in $E_{\mathcal{X}(w)_{\overline{K}}}$ is a pair (α, β), where $\alpha\colon \mathcal{U}' \longrightarrow \mathcal{U}$ is a morphism in $\mathcal{U}^{\rm ket}$ and $\beta\colon W' \longrightarrow W \times_{\mathcal{U}_K} \mathcal{U}'_K$ is a morphism in $\big(\mathcal{U}'\big)_{\overline{K}}^{fket}$.

The pair $(\mathcal{X}, X)$ is a final object in $E_{\mathcal{X}_{\overline{K}}}$ and moreover in this category finite projective limits are representable and in particular fiber products exist (see [AI] section §1.2.3 and [Err] proposition 2.6 for an explicit description of the fiber product).

A family of morphisms $\{(\mathcal{U}_i, W_i) \longrightarrow (\mathcal{U}, W)\}_{i\in I}$ is a covering family if either

(α) $\{\mathcal{U}_i \longrightarrow \mathcal{U}\}_{i\in I}$ is a covering family in $\mathcal{X}^{\rm ket}$ and $W_i \cong W \times_{\mathcal{U}_K} \mathcal{U}i, K$ for every $i \in I$

or

(β) the morphism $\mathcal{U}_i \to \mathcal{U}$ is an isomorphism for all $i \in I$ and $\{W_i \longrightarrow W\}_{i\in I}$ defines a covering in $\mathcal{U}_{\overline{K}}^{\rm fket}$.

We endow $E_{\mathcal{X}_{\overline{K}}}$ with the topology generated by the covering families defined above and denote by $\mathfrak{X}$ the associated site.

Finally, the basic commutative diagram of log formal schemes and log rigid spaces

$$(*) \quad \begin{array}{ccc} \mathcal{X}(w) & \stackrel{\nu}{\longrightarrow} & \mathcal{X}(N,p) \\ u\uparrow & & u\uparrow \\ X(w) & \subset & X(N,p) \end{array}$$

defines a natural functor $\nu\colon \mathfrak{X}(N,p) \longrightarrow \mathfrak{X}(w)$ defined, say on objects by: $\nu(\mathcal{U}, W) := \big(\mathcal{U} \times_{\mathcal{X}(N,p)} \mathcal{X}(w), W \times_{X(N,p)} X(w)\big)$. This functor sends covering families to covering families and final objects to final objects therefore it is a continuous functor of sites.

Remark 2.3. Due to the mild singularities of the special fiber of the formal scheme $\underline{\mathcal{X}}(w)$, the site $\mathfrak{X}(w)$ and the sheaves on it were studied carefully and were well understood in [AI]. In the present paper however we would need to study Faltings' site associated to $\mathcal{X}^{(r)}(w)$, for various r's. Unfortunately we do not understand well enough the geometry of $\underline{\mathcal{X}}^{(r)}(w)$ to be able to work with this site directly, so instead we'll use a trick. Let us observe that $(\mathcal{X}(w), X^{(r)}(w))$ is an object of $E_{\mathcal{X}(w)_{\overline{K}}}$, for all $r \geq 1$ (while $(\mathcal{X}^{(r)}(w), X^{(r)}(w))$ is not) so we define the induced (or localized) site $\mathfrak{X}^{(r)}(w) := \mathfrak{X}(w)_{/\left(\mathcal{X}(w), X^{(r)}(w)\right)}$ and the sheaves on it and this will be our substitute for Faltings' site attached to $\mathcal{X}^{(r)}(w)$. Everything will be defined and explained in the next two sections.

2.4 Generalities on induced topoi

In this section we'll recall some fundamental constructions and results from [SGA4], Exposé IV, §5 (Topos induit), in the restricted generality that we need.

Let E denote a topos, namely the category of sheaves of sets on a site S, whose underlying category will be henceforth denoted C and let X be an object of C. We denote by $C_{/X}$ the category of pairs (Y, u) where Y is an object of C and $u \colon Y \longrightarrow X$ is a morphism in C. A morphism $(Y, u) \longrightarrow (Y', u')$ in $C_{/X}$ is a morphism $\gamma \colon Y \longrightarrow Y'$ in C such that $u' \circ \gamma = u$.

Let $\alpha_X \colon C_{/X} \longrightarrow C$ be the functor forgetting the morphism to X, i.e., for example, on objects it is defined by $\alpha_X(Y, u) = Y$. We endow the category $C_{/X}$ with the topology induced from C via α_X and denote the site thus obtained $S_{/X}$. We denote by $E_{/X}$ the topos of sheaves on $S_{/X}$ and call $S_{/X}$ and $E_{/X}$ the site and respectively the topos induced by X. We have natural functors $\alpha_X^* \colon E_{/X} \longrightarrow E$, $\alpha_{X,*} \colon E \longrightarrow E_{/X}$ such that α_X^* is left adjoint to $\alpha_{X,*}$.

Suppose that C has a final object f and that it has fiber products. Then we have another functor $j_X \colon C \longrightarrow C_{/X}$ defined by $j_X(Z) := \bigl(X \times_f Z, \mathrm{pr}_1\bigr)$, i.e., j_X is the base change to X-functor. Then j_X defines a continuous functor of sites $j_X \colon S \longrightarrow S_{/X}$ sending final object to final object and so it defines a morphism of topoi $j_X^* \colon E \longrightarrow E_{/X}$ and $j_{X,*} \colon E_{/X} \longrightarrow E$. In particular, j_X^* is left adjoint to $j_{X,*}$. Moreover by loc. cit. we have

$$j_X^*(\mathcal{F})(Y, u) = \mathcal{F}\bigl(Y\bigr) = \mathcal{F}(\alpha_X(Y, u)) = \alpha_{X,*}(\mathcal{F})(Y, u) \text{ for every } \mathcal{F} \in E, (Y, u) \in C_{/X}.$$

Therefore we have a canonical isomorphism of functors $j_X^* \cong \alpha_{X,*}$ which implies that j_X^* has a canonical left adjoint, namely α_X^*. This left adjoint of j_X^* is denoted $j_{X,!}$ and we have an explicit description of it. Namely, for every $\mathcal{F} \in E_{/X}$ we have $j_{X,!}(\mathcal{F}) = \alpha_X^*(\mathcal{F})$ is the sheaf associated to the presheaf on C given by $Z \longrightarrow \varinjlim \mathcal{F}(Y, u)$, where the limit is over the category of triples (Y, u, v) where (Y, u) is an object of $C_{/X}$ and $v \colon Z \longrightarrow Y$ is a morphism in C. As the limit is isomorphic to $\amalg_{g \in Hom_C(Z,X)} \mathcal{F}(Z, g)$ we conclude that $j_{X,!}(\mathcal{F})$ is the sheaf associated to the presheaf

$$Z \longrightarrow \amalg_{g \in Hom_C(Z,X)} \mathcal{F}(Z, g).$$

2.5 The site $\mathfrak{X}_{/(\mathcal{X},Z)}$

Our main application of the theory in section 2.4 is the following. Let us recall that we denoted in section 2.3 by $(\mathcal{X}, X)$ any one of the two pairs $\bigl(\mathcal{X}(w), X(w)\bigr)$ and $\bigl(\mathcal{X}(N,p), X(N,p)\bigr)$ and let $Z \longrightarrow X$ be a finite Kummer étale morphism of log rigid spaces, i.e., a morphism in the category $\mathcal{X}_{\overline{K}}^{\rm fket}$. Therefore the pair $(\mathcal{X}, Z)$ is an object of $E_{\mathcal{X}_{\overline{K}}}$. We denote by $\bigl(E_{\mathcal{X}_{\overline{K}}}\bigr)_{/(\mathcal{X},Z)}$ the induced category and by $\mathfrak{Z} := \mathfrak{X}_{/(\mathcal{X},Z)}$ the associated induced site.

As pointed out in section 2.4 we have a functor $\alpha := \alpha_{(\mathcal{X},Z)}\colon \mathfrak{Z} \longrightarrow \mathfrak{X}$ and the associated adjoint functors α^*, α_*. As in the category $E_{\mathcal{X}_{\overline{K}}}$ fiber products exist and the category has a final element $(\mathcal{X}, X)$, we also have a base change functor $j := j_{(\mathcal{X},Z)}\colon \mathfrak{X} \longrightarrow \mathfrak{Z}$ defined by $j(\mathcal{U}, W) := \bigl(\mathcal{U}, Z\times_X W, {\rm pr}_1\bigr)$. This functor commutes with fiber products, final elements and maps covering families to covering families so it it induces a morphism of topoi $j^*\colon {\rm Sh}(\mathfrak{X}) \longrightarrow {\rm Sh}(\mathfrak{Z})$ and $j_*\colon {\rm Sh}(\mathfrak{Z}) \longrightarrow {\rm Sh}(\mathfrak{X})$ such that j^* is left adjoint to j_*. We further have a left adjoint $j_!\colon {\rm Sh}(\mathfrak{Z}) \longrightarrow {\rm Sh}(\mathfrak{X})$ to j^*. More precisely, for every sheaf of abelian groups $\mathcal{F}$ on $\mathfrak{Z}$, the sheaf $j_!(\mathcal{F})$ on $\mathfrak{X}$ is the sheaf associated to the presheaf

$$(\mathcal{U}, W) \longrightarrow \oplus_{g\in Hom_{\mathcal{X}(w)_{\overline{K}}^{\rm fket}}(W,Z)}\mathcal{F}(\mathcal{U}, W, g).$$

We have the following fundamental facts:

Proposition 2.4. *For all Z as above there is a natural isomorphism of functors $j_{(\mathcal{X},Z),!} \to j_{(\mathcal{X},Z),*}$.*

Proof. We divide the proof in two steps. First we construct a natural transformation of functors $j_{(\mathcal{X},Z),!} \to j_{(\mathcal{X},Z),*}$. then we prove that it is an isomorphism.

Claim 1 Let $(\mathcal{U}, W)$ be an object of $\mathfrak{X}$ and let $g\colon W \longrightarrow Z$ be a morphism in $\mathcal{X}_{\overline{K}}^{\rm fket}$. Then we have a canonical isomorphism: $Z\times_X W \cong W \amalg Z'_g$ for some object Z'_g in $\mathcal{X}_{\overline{K}}^{\rm fket}$, such that this isomorphism composed with the morphism induced by g is the natural inclusion $W \hookrightarrow W \amalg Z'_g$.

Let us first point out that Claim 1 implies the existence of a canonical isomorphism

$$(*)\ Z\times_X W \cong \bigl(\amalg_{g\colon W\to Z} W\bigr) \amalg Z'_W,\ \text{ for some object } Z'_W \text{ of } \mathcal{X}_{\overline{K}}^{\rm fket}.$$

Thus for every sheaf $\mathcal{F} \in {\rm Sh}(\mathfrak{Z})$ we have a canonical morphism

$$j_!(\mathcal{F})(\mathcal{U}, W) = \oplus_{g\colon W\to Z}\mathcal{F}(\mathcal{U}, W, g) \longrightarrow \mathcal{F}(\mathcal{U}, W\times_X Z, {\rm pr}_1) =: j_*(\mathcal{F})(\mathcal{U}, W).$$

Now let us prove Claim 1. We first prove the following

Lemma 2.5. *Suppose $f\colon U \longrightarrow V$ is a finite Kummer log étale map of log affinoid spaces given by a chart of the form*

$$\begin{array}{ccc} P & \longrightarrow & B \\ \uparrow & & \uparrow f \\ Q & \longrightarrow & A \end{array}$$

with P, Q fine saturated monoids. We also suppose that A, B are normal K-algebras, A is an integral domain and the images of the elements of P in B are not zero divisors. Then if

$g\colon V \longrightarrow U$ is a morphism of log affinoids over f, there is an object W and an isomorphism $U \cong V \amalg W$ in the category $V^{\rm fket}$ such that the following diagram is commutative

$$\begin{array}{ccc} U & \cong & V \amalg W \\ g \uparrow & & \iota \uparrow \\ V & = & V \end{array}$$

where $\iota\colon V \hookrightarrow V \amalg W$ is the natural map.

Proof. Let $s \in A$ be the product of the images in A of a set of generators of Q and let us remark that the image of s in B is not a zero divisor. Then $f_s\colon A[1/s] \longrightarrow B[1/s]$ is a finite and étale morphism of K-algebras such that we have a morphism of K-algebras $g_s\colon B[1/s] \longrightarrow A[1/s]$ which is a section of f_s. Then there is an $A[1/s]$-algebra C', finite and étale and an isomorphism of K-algebras $B[1/s] \cong A[1/s] \times C'$ such that the following diagram is commutative

$$\begin{array}{ccc} B[1/s] & \cong & A[1/s] \times C' \\ f_s \downarrow & & {\rm pr}_1 \downarrow \\ A[1/s] & = & A[1/s] \end{array}$$

This is a well known fact but let us briefly recall the idea. As $B[1/s]$ is a finite $A[1/s]$-algebra, it is a finite projective and separable $A[1/s]$-algebra. Then there is a unique element $e \in B[1/s]$ such that $g_s(x) = {\rm Tr}_{B[1/s]/A[1/s]}(ex)$, for every $x \in B[1/s]$. Now it is not difficult to prove that e is an idempotent which gives a decomposition first as $A[1/s]$-modules $B[1/s] \cong A[1/s] \times {\rm Ker}(g_s)$. Now one proves that $C' := {\rm Ker}(g_s)$ has a natural structure of K-algebra and the isomorphism is as K-algebras.

Let as above e and $1-e$ be the idempotents which give the decomposition $B[1/s] \cong A[1/s] \times C'$. Then e satisfies $e^2 - e = 0$, i.e., e is an element of $B[1/s]$ integral over A therefore $e \in B$. Therefore e, $1-e$ give an isomorphism as K-algebras $B \cong B' \times C$, where $A \subset B' \subset A[1/s]$. But B' is finite over A therefore $B' = A$ as A was supposed normal. Moreover C is a finite A-algebra, which is an affinoid algebra as it is a quotient of B.

Now we endow C with the prelog structure: $P \longrightarrow B \xrightarrow{{\rm pr}_2} C$ and notice that the log rigid space $W := \big({\rm Spm}(C), P^a\big)$ satisfies $U \cong V \times W$ and makes the diagram of the lemma commutative.

Moreover as $U \longrightarrow V$ is a finite Kummer log étale map, therefore $W \longrightarrow V$ is also Kummer log étale, as this can be read on stalks of geometric points.

□

Now let, as in Claim 1, $W \to X, Z \to X$ be morphisms in $\mathcal{X}_{\overline{K}}^{\rm fket}$, i.e., there is a finite extension L of K such that W and Z are both defined over L and we have finite Kummer log étale morphisms $W \longrightarrow X_L$ and $Z \longrightarrow X_L$. In fact it is enough to assume that $\underline{Z}$ and $\underline{W}$ are both affinoids (if $\mathfrak{X} = \mathfrak{X}(w)$ this is always the case: as $\underline{X}(w)$ is an affinoid and both maps $W \to X(w)$ and $Z \to X(w)$ are finite it follows that $\underline{W}$ and $\underline{Z}$ are affinoids). Moreover, as the morphism $X \longrightarrow {\rm Spm}(K)$ (with trivial log structure on ${\rm Spm}(K)$) is log smooth, it follows that $Z \longrightarrow {\rm Spm}(K)$ and $W \longrightarrow {\rm Spm}(K)$ are both log smooth and so $\underline{X}$, $\underline{Z}$ and $\underline{W}$ are all normal affinoids.

If $g\colon W \longrightarrow Z$ is a morphism over X, we have a natural morphism $g'\colon W \longrightarrow Z \times_X W$ which is a section of the projection $Z \times_X W \longrightarrow W$. We apply the lemma 2.5 and we get Claim 1.

To conclude the proof of proposition 2.4 we make the following

Claim 2 For every $(\mathcal{U}, W)$ object of $\mathfrak{X}$, there is $W' \longrightarrow W$ a surjective morphism in $\mathcal{X}_{\overline{K}}^{\rm fket}$ with the property $Z \times_X W' \cong \amalg_{g\colon W' \longrightarrow Z} W'$, i.e., in formula $(*)$ we have $Z'_{W'} = \phi$.

Clearly, Claim 2 implies that for a sheaf $\mathcal{F}$ on $\mathfrak{X}$ the natural morphism $j_!(\mathcal{F}) \longrightarrow j_*(\mathcal{F})$ is an isomorphism. So we are left with the task of proving Claim 2.

Here and elsewhere if W is a log rigid space or a log formal scheme, we denote by $W^{\rm triv}$ the sub-space (or sub-formal scheme) on which the log structure is trivial. Given our finite, Kummer étale morphism $Z \longrightarrow X$ let $\deg_{Z/X}\colon Z^{\rm triv} \longrightarrow \mathbb{Z}$ denote the degree of $Z^{\rm triv}$ over $X^{\rm triv}$. By restricting to a connected component of Z we may suppose that $\deg_{Z/X}$ is constant equal to n. We prove Claim 2 by induction on $n = \deg_{Z/X}$.

If $n = 0$ there is nothing to prove so let us suppose $n > 0$. As the morphism $Z \longrightarrow X$ is a morphism of normal affinoids, if we regard $Z \times_X Z$ as a log rigid space over Z via the second projection then the diagonal $\Delta\colon Z \longrightarrow Z \times_X Z$ provides a section as in lemma 2.5. Therefore there is an object Z' in $\mathcal{X}_{\overline{K}}^{\rm fket}$ such that $Z \times_X Z \cong Z \amalg Z'$. It follows that $\deg_{Z'/Z} = n - 1$ and applying the induction hypothesis we find an object $W \longrightarrow Z$ in $Z^{\rm fket}$ such that $Z' \times_Z W = \amalg_{i=1}^m W_i$, where $W_i = W$ for all $1 \le i \le m$. Then the composition $W \longrightarrow Z' \longrightarrow X$ makes W an object in $\mathcal{X}_{\overline{K}}^{\rm fket}$ and we have $Z \times_X W$ is isomorphic to a disjoint union of objects isomorphic to W.

□

Proposition 2.4 has the following immediate consequence.

Corollary 2.6. *Suppose $Z \longrightarrow X$ is a morphism in $\mathcal{X}_{\overline{K}}^{\rm fket}$. Then we have*

a) The functor j_* is an exact functor.

b) $R^i j_* = 0$ for all $i \ge 1$.

Proof. For a) we remark that $j_* \cong j_!$ by proposition 2.4 and $j_! \cong \alpha^*$. It follows that $j_!$, and so also j_*, is right exact. As j_* is left exact, it is exact. This immediately implies b). □

As j^* admits a left adjoint $j_!$ by adjunction we get a morphism

$$S_Z\colon j_*\big(j^*(\mathcal{F})\big) \cong j_!\big(j^*(\mathcal{F})\big) \longrightarrow \mathcal{F}, \tag{1}$$

functorial on the category of sheaves of abelian groups on the site $E_{\mathcal{X}_{\overline{K}}}$. We call it the *trace map* relative to Z. More explicitly, given a sheaf of abelian groups $\mathcal{F}$ on $E_{\mathcal{X}_{\overline{K}}}$ it is the map of sheaves associated to the map of presheaves:

$$j_!\big(j^*(\mathcal{F})\big)(\mathcal{U}, W) = \oplus_{g \in Hom_{\mathcal{X}_{\overline{K}}^{\rm fket}}(W,Z)} j^*(\mathcal{F})(\mathcal{U}, W, g) = \oplus_{g \in Hom_{\mathcal{X}_{\overline{K}}^{\rm fket}}(W,Z)} \mathcal{F}(\mathcal{U}, W) \longrightarrow \mathcal{F}(\mathcal{U}, W),$$

given by the sum.

2.6 Sheaves on $\mathfrak{X}$

We continue, as in the previous section, to denote by $\mathfrak{X}$ any one of the sites $\mathfrak{X}(w)$ or $\mathfrak{X}(N,p)$ and we will describe certain sheaves on this site.

We denote by $\mathcal{O}_{\mathfrak{X}}$ the presheaf of $\mathcal{O}_{\overline{K}}$-algebras on $\mathfrak{X}$ defined by

$$\mathcal{O}_{\mathfrak{X}}(\mathcal{U},W) := \text{ the normalization of } \mathrm{H}^0(\underline{\mathcal{U}},\mathcal{O}_{\underline{\mathcal{U}}}) \text{ in } \mathrm{H}^0(\underline{W},\mathcal{O}_{\underline{W}}).$$

We also define by $\mathcal{O}_{\mathfrak{X}}^{\mathrm{un}}$ the sub-presheaf of $\mathbb{W}(k)$-algebras of $\mathcal{O}_{\mathfrak{X}}$ whose sections over $(\mathcal{U},W)$ consist of the elements $x \in \mathcal{O}_{\mathfrak{X}}(\mathcal{U},W)$ such that there exist a finite unramified extension M of K contained in $\overline{K}$, a Kummer log étale morphism $\mathcal{V} \longrightarrow \mathcal{U}\times_{\mathcal{O}_K}\mathcal{O}_M$ and a morphism $W \longrightarrow \mathcal{V}_K$ over $\mathcal{U}_K$ such that x, viewed as an element of $\mathrm{H}^0(\underline{W},\mathcal{O}_{\underline{W}})$ lies in the image of $\mathrm{H}^0(\underline{\mathcal{V}},\mathcal{O}_{\underline{\mathcal{V}}})$.

We also have the fundamental functor $v_{\mathfrak{X}}\colon \mathcal{X}^{\mathrm{ket}} \longrightarrow \mathfrak{X}$, defined by $v_{\mathfrak{X}}(\mathcal{U}) := (\mathcal{U},\mathcal{U}_K)$, which induces a morphism of sites. We then have

Proposition 2.7 ([AI], Proposition 1.10)**.** *The presheaves $\mathcal{O}_{\mathfrak{X}}$ and $\mathcal{O}_{\mathfrak{X}}^{\mathrm{un}}$ are sheaves and $\mathcal{O}_{\mathfrak{X}}^{\mathrm{un}}$ is isomorphic to the sheaf $v_{\mathcal{X}}^*\big(\mathcal{O}_{\mathcal{X}^{\mathrm{ket}}}\big)$*

We denote by $\widehat{\mathcal{O}}_{\mathfrak{X}}$ and $\widehat{\mathcal{O}}_{\mathfrak{X}}^{\mathrm{un}}$ the continuous sheaves on $\mathfrak{X}$ given by the projective systems of sheaves $\{\mathcal{O}_{\mathfrak{X}}/p^n\mathcal{O}_{\mathfrak{X}}\}_{n\geq 0}$ and respectively $\{\mathcal{O}_{\mathfrak{X}}^{\mathrm{un}}/p^n\mathcal{O}_{\mathfrak{X}}^{\mathrm{un}}\}_{n\geq 0}$.

In the notations of section 2.5 let $r \geq 1$ and $w \in \mathbb{Q}$ adapted to r and let $Z := X^{(r)}(w) \longrightarrow X(w)$ if $\mathfrak{X}$ denotes $\mathfrak{X}(w)$ or $Z := X^{(r)} \longrightarrow X(N,p)$ if $\mathfrak{X}$ denotes $\mathfrak{X}(N,p)$ and let us denote by $\mathfrak{X}^{(r)} := \mathfrak{X}_{/(\mathcal{X},Z)}$ the site induced by $(\mathcal{X},Z)$ and by j_r^*, $j_{r,*}(\cong j_{r,!})$ the associated morphism of topoi. We have the functor

$$v_r\colon \mathcal{X}^{\mathrm{ket}} \longrightarrow \mathfrak{X}^{(r)}$$

defined by $v_r := j_r \circ v_{\mathfrak{X}}$. More explicitly, $v_r(\mathcal{U}) := j(\mathcal{U},\mathcal{U}_K) = \big(\mathcal{U}, Z\times_X \mathcal{U}_K, \mathrm{pr}_1\big)$. This functor sends covering families to covering families, commutes with fiber products and sends final objects to final objects. In particular it defines a morphism of topoi. Corollary 2.6 implies that the Leray spectral sequence for $v_{r,*} = v_{\mathfrak{X},*}\circ j_{r,*}$ degenerates and we have $R^i v_{r,*} \cong R^i v_{\mathfrak{X},*}\circ j_{r,*}$.

We denote by $\mathcal{O}_{\mathfrak{X}^{(r)}} := j_r^*(\mathcal{O}_{\mathfrak{X}})$ and by $\widehat{\mathcal{O}}_{\mathfrak{X}^{(r)}} := j_r^*(\widehat{\mathcal{O}}_{\mathfrak{X}})$. Let us recall the morphism $\theta_r\colon \underline{\mathcal{X}}^{(r)} \longrightarrow \underline{\mathcal{X}}$ which is finite and defines $\underline{\mathcal{X}}^{(r)}$ as the normalization of $\underline{\mathcal{X}}$ in $X^{(r)}$ and let $G_r \cong (\mathbb{Z}/p^r\mathbb{Z})^\times$ denote the Galois group of $X^{(r)}/X$. Then G_r acts naturally on $\underline{\mathcal{X}}^{(r)}$ over $\underline{\mathcal{X}}$ and $\underline{\mathcal{X}} \cong \underline{\mathcal{X}}^{(r)}/G_r$.

Lemma 2.8. *We have a natural isomorphism of sheaves on $\mathcal{X}^{\mathrm{ket}}$*

$$\big(v_{r,*}(\mathcal{O}_{\mathfrak{X}^{(r)}})\big)^{G_r} \cong \mathcal{O}_{\underline{\mathcal{X}}} \textit{ and similarly } \big(v_{r,*}(\widehat{\mathcal{O}}_{\mathfrak{X}^{(r)}})\big)^{G_r} \cong \widehat{\mathcal{O}}_{\underline{\mathcal{X}}}.$$

Proof. Let $\mathcal{U} \longrightarrow \mathcal{X}$ be a morphism in $\mathcal{X}^{\mathrm{ket}}$. Then we have

$$v_{r,*}(\mathcal{O}_{\mathfrak{X}^{(r)}})(\mathcal{U}) = \mathcal{O}_{\mathfrak{X}}(\mathcal{U}, X^{(r)}\times_X \mathcal{U}_K) = \mathrm{H}^0(\mathcal{X}^{(r)}\times_{\mathcal{X}}\mathcal{U}, \mathcal{O}_{\underline{\mathcal{X}}^{(r)}}) = \theta_{r,*}(\mathcal{O}_{\underline{\mathcal{X}}^{(r)}})(\mathcal{U}).$$

Form this the claim follows.

□

2.7 The localization functors

As in the previous section, $\mathfrak{X}$ denotes any one of the sites $\mathfrak{X}(w)$ or $\mathfrak{X}(N,p)$. We recall here the localization of a sheaf or a continuous sheaf on $\mathfrak{X}$ to a "small affine of $\mathcal{X}^{\rm ket}$" (for more details see [AI] section §1.2.6). Let $\mathcal{U}=\big(\mathrm{Spf}(R_\mathcal{U}),N_\mathcal{U}\big)$ be a connected small affine object of $\mathcal{X}^{\rm ket}$ and we denote by $U:=\mathcal{U}_K$ the log rigid analytic generic fiber of $\mathcal{U}$. Let us recall that under the above hypothesis $\mathcal{U}$ is a log formal scheme whose log structure is given by the sheaf of monoids denoted $N_\mathcal{U}$.

