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Localization techniques for renorming

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Contents

Chapter 1

Introduction

Most of our notation and terminology are standard, otherwise it is either explained here or when needed. Classical references for Banach spaces theory are [Con90], [DGZ93], [JL01] and [FHH$^+$11]; for locally convex spaces is [Köt69]; for general topology are [Eng89], [Nag74], [Kel75] and [Wil04].

By capital letters X, Y, Z, F, etc. we denote sets and sometimes topological spaces. All vector spaces X are assumed to be real. Sometimes X is a normed space with the norm $\|\cdot\|$. Given a subset F of a vector space, we write $\mathrm{conv}(F)$, $\mathrm{aconv}(F)$ and $\mathrm{span}(F)$ to denote respectively, the convex, absolutely convex and the linear hull of F. If $(X, \|\cdot\|)$ is a normed space, then X^* denotes its topological dual. If F is a subset of X^*, then $\sigma(X, F)$ denotes the weakest topology on X that makes each member of F continuous or, equivalently, the topology of pointwise convergence on F. Analogously, if E is a subset of X, then $\sigma(X^*, E)$ is the topology for X^* of pointwise convergence on E. In particular $\sigma(X, X^*)$ and $\sigma(X^*, X)$ are the weak (w) and the weak-star (w^*) topology respectively. If $x \in X$ and $\delta > 0$ we denote by $\mathscr{B}(x, \delta)$ ($\mathscr{B}[x, \delta]$, respectively) the open (closed, respectively) ball centred at x of radius δ. If $x = 0$ and $\delta = 1$ we simply write $\mathscr{B}_X = \mathscr{B}[0, 1]$, the unit sphere will be denote by $\mathcal{S}_X$. Recall that a subset B of $\mathscr{B}_{X^*}$ is said to be *norming* (1-*norming*, respectively) if

$$\|x\|_B = \sup_{b^* \in B} |b^*(x)|$$

is a norm on X equivalent (equal, respectively) to the original norm of X. Observe that the definition of $\|\cdot\|_B$ is plenty of sense also for element x^{**} in the bidual space X^{**}. A subspace $F \subseteq X^*$ is norming (1-norming, respectively) if $F \cap \mathscr{B}_{X^*}$ is norming (1-norming, respectively).

1.1 Development of the dissertation

Renorming theory involves finding isomorphisms in order to improve the norm of a normed space X. That means to make the geometrical and topological properties of the unit ball of a given normed space as close as possible to those of the unit ball of an Hilbert space. An excellent monograph of renorming theory up to 1993 is [DGZ93], in order to have an up-to-date account of the theory we should add [Hay99], [God01], [Ziz03], [MOTV09] and [ST10]. In this work we will study different types of geometrical properties, we will state the fundamental ones in the following list: let $(X, \|\cdot\|)$ a normed space and τ a topology on X, the norm $\|\cdot\|$ is said to be

- *rotund*, or *strictly convex*, if for every $x, y \in X$ we have $x = y$, whenever $2\|x\|^2 + 2\|y\|^2 - \|x + y\|^2 = 0$.
- *asymptotically rotund*, or *asymptotically strictly convex*, if for every $y_1, y_2 \in X$ and $(x_n)_{n\in\mathbb{N}} \subseteq X$ we have $y_1 = y_2$, whenever for every $i = 1, 2$

$$\lim_{n\in\mathbb{N}}(2\|y_i\|^2 + 2\|x_n\|^2 - \|y_i + x_n\|^2) = 0.$$

- *τ-Kadec*, if the norm and the τ topologies agree on the unit sphere. If τ is the w-topology, we will simply say that the norm $\|\cdot\|$ is Kadec.
- *τ-locally uniformly rotund* (*τ-LUR*, for short), if for every sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ and $x \in X$ we have $\tau\text{-}\lim_{n\in\mathbb{N}} x_n = x$, whenever

$$\lim_{n\in\mathbb{N}}(2\|x\|^2 + 2\|x_n\|^2 - \|x + x_n\|^2) = 0.$$

  If τ is the norm topology we will simply say that the norm $\|\cdot\|$ is locally uniformly rotund (LUR, for short).

- *τ-uniformly rotund* (*τ-UR*, for short), if for every pair of sequence $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \subseteq X$ we have $\tau\text{-}\lim_{n\in\mathbb{N}}(x_n - y_n) = 0$, whenever

$$\lim_{n\in\mathbb{N}}(2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n + y_n\|^2) = 0.$$

  If τ is the norm topology we will simply say that the norm $\|\cdot\|$ is uniformly rotund (UR, for short).

By means of probabilistic techniques, in 1979 Troyanski has proved that a Banach space X has an equivalent LUR renorming if, and only if, it admits an equivalent Kadec renorming and an equivalent rotund renorming (see

[Tro79]). In 1999 Raja proved, in a more geometrical fashion, the same result of Troyanski, but his proof add something more: the existence of an equivalent, $\sigma(X,F)$-lower semicontinuous and LUR renorming, for some norming subspace $F \subseteq X^*$, is equivalent to the existence of an equivalent $\sigma(X,F)$-Kadec renorming and and equivalent rotund renorming (see [Raj99b, Theorem 2]).

One of the most outstanding tool in LUR renorming theory is Deville's master lemma (see [DGZ93, Lemma VII.1.1]). It was widely use in the study of this particular geometrical property.

Lemma 1.1 (Deville's master lemma) *Let $(\varphi_i)_{i\in I}$ and $(\psi_i)_{i\in I}$ be two families of real valued, convex and nonnegative functions defined on a normed space X, which are both uniformly bounded on bounded subsets of X. For every $i \in I$ and $k \in \mathbb{N}$, let us denote*

$$\begin{aligned} \theta_{i,k}(x) &= \varphi_i^2(x) + \frac{1}{k}\psi_i^2(x); \\ \theta_k(x) &= \sup_{i\in I} \theta_{i,k}(x); \\ \theta(x) &= \|x\|^2 + \sum_{k\in\mathbb{N}} 2^{-k}(\theta_k(x) + \theta_k(-x)), \end{aligned}$$

where $\|\cdot\|$ is the norm of X. If $\|\cdot\|_\theta$ denotes the Minkowski functional of the set $B = \{x \in X \mid \theta(x) \leq 1\}$, then $\|\cdot\|_\theta$ is an equivalent norm on X with the following property: if $(x_n)_{n\in\mathbb{N}} \subseteq X$ and $x \in X$ satisfy

$$\lim_{n\in\mathbb{N}} \left(2\|x\|_\theta^2 + 2\|x_n\|_\theta^2 - \|x + x_n\|_\theta^2\right) = 0,$$

then there exists a sequence $(i_n) \subseteq I$ such that

1. $\lim_{n\in\mathbb{N}} \left(\frac{1}{2}\psi_{i_n}^2(x) + \frac{1}{2}\psi_{i_n}^2(x_n) - \psi_{i_n}^2\left(\frac{x+x_n}{2}\right)\right) = 0;$
2. $\lim_{n\in\mathbb{N}} \varphi_{i_n}(x) = \lim_{n\in\mathbb{N}} \varphi_{i_n}(x_n) = \lim_{n\in\mathbb{N}} \varphi_{i_n}\left(\frac{x+x_n}{2}\right) = \sup_{i\in I} \varphi_i(x).$

This result is very powerful: in [DGZ93, Theorem VII.1.4] is showed how to use this result in order to prove a classical theorem of Day: for every set Γ, $c_0(\Gamma)$ admits an equivalent LUR norm (see [DGZ93, Theorem II.7.3]), and to generalize a result of Troyanski [Tro71, Proposition 1] to obtain that X admits an equivalent LUR norm, whenever it admits a particular projectional resolution of the identity (see [DGZ93, Theorem VII.1.8]). Deville's master lemma gives also [FG88, Theorem 2.(iii)]: the dual of an Asplud space admits an equivalent (in general non-dual) LUR norm. Haydon used Deville's master lemma in both [Hay99], where he proved powerful results on the renorming of $\mathscr{C}(\Upsilon)$, where Υ is a tree, and [Hay08] where he proved that a Banach space such that his dual admits an equivalent dual LUR norm admits an equivalent LUR norm. For more information about LUR renorming see also [GTWZ83], [GTWZ85], [HR90], [Fab91] and [HJNR00].

In [OT09b, Theorem 3] the authors get some characterizations of the existence of LUR renormings via a suitable localization result, that in turn is obtaines with the help of Deville's master lemma (lemma 1.1).

Theorem 1.2 (Slice localization theorem) *Let X a normed space with a norming subspace F in X^*. Let A be a bounded subset in X and $\mathcal{H}$ a family of $\sigma(X,F)$-open half-spaces such that for every $H \in \mathcal{H}$ the set $A \cap H$ is nonempty. Then there exists an equivalent $\sigma(X,F)$-lower semicontinuous norm $\|\cdot\|_{A,\mathcal{H}}$ such that for every sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ and $x \in A \cap H$ for some $H \in \mathcal{H}$, if*

$$\lim_{n\in\mathbb{N}}(2\|x\|^2_{A,\mathcal{H}} + 2\|x_n\|^2_{A,\mathcal{H}} - \|x + x_n\|^2_{A,\mathcal{H}}) = 0,$$

then there exists a sequence of $\sigma(X,F)$-open half-spaces $\{H_n\}_{n\in\mathbb{N}} \subseteq \mathcal{H}$ so that

1. *there exists $n_0 \in \mathbb{N}$ such that $x, x_n \in H_n$ for $n \geq n_0$, if $x_n \in A$;*
2. *for every $\delta > 0$ there exists $n_\delta \in \mathbb{N}$ such that*

$$x, x_n \in \overline{\operatorname{conv}(A \cap H_n) + \mathscr{B}(0,\delta)}^{\sigma(X,F)},$$

for every $n \geq n_\delta$.

This result was later used in [OST12], in order to characterize rotund renorming in linear topological terms. In what follows we will use or adapt Deville's master lemma and the slice localization theorem in order to obtain result about rotund, Kadec and other renormings.

In 1989 Hansell introduced, see [Han01], the notion of descriptive topological space, we will not state his definition here, since it is rather technical. Hansell pointed out the role played by the existence of a σ-isolated network in these spaces, replacing a σ-discrete topological basis which are exclusive of metrizable spaces after the Bing–Nagata–Smirnov theorem (see [Eng89]), and proved that a Banach space is descriptive with respect to the w-topology if, and only if, the norm topology has a σ-isolated network with respect to the w-topology. Hansell also proved that if a Banach space has a Kadec norm, then it is descriptive with respect to the w-topology (see [Han01, Theorem 1.5]). The main problem in Kadec renorming theory is if it is possible to prove the converse of the previous statement, this is due to the fact that there are no known examples of descriptive Banach space, with respect to the w-topology, without an equivalent Kadec norm (see [Ori07] and [MOTV09, Chapter 3]). We will develop a study of this problem in chapter 2 of this work; we will start from the following theorem of Raja [Raj99a, Theorem 1].

Theorem 1.3 *Let X a Banach space and Z in X^* a norming subspace. The following are equivalent:*

1. *X is descriptive with respect to the $\sigma(X,Z)$-topology;*
2. *there exists a nonnegative, symmetric, homogeneous and $\sigma(X,Z)$-lower semicontinuous function φ on X with $\|\cdot\| \leq \varphi(\cdot) \leq 3\|\cdot\|$, such that the norm and the $\sigma(X,Z)$-topology agree on the "unit sphere" $S = \{x \in X \,|\, \varphi(x) = 1\}$.*

The norm continuity of Raja's function does not follow immediately from his construction and it was asked by different people if it can be done (see [Ori07] and [MOTV09]). In the unpublished note of Raja [Raj03a] appears a previuos estimate of the continuity of φ, here we include the proof for the sake of completeness.

Theorem 1.4 *The function φ in statement 2 of theorem 1.3 can be made also norm continuous.*

Proof We may assume, without loss of generality, that X is a subspace of Z^*, thus the $\sigma(X,Z)$-topology is induced on X by the w^*-topology of Z^*. Let φ the function given in statement 2 of theorem 1.3. In this proof we will use the following notation: $\mathscr{B}_{w^*}[x,\varepsilon] := \overline{\mathscr{B}(x,\varepsilon)}^{w^*}$.