We write $R_\mathcal{U}\otimes\overline{K}=\prod_{i=1}^n R_{\mathcal{U},i}$ with $\mathrm{Spf}(R_{\mathcal{U},i})$ connected, we let $N_{\mathcal{U},i}$ denote the monoids which give the respective log structures and we let U_i denote the respective log rigid analytic generic fiber. Then each $R_{\mathcal{U},i}$ is an integral domain, so we let $\mathbb{C}_{\mathcal{U},i}$ denote an algebraic closure of the fraction field of $R_{\mathcal{U},i}$ for all $1\le i\le n$ and let $\mathbb{C}_{\mathcal{U},i}^{\log}:=\mathrm{Spec}\big(\mathbb{C}_{\mathcal{U},i}),N_{\mathcal{U},i}\big)$ denote the log geometric point of $\mathcal{U}_i:=\big(\mathrm{Spf}(R_{\mathcal{U},i}),N_{\mathcal{U},i})\big)$ over $\mathbb{C}_{\mathcal{U},i}$ (see [Il] definition 4.1 or [AI] section 1.2.5 for the definition of a log geometric point). We denote by $\mathcal{G}_{U,i}:=\pi_1^{\log}\big(U_i,\mathbb{C}_{\mathcal{U},i}^{\log}\big)$ the Kummer étale fundamental group of U_i. We have then that the category $U_i^{\rm fket}$ is equivalent to the category of finite sets with continuous $\mathcal{G}_{U,i}$-action. We write $(\overline{R}_{\mathcal{U},i},\overline{N}_{\mathcal{U},i})$ for the direct limit over all finite normal extensions $R_{\mathcal{U},i}\subset S\subset\mathbb{C}_{\mathcal{U},i}$, all log structures N_S on $\mathrm{Spm}(S_K)$ such that there are Kummer étale morphisms $\mathbb{C}_{\mathcal{U},i}\longrightarrow\big(\mathrm{Spm}(S_K),N_S\big)\longrightarrow U_i$ compatible with the one between the underlying formal schemes. Finally we denote $\overline{R}_\mathcal{U}:=\prod_{i=1}^n\overline{R}_{\mathcal{U},i}$, $\overline{N}_\mathcal{U}:=\prod_{i=1}^n\overline{N}_{\mathcal{U},i}$ and $\mathcal{G}_{U_{\overline{K}}}:=\prod_{i=1}^n\mathcal{G}_{U,i}$.

We denote by $\mathrm{Rep}\big(\mathcal{G}_{U_{\overline{K}}}\big)$ and $\mathrm{Rep}\big(\mathcal{G}_{U_{\overline{K}}}\big)^{\mathbb{N}}$ the category of discrete abelian groups with continuous action by $\mathcal{G}_{U_{\overline{K}}}$, respectively the category of projective systems of such. It follows from [Il] section §4.5 that we have an equivalence of categories

$$\mathrm{Sh}(U_{\overline{K}}^{\rm fket})\cong\mathrm{Rep}\big(\mathcal{G}_{U_{\overline{K}}}\big)$$

sending $\mathcal{F}\to\lim\limits_{\to}\mathcal{F}(\mathrm{Spm}(S_K),N_S)$. Therefore composing with the restriction $\mathrm{Sh}(\mathfrak{X})\longrightarrow\mathrm{Sh}(U_{\overline{K}}^{\rm fket})$ defined by $\mathcal{F}\to\big(W\to\mathcal{F}(\mathcal{U},W)\big)$, we obtain a functor, called localization functor

$$\mathrm{Sh}(\mathfrak{X})\longrightarrow\mathrm{Rep}\big(\mathcal{G}_{U_{\overline{K}}}\big)\ \text{denoted}\ \mathcal{F}\to\mathcal{F}(\overline{R}_\mathcal{U},\overline{N}_\mathcal{U}).$$

We consider the following variant. Let $Z\to X$, with $X=X(w)$ or $X=X(N,p)$, be a finite Kummer étale morphism in $X_{\overline{K}}^{\rm fket}$. Consider the associated site $\mathfrak{Z}:=\mathfrak{X}_{/(\mathcal{X},Z)}$ as in section §2.5 and let $j_{(\mathcal{X},Z)}\colon\mathfrak{X}\to\mathfrak{Z}$ be the induced morphism of sites. Consider a sheaf $\mathcal{F}\in\mathrm{Sh}(\mathfrak{Z})$ and fix a connected small affine object $\mathcal{U}=\big(\mathrm{Spf}(R_\mathcal{U}),N_\mathcal{U}\big)$ of $\mathcal{X}^{\rm ket}$ as before. Denote by $\Upsilon_\mathcal{U}$ the set of homomorphisms of $R_\mathcal{U}\otimes\overline{K}$-algebras $\Gamma\big(Z\times_X U,\mathcal{O}_{Z\times_X U}\big)\to\overline{R}_\mathcal{U}[1/p]$. For any $g\in\Upsilon_\mathcal{U}$ we write $\mathcal{F}\big(\overline{R}_\mathcal{U},\overline{N}_\mathcal{U},g\big):=\lim\mathcal{F}\big(\mathcal{U},W\big)$, where the limit is taken over all finite and Kummer étale maps $\mathrm{Spm}(S_K)=W\to Z\times_X U$ with $S_K\subset\overline{R}_\mathcal{U}[1/p]$ a $\Gamma\big(Z\times_X U,\mathcal{O}_{Z\times_X U}\big)$-subalgebra (using g). Let $\mathcal{G}_{U_{\overline{K}},Z,g}$ be the subgroup of $\mathcal{G}_{U_{\overline{K}}}$ fixing $\Gamma\big(Z\times_X U,\mathcal{O}_{Z\times_X U}\big)$. Then $\mathrm{Sh}\big((Z\times_X U)^{\rm fket}\big)\cong\mathrm{Rep}\big(\mathcal{G}_{\mathcal{U}_{\overline{K}},Z,g}\big)$ and we obtain as before a localization functor:

$$\mathrm{Sh}(\mathfrak{Z})\longrightarrow\mathrm{Rep}\big(\mathcal{G}_{\mathcal{U}_{\overline{K}},Z,g}\big),\qquad\mathcal{F}\to\mathcal{F}\big(\overline{R}_\mathcal{U},\overline{N}_\mathcal{U},g\big).$$

If $\{\mathcal{U}_i\}_i$ is a covering of $\mathcal{X}^{\rm ket}$ and for every i we choose $g_i\in\Upsilon_{\mathcal{U}_i}$, it follows from the definition of coverings in the site $\mathfrak{Z}$ that the map $\mathrm{Sh}(\mathfrak{Z})\longrightarrow\prod_i\mathrm{Rep}\big(\mathcal{G}_{U_{i,\overline{K}},Z,g_i}\big)$ is faithful. It also follows from proposition 2.4 that

$$j_{(\mathcal{X},Z),*}(\mathcal{F})\big(\overline{R}_{\mathcal{U}},\overline{N}_{\mathcal{U}}\big)\cong\oplus_{g\in\Upsilon_{\mathcal{U}}}\mathcal{F}\big(\overline{R}_{\mathcal{U}},\overline{N}_{\mathcal{U}},g\big). \tag{2}$$

3 Analytic Modular Symbols

3.1 Analytic functions and distributions

In this section we recall a number of definitions and results from [AS] and [HIS] and also define some new objects. Let $T_0:=\mathbb{Z}_p^\times\times\mathbb{Z}_p$, which we regard as a compact open subset of the space of row vectors $(\mathbb{Z}_p)^2$. We have the following structures on T_0:

a) a natural left action of $\mathbb{Z}_p^\times$ by scalar multiplication;

b) a natural right action of the semigroup

$$\Xi(\mathbb{Z}_p)=\left\{\left(\begin{array}{cc} a & b\\ c & d\end{array}\right)\in M_2(\mathbb{Z}_p)\cap\mathbf{GL}_2(\mathbb{Q}_p)\;\middle|\;(a,c)\in\mathbb{Z}_p^\times\times p\mathbb{Z}_p\right\}$$

and its subgroup

$$\mathrm{Iw}(\mathbb{Z}_p):=\Xi(\mathbb{Z}_p)\cap\mathbf{GL}_2(\mathbb{Z}_p)$$

given by matrix multiplication on the right.

The two actions obviously commute.

Let us denote by $\mathcal{W}^*$ the rigid subspace of $\mathcal{W}$ of accessible weights, i.e., weights k such that $|k(t)^{p-1}-1|<p^{-1/(p-1)}$. Let $U\subset\mathcal{W}^*$ be a wide open disk which is an admissible open of $\mathcal{W}^*$. Write A_U for the O_K-algebra of rigid functions on U and denote by

$$\Lambda_U=\{f\text{ rigid function on }U\text{ such that }|f(x)|\le 1\text{ for every point }x\in U\},$$

the $\mathcal{O}_K$-algebra of bounded rigid functions on U. Then, as remarked in section 4, Λ_U is a complete, regular, local, noetherian $\mathcal{O}_K$-algebra, in fact Λ_U is (non canonically) isomorphic to the $\mathcal{O}_K$-algebra $\mathcal{O}_K[\![T]\!]$. The completeness refers to the $\underline{m}_U$-adic topology (called the **weak topology** of Λ_U), for $\underline{m}_U$ the maximal ideal of Λ_U. As remarked in section 4, Λ_U is also complete for the p-adic topology.

Let now B denote one of the complete, regular, local, noetherian rings: $\mathcal{O}_K$ or Λ_U, for $U\subset\mathcal{W}^*$ a wide open disk as above. Let also $k\in\mathcal{W}^*(B_K)$ be as follows: if $B=\mathcal{O}_K$, $k\in\mathcal{W}^*(K)$ and if $B=\Lambda_U$ we set $k=k_U:\mathbb{Z}_p^\times\longrightarrow\Lambda_U^\times$ defined by $t^{k_U}(x)=t^x$ for all $x\in U(K)$.

Definition 3.1. We set

$$A_k^o:=\left\{f:T_0\longrightarrow B\;\middle|\;i)\ \forall a\in\mathbb{Z}_p^\times,\,t\in T_0,\quad\text{we have }f(at)=k(a)f(t)\right.$$

and

$$\left.ii)\text{ the function }z\to f(1,z)\text{ extends to a rigid analytic function on the closed unit disk }B[0,1]\right\}.$$

Let us make ii) of the above definition more precise. Let $\mathrm{ord}_p\colon B-\{0\}\longrightarrow\mathbb{Z}$ be defined by $\mathrm{ord}_\pi(\alpha)=\sup\{n\in\mathbb{Z}\quad|\quad \alpha\in\pi^nB\}$. Then we say that the function $z\to f(1,z)$ in the definition above is rigid analytic on $B[0,1]$ if there exists a power series $F(X)=\sum_{n=0}^{\infty}a_nX^n\in B[\![X]\!]$ with $\mathrm{ord}_\pi(a_n)\overset{n\to\infty}{\longrightarrow}\infty$ and such that $f(1,z)=F(z)$ for all $z\in\mathbb{Z}_p$. Let us denote by $A_k:=A_k^o\otimes_{\mathcal{O}_K}K$, which is naturally a B_K-module. As B_K is a K-Banach space (for its p-adic topology) let us point out that A_k is an orthonormalizable Banach B_K-module, where an orthonormal basis is given by: $\{f_n\}_{n\geq 0}$ where $f_n\in A_k^o$ are the unique elements such that $f_n(1,z)=z^n$ for all $z\in\mathbb{Z}_p$. In other words $f_n(x,y)=k(x)(y/x)^n$ for all $(x,y)\in T_0$.

For every $\gamma\in\Xi(\mathbb{Z}_p)$ and function $f\colon T_0\longrightarrow B$ we define $(\gamma f)(v):=f(v\gamma)$. We have

Lemma 3.2. *If $f\in A_k^o$ and $\gamma\in\Xi(\mathbb{Z}_p)$ then $\gamma f\in A_k^o$.*

Proof. Let $\gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$. Then for every $v\in T_0$ and $a\in\mathbb{Z}_p^\times$ we have $(\gamma f)(av)=f\big((av)\gamma\big)=k(a)f(v\gamma)k(a)(\gamma f)(v)$.

Moreover,

$$(\gamma f)(1,z)=f(a+cz,b+dz)=k(a+cz)f\Big(1,\frac{b+dz}{a+cz}\Big)=k(a)k(1+ca^{-1}x)\sum_{n=0}^{\infty}a_n\Big(\frac{b+dz}{a+cz}\Big)^n.$$

Using the fact that k is analytic and $a_n\overset{n\to\infty}{\longrightarrow}0$ we deduce that the function $z\to(\gamma f)(1,z)$ extends to an analytic function on the closed unit disk $B[0,1]$. $\square$

Definition 3.3. a) Let $k\in\mathcal{W}^*(K)$ be a weight. We define $D_k^o:=\mathrm{Hom}_{\mathrm{cont},\mathcal{O}_K}\big(A_k^o,\mathcal{O}_K\big)$, i.e., the $\mathcal{O}_K$-module of continuous, $\mathcal{O}_K$-linear homomorphisms from A_k^o to $\mathcal{O}_K$. We also denote by $D_k:=D_k^o\otimes_{\mathcal{O}_K}K$.

b) If $U\subset\mathcal{W}^*$ is a wide open disk defined over K, we define $A_U^o:=A_{k_U}^o$ and we set $D_U^o:=\mathrm{Hom}_{\Lambda_U}\big(A_U^o,\Lambda_U\big)$, i.e., the Λ_U-module of continuous (for the $\underline{m}_U$-adic topology) λ_U-linear homomorphisms from A_U^o to Λ_U. We denote by $D_U:=D_U^o\otimes_{\mathcal{O}_K}K$.

Remark 3.4. i) The (left) action of the semigroup $\Xi(\mathbb{Z}_p)$ on A_k^o induces a (right) action on D^o by $(\mu|\gamma)(f):=\mu(\gamma f)$ for all $\gamma\in\Xi(\mathbb{Z}_p)$, $f\in A_k^o$ and $\mu\in D^o$.

ii) We have a natural, fundamental homomorphism of B-modules

$$\psi\colon D^o\longrightarrow\prod_{n\in\mathbb{N}}B,\ \text{defined by}\ \mu\to\big(\mu(f_n)\big)_{n\in\mathbb{N}}.$$

As the family $\big(f_n\big)_n$ is an orthonormal basis of A_k over B_K, the above morphism is a B-linear isomorphism. Moreover, under this isomorphism, the weak topology on D^o corresponds to the weak topology (i.e., the product of the $\underline{m}_B$-adic topologies on the product).

iii) A more common definition in the literature (see [AS]) would be:

$$\widetilde{D}_U^o=\mathrm{Hom}_{p,\Lambda_U}^{\mathrm{cpt}}(A_U^o,\Lambda_U)\subset D_U^o,$$

i.e., $\widetilde{D}_U^o$ consists of the continuous and compact (or completely continuous) in the p-adic topology, Λ_U-linear homomorphisms. Then $\widetilde{D}_U:=\widetilde{D}_U^o\otimes_{\mathcal{O}_K}K$ is an orhtonormalizable $\Lambda_{U,K}$-module.

Our D_U^o has a property very similar to being orthonormalizable, which will be sufficient to define Fredholm characteristic series of compact Λ_U-linear operators on it. More precisely we have the following.

Lemma 3.5. *Let $\left(e_i\right)_{i\in I}$ be a sequence of elements in D_U^o such that their image $\left(\overline{e}_i\right)_{i\in I}$ in $D_U^o/\underline{m}_U D_U^o$ is a basis of this vector space over $\mathbb{F} := \Lambda_U/\underline{m}_U \cong \mathcal{O}_K/\pi\mathcal{O}_K$. Then for every $n \in \mathbb{N}$ the natural map*

$$\Psi_n\colon\ \oplus_{i\in I}\Big(\Lambda_U/\underline{m}_U^n\Big)e_i \longrightarrow D_U^o/\underline{m}_U^n D_U^o$$

is an isomorphism of Λ_U-modules.

Proof. Let us first recall that we have a natural isomorphism of Λ_U-modules $D_U^o \cong \prod_{i\in\mathbb{N}} \Lambda_U$. In particular it follows that for every $n \geq 0$

$$\underline{m}^n D_U^o/\underline{m}_U^{n+1}D_U^o \cong \prod_{i\in\mathbb{N}} \underline{m}_U^n/\underline{m}_U^{n+1} \cong \underline{m}_U^n/\underline{m}_U^{n+1} \otimes_{\mathbb{F}} D_U^o/\underline{m}_U D_U^o \cong \oplus_{i\in I}\big(\underline{m}_U^n/\underline{m}_U^{n+1}\big)e_i.$$

The second isomorphism follows from the fact that $\underline{m}_U^n/\underline{m}_U^{n+1}$ is a finite dimensional $\mathbb{F}$-vector space.

Now we prove the lemma by induction on n. The case $n = 1$ is clear, therefore let us suppose that the property is true for $n \geq 1$ and we'll prove that Ψ_{n+1} is an isomorphism. We have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \underline{m}_U^n D_U^o/\underline{m}_U^{n+1}D_U^o & \longrightarrow & D_U^o/\underline{m}_U^{n+1}D_U^o & \longrightarrow & D_U^o/\underline{m}_U^n & \longrightarrow & 0 \\
 & & \uparrow \varphi & & \uparrow \Psi_{n+1} & & \uparrow \Psi_n & & \\
0 & \longrightarrow & \oplus_{i\in I}\big(\underline{m}_U^n/\underline{m}_U^{n+1}\big)e_i & \longrightarrow & \oplus_{i\in I}\big(\Lambda_U/\underline{m}_U^{n+1}\big)e_i & \longrightarrow & \oplus_{i\in I}\big(\Lambda_U/\underline{m}_U^n\big)e_i & \longrightarrow & 0
\end{array}$$

By inductive hypothesis Ψ_n is an isomorphism and by the comment before the diagram, φ is an isomorphism as well. Therefore Ψ_{n+1} is an isomorphism.

□

Corollary 3.6. *Let us fix a family $\left(e_i\right)_{i\in I}$ as in lemma 3.5. Then for every $x \in D_U^o$ there is a unique sequence $a_i \in \Lambda_U$, $i \in I$ such that*

i) $a_i \to 0$ in the filter of complements of finite sets in I, in the $\underline{m}_U$-topology, i.e., for every $h \in \mathbb{N}$ the subset $i \in I$ with the property $a_i \notin \underline{m}_U^h$ is finite.

ii) $x = \sum_{i\in I} a_i e_i$.

Proof. The corollary follows immediately from lemma 3.5. □

Remark 3.7. 1) A family of elements $\left(e_i\right)_{i\in I}$ as in corollary 3.6 plays the role of an orthonormal basis of a Banach Λ_U-module. In particular it can be used to define the Fredholm series of a compact Λ_U-linear operator on $\mathrm{H}^1\big(\Gamma, D_U\big)$.

2) If $k \in U(K)$ is a weight, the image of a family of elements $\left(e_i\right)_{i\in I}$ of D_U^o as at 1) above in D_k^o is a true ON basis of D_k over K.

Keeping the notations above, let $V \subset U$ be an affinoid disk with affinoid algebra B_V and let B_V^o denote the bounded by 1 rigid functions on V. We have

Lemma 3.8. *$D_U|_V := D_U\hat{\otimes}_{\Lambda_U}B_V$ is an ON-able B_V-module with orthorormal basis $(e_i\hat{\otimes}1)_{i\in I}$ and in fact coincides with the B_V-module of completely continuous B_V-linear maps from A_V to B_V.*

Proof. Let us first make precise the completed tensor product in the statement of the lemma. We have $D_U|_V = \big(D_U^o|_V\big)\otimes_{\mathcal{O}_K} K$ where we define

$$D_U^o|_V := \lim_{\infty\leftarrow n}\Big(D_U^o\otimes_{\Lambda_U} B_V^o/p^nB_V^o\Big).$$

Let us remark that for every $n\in\mathbb{N}$ there is $N := N(n)\in\mathbb{N}$ such that the image of $\underline{m}_U^N$ in $B_V^o/p^nB_V^o$ is 0. Therefore we may write

$$D_U^o|_V = \lim_{\infty\leftarrow n}\Big(D_U^o/\underline{m}_U^{N(n)}\otimes_{\Lambda_U} B_V^o/p^nB_V^o\Big).$$

Therefore the first claim follows from lemma 3.5 and the second is clear.

□

Let now $(B,\underline{m})$ be any of the pairs $(\mathcal{O}_K,\underline{m}_K)$ or $(\Lambda_U,\underline{m}_U)$, for $U\subset\mathcal{W}^*$ a wide open disk defined over K and let A^o, D^o be either A_k^o, D_k^o, for some $k\in\mathcal{W}^*(K)$ in the first case or A_U^o, D_U^o in the second.

Definition 3.9. i) We define for every $n\in\mathbb{N}$ the following $B/\underline{m}^n$-submodule $\mathrm{Fil}_n(A^o/\underline{m}^nA^o)$ of $A^o/\underline{m}^nA^o$ by setting

$$\mathrm{Fil}_n(A^o/\underline{m}^nA^o) := \oplus_{j=0}^n\big(\underline{m}^j/\underline{m}^n\big)f_j,$$

where $\{f_i\}_{i\in\mathbb{N}}$ is the orthornomal basis of A^o described above.

ii) We define the following filtration $\mathrm{Fil}^\bullet(D^o)$ of D^o. For $n\in\mathbb{N}$ we set

$$\mathrm{Fil}^n(D^o) := \{\mu\in D^o \quad | \quad \mu(f_j)\in\underline{m}_B^{n-j} \text{ for all } 0\le j\le n.\}$$

Proposition 3.10. *i) $D^o/\mathrm{Fil}^n(D^o)$ is a finite $B/\underline{m}^n$-module and consequently an artinian $\mathcal{O}_K$-module. Moreover the image of $\mathrm{Fil}^n(D^o)$ in $D^o/\underline{m}^nD^o$, which we identify with the $B/\underline{m}^n$-dual of $A^o/\underline{m}^nA^o$, is the orthogonal complement of $\mathrm{Fil}_n(A^o/\underline{m}^nA^o)$.*

ii) For every $\gamma\in\Xi(\mathbb{Z}_p)$ and $\mu\in\mathrm{Fil}^n(D^o)$ we have $\mu|\gamma\in\mathrm{Fil}^n(D^o)$. In particular $D^o/\mathrm{Fil}^n(D^o)$ is an artinian $\mathcal{O}_K$ and $\Xi(\mathbb{Z}_p)$ module for every $n\ge 0$.

iii) The natural B-linear morphism $D^o\longrightarrow \lim\limits_{\infty\leftarrow n} D^o/\mathrm{Fil}^n(D^o)$ is an isomorphism.

Proof. Let us recall the B-linear map $\psi\colon D^o\longrightarrow\prod_{n\in\mathbb{N}}B$ of remark 3.4 defined by $\psi(\mu) = \big(\mu(f_n)\big)_{n\in\mathbb{N}}$. Let us remark that

$$\psi\big(\mathrm{Fil}^n(D^o)\big) = \prod_{j=0}^{n-1}\underline{m}_B^{n-j}\times\prod_{m\ge n}B,$$

and therefore ψ induces a B-linear isomorphism $D^o/\mathrm{Fil}^n(D^o)\cong\prod_{j=0}^{n-1}B/\underline{m}_B^{n-j}$. This proves the first statement in i) and also iii) because it shows that D^o is separated and complete in the topology given by $\mathrm{Fil}^\bullet(D^o)$. For the second statement of i) let us remark that $\mathrm{Ann}_{B/\underline{m}^n}\big(\underline{m}^i/\underline{m}^n\big) = \underline{m}^{n-i}/\underline{m}^n$ for all $0\le i\le n$.

In order to prove ii) we could proceed as in the proof of lemma 3.2, or better let us recall that we have a natural decomposition $\Xi(\mathbb{Z}_p) = N^{\rm opp}T^+N$ where N is the subgroup of $\mathbf{GL}_2(\mathbb{Z}_p)$ of upper triangular matrices, $N^{\rm opp}$ is the subgroup of lower triangular matrices which reduce to the identity modulo p and T^+ is the semigroup of matrices $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ with $a \in \mathbb{Z}_p^\times$ and $d \in \mathbb{Z}_p - \{0\}$. In order to show that $\mathrm{Fil}^n(D^o)$ is preserved by the matrices in N it is enough to do it for $\gamma := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (which is a topological generator of N). Let $\mu \in \mathrm{Fil}^n(D^o)$, then we have $(\mu|\gamma)(f_j) = \mu(\gamma f_j)$ and moreover

$$(\gamma f_j)(x,y) = f_j(x, y+x) = k(x)(1+y/x)^j = \sum_{k=0}^{j} \begin{pmatrix} n \\ k \end{pmatrix} k(x)(y/x)^k = \sum_{k=0}^{j} \begin{pmatrix} n \\ k \end{pmatrix} f_k(x,y).$$

Therefore $(\mu|\gamma)(f_j) \in \sum_{k=0}^{j} \underline{m}_B^{n-k} \subset \underline{m}_B^{n-j}$.

To show that the matrices in T^+ preserve $\mathrm{Fil}^n(D^o)$ is is enough to show it for $\delta := \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \in T^+$. We have $(\delta f_j)(x,y) = f_j(x, py) = k(x)p^j(y/x)^j = p^j f_j(x,y)$. Therefore for all $\mu \in \mathrm{Fil}^n(D^o)$ we have $(\mu|\delta)(f_j) = p^j\mu(f_j) \in p^j\underline{m}_B^{n-j} \subset \underline{m}_B^{n-j}$.

Finally we leave it to the reader to check that for $\epsilon := \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix} \in N^{\rm opp}$ (which topologically generates this group), if $\mu \in \mathrm{Fil}^n(D^o)$ then $(\mu|\epsilon)(f_j) \in \underline{m}_B^{n-j}$. □

We'd now like to show that the formation of the above defined filtrations commutes with base change. More precisely, let as at the beginning of this section $U \subset \mathcal{W}^*$ denote a wide open disk and Λ_U the $\mathcal{O}_K$-algebra of bounded rigid functions on U. Let $k \in U(K)$. Then if we denote by $t_k \in \Lambda_U$ a uniformizer at k, i.e., an element which vanishes of order 1 at k and nowhere else, then on the one hand $(\pi, t_k) = \underline{m}_U$ (let us recall that we denoted by π a fixed uniformizer of K) and we have an exact sequence

$$0 \longrightarrow \Lambda_U \xrightarrow{t_k} \Lambda_U \xrightarrow{\rho_k} \mathcal{O}_K \longrightarrow 0$$

which we call the specialization exact sequence. Moreover the weak topology on Λ_U induces the p-adic topology on $\mathcal{O}_K \cong \Lambda_U/t_k\Lambda_U$.

We have natural specialization maps

$$\begin{array}{ccccccc} A_U^o & \longrightarrow & A_k^o & \textit{and} & \eta_k\colon D_U^o & \longrightarrow & D_k^o \\ f & \longmapsto & f_k & & \mu & \longmapsto & \mu_\kappa \end{array} \tag{3}$$

where $f_U(x,y) := f(x,y)(k)$ and $\mu_k \in D_k^o$ is given by $\mu_k\colon f \in A_k^o \longmapsto \mu(f_U)(k)$, where $f_U \in A_U^o$ is given by $f_U(x,y) := k_U(x)f(1, y/x)$.

Proposition 3.11. *Let $U \subseteq \mathcal{W}^*$ be a wide open disk, let $k \in U(K)$, and let $t_k \in \Lambda_U$ be a uniformizer at k. Then we have canonical exact sequences of $\Xi(\mathbb{Z}_p)$-modules*

$$0 \longrightarrow A_U^o \xrightarrow{t_k} A_U^o \longrightarrow A_k^o \to 0$$

$$0 \longrightarrow D_U^o \xrightarrow{t_k} D_U^o \xrightarrow{\eta_k} D_k^o \to 0.$$

For the proof if Λ_U is replaced by an affinoid algebra see [HIS], section §3. The arguments are the same in this case.

Lemma 3.12. *With notations as above we have:* $\eta_k\big(\mathrm{Fil}^n(D_U^o)\big) = \mathrm{Fil}^n(D_k^o)$.

Proof. Let us recall that we have two commutative diagrams

$$\begin{array}{ccc} D_U^o & \overset{\psi_U}{\cong} & \prod_{n\in\mathbb{N}} \Lambda_U \\ \cup & & \cup \\ \mathrm{Fil}^n(D_U)^o & \cong & \prod_{j=0}^{n-1} \underline{m}_U^{n-j} \times \prod_{m\geq n} \Lambda_U \end{array} \quad \text{and} \quad \begin{array}{ccc} D_k^o & \overset{\psi_k}{\cong} & \prod_{n\in\mathbb{N}} \mathcal{O}_K \\ \cup & & \cup \\ \mathrm{Fil}^n(D_k^o) & \cong & \prod_{j=0}^{n-1} \underline{m}_K^{n-j} \times \prod_{m\geq n} \mathcal{O}_K \end{array}$$

The lemma follows observing that the diagram

$$\begin{array}{ccc} D_U^o & \overset{\eta_k}{\longrightarrow} & D_k^o \\ \downarrow \psi_{k_U} & & \downarrow \psi_k \\ \prod_{n\in\mathbb{N}} \Lambda_U & \overset{\prod_{n\in\mathbb{N}} \rho_k}{\longrightarrow} & \prod_{n\in\mathbb{N}} \mathcal{O}_K \end{array}$$

is commutative and that for every $n \in \mathbb{N}$ we have $\rho_k(\underline{m}_U^n) = \underline{m}_K^n$. □

Let us now suppose that $k \in \mathcal{W}^*(K)$ is a classical weight, i.e., k is associated to a pair (k, k) with $k \in \mathbb{N}$. We define $\mathcal{P}_k^o \subset A_k^o$ as the subset of functions $f\colon T_0 \to \mathcal{O}_K$ which are homogeneous polynomials of degree k. It is an $\mathcal{O}_K$-submodule invariant for the action of the semigroup $\Xi(\mathbb{Z}_p) \cap \mathbf{GL}_2(\mathbb{Q}_p)$. Dualizing we obtain a $\Xi(\mathbb{Z}_p) \cap \mathbf{GL}_2(\mathbb{Q}_p)$-equivariant, surjective, $\mathcal{O}_K$-linear map

$$\rho_k\colon D_k^o \longrightarrow V_k^o := \mathrm{Hom}_{\mathcal{O}_K}\big(\mathcal{P}_k^o, \mathcal{O}_K\big).$$

Let us remark that we may identify V_k^o with $\mathrm{Sym}^k(T) \otimes_{\mathbb{Z}_p} \mathcal{O}_K$ with its natural right action of $\Xi(\mathbb{Z}_p) \cap \mathbf{GL}_2(\mathbb{Q}_p)$. For every $n \geq 1$ we set $\mathrm{Fil}^n(V_k^o) := \rho_k(\mathrm{Fil}^n(D_k^o))$. We get a filtration inducing the p-adic topology on V_k^o. We view V_k^o as the continuous representation of Γ defined by the projective system $\big(V_{k,m}^o\big)_{m\in\mathbb{N}}$ with $V_{k,m}^o := V_k^o/\mathrm{Fil}^m V_k^o$ and we set $V_k := V_k^o \otimes_{\mathcal{O}_K} K$.

Therefore, if $U \subset \mathcal{W}^*$ is a wide open disk which contains the classical weight k, we have natural $\Xi(\mathbb{Z}_p) \cap \mathbf{GL}_2(\mathbb{Q}_p)$-equivariant maps

$$D_U \longrightarrow D_k \overset{\rho_k}{\longrightarrow} V_k, \tag{4}$$

and the maps are compatible with the filtrations.

3.2 Overconvergent and p-Adic Families of Modular Symbols

Let us fix an integer $N \geq 3$ and let $\Gamma = \Gamma_1(N) \cap \Gamma_0(p) \subset \mathrm{Iw} \subset \Xi(\mathbb{Z}_p)$. Let us also fix a wide open disk $U \subset \mathcal{W}^*$ and its associated universal character k_U and let $k \in U(K)$ be a weight. Let us also recall that we have natural orthonormalizable $\Lambda_{U,K}$-Banach-modules $D_U := D_{k_U}$ with continuous action of the monoid $\Xi(\mathbb{Z}_p)$ and specialization maps $D_U \longrightarrow D_k$ which are $\Xi(\mathbb{Z}_p)$-equivariant. In particular we'll be interested in the $\Lambda_{U,K}$-module $\mathrm{H}^1\big(\Gamma, D_U\big)$, which we call **module of p-adic families of modular symbols**, the K-vector space $\mathrm{H}^1\big(\Gamma, D_k\big)$, which we call **overconvergent modular symbols of weight** k and the specialization map $\mathrm{H}^1\big(\Gamma, D_U\big) \longrightarrow \mathrm{H}^1\big(\Gamma, D_k\big)$.

Moreover, if $k \in U(K)$ happens to be a classical weight we also have the module $\mathrm{H}^1\big(\Gamma, V_k\big)$, which we call **module of classical modular symbols**, and maps $\rho_k \colon \mathrm{H}^1\big(\Gamma, D_k\big) \longrightarrow \mathrm{H}^1\big(\Gamma, V_k\big)$ obtained from (4) in §3.1.

Relationship with continuous Γ*-cohomology.*

In the notations above let D be any of the $\Xi(\mathbb{Z}_p)$-modules D_U^o, D_k^o, with $k \in U(K)$ and if $k \in U(K)$ is a classical weight V_k^o. We consider the pair $\big(D, \mathrm{Fil}^\bullet(D)\big)$ and wish to study the relationship between $\mathrm{H}^1(\Gamma, D)$ and $\mathrm{H}^1_{\mathrm{cont}}\Big(\Gamma, \big(D/\mathrm{Fil}^n(D)\big)_{n\in\mathbb{N}}\Big)$.

Let us recall that if we denote by $\mathrm{Cont}(\Gamma)$ the category of projective systems $(M_n)_{n\in\mathbb{N}}$ where each M_n is a discrete, torsion Γ-module, then $\mathrm{H}^i_{\mathrm{cont}}(\Gamma, -)$ is the defined as the i-th right derived functor of the functor

$$(M_n)_{n\in\mathbb{N}} \longrightarrow \mathrm{H}^0\big(\Gamma, \lim_{\infty\leftarrow n} M_n\big).$$

The degeneration of the Leray spectral sequence for the composition of the two left exact functors in the above definition gives for every object $(M_n)_{n\in\mathbb{N}} \in \mathrm{Cont}(\Gamma)$ an exact sequence of abelian groups

$$0 \longrightarrow \lim{}^{(1)}\mathrm{H}^0(\Gamma, M_n) \longrightarrow \mathrm{H}^1_{\mathrm{cont}}\big(\Gamma, (M_n)_{n\in\mathbb{N}}\big) \longrightarrow \lim_{\infty\leftarrow n} \mathrm{H}^1(\Gamma, M_n) \longrightarrow 0.$$

For a pair $\big(D, \mathrm{Fil}^\bullet(D)\big)$ as above we have the projective system of **artinian** $\mathcal{O}_K$ and Γ-modules $(D_n)_{n\in\mathbb{N}}$, where $D_n := D/\mathrm{Fil}^n(D)$ and the morphisms in the projective limit are the natural projections. Therefore the projective system $\Big(\mathrm{H}^0(\Gamma, D_n)\Big)_{n\in\mathbb{N}}$ satisfies the Mittag-Leffler condition and consequently we have a natural isomorphism

$$\mathrm{H}^1_{\mathrm{cont}}\big(\Gamma, (D_n)_{n\in\mathbb{N}}\big) \cong \lim_{\infty\leftarrow n} \mathrm{H}^1(\Gamma, D_n).$$

On the other hand we have:

Lemma 3.13. *The natural map*

$$\mathrm{H}^1(\Gamma, D) \longrightarrow \lim_{\infty\leftarrow n} \mathrm{H}^1(\Gamma, D_n)$$

is an isomorphism.

Proof. If M is a Γ-module we denote by $B^1(\Gamma, M)$ the group of 1-coboundaries with coefficients in M and by $Z^1(\Gamma, M)$ the group of 1-cocycles with values in M (no continuity condition is involved in the definition of either coboundaries or cocycles). We have a natural commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
\mathrm{H}^0(\Gamma, D) & \hookrightarrow & B^1(\Gamma, D) & \stackrel{d}{\longrightarrow} & Z^1(\Gamma, D) & \longrightarrow & \mathrm{H}^1(\Gamma, D) & \longrightarrow & 0\\
\downarrow f & & \downarrow g & & \downarrow h & & \downarrow u & & \\
\lim_{\leftarrow} \mathrm{H}^0(\Gamma, D_n) & \hookrightarrow & \lim_{\leftarrow} B^1(\Gamma, D_n) & \stackrel{\lim_{\leftarrow} d_n}{\longrightarrow} & \lim_{\leftarrow} Z^1(\Gamma, D_n) & \stackrel{\alpha}{\longrightarrow} & \lim_{\leftarrow} \mathrm{H}^1(\Gamma, D_n) & &
\end{array}$$

We first claim that the map α in the above diagram is surjective. For this, let us recall that for every $n \in \mathbb{N}$ we have an exact sequence of abelian groups

$$0 \longrightarrow B^1(\Gamma, D_n)/\mathrm{H}^0(\Gamma, D_n) \longrightarrow Z^1(\Gamma, D_n) \longrightarrow \mathrm{H}^1(\Gamma, D_n) \longrightarrow 0.$$

Then let us remark that $B^1(\Gamma, D_n)$ is an artinian $\mathcal{O}_K$-module and so is $B^1(\Gamma, D_n)/H^0(\Gamma, D_n)$. This implies that the projective system $\big(B^1(\Gamma, D_n)/\mathrm{H}^0(\Gamma, D_n)\big)_{n\in\mathbb{N}}$ satisfies the Mittag-Leffler condition and so α is indeed surjective.

Moreover, let us remark that in the above diagram the maps f, g, h are all isomorphisms. For f and g this follows immediately from the definition and the fact that $D = \lim\limits_{\infty\leftarrow n} D_n$. The injectivity of h follows from the fact that $\cap_{n\geq 0}\mathrm{Fil}^n(D) = \{0\}$. On the other hand let us first remark that Γ is a finitely generated group and moreover that a 1-cocycle of Γ is determined by its values on a family of generators. This implies that h is surjective.

Now by the five lemma, u is then an isomorphism as well. □

Remark 3.14. In the notations of the proof of lemma 3.13, let us remark that for every $n \in \mathbb{N}$, $Z^1(\Gamma, D_n)$ is also an artinian $\mathcal{O}_K$-module. The reason is: Γ is a finitely generated group and a 1-cocycle is determined on its values on a set of generators of Γ. It follows that $\mathrm{H}^1(\Gamma, D_n)$ is an artinian $\mathcal{O}_K$-module for all $n \in \mathbb{N}$ and therefore $\mathrm{H}^1(\Gamma, D)$ has a natural structure as profinite $\mathcal{O}_K$-module. In particular it is compact. Moreover, if $D = D_U$ for a certain $U \subset \mathcal{W}^*$, then $\mathrm{H}^1(\Gamma, D_U)$ is naturally a Λ_U-module and its profinite topology is the same as its $\underline{m}_U$-adic topology. In particular $\mathrm{H}^1(\Gamma, D_U)$ is complete and separated for the $\underline{m}_U$-adic topology.

Let us remark that we proved the following theorem.

Theorem 3.15. *Let D be one of D_U^o, D_k^o or if k is a classical weight, V_k^o.*

a) We have canonical isomorphisms

$$\mathrm{H}^1_{\mathrm{cont}}\big(\Gamma, (D_n)_{n\in\mathbb{N}}\big) \cong \lim_{\infty\leftarrow n} \mathrm{H}^1(\Gamma, D_n) \cong \mathrm{H}^1(\Gamma, D).$$

b) The isomorphisms at a) above are compatible with specializations.

Proof. a) follows from lemma 3.13 and b) from the discussion on specialization in the previous section. □

Hecke operators.

Let M be any one of the modules $\mathrm{H}^1\big(\Gamma, D_U\big)$, $\mathrm{H}^1\big(\Gamma, D_k\big)$, $\mathrm{H}^1\big(\Gamma, V_k\big)$. The action of $\Xi(\mathbb{Z}_p)$ on the coefficients defines actions of Hecke operators T_ℓ for ℓ not dividing pN and U_ℓ for ℓ dividing pN on the module M (see [AS] and [HIS]). Moreover U_p is completely continuous on M. Let now fix $h \in \mathbb{Q}$, $h \geq 0$ and denote by D one of D_k or V_k. Then we have a natural direct sum decomposition

$$\mathrm{H}^1\big(\Gamma, D\big) \cong \mathrm{H}^1\big(\Gamma, D\big)^{(h)} \oplus \mathrm{H}^1\big(\Gamma, D\big)^{(>h)},$$

where the decomposition is characterized by the following properties:

a) $\mathrm{H}^1\big(\Gamma, D\big)^{(h)}$ is a finite K-vector space and for every $x \in \mathrm{H}^1\big(\Gamma, D\big)^{(h)}$ there is a non-zero polynomial $Q(t) \in K[t]$ of slope smaller or equal to h such that $Q^*(U_p)\cdot x = 0$, where $Q^*(t) = t^{\deg Q}Q(1/t)$.

b) For every polynomial $Q(t)$ as at a) above the linear map $Q^*(U_p)\colon \mathrm{H}^1\big(\Gamma, D\big)^{(>h)} \longrightarrow \mathrm{H}^1\big(\Gamma, D\big)^{(>h)}$ is an isomorphism.

Moreover we have the following two results (see [AS])

Theorem 3.16. *Suppose $k = (k_0, i) \in \mathcal{W}^*(K)$ a classical weight, $h \geq 0$ is a slope such that $h < k_0 - 1$. Then ρ_k induces an isomorphisms*

$$\mathrm{H}^1\big(\Gamma, D_k\big)^{(h)} \cong \mathrm{H}^1\big(\Gamma, V_k\big)^{(h)}.$$

and

Theorem 3.17. *Let $k \in \mathcal{W}^*(K)$ be an accessible weight and $h \geq 0$ a slope. Then there is a wide open disk defined over K, $U \subset \mathcal{W}^*$, containing k such that*

i) We have a natural, B_U-linear slope $\leq h$-decomposition

$$\mathrm{H}^1\big(\Gamma, D_U\big) \cong \mathrm{H}^1\big(\Gamma, D_U\big)^{(h)} \oplus \mathrm{H}^1\big(\Gamma, D_U\big)^{(>h)}$$

satisfying analogue properties as in a) and b) above.

ii) The slope decomposition at i) above is compatible with specialization, i.e., the map

$$\psi_k \colon \mathrm{H}^1\big(\Gamma, D_U\big) \longrightarrow \mathrm{H}^1\big(\Gamma, D_k\big)$$

satisfies $\psi_k\Big(\mathrm{H}^1\big(\Gamma, D_U\big)^{(\&)}\Big) \subset \mathrm{H}^1\big(\Gamma, D_k\big)^{(\&)}$, where $(\&) \in \{(h), (> h)\}$

Proof. Claim ii) follows from the functoriality of the slope $\leq h$-decompositions proved in [AS].

To show i) it suffices to produce a slope $\leq h$-decomposition at the level of cochains $C^\bullet(\Gamma, D_U)$ using a finite resolution of $\mathbb{Z}$ by finite and free $\mathbb{Z}[\Gamma]$-modules as explained in [AS] section §4.

We start by choosing a wide open disk $U' \subset \mathcal{W}^*$ defined over K such that $k \in U'(K)$. Let $(e_i)_{i \in I}$ be a $B_{U'}$-orthonormal basis of $D_{U'}$ defined in lemma 3.5) and let $V = \mathrm{Spm}(K\langle T \rangle)$ be a closed disk centered at k contained in U'. Thanks to lemma 3.8 and the theory in [AS] section §4, possibly after shrinking V to a smaller affinoid disk, we may assume that the Fredholm determinant of the U_p-operator $F_V^\bullet$ on the complex of group cochains $C^\bullet\big(\Gamma, D_{U'}|_V\big)$ admits a slope $\leq h$-decomposition. Let U be the wide open disk associated to the noetherian local $\mathcal{O}_K$-algebra $\Lambda_U := \mathcal{O}_K[\![T]\!]$. Let us remark that the Fredholm determinant of U_p acting on $C^\bullet\big(\Gamma, D_{U'}|_V\big)$, $F_V^\bullet$, is the same as the Fredholm determinant of U_p acting on $C^\bullet\big(\Gamma, D_U\big)$, $F_U^\bullet$, as they are both computed using the same (weak) ON basis. Because the Banach norm of $\Lambda_U[1/p] = B_U$ restricts to the Gauss norm of B_V it follows that the slope $\leq h$-decomposition of $F_V^\bullet$ determines a slope $\leq h$-decomposition of $F_U^\bullet$. Therefore we obtain the slope $\leq h$-decomposition of $C^\bullet(\Gamma, D_U)$. $\square$

3.3 The geometric picture

Let us recall from the beginning of section §2 the modular curves $X(N,p) \longrightarrow X_1(N)$, their natural formal models $\mathcal{X}(N,p) \longrightarrow \mathcal{X}_1(N)$ and if $w \in \mathbb{Q}$, $0 \leq w \leq p/(p+1)$ we also had a rigid analytic space $X(w) \subset X(N,p)$ and its formal model $\mathcal{X}(w)$, with its natural morphism $\mathcal{X}(w) \longrightarrow \mathcal{X}(N,p)$. All these rigid spaces and formal schemes are in fact log rigid spaces and respectively log formal schemes, which are all log smooth and all the maps described are maps of log formal schemes or log rigid spaces.

Sheaves on $X(N,p)^{\mathrm{ket}}_{\overline{K}}$ associated to modular symbols: Let $\mathcal{E} \longrightarrow X(N,p)$ be the universal generalised elliptic curve, and let us denote, as in section §3.1, by $\mathcal{T}$ the p-adic Tate-module of $\mathcal{E}$, seen as a continuous sheaf on the Kummer étale site of $X(N,p)$, denoted $X(N,p)^{\rm ket}$. If $\eta = \mathrm{Spec}(\mathbb{K})$ denotes a geometric generic point of $X(N,p)$, let $\mathcal{G}$ denote the geometric Kummer étale fundamental group associated to $\big(X(N,p),\eta\big)$ and let $T := \mathcal{T}_\eta$. One can easily see that T is a free $\mathbb{Z}_p$-module of rank 2 with continuous action of $\mathcal{G}$. Let us choose a $\mathbb{Z}_p$-basis $\{\epsilon_1,\epsilon_2\}$ of T satisfying the properties: $\langle \epsilon_1,\epsilon_2\rangle = 1$ and $\epsilon_1(\bmod\ pT) \in \mathcal{E}_\eta[p]$ does not belong to the universal level p-subgroup C. We let $T_0 := \{a\epsilon_1 + b\epsilon_2 \quad | \quad a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p\}$. Then T_0 is a compact subset of T preserved by $\mathcal{G}$ which can be identified to $\mathbb{Z}_p^\times \times \mathbb{Z}_p$. Moreover the right action of $\mathcal{G}$ on the above chosen basis defines a continuous group homomorphism

$$\gamma\colon \mathcal{G} \longrightarrow \mathrm{Iw} \text{ defined by } (\epsilon_1\sigma,\epsilon_2\sigma) = (\epsilon_1,\epsilon_2)\gamma(\sigma) \text{ for } \sigma \in \mathcal{G}.$$

Therefore, if $k \in U \subset \mathcal{W}^*$ and $n \geq 1$ as in section 3.1 then via the homomorphism γ above, the Iw-modules A_U, D_U, A_k, D_k and if k is classical V_k, can be seen as ind-continuous representations of $\mathcal{G}$. More precisely, for example $A_U = \lim\limits_{\rightarrow}\Big(\lim\limits_{\infty\leftarrow n}(A_U^o/\underline{m}_U^n A_U^o)\Big)$ and $D_U = \lim\limits_{\rightarrow}\Big(\lim\limits_{\infty\leftarrow n}(D_U^o/\mathrm{Fil}^n(D_U^o))\Big)$, where the inductive limits are taken with respect to the multiplication by p-map. We denote by $\mathcal{A}_U$, $\mathcal{D}_U$, $\mathcal{A}_k$, $\mathcal{D}_k$ and if k is classical by $\mathcal{V}_k$ the ind-continuous shaves on $X(N,p)^{\rm ket}_{\overline{K}}$ associated to these representations. Let us remark that if the classical weight is associated to the pair $\big(k, k(\mathrm{mod}(p-1)\big)$, for $k \in \mathbb{Z}$, $k \geq 0$ in fact we have $\mathcal{V}_k \cong \mathrm{Sym}^k(\mathcal{T}) \otimes_{\mathbb{Z}_p} K$, as ind-continuous sheaves on $X(N,p)^{\rm ket}_{\overline{K}}$.

Notation: For later use, for $A = A_U$ or $A = A_k$ we write $\mathcal{A}^o := \big(\mathcal{A}_n^o\big)_{n\in\mathbb{N}}$ for the continuous sheaf on Faltings' site $\mathfrak{X}(N,p)$ associated to the continuous representation of $A^o = \big(A^o/\underline{m}^n A^o\big)_{n\in\mathbb{N}}$ of the Kummer étale fundamental group $\mathcal{G}$ of $X(N,p)$. The ind-continuous sheaf $\mathcal{A}$ is simply $\mathcal{A}^o[1/p]$.

Analogously, for $D = D_U$ or D_k, we write $\mathcal{D}^o := \big(\mathcal{D}_n\big)_{n\in\mathbb{N}}$ for the continuous sheaf associated to $D^o/\mathrm{Fil}^n(D^o)$. Then, $\mathcal{D}$ is the ind-continuous sheaf $\mathcal{D}^o[1/p]$.

We proceed as in section 5 in order to define Hecke operators on $\mathrm{H}^1\big(X(N,p)^{\rm ket}_{\overline{K}}, \mathcal{D}_U\big)$ and $\mathrm{H}^1\big(X(N,p)^{\rm ket}_{\overline{K}}, \mathcal{D}_k\big)$ and if k is classical on $\mathrm{H}^1\big(X(N,p)^{\rm ket}_{\overline{K}}, \mathcal{V}_k\big)$. Let ℓ denote a prime integer. Let $X(N,p)_\ell$ be the modular curve classifying generalized elliptic curves $\mathcal{E}/S$ with a $\Gamma_1(N)$-level structure $\psi_N\colon \mathbb{Z}/N\mathbb{Z} \subset \mathcal{E}$, a subgroup scheme $C \subset \mathcal{E}$ defining a $\Gamma_0(p)$-level structure and a subgroup scheme $H \subset \mathcal{E}$ defining a $\Gamma_0(\ell)$-level structure such that H does not intersect the image of ψ_N and similarly $C \cap H = \{0\}$. We have morphisms

$$X(N,p) \xleftarrow{\pi_1} X(N,p)_\ell \xrightarrow{\pi_2} X(N,p),$$

where π_1 forgets H while π_2 is defined by taking the quotient by H. They are finite and Kummer log étale. The dual $\pi_\ell^\vee\colon \big(\mathcal{E}/H\big)^\vee \to \mathcal{E}^\vee$ of the universal isogeny $\pi_\ell\colon \mathcal{E} \to \mathcal{E}/H$ over $X(N,p)_\ell$ provides a map $\pi_\ell^\vee\colon p_2^*(T) \to p_2^*\big(T\big)$. Here we identify $\mathcal{E} \cong \mathcal{E}^\vee$ and $\mathcal{E}/H \cong \big(\mathcal{E}/H\big)^\vee$ via the principal polarizations. The condition $H \cap C = \{0\}$ implies that π_ℓ^* restricts to a map $p_2^*\big(T_0\big) \to p_1^*\big(T_0\big)$. Proceeding as in section 5 we get Hecke operators acting on $\mathrm{H}^1\big(X(N,p)^{\rm ket}_{\overline{K}}, \mathcal{D}_U\big)$ and on $\mathrm{H}^1\big(X(N,p)^{\rm ket}_{\overline{K}}, \mathcal{D}_k\big)$, which commute with the action of G_K and are compatible with specializations. As customary we denote them by T_ℓ, for ℓ prime to pN, and U_ℓ for ℓ dividing pN.

Proposition 3.18. *We have natural isomorphisms as Hecke modules, compatible with specializations*

$$\mathrm{H}^1\big(\Gamma, D_U\big) \cong \mathrm{H}^1\big(X(N,p)^{\mathrm{ket}}_{\overline{K}}, \mathcal{D}_U\big) \text{ and } \mathrm{H}^1\big(\Gamma, D_k\big) \cong \mathrm{H}^1\big(X(N,p)^{\mathrm{ket}}_{\overline{K}}, \mathcal{D}_k\big).$$

Here on the left modules the Hecke operators are defined by the action of the monoid $\Xi(\mathbb{Z}_p)$ on the coefficients.

Proof. We first prove that, given a finite representation F of $\mathcal{G}$ and considering the associated locally constant sheaf $\mathcal{F}$ on $X(N,p)^{\mathrm{ket}}_{\overline{K}}$, we have an isomorphism $\mathrm{H}^1\big(\Gamma, F\big) \cong \mathrm{H}^1\big(X(N,p)^{\mathrm{ket}}_{\overline{K}}, \mathcal{F}\big)$, functorial in F. First of all notice that restriction to the open modular curve where the log-structure is trivial, i.e., to $Y(N,p) \subset X(N,p)$, induces an isomorphism $\mathrm{H}^1\big(X(N,p)^{\mathrm{ket}}_{\overline{K}}, \mathcal{F}\big) \cong \mathrm{H}^1\big(Y(N,p)^{\mathrm{et}}_{\overline{K}}, \mathcal{F}\big)$ thanks to [Il, Cor. 7.5]. As $Y(N,p)_{\overline{K}}$ is a smooth affine curve and for every embedding $\overline{K} \subset \mathbb{C}$ the fundamental group of $Y(N,p)_{\mathbb{C}}$ is Γ, it is a classical result that $\mathrm{H}^1\big(Y(N,p)^{\mathrm{et}}_{\overline{K}}, \mathcal{F}\big) \cong \mathrm{H}^1\big(Y(N,p)^{\mathrm{et}}_{\mathbb{C}}, \mathcal{F}\big) \cong \mathrm{H}^1\big(\Gamma, F\big)$.

Therefore, if we denote by D any one of D_U, D_k or if k is classical V_k and by $\big((D^o_n)_{n\in\mathbb{N}}\big) \otimes_{\mathcal{O}_K} K$ with $D^o_n := D^o/\mathrm{Fil}^n(D^o)$, the ind-continuous representation of $\mathcal{G}$ associated to D, let $\mathcal{D} := \big((\mathcal{D}^o_n)_{n\in\mathbb{N}}\big) \otimes_{\mathcal{O}_K} K$ be the ind-continuous étale sheaf on $X(N,p)^{\mathrm{ket}}_{\overline{K}}$ associated to it. By theorem 3.15 and the discussion above we have natural isomorphisms

$$\mathrm{H}^1\big(\Gamma, D\big) \cong \mathrm{H}^1_{\mathrm{cont}}\Big(\Gamma, \big((D^o_n)_n \otimes K\Big) \cong \mathrm{H}^1_{\mathrm{cont}}\Big(X(N,p)^{\mathrm{ket}}_{\overline{K}}, \big((\mathcal{D}^o_n)_n\big) \otimes K\Big) =: \mathrm{H}^1\big(X(N,p)^{\mathrm{ket}}_{\overline{K}}, \mathcal{D}\big).$$

As the definition of the Hecke operators uses the Hecke correspondence $X(N,p)_\ell$, it is clear that for D one of D_U, D_k or V_k the isomorphism $\mathrm{H}^1\big(X(N,p)^{\mathrm{ket}}_{\overline{K}}, \mathcal{D}\big) \cong \mathrm{H}^1\big(Y(N,p)^{\mathrm{et}}_{\mathbb{C}}, \mathcal{D}\big)$ is Hecke equivariant and the claim follows.