We will make another function with the same properties of φ which is also norm continuous. Let K the w^*-closure of the star-shaped set $\{x \in X \,|\, \varphi(x) \leq 1\}$. It is easy to verify that K is also star-shaped. Let φ_n the Minkowski functional of the sets $K + \mathscr{B}_{w^*}[0, 1/n]$ for $n \geq 2$. Since these sets are w^*-closed, then φ_n are w^*-lower semicontinuous. It is easy to realize that φ_n is also symmetric and verifies the inequality

$$\left(1 - \frac{1}{n}\right)\|\cdot\| \leq \varphi_n(\cdot) \leq 3\|\cdot\|.$$

We claim that every φ_n is norm continuous. Indeed, it is clear that φ_n is norm lower semicontinuous. By homogeneity is enough to show that the set

$$U = \{z^* \in Z^* \,|\, \varphi_n(z^*) < 1\}$$

is norm open. Take $z^* \in U$, we know $\varphi_n(z^*) < 1$, then take $\lambda \in (0,1)$ such that $\varphi_n(z^*) < \lambda^2$. This implies that $\lambda^{-2}z^* \in K + \mathscr{B}_{w^*}[0, 1/n]$ and thus $z^* \in$

$\lambda^2 K + \mathscr{B}_{w^*}[0, \lambda^2/n]$. In particular $z^* \in K + \mathscr{B}(0, \lambda/n)$ which is norm open and contained in $K + \mathscr{B}_{w^*}[0, 1/n]$. Consider the function

$$\Phi(z^*) = \|z^*\| + \sum_{n \geq 2} 2^{-n} \varphi_n(z^*)$$

which is homogeneous, symmetric, w^*-lower semicontinuous, norm continuous and satisfies $\|\cdot\| \leq \Phi(\cdot) \leq 3\|\cdot\|$. We claim that Φ has the Kadec property at the points of X, that is, if $(z^*_\omega)_{\omega\in\Omega}$ is a net w^*-converging to $x \in X$ such that $\Phi(z^*_\omega)$ converges to $\Phi(x)$, then $(z^*_\omega)_{\omega\in\Omega}$ is norm convergent to x. Clearly, we may assume $x \neq 0$, and by homogeneity also assume that $\varphi(x) = 1$. If $(z^*_\omega)_{\omega\in\Omega}$ is a net as above, using the lower semicontinuity in a standard way we obtain that $\varphi_n(z^*_\omega)$ converges to $\varphi_n(x)$. As $\varphi_n(x) < 1$, for $\omega \in \Omega$ large enough we have $\varphi_n(z^*_\omega) < 1$ and thus $z^*_\omega \in K + \mathscr{B}_{w^*}[0, 1/n]$. Given any $\varepsilon > 0$ it is possible to take a $\sigma(X, Z)$-open neighbourhood U of x such that $U \cap \{x \in X \,|\, \varphi(x) \leq 1\}$ has diameter less than ε. We may assume that U is w^*-open in Z^* and passing to the closure we obtain that $\operatorname{diam}(U \cap K) \leq \varepsilon$. By [Raj99a, Lemma 1], given $\varepsilon > 0$ there exists $r > 0$ and a neighbourhood U of x such that $\operatorname{diam}(U \cap (K + \mathscr{B}(0, r))) < \varepsilon$. If we take $n \geq 2$ such that $1/n < r$, then

$$\operatorname{diam}(U \cap \{z^* \in Z^* \,|\, \varphi_n(z^*) \leq 1\}) < \varepsilon.$$

For $\omega \in \Omega$ large enough, $z^*_\omega \in U$ by the w^*-convergence and $\varphi_n(z^*_\omega) < 1$, so $z^* \in U \cap \{z^* \in Z^* \,|\, \varphi_n(z^*) \leq 1\}$ and this implies $\|z^* - x\| < \varepsilon$. Now it is clear that the restriction of Φ to X will satisfy all the properties required and this ends the proof of the lemma. □

In what follows we will prove the following theorem which gives the result of Raja and something more. Remember that a family $\mathcal{A}$ of set of a topological space X is called *discrete* (*isolated*) if for every $x \in X$ ($x \in \bigcup \mathcal{A}$) there exists U_x such that at most one element of $\mathcal{A}$ has non-empty intersection with U_x.

Theorem 1.5 *Let X a Banach space and Z in X^* a norming subspace. The following are equivalent:*

1. *X is descriptive with respect to the $\sigma(X, Z)$-topology;*
2. *there exists an equivalent $\sigma(X, Z)$-lower semicontinuous and $\sigma(X, Z)$-Kadec quasinorm $q(\cdot)$, i.e. a quasinorm such that the $\sigma(X, Z)$ and the norm topologies agree on the set $\{x \in X \,|\, q(x) = 1\}$, and such that*

$$\mu\|\cdot\| \leq q(\cdot) \leq \xi\|\cdot\|$$

for some positive constants μ and ξ.

3. *the norm topology admits a basis $\mathcal{B} = \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$ such that every one of the families $\mathcal{B}_n$ is isolated, with respect to the $\sigma(X,Z)$-topology, and norm discrete.*

Remember that a quasinorm (see definition 2.10) is a generalization of the concept of norm, which is useful in the study of locally bounded space (see [Köt69]). The quasinorm we construct in theorem 1.5 does not depend on the constuction of Raja's function. The triangle inequality tell us that our quasinorm is a Lipschitz function with respect to the metric associated with q, thus uniformly continuous for the original norm. If we agree with the loss of the homogeneity of Raja's function, then we can obtain the following result:

Theorem 1.6 *Let X a normed space and Z in X^* a norming subspace. If X is descriptive with respect to the $\sigma(X,Z)$-topology, then there exists an F-norm F such that it is LUR and $\sigma(X,Z)$-Kadec, i.e. the norm and the $\sigma(X,Z)$ topologies agree on the set $\{x \in X \mid F(x) = 1\}$, and for every $(x_n)_{n\in\mathbb{N}} \subseteq X$ and $x \in X$ we have $\|\cdot\|$-$\lim_{n\in\mathbb{N}} x_n = x$, whenever*

$$\lim_{n\in\mathbb{N}}(2F^2(x) + 2F^2(x_n) - F^2(x + x_n)) = 0.$$

Remember that a F-norm (see definition 2.9) is a tool useful for the study of the uniform structure of a topological vector space.

In chapter 3 we will state some results on rotund renormings. Our starting points are the papers [Smi09], [ST10] and [OST12]. Of particular interest is the following topological definition (see [OST12, Definition 2.6]):

Definition 1.7 *A topological space A has* property (*) *if, and only if, it admits a sequence $(\mathcal{U}_n)_{n\in\mathbb{N}}$ of families of open sets such that for every $x, y \in A$ there exists $n_0 \in \mathbb{N}$ such that $\mathcal{U}_{n_0}$ (*)-separates x and y, i.e.*

1. *$\{x, y\} \cap \bigcup \mathcal{U}_{n_0} \neq \emptyset$;*

2. *for every $U \in \mathcal{U}_{n_0}$ the set $\{x, y\} \cap U$ is at most a singleton.*

We will call $(\mathcal{U}_n)_{n\in\mathbb{N}}$ a (*)-sequence *for A. If A is a subset of a topological vector space and every family $\mathcal{U}_n$ is formed by open slices of A, then we say that A has* (*) with slices.

What follows is the main theorem of [OST12, Theorem 2.7]:

Theorem 1.8 *Let X a normed space and F in X^* a norming subspace, then the following are equivalent:*

1. *X admits an equivalent, $\sigma(X,F)$-lower semicontinuous and rotund norm;*

2. *$(X, \sigma(X,F))$ has (*) with slices;*

3. *$(\mathcal{S}_X, \sigma(X,F))$ has (*) with slices.*

The definition of the property (*) is actually a generalization of the definition of a topological space with a G_δ-diagonal (see [Gru84, Section 2]). In this work we will give other characterizations of rotund renorming in term of the G_δ-diagonal property.

Theorem 1.9 *Let X a normed space and $F \subseteq X^*$ a norming subspace. The following are equivalent:*

1. *X admits an equivalent, $\sigma(X,F)$-lower semicontinuous and rotund norm;*

2. *X admits an equivalent, $\sigma(X,F)$-lower semicontinuous norm $\|\cdot\|_\delta$ such that the set $\{x \in X \mid \|x\|_\delta = 1\}$ has a G_δ-diagonal with slices;*

3. *there exist a symmetric ρ on X and an equivalent, $\sigma(X,F)$-lower semicontinuous norm $\|\cdot\|_\rho$ such that for every point $x \in X$, with $\|x\|_\rho = 1$, and $\varepsilon > 0$ there exists a $\sigma(X,F)$-open halfspace H such that $x \in H$ and*
$$\rho\text{-}\operatorname{diam}(H \cap \{x \in X | \|x\|_\rho \leq 1\}) < \varepsilon.$$

For a definition of the topological concepts involved see [Gru84]. In a, by now, failed attempt to improve theorem 1.8 in the dual case we develop an eating process (see lemma 3.4), which enable us to prove the following result:

Theorem 1.10 *Let X^* a dual Banach space. The following are equivalent:*

1. *X^* admits an equivalent dual rotund norm;*

2. *X^* admits an equivalent dual norm $\|\cdot\|_\delta$ such that $\{x \in X \mid \|x\|_\delta = 1\}$ has a G_δ-diagonal;*

3. *there exist a symmetric ρ on X^* and an equivalent dual norm $\|\cdot\|_\rho$ such that for every point $x \in X^*$, with $\|x^*\|_\rho = 1$, and $\varepsilon > 0$ there exists a w^*-open set U such that $x \in U$ and*
$$\rho\text{-}\operatorname{diam}(U \cap \{x^* \in X^* | \|x^*\|_\rho \leq 1\}) < \varepsilon.$$

We will also give the following nonconvex characterization result.

Theorem 1.11 *Let X^* a dual Banach space. X^* admits an equivalent dual rotund norm if, and only if, there exists a w^*-compact, cirlced and absorbing set $A \subseteq X^*$ such that ∂A admits a G_δ with respect to the w^*-topology, where ∂A is the w^*-boundary of A.*

In the second part of chapter 3, we will prove, using the slice localization theorem, some transfer results, in the same spirits of the ones in [MOTV09].

Theorem 1.12 *Let X, Y normed spaces, $F \subseteq X^*$ and $G \subseteq Y^*$ norming subspaces. Suppose that Y admits an equivalent $\sigma(Y, G)$-lower semicontinuous rotund norm. If $\Phi : X \to Y$ is a $\sigma(X, F)$-$\sigma(Y, G)$-continuous and one-to-one function, such that there exists a family $\{A_p\}_{p\in\mathbb{N}}$ of convex sets such that for every $x \in X$ and K $\sigma(Y, G)$-open half-space with $\Phi(x) \in K$, there exists $p \in \mathbb{N}$ and a $\sigma(X, F)$-open halfspace H such that $x \in A_p \cap H$ and $\Phi(A_p \cap H) \subseteq K$. Then X admits an equivalent, $\sigma(X, F)$-lower semicontinuous and rotund norm.*

The next theorem shows how well a function must behave in order to transfer a rotund norm.

Theorem 1.13 *Let X and Y normed spaces with norming subspaces $F \subseteq X^*$ and $G \subseteq Y^*$. Let $\Phi : X \to Y$ a one to one map such that for every $g \in G$ we have that $(g \circ \Phi)^+$ is quasiconvex and $\sigma(X, F)$-lower semicontinuous. Assume that Y has a $\sigma(Y, G)$-lower semicontinuous and rotund norm. Then X admits an equivalent rotund and $\sigma(X, F)$-lower semicontinuous norm.*

Remember that a quasiconvex function is a real-valued function such that the inverse image of any sets of the form $(-\infty, a)$ is a convex set.

In the fourth chapter we will start a study of uniformly rotund norm. The main result is a generalization of [MOTV09, Theorem 1.1].