□

Sheaves on Faltings' site associated to modular symbols: Let us now denote by $\mathfrak{X}(N,p)$, Faltings' site associated to the pair $\big(\mathcal{X}(N,p), X(N,p)\big)$. The map of sites $u\colon \mathfrak{X}(N,p) \longrightarrow X^{\mathrm{ket}}_{\overline{K}}$, given by $(U,W) \mapsto W$, sends covering families to covering families, commutes with fibre products and sends the final object to the final object. It defines a morphism of topoi $u_*\colon \mathrm{Sh}(X^{\mathrm{ket}}_{\overline{K}}) \longrightarrow \mathrm{Sh}(\mathfrak{X}(N,p))$ which extends to inductive systems of continuous sheaves. In particular all the ind-continuous Kummer étale sheaves $\mathcal{D}_U$, $\mathcal{D}_k$ and, if k is a classical weight, $\mathcal{V}_k$ can be seen as ind-continuous sheaves on $\mathfrak{X}(N,p)$ by applying u_*. For simplicity we omit u_* from the notation.

Proposition 3.19. *The natural morphisms*

$$\mathrm{H}^1\big(\mathfrak{X}(N,p), \mathcal{D}_U\big) \longrightarrow \mathrm{H}^1\big(X(N,p)^{\mathrm{ket}}_{\overline{K}}, \mathcal{D}_U\big) \text{ and } \mathrm{H}^1\big(\mathfrak{X}(N,p), \mathcal{D}_k\big) \longrightarrow \mathrm{H}^1\big(X(N,p)^{\mathrm{ket}}_{\overline{K}}, \mathcal{D}_k\big)$$

are isomorphisms of G_K-modules,compatible with specializations and action of the Hecke operators.

Proof. As $\mathcal{X}(N,p)$ is proper and log smooth over $\mathrm{Spf}(\mathcal{O}_K)$ it follows from [F2, Thm. 9] that for $\mathcal{F}$ a finite locally constant sheaf the natural maps $\mathrm{H}^1\big(\mathfrak{X}(N,p), \mathcal{F}\big) \longrightarrow \mathrm{H}^1\big(X(N,p)^{\mathrm{ket}}_{\overline{K}}, \mathcal{F}\big)$ are isomorphisms. As $\mathcal{D}_U$ and $\mathcal{D}_k$ are inductive limits of projective limits of finite locally constant sheaves in a compatible way by 3.10, the maps in the proposition are isomorphisms. The Hecke operators are defined in terms of the Hecke correspondence $p_1, p_2\colon X(N,p)_\ell \longrightarrow X(N,p)$ and the trace maps $p_{1,*}\big(p_1^*(\mathcal{F})\big) \to \mathcal{F}$ on $X(N,p)^{\mathrm{ket}}_{\overline{K}}$ and on $\mathfrak{X}(N,p)$ defined in (1). As they are compatible via u_*, the displayed isomorphisms are equivariant for the action of the Hecke operators. □

Let us fix $0 \leq w < p/(p+1)$ and let us recall that we have a natural morphism of log formal schemes $\mathcal{X}(w) \longrightarrow \mathcal{X}(N,p)$ whose generic fiber is the inclusion of log rigid spaces $X(w) \subset X(N,p)$. Base change induces a natural functor $\nu\colon \mathfrak{X}(N,p) \longrightarrow \mathfrak{X}(w)$ which sends final objects to final objects and respects the coverings. We obtain ind-continuous sheaves $\mathbb{A}_U$, $\mathbb{A}_k$, $\mathbb{D}_U$, $\mathbb{D}_k$ as ν^* of the sheaves $\mathcal{A}_U$, $\mathcal{D}_U$, $\mathcal{A}_k$, $\mathcal{D}_k$. Note that we have a natural map of continuous sheaves of rings $\widehat{\mathcal{O}}_{\mathfrak{X}(N,p)} \longrightarrow \nu_*\big(\widehat{\mathcal{O}}_{\mathfrak{X}(w)}\big)$ inducing by adjunction a morphism $\nu^*\big(\widehat{\mathcal{O}}_{\mathfrak{X}(N,p)}\big) \longrightarrow \widehat{\mathcal{O}}_{\mathfrak{X}(w)}$. As ν^* commutes with tensor products we deduce the following

Lemma 3.20. *We have natural morphisms of ind-continuous sheaves on $\mathfrak{X}(w)$:*

$$\nu^*\big(\mathcal{D}_U\hat{\otimes}\widehat{\mathcal{O}}_{\mathfrak{X}(N,p)}\big) \longrightarrow \mathbb{D}_U\hat{\otimes}\widehat{\mathcal{O}}_{\mathfrak{X}(w)}$$

and

$$\nu^*\big(\mathcal{D}_k\hat{\otimes}\widehat{\mathcal{O}}_{\mathfrak{X}(N,p)}\big) \longrightarrow \mathbb{D}_k\hat{\otimes}\widehat{\mathcal{O}}_{\mathfrak{X}(w)}.$$

4 Modular sheaves

Modular sheaves are only defined on the site $\mathfrak{X}(w)$, $0 \leq w < p/(p+1)$. Let $\mathcal{W}$ denote the weight space for $\mathbf{GL}_{2/\mathbb{Q}}$, i.e., the rigid analytic space over $\mathbb{Q}_p$ associated to the noetherian $\mathbb{Z}_p$-algebra $\mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]$ and let us fix $(B,\underline{m})$, a complete, local, regular, noetherian $\mathcal{O}_K$-algebra. Let us recall that B is complete and separated for its $\underline{m}$-adic toplogy (the weak topology) and therefore also for the p-adic toplogy. We denote by $B_K := B\otimes_{\mathcal{O}_K} K$ and by $|| \ ||$ the Gauss norm on the Banach K-algebra B_K. Let $k \in \mathcal{W}(B_K)$ be a B_K-valued weight, i.e., a continuous group homomorphism $k\colon \mathbb{Z}_p^\times \longrightarrow B^\times$. We embed $\mathbb{Z}$ in $\mathcal{W}(\mathbb{Q}_p)$ by sending $k\in\mathbb{Z}$ to the character $a \to a^k$ and in general if $k \in \mathcal{W}(B_K)$ as above and $t \in \mathbb{Z}_p^\times$ we use the additive notation $t^k := k(t)$.

Once we fixed B and $k \in \mathcal{W}(B_K)$ we define

$$r := \min\{n \in \mathbb{N} \quad | \quad n > 0 \text{ and } ||k(1+p^n\mathbb{Z}_p) - 1|| < p^{\frac{-1}{p-1}}\},$$

and we fix $w \in \mathbb{Q}$, $w \geq 0$ and $w < 2/(p^r-1)$ if $p > 3$ and $w < 1/3^r$ if $p = 3$. We say that w is **adapted** to r (and to k). Let us also note that given B, k, r as above there is a unique $a \in B_K$ such that for all $t \in 1+p^r\mathbb{Z}_p$ we have $t^k = \exp(a\log(t))$.

There are two instances of the above general situation which will be relevant in what follows:

a) $B = \mathcal{O}_K$ so $B_K = K$, in that case $k \in \mathcal{W}(K)$ is a K-valued weight.

b) We fix first $r > 0$, $r\in\mathbb{N}$ and denote $\mathcal{W}_r := \{k \quad | \quad ||k(1+p^r\mathbb{Z}_p) - 1|| < p^{-1/(p-1)}\}$. It is a wide open rigid subspace of $\mathcal{W}$. Let now $U \subset \mathcal{W}_r$ be a wide open disk of $\mathcal{W}_r$, let A_U be the affinoid algebra of U and we define

$$\Lambda_U = A_U^{\mathfrak{b}} := \{f \in A_U \text{ such that } |f(x)| \leq 1 \text{ for all points } x \in U\}$$

the $\mathcal{O}_K$-algebra of the **bounded rigid functions** on U. Then Λ_U is a complete, local, noetherian $\mathcal{O}_K$-algebra non-canonically isomorphic to $\mathcal{O}_K[\![T]\!]$. Let also: $k_U\colon \mathbb{Z}_p^\times \longrightarrow \Lambda_U^\times$ be the universal character, i.e., if $t\in\mathbb{Z}_p^\times$ and $x \in U$ we have $t^{k_U}(x) = t^x$.

Remark 4.1. Let $(B, \underline{m})$ be as above and let R be a p-adically complete and separated $\mathcal{O}_K$-algebra in which p is not a zero divisor. We denote by $R\hat{\otimes}B$ the ring

$$R\hat{\otimes}B := \lim_{\infty \leftarrow n}\big(R/p^nR \otimes_{\mathcal{O}_K} B/\underline{m}^n\big) \cong \lim_{\leftarrow, n}\big(R/p^nR\hat{\otimes}_{\mathcal{O}_K} B/p^nB\big),$$

where we denoted by $R/p^nR\hat{\otimes}_{\mathcal{O}_K} B/p^nB$ the completion of the usual tensor product with respect to the ideal generated by the image of $R \otimes_{\mathcal{O}_K} \underline{m}$.

In particular, if $B = \mathcal{O}_K$ then $R\hat{\otimes}B = R$ and if $B \cong \mathcal{O}_K[\![T]\!]$ then $R\hat{\otimes}B \cong R[\![T]\!]$.

4.1 The sheaves $\Omega^k_{\mathfrak{X}}(w)$

In this section we recall the main constructions of chapter 3 of [AIS] in a slightly different context. Let $w \in \mathbb{Q}$ be such that $0 \le w < p/(p+1)$.

Let $f\colon \mathcal{E} \longrightarrow \mathcal{X}(w)$ and $f_K\colon \mathcal{E}_K \longrightarrow X(w)$ denote the universal semi-abelian scheme over $\mathcal{X}(w)$ and respectively its generic fiber and let $\mathcal{T} \longrightarrow X(w)$ denote the p-adic Tate module of $\mathcal{E}_K^\vee \longrightarrow X(w)$ seen as a continuous sheaf on $\mathcal{X}(w)^{\rm ket}_{\overline{K}}$. Notice that $\mathcal{E}$ admits a canonical subgroup $\mathcal{C}_1 \subset \mathcal{E}$. Let $\mathcal{T}_0 \subset \mathcal{T}$ be the inverse image of $\mathcal{C}_{1,K}^\vee \backslash \{0\}$ in $\mathcal{T}$ via the natural map $\mathcal{T} \to \mathcal{E}_K^\vee[p] \to \mathcal{C}_{1,K}^\vee$. Then $\mathcal{T}$ and $\mathcal{T}_0$ are continuous, locally constant sheaves on $\mathcal{X}(w)^{\rm ket}_{\overline{K}}$ and as such can be seen as a continuous sheaf on $\mathfrak{X}(w)$. Notice that the sheaf $\mathcal{T}$ is a continuous sheaf of abelian groups, while $\mathcal{T}_0$ is a continuous sheaves of sets and it is endowed with a natural action of $\mathbb{Z}_p^*$.

Let $e\colon \mathcal{X}(w) \longrightarrow \mathcal{E}$ denote the identity section of f and let $\omega_{\mathcal{E}/\mathcal{X}(w)} := e^*\big(\Omega^1_{\mathcal{E}/\mathcal{X}(w)}\big)$. It is a locally free $\mathcal{O}_{\mathcal{X}(w)}$-module of rank 1 and we denote by $\omega_{\mathcal{E}/\mathfrak{X}(w)} := v^*_{\mathcal{X}(w)}\big(\omega_{\mathcal{E}/\mathcal{X}(w)}\big)$. Then $\omega_{\mathcal{E}/\mathfrak{X}(w)}$ is a a continuous sheaf on $\mathfrak{X}(w)$, a locally free $\widehat{\mathcal{O}}^{\rm un}_{\mathfrak{X}(w)}$-module of rank 1.

We have a natural sequence of sheaves and morphisms of sheaves on $\mathfrak{X}(w)$ called the Hodge-Tate sequence of sheaves for $\mathcal{E}/\mathcal{X}(w)$

$$0 \longrightarrow \omega^{-1}_{\mathcal{E}/\mathfrak{X}(w)} \otimes_{\widehat{\mathcal{O}}^{\rm un}_{\mathfrak{X}(w)}} \widehat{\mathcal{O}}_{\mathfrak{X}(w)}(1) \longrightarrow \mathcal{T} \otimes \widehat{\mathcal{O}}_{\mathfrak{X}(w)} \stackrel{\rm dlog}{\longrightarrow} \omega_{\mathcal{E}/\mathfrak{X}(w)} \otimes_{\widehat{\mathcal{O}}^{\rm un}_{\mathfrak{X}(w)}} \widehat{\mathcal{O}}_{\mathfrak{X}(w)} \longrightarrow 0.$$

Lemma 4.2. *For every connected, small affine object $\mathcal{U} = \big(Spf(R_{\mathcal{U}}), N_{\mathcal{U}}\big)$ of $\mathcal{X}(w)^{\rm ket}$, the localization of the Hodge-Tate sequence of sheaves at $\mathcal{U}$ is the Hodge-Tate sequence of continuous $\mathcal{G}_{\mathcal{U}}$-representations which appears in [AIS] section §2*

$$0 \longrightarrow \omega^{-1}_{\mathcal{E}/\mathcal{X}(w)}(\mathcal{U}) \otimes_{R_{\mathcal{U}}} \widehat{\overline{R}}_{\mathcal{U}}(1) \longrightarrow T_p\big(\mathcal{E}_{\mathcal{U}}^\vee\big) \otimes \widehat{\overline{R}}_{\mathcal{U}} \stackrel{{\rm dlog}_{\mathcal{U}}}{\longrightarrow} \omega_{\mathcal{E}/\mathcal{X}(w)}(\mathcal{U}) \otimes_{R_{\mathcal{U}}} \widehat{\overline{R}}_{\mathcal{U}} \longrightarrow 0$$

Proof. The proof is clear. □

Lemma 4.3. *Let $\mathcal{F}^0 := {\rm Im}({\rm dlog})$ and $\mathcal{F}^1 := {\rm Ker}({\rm dlog})$. Then*

i) $\mathcal{F}^0, \mathcal{F}^1$ are locally free sheaves of $\widehat{\mathcal{O}}_{\mathfrak{X}(w)}$-modules on $\mathfrak{X}(w)$ of rank 1. We denote by $\mathcal{F}^{i,(r)} := j_r^*(\mathcal{F}^i)$ for $i = 0, 1$, they are locally free $\widehat{\mathcal{O}}_{\mathfrak{X}^{(r)}(w)}$-modules of rank 1.

ii) We set $v := w/(p-1)$ and let us suppose that w is adapted to r, for a certain $r \ge 1$, $r \in \mathbb{N}$. We denote $\mathcal{C}_r \subset \mathcal{E}[p^r]$ the canonical subgroup of level p^r of $\mathcal{E}[p^r]$ over $X^{(r)}(w)$ (which exists by the assumption on w), denote by $\mathcal{C}_r^\vee$ its Cartier dual and we also denote by $\mathcal{C}_r$ and $\mathcal{C}_r^\vee$ the groups of points of these group-schemes over $X^{(r)}(w)$, and by the same symbols the constant *abelian sheaves on* $\big(X^{(r)}(w)\big)^{\rm ket}$. *We have natural isomorphisms as* $\widehat{\mathcal{O}}_{\mathfrak{X}(w)}$*-modules on* $\mathfrak{X}(w)$*:* $\mathcal{F}^0/p^{(1-v)r}\mathcal{F}^0 \cong \mathcal{C}_r^\vee \otimes \mathcal{O}_{\mathfrak{X}(w)}/p^{(1-v)r}\mathcal{O}_{\mathfrak{X}(w)}$ *and* $\mathcal{F}^1/p^{(1-v)r}\mathcal{F}^1 \cong \mathcal{C}_r \otimes \mathcal{O}_{\mathfrak{X}(w)}/p^{(1-v)r}\mathcal{O}_{\mathfrak{X}(w)}$.

iii) we have natural isomorphisms of $\mathcal{O}_{\mathcal{X}^{(r)}(w)}$*-modules with* G_r*-action:*

$$v_{\mathfrak{X}^{(r)}(w),*}\big(\mathcal{F}^{i,(r)}\big) \cong \mathcal{F}_i^{(r)} \otimes_{\mathcal{O}_K} \mathcal{O}_{\mathbb{C}_p},$$

for $i = 0, 1$. *Here* $\mathcal{F}_i^{(r)}$ *are the sheaves on* $\mathcal{X}^{(r)}(w)$ *defined in section §2 of [AIS]. Moreover* $\mathcal{F}^{i,(r)} \cong v^*_{\mathfrak{X}^{(r)}(w)}(\mathcal{F}_i^{(r)}) \otimes_{\widehat{\mathcal{O}}^{\rm un}_{\mathfrak{X}^{(r)}(w)}} \widehat{\mathcal{O}}_{\mathfrak{X}^{(r)}(w)}$.

Proof. We consider a connected, small affine object $\mathcal{U} = \big(\mathrm{Spf}(R_{\mathcal{U}}), N_{\mathcal{U}}\big)$ of $\mathcal{X}^{\rm ket}$ as in lemma 4.2. Then the localizations of $\mathcal{F}^i$ at $\mathcal{U}$ are: $\mathcal{F}^0\big(\overline{R}_{\mathcal{U}}, \overline{N}_{\mathcal{U}}\big) = \mathrm{Im}(\mathrm{dlog}_{\mathcal{U}}) = F^0$ and $\mathcal{F}^1\big(\overline{R}_{\mathcal{U}}, \overline{N}_{\mathcal{U}}\big) = \mathrm{Ker}(\mathrm{dlog}_{\mathcal{U}}) = F^1$ and we apply proposition 2.4 of [AIS]. This proves i) and ii). Now we apply proposition 2.6 of [AIS] and iii) follows. □

Let now B, k, r, w be as at the beginning of section 4, i.e., B is a complete, regular, local, noetherian $\mathcal{O}_K$-algebra, $k \in \mathcal{W}(B_K)$ and $r \in \mathbb{N}$ and $w \in \mathbb{Q}$ such that w is adapted to r and k. Let us also recall that we denoted $v := w/(p-1)$.

Let us denote $S_{\mathfrak{X}^{(r)}(w)} := \mathbb{Z}_p^\times\big(1 + p^{(1-v)r}\widehat{\mathcal{O}}_{\mathfrak{X}^{(r)}(w)}\big)$, it is a sheaf of abelian groups on $\mathfrak{X}^{(r)}(w)$ which acts on $\widehat{\mathcal{O}}_{\mathfrak{X}^{(r)}(w)}\hat{\otimes}B := \lim\limits_{\infty\leftarrow n}\big(\widehat{\mathcal{O}}_{\mathfrak{X}^{(r)}(w)}/\pi^n\widehat{\mathcal{O}}_{\mathfrak{X}^{(r)}(w)} \otimes_{\mathcal{O}_K} B/\underline{m}^n\big)$ as follows: let $s = c \cdot x \in S_{\mathfrak{X}^{(r)}(w)}(\mathcal{U}, W, \alpha) = \mathbb{Z}_p^\times\big(1+p^{(1-v)r}\widehat{\mathcal{O}}_{\mathfrak{X}(w)}(\mathcal{U}, W)\big)$ and $y \in \widehat{\mathcal{O}}_{\mathfrak{X}^{(r)}(w)}(\mathcal{U}, W, \alpha)\hat{\otimes}B = \widehat{\mathcal{O}}_{\mathfrak{X}(w)}(\mathcal{U}, W)\hat{\otimes}B$. Then we define

$$s * y := \exp(a\log(x)) \cdot c^k \cdot y, \text{ where } a \in B_K \text{ is such that } t^k = \exp(a\log(t)), t \in 1 + p^r\mathbb{Z}_p.$$

Let us remark that $s * y \in \widehat{\mathcal{O}}_{\mathfrak{X}^{(r)}(w)}(\mathcal{U}, W, \alpha)\hat{\otimes}B$. We denote by $\big(\widehat{\mathcal{O}}_{\mathfrak{X}^{(r)}(w)}\hat{\otimes}B\big)^{(k)}$ the continuous sheaf $\widehat{\mathcal{O}}_{\mathfrak{X}^{(r)}(w)}\hat{\otimes}B$ with the above defined action of $S_{\mathfrak{X}^{(r)}(w)}$.

Thanks to lemma 4.3 ii) that we have an isomorphism of sheaves

$$\varphi\colon \mathcal{F}^{0,(r)}/p^{(1-v)r}\mathcal{F}^{0,(r)} \cong \big(\mathcal{C}_r\big)^\vee \otimes \widehat{\mathcal{O}}_{\mathfrak{X}^{(r)}(w)}/p^{(1-v)r}\widehat{\mathcal{O}}_{\mathfrak{X}^{(r)}(w)}.$$

Let $\mathcal{F}^{(r)'}$ denote the inverse image under the isomorphism φ above of the sheaf of sets $(\mathcal{C}_r)^\vee - (\mathcal{C}_r)^\vee[p^{r-1}]$. It is endowed with an action of $S_{\mathfrak{X}^{(r)}(w)}$.

Recall from §2.6 that we have define a morphism of sites $j_r\colon \mathfrak{X}(w) \to \mathfrak{X}^{(r)}(w)$. It then follows from the construction that dlog induces a map

$$\mathrm{dlog}\colon j_r^*(\mathcal{T}_0) \longrightarrow \mathcal{F}^{(r)'},$$

compatible with the actions of $\mathbb{Z}_p^*$ on the two sides.

Lemma 4.4. *The sheaf* $\mathcal{F}^{(r)'}$ *is an* $S_{\mathfrak{X}^{(r)}(w)}$*-torsor and there exists a covering of* $\mathcal{X}(w)$ *by small affine objects* $\{U_i\}$ *such that* $\mathcal{F}^{(r)'}|_{(U_i, U_i\times_{\mathcal{X}(w)}X^{(r)}}$ *is the trivial torsor for every* i.

Proof. We localize at a connected, small affine object $\mathcal{U}$ of $(\mathcal{X}^{(r)}(w))^{\rm ket}$ and apply lemma 4.3(iii) and [AIS] section §3. □

Let us now consider the $\mathcal{O}_{\mathfrak{X}^{(r)}(w)}\hat{\otimes}B$-module

$$\mathcal{M}_k^{(r)}(w) := \mathfrak{H}om_{S_{\mathfrak{X}^{(r)}(w)}}\big(\mathcal{F}^{(r)'}, (\widehat{\mathcal{O}}_{\mathfrak{X}^{(r)}(w)}\hat{\otimes}B)^{(-k)}\big).$$

Thanks to lemma 4.4 it is a locally free $\mathcal{O}_{\mathfrak{X}^{(r)}(w)}\hat{\otimes}B$-module of rank 1 and we have a natural isomorphism of $\mathcal{O}_{\mathfrak{X}^{(r)}(w)}\hat{\otimes}B$-modules

$$\mathfrak{H}om_{\widehat{\mathcal{O}}_{\mathfrak{X}^{(r)}(w)}\hat{\otimes}B}\left(\mathcal{M}_k^{(r)}(w), \widehat{\mathcal{O}}_{\mathfrak{X}^{(r)}(w)}\hat{\otimes}B\right) \longrightarrow \mathcal{M}_{-k}^{(r)}(w).$$

We also have the continuous sheaf of $\mathcal{O}_{\mathfrak{X}^{(r)}(w)}\hat{\otimes}B$-modules

$$\mathbb{A}_k^{(r)} := \mathfrak{H}om_{\mathbb{Z}_p^*}\big(j_r^*(\mathcal{T}_0), (\widehat{\mathcal{O}}_{\mathfrak{X}^{(r)}(w)}\hat{\otimes}B)^{(-k)}\big)$$

and a map of continuous sheaves of $\mathcal{O}_{\mathfrak{X}^{(r)}(w)}\hat{\otimes}B$-modules

$$\mathrm{dlog}^{\vee,k}\colon \mathcal{M}_k^{(r)}(w) \longrightarrow \mathbb{A}_k^{(r)}$$

induced by dlog.

For every element $\sigma \in G_r$ we denote also by σ the functor $\big(E_{\mathcal{X}(w)_{\overline{K}}}\big)_{/(\mathcal{X}(w),X^{(r)}(w))} \longrightarrow \big(E_{\mathcal{X}(w)_{\overline{K}}}\big)_{/(\mathcal{X}(w),X^{(r)}(w))}$ defined on objects by $(\mathcal{U}, W, \alpha) \to (\mathcal{U}, W, \sigma\circ\alpha)$ and by identity on the morphisms. This functor induces a continuous functor on the site $\mathfrak{X}^{(r)}(w)$. If $\mathcal{H}$ is a sheaf (or continuous sheaf) on $\mathfrak{X}^{(r)}(w)$ we denote by $\mathcal{H}^\sigma$ the sheaf: $\mathcal{H}^\sigma(\mathcal{U}, W, \alpha) := \mathcal{H}\big(\sigma(\mathcal{U}, W, \alpha)\big) = \mathcal{H}(\mathcal{U}, W, \sigma\circ\alpha)$.

Lemma 4.5. *a) Let us suppose that $\mathcal{G}$ is a sheaf of abelian groups on $\mathfrak{X}(w)$ and $\mathcal{H} := j_r^*(\mathcal{G})$. Then $\mathcal{H}^\sigma = \mathcal{H}$ for all $\sigma \in G_r$.*

b) For $\mathcal{H} = \mathcal{F}^{(r)'}$, $(\widehat{\mathcal{O}}_{\mathfrak{X}^{(r)}(w)}\hat{\otimes}B)^{(-k)}$, or $j_r^*(\mathcal{T}_0)$ we have $\big(\mathcal{H}\big)^\sigma = \mathcal{H}$ for every $\sigma \in G_r$. Hence, the same applies for $\mathcal{M}_k^{(r)}(w)$ and $\mathbb{A}_k^{(r)}$ compatibly with $\mathrm{dlog}^{\vee,k}$.

c) Suppose that $\mathcal{H}$ is a sheaf on $\mathfrak{X}^{(r)}(w)$ such that $\mathcal{H}^\sigma = \mathcal{H}$ for all $\sigma \in G_r$. Then each element $\sigma \in G_r$ defines a canonical automorphism of the sheaf $j_{r,*}(\mathcal{H})$, i.e., we have a canonical action of the group G_r on the sheaf $j_{r,*}(\mathcal{H})$.

Proof. a) follows immediately as $j_r^*(\mathcal{G})(\mathcal{U}, W, \alpha) = \mathcal{G}(\mathcal{U}, W)$.

b) As $\mathcal{O}_{\mathfrak{X}^{(r)}(w)} = j_r^*(\mathcal{O}_{\mathfrak{X}(w)})$ we only need to verify the property for the sheaf $\mathcal{F}^{(r)'}$ which is clear from its definition.

c) Let us recall that $j_{r,*}(\mathcal{H})(\mathcal{U}, W) := \mathcal{H}\big(\mathcal{U}, W \times_{X(w)_{\overline{K}}} X^{(r)}(w)_{\overline{K}}, \mathrm{pr}_1\big)$. We define the automorphism

$$\sigma\colon j_{r,*}(\mathcal{H})(\mathcal{U}, W) = \mathcal{H}\big(\mathcal{U}, W \times_{X(w)_{\overline{K}}} X^{(r)}(w)_{\overline{K}}, \mathrm{pr}_1\big) \longrightarrow$$

$$\longrightarrow j_{r,*}(\mathcal{H})(\mathcal{U}, W) = \mathcal{H}\big(\mathcal{U}, W \times_{X(w)_{\overline{K}}} X^{(r)}(w)_{\overline{K}}, \sigma^{-1}\circ\mathrm{pr}_1\big)$$

by the fact that $\sigma\colon X^{(r)}(w) \longrightarrow X^{(r)}(w)$ is an automorphism over $X(w)$. □

Definition 4.6. We define the sheaves $\Omega^k_{\mathfrak{X}(w)}$ and $\omega^{\dagger,k}_{\mathfrak{X}(w)}$ on $\mathfrak{X}(w)$ by

$$\Omega^k_{\mathfrak{X}(w)} := \Big(j_{r,*}\big(\mathfrak{Hom}_{S_{\mathfrak{X}^{(r)}(w)}}(\mathcal{F}^{(r)\prime}, (\widehat{\mathcal{O}}_{\mathfrak{X}^{(r)}(w)}\hat{\otimes}B)^{(-k)})\big)\Big)^{G_r}$$

and

$$\omega^{\dagger,k}_{\mathfrak{X}(w)} := \Big(j_{r,*}\big(\mathfrak{Hom}_{S_{\mathfrak{X}^{(r)}(w)}}(\mathcal{F}^{(r)\prime}, (\widehat{\mathcal{O}}_{\mathfrak{X}^{(r)}(w)}\hat{\otimes}B)^{(-k)})\big)[1/p]\Big)^{G_r}.$$

The sheaves thus defined enjoy the following properties.

Lemma 4.7. *For every B, k and w as above we have*

i) $\omega^{\dagger,k}_{\mathfrak{X}(w)}$ is a locally free $(\widehat{\mathcal{O}}_{\mathfrak{X}(w)}\hat{\otimes}B)[1/p]$-module of rank 1.

ii) $v_{\mathfrak{X}(w),*}(\omega^{\dagger,k}_{\mathfrak{X}}) \cong \omega^{\dagger,k}_w \otimes_K \mathbb{C}_p$, where $\omega^{\dagger,k}_w$ is the sheaf on $X(w)$ given in definition 3.2 of [AIS].

iii) $\omega^{\dagger,k}_{\mathfrak{X}(w)} \cong \omega^{\dagger,k}_w\hat{\otimes}_{\widehat{\mathcal{O}}_{\mathcal{X}(w)}}\widehat{\mathcal{O}}_{\mathfrak{X}(w)}$.

Proof. i) is a consequence of the fact that $\mathcal{F}^{(r)\prime}$ is a locally trivial $S_{\mathfrak{X}^{(r)}(w)}$-torsor and ii) and iii) follow by localization at a connected small affine $\mathcal{U}$ of $\mathcal{X}(w)^{\rm ket}$ using lemma 4.4. □

As mentioned at the beginning of this section we shall be most interested in two instances of these constructions corresponding to choices of pairs (B,k) as above.

1) The first is the simplest, i.e., when $B = \mathcal{O}_K$ and so an $B_K = K$-valued weight is simply an element $k \in \mathcal{W}(K)$.

2) The second instance appears as follows. Let $U \subset \mathcal{W}^*$ be a wide open disk. Let us recall that Λ_U is a $\mathcal{O}_K$-algebra of bounded rigid functions on U and let us denote by $|| \ ||$ the norm on the K-Banach algebra $\Lambda_{U,K}$.

We denote by $k_U\colon \mathbb{Z}_p^\times \longrightarrow \Lambda_U^\times$ the universal character of U defined by the relation $k_U(t)(x) = t^x$ for $t \in \mathbb{Z}_p^\times, x \in U$.

The constructions for the two instances above are connected as follows. Let U, Λ_U, k_U be as at 2) above. Let also $k \in U(K)$ be a K-valued weight and t_k a uniformizer at k, i.e., an element of Λ_U which vanishes of order 1 at k and nowhere else. Then we have an exact sequence of K-algebras

$$0 \longrightarrow \Lambda_U \xrightarrow{t_k} \Lambda_U \longrightarrow \mathcal{O}_K \longrightarrow 0$$

which induces an exact sequence of sheaves on $\mathfrak{X}(w)$:

$$0 \longrightarrow \omega^{\dagger,k_U}_{\mathfrak{X}(w)} \xrightarrow{t_k} \omega^{\dagger,k_U}_{\mathfrak{X}(w)} \longrightarrow \omega^{\dagger,k}_{\mathfrak{X}(w)} \longrightarrow 0$$

which will be called the specialization exact sequence.

4.2 The map $\mathrm{dlog}^{\vee,k}$.

We start by fixing a triple B, k as in the previous section such that the associated $r = 1$ and let w be adapted to k. We have explained in section §3.3 how to construct a continuous sheaf $\mathcal{A}^o_k(w) = \big(\mathcal{A}^o_{k,n}(w)\big)_{n\in\mathbb{N}} := \big(\nu^*(\mathcal{A}^o_{k,n})\big)_{n\in\mathbb{N}}$ on Faltings' site $\mathfrak{X}(w)$ associated to the continuous representation of $A^o_k = \big(A^o_k/\underline{m}^n A^o_k\big)_{n\in\mathbb{N}}$ (see definition 3.1) of the Kummer étale fundamental group $\mathcal{G}$ of $X(N,p)$. Similarly we have the sheaves $\mathcal{D}^o_k(w)=\big(\mathcal{D}^o_{k,n}(w)\big)_{n\in\mathbb{N}}:=\big(\nu^*(\mathcal{D}^o_{k,n})\big)_{n\in\mathbb{N}}$. By construction and proposition 3.10, the sheaf $\mathcal{D}^o_{k,n}(w)$ is a quotient of $\mathfrak{Hom}_B\big(\mathcal{A}^o_{k,n}(w),B/\underline{m}^m\big)$.

Write $\mathcal{T}_0$ as the continuous sheaf on $\mathfrak{X}(w)$ obtained similarly from the $\mathcal{G}$-representation T_0. Then we have an inclusion of sheaves

$$\mathcal{A}^o_{k,n}(w)\subset\mathfrak{Hom}_{\mathbb{Z}_p^*}\big(\mathcal{T}_0,\big(B/\underline{m}^n\big)^{(k)}\big)$$

on $\mathfrak{X}(w)$, which for every r and $n\in\mathbb{N}$ provides a map of sheaves of $\mathcal{O}_{\mathfrak{X}^{(r)}(w)}\otimes B/\underline{m}^n$-modules

$$\beta_n^{(r)}\colon j_r^*\big(\mathcal{A}^o_{k,n}(w)\big)\otimes_{\mathcal{O}_K}\big(\mathcal{O}_{\mathfrak{X}^{(r)}(w)}/p^n\mathcal{O}_{\mathfrak{X}^{(r)}(w)}\big)\longrightarrow\mathfrak{Hom}_{\mathbb{Z}_p^*}\big(j_r^*\big(\mathcal{T}_0\big),\big(\mathcal{O}_{\mathfrak{X}^{(r)}(w)}\otimes B/\underline{m}^n\big)^{(-k)}\big).$$

These maps are compatible for varying n and define a map of continuous sheaves

$$\beta^{(r)}\colon j_r^*\big(\mathcal{A}^o_k(w)\big)\widehat{\otimes}_{\mathcal{O}_K}\widehat{\mathcal{O}}_{\mathfrak{X}^{(r)}(w)}\longrightarrow\mathbb{A}_k^{(r)}.$$

Proposition 4.8. *(1) The map $\beta^{(r)}$ is injective and G_r-invariant.*

(2) The map $\mathrm{dlog}^{\vee,k}$ *is G_r-invariant and factors via $\beta^{(r)}$.*

Proof. The fact that $\beta^{(r)}$ is G_r-invariant is clear as it is defined already over $\mathfrak{X}(w)$. The G_r-invariance of $\mathrm{dlog}^{\vee,k}$ follows from the G_r-invariance of dlog which is clear from its definition. We prove that $\beta_n^{(r)}$ is injective for every $n\in\mathbb{N}$ and that the map

$$\mathfrak{H}om_{S_{\mathfrak{X}^{(r)}(w)}}\big(\mathcal{F}^{(r)'},(\mathcal{O}_{\mathfrak{X}^{(r)}(w)}\otimes B/\underline{m}^n)^{(-k)}\big)\longrightarrow\mathfrak{H}om_{\mathbb{Z}_p^*}\big(j_r^*(\mathcal{T}_0),(\mathcal{O}_{\mathfrak{X}^{(r)}(w)}\otimes B/\underline{m}^n)^{(-k)}\big)$$

factors via $\beta_n^{(r)}$. It suffices to show this on localizations after localizing at small affine objects of $\mathcal{X}(w)^{\mathrm{ket}}$ covering $\mathcal{X}(w)$; see §2.7. Let $\mathcal{U}=(\mathrm{Spf}(R_\mathcal{U},N_\mathcal{U})$ be any small affine.

The localization of $\mathcal{A}^o_k(w)$: Form now until the end of this section we set $r=1$. Using the notation of section §3.3 choose a $\mathbb{Z}_p$-basis $\{\epsilon_0,\epsilon_1\}$ of T such that $\langle\epsilon_0,\epsilon_1\rangle=1$ and $\epsilon_1(\bmod pT)\in\mathcal{E}_\eta[p]$ belongs to the canonical subgroup $C=C_1$ of level p. Then $T_0:=\{a\epsilon_0+b\epsilon_1|a\in\mathbb{Z}_p^*,b\in\mathbb{Z}_p\}$. Let x be the $\mathbb{Z}_p$-dual of ϵ_0 and y the $\mathbb{Z}_p$-dual of ϵ_1. We deduce from the discussion after definition 3.1 that

$$j_1^*\big(\mathcal{A}^o_{k,n}(w)\big)(\overline{R}_\mathcal{U},\overline{N}_\mathcal{U},g):=\oplus_{h\in\mathbb{N}}(B/\underline{m}^n)x^k\big(y/x\big)^h.$$

Thus, if we let $D:=\big(\mathcal{O}_{\mathfrak{X}^{(1)}(w)}\otimes B/\underline{m}^n\big)(\overline{R}_\mathcal{U},\overline{N}_\mathcal{U},g)$, then

$$\big(j_1^*\big(\mathcal{A}^o_{k,n}(w)\big)\otimes_{\mathcal{O}_K}\mathcal{O}_{\mathfrak{X}^{(1)}(w)}\big)(\overline{R}_\mathcal{U},\overline{N}_\mathcal{U},g)=\oplus_{h\in\mathbb{N}}D\cdot x^k\big(y/x\big)^h.$$

Similarly $\mathfrak{Hom}_{\mathbb{Z}_p^*}\big(j_1^*\big(\mathcal{T}_0\big),(\widehat{\mathcal{O}}_{\mathfrak{X}^{(1)}(w)}\hat{\otimes}B/\underline{m}^n)^{(-k)}\big)(\overline{R}_\mathcal{U},\overline{N}_\mathcal{U},g)$ is the D-module of continuous maps $\mathrm{Hom}_{\mathbb{Z}_p^*}\big(\mathbb{Z}_p^*\epsilon_0+\mathbb{Z}_p\epsilon_1,D^{(-k)}\big)$. To an element $f(x,y):=\sum\alpha_hx^k(y/x)^h$ we associate the function $\mathbb{Z}_p^*\epsilon_0+\mathbb{Z}_p\epsilon_1\to D$ sending $a\epsilon_0+b\epsilon_1\mapsto f(a,b)=\sum\alpha_hk(a)(b/a)^h$. If such function is zero then $f(x,y)$ is zero, proving the first claim.

The localization of $\mathcal{M}_k^{(1)}(w)$: Using the notation of lemma 4.2 we let e_0, e_1 denote an $\widehat{\overline{R}}_\mathcal{U}$-basis of $T\otimes\widehat{\overline{R}}_\mathcal{U}$ such that e_1 is a basis of F^1 over $\widehat{\overline{R}}_\mathcal{U}$ reducing to ϵ_1 modulo p^{1-v} and $\mathrm{dlog}_\mathcal{U}(e_0)$ is a basis of F^0 over $\widehat{\overline{R}}_\mathcal{U}$ reducing to ϵ_0 modulo p^{1-v}. Let X, Y denote the basis of $T\otimes\widehat{\overline{R}}_\mathcal{U}$ which is $\widehat{\overline{R}}_\mathcal{U}$-dual to e_0, e_1 respectively (i.e., $X(e_1)=Y(e_0)=0$ and $X(e_0)=Y(e_1)=1$). Then,

$$\mathfrak{Hom}_{S_{\mathfrak{X}^{(1)}(w)}}\big(\mathcal{F}^{(1)'},(\mathcal{O}_{\mathfrak{X}^{(1)}(w)}\otimes B/\underline{m}^n)^{(-k)}\big)(\overline{R}_\mathcal{U},\overline{N}_\mathcal{U},g)=D\cdot X^k.$$

As $X = ux + vy$ with $u \in \widehat{\overline{R}}_{\mathcal{U}}$ congruent to 1 modulo $p^{1-v}\widehat{\overline{R}}_{\mathcal{U}}$ and $v \in \widehat{\overline{R}}_{\mathcal{U}}$ congruent to 0 modulo $p^{1-v}\widehat{\overline{R}}_{\mathcal{U}}$, it follows that $X^k = x^k\gamma$ with $\gamma \in 1 + p^{1-v}\widehat{\overline{R}}_{\mathcal{U}}\langle(y/x)\rangle$. Here, $\widehat{\overline{R}}_{\mathcal{U}}\langle y/x\rangle$ denotes the p-adically convergent power series in the variable y/x. The second claim follows.

□

In particular, $\mathrm{dlog}^{\vee,k}$ induces a G_1-invariant morphism of $\widehat{\mathcal{O}}_{\mathfrak{X}^{(1)}(w)}\hat{\otimes}B$-modules

$$\mathcal{M}_k^{(1)}(w) \longrightarrow j_1^*\big(\mathcal{A}_k^o(w)\big)\widehat{\otimes}_{\mathcal{O}_K}\widehat{\mathcal{O}}_{\mathfrak{X}^{(1)}(w)}.$$

Taking $\mathfrak{Hom}_{\mathcal{O}_{\mathfrak{X}^{(1)}(w)}\hat{\otimes}B}\big(-, \mathcal{O}_{\mathfrak{X}^{(1)}(w)}\hat{\otimes}B\big)$ and using the identification

$$\mathfrak{Hom}_{\mathcal{O}_{\mathfrak{X}^{(1)}(w)}\hat{\otimes}B}\big(\mathcal{M}_k^{(1)}(w), \mathcal{O}_{\mathfrak{X}^{(1)}(w)}\hat{\otimes}B\big) \cong \mathcal{M}_{-k}^{(1)}(w),$$

we get an induced G_1-invariant morphism $\widehat{B}$-modules

$$\delta\colon \mathfrak{Hom}_{\widehat{B}}\big(j_1^*\big(\mathcal{A}_k^o(w)\big), \widehat{B}\big) \longrightarrow \mathcal{M}_{-k}^{(1)}(w).$$

Then,

Lemma 4.9. *For every $n \in \mathbb{N}$ there exists $m \geq n$ such that the map*

$$\mathfrak{Hom}_B\big(j_1^*\big(\mathcal{A}_{k,m}^o(w)\big), B/\underline{m}^m\big) \longrightarrow \mathfrak{H}om_{\mathcal{O}_{\mathfrak{X}^{(1)}(w)}\otimes B}\big(\mathcal{M}_k^{(1)}(w), (\mathcal{O}_{\mathfrak{X}^{(1)}(w)} \otimes B/\underline{m}^n)^{(-k)}\big),$$

induced by δ, factors via $j_1^*\big(\mathcal{D}_{k,m}^o(w)\big)$.

Proof. As $\mathcal{X}(w)^{\mathrm{ket}}$ can be covered by finitely many small affines, it suffices to show the claim on localizations at a small affine $\mathcal{U} = (\mathrm{Spf}(R_{\mathcal{U}}, N_{\mathcal{U}})$ of $\mathcal{X}(w)^{\mathrm{ket}}$ (see section §2.7). We use the notation of the proof of proposition 4.8. Thanks to proposition 3.10 the quotient map

$$\mathfrak{H}om_B\big(j_1^*\big(\mathcal{A}_{k,m}^o(w)\big), B/\underline{m}^m\big)(\overline{R}_{\mathcal{U}}, \overline{N}_{\mathcal{U}}, g) = \oplus_{h\in\mathbb{N}}(B/\underline{m}^m)\cdot\big(x^k(y/x)^h\big)^\vee \longrightarrow j_r^*\big(\mathcal{D}_{k,m}^o(w)\big)(\overline{R}_{\mathcal{U}}, \overline{N}_{\mathcal{U}}, g)$$

identifies the latter with the B-module $\oplus_{0\leq h\leq m}(B/\underline{m}^{m-h}) \cdot \big(x^k(y/x)^h\big)^\vee$. Recall from the proof of proposition 3.10 that, setting $D := \big(\mathcal{O}_{\mathfrak{X}(w)} \otimes B/\underline{m}^n\big)(\overline{R}_{\mathcal{U}}, \overline{N}_{\mathcal{U}})$, we have defined a generator X^k of $\mathcal{M}_{-k}^{(r)}(w)(\overline{R}_{\mathcal{U}}, \overline{N}_{\mathcal{U}}, g)$ as D-module. We conclude that

$$\mathfrak{H}om_{\mathcal{O}_{\mathfrak{X}^{(1)}(w)}\otimes B}\left(\mathcal{M}_k^{(1)}(w), (\mathcal{O}_{\mathfrak{X}^{(1)}(w)} \otimes B/\underline{m}^n)^{(-k)}\right)(\overline{R}_{\mathcal{U}}, \overline{N}_{\mathcal{U}}, g) \cong D \cdot (X^k)^\vee.$$

Let $N(n)$ be the degree of X^k in $\oplus_{h\in\mathbb{N}}D \cdot x^k\big(y/x\big)^h$, i.e., the maximal N such that the coordinate of X^k with respect to $x^k\big(y/x\big)^N$ is non zero. If we take m so that $m - N \geq n$, then the map δ localized at $\mathcal{U}$ factors via $\mathfrak{Hom}_{\mathcal{O}_{\mathfrak{X}^{(1)}(w)}\otimes B}\big(\mathcal{M}_k^{(1)}(w), (\mathcal{O}_{\mathfrak{X}^{(1)}(w)} \otimes B/\underline{m}^n)^{(-k)}\big)(\overline{R}_{\mathcal{U}}, \overline{N}_{\mathcal{U}}, g)$ as wanted. □

It follows from the lemma 4.9 that we get a map of continuous sheaves of $\widehat{\mathcal{O}}_{\mathfrak{X}^{(r)}(w)}\hat{\otimes}B$-modules

$$\mathcal{D}_k^o(w) \longrightarrow \big(j_{1,*}\big(j_1^*\big(\mathcal{D}_{k,m}^o(w)\big)\big)\big)^{G_1} \longrightarrow \left(j_{1,*}\big(\mathcal{M}_{-k}^{(1)}(w)\big)\right)^{G_1} = \Omega_{\mathfrak{X}(w)}^{\dagger,k}.$$

Passing to ind-sheaves and using lemma 4.7 we obtain a map

$$\delta_k^\vee(w)\colon \nu^*(\mathcal{D}_k) = \mathcal{D}_k^o(w)[1/p] \longrightarrow \omega_{\mathfrak{X}(w)}^{\dagger,k} \cong \omega_w^{\dagger,k}\hat{\otimes}_{\widehat{\mathcal{O}}_{\mathcal{X}(w)}}\widehat{\mathcal{O}}_{\mathfrak{X}(w)}. \tag{5}$$

In the next section we will calculate the cohomology of the ind-continuous sheaves $\omega_{\mathfrak{X}(w)}^{\dagger,k} \cong \omega_w^{\dagger,k}\hat{\otimes}_{\widehat{\mathcal{O}}_{\mathcal{X}(w)}}\widehat{\mathcal{O}}_{\mathfrak{X}(w)}$.

4.3 The cohomology of the sheaves $\omega^{\dagger,k}_{\mathfrak{X}(w)}$

Let $\iota\colon Z \longrightarrow X(w)$ be a morphism in $\mathcal{X}(w)^{\rm fket}_{\overline{K}}$. Let $\mathfrak{Z} := \mathfrak{X}(w)_{/(\mathcal{X}(w),Z)}$ the associated induced site and $j := j_{(\mathcal{X}(w),Z)}\colon \mathfrak{X}(w) \longrightarrow \mathfrak{Z}$ the map $j(\mathcal{U}, W) := \big(\mathcal{U}, Z\times_{X(w)} W, {\rm pr}_1\big)$; see 2.5. It induces a morphism of topoi. For $i \geq 0$ we shall calculate $\mathrm{H}^i\big(\mathcal{Z}, j^*\big(\omega^{\dagger,k}_{\mathfrak{X}(w)}\big)\big)$. For $\iota = \mathrm{id}$ we get in particular the calculation of $\mathrm{H}^i\big(\mathfrak{X}(w), \omega^{\dagger,k}_{\mathfrak{X}(w)}\big)$. We will need the following:

Lemma 4.10. *Let $\mathcal{F}$ be a locally free $(\widehat{\mathcal{O}}_{\mathfrak{X}(w)}\hat{\otimes}B)[1/p]$-module of finite rank. The sheaf $R^b v_{\mathfrak{X}(w),*}\big(\mathcal{F}\big)$ is the sheaf associated to the presheaf on $\mathcal{X}(w)^{\rm ket}$:*

$$\mathcal{U} = (Spf(R_{\mathcal{U}}), N_{\mathcal{U}}) \to \mathrm{H}^b\big(\mathcal{G}_{\mathcal{U}}, \mathcal{F}(\overline{R}_{\mathcal{U}}, \overline{N}_{\mathcal{U}})\big),$$

where $\mathcal{G}_{\mathcal{U}}$ is the Kummer-étale geometric fundamental group of $\mathcal{U}$, for a choice of a geometric generic point, i.e., $\mathcal{G}_{\mathcal{U}} = \mathrm{Gal}\big(\overline{R}_{\mathcal{U}}[1/p]/(R_{\mathcal{U}}\overline{K})\big)$.

Proof. The lemma follows arguing as in [AI, Prop. 2.12 and Lemma 2.24].

□

Theorem 4.11. *We have isomorphisms as G_K-modules*

a) $\mathrm{H}^0\big(\mathfrak{Z}, j^*\big(\omega^{\dagger,k}_{\mathfrak{X}(w)}(1)\big)\big) \cong \mathrm{H}^0\big(Z, \iota^*\big(\omega^{\dagger,k}_{w}\big)\big)\hat{\otimes}_K\mathbb{C}_p(1)$*;*

b) $\mathrm{H}^1\big(\mathfrak{Z}, j^*\big(\omega^{\dagger,k}_{\mathfrak{X}(w)}(1)\big)\big) \cong \mathrm{H}^0\big(Z, \iota^*\big(\omega^{\dagger,k+2}_{w}\big)\big)\hat{\otimes}_K\mathbb{C}_p$*;*

c) $\mathrm{H}^i\big(\mathfrak{Z}, j^*\big(\omega^{\dagger,k}_{\mathfrak{X}(w)}(1)\big)\big) = 0$ *for $i \geq 2$.*

Proof. As $R^i j_* = 0$ for all $i \geq 1$ by 2.6 we have $\mathrm{H}^i\big(\mathfrak{Z}, j^*\big(\omega^{\dagger,k}_{\mathfrak{X}(w)}(1)\big)\big) \cong \mathrm{H}^1\big(\mathfrak{X}(w), j_*\big(j^*(\omega^{\dagger,k}_{\mathfrak{X}(w)}(1))\big)\big)$. Set $\mathcal{F} := j_*\big(j^*\big(\omega^{\dagger,k}_{\mathfrak{X}(w)}(1)\big)\big)$. Recall that $\omega^{\dagger,k}_{\mathfrak{X}(w)}$ is isomorphic to $\omega^{\dagger,k}_{w}\hat{\otimes}_{\widehat{\mathcal{O}}_{\mathcal{X}(w)}}\widehat{\mathcal{O}}_{\mathfrak{X}(w)}$ by 4.7. Thus, $j^*\big(\omega^{\dagger,k}_{\mathfrak{X}(w)}(1)\big) \cong \omega^{\dagger,k}_{w}\hat{\otimes}_{\widehat{\mathcal{O}}_{\mathcal{X}(w)}}\widehat{\mathcal{O}}_{\mathfrak{Z}}(1)$ as $j^*\big(\widehat{\mathcal{O}}_{\mathfrak{X}(w)}\big) \cong \widehat{\mathcal{O}}_{\mathfrak{Z}}$ and $\mathcal{F} \cong \omega^{\dagger,k}_{w}\hat{\otimes}_{\widehat{\mathcal{O}}_{\mathcal{X}(w)}} j_*\big(\widehat{\mathcal{O}}_{\mathfrak{Z}}\big)(1)$. Due to 2.4 the natural map $\widehat{\mathcal{O}}_{\mathfrak{X}(w)}\hat{\otimes}_{\widehat{\mathcal{O}}_{\mathcal{X}(w)}}\iota_*\big(\widehat{\mathcal{O}}_Z\big) \longrightarrow j_*\big(\widehat{\mathcal{O}}_{\mathfrak{Z}}\big)\big[p^{-1}\big]$ is an isomorphism. Hence, $\mathcal{F} \cong \omega^{\dagger,k}_{w}\hat{\otimes}_{\widehat{\mathcal{O}}_{\mathcal{X}(w)}}\iota_*\big(\widehat{\mathcal{O}}_{\mathcal{Z}}\big)(1)$ is a locally free $(\widehat{\mathcal{O}}_{\mathfrak{X}(w)}\hat{\otimes}B)[1/p]$-module.