Theorem 1.14 *Let X a normed space. X admits an equivalent uniformly rotund norm if, and only if, for every $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that it is possible to write the unit ball as*

$$\mathscr{B}_X = \bigcup_{n=1}^{N_\varepsilon} B_i^\varepsilon,$$

and for every $n = 1, \ldots, N_\varepsilon$ there exist $\delta \in (0, \varepsilon)$ and a family of open half-spaces $\mathcal{H}_{n,\varepsilon}$ which cover B_n such that for every $H \in \mathcal{H}_{n,\varepsilon}$

$$\operatorname{diam}(B_n \cap (H - \delta)) < \varepsilon.$$

Where if $H = \{x \in X \mid f(x) > \mu\}$, then $H - \delta := \{x \in X \mid f(x) > \mu - \delta\}$.

Chapter 2

Quasinorms with the Kadec property

Of particular interest in the study of topological properties related to renorming theory is the definition of paracompact space (see [Dug66, Pag. 162] or [Eng89, Chapter 5]).

Definition 2.1 *A topological space X is said to be* paracompact *if for every open cover $\mathcal{U}$ of X, there exists a locally finite open cover $\mathcal{V}$ which is a refinement of $\mathcal{U}$; i.e. for every $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subseteq U$ and for $x \in X$ there exists a neighbourhood of x that meets only finitely many members of $\mathcal{V}$.*

The paracompactness is a generalization of the concept of compactness and it belongs to the class of concepts related with covering properties of a topological spaces. On the other hand, the concept of full normality can be regarded as belonging to another genealogy of concepts, the separation axioms which include regularity, normality, etc.

Definition 2.2 *A topological space X is* full normal *if every open cover $\mathcal{U} = \{U_i\}_{i\in I}$ admits an open star refinement $\mathcal{V} = \{V_j\}_{j\in J}$, i.e. if for every $j \in J$ there exists $i \in I$ such that*

$$\mathrm{st}(V_j, \mathcal{V}) := \bigcup \{V \in \mathcal{V} \,|\, V_j \cap V \neq \emptyset\} \subseteq U_j.$$

The Stone's theorem says that those two concepts, belonging to different categories, coincide for Hausdorff topological spaces (see [Nag74, Theorem V.2]).

Theorem 2.3 (Stone's theorem) *An Hausdorff space is paracompact if, and only if, it is fully normal.*

In particular, the fact that every metrizable space is paracompact is going to be a fundamental one when we are looking for convex renorming properties in a Banach space. Indeed the use of Stone's theorem has been extensively considered in order to build up new techniques to construct equivalent locally uniformly rotund norms on a given normed space X in [MOTV09]. The σ-discreteness of the basis for the metric topologies gives the necessary rigidity condition that appears in all the known cases of existence of such a renorming property (see [Hay99] and [MOTV99]). It is our aim here to study the impact of Stone's theorem for Kadec renormings.

The following general problem has been asked in different contexts by different people, [MOTV09].

Problem 2.4 Find a class of nonlinear maps $\Phi : X \to Y$ which transfer a Kadec norm from a normed space Y with a Kadec norm to X.

The main reason of the difficulties of the problem above is that there is no example of a normed space with a σ-isolated network, with respect to the w-topology, without admitting an equivalent Kadec norm, [MOTV09]. Indeed, transferring results for normed spaces with a σ-isolated network,with respect to the w-topology, are obtained in chapter 3 of [MOTV09]. Nevertheless, the convexification problem in the core of the matter seems to be very difficult to deal with. Let us now introduce some ideas for the study of this question.

2.1 Basic definitions and main results

After 1989 when first appeared a seminal paper of Hansell, [Han01], the notion of descriptive space has assume a lot of importance in the theory of LUR and Kadec renorming of normed spaces. Remember that a family $\mathcal{U} = \{U_i\}_{i\in I}$ is *isolated, with respect to the τ-topology*, if it is discrete in $\bigcup\mathcal{U}$, i.e. for every $x \in \bigcup\mathcal{U}$ there exists a τ-neighbourhood V of x such that

$$\text{card}\,\{i \in I \mid U_i \cap V \neq \emptyset\} = 1.$$

We say that $\mathcal{U}$ is *σ-isolatedly decomposable, with respect to the τ-topology*, if for every $i \in I$ we can write $U_i = \bigcup_{n\in\mathbb{N}} U_i^n$ such that $\{U_i^n\}_{i\in I}$ is disjoint and isolated, with respect to the τ-topology.

Definition 2.5 *An Hausdorff space (X, τ) is called* descriptive *if there exists a complete metric space T and a continuous surjective map $f : T \to X$ such that if $\{E_\lambda\}_{\lambda\in\Lambda}$ is an isolated family, with respect to the τ-topology, in T, then $\{f(E_\lambda)\}_{\lambda\in\Lambda}$ is σ-isolatedly decomposable, with respect to the τ-topology, in X.*

This complicated and technical definition can be simplified in normed space case (see [Han01, Theorem 6.5]). Recall that a family $\mathcal{N}$ of subsets of a topological space X is said to be a *network*, if for every $x \in X$ and U neighbourhood of x there exists $N \in \mathcal{N}$ such that $x \in N \subseteq U$.

Theorem 2.6 *A Banach space X is descriptive with respect to the w-topology if, and only if, the norm topology admits a network $\mathcal{N}$ that can be written as a countable union of subfamilies $\mathcal{N} = \bigcup_{n\in\mathbb{N}} \mathcal{N}_n$, where every one of the subfamilies $\mathcal{N}_n$ is isolated, with respect to the w-topology.*

Remember that if we have a Kadec norm $\|\cdot\|$ on the normed space X the identity map from $(\mathcal{S}_X, w)$ to $(X, \|\cdot\|)$ is continuous.

Definition 2.7 *Let X, Y normed space and $C \subseteq X$. A map $\Phi : (C, w) \to (Y, \|\cdot\|_Y)$ is called* piecewise continuous*, if there exists a countable cover*

$$C = \bigcup_{n\in\mathbb{N}} C_n$$

such that every one of the restrictions $\Phi_{|C_n}$ is weak to norm continuous. A pointwise norm limit of a sequence of piecewise continuous maps is called a σ-continuous *map (see [MOTV06]).*

In [MOTV06, Theorem 1] the following result is proved:

Proposition 2.8 *Let C a subset of a normed space X. A map ϕ from (C, w) into a normed space $(Y, \|\cdot\|_Y)$ is σ-continuous if, and only if, for every $\varepsilon > 0$ we have $C = \bigcup_{n\in\mathbb{N}} C_{n,\varepsilon}$ in such a way that for every $n \in \mathbb{N}$ and every $x \in C_{n,\varepsilon}$ there is a w-neighbourhood U of x with*

$$\operatorname{osc}(\phi_{|U\cap C_{n,\varepsilon}}) := \sup_{x,y\in U\cap C_{n,\varepsilon}} \|\phi(x) - \phi(y)\|_Y < \varepsilon.$$

In a normed space $(X, \|\cdot\|)$ with a Kadec norm the identity map in X from the w to the norm topology is σ-continuous and the norm topology has a network $\mathcal{N}$ that can be written as a countable union of subfamilies, $\mathcal{N} = \bigcup_{n\in\mathbb{N}} \mathcal{N}_n$, where every one of the subfamilies $\mathcal{N}_n$ is isolated, with respect to the w-topology, so every Kadec renormable space is descriptive with respect to the w-topology (see [Han01, Theorem 1.5]).

In the classical theory of Banach spaces, not only normed spaces have been considered, but also the so called metric linear spaces, i.e. linear spaces equipped with a metric that turn out to be compatible with the vector space operations. The uniform structure of a metrizable topological vector spaces is usually described with the following notion, [Köt69, pag. 163]:

Definition 2.9 (*F*-norm) *An* F-norm *on a vector space* X *is function* $F : X \longrightarrow [0, +\infty)$ *such that:*

1. $x = 0$ *if, and only if,* $F(x) = 0$*;*
2. $F(\lambda x) \leq F(x)$*, for every* $|\lambda| \leq 1$ *and* $x \in X$*;*
3. $F(x + y) \leq F(x) + F(y)$ *for every* $x, y \in X$*;*
4. $\lim_{n\in\mathbb{N}} F(\lambda x_n) = 0$*, if* $\lim_{n\in\mathbb{N}} F(x_n) = 0$ *for every* $(x_n)_{n\in\mathbb{N}} \subseteq X$ *and* $\lambda \in \mathbb{R}$*;*
5. $\lim_{n\in\mathbb{N}} F(\lambda_n x) = 0$*, if* $\lim_{n\in\mathbb{N}} \lambda_n = 0$ *for every* $(\lambda_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ *and* $x \in X$*.*

This definition was already in [Ban55], but for a more update account of the basic results one can refer to [Mus83] and [KPR84]. Another relevant notion is the following, see [Köt69, pag. 159]:

Definition 2.10 (Quasinorm) *A* quasinorm *on a vector space* X *is a function* $q : X \longrightarrow [0, +\infty)$ *such that:*

1. $x = 0$ *if, and only if,* $q(x) = 0$*;*
2. $q(\alpha x) = |\alpha| q(x)$ *for every* $\alpha \in \mathbb{R}$ *and* $x \in X$*;*
3. *there exists* $k \geq 1$ *such that* $q(x+y) \leq k(q(x)+q(y))$ *for every* $x, y \in X$*.*

An account of the basic results can be found in [KPR84], [Rol85] and [Kal03]. This definition is actually important in the locally bounded spaces case since the following result (see [Aok42] and [Rol57]):

Theorem 2.11 (Aoki–Rolewicz) *A topological vector space* X *is locally bounded if, and only if, there exists a quasinorm which generates an equivalent topology.*

In this chapter we are going to prove this version of theorem 1.5:

Theorem 2.12 (Kadec quasi-renorming) *Let* $(X, \|\cdot\|)$ *a normed space with a norming subspace* Z *in* X^**, the following conditions are equivalent:*

1. *There is an equivalent* $\sigma(X, Z)$*-lower semicontinuous and* $\sigma(X, Z)$*-Kadec quasinorm* $q(\cdot)$*, i.e. a quasinorm such that* $\sigma(X, Z)$ *and the norm-topology agree on the unit "sphere"* $\{x \in X \mid q(x) = 1\}$*, and such that*

$$\mu\|\cdot\| \leq q(\cdot) \leq \xi\|\cdot\|$$

for some constant μ *and* ξ*.*

2. *For every* $\varepsilon > 0$ *there is an equivalent* $\sigma(X, Z)$*-lower semicontinuous quasinorm* $q_\varepsilon(\cdot)$ *such that*

$$(1-\varepsilon)\|x\| \leq q_\varepsilon(x) \leq (1+\varepsilon)\|x\|$$

and

$$q_\varepsilon(x+y) \leq \frac{1+\varepsilon}{1-\varepsilon}(q_\varepsilon(x) + q_\varepsilon(y))$$

for every $x, y \in X$*, and such that* $\sigma(X, Z)$ *and the norm-topology which agree on the unit "sphere"* $\{x \in X : q_\varepsilon(x) = 1\}$*.*

3. *There are isolated families for the* $\sigma(X, Z)$*-topology*

$$\{\mathcal{B}_n \,|\, n = 1, 2, \ldots\}$$

in the unit sphere $\mathcal{S}_X$ *such that for every* x *in* $\mathcal{S}_X$ *and every* $\varepsilon > 0$ *there is some positive integer* n *and a set* $B \in \mathcal{B}_n$ *with the property that* $x \in B$ *and* $\|\cdot\|\text{-}\operatorname{diam}(B) < \varepsilon$*.*

4. *The identity map from the unit sphere* $(\mathcal{S}_X, \sigma(X, Z))$ *into the normed space* $(X, \|\cdot\|)$ *is* σ*-continuous.*

5. $(X, \|\cdot\|)$ *is* $\sigma(X, Z)$*-descriptive, i.e. there exists a network* $\mathcal{N}$ *that can be written as a countable union of subamilies,* $\mathcal{N} = \bigcup_{n\in\mathbb{N}} \mathcal{N}_n$*, where every one of the subfamilies* $\mathcal{N}_n$ *is isolated*

$$\bigcup \mathcal{N}_n := \bigcup \{N \,|\, N \in \mathcal{N}_n\}$$

endowed with the $\sigma(X, Z)$*-topology.*

The question that remains unsolved is the following:

Problem 2.13 Is it possible to convexify the construction in theorem 2.12 in order to get an equivalent $\sigma(X, Z)$-lower semicontinuous norm $\|\cdot\|$ on X such that the $\sigma(X, Z)$ and norm topologies agree on the unit sphere $\{x \in X \,|\, \|x\| = 1\}$?