To prove the theorem we will first calculate the sheaves $R^b v_{\mathfrak{X}(w),*}\big(\mathcal{F}\big)$ using lemma 4.10 and then we'll use the Leray spectral sequence (see [AI]):

$$\mathrm{H}^a\Big(\mathcal{X}(w)^{\rm ket}, R^b v_{\mathfrak{X}(w),*}\big(\mathcal{F}\big)\Big) \Longrightarrow \mathrm{H}^i\big(\mathfrak{X}(w), \mathcal{F}\big).$$

We compute the Galois cohomology of the localization of the ind-continuous sheaf $\mathcal{F}$

$$\mathcal{F}(\overline{R}_{\mathcal{U}}, \overline{N}_{\mathcal{U}}) = \omega^{\dagger,k}_{w}(\mathcal{U})\hat{\otimes}_{R_{\mathcal{U},K}}\iota_*(\mathcal{O}_Z)(\mathcal{U}_K)\hat{\otimes}_{R_{\mathcal{U}}}\widehat{\overline{R}}_{\mathcal{U}}(1).$$

for $\mathcal{U} = (\mathrm{Spf}(R_{\mathcal{U}}, N_{\mathcal{U}})$ a small affine open of $\mathcal{X}(w)^{\rm ket}$. Using that $\omega^{\dagger,k}_{\mathfrak{X}(w)}$ is a locally free $(\widehat{\mathcal{O}}_{\mathfrak{X}(w)}\hat{\otimes}A)[1/p]$-module of rank 1, it follows from the main result of [F1] that $\mathrm{H}^0\big(\mathcal{G}_{\mathcal{U}}, \widehat{\overline{R}}_{\mathcal{U},K}\big) = R_{\mathcal{U},K}\hat{\otimes}_K\mathbb{C}_p$ so that

$$\mathrm{H}^0\big(\mathcal{G}_{\mathcal{U}}, \mathcal{F}(\overline{R}_{\mathcal{U}}, \overline{N}_{\mathcal{U}})\big) = \omega^{\dagger,k}_{w}(\mathcal{U})\hat{\otimes}_{R_{\mathcal{U},K}}\iota_*(\mathcal{O}_Z)(\mathcal{U}_K)\hat{\otimes}_K\mathbb{C}_p(1).$$

Moreover $\mathrm{H}^1\big(\mathcal{G}_{\mathcal{U}}, \widehat{\overline{R}}_{\mathcal{U},K}\big) \cong \Omega^1_{\mathcal{U}_K/K}\hat{\otimes}_K\mathbb{C}_p(-1)$ so that

$$\mathrm{H}^1\big(\mathcal{G}_{\mathcal{U}}, \mathcal{F}(\overline{R}_{\mathcal{U}}, \overline{N}_{\mathcal{U}})\big) \cong \big(\Omega^1_{\mathcal{U}_K/K}\hat{\otimes}_K\mathbb{C}_p(-1)\big)\hat{\otimes}_{R_{\mathcal{U},K}}\iota_*(\mathcal{O}_Z)(\mathcal{U}_K)\hat{\otimes}_K\omega_w^{\dagger,k}(\mathcal{U})(1).$$

The Kodaira-Spencer isomorphism gives $\Omega^1_{\mathcal{U}_K/K}\hat{\otimes}_K B_K \cong \omega^{\otimes 2}_{\mathcal{E}_{\mathcal{U}_K}/\mathcal{U}_K}\hat{\otimes}_K B_K \cong \omega_w^{\dagger,2}(\mathcal{U})$. Therefore we obtain

$$\mathrm{H}^1\big(\mathcal{G}_{\mathcal{U}}, \mathcal{F}(\overline{R}_{\mathcal{U}}, \overline{N}_{\mathcal{U}})\big) \cong \omega_w^{\dagger,k+2}(\mathcal{U})\hat{\otimes}_{R_{\mathcal{U},K}}\iota_*(\mathcal{O}_Z)(\mathcal{U}_K)\hat{\otimes}_K\mathbb{C}_p.$$

Finally $\mathrm{H}^i\big(\mathcal{G}_{\mathcal{U}}, \mathcal{F}(\overline{R}_{\mathcal{U}}, \overline{N}_{\mathcal{U}})\big) = 0$ for $i \geq 2$ because $\mathcal{G}_{\mathcal{U}}$ has cohomological dimension 1. It follows that we have

$$R^0 v_{\mathfrak{X}(w),*}\mathcal{F} \cong \omega_w^{\dagger,k}\hat{\otimes}_{R_{\mathcal{U},K}}\iota_*(\mathcal{O}_Z)(\mathcal{U}_K)\hat{\otimes}_K\mathbb{C}_p(1),$$

where the isomorphism is as sheaves on $\mathcal{X}(w)^{\mathrm{ket}}$. Similarly we have

$$R^1 v_{\mathfrak{X}(w),*}\mathcal{F} = \omega_w^{\dagger,k+2}\hat{\otimes}_{R_{\mathcal{U},K}}\iota_*(\mathcal{O}_Z)(\mathcal{U}_K)\hat{\otimes}_K\mathbb{C}_p,$$

and $R^b v_{\mathfrak{X}(w),*}\mathcal{F} = 0$ for $b \geq 2$. Now let us observe that $\omega^{\dagger,k}\hat{\otimes}_{R_{\mathcal{U},K}}\iota_*(\mathcal{O}_Z)(\mathcal{U}_K)\hat{\otimes}_K\mathbb{C}_p(1)$ is a sheaf of K-Banach modules on $X(w)$, as it is locally isomorphic to $\iota_*(\mathcal{O}_Z)\hat{\otimes}B_K\hat{\otimes}\mathbb{C}_p$. As $X(w)$ is an affinoid we obtain that

$$\mathrm{H}^1\big(\mathcal{X}(w)^{\mathrm{ket}}, \omega_w^{\dagger,k}\hat{\otimes}_{\mathcal{O}_{\mathcal{X}(w)}}\iota_*(\mathcal{O}_Z)\hat{\otimes}_K\mathbb{C}_p(1)\big) = \mathrm{H}^1\big(X(w), \omega_w^{\dagger,k}\hat{\otimes}_{\mathcal{O}_{\mathcal{X}(w)}}\iota_*(\mathcal{O}_Z)\hat{\otimes}\mathbb{C}_p(1)\big) = 0,$$

by Kiehl's vanishing theorem. Therefore the Leray spectral sequence gives now the result of the theorem. □

5 Hecke Operators

Let ℓ denote a prime integer and $w \in \mathbb{Q}$ be such that $0 \leq w < p/(p+1)$. We assume that w is adapted to some integer $r \geq 1$ (see the beginning of section §3). We denote (see section §3.1.1 of [AIS]) by $X_\ell^{(r)}(w)$ the rigid analytic space over K which represents the functor associating to a K-rigid space S a quadruple $\big(\mathcal{E}/S, \psi_S, H, Y\big)$, where $\mathcal{E} \longrightarrow S$ is a semiabelian scheme of relative dimension 1 and Y is a global section of $\omega^{1-p}_{\mathcal{E}/S}$ such that $Yh(\mathcal{E}/S) = p^w$, where we have denoted by h a lift to characteristic 0 of the Hasse-invariant. Let us notice that the existence of Y as above implies that there is a canonical subgroup $C_r \subset \mathcal{E}[p^r]$, of order p^r defined over S. We continue to describe the quadruple $(\mathcal{E}/S, \psi_S, H, Y)$: ψ_S is a $\Gamma_1(Np^r)$-level structure of $\mathcal{E}/S$, more precisely $\psi_S = \psi_N \cdot \psi_{p^r}$, where $\psi_N \colon \mathbb{Z}/N\mathbb{Z} \to \mathcal{E}$ is a closed immersion of group schemes over S and ψ_{p^r} is a generator of C_r. Furthermore $H \subset \mathcal{E}$ is locally free subgroup scheme, finite of order ℓ defining a $\Gamma_0(\ell)$-level structure such that H has empty intersection with the image of ψ_N (this condition is automatic if ℓ does not divide N) and $H \cap C_r = \{0\}$ (this condition is automatic if $\ell \neq p$). We consider on $X_\ell^{(r)}(w)$ the log structure given by the divisor of cusps and denote the resulting log rigid space by the same notation: $X_\ell^{(r)}(w)$.

We have natural morphisms $p_1 \colon X_\ell^{(r)}(w) \longrightarrow X^{(r)}(w)$ and $p_2 \colon X_\ell^{(r)}(w) \longrightarrow X^{(r)}(w')$, where $w' = w$ if $\ell \neq p$ and $w' = p^r w$ if $\ell = p$, in which case we will assume that $0 \leq w \leq 2/p^{2r}$. These morphisms are defined on points as follows: $p_1(\mathcal{E}, \psi, H, Y) := (\mathcal{E}, \psi, Y) \in X^{(r)}(w)$ and $p_2(\mathcal{E}, \psi, H, Y) = \big(\mathcal{E}/H, \psi', Y'\big) \in X^{(r)}(w')$ where ψ', Y' are the induced level structure and global section associated to $\mathcal{E}/H$. The morphism p_1 is finite and Kummer log étale and if $\ell = p$ then p_2 is an isomorphism of K-rigid spaces.

Let us recall that we have denoted $\mathfrak{X}(w)$ Faltings' site associated to the log formal scheme $\mathcal{X}(w)$ and with $\mathfrak{X}^{(r)}(w)$ the site $\mathfrak{X}(w)$ localized at its object $\big(\mathcal{X}(w), X^{(r)}(w)\big)$. Let us observe that $\big(\mathcal{X}(w), X^{(r)}_\ell(w)\big)$ is also an object of $\mathfrak{X}(w)$ therefore we will denote by $\mathfrak{X}^{(r)}_\ell(w) := \mathfrak{X}(w)_{/(\mathcal{X}(w), X^{(r)}_\ell(w))}$, i.e., the localized site (see sections 2.3 and 2.4).

The morphisms p_1, p_2 defined above induce continuous morphisms of sites:

$$\begin{array}{ccccc} & & \mathfrak{X}^{(r)}_\ell(w) & & \\ & \nearrow p_1 & & p_2 \nwarrow & \\ \mathfrak{X}^{(r)}(w) & & & & \mathfrak{X}^{(r)}(w') \end{array}$$

We denote by $\mathcal{E}^{(r)}_w$ the universal generalised elliptic curve over $X^{(r)}(w)$ and by $\pi_\ell\colon \mathcal{E} \longrightarrow \mathcal{E}/H$ the natural universal isogeny over $X^{(r)}_\ell(w)$. Let $\mathcal{T}(\mathcal{E})$, $\mathcal{T}(\mathcal{E}/H)$, $\mathcal{T}(\mathcal{E}^{(r)}_w)$ denote the p-adic Tate modules of $\mathcal{E}, \mathcal{E}/H, \mathcal{E}^{(r)}_w$ seen as continuous sheaves on $\mathfrak{X}^{(r)}_\ell(w)$ and $\mathfrak{X}^{(r)}(w)$ respectively. Then we have maps

$$p_2^*\big(\mathcal{T}(\mathcal{E}^{(r)}_{w'})\big) = \mathcal{T}\big((\mathcal{E}/H)\big) \stackrel{\pi_\ell}{\longleftarrow} p_1^*\big(\mathcal{T}(\mathcal{E}^{(r)}_w)\big) = \mathcal{T}\big(\mathcal{E}\big).$$

which induce the following commutative diagram

$$\begin{array}{ccccc} p_2^*\Big(\mathcal{T}\big((\mathcal{E}^{(r)}_{w'})^\vee\big) \otimes \widehat{\mathcal{O}}_{\mathfrak{X}^{(r)}(w')}\Big) & \cong & \mathcal{T}\big((\mathcal{E}/H)^\vee\big) \otimes \widehat{\mathcal{O}}_{\mathfrak{X}^{(r)}_\ell(w)} & \stackrel{\mathrm{dlog}}{\longrightarrow} & \omega_{\mathcal{E}/H} \otimes \widehat{\mathcal{O}}_{\mathfrak{X}^{(r)}_\ell(w)} \\ & & \downarrow \pi_\ell^\vee \otimes \mathrm{Id} & & \downarrow d(\pi_\ell) \otimes \mathrm{Id} \\ p_1^*\Big(\mathcal{T}\big((\mathcal{E}^{(r)}_w)^\vee\big) \otimes \widehat{\mathcal{O}}_{\mathfrak{X}^{(r)}_\ell(w)}\Big) & \cong & \mathcal{T}\big(\mathcal{E}^\vee\big) \otimes \widehat{\mathcal{O}}_{\mathfrak{X}^{(r)}_\ell(w)} & \stackrel{\mathrm{dlog}}{\longrightarrow} & \omega_{\mathcal{E}} \otimes \widehat{\mathcal{O}}_{\mathfrak{X}^{(r)}_\ell(w)} \end{array}$$

Until the rest of this section we suppose $r = 1$.

Lemma 5.1. *Let now $k \in \mathcal{W}^*(B_K)$ be a weight with associated integer $r = 1$ such that w as above is associated to k. Then the above diagram induces morphisms:*

$$\pi_\ell\colon p_2^*\big(\mathcal{M}^{(1)}_{-k}(w')\big) \longrightarrow p_1^*\big(\mathcal{M}^{(1)}_{-k}(w)\big), \quad \pi_\ell\colon p_2^*\big(j_1^*(\mathcal{D}^o_k(w'))\big) \longrightarrow p_1^*\big(j_1^*(\mathcal{D}^o_k(w'))\big),$$

where $\mathcal{M}^{(1)}_{-k}(w)$ is the sheaf $\mathfrak{Hom}_{S_{\mathfrak{X}^{1r)}(w)}}\big(\mathcal{F}^{(1)\prime}, (\widehat{\mathcal{O}}_{\mathfrak{X}^{(1)}(w)}\hat{\otimes} B)^{(-k)}\big)$ on $\mathfrak{X}^{(1)}(w)$ defined in section §4.1, such that the diagram

$$\begin{array}{ccc} p_2^*\big(j_1^*(\mathcal{D}^o_k(w'))\big) & \stackrel{p_2^*(\delta)}{\longrightarrow} & p_2^*\big(\mathcal{M}^{(1)}_{-k}(w')\big) \\ \downarrow \pi_\ell & & \downarrow \pi_\ell \\ p_1^*\big(j_1^*(\mathcal{D}^o_k(w'))\big) & \stackrel{p_1^*(\delta)}{\longrightarrow} & p_1^*\big(\mathcal{M}^{(1)}_{-k}(w)\big), \end{array}$$

where δ is the map defined in lemma 4.9, is commutative.

Proof. Let $\mathcal{F}^0(\mathcal{E}) := \mathrm{Im}\Big(\mathrm{dlog}\colon \mathcal{T}(\mathcal{E}^\vee) \otimes \widehat{\mathcal{O}}_{\mathfrak{X}^{(1)}_\ell(w)} \longrightarrow \omega_{\mathcal{E}/\mathfrak{X}^{(1)}_\ell(w)}\Big)$ as in lemma 4.3 and we denote $\mathcal{F}^0(\mathcal{E}/H)$ the analogue object constructed with $\mathcal{E}/H$ instead of $\mathcal{E}$. Then, if C_1 is the canonical subgroup of $\mathcal{E}$ and C_1' is the canonical subgroup of $\mathcal{E}/H$ we have a natural commutative diagram

$$\begin{array}{ccc} \mathcal{E}[p] & \xrightarrow{\pi_\ell} & (\mathcal{E}/H)[p] \\ \cup & & \cup \\ C_1 & \cong & C_1' \end{array}$$

where the map on canonical subgroups is an isomorphism. It follows that the dual isogeny $\pi_\ell^\vee\colon (\mathcal{E}/H)^\vee \to \mathcal{E}^\vee$ induces an isomorphism of torsors $\mathcal{F}'\big((\mathcal{E}/H)^\vee\big) \longrightarrow \mathcal{F}'(\mathcal{E}^\vee)$ and, hence, an isomorphism

$$\pi_\ell^\vee\colon p_1^*\big(\mathcal{M}_k^{(1)}(w)\big) \longrightarrow p_2^*\big(\mathcal{M}_k^{(1)}(w')\big).$$

Dualizing with respect to $\widehat{\mathcal{O}}_{\mathfrak{X}^{(1)}(w)}\hat{\otimes}B$ and identifying the dual of $\mathcal{M}_k^{(1)}(w)$ with $\mathcal{M}_{-k}^{(1)}(w)$ we get the first map π_ℓ, which is an isomorphism.

The map $\pi_\ell^\vee\colon \mathcal{T}\big((\mathcal{E}/H)^\vee\big) \longrightarrow \mathcal{T}(\mathcal{E}^\vee)$ induces a map $\mathcal{T}_0\big((\mathcal{E}/H)^\vee\big) \longrightarrow \mathcal{T}_0(\mathcal{E}^\vee)$ and, hence, a morphism $\pi_\ell^\vee\colon p_1^*\Big(j_1^*\big(A_{k,m}(w)\big)\Big) \longrightarrow p_2^*\Big(j_1^*\big(A_{k,m}(w')\big)\Big)$ for every $m \in \mathbb{N}$ which dualized induces the second morphism

$$\pi_\ell\colon p_2^*\Big(j_1^*\big(\mathcal{D}_{k,m}^o(w')\big)\Big) \longrightarrow p_1^*\Big(j_1^*\big(\mathcal{D}_{k,m}^o(w)\big)\Big).$$

As the diagram

$$\begin{array}{ccc} \mathcal{T}_0\big((\mathcal{E}/H)^\vee\big) & \longrightarrow & \mathcal{F}'\big((\mathcal{E}/H)^\vee\big) \\ \downarrow \pi_\ell^\vee & & \downarrow \pi_\ell^\vee \\ \mathcal{T}_0(\mathcal{E}^\vee) & \longrightarrow & \mathcal{F}'(\mathcal{E}^\vee) \end{array}$$

is commutative, the above maps $\pi_\ell^\vee$ are compatible with the morphisms

$$p_2^*\big(\mathcal{M}_k^{(1)}(w')\big) \longrightarrow p_2^*\big(j_1^*\big(\mathcal{A}_k^o(w')\big)\big)\widehat{\otimes}_{\mathcal{O}_K}\widehat{\mathcal{O}}_{\mathfrak{X}^{(1)}(w')}$$

and

$$p_1^*\big(\mathcal{M}_k^{(1)}(w)\big) \longrightarrow p_1^*\big(j_1^*\big(\mathcal{A}_k^o(w')\big)\big)\widehat{\otimes}_{\mathcal{O}_K}\widehat{\mathcal{O}}_{\mathfrak{X}^{(1)}(w)}$$

defined using $\mathrm{dlog}^{\vee,k}$ (see lemma 4.8 and the following discussion). Dualizing the compatibility of the two maps π_ℓ in the statement via δ follows. □

Now we define the Hecke operators T_ℓ for ℓ not dividing pN and U_ℓ for ℓ dividing pN on modular forms and cohomology. More precisely T_ℓ for ℓ not dividing pN, respectively U_ℓ for ℓ dividing pN on overconvergent modular forms are defined as the maps

$$T_\ell, \quad U_\ell\colon \mathrm{H}^0\big(\mathfrak{X}(w'), \omega_{\mathfrak{X}(w')}^{\dagger,k}\big) \longrightarrow \mathrm{H}^0\big(\mathfrak{X}(w), \omega_{\mathfrak{X}(w)}^{\dagger,k}\big)$$

as follows. Recall that by the definition $\omega_{\mathfrak{X}(w)}^{\dagger,k} := \Big(j_{1,*}\big(\mathcal{M}_{-k}^{(1)}(w)\big)\Big)^{G_1}[1/p]$, see definition 4.6. Using the fact that $R^i j_{1,*} = 0$ for $i \geq 1$ by corollary 2.6 we may identify

$$\mathrm{H}^i\big(\mathfrak{X}(w'), \omega_{\mathfrak{X}(w')}^{\dagger,k}\big) \cong \mathrm{H}^i\big(\mathfrak{X}(w'), j_{1,*}\big(\mathcal{M}_{-k}^{(1)}(w')\big)[1/p]\big)^{G_1} \cong \mathrm{H}^i\big(\mathfrak{X}^{(1)}(w'), \mathcal{M}_{-k}^{(1)}(w')[1/p]\big)^{G_1}$$

and similarly $\mathrm{H}^i(\mathfrak{X}(w), \omega_{\mathfrak{X}(w)}^{\dagger,k}\big) \cong \mathrm{H}^i\big(\mathfrak{X}^{(1)}(w), \mathcal{M}_{-k}^{(1)}(w')[1/p]\big)^{G_1}$ for every $i \in \mathbb{N}$. The maps T_ℓ, U_ℓ are defined using these identifications and taking G_1-invariants and inverting p in

$$\mathrm{H}^i\big(\mathfrak{X}^{(1)}(w'), \mathcal{M}^{(1)}_{-k}(w')\big) \longrightarrow \mathrm{H}^i\big(\mathfrak{X}^{(1)}_\ell(w), p_2^*(\mathcal{M}^{(1)}_{-k}(w'))\big) \stackrel{\pi_\ell}{\longrightarrow}$$

$$\stackrel{\pi_\ell}{\longrightarrow} \mathrm{H}^i\big(\mathfrak{X}^{(1)}_\ell(w), p_1^*(\mathcal{M}^{(1)}_{-k}(w)[1/p])\big) = \mathrm{H}^0\big(\mathfrak{X}^{(1)}(w), p_{1,*}p_1^*(\mathcal{M}^{(1)}_{-k}(w))\big) \longrightarrow \mathrm{H}^i\big(\mathfrak{X}^{(1)}(w), \mathcal{M}^{(1)}_{-k}(w))\big).$$

The equality $\mathrm{H}^i\big(\mathfrak{X}^{(1)}_\ell(w), p_1^*(j_1^*(\mathcal{M}^{(1)}_{-k}(w)\big) = \mathrm{H}^i\big(\mathfrak{X}^{(1)}(w), p_{1,*}p_1^*(\mathcal{M}^{(1)}_{-k}(w)\big)$ follows from a Leray spectral sequence argument using the vanishing of $R^h p_{1,*}$, for $h \geq 1$, proven in corollary 2.6. The last map

$$\mathrm{H}^i\big(\mathfrak{X}^{(1)}(w), p_{1,*}p_1^*(\mathcal{M}^{(1)}_{-k}(w)[1/p])\big) \longrightarrow \mathrm{H}^i\big(\mathfrak{X}^{(1)}(w), \mathcal{M}^{(1)}_{-k}(w)[1/p]\big)$$

is the map on cohomology associated to the trace map $p_{1,*}p_1^*(\mathcal{F}) \to \mathcal{F}$ defined in (1) and can be seen as the trace map for Faltings' cohomology (H^i).

Now we assume that $k \in \mathcal{W}^*(B_K)$. We have a G_K-equivariant isomorphism

$$\mathrm{H}^0\big(\mathfrak{X}(w), \omega^{\dagger,k}_{\mathfrak{X}(w)}\big) \cong \mathrm{H}^0\big(X(w), \omega^{\dagger,k}_w \hat{\otimes}_K \mathbb{C}_p\big)$$

and that the latter is provided with Hecke operators given in [AIS] (strictly speaking only the operators T_ℓ, for ℓ not dividing pN, and U_p have been defined in loc. cit., but the same approach provides also operators U_ℓ for ℓ dividing N). Recall that in (5) of section §4.2 we defined a map of sheaves on $\mathfrak{X}(w)$

$$\delta^\vee_k(w)\colon \nu^*(\mathcal{D}_k) \longrightarrow \omega^{\dagger,k}_{\mathfrak{X}(w)},$$

for w adapted to k and p^w is a uniformizer of K. Moreover in section 4.3 we have calculated the cohomology group

$$\mathrm{H}^1\big(\mathfrak{X}(w), \omega^{\dagger,k}_{\mathfrak{X}(w)}(1)\big) \cong \mathrm{H}^0\big(X(w), \omega^{\dagger,k+2}_w\big)\hat{\otimes}_K \mathbb{C}_p.$$

Combining the remarks above we have an $B_K\hat{\otimes}_K\mathbb{C}_p$-linear map, G_K-equivariant

$$\Psi_{k,w}\colon \mathrm{H}^1\big(\mathfrak{X}(w), \nu^*(\mathcal{D}_k)(1)\big) \longrightarrow \mathrm{H}^0\big(X(w), \omega^{\dagger,k+2}_w\big)\hat{\otimes}_K \mathbb{C}_p.$$

We have

Theorem 5.2. *The isomorphism $\mathrm{H}^0\big(\mathfrak{X}(w), \omega^{\dagger,k}_{\mathfrak{X}(w)}\big) \cong \mathrm{H}^0\big(X(w), \omega^{\dagger,k}_w \hat{\otimes}_K \mathbb{C}_p\big)$ and the map $\Psi_{k,w}$ commute with the Hecke operators T_ℓ and U_ℓ defined above.*

Proof. Due to the compatibility proven in 5.1 the map on cohomology induced by $\delta^\vee_k(w)$ is compatible with Hecke operators. It suffices to prove that the isomorphisms $\mathrm{H}^0\big(\mathfrak{X}(w), \omega^{\dagger,k}_{\mathfrak{X}(w)}\big) \cong \mathrm{H}^0\big(X(w), \omega^{\dagger,k}_w \hat{\otimes}_K \mathbb{C}_p\big)$ and $\mathrm{H}^1\big(\mathfrak{X}(w), \omega^{\dagger,k}_{\mathfrak{X}(w)}(1)\big) \cong \mathrm{H}^0\big(X(w), \omega^{\dagger,k+2}_w\big)\hat{\otimes}_K \mathbb{C}_p$ are compatible with the Hecke operators on $\omega^{\dagger,k}$ defined in [AIS] (considering also the variant for U_ℓ with ℓ dividing N not explicitly treated in loc. cit.). By theorem 4.11 this amounts to prove that the trace map $p_{1,*} \circ p_1^*\big(\widehat{\mathcal{O}}_{\mathfrak{X}^{(1)}(w)}\big) = p_{1,*}\big(\widehat{\mathcal{O}}_{\mathfrak{X}^{(1)}_\ell(w)}\big) \to \widehat{\mathcal{O}}_{\mathfrak{X}^{(1)}(w)}$ is compatible with the trace map $p_{1,*}\big(\mathcal{O}_{\mathcal{X}^{(r)}_\ell(w)}\big) \to \mathcal{O}_{\mathcal{X}^{(1)}(w)}$. This follows, for example, from the explicit description of the trace given after formula (1) in section §2.5. □

6 The Eichler-Shimura isomorphism

Let $U \subset \mathcal{W}^*$ be a wide open disk with universal weight k_U and ring of bounded analytic functions Λ_U. We have described in lemma 3.20 the following sequence of maps

$$\mathrm{H}^1\big(\Gamma, D_U\big)\hat{\otimes}_K \mathbb{C}_p(1) \cong \mathrm{H}^1\big(\mathfrak{X}(N,p), \mathcal{D}_U\big)\hat{\otimes}_K \mathbb{C}_p(1) \longrightarrow \mathrm{H}^1\Big(\mathfrak{X}(w), \nu^*\big(\mathcal{D}_U(1)\big)\Big).$$

These maps are equivariant for the action of the Galois group G_K and the Hecke operators and (1) denotes the usual Tate twist. Now let us recall that in (5) of section §4.2 we defined a map of sheaves

$$\delta_{k_U}^\vee(w)\colon \nu^*(\mathcal{D}_U) \longrightarrow \omega_{\mathfrak{X}(w)}^{\dagger, k_U}$$

on $\mathfrak{X}(w)$. Therefore, by theorem 4.11 we have maps:

$$\mathrm{H}^1\big(\mathfrak{X}(w), \nu^*\big(\mathcal{D}_U\big)\otimes K)(1)\big) \longrightarrow \mathrm{H}^1\big(\mathfrak{X}(w), \omega_{\mathfrak{X}(w)}^{\dagger, k_U}(1)\big) \cong \mathrm{H}^0\big(X(w), \omega_w^{\dagger, k_U+2}\big)\hat{\otimes}_K \mathbb{C}_p.$$

These maps are also equivariant for the action of G_K and the Hecke operators due to theorem 5.2. Putting everything together we have a map:

$$\Psi_U\colon \mathrm{H}^1\big(\Gamma, D_U\big)\hat{\otimes}_K \mathbb{C}_p(1) \longrightarrow \mathrm{H}^0\big(X(w), \omega_w^{\dagger, k_U+2}\big)\hat{\otimes}_K \mathbb{C}_p.$$

This map is equivariant for the action of G_K and the Hecke operators and commutes with specializations. In other words if $k \in U(K)$ is a weight, in similar way we obtain a map

$$\Psi_k\colon \mathrm{H}^1\big(\Gamma, D_k\big)\otimes_K \mathbb{C}_p(1) \longrightarrow \mathrm{H}^0\big(X(w), \omega_w^{\dagger, k+2}\big)\otimes_K \mathbb{C}_p,$$

such that the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{H}^1\big(\Gamma, D_U\big)\hat{\otimes}_K \mathbb{C}_p(1) & \xrightarrow{\Psi_U} & \mathrm{H}^0\big(X(w), \omega_w^{\dagger, k_U+2}\big)\hat{\otimes}_K \mathbb{C}_p \\ \downarrow \rho_k & & \downarrow \rho_k \\ \mathrm{H}^1\big(\Gamma, D_k\big)\otimes_K \mathbb{C}_p(1) & \xrightarrow{\Psi_k} & \mathrm{H}^0\big(X(w), \omega_w^{\dagger, k+2}\big)\otimes_K \mathbb{C}_p \end{array}$$

The goal of this section is to study the maps Ψ_k using the map Ψ_U.

6.1 The main result

Let us fix a slope $h \in \mathbb{Q}$, $h \geq 0$ and an integer k_0 such that $h < k_0 + 1$ and think about k_0 as a point in $\mathcal{W}^*(K)$. We let $U \subset \mathcal{W}^*$ be a wide open disk defined over K such that

a) $k_0 \in U(K)$

b) Both $\mathrm{H}^1\big(\Gamma, D_U\big)$ and $\mathrm{H}^0\big(X(w), \omega_w^{\dagger, k_U+2}\big)$ have slope h decompositions and $\mathrm{H}^1\big(\Gamma, D_U\big)^{(h)} \neq 0$.

We denote by k_U the universal weight of U, by Λ_U the ring of bounded rigid functions on U and by $B_U := \Lambda_U \otimes_{\mathcal{O}_K} K$. Then B_U is a K-Banach algebra and moreover it is a principal ideal domain.