If the former question has a positive answer, then problem 2.4 for Kadec renormings has a similar solution to the one given in [MOTV09] for LUR renormings.

2.2 Needed tools

In what follows we develop the study of two tools that will be used in the sequel. In particular, we investigate a generalization of the notion of convexity and construct a family of functions which behaves as a biorthogonal system in some useful cases.

2.2.1 p-convex constructions

We state here some propositions regarding generalized convexity. First of all let us remember the following definition, [Köt69, pag. 160]:

Definition 2.14 (p-convex set and hull) *Let A a subset of a vector space X and $p \in (0, 1]$. A is said to be* p-convex *if for every $x, y \in A$ and $\tau, \mu \in [0, 1]$ such that $\tau^p + \mu^p = 1$ we have $\tau x + \mu y \in A$. We denote with* $\mathrm{conv}_p(A)$ *the* p-convex hull *of a set A, i.e. the intersection of all the p-convex sets of X containing A.*

Obviously if $p = 1$, then the 1-convex sets are exactly the convex sets. If we have a p-convex and absorbent subset A in a vector space X, we can define the p-Minkowski functional of it as

$$p_A(x) := \inf \{\lambda^p \,|\, x \in \lambda A\}$$

and p_A is a *p-seminorm* in the terminology of [Köt69, pag.160], i.e we have $p_A(\lambda x) = |\lambda|^p p_A(x)$ and $p_A(x + y) \leq p_A(x) + p_A(y)$. The usual Minkowski functional is defined as always:

$$q_A(x) := \inf \{\lambda \,|\, x \in \lambda A\}$$

and we obviously have $q_A(x) = p_A(x)^{1/p}$ for every $x \in X$. The functional q_A is a quasinorm and we have that $q_A(x + y) \leq 2^{(1/p)-1}(q_A(x) + q_A(y))$.

Now we pass to study some fundamental properties of the functions whose epigraph is p-convex.

Definition 2.15 (p-convex function) *A function ϕ from a vector space X to the real line $\mathbb{R}$ is said to be* p-convex *(to satisfy the* p-property*), for $p \in (0, 1]$, if*

$$\phi(\tau x + \mu y) \leq \tau\phi(x) + \mu\phi(y) \qquad (\phi(\tau x + \mu y) \leq \tau^p\phi(x) + \mu^p\phi(y)),$$

whenever $\tau, \mu \in [0, 1]$ and $\tau^p + \mu^p = 1$.

We will need the following observations:

- the epigraph of ϕ is p-convex if, and only if, ϕ is p-convex;
- if ϕ is convex and $\phi(0) = 0$, then ϕ is p-convex for every $p \in (0, 1]$;
- if ϕ is p-convex and non-negative, then ϕ satisfies the p-property;
- if ϕ_p is p-convex, ϕ_q is q-convex, with $0 < p \leq q < 1$ and both of them are non-negative, then $\phi_p + \phi_q$ is p-convex;
- if $\phi : X \to [0, +\infty)$ is bounded from above and satisfies the p-property, then ϕ is continuous on X.

Last, but not least, we state some inequalities and facts concerning functions which satisfy the p-property.

Proposition 2.16 *Suppose that a map ϕ satisfies the p-property for some $p \in (0, 1]$, then for every $x, y \in X$,*

$$\tau^p \mu^p (\phi(x) - \phi(y))^2 \leq \tau^p \phi(x)^2 + \mu^p \phi(y)^2 - \phi(\tau x + \mu y)^2,$$

whenever $\tau^p + \mu^p = 1$ and $\tau, \mu \in [0, 1]$.

Proof We have

$$\begin{aligned}
\tau^p \phi(x)^2 + \mu^p \phi(x)^2 - \phi(\tau x + \mu y)^2 &\geq \tau^p \phi(x)^2 + \mu^p \phi(x)^2 - (\tau^p \phi(x) + \mu^p \phi(y))^2 = \\
&= (\tau^p - \tau^{2p})\phi(x)^2 + (\mu^p - \mu^{2p})\phi(y)^2 - 2\tau^p \mu^p \phi(x)\phi(y) = \\
&= \tau^p (1 - \tau^p)\phi(x)^2 + \mu^p (1 - \mu^p)\phi(y)^2 - 2\tau^p \mu^p \phi(x)\phi(y) = \\
&= \tau^p \mu^p (\phi(x) - \phi(y))^2.
\end{aligned}$$

□

Corollary 2.17 *For a p-seminorm $\|\cdot\|_p$ on the vector space X we have*

$$0 \leq (\|x\|_p - \|y\|_p)^2 \leq 2\|x\|_p^2 + 2\|y\|_p^2 - \|x + y\|_p^2.$$

Proof A p-seminorm is a nonnegative function that satisfied the p-property, we can apply the former proposition for $\tau = \mu = (1/2)^{1/p}$. □

What follows is a p-version of [DGZ93, Fact II.2.3].

Proposition 2.18 *Let X a vector space, $x \in X$ and $(x_j)_{j\in\mathbb{N}} \subseteq X$. The following hold:*

1. *If $\|\cdot\|_p$ is a p-seminorm on X, then the following are equivalent:*
   (a) $\lim_{j\in\mathbb{N}} \|x_j\|_p = \|x\|_p$ *and* $\lim_{j\in\mathbb{N}} \left\|\frac{x+x_j}{2^{1/p}}\right\|_p = \|x\|_p$;

(b) $\lim_{j\in\mathbb{N}}(2\|x\|_p^2 + 2\|x_j\|_p^2 - \|x + x_j\|_p^2) = 0$

2. *If* $\alpha_n > 0$, $\|\cdot\|_{p_n}$ *are* p_n*-seminorm on* X *for some sequence* $(p_n) \subseteq (0,1)$ *and*

$$\lim_{j\in\mathbb{N}} \big(2F^2(x) + 2F^2(x_j) - F^2(x + x_j)\big) = 0,$$

where $F^2(x) = \sum_{n\in\mathbb{N}} \alpha_n \|x\|_{p_n}^2$*, then for every* $n \in \mathbb{N}$

$$\lim_{j\in\mathbb{N}}(2\|x\|_{p_n}^2 + 2\|x_j\|_{p_n}^2 - \|x + x_j\|_{p_n}^2) = 0.$$

Proof Both follow from corollary 2.17. □

2.2.2 Similarities with biorthogonal systems

When on a Banach space there exists a good system of coordinates, for example a biorthogonal systems

$$\{(x_i, f_i) \in X \times X^* \mid i \in I\}$$

with some additional properties such as being a strong Markushevich basis (see [HMSVZ08]), it is possible to construct an equivalent Kadec norm (see [Ziz03, Chapter 4]). In [OT09a, Proposition 2.1] a result is proved that glues the discreteness of Stone's theorem with the linear topological structure of a dual space to X; here we will prove a p-convex verson of such a result

Proposition 2.19 (p-distance) *Let* X *a normed space and* Z *a norming subspace in* X^**. If* C *is a* w^**-compact and* p*-convex subset of* X^{**}, $0 < p \leq 1$, *and define*

$$\varphi(x) := \inf_{c^{**}\in C} \|x - c^{**}\|_Z,$$

then φ *is* p*-convex,* $\sigma(X, Z)$*-lower semicontinuous and 1-Lipschitz map from* X *to* $[0, +\infty)$*. We call such a function* p-distance *to the set* C*.*

Proof The fact that C is p-convex implies that φ is a p-convex function. Indeed, let us take $x, y \in X$ and fix $\tau, \mu \in [0,1]$ with $\tau^p + \mu^p = 1$, and $\varepsilon > 0$. If we choose c_x^{**} and c_y^{**} such that

$$\|x - c_x^{**}\|_Z \leq \varphi(x) + \varepsilon \quad \text{and} \quad \left\|x - c_y^{**}\right\|_Z \leq \varphi(y) + \varepsilon,$$

then

$$\left\|\tau x + \mu y - (\tau c_x^{**} + \mu c_y^{**})\right\|_Z \leq \|\tau x - \tau c_x^{**}\|_Z + \left\|\mu y - \mu c_y^{**}\right\|_Z \leq$$
$$\leq \tau(\varphi(x) + \epsilon) + \mu(\varphi(y) + \epsilon) \leq \tau\varphi(x) + \mu\varphi(y) + (\tau + \mu)\epsilon,$$

since $\tau c_x^{**} + \mu c_y^{**} \in C$ we have that

$$\varphi(\tau x + \mu y) \leq \tau\varphi(x) + \mu\varphi(y) + \varepsilon,$$

for every $\varepsilon > 0$ and $\tau, \mu \in [0, 1]$ with $\tau^p + \mu^p = 1$.

Let us prove the $\sigma(X, Z)$-lower semicontinuity. Fix $r \geq 0$ and take a net $\{x_\alpha\}_{\alpha \in A}$ in X with $\varphi(x_\alpha) \leq r$ for every $\alpha \in A$ and let $x \in X$ the $\sigma(X, Z)$-limit of the net $\{x_\alpha\}_{\alpha \in A}$. We will see that $\varphi(x) \leq r$ too. Let us fix an $\varepsilon > 0$ and choose $c_\alpha^{**} \in C$ such that

$$\|x_\alpha - c_\alpha^{**}\| \leq r + \varepsilon$$

for every $\alpha \in A$. Since C is w^*-compact we can find a cluster point (x^{**}, c^{**}) of the net $\{(x_\alpha, c_\alpha^{**})\}_{\alpha \in A}$ in $X^{**} \times X^{**}$ for the topology $\sigma(X^{**}, X^*)$. Then we have that x^{**} does coincide with x when both linear functionals are restricted to Z and thus for every $f \in \mathscr{B}_{X^*} \cap Z$

$$f(x^{**} - c^{**}) = f(x - c^{**}) \leq r + \varepsilon$$

and so $\varphi(x) \leq r + \varepsilon$. Since the reasoning is valid for every $\varepsilon > 0$ we have got $\varphi(x) \leq r$ as required.

The Lipschitz condition follows from the triangle inequality of the seminorm $\|\cdot\|_Z$ on X^{**}. Indeed, for every $x, y \in X$ and $c^{**} \in \overline{C}^{\sigma(X^{**}, X^*)}$ we have $\|x - c^{**}\|_Z \leq \|x - y\|_Z + \|y - c^{**}\|_Z$, thus $\varphi(x) \leq \|x - y\|_Z + \varphi(y)$ and we see that

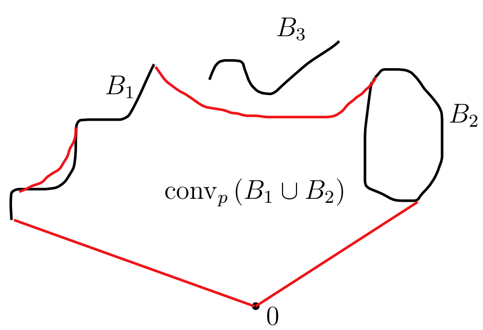
$$|\varphi(x) - \varphi(y)| \leq \|x - y\|_Z. \qquad \square$$

Looking for the "scalpel parameter" to get our renormings we introduce the following:

Definition 2.20 *Let $(X, \|\cdot\|)$ a normed space, Z a norming subspace in X^* and $p \in (0, 1]$. A family $\mathcal{B} := \{B_i \,|\, i \in I\}$ of subsets in the normed space X is said to be p-*isolated, with respect to the $\sigma(X, Z)$-topology*, when for every $i \in I$*

$$B_i \cap \overline{\operatorname{conv}_p}^{\sigma(X,Z)}\{B_j \,|\, j \neq i,\ j \in I\} = \emptyset.$$

Here I will draw an example of what does the above formula means.



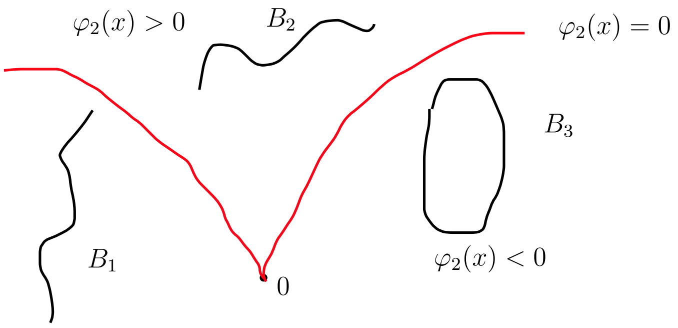
Let us observe that the definition of 1-isolated familiy, with respect to the $\sigma(X,Z)$-topology, corresponds to the definition of $\sigma(X,Z)$-slicely isolated family, as defined in [OT09a, Definition 1.13], by the Hanh-Banach theorem. We now can state the following interplay result:

Lemma 2.21 *Let $(X, \|\cdot\|)$ a normed space and Z a norming subspace in X^*. Let $\mathcal{B} := \{B_i \,|\, i \in I\}$ an uniformly bounded family of subsets of X. The following are equivalent:*

1. *The family $\mathcal{B}$ is p-isolated, with respect to the $\sigma(X,Z)$-topology;*
2. *There exists a family $\mathcal{L} := \{\varphi_i : X \to [0, +\infty) \,|\, i \in I\}$ of p-convex and $\sigma(X,Z)$-lower semicontinuous functions such that for every $i \in I$*

$$\{x \in X \,|\, \varphi_i(x) > 0\} \cap \bigcup_{j \in I} B_j = B_i.$$

To make clear what happens a drawing here is added:



3. *There exist a family $\mathcal{L} := \{\psi_i : X \to [0, +\infty) \,|\, i \in I\}$ of p-convex and $\sigma(X,Z)$-lower semicontinuous functions and numbers $0 \leq \alpha \leq \beta$ such that for every $i, j \in I$*

$$\psi_i(B_i) > \beta \geq \alpha \geq \psi_i(B_j).$$

Proof Assume that the family $\mathcal{B}$ is p-isolated, with respect to the $\sigma(X,Z)$-topology. Applying proposition 2.19 we may consider φ_i to be the p-distance from

$$\overline{\mathrm{conv}_p}^{\sigma(X^{**},X^*)}\{B_j \,|\, j \neq i, \; j \in I\}$$

for every $i \in I$. Our hypothesis on the p-isolated character of the family $\mathcal{B}$ tells us that when a point x belongs to the set B_{i_0} of the family $\mathcal{B}$, then there exist $\sigma(X,Z)$-open half-spaces

$$H_s := \{z \in X : f_s(z) > \mu_s\}, \quad f_s \in \mathscr{B}_{X^*} \cap Z, \quad s = 1, \ldots, N$$

in X with $x \in W := \bigcap_{s=1}^{N} H_s$ and

$$W \cap \overline{\mathrm{conv}_p}^{\sigma(X^{**},X^*)}\{B_j \,|\, j \neq i_0,\ j \in I\} = \emptyset.$$

Then we have $\|y - z\|_Z \geq f_s(y - z) \geq f_s(y) - \mu_s$ whenever $z \notin H_s$. Thus

$$\varphi_{i_0}(y) \geq \min_{s=1,\ldots,N}(f_s(y) - \mu_s) > 0,$$

whenever $y \in W$ since

$$\varphi_{i_0}(y) = \inf\Big\{\|y - z\|_Z : z \in \overline{\mathrm{conv}_p}^{\sigma(X^{**},X^*)}\{B_j \,|\, j \neq i_0,\ j \in I\}\Big\}.$$

Thus $\varphi_{i_0}(x) > 0$, and $\varphi_i(x) = 0$ for every $i \in I$ with $i \neq i_0$. The condition 2 clearly implies 3 with $\alpha = \beta = 0$. Finally, if we assume 3, given a family $\mathcal{L} := \{\psi_i : X \to [0, +\infty) \,|\, i \in I\}$ of p-convex and $\sigma(X, Z)$-lower semicontinuous functions such that the conditions in 3 are satisfied we will have that $\psi_i(y) \leq \alpha$ for every $y \in \mathrm{conv}_p\{B_j \,|\, j \neq i,\ j \in I\}$ by the p-convexity of the function ψ_i, and also for every $y \in \overline{\mathrm{conv}_p}^{\sigma(X,Z)}\{B_j \,|\, j \neq i,\ j \in I\}$ by the $\sigma(X, Z)$-lower semicontinuity of ψ_i. Consequently we have $x \notin \overline{\mathrm{conv}_p}^{\sigma(X,Z)}\{B_j \,|\, j \neq i,\ j \in I\}$ for every $x \in B_i$ and every $i \in I$ and it finishes the proof of the p-isolated property, with respect to the $\sigma(X, Z)$-topology, of the family $\mathcal{B}$. □

2.3 Constructing a Kadec quasinorm

In this section we are going to prove theorem 2.12, but we need two fundamental lemmata in order to do so. The first lemma is a decomposition lemma, which says how to decompose an isolated family of sets in order to obtain countable many p_n-isolated families, the second lemma is a connection lemma between the existence of a p-isolated family and the Kadec property.

Lemma 2.22 (Decomposition lemma) *Let $(X, \|\cdot\|)$ a normed space, Z a norming subspace in X^* and (q_n) a decreasing sequence going to zero in $(0, 1]$. Let $\mathcal{B}$ an isolated family, with respect to the $\sigma(X, Z)$-topology. Then there exists a decomposition of every $B \in \mathcal{B}$ such that*

$$B = \bigcup_{n \in \mathbb{N}} B_n$$

and $\{B_n \,|\, B \in \mathcal{B}\}$ is a q_m-isolated family, with respect to the $\sigma(X, Z)$-topology, for $m \in \mathbb{N}$ big enough and every $n \in \mathbb{N}$.

Proof Given a neighbourhood W of the origin in the $\sigma(X, Z)$-topology let us define the width of W as:

$$\mathrm{wd}(W) = \sup\{\delta > 0 \,|\, \mathscr{B}(0, \delta) \subseteq W\}.$$

The isolated family $\mathcal{B}$ for the $\sigma(X,Z)$-topology can be decomposed as follows. Let us denote with $\mathcal{U}$ the family of all convex and $\sigma(X,Z)$-open neigbourhoods of the origin in X. We set $B = \bigcup_{n\in\mathbb{N}} B_n$ where

$$B_n := \left\{x \in B \,\middle|\, \exists W \in \mathcal{U}, \mathrm{wd}(W) > 1/n, (x+W)\cap B' = \emptyset \ \forall B' \neq B\right\}.$$

Let us see that the family $\{B_n \,|\, B \in \mathcal{B}\}$ is p_m-isolated whenever p_m satisfies the inequality

$$\left(\frac{1}{2}\right)^{1/p_m} < 1/4n.$$

Indeed, for $x \in B_n$ we have W open neigbourhood of the origin in the $\sigma(X,Z)$-topology, with $\mathscr{B}(0,1/n) \subseteq W$, and $(x+W)\cap B' = \emptyset$ for every $B' \in \mathcal{B}, B' \neq B$. In particular we see that

$$(x + 1/2W) \cap [(B' + \mathscr{B}(0,1/2n)] = \emptyset$$

for every $B' \neq B \in \mathcal{B}$. Then we have

$$(x+W/4)\cap\left(X \setminus \overline{\mathscr{B}(0,1-1/4n)}^{\sigma(X,Z)}\right)\cap\overline{\mathrm{conv}_{p_m}\{B' \,|\, B' \in \mathcal{B},\ B' \neq B\}}^{\sigma(X,Z)} = \emptyset$$

since

$$\mathrm{conv}_{p_m}\left\{B' \,\middle|\, B' \in \mathcal{B},\ B' \neq B\right\} \subseteq \bigcup_{B'\neq B}\left\{\lambda x' \,\middle|\, 0 \le \lambda \le 1,\ x' \in B'\right\} + \mathscr{B}(0,1/4n).$$

The last assertion holds because for $\tau, \mu \in [0,1]$ such that $\tau^{p_m} + \mu^{p_m} = 1$ we have that $\tau \le \mu \Leftrightarrow \tau \le (1/2)^{1/p_m}$ and so

$$(1/2)^{1/p_m} = \max_{\tau\in[0,1]}\left\{\min\left\{\tau, (1-\tau^{p_m})^{1/p_m}\right\}\right\},$$

then for z and y in $\bigcup\{B' \,|\, B' \in \mathcal{B},\ B' \neq B\}$ we have

$$\|\cdot\|\text{-}\operatorname{dist}\left(\tau z + \mu y, \bigcup_{B'\neq B}\left\{\lambda x' \,\middle|\, 0 \le \lambda \le 1,\ x' \in B'\right\}\right) \le (1/2)^{1/p_m}$$

from where the conclusion follows. □

The following variant of Deville's master lemma has been stated, for sequences, by Haydon (see [Hay99, Proposition 1.2]) for the construction of Kadec norms in spaces of the type $C(\Upsilon)$, where Υ is a tree. The following is a net version which was already stated in [BKT06, Lemma 5.3], we will use it in order to describe the connection between Haydon's approach and Stone's discreteness.

Lemma 2.23 *Let X a topological space, S a nonempty set and $\varphi_s, \psi_s : X \to [0, +\infty)$ be lower semicontinuous functions such that $\sup_{s\in S}(\varphi_s(x)+\psi_s(x)) < +\infty$ for every $x \in X$. Define*

$$\theta_{s,m}(x) = \varphi_s(x) + \frac{1}{2^m}\psi_s(x);$$
$$\theta_m(x) = \sup_{s\in S} \theta_{s,m}(x);$$
$$\theta(x) = \sum_{m\in\mathbb{N}} 2^{-m}\theta_m(x).$$

Assume that $\{x_\sigma\}_{\sigma\in\Sigma}$ is a net converging to $x \in X$ and $\theta(x_\sigma) \to \theta(x)$. Then there exists a finer net $\{x_\gamma\}_{\gamma\in\Gamma}$ and a net $\{i_\gamma\}_{\gamma\in\Gamma} \subseteq S$ such that

$$\lim_{\gamma\in\Gamma} \varphi_{i_\gamma}(x_\gamma) = \lim_{\gamma\in\Gamma} \varphi_{i_\gamma}(x) = \lim_{\gamma\in\Gamma} \varphi(x_\gamma) = \sup_{s\in S} \varphi_s(x)$$

and $\lim_{\gamma\in\Gamma}(\psi_{i_\gamma}(x_\gamma) - \psi_{i_\gamma}(x)) = 0$.

We can now state a connection lemma between Haydon's approach and Stone's discreteness.

Lemma 2.24 (Connection lemma) *Let $(X, \|\cdot\|)$ a normed space and Z a norming subspace in X^*. Let $\mathcal{B} := \{B_i \mid i \in I\}$ an uniformly bounded and p-isolated family of subsets of X, with respect to the $\sigma(X,Z)$-topology, for some $p \in (0,1]$. Then there exists an equivalent quasinorm, with p-power a p-norm, $\|\cdot\|_{\mathcal{B}}$ on X such that: for all net $\{x_\alpha\}_{\alpha\in A}$ and x in X with $x \in B_{i_0}$ for $i_0 \in I$, the conditions $\sigma(X,Z)$-$\lim_{\alpha\in A} x_\alpha = x$ and $\lim_{\alpha\in A} \|x_\alpha\|_{\mathcal{B}} = \|x\|_{\mathcal{B}}$ imply that*

1. *there exists $\alpha_0 \in A$ such that x_α is not in $\overline{\mathrm{conv}_p}^{\sigma(X,Z)}\{B_i \mid i \neq i_0,\ i \in I\}$ for $\alpha \geq \alpha_0$;*
2. *for every $\delta > 0$ there exists $\alpha_\delta \in A$ such that*

$$x, x_\alpha \in \overline{(\mathrm{conv}\,(B_{i_0} \cup \{0\}) + \mathscr{B}(0,\delta))}^{\sigma(X,Z)},$$

whenever $\alpha \geq \alpha_\delta$.