We let $\mathrm{H}^1\big(\Gamma, D_U\big)^{(h)}$ and $\mathrm{H}^0\big(X(w), \omega_w^{\dagger, k_U}\big)^{(h)}$ be the parts corresponding to slopes smaller or equal to h of the respective cohomology groups. Both these modules are free B_U-modules of finite rank and Ψ_U induces an $(B_U\hat{\otimes}_K \mathbb{C}_p)$-linear map

$$\Psi_U^{(h)}\colon \mathrm{H}^1\big(\Gamma, D_U\big)^{(h)}\hat{\otimes}_K \mathbb{C}_p(1) \longrightarrow \mathrm{H}^0\big(X(w), \omega_w^{\dagger, k_U+2}\big)^{(h)}\hat{\otimes}\mathbb{C}_p$$

compatible with specializations and equivariant for the action of G_K and the Hecke operators T_ℓ, for $(\ell, Np) = 1$, and U_ℓ for ℓ dividing N. We denote by M_U the kernel of Ψ_U and by $M_U^{(h)}$ the kernel of $\Psi_U^{(h)}$. Then M_U is an $(B_U\hat{\otimes}_K\mathbb{C}_p)$-submodule of $\mathrm{H}^1\big(\Gamma, D_U\big)\hat{\otimes}_K\mathbb{C}_p(1)$, preserved by G_K and the Hecke operators T_ℓ, for $(\ell, Np) = 1$, and U_ℓ, for ℓ dividing N, and $M_U^{(h)}$ is a $(B_U\hat{\otimes}\mathbb{C}_p)$-submodule on which U_p-acts by slopes smaller or equal to h. We have

Theorem 6.1. *a) There is a non-zero element $b \in B := (B_U\hat{\otimes}\mathbb{C}_p)$ such that b annihilates* $\mathrm{Coker}(\Psi_U^{(h)})$.

b)Let $Z \subset U(\mathbb{C}_p)$ be the (finite) set of zeroes of $b \in B$ at a) above and let $V \subset U$ be a wide open disk defined over K satisfying: $V(K)$ contains an integer k such that $k > h-1$ and $V(\mathbb{C}_p)\cap Z = \phi$. Then restriction to V induces an exact sequence

$$0 \longrightarrow M_V^{(h)} \longrightarrow \mathrm{H}^1\big(\Gamma, D_V\big)\hat{\otimes}_K\mathbb{C}_p(1) \xrightarrow{\Psi_V^{(h)}} \mathrm{H}^0\big(X(w), \omega_w^{\dagger, k_V+2}\big)\hat{\otimes}_K\mathbb{C}_p \longrightarrow 0.$$

c) For a wide open disk V as at b) above let us denote by χ_V^{univ} the following composition:

$$G_K \xrightarrow{\chi} \mathbb{Z}_p^\times \xrightarrow{k_V} B_V^\times \longrightarrow (B_V\hat{\otimes}\mathbb{C}_p)^\times,$$

where χ is the cyclotomic character of K. We call χ_V^{univ} the universal cyclotomic character attached to V. Then the semilinear action of G_K on the module $S_V := M_V^{(h)}\big(\chi^{-1}(\chi_V^{\mathrm{univ}})^{-1}\big)$ is trivial. Moreover S_V is a finite, projective $(B_V\hat{\otimes}\mathbb{C}_p)$-module with trivial semilinear G_K-action. Obviously $M_V^{(h)} = S_V(\chi\cdot\chi_V^{\mathrm{univ}})$ as semilinear G_K-modules.

d) For each V as at b) and c) above there is a non-zero element $0 \neq \beta \in B_V$ such that the localized exact sequence

$$0 \longrightarrow \big(S_V(\chi\cdot\chi_U^{\mathrm{univ}})\big)_\beta \longrightarrow \Big(\mathrm{H}^1\big(\Gamma, D_V\big)^{(h)}\hat{\otimes}\mathbb{C}_p(1)\Big)_\beta \longrightarrow \Big(\mathrm{H}^0\big(X(w), \omega_w^{\dagger, k_V}\big)^{(h)}\hat{\otimes}\mathbb{C}_p\Big)_\beta \longrightarrow 0$$

is naturally and uniquely split as as a sequence of G_K-modules.

Before proving the theorem let us point out some of its consequences.

Corollary 6.2. *Assume we have U, k_0, h as at the beginning of section 6.1.*

a) There exists a finite set of weights $Z' \subset U(\mathbb{C}_p)$ such that for every $k \in U(K) - Z'$ we have a natural isomorphism as $\mathbb{C}_p$-vector spaces equivariant for the semilinear G_K-action and the action of the Hecke operators T_ℓ for ℓ not dividing Np and U_ℓ for ℓ dividing Np:

$$\Psi_k^{\mathrm{ES}}\colon \mathrm{H}^1\big(\Gamma, D_k\big)^{(h)}\otimes_K\mathbb{C}_p(1) \cong S_k(k+1)\oplus \mathrm{H}^0\big(X(w), \omega_w^{\dagger, k+2}\big)^{(h)}\hat{\otimes}_K\mathbb{C}_p.$$

Here S_k is a finite $\mathbb{C}_p$-vector space with trivial, semilinear action of G_K and an action of the Hecke operators.

b) The set Z' at a) above contains the integers $\kappa \in U\cap\mathbb{Z}$ such that $0 \le \kappa \le h-1$.

Proof of theorem 6.1. Let $k \in U(K)\cap\mathbb{Z}$ such that $k \geq 0$.

We will first recall Faltings' version of the classical Eichler-Shimura isomorphism, see [F1]. Let us recall the ind-continuous sheaf $\mathcal{V}_k \cong \mathrm{Sym}^k(\mathcal{T})\otimes_{\mathbb{Z}_p} K$ on $X(N,p)^{\mathrm{ket}}$, which can also be seen as an ind-continuous sheaf on $\mathfrak{X}(N,p)$. Let $k\geq 0$ be an integer. The main result of [F1] is that there is a $\mathbb{C}_p$-linear, G_K-equivariant isomorphism (the Eichler-Shimura isomorphism)

$$\Phi_k\colon \mathrm{H}^1\big(X(N,p)^{\mathrm{ket}}_{\overline{K}},\mathcal{V}_k\big)\otimes_K\mathbb{C}_p(1)\cong \mathrm{H}^0\big(X(N,p),\omega^{k+2}\big)\otimes_K\mathbb{C}_p\oplus \mathrm{H}^1\big(X(N,p),\omega^{-k}\big)\otimes_K\mathbb{C}_p(k+1).$$

We have a natural isomorphism $\mathrm{H}^1\big(X(N,p)^{\mathrm{ket}}_{\overline{K}},\mathcal{V}_k(1)\big)\cong \mathrm{H}^1\big(\Gamma,V_k(1)\big)$ compatible with all structure and therefore we have a natural diagram

$$\begin{array}{ccc}
\mathrm{H}^1\big(\Gamma,D_U\big)\hat{\otimes}_K\mathbb{C}_p(1) & \stackrel{\Psi_U}{\longrightarrow} & \mathrm{H}^0\big(X(w),\omega_w^{\dagger,k_U+2}\big)\hat{\otimes}\mathbb{C}_p\\
\downarrow & & \downarrow\\
\mathrm{H}^1\big(\Gamma,D_k\big)\otimes_K\mathbb{C}_p(1) & \stackrel{\Psi_k}{\longrightarrow} & \mathrm{H}^0\big(X(w),\omega_w^{\dagger,k+2}\big)\otimes_K\mathbb{C}_p\\
\downarrow & & \uparrow\\
\mathrm{H}^1\big(\Gamma,V_k(1)\big)\otimes_K\mathbb{C}_p & \stackrel{p_2\circ\Phi_k}{\longrightarrow} & \mathrm{H}^0\big(X(N,p),\omega^{k+2}\big)\otimes_K\mathbb{C}_p
\end{array}$$

where the left vertical maps are induced by the specializations $D_U\longrightarrow D_k\longrightarrow V_k$ (see (4) in §3.1), same as the top right map. The lower right map is restriction (let us recall that if $m>0$ is an integer $\omega^m|_{X(w)}\cong\omega_w^{\dagger,m}$).

Claim 1 The above diagram is commutative. In fact we know that the upper rectangle is commutative so it would be enough to show that the lower rectangle is also commutative.

Proof. For this we have to briefly recall (in a slightly different formulation) the proof of Faltings' result, namely the definition of the map $p_2\circ\Phi_k$. We first notice that arguing as in the proof of proposition 3.19 we have a natural isomorphism (as $\mathcal{X}(N,p)$ is proper and semistable)

$$\mathrm{H}^1\big(X(N,p)^{\mathrm{ket}}_{\overline{K}},\mathcal{V}_k(1)\big)\otimes_K\mathbb{C}_p\longrightarrow \mathrm{H}^1\big(\mathfrak{X}(N,p),\mathrm{Sym}^k(\mathcal{T})\hat{\otimes}\widehat{\mathcal{O}}_{\mathfrak{X}(N,p)}\otimes_{\mathcal{O}_K}K(1)\big).$$

As in definition 4.6 we denote by

$$\omega_{\mathfrak{X}(N,p)}:=v^*_{\mathfrak{X}(N,p)}\big(\omega\big)\hat{\otimes}_{\widehat{\mathcal{O}}^{\mathrm{un}}_{\mathfrak{X}(N,p)}}\widehat{\mathcal{O}}_{\mathfrak{X}(N,p)}.$$

We then have the Hodge-Tate sequence of sheaves of $\widehat{\mathcal{O}}_{\mathfrak{X}(N,p)}$-modules on $\mathfrak{X}(N,p)$

$$0\longrightarrow\omega^{-1}_{\mathfrak{X}(N,p)}(1)\longrightarrow\mathcal{T}\hat{\otimes}\widehat{\mathcal{O}}_{\mathfrak{X}(N,p)}\longrightarrow\omega_{\mathfrak{X}(N,p)}\longrightarrow 0.$$

This sequence is not exact but it becomes exact if we invert p, or if we tensor with K. We obtain

$$0\longrightarrow(\omega^{-1}_{\mathfrak{X}(N,p)}\otimes K)(1)\longrightarrow\mathcal{T}\hat{\otimes}\widehat{\mathcal{O}}_{\mathfrak{X}(N,p)}\otimes_{\mathcal{O}_K}K\longrightarrow(\omega_{\mathfrak{X}(N,p)}\otimes K)\longrightarrow 0.$$

One shows by induction that, for every $m\geq 1$ we have a surjective map of $(\widehat{\mathcal{O}}_{\mathfrak{X}(N,p)}\otimes K)$-modules:

$$\mathrm{Sym}^k(\mathcal{T})\hat{\otimes}\widehat{\mathcal{O}}_{\mathfrak{X}(N,p)}\otimes_{\mathcal{O}_K}K(1)\longrightarrow\omega^k_{\mathfrak{X}(N,p)}\otimes K(1),$$

which induces the morphism

$$\mathrm{H}^1\big(\mathfrak{X}(N,p),\mathrm{Sym}^k(\mathcal{T})\hat{\otimes}\widehat{\mathcal{O}}_{\mathfrak{X}(N,p)}\otimes_{\mathcal{O}_K}K(1)\big)\longrightarrow\mathrm{H}^1\big(\mathfrak{X}(N,p),\omega^k_{\mathfrak{X}(N,p)}\otimes K(1)\big).$$

Similarly to the calculations in the proof of theorem 4.11, we calculate $\mathrm{H}^1\big(\mathfrak{X}(N,p), \omega^k_{\mathfrak{X}(N,p)} \otimes K(1)\big)$ using the spectral sequence

$$\mathrm{H}^a\Big(\mathcal{X}(N,p)^{\mathrm{ket}}, R^b v_{\mathfrak{X}(N,p),*}\big(\omega^k_{\mathfrak{X}(N,p)} \otimes K(1)\big)\Big) \Longrightarrow \mathrm{H}^{a+b}\big(\mathfrak{X}(N,p), \omega^k_{\mathfrak{X}(N,p)} \otimes K(1)\big).$$

For $a+b=1$ we have an edge morphism

$$\mathrm{H}^1\big(\mathfrak{X}(N,p), \omega_{\mathfrak{X}(N,p)} \otimes K(1)\big) \longrightarrow \mathrm{H}^0\Big(\mathcal{X}(N,p)^{\mathrm{ket}}, R^1 v_{\mathfrak{X}(N,p),*}\big(\omega^k_{\mathfrak{X}(N,p)} \otimes K(1)\big)\Big).$$

As in the proof of theorem 4.11 we have:

$$R^1 v_{\mathfrak{X}(N,p),*}\big(\omega^k_{\mathfrak{X}(N,p)} \otimes K(1)\big) \cong \omega^k \hat{\otimes} R^1 v_{\mathfrak{X}(N,p),*}\big(\widehat{\mathcal{O}}_{\mathfrak{X}(N,p)} \otimes K(1)\big) \cong \omega^{k+2} \hat{\otimes} \mathbb{C}_p.$$

Therefore we obtain a natural composition

$$\mathrm{H}^1\big(\mathfrak{X}(N,p), \mathcal{V}_k \hat{\otimes} \widehat{\mathcal{O}}_{\mathfrak{X}(N,p)}(1)\big) \longrightarrow \mathrm{H}^1\big(\mathfrak{X}(N,p), \omega^k_{\mathfrak{X}(N,p)} \otimes K(1)\big) \longrightarrow \mathrm{H}^0\big(X(N,p), \omega^{k+2}\big) \otimes_K \mathbb{C}_p,$$

which is the map $p_2 \circ \Phi_k$ appearing in [F1].

Let us now see what happens on $\mathfrak{X}(w)$. We denote by $\mathcal{T}_w := \nu^*(\mathcal{T})$. We have a natural map $\nu^*\big(\mathcal{D}_k\big) \longrightarrow \mathrm{Sym}^k(\mathcal{T}_w) = \nu^*\big(\mathcal{V}_k\big)$ (associated to (4) in §3.1) and the composite with $\mathrm{Sym}^m(\mathcal{T}) \hat{\otimes} \widehat{\mathcal{O}}_{\mathfrak{X}(N,p)} \otimes_{\mathcal{O}_K} K \longrightarrow \omega^k_{\mathfrak{X}(N,p)} \otimes K$ is the morphism $\delta_k^\vee(w) \colon \nu^*(\mathcal{D}_k) \longrightarrow \omega^k_{\mathfrak{X}(N,p)} \otimes K$ of (5) of section §4.2. Moreover we have the following natural commutative diagram of sites and continuous functors, inducing morphisms of topoi:

$$\begin{array}{ccc} \mathcal{X}(N,p)^{\mathrm{ket}} & \stackrel{v_{\mathfrak{X}(N,p)}}{\longrightarrow} & \mathfrak{X}(N,p) \\ \downarrow \mu & & \downarrow \nu \\ \mathcal{X}(w)^{\mathrm{ket}} & \stackrel{v_{\mathfrak{X}(w)}}{\longrightarrow} & \mathfrak{X}(w) \end{array}$$

Here μ, ν are induced by the natural morphism of log formal schemes $\mathcal{X}(w) \longrightarrow \mathcal{X}(N,p)$.

Let us recall from theorem 4.11 that we have isomorphisms:

$$\mathrm{H}^1\big(\mathfrak{X}(w), \omega^{\dagger,k}_{\mathfrak{X}(w)}(1)\big) \cong \mathrm{H}^0\big(\mathcal{X}(w)^{\mathrm{ket}}, R^1 v_{\mathfrak{X}(w),*}(\omega^{\dagger,k}_{\mathfrak{X}(w)}(1))\big) \cong \mathrm{H}^0\big(X(w), \omega_w^{\dagger,k+2}\big) \otimes_K \mathbb{C}_p.$$

We also have the following sequence of isomorphisms of sheaves on $\mathfrak{X}(w)$:

$$R^1 v_{\mathfrak{X}(w),*}\Big(\nu^*\big(\omega^{k+2}_{\mathfrak{X}(N,p)} \otimes K(1)\big)\Big) \cong R^1 v_{\mathfrak{X}(w),*}\big(\omega^{\dagger,k+2}_{\mathfrak{X}(w)}(1)\big) \cong$$

$$\cong \omega_w^{\dagger,k+2} \otimes \mathbb{C}_p \cong \mu^*\big(\omega^{k+2} \otimes \mathbb{C}_p\big) \cong \mu^*\Big(R^1 v_{\mathfrak{X}(N,p),*}\big(\omega^{k+2}_{\mathfrak{X}(N,p)} \otimes K(1)\big)\Big).$$

Finally putting together what we have done so far we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{H}^1\big(\mathfrak{X}(N,p), \mathcal{V}_k \hat{\otimes} \widehat{\mathcal{O}}_{\mathfrak{X}(N,p)}(1)\big) & \longrightarrow & \mathrm{H}^1\big(\mathfrak{X}(w), \mathrm{Sym}^k(\mathcal{T}_w) \hat{\otimes} \widehat{\mathcal{O}}_{\mathfrak{X}(w)} \otimes_{\mathcal{O}_K} K(1)\big) \\ \downarrow & & \downarrow \\ \mathrm{H}^1\big(\mathfrak{X}(N,p), \omega^k_{\mathfrak{X}(N,p)} \otimes K(1)\big) & \longrightarrow & \mathrm{H}^1\big(\mathfrak{X}(w), \omega^{\dagger,k}_{\mathfrak{X}(w)}(1)\big) \\ \downarrow & & \downarrow\cong \\ \mathrm{H}^0\big(X(N,p), \omega^{k+2} \otimes K\big) \otimes_K \mathbb{C}_p & \stackrel{\varphi}{\longrightarrow} & \mathrm{H}^0\big(X(w), \omega_w^{\dagger,k+2}\big) \otimes_K \mathbb{C}_p \end{array}$$

where φ is the restriction map. This proves the claim 1. $\square$

Remark 1 Let us suppose now that $h \geq 0$, $h \in \mathbb{Q}$ is a slope such that both $\mathrm{H}^1\big(\Gamma, D_U\big)$ and $\mathrm{H}^0\big(X(w), \omega_w^{\dagger, k_U+2}\big)$ have slope decompositions. Let $k \in U(K) \cap \mathbb{Z}$, $k \geq 0$. Then the diagram

$$\begin{array}{ccc} \mathrm{H}^1\big(\Gamma, D_U\big)^{(h)} \hat{\otimes}_K \mathbb{C}_p(1) & \xrightarrow{\Psi_U^{(h)}} & \mathrm{H}^0\big(X(w), \omega_w^{\dagger, k_U+2}\big)^{(h)} \hat{\otimes} \mathbb{C}_p \\ \downarrow & & \downarrow \\ \mathrm{H}^1\big(\Gamma, D_k\big)^{(h)} \otimes_K \mathbb{C}_p(1) & \xrightarrow{\Psi_k^{(h)}} & \mathrm{H}^0\big(X(w), \omega_w^{\dagger, k+2}\big)^{(h)} \otimes_K \mathbb{C}_p \\ \downarrow & & \uparrow \varphi \\ \mathrm{H}^1\big(\Gamma, V_k(1)\big)^{(h)} \otimes_K \mathbb{C}_p & \xrightarrow{p_2 \circ \Phi_k} & \mathrm{H}^0\big(X(N,p), \omega^{k+2}\big)^{(h)} \otimes_K \mathbb{C}_p \end{array}$$

is commutative.

Claim 2 Let $U \subset \mathcal{W}$ be a wide open disk, $k_U \colon \mathbb{Z}_p^\times \longrightarrow B_U^\times$ the universal weight, $w > 0, w \in \mathbb{Q}$ adapted to k_U and $k \in U(K)$. Let $t_k \in B_U$ be a rigid analytic function on U which vanishes with order 1 at k and nowhere else on U. The specialization maps $D_U \longrightarrow D_k$ and $\omega_w^{\dagger, k_U} \longrightarrow \omega_w^{\dagger, k}$ induce the following exact sequences

$$\mathrm{H}^1\big(\Gamma, D_U\big) \xrightarrow{t_k} \mathrm{H}^1\big(\Gamma, D_U\big) \longrightarrow \mathrm{H}^1\big(\Gamma, D_k\big) \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{H}^0\big(X(w), \omega_w^{\dagger, k_U}\big) \xrightarrow{t_k} \mathrm{H}^0\big(X(w), \omega_w^{\dagger, k_U}\big) \longrightarrow \mathrm{H}^0\big(X(w), \omega_w^{\dagger, k}\big) \longrightarrow 0.$$

Proof. In fact the specialization maps are part of the following exact sequences:

$$0 \longrightarrow D_U \xrightarrow{t_k} D_U \longrightarrow D_k \longrightarrow 0 \text{ and } 0 \longrightarrow \omega_w^{\dagger, k_U} \xrightarrow{t_k} \omega_w^{\dagger, k_U} \longrightarrow \omega_w^{\dagger, k} \longrightarrow 0.$$

It follows that we have an exact sequence of B_U-modules

$$\mathrm{H}^1\big(\Gamma, D_U\big) \xrightarrow{t_k} \mathrm{H}^1\big(\Gamma, D_U\big) \longrightarrow \mathrm{H}^1\big(\Gamma, D_k\big) \longrightarrow \mathrm{H}^2\big(\Gamma, D_U\big).$$

Now let us recall that $\Gamma = \Gamma_1(N) \cap \Gamma_0(p)$ is a torsion free group, therefore it is the fundamental group of the complement $Y(N,p)_{/\mathbb{C}}$ of a non-void, finite set of points in a compact Riemann surface $X(N,p)_{/\mathbb{C}}$ and so it has cohomological dimension 1. It follows that $\mathrm{H}^2\big(\Gamma, D_U\big) = 0$ and the first claim follows.

Let us also remark that we have an exact sequence of B_U-modules

$$0 \longrightarrow \mathrm{H}^0\big(X(w), \omega^{\dagger, k_U}\big) \xrightarrow{t_k} \mathrm{H}^0\big(X(w), \omega^{\dagger, k_U}\big) \longrightarrow \mathrm{H}^0\big(X(w), \omega^{\dagger, k}\big) \longrightarrow \mathrm{H}^1\big(X(w), \omega^{\dagger, k_U}\big).$$

As $X(w)$ is an affinoid subdomain and $\omega_w^{\dagger, k_U}$ is a sheaf of B_U-Banach modules by the Appendix of [AIP] it follows that $\mathrm{H}^1\big(X(w), \omega^{\dagger, k_U}\big) = 0$ (note that we can not appeal directly to Tate's acyclicity theorem as $\omega_w^{\dagger, k_U}$ is not a coherent sheaf on $X(w)$). □

Remark 2 Let $U \subset \mathcal{W}$ be a wide open disk defined over K, $k_U \colon \mathbb{Z}_p^\times \longrightarrow B_U^\times$ the universal character, $w > 0, w \in \mathbb{Q}$ adapted to k_U and $h \geq 0, h \in \mathbb{Q}$ a slope. We suppose that both

$\mathrm{H}^1\big(\Gamma, D_U\big)$ and $\mathrm{H}^0\big(X(w), \omega_w^{\dagger,k_U}\big)$ have slope decompositions. Let $k \in U(K) \cap \mathbb{Z}$, $k \geq 0$ and let us recall the commutative diagram of the previous remark.

$$\begin{array}{ccc}
\mathrm{H}^1\big(\Gamma, D_U\big)^{(h)} \hat{\otimes}_K \mathbb{C}_p(1) & \stackrel{\Psi_U^{(h)}}{\longrightarrow} & \mathrm{H}^0\big(X(w), \omega_w^{\dagger, k_U+2}\big)^{(h)} \hat{\otimes} \mathbb{C}_p \\
\downarrow & & \downarrow \\
\mathrm{H}^1\big(\Gamma, D_k\big)^{(h)} \otimes_K \mathbb{C}_p(1) & \stackrel{\Psi_k^{(h)}}{\longrightarrow} & \mathrm{H}^0\big(X(w), \omega_w^{\dagger, k+2}\big)^{(h)} \otimes_K \mathbb{C}_p \\
\downarrow \psi & & \uparrow \varphi \\
\mathrm{H}^1\big(\Gamma, V_k(1)\big)^{(h)} \otimes_K \mathbb{C}_p & \stackrel{p_2 \circ \Phi_k}{\longrightarrow} & \mathrm{H}^0\big(X(N,p), \omega^{k+2}\big)^{(h)} \otimes_K \mathbb{C}_p
\end{array}$$

First let us remark that the surjective map $p_2 \circ \Phi_k$ induces a surjective $\mathbb{C}_p$-linear map denoted by the same symbols $\mathrm{H}^1\big(\Gamma, V_k(1)\big)^{(h)} \otimes_K \mathbb{C}_p \stackrel{p_2\circ\Phi_k}{\longrightarrow} \mathrm{H}^0\big(X(N,p), \omega^{k+2}\big)^{(h)} \otimes_K \mathbb{C}_p$.

Then, there are two cases:

i) $k+1 \geq h$. Then the classicity theorems both for overconvergent modular symbols and for overconvergent modular forms imply that ψ and φ are isomorphisms. It follows that in this case $\Psi_k^{(h)}$ is surjective.

ii) $k+1 < h$. In this case the commutativity of the lower rectangle implies that the image of Ψ_k is contained in the image of the classical forms inside $\mathrm{H}^0\big(X(w), \omega_w^{\dagger,k+2}\big)^{(h)} \hat{\otimes} \mathbb{C}_p$. The relationship between h and k implies that in this case $\Psi_k^{(h)}$ is not surjective in general.

Now we prove a) of theorem 6.1. We assume that U, w, h are as at the beginning of section 6.1. Let $k \in U(K) \cap \mathbb{Z}$ be such that $h < k+1$ (there are infinitely many such k's).

Let us recall that both $\mathrm{H}^1\big(\Gamma, D_U\big)^{(h)} \hat{\otimes}_K \mathbb{C}_p(1)$ and $\mathrm{H}^0\big(X(w), \omega_w^{\dagger,k_U}\big)^{(h)} \hat{\otimes}_K \mathbb{C}_p$ are finite free $B := (B_U \hat{\otimes}_K \mathbb{C}_p)$-modules of ranks n and m respectively. By choosing a basis, we can write $\Phi_U^{(h)}$ as a matrix $\Phi_U^{(h)} = \big(a_{ij}\big)_{1\leq i\leq n, 1\leq j\leq m}$ with $a_{ij} \in B$ for all i, j. By Claim 2 we have:

$$\mathrm{H}^1\big(\Gamma, D_k\big)^{(h)} \otimes_K \mathbb{C}_p \cong \Big(\mathrm{H}^1\big(\Gamma, D_U\big)^{(h)} \hat{\otimes}_K \mathbb{C}_p\Big)/t_k\Big(\mathrm{H}^1\big(\Gamma, D_U\big)^{(h)} \hat{\otimes}_K \mathbb{C}_p\Big),$$

and

$$\mathrm{H}^0\big(X(w), \omega_w^{\dagger,k+2}\big)^{(h)} \hat{\otimes}_K \mathbb{C}_p \cong \Big(\mathrm{H}^0\big(X(w), \omega_w^{\dagger,k_U+2}\big)^{(h)} \hat{\otimes}_K \mathbb{C}_p\Big)/t_k\Big(\mathrm{H}^0\big(X(w), \omega_w^{\dagger,k_U+2}\big)^{(h)} \hat{\otimes}_K \mathbb{C}_p\Big),$$

where let us recall that we have denoted by t_k an element of B which vanishes with order 1 at k and nowhere else in U.

Moreover $\Psi_k^{(h)} = \big(a_{ij}(k)\big)_{i,j}$. The second remark above implies that $n \geq m$ and the matrix $\Psi_k^{(h)}$ has rank exactly m, i.e., there is an $m \times m$-minor of $\Psi_U^{(h)}$, Q, whose determinant has the property $\det(Q)(k) \neq 0$. Therefore $b := \det(Q) \neq 0$, $b \in B$ has the property that $b\mathrm{Coker}\big(\Psi_U^{(h)}\big) = 0$.