Proof Fix an index $i \in I$ and define the function φ_i as the p-distance from the set
$$\overline{\mathrm{conv}_p}^{\sigma(X^{**},X^*)}\{B_j \mid j \neq i,\ j \in I\},$$
for every $i \in I$. Consider the convex sets $D_i = \mathrm{conv}(B_i \cup \{0\})$, together with $D_i^\delta := D_i + \mathscr{B}(0,\delta)$, where $\mathscr{B}(0,\delta) := \{x \in X \mid \|x\|_Z < \delta\}$ for every $\delta > 0$ and

$i \in I$. We are going to denote with p_i^δ the Minkowski functional of the convex bodies $\overline{D_i^\delta}^{\sigma(X,Z)}$. Then we define the $\sigma(X,Z)$-lower semicontinuous norms ψ_i by the formula

$$\psi_i(x) = \|x\|_Z + \sum_{n\in\mathbb{N}} \frac{1}{n2^n} p_i^{1/n}(x),$$

for every $x \in X$. We are now in position to apply lemma 2.23 in order to get an equivalent quasinorm $\|\cdot\|_{\mathcal{B}}$ on X such that the condition $\lim_{\alpha\in A} \|x_\alpha\|_{\mathcal{B}} = \|x\|_{\mathcal{B}}$ together with $\sigma(X,Z)$-$\lim_{\alpha\in A} x_\alpha = x$ for a net $\{x_\alpha\}_{\alpha\in A}$ and x in X imply that there are a finer net $\{x_\beta\}_{\beta\in B}$ and a net $(i_\beta)_{\beta\in B}$ in I such that

1. $\lim_{\beta\in B} \varphi(x_\beta) = \lim_{\beta\in B} \varphi_{i_\beta}(x) = \lim_{\beta\in B} \varphi_{i_\beta}(x_\beta) = \sup_{i\in I} \varphi_i(x)$;
2. $\lim_{\beta\in B}(\psi_{i_\beta}(x_\beta) - \psi_{i_\beta}(x)) = 0$.

Indeed, using the definitions in lemma 2.23 we introduce the functions:

$$\begin{aligned}
\theta_{i,m}(x) &:= \varphi_i(x) + \frac{1}{2^m}\psi_i(x);\\
\theta_m(x) &:= \sup_{i\in I} \theta_{i,m}(x);\\
\theta(x) &:= \sum_{m\in\mathbb{N}} 2^{-m}(\theta_m(x) + \theta_m(-x)).
\end{aligned}$$

It is not difficult to prove that θ is a p-convex and $\sigma(X,Z)$-lower semicontinuous function such that $\lim_{\alpha\in A} \theta(x_\alpha) = \theta(x)$ together with $\sigma(X,Z)$-$\lim_{\alpha\in A} x_\alpha = x$ imply the conditions 1 and 2 above by lemma 2.23. The Minkowski functional of the p-convex set

$$\{x \in X \mid \theta(x) \leq 1\}$$

provide us with the quasinorm $\|\cdot\|_{\mathcal{B}}$ we are looking for. Let us take the net $\{x_\alpha\}_{\alpha\in A}$ and x in X verifying that $\lim_{\alpha\in A} \|x_\alpha\|_{\mathcal{B}} = \|x\|_{\mathcal{B}}$ and x is the $\sigma(X,Z)$-limit of $\{x_\alpha\}_{\alpha\in A}$. Then we also have $\lim_{\alpha\in A} \theta(x_\alpha) = \theta(x)$ since the following equality holds:

$$\{x \in X \mid \|x\|_{\mathcal{B}} = 1\} = \{x \in X \mid \theta(x) = 1\}.$$

Our hypothesis on the p-isolated character of the family $\mathcal{B}$ tell us that

$$x \notin \overline{\mathrm{conv}_p}^{\sigma(X,Z)}\{B_i \mid i \neq i_0,\ i \in I\}$$

whenever $x \in B_{i_0}$, and so $\varphi_{i_0}(x) > 0$ but $\varphi_i(x) = 0$ for every $i \in I$ with $i \neq i_0$, see lemma 2.21. From condition 1 there exists a β_0 such that $i_\beta = i_{\beta_0}$ and $\varphi_{i_{\beta_0}}(x_\beta) > 0$ for all $\beta \geq \beta_0$, from where the conclusion 1 of the lemma follows. Moreover, the condition 2 above implies that for every positive integer q, we have

$$\lim_{\beta\in B} p_{i_{\beta_0}}^{1/q}(x_\beta) = p_{i_{\beta_0}}^{1/q}(x).$$

If we fix a positive number δ and we set the integer q such that $1/q < \delta$, since $x \in D_{i_0}^{1/q}$ we have that $p_{i_{\beta_0}}^{1/q}(x) < 1$ because $D_{i_0}^{1/q}$ is norm open and therefore, there exists $\beta_\delta \in B$ such that for $\beta \geq \beta_\delta$ we have that $p_{i_{\beta_0}}^{1/q}(x_\beta) < 1$ and thus $x_\beta \in \overline{D_{i_{\beta_0}}^{\delta}}^{\sigma(X,Z)}$, and indeed $x_\beta \in \overline{(\mathrm{conv}(B_{i_{\beta_0}} \cup \{0\}) + \mathscr{B}(0,\delta))}^{\sigma(X,Z)}$. □

Remark 2.25 The following observations will be useful.

1. The quasinorm constructed here is the Minkowski functional $\|\cdot\|_{\mathcal{B}}$ of the p-convex set $\{x \in X \,|\, \theta(x) \leq 1\}$. Since the φ_i's are p-convex and $\sigma(X,Z)$-lower semicontinuous and the ψ_i's are convex and $\sigma(X,Z)$-lower semicontinuous norms, it follows that function θ is p-convex and $\sigma(X,Z)$-lower semicontinuous; so the Minkowski functional $\|\cdot\|_{\mathcal{B}}$ is a $\sigma(X,Z)$-lower semicontinuous equivalent quasinorm, with p-power a p-norm.

2. If we take the p-Minkowski functional of the set $\{x \in X \,|\, \theta(x) \leq 1\}$, instead of its Minkowski functional, we get a p-norm wich satisfies the same conclusion of the lemma.

3. For every $\beta > 1$ and $\alpha > 0$ is possible to construct the former quasinorm $\|\cdot\|_{\mathcal{B}}$ such that:
$$\frac{\alpha}{4+\alpha}\beta^p\|x\|_Z \leq \|x\|_{\mathcal{B}} \leq \beta\|x\|_Z$$
for every $x \in X$.

   Proof We use the same notations as above and define
$$\psi_i^\alpha(x) = \alpha\|x\|_Z + \sum_{n\in\mathbb{N}} \frac{1}{n2^n} p_i^{1/n}(x)$$
for $\alpha > 0$. If we do the calculations with ψ_i^α instead of ψ_i we see that: given $x \in X$ with $\|x\|_Z \leq 1$, we have $\varphi_i(x) \leq 1$ and $\psi_i^\alpha(x) \leq 1+\alpha$, for every $i \in I$, thus $\theta_m(x) \leq (1 + (\alpha+1)2^{-m})$ for every $m \in \mathbb{N}$, and therefore
$$\theta(x) \leq \sum_{m\in\mathbb{N}} 2^{1-m}\big(1 + (1+\alpha)2^{-m}\big) = \frac{2(4+\alpha)}{3} =: k_\alpha.$$
It follows $\|x\|_{\mathcal{B}} \leq k_\alpha^{1/p}\|x\|_Z$, for every $x \in X$. If we have $\beta > 1$ we can consider the functions $\frac{1}{k_\alpha}\beta^p\varphi_i$ and $\frac{1}{k_\alpha}\beta^p\psi_i^\alpha$ instead of φ_i and ψ_i^α respectively and we will have now
$$\|x\|_{\mathcal{B}} \leq \beta\|x\|_Z$$
for every $x \in X$. On the other hand, $\alpha\|\cdot\|_Z \leq \psi_i^\alpha(\cdot)$, thus $\theta_m(\cdot) \geq \alpha\frac{1}{2^m}\|\cdot\|_Z$, so
$$\theta(\cdot) \geq \alpha\frac{2}{3}\|\cdot\|_Z$$

that gives $\alpha\frac{2}{3}\|\cdot\|_Z \leq \|\cdot\|_{\mathcal{B}}$. If we are using the factor $\frac{1}{k_\alpha}\beta^p$ in both functions φ_i and ψ_i^α too, we will have that

$$\frac{2}{3}\alpha\frac{1}{k_\alpha}\beta^p\|\cdot\|_Z \leq \|\cdot\|_{\mathcal{B}} \leq \beta\|\cdot\|_Z.$$

Thus we obtain $\frac{\alpha}{4+\alpha}\beta^p\|\cdot\|_Z \leq \|\cdot\|_{\mathcal{B}} \leq \beta\|\cdot\|_Z$. □

4. Let $\varepsilon \in (0,1)$ and chose $\alpha > 0$ such that $\frac{\alpha}{4+\alpha} > 1-\varepsilon$ and $\beta \in (1, 1+\varepsilon)$. Then we see that $(1-\varepsilon)\|\cdot\|_Z \leq \|\cdot\|_{\mathcal{B}} \leq (1+\varepsilon)\|\cdot\|_Z$. Consequently the quasinorm constructed verifies

$$\|x+y\|_{\mathcal{B}} \leq \frac{1+\varepsilon}{1-\varepsilon}(\|x\|_{\mathcal{B}} + \|y\|_{\mathcal{B}})$$

for every $x, y \in X$.

Now we can present the main resut of this section:

Theorem 2.26 *A normed space $(X, \|\cdot\|)$ with a norming subspace Z in X^* admits an equivalent $\sigma(X,Z)$-lower semicontinuous and $\sigma(X,Z)$-Kadec quasinorm if, and only if, there exist, in the unit sphere $\mathcal{S}_X$, families isoltated, with respect to the $\sigma(X,Z)$-topology,*

$$\{\mathcal{B}_n \,|\, n = 1, 2, \dots\}$$

such that for every x in $\mathcal{S}_X$ and every $\varepsilon > 0$ there exist $n \in \mathbb{N}$ and a set $B \in \mathcal{B}_n$, such that $x \in B$ and $\|\cdot\|$-$\operatorname{diam}(B) < \varepsilon$.

Proof The decomposition lemma (lemma 2.22) says that for a fixed sequence $q_n \searrow 0$ in $(0,1]$ we will have a decomposition of the sets in the family $\mathcal{B}_n$ producing families $\mathcal{B}_n^m$, $m = 1, 2, \dots$, with $\mathcal{B}_n^m$ being q_{s_m}-isolated, with respect to the $\sigma(X,Z)$-topology, for all $m, n = 1, 2, \dots$. Therefore without loss of generality it is possible to reorder the sequence and to assume that for $n = 1, 2, \dots$ the family $\mathcal{B}_n$ is already p_n-isolated (with respect to the $\sigma(X,Z)$-topology). We can now consider the equivalent quasinorms $\|\cdot\|_{\mathcal{B}_n}$ constructed using the connection lemma (lemma 2.24) for every one of the families $\mathcal{B}_n$. We define now an equivalent quasinorm on X as follows:

$$|||x||| := \sum_{n\in\mathbb{N}} c_n\|x\|_{\mathcal{B}_n}, \qquad x \in X$$

where the sequence $(c_n)_{n\in\mathbb{N}}$ is chosen so that the series converges uniformly on bounded sets. That is possible since we can, and do assume, that the following inequality holds

$$(1-\delta)\|x\|_Z \leq \|x\|_{\mathcal{B}_n} \leq (1+\delta)\|x\|_Z$$

for the same fixed $\delta > 0$ and for every $n \in \mathbb{N}$, after point 4 of remark 2.25. We will now prove the Kadec property. Take a net $\{x_\alpha\}_{\alpha\in(A,\succ)}$ and x with $\|x\|_Z = 1$, $\lim_{\alpha\in A} |||x_\alpha||| = |||x|||$ and (x_α) being $\sigma(X,Z)$-convergent to x. Then we have

$$\lim_{\alpha\in A} \|x_\alpha\|_{\mathcal{B}_q} = \|x\|_{\mathcal{B}_q}$$

for every $q \in \mathbb{N}$ by the $\sigma(X,Z)$-lower semicontinuity of the quasinorms $\|\cdot\|_{\mathcal{B}_q}$. Given $\varepsilon > 0$ consider $q \in \mathbb{N}$ such that for some $B \in \mathcal{B}_q$ we have $x \in B$ and $\|\cdot\|\text{-}\operatorname{diam}(B) < \varepsilon/2$. Then the connection lemma (lemma 2.24) tell us that for some $\alpha_{\varepsilon/2}$ we have

$$x_\alpha \in \overline{\operatorname{conv}(B \cup \{0\}) + \mathscr{B}(0,\varepsilon/2)}^{\sigma(X,Z)}$$

whenever $\alpha \succ \alpha_{\varepsilon/2}$. We have that $\|\cdot\|\text{-}\operatorname{dist}(x_\alpha, I_x) \leq \varepsilon$ for $\alpha \succ \alpha_{\varepsilon/2}$ where I_x is the segment joining x with the origin, and so there exist numbers $r_{(\alpha,\varepsilon)} \in [0,1]$ such that

$$\left\|x_\alpha - r_{(\alpha,\varepsilon)} x\right\| \leq \varepsilon,$$

for every $\alpha \succ \alpha_{\epsilon/2}$. Consider the directed set $A \times (0,1]$ with the product order and the subset $D := \left\{(\alpha,\varepsilon) \in A \times (0,1] \,\middle|\, \alpha \succ \alpha_{\varepsilon/2}\right\}$ which is a directed set with the induced order. Then for the net $\left\{r_{(\alpha,\varepsilon)} \,\middle|\, (\alpha,\varepsilon) \in D\right\}$ there exists a subnet map $\sigma : B \to D$ for some directed set $(B,\succ)$ such that the limit $r := \lim_{\beta\in B} r_{\sigma(\beta)}$ exists by the compactness of the unit interval $[0,1]$. Let us denote with $\overline{\sigma}$ the composition of the map σ with the projection from $A \times (0,1]$ onto A, which is a subnet map too, and we have:

$$\|\cdot\|\text{-}\lim_{\beta\in B} x_{\overline{\sigma}(\beta)} = rx.$$

The assumption $\lim_{\alpha\in A} |||x_\alpha||| = |||x|||$ guarantees that $|||rx||| = |||x||| \neq 0$, so $r = 1$. We are done since the former reasoning is valid for every subnet of the given net, so

$$\|\cdot\|\text{-}\lim_{\alpha\in A} x_\alpha = x.$$

If the given point x doesn't lie in the unit sphere $\mathcal{S}_X$ and different from the origin it is enough to consider $\frac{x}{\|x\|_Z}$ and the net $(\frac{x_\alpha}{\|x\|_Z})$ to obtain the same conclusion. To see that $|||\cdot|||$ is a quasinorm just use the fact that

$$\|x+y\|_{\mathcal{B}_n} \leq \frac{1+\delta}{1-\delta}\left(\|x\|_{\mathcal{B}_n} + \|y\|_{\mathcal{B}_n}\right)$$

to obtain $|||x+y||| \leq \frac{1+\delta}{1-\delta}(|||x\,||| + |||\,y|||)$. □

Remarks Observe that if we had taken p_n-norm $p_{\mathcal{B}_n}(\cdot)$ instead of the quasinorm, then the function

$$F(x) = \sum_{n\in\mathbb{N}} c_n p_{\mathcal{B}_n}(x),$$

where c_n are chosen in order to guarantee the convergence of the series on bounded set (this can be done due to the equivalence), is an equivalent $\sigma(X,Z)$-lower semicontinouos and $\sigma(X,Z)$-Kadec F-norm.

Theorem 2.12 follows easily now.

Proof (Theorem 2.12) The implication (2⇒1) is obvious, (3⇒2) is theorem 2.26 and (4⇒3) is proposition 2.8.

(1⇒5) Observing that theorem 7.2 of [Han01] can be proved with a radial set in the place of the unit sphere and knowing that the set $\{x \in X \mid q(x) = 1\}$ admits a basis for the strong topology that is σ-discrete for the $\sigma(X, Z)$-topology, we obtain that the space X admits a newtork for the strong topology which is σ-isolated for the $\sigma(X, Z)$-topology. This means that X is $\sigma(X, Z)$-descriptive.

(5⇒4) By proposition 2.7 of [MOTV09] we obtain that $id : (X, \sigma(X, Z)) \to (X, \|\cdot\|)$ is σ-continuous, then also $id : (\mathcal{S}_X, \sigma(X, Z)) \to (X, \|\cdot\|)$ is σ-continuous. □

2.4 Applications

Now we are going to state some applications of our results. In particular an improvement of a result [Rib00] (already in [Raj03a]) and a version with Kadec quasinorm of [OT09a, Section 2].

2.4.1 On a result of Ribarska

In this section we prove the results answered in [Raj03a]. First of all, let us remember the following definition related to descriptiveness (see [JNR92a]).

Definition 2.27 (d-SLD) *Let (X, τ) a topological space and d a metric on X. It is said that X has* countable cover by sets of small local diameter *(d-SLD, for short) if for every $\varepsilon > 0$ there exists a decomposition*

$$X = \bigcup_{n \in \mathbb{N}} X_n^\varepsilon$$

such that for each $n \in \mathbb{N}$ every point of X_n^ε has a relatively non-empty τ-neighbourhood of d-diameter less than ε.

If (X, τ) is of the kind $\mathscr{C}_p(K)$ or a Banach space endowed with its weak topology, then X has $\|\cdot\|$-SLD if and only if (X, τ) is descriptive (see [Onc00]). Ribarska and Babev have proved in [RB09] that the function space $\mathscr{C}(K \times L)$ has an equivalent **LUR** norm provided that both $\mathscr{C}(K)$ and $\mathscr{C}(L)$ are **LUR** renormable, where K and L are Hausdorff compacta. An analogous result holds for **LUR** norms which are pointwise lower semicontinuous. The main result in [Rib00] is the following:

Theorem 2.28 *If K and L are Hausdorff compacta such that $\mathscr{C}_p(K)$ admits a pointwise Kadec norm and $\mathscr{C}_p(L)$ have $\|\cdot\|$-SLD, then $\mathscr{C}_p(K \times L)$ has $\|\cdot\|$-SLD.*

Ribarska observed that theorem 2.28 holds when the existence of a pointwise Kadec norm is repleced by the following assumption: there exists a nonnegative, homogeneous, norm continuous and pointwise lower semicontinuous function φ on $\mathscr{C}_p(K)$, with $\|h\| \leq \varphi(h) \leq 2\|h\|$ whenever $h \in \mathscr{C}(K)$ and such that the norm and the pointwise topology agree on the set $S = \{h \in \mathscr{C}(K) \mid \varphi(h) = 1\}$. So, using theorem 2.12, we obtain the following result:

Theorem 2.29 *If K and L are Hausdorff compacta such that both $\mathscr{C}_p(K)$ and $\mathscr{C}_p(L)$ have $\|\cdot\|$-SLD, then $\mathscr{C}_p(K \times L)$ has $\|\cdot\|$-SLD.*

Our second aim is to prove some permanence results for the class of compact Hausdorff spaces K such that $\mathscr{C}_p(K)$ has $\|\cdot\|$-SLD. In that context, theorem 2.29 is the staring point. Similar results can be obtain for the class of compact Hausdorff spaces K such that $\mathscr{C}(K)$ has an equivalent **LUR** norm. Among them, let us mention the following one:

Proposition 2.30 *Let K a norm fragmented w^*-compact subset of a dual Banach space X^* such that $\mathscr{C}(K)$ has an equivalent **LUR** norm. Then $\mathscr{C}(H)$ has an equivalent **LUR** norm, where $H = \overline{\text{conv}}^{w^*}(K)$ is consider endowed with the w^*-topology.*

In the following by property (R) we shall denote one of the following three properties: “having $\|\cdot\|$-SLD with the pointwise topology”, “having an equivalent **LUR** norm” or “having an equivalent pointwise lower semicontinuous **LUR** norm”. The following generalizes corollary 8 of [MOT97]:

Proposition 2.31 *Let K a compact space and $K_n \subseteq K$ compact subsets of K such that every space $\mathscr{C}(K_n)$ has the property (R). If there is a lower semicontinuous metric d on K such that*

$$K = \overline{\bigcup_{n\in\mathbb{N}} K_n}^{d},$$

then $\mathscr{C}(K)$ has the property (R).

Proof We shall prove the result when the property (R) is the **LUR** renormability of the space and we shall give hints to modify the proof for the other properties.

Let $\|\cdot\|_n$ an equivalent **LUR** norm on $\mathscr{C}(K_n)$ bounded by the supremum norm. For every $n \in \mathbb{N}$ define

$$O_n(f) = \sup\left\{|f(x) - f(y)| \,\middle|\, x, y \in K,\ d(x,y) \leq \frac{1}{n}\right\}$$

and consider the equivalent norm $|||\cdot|||$ on $\mathscr{C}(K)$ defined by the formula

$$|||f|||^2 = \|f\|^2 + \sum_{n\in\mathbb{N}} 2^{-n}\left\|f_{|K_n}\right\|_n^2 + \sum_{n\in\mathbb{N}} 2^{-n} O_n(f)^2.$$

If we prove that $|||\cdot|||$ is a w-**LUR** norm, then the result will follow from [MOTV99]. To see that, suppose that $|||f_k||| = |||f|||$ and $\lim_k |||f_k + f||| = 2\,|||\,f|||$. A standard convexity argument [DGZ93, Fact II.2.3] gives us that (f_k) converges to f uniformly on every K_n. We claim that $(f_k(x))$ converges to $f(x)$ for every $x \in X$. Fix $\varepsilon > 0$ and take n big enough to have $O_n(f) < \varepsilon/3$ (this is possible because continuous function on K are d-uniformly continuous, see the proof of [Raj99a, Theorem 4]). Now take $y \in \bigcup_{m\in\mathbb{N}} K_m$ such that $d(x,y) < 1/n$. If k is big enough, then $O_n(f_k) < \varepsilon/3$ and $|f_k(y) - f(y)| < \varepsilon/3$. We have that

$$|f_k(x) - f(x)| \leq |f_k(x) - f_k(y)| + |f_k(y) - f(y)| + |f(y) - f(x)| < \varepsilon$$

and this end the proof of the claim. Thus we have that (f_k) converges to f weakly by Lebesgue's theorem and $|||\cdot|||$ is w-**LUR** .

For t_p-lower semicontinuous **LUR** renormability, the proof is the same if we notice that the norm $|||\cdot|||$ built above is t_p-lower semicontinuous. For $\|\cdot\|$-SLD consider the formula

$$\Phi(f) = \sum_{n\in\mathbb{N}} 2^{-n}\varphi_n(f_{|K_n}) + \sum_{n\in\mathbb{N}} 2^{-n} O_n(f)$$

where φ_n are Kadec functions on $\mathscr{C}(K_n)$. The convexity argument above can be replaced by an argument of lower semicontinuity in order to obtain that Φ is a Kadec function on $\mathscr{C}(K)$. □

Corollary 2.32 *Let K a norm fragmented w^*-compact subset of X^* and $H = \overline{\mathrm{conv}}^{w^*}(K)$. If $\mathscr{C}(K)$ has the property (R), then $\mathscr{C}(H)$ also has the property (R).*

Proof First notice that if K is a norm fragmented w^*-compact subset of X^* then

$$\overline{\mathrm{conv}}^{w^*}(K) = \overline{\mathrm{conv}}^{\|\cdot\|}(K)$$

by a result of Namioka [Nam87]. Also notice that if L is a compact Hausdorff space such that $\mathscr{C}(L)$ has the property (R), then $\mathscr{C}(L')$ has the property (R) for any compact L' which is continuous image of L. Let K_n the set of convex combinations of at most n points of K. It is easy to see that K_n is compact and continuous image of $L = \Delta \times K^n$, where

$$\Delta = \left\{ (\lambda_i)_{i=1}^n \,\middle|\, \lambda_i \geq 0,\ \sum_{i=1}^n \lambda_i = 1 \right\}.$$

By Ribarska's result $\mathscr{C}(L)$ has the property (R), and so $\mathscr{C}(K_n)$ does also. Now we have that $H = \overline{\bigcup_{n\in\mathbb{N}} K_n}$ and the result follows from proposition 2.31. □

Under the hypothesis of the previous corollary the **LUR** norm can be made pointwise lower semicontinuous because for a Radon-Nikodým compact K, $\mathscr{C}(K)$ has convex-$P(t_p, w)$ as a consequence of [Raj99a, Theorem 4], and then is possible to apply [Raj99b, Theorem A].