Let us now prove b) of theorem 6.1. Let Z denote the set of zeroes of b and let $V \subset U$ be a connected affinoid subdomain defined over K such that $V(\mathbb{C}_p) \cap Z = \phi$ and such that $V(K)$ contains an integer $k > h-1$. Then $b|_{V\times_K \mathbb{C}_p} \in (B_V \hat{\otimes}_K \mathbb{C}_p)^\times$, therefore the following sequence

$$0 \longrightarrow M_V^{(h)} \longrightarrow \mathrm{H}^1\big(\Gamma, D_V\big)^{(h)} \hat{\otimes}_K \mathbb{C}_p(1) \stackrel{\Psi_V^{(h)}}{\longrightarrow} \mathrm{H}^0\big(X(w), \omega_w^{\dagger,k_V+2}\big)^{(h)} \hat{\otimes}_K \mathbb{C}_p \longrightarrow 0$$

is exact, where we have denoted by $M_V^{(h)}$ the kernel of $\Psi_V^{(h)}$. As $\mathrm{H}^0\big(X(w),\omega_w^{\dagger,k_V+2}\big)^{(h)}\hat{\otimes}_K\mathbb{C}_p$ is a free $(B_V\hat{\otimes}_K\mathbb{C}_p)$-module of finite rank the above exact sequence is split (as sequence of $(B_V\hat{\otimes}_K\mathbb{C}_p)$-modules ignoring for the moment the G_K-action). Therefore $M_V^{(h)}$ is a finite projective $(B_V\hat{\otimes}_K\mathbb{C}_p)$-module and because $(B_V\hat{\otimes}_K\mathbb{C}_p)$is a PID, $M_V^{(h)}$ is a finite and free $(B_V\hat{\otimes}_K\mathbb{C}_p)$-module of finite rank.

Remark 3 In fact we also have a localized exact sequence of B_b-modules ($0\neq b\in B$ is the element chosen at a) above)

$$0\longrightarrow \big(M_U^{(h)}\big)_b\longrightarrow \Big(\mathrm{H}^1\big(\Gamma,D_U\big)^{(h)}\hat{\otimes}_K\mathbb{C}_p(1)\Big)_b\longrightarrow \Big(\mathrm{H}^0\big(X(w),\omega_w^{\dagger,k_U+2}\big)^{(h)}\hat{\otimes}_K\mathbb{C}_p\Big)_b\longrightarrow 0.$$

As in general b is not invariant under G_K, the above exact sequence is not G_K-equivariant.

Now we prove c) of theorem 6.1. Let V be as in the theorem and we denote by $S_V := M_V^{(h)}\big(\chi^{-1}(\cdot\chi_U^{\mathrm{univ}})^{-1}\big)$. It is a finite free $(B_V\hat{\otimes}_K\mathbb{C}_p)$-module of rank say $q=n-m$ endowed with a continuous, semilinear action of G_K. Let us briefly recall the so called *Sen's theory in families*, see [Se1], [Se2] and also the section §2 of [Ki].

We assume (to simplify the exposition) that K contains a non trivial p-th root of 1. Let R be an affinoid K-algebra, M a finite free $R\hat{\otimes}_K\mathbb{C}_p$-module of rank q with a continuous, semilinear action of G_K. The action of G_K on $R\hat{\otimes}_K\mathbb{C}_p$ is via its natural action on $\mathbb{C}_p$.

Let $K'\subset\mathbb{C}_p$ be a finite extension, we denote by $H_{K'} := \mathrm{Ker}\big(\chi\colon G_K\longrightarrow(1+p\mathbb{Z}_p)\big)$ and $\Gamma_{K'} := G_{K'}/H_{K'}$. Also $K'_\infty := \overline{K}^{H_{K'}}$ and $\widehat{K}'_\infty := \mathbb{C}_p^{H_{K'}}$.

We denote by

$$\widehat{W}_{K'_\infty}\big(M\big) := M^{H_{K'}}.$$

Then, if K' is large enough (but still a finite extension of K) then $\widehat{W}_{K'_\infty}\big(M\big)$ is a free $\widehat{K}'_\infty\hat{\otimes}_K R$-module of rank q and the natural map $\mathbb{C}_p\hat{\otimes}_{\widehat{K}'_\infty}\widehat{W}_{K'_\infty}\big(M\big)\longrightarrow M$ is an isomorphism.

There is a finite extension K' of K (possibly larger then at the previous step) such that $\widehat{W}_{K'_\infty}\big(M\big)$ has a basis $\{e_1,e_2,\ldots,e_q\}$ over $\widehat{K}'_\infty\hat{\otimes}_K R$ such that the $K'\otimes_K R$-submodule W_* generated by this basis in $\widehat{W}_{K'_\infty}\big(M\big)$ is stable by $\Gamma_{K'}$. If we denote by γ a topological generator of $\Gamma_{K'}$, we define the linear endomorphism $\phi\in\mathrm{End}_{K'\otimes_K R}\big(W_*\big)$ by

$$\phi := \frac{\log(\gamma^{p^r})}{\log\big(\chi(\gamma^{p^r})\big)} \text{ for some } r >> 0.$$

We extend ϕ by linearity to $\widehat{W}_{K'_\infty}\big(M\big)$, where it is independent of all the choices and whose characteristic polynomial has coefficients in R. It is called **the Sen operator** associated to M. Its importance consists in that its formation commutes with base change and for r large enough the action of γ^{p^r} on W_* is determined by

$$\gamma^{p^r}|_{W_*} = \exp\Big(\log\big(\chi(\gamma^{p^r})\big)\phi\Big).$$

We will apply the theory above as follows. Let $Z\subset U(\mathbb{C}_p)$ be the finite set of zeroes of the function $b\in B$ above and let $V\subset U$ be a wide open disk defined over K which contains an integer $k > h-1$ and such that $V(\mathbb{C}_p) \cap Z = \phi$. Let $B_V := \Lambda_V \otimes_{\mathcal{O}_K} K$, where as usual Λ_V denotes the algebra of bounded rigid functions on V. Let

$$S_V := (M_V^{(h)}\big(\chi^{-1} \cdot (\chi_U^{\rm univ})^{-1}\big).$$

Let us recall that S_V is a free $(B_V \hat{\otimes}_K \mathbb{C}_p)$-module of rank q with continuous, semilinear action of G_K.

Let ϕ_V denote the Sen operator attached to S_V and let K' be a finite, Galois extension of K in $\mathbb{C}_p$ such that:

i) $\widehat{W}_{K'_\infty}\big(S_V\big)$ is a free $(A_V \hat{\otimes}_K \widehat{K}'_\infty)$-module of rank q,

ii) There is a basis $\{e_1, e_2, , e_q\}$ of $\widehat{W}_{K'_\infty}\big(S_V\big)$ over $(A_V \hat{\otimes}_K \widehat{K}'_\infty)$ such that $W_* := (K' \otimes_K B_V)e_1 + \ldots + (K' \otimes_K B_V)e_q$ is stable under $\Gamma_{K'}$

and

iii) The action of γ on this basis is given by:

$$\gamma(e_i) = \exp\Big(\log\big(\chi(\gamma)\big)\phi\Big)(e_i) \text{ for every } .1 \le i \le q,$$

where γ is a topological generator of $\Gamma_{K'}$.

Let us write the matrix of ϕ_V in the basis $\{e_1, e_2, \ldots, e_q\}$ as $\big(\alpha_{ij}\big)_{1 \le i,j \le q} \in M_{q\times q}\big(\widehat{K}'_\infty \hat{\otimes}_K B_V\big)$.

Let now $k \in V(K)$ be an integer such that $k > h-1$ (there are infinitely many such weights). We have an exact sequence of $(B_V \hat{\otimes}_K \mathbb{C}_p)$-modules, with G_K and Hecke actions

$$0 \longrightarrow S_V \longrightarrow \mathrm{H}^1\big(\Gamma, D_V\big)^{(h)} \hat{\otimes}_K \mathbb{C}_p(1)\big(\chi^{-1}(\chi_V^{\rm univ})^{-1}\big) \longrightarrow$$

$$\longrightarrow \mathrm{H}^0\big(X(w), \omega_w^{\dagger, k_V+2}\big)^{(h)} \hat{\otimes}_K \mathbb{C}_p\big(\chi^{-1}(\chi_V^{\rm univ})^{-1}\big) \longrightarrow 0.$$

We now specialize the sequence at the weight k, i.e., tensor over B_V with K, for the map $B_V \longrightarrow K$ sending $\alpha \to \alpha(k)$. As usual we denote by t_k a generator of the kernel of the above map which does not vanish anywhere else in V. Because in the above exact sequence all modules are free $(B_V \hat{\otimes} \mathbb{C}_p)$-modules, specialization gives an exact sequence. Comparing with Faltings' result above we obtain the following commutative diagram with exact rows

$$\begin{array}{ccccc}
\mathrm{H}^1\big(\Gamma, D_k(1)\big)^{(h)} \otimes_K \mathbb{C}_p(-k-1) & \xrightarrow{\Psi_k^{(h)}} & \mathrm{H}^0\big(X(w), \omega_w^{k+2}\big)^{(h)} \hat{\otimes} \mathbb{C}_p(-k-1) & \longrightarrow & 0 \\
\downarrow\cong & & \uparrow\cong & & \\
\mathrm{H}^1\big(\Gamma, V_k(1)\big)^{(h)} \otimes_K \mathbb{C}_p(-k-1) & \xrightarrow{(p_2 \circ \Phi_k^{(h)})} & \mathrm{H}^0\big(X(N,p), \omega^{k+2}\big)^{(h)} \otimes_K \mathbb{C}_p(-k-1) & \longrightarrow & 0
\end{array}$$

Therefore we have a natural isomorphism, G_K and Hecke equivariant

$$S_k = S_V/t_k S_V = \mathrm{Ker}\big(\Psi_k^{(h)}\big) \cong \mathrm{Ker}\big((p_2 \circ \Phi_k)\big) = \mathrm{H}^1\big(X(N,p), \omega^{-k}\big)^{(h)} \otimes \mathbb{C}_p.$$

The exact sequence $0 \longrightarrow S_V \xrightarrow{t_k} S_V \longrightarrow S_k \longrightarrow 0$ induces the exact sequence

$$0 \longrightarrow \widehat{W}_{K'_\infty}\big(S_V\big) \xrightarrow{t_k} \widehat{W}_{K'_\infty}\big(S_V\big) \longrightarrow \widehat{W}_{K'_\infty}\big(S_k\big) \longrightarrow \mathrm{H}^1\big(H_{K'}, S_V\big).$$

The theory of almost étale extensions implies that $\mathrm{H}^1\big(H_{K'}, S_V\big) = 0$ and therefore if we denote by ϕ_k the Sen operator attached to S_k, then the image of $\{e_1, e_2, \ldots, e_q\}$ is a basis of $\widehat{W}_{K'_\infty}\big(S_k\big)$

in which ϕ_k has matrix $\big(\alpha_{ij}(k)\big)_{1\le i,j\le q}$. But $\widehat{W}_{K'_\infty}\big(S_k\big)\cong \widehat{K}'_\infty\hat{\otimes}_K \mathrm{H}^1\big(X(N,p),\omega^{-k}\big)$, therefore $\phi_k=0$. It follows that $\alpha_{ij}(k)=0$ for infinitely many $k\in V(K)$, therefore $\alpha_{ij}=0$ for all $1\le i,j\le q$. It follows that $\phi_V(e_i)=0$ which implies that $\gamma(e_i)=e_i$ for all $1\le i\le q$. Therefore the free $K'\otimes_K B_V$-module of rank q, W_*, is equal to $(S_V)^{G_{K'}}$, i.e., S_V is a trivial $G_{K'}$-module. We supposed that K'/K was a finite Galois extension, therefore by étale descent S_V is trivial as G_K-module. It follows that $M_V^{(h)}\cong S_V\big(\chi\cdot\chi_V^{\mathrm{univ}}\big)$ as G_K-modules, where S_V is a free $(B_V\hat{\otimes}_K\mathbb{C}_p)$-module of rank q with trivial G_K-action.

Finally let us prove d) of theorem 6.1. We denote by

$$\mathcal{H}:=\mathrm{Hom}_{(B_V\hat{\otimes}\mathbb{C}_p)}\Big(\mathrm{H}^0\big(X(w),\omega_w^{\dagger,k_V+2}\big)\hat{\otimes}_K\mathbb{C}_p, S_V\big(\chi\cdot\chi_V^{\mathrm{univ}}\big)\Big).$$

Then $\mathcal{H}$ is a free $(B_V\hat{\otimes}\mathbb{C}_p)$-module of finite rank with continuous, semilinear action of G_K. Moreover, the extension class of the exact sequence at b) corresponds to a cohomology class in $\mathrm{H}^1\big(G_K,\mathcal{H}\big)$. If we denote by ϕ the Sen operator of $\mathcal{H}$, a result of [Se1] implies that $\det(\phi)\in B_V$ annihilates this cohomology group. Moreover $\det(\phi)\ne 0$, so if we localize the sequence at this element it will split naturally as a short exact sequence of G_K-modules. This finally ends the proof of theorem 6.1.

Proof of corollary 6.2. Let us assume the hypothesis of the corollary, i.e., we have U,h satisfying the assumption there. Let $Z\subset U(\mathbb{C}_p)$ be the finite set defined in theorem 6.1 b) and let first $k\in U(K)-Z$. Then there exists a wide open disk $V\subset U$, defined over K such that $V(\mathbb{C}_p)\cap Z=\phi$ and $k\in V(K)$. By theorem 6.1 c) we have an exact sequence of $(B_V\hat{\otimes}_K\mathbb{C}_p)$-modules with continuous semilinear G_K-action

$$0\longrightarrow S_V\big(\chi\cdot\chi_V^{\mathrm{univ}}\big)\longrightarrow \mathrm{H}^1\big(\Gamma,D_U^{(h)}\big)\hat{\otimes}_K\mathbb{C}_p(1)\longrightarrow \mathrm{H}^0\big(X(w),\omega_w^{\dagger,k_V+2}\big)^{(h)}\hat{\otimes}_K\mathbb{C}_p\longrightarrow 0.$$

As $k\in V(K)$ we may specialize this sequence and we obtain an exact sequence of $\mathbb{C}_p$-vector spaces with continuous, semilinear action of G_K

$$0\longrightarrow S_k(k+1)\longrightarrow \mathrm{H}^1\big(\Gamma,D_k\big)^{(h)}\hat{\otimes}_K\mathbb{C}_p(1)\longrightarrow \mathrm{H}^0\big(X(w),\omega_w^{\dagger,k+2}\big)\hat{\otimes}_K\mathbb{C}_p\longrightarrow 0.$$

Now $k\in V(K)\subset U(K)\subset\mathcal{W}^*(K)$ so k is an accessible weight associated to a pair (s,i). Then it follows that if $s\ne -1$, the character χ^{s+1} is a character of infinite order, therefore by the main result of [Ta], the above sequence is naturally and uniquely split as a sequence of G_K-modules. Therefore we choose $Z':=Z\cup\{k\in U(K)-Z \quad|\quad k=(s,i), s\ne -1\}$. Then Z' is finite and the corollary follows.

Finally, in this article we do not give a precise geometric interpretation of the B_V-module S_V of theorem 6.1 but we prove the following lemma. With U,h,Z' and V as in theorem 6.1, we denote by

$$\mathbb{S}_V:=H^0\big(X(w),\omega_w^{\dagger,k_V}\otimes\Omega^1_{X(w)/K}\big)\subset H^0\big(X(w),\omega_w^{\dagger,k_V+2}\big),$$

the A_V-module of families of overconvergent cuspforms over V. It is a Hecke-submodule of the B_V-module of families of overconvergent modular forms over V, $\mathbb{M}_V:=H^0\big(X(w),\omega_w^{\dagger,k_V+2}\big)$ and we have:

Lemma 6.3. *Let ℓ be a positive prime integer. Then the characteristic polynomials of T_ℓ, if ℓ does not divide Np and of U_ℓ if ℓ divides Np acting on $S_V^{(h)}$ and on $\mathbb{S}_V^{(h)}$ are equal.*

Proof. Let us first consider an integer weight $k \in V$ such that $k > h-1$ and let us recall ([F1]) that the natural Poincaré pairing between $H^1\big(\Gamma, V_k(1)\big)$ and $H^1_c\big(\Gamma, V_k(1)\big)$ induces Serre-duality between $H^1\big(X(N,p), \omega^{-k}\big)$ and $H^0\big(X(N,p), \omega^k \otimes \Omega^1_{X(N,p)/K}\big)$. Therefore we can identify $H^1\big(X(N,p), \omega^{-k}\big)$ with the K-dual of $H^0\big(X(N,p), \omega^k \otimes \Omega^1_{X(N,p)/K}\big)$, and so the characteristic polynomials of T_ℓ, respectively U_ℓ acting on $H^1\big(X(N,p), \omega^{-k}\big)$ and $H^0\big(X(N,p), \omega^k \otimes \Omega^1_{X(N,p)/K}\big)$ are equal.

Now let $P_{\ell,i}(T) \in B_U[T]$, for $i = 1,2$ be the characteristic polynomials of T_ℓ, if ℓ does not divide Np and respectively of U_ℓ if ℓ divides Np acting on $S_V^{(h)}$ and respectively $\mathbb{S}_V^{(h)}$. If $k \in V$ is an integer weight such that $k > h-1$, then the characteristic polynomials of T_ℓ, respectively U_ℓ acting on:

$$S_V^{(h)}/t_k S_V^{(h)} \cong H^1\big(X(N,p), \omega^{-k}\big)^{(h)} \text{ and on } \mathbb{S}_V^{(h)}/t_k \mathbb{S}_V^{(h)} \cong H^0\big(X(N,p), \omega^k \otimes \Omega^1_{X(N,p)/K}\big)^{(h)}$$

are $P_{\ell,1}(k)$ and $P_{\ell,2}(k)$. By the above argument $P_{\ell,1}(T)(k) = P_{\ell,2}(T)(k)$ for infinitely many $k \in V$ therefore $P_{\ell,1}(T) = P_{\ell,2}(T)$. Here by $P_{\ell,1}(T)(k)$ we mean the polynomial obtained by evaluating the coefficients of $P_{\ell,1}(T)$ at k. □

6.2 On the global Galois representations attached to overconvergent eigenforms

In this section we give a geometric interpretation of the $G_\mathbb{Q} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-representation attached to a *generic* overconvergent cuspidal eigenform.

Let us start by fixing a slope $h \geq 0$, $h \in \mathbb{Q}$ and a wide open disk $U \subset \mathcal{W}^*$ as in the statement of theorem 6.1.

Let us recall (see section §5) that we may think of Γ as the fundamental group of $Y(N,p)_\mathbb{C}$ for a choice of a geometric generic point, therefore we have canonical isomorphisms of topological B_U-modules

$$H^1\big(\Gamma, D_U\big) \cong H^1\big(X(N,p)^{\mathrm{ket}}_\mathbb{C}, \mathcal{D}_U\big) \cong H^1\big(X(N,p)^{\mathrm{ket}}_{\bar{\mathbb{Q}}}, \mathcal{D}\big).$$

Let us remark that the last B_U-module has a natural, continuous, B_U-linear action of $G_\mathbb{Q}$ with the property that its restriction to G_K (seen as an open subgroup of a decomposition group of $G_\mathbb{Q}$ at p) is what we denoted in section §5 by $H^1\big(X(N,p)^{\mathrm{ket}}_{\bar{K}}, \mathcal{D}\big)$. Moreover, as the $G_\mathbb{Q}$-action commutes with the action of the Hecke operators, in particular with the action of U_p, it induces a $G_\mathbb{Q}$-module structure on the finite free B_U-module $H^1(\Gamma, D_U)^{(h)}$. We also denote by $H^1_p\big(\Gamma, D_U\big)$ the image of the natural map $H^1_c\big(\Gamma, D_U\big) \longrightarrow H^1\big(\Gamma, D_U\big)$. Then all these cohomology groups have natural interpretations as étale cohomology groups with compact support, respectively étale parabolic cohomology, they are naturally $G_\mathbb{Q}$ and Hecke modules. Moreover, U will now be chosen such that both $H^1\big(\Gamma, D_U\big)$ and $H^1_p\big(\Gamma, D_U\big)$ have slope $\leq h$-decompositions.

We have

Theorem 6.4. *a) For every positive prime integer ℓ with $(\ell, Np) = 1$ the $G_\mathbb{Q}$-representations $H^1\big(\Gamma, D_U(1)\big)$ and $H^1_p(\Gamma, D_U(1))$ are unramified at ℓ.*

b) Let us fix ℓ as at a) above and denote by φ_ℓ a geometric Frobenius at ℓ and by T_ℓ the Hecke operator, both acting on $H^1_p\big(\Gamma, D_U(1)\big)^{(h)}$. Then the characteristic polynomials of φ_ℓ and T_ℓ are equal.

Proof. a) is clear as $X(N,p)_{\mathbb{Q}_\ell}$ has a smooth proper model over $\mathrm{Spec}(\mathbb{Z}_\ell)$. For b), let us denote by $P_i(T) \in B_U[T], i = 1, 2$ the characteristic polynomials of φ_ℓ and T_ℓ respectively. For every $k \in U \cap \mathbb{Z}$ with $k > h + 1$ we have natural isomorphisms, equivariant for the $G_{\mathbb{Q}}$ and Hecke actions

$$H^1_p\big(\Gamma, D_U(1)\big)^{(h)}/t_k H^1_p\big(\Gamma, D_U(1)\big)^{(h)} \cong H^1_p\big(\Gamma, D_k(1)\big)^{(h)} \cong H^1_p\big(\Gamma, V_k(1)\big)^{(h)}.$$

Moreover the characteristic polynomials of φ_ℓ and T_ℓ on the last group are $P_1(T)(k)$ and $P_2(T)(k)$ and by theorem 4.9 of [D] they are equal: $P_1(T)(k) = P_2(T)(k)$. As there are infinitely many weights k as above in U, it follows that $P_1(T) = P_2(T)$. □

Let now $Z' \subset U(\mathbb{C}_p)$ be the finite set of weights of corollary 6.2.

Corollary 6.5. *Let $k \in U(\mathbb{C}_p) - Z'$ and let f be an overconvergent cuspidal eigenform (for all the Hecke operators) of weight $k+2$ and slope smaller or equal to h. Let K_f denote the finite extension of K generated by the eigenvalues of f for all the Hecke operators. Then $H^1\big(\Gamma, D_k(1)\big)^{(h)}_f$ is a K_f-vector space of dimension 2 with a continuous action of the absolute Galois group of $\mathbb{Q}$, $G_{\mathbb{Q}}$, isomorphic to the p-adic $G_{\mathbb{Q}}$-representation attached to f by the theory of pseudo-representations.*

Before staring the proof of this corollary, let us explain its notations: we denote by $\mathbb{T}$ the K sub-algebra of the K-endomorphism algebra of $H^1\big(\Gamma, D_k(1)\big)^{(h)}$ generated by the images of T_ℓ for all positive prime integers ℓ such that $(\ell, Np) = 1$ and the images of U_ℓ for ℓ dividing Np. The overconvergent cuspidal eigenform f determines a surjective K-algebra homomorphism $\mathbb{T} \longrightarrow K_f$ sending T_ℓ and respectively U_ℓ to its f eigenvalue. Then we denote by $\mathrm{H}^1\big(\Gamma, D_k(1)\big)^{(h)}_f := H^1\big(\Gamma, D_k(1)\big)^{(h)} \otimes_{\mathbb{T}} K_f$.

Proof. As f is an overconvergent cuspform we have $W_f := H^1\big(\Gamma, D_k(1)\big)^{(h)}_f \cong H^1_p\big(\Gamma, D_k(1)\big)^{(h)}_f$.

In order to prove that W_f is isomorphic to the p-adic $G_{\mathbb{Q}}$-representation associated to f by the theory of pseudo-representations it would be enough to show that W_f has dimension 2 as K_f-vector space and that for all the primes ℓ not dividing Np, at which W_f is unramified (see theorem 6.4) the characteristic polynomials of the geometric Frobenius φ_ℓ at ℓ and of the Hecke operator T_ℓ, acting on W_f are equal.

The statement on chararcteristic polynomials follows from theorem 6.4 by specialization at weight k.

For the statement refering to the dimension of W_f, we consider $W_f \otimes_K \mathbb{C}_p$ and forget the global Galois action. Let us recall that the Hecke eigenvalues of f determine its q-expansion up to multiplication by a constant (in K) and therefore by the q-expansion principle for overconvergent modular forms, the eigenspace for $\mathbb{T}$ in $\mathrm{H}^0\big(X(w), \omega_w^{\dagger,k+2}\big)^{(h)}$ determined by the eigenvalues of f is one dimensional. Corollary 6.2 and lemma 6.3 imply that $W_f \otimes_K \mathbb{C}_p$ is a free two dimensional $K_f \otimes_K \mathbb{C}_p$-module which means that W_f is two dimensional over K_f. This ends the argument. □

Remark 6.6. As pointed out in the introduction, corollary 6.5 can be proved using different methods. For example it follows from the isomorphism between the various reduced eigencurves. The interest in this new approach is that it seems to work in higher dimensions while the isomorphism between the relevant eigenvarieties is not known beyond curves.

References

[Err] F. Andreatta, A. Iovita: *Erratum to the article "Global applications to relative* (φ, Γ)*-modules, I"*, Astérisque 330, Représentations p-adiques de groupes p-adiques II: Représentations de $\mathbf{GL}_2(\mathbb{Q}_p)$ et (φ, Γ)-modules, 543–554 (2010).

[AI] F. Andreatta, A. Iovita: *Semi-stable sheaves and the comparison isomorphisms for semi-stable formal schemes*, Rendiconti del Seminario Matematico dell'Università di Padova **128**, 131–285 (2012).

[AIP] F. Andreatta, A. Iovita, V. Pilloni: *p-Adic families of Siegel cuspforms*, to appear in Annals of Math.

[AIS] F. Andreatta, A. Iovita, G. Stevens: *Overconvergent modular sheaves and modular forms for* $\mathbf{GL}_{2/F}$, to appear in Israel J. of Math.

[AS] A. Ash, G. Stevens, *p-Adic deformations of arithmetic cohomology*, preprint available at *https://www2.bc.edu/ ashav/*, (2008)

[Co] R. Coleman: *p-Adic Banach spaces and families of modular forms*, Invent. Math. **127**, 417–479 (1997).

[D] P. Deligne: *Formes modulaires et représentations ℓ-adiques*, Sem. Bourbaki, exp. **355**, 139-172, (1968-1969).

[E] M. Emerton: *On the interpolation of systems of eigenvalues attached to automorphic Hecke eigenforms*, Invent. Math.164, No 1, (2006), 1-84.

[F1] G. Faltings: *Hodge-Tate structures of modular forms*, Math. Ann. **278**, 133–149 (1987).

[F2] G. Faltings: *Almost étale extensions*, In "Cohomologies p-adiques et applications arithmétiques," vol. II. P. Berthelot, J.-M. Fontaine, L. Illusie, K. Kato, M. Rapoport eds. Astérisque **279**, 185–270 (2002).

[HIS] M. Harris, A. Iovita, G. Stevens: *A geometric Jacquet-Langlands corespondence*, preprint.

[Hy] O. Hyodo: *On variation of Hodge-Tate structures*, Math. Ann. **283**, 7–22 (1989).

[Il] L. Illusie, *An overview of the work of the work of K. Fujiwara, K. Kato and C. Nakamura on logarithmic étale cohomology*, Astérisque **279**, 271–322 (2002).

[Ka1] K. Kato: *Semi-stable reduction and p-adic étale cohomology.* In Périodes p-adiques (Bures-sur-Yvette, 1988). Astérisque **223**, 269–293 (1994).

[Ka2] K. Kato: *Toric singularities*, AJM **116**, 1073–1099 (1994).

[Ki] M. Kisin: *Overconvergent modular forms and the Fontaine-Mazur conjecture*, Invent. Math. **153**, 373-454 (2003)

[Se1] S. Sen: *The analytic variation of p-adic Hodge Structures*, Ann. Math. **127**, 647–661 (1988).

[Se2] S. Sen: *An infinite dimensional Hodge-Tate theory*, Bull. Soc. Math. Fr. **121**, 13–34 (1993).

[SGA4] *Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos.* Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4). Dirigé par M. Artin, A. Grothendieck, et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat. LNM **269** (1972).

[Ta] J. Tate: *p-Divisible groups*. In Proc. Conf. Local Fields (Driebergen, 1966), 158–183 (1967).