2.4.2 Similarity with Stone's theorem

In theorem 2.26 we have proved that a Kadec quasinorm exists if, and only if, a σ-isolated network exists. What we want to prove now if it is possible to obtain a σ-isolated and norm discrete base for the norm topology. Let us begin with the following fattening lemma. Through this section we set $\mathscr{B}_Z(x, \varepsilon) := \{x \in X \mid \|x\|_Z < \varepsilon\}$.

Lemma 2.33 *Let X a normed space with a norming subspace $Z \subseteq X^*$. Given a uniformly bounded and p-isolated family, with respect to the $\sigma(X, Z)$-topology, $\mathcal{A} := \{A_i\}_{i\in I}$ there exist decompositions $A_i = \bigcup_{n\in\mathbb{N}} A_i^n$ with*

$$A_i^1 \subseteq A_i^2 \subseteq \cdots \subseteq A_i^n \subseteq A_i^{n+1} \subseteq \cdots \subseteq A_i$$

for every $i \in I$ and such that the families $\{A_i^n + \mathscr{B}_Z(0, 1/4n) \mid i \in I\}$ are p-isolated, with respect to the $\sigma(X, Z)$-topology, and norm discrete for every $n \in \mathbb{N}$.

Proof Denote by φ_i the p-distance to $\overline{\mathrm{conv}_p}^{\sigma(X^{**}.X^*)}\{A_j \mid j \neq i\}$. Theorem 2.21 gives us the scalpel to split up the sets of the family using these p-convex functions. Indeed, let us define $A_i^n := \{x \in A_i \mid \varphi_i(x) > 1/n\}$ and we have $A_i = \bigcup_{n\in\mathbb{N}} A_i^n$. Moreover, if $x \in A_i^n + \mathscr{B}_Z(0, 1/4n)$, then we have

$$\varphi_i(x) > 3/4n.$$

Indeed, let us write $x = y + z$, $y \in A_i^n, \|z\|_Z < 1/4n$, since $\varphi_i(y) > 1/n$ we can select a number ρ with $\varphi_i(y) > \rho > 1/n$ and we will have for every fixed $c^{**} \in \overline{\mathrm{conv}_p}^{\sigma(X^{**}.X^*)}\{A_j \mid j \neq i\}$ that $\|y - c^{**}\|_Z > \rho$. So we can find some $f \in \mathscr{B}_{X^*} \cap Z$ with $f(y - c^{**}) > \rho$. Now we see that $f((y + z) - c^{**}) > \rho - 1/4n$ and so $\|x - c^{**}\|_Z > \rho - 1/4n$ for every $c^{**} \in \overline{\mathrm{conv}_p}^{\sigma(X^{**}.X^*)}\{A_j \,|\, j \neq i\}$. Consequently we see that $\varphi_i(x) \geq \rho - 1/4n > 3/4n$.

On the other hand for $y \in A_j$ with $j \neq i$, we know that $\varphi_i(y) = 0$, then for $x \in A_j^n + \mathscr{B}_Z(0, 1/4n)$ if we write $x = y + z$, with $y \in A_j^n$ and $\|z\|_Z < 1/4n$ we have, for fixed $c^{**} \in \overline{\mathrm{conv}_p}^{\sigma(X^{**}.X^*)}\{A_j \,|\, j \neq i\}$;

$$\|x - c^{**}\|_Z < \|y - c^{**}\|_Z + 1/4n$$

from where follows that

$$\varphi_i(x) = \inf\left\{\|x - c^{**}\|_Z \,\middle|\, c^{**} \in \overline{\mathrm{conv}_p}^{\sigma(X^{**}.X^*)}\{A_j \,|\, j \neq i\}\right\} \leq 1/4n,$$

since $\varphi_i(y) = 0$. This means that the family $\{A_i^n + \mathscr{B}_Z(0, 1/4n)\}_{i \in I}$ verifies the conditions 3 of theorem 2.21 with the functions $(\varphi_i)_{i \in I}$ and constants $\alpha = 1/4n$, $\beta = 3/4n$. Thus it is p-isolated, with respect to the $\sigma(X, Z)$-topology, as we wanted to prove. Moreover, if we fix $\delta > 0$ and such that

$$1/4n + \delta < 3/4n - \delta$$

we can prove that the former family is discrete for the norm topology. Indeed for any $z \in X$ we have that

$$\mathscr{B}_Z(z, \delta) \cap \bigcup_{i \in I}(A_i^n + \mathscr{B}_Z(0, 1/4n))$$

has nonempty intersection with at most one member of the family because every time the intersection is nonempty we can see that $\varphi_i(z) > 3/4n - \delta$ if

$$\mathscr{B}_Z(z, \delta) \cap (A_i^n + \mathscr{B}_Z(0, 1/4n)) \neq \emptyset,$$

but $\varphi_i(z) < 1/4n + \delta$ when

$$\mathscr{B}_Z(z, \delta) \cap \left(A_j^n + \mathscr{B}_Z(0, 1/4n)\right) \neq \emptyset,$$

for any $j \neq i$ and $j \in I$. This fact can be seen as above writing now $z = x + y$ with $x \in \mathscr{B}_Z(z, \delta) \cap (A_i^n + \mathscr{B}_Z(0, 1/4n))$ and $\|y\|_Z < \delta$ in the first case and $x \in \mathscr{B}_Z(z, \delta) \cap (A_j^n + \mathscr{B}_Z(0, 1/4n))$ with $\|y\|_Z < \delta$ for the second one. □

Proposition 2.34 *Let X a normed space and Z a norming subspace in the dual space X^*. Let us assume the space X admits an equivalent $\sigma(X, Z)$-lower semicontinuous and $\sigma(X, Z)$-Kadec quasinorm. Then the norm topology admits a network $\mathcal{N}$, such that*

$$\mathcal{N} = \bigcup_{n \in \mathbb{N}} \mathcal{N}_n$$

where for every $n \in \mathbb{N}$ there exists $p_n \in (0, 1]$ such that the family $\mathcal{N}_n$ is p_n-isolated, with respect to the $\sigma(X, Z)$-topology and it consists of sets which are the difference of $\sigma(X, Z)$-closed and p_n-convex subsets of X. Moreover, there exists $\delta_n > 0$ such that $\mathcal{N}_n + \mathscr{B}_Z(0, \delta_n)$ is norm discrete for every $n \in \mathbb{N}$.

Proof Consider the network $\mathcal{M} = \bigcup_{n\in\mathbb{N}} \mathcal{M}_n$ of the norm topology such that every one of the families $\mathcal{M}_r := \{M_{r,i} \mid i \in I_r\}$ is isolated, with respect to the $\sigma(X,Z)$-topology. The decomposition lemma (lemma 2.22) says that for a fixed sequence $p_n \searrow 0$ in $(0,1]$ we will have a decomposition of the sets in the family $\mathcal{M}_r$ producing families $\mathcal{M}_r^n$, $n = 1, 2, \dots$ with $\mathcal{M}_r^n$ being p_{n_m}-isolated, with respect to the $\sigma(X,Z)$-topology, for all $r, m = 1, 2, \dots$. Therefore without loss of generality it is possible to reorder the sequence and to assume that for $r = 1, 2, \dots$ the family $\mathcal{M}_r$ is already p_r-isolated (with respect to the $\sigma(X,Z)$-topology). Let us perform the following decomposition: denote by $\varphi_{r,i}$ the p_r-distance to $\overline{\text{conv}_{p_r}}^{\sigma(X^{**},X^*)}\{M_{r,j} \mid j \neq i\}$ and define

$$N_{r,i}^n := \left\{x \in \overline{\text{conv}_{p_r}}^{\sigma(X,Z)}(M_{r,i}) \,\middle|\, \varphi_{r,i}(x) > 3/4n\right\}.$$

The fact that each one of the families $\mathcal{N}_r^n := \{N_{r,i}^n | i \in I_r\}$ is $\sigma(X,Z)$-p_r-isolated follows from theorem 2.21 since the lower semicontinuity and p_r-convexity of the functions $\varphi_{r,i}$ tell us that $\varphi_{r,j}(y) = 0$ for every $y \in \overline{\text{conv}_{p_r}}^{\sigma(X,Z)}(M_{r,i})$ and $j \neq i$, $j \in I_r$. Moreover, as in lemma 2.33, we have here that

$$\varphi_{r,i}(z) > 3/4n - \mu$$

whenever $z \in N_{r,i}^n + \mathscr{B}_Z(z,\mu)$ and $\varphi_{r,i}(z) < \mu$. Choose δ_n such that $0 < 2\delta_n < 3/4n - \delta_n$, then we have that the norm open sets $\{N_{r,i}^n + \mathscr{B}_Z(0,\delta_n) | i \in I_r\}$ are disjoint and they form a norm discrete and p_r-isolated family, with respect to the $\sigma(X,Z)$-topology. Moreover, each one of the sets $N_{r,i}^n$ is the difference of p_r-convex and $\sigma(X,Z)$-closed subsets of X: $\overline{\text{conv}_{p_r}}^{\sigma(X,Z)}(M_{r,i})$ and $\{x \in X \mid \varphi_{r,i}(x) \leq 3/4n\}$. The union of all these families

$$\bigcup_{r,n\in\mathbb{N}} \mathcal{N}_r^n$$

is the network we are looking for. Indeed, given $x \in X$ there exists $r \in \mathbb{N}$ and $i \in I_r$ such that $x \in M_{r,i} \subseteq x + \mathscr{B}_Z(0,\varepsilon/3)$. Then for $n \in \mathbb{N}$ big enough we have $x \in N_{r,i}^n$, $x \in \overline{\text{conv}_{p_r}}^{\sigma(X,Z)}(M_{r,i}) \subseteq x + \mathscr{B}_Z(0,2\varepsilon/3)$ and we have $x \in \overline{\text{conv}_{p_r}}^{\sigma(X,Z)}(N_{r,i}^n) + \mathscr{B}_Z(0,\delta_n) \subseteq x + \mathscr{B}_Z(0,\varepsilon)$ if we take $n \in \mathbb{N}$ big enough. □

We now arrive to:

Theorem 2.35 *Let X a normed space with a norming subspace $Z \subseteq X^*$. X admits an equivalent $\sigma(X,Z)$-lower semicontinuous and $\sigma(X,Z)$-Kadec quasinorm, if and only if, the norm topology admits a σ-discrete basis $\mathcal{B} = \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$ such that every one of the families $\mathcal{B}_n$ is isolated, with respect to the $\sigma(X,Z)$-topology, and norm discrete.*

Proof Starting from the network constructed in the previous proposition, let us continue with the same notation and observe that when we add open balls of suitable small radii to the network we get the basis of the norm topology we are looking for. In fact

$$\bigcup_{n,r\in\mathbb{N}} \left\{N^n_{r,i} + \mathscr{B}_Z(0,\delta_n) \,\middle|\, i \in I_r\right\}$$

is a basis of the norm topology. Indeed, for given $x \in X$ and $\varepsilon > 0$ we can find $p \in \mathbb{N}$ and $i \in I_p$ with $x \in N_{p,i} \subseteq \mathscr{B}_Z(x, \varepsilon/2)$. There is $m_0 \in \mathbb{N}$ such that $x \in N^m_{p,i}$ whenever $m \geq m_0$. It follows that, for $m \in \mathbb{N}$ big enough we have $N^m_{p,i} + \mathscr{B}_Z(0,\delta_m) \subseteq \mathscr{B}_Z(x,\varepsilon)$ since $x \in N_{p,i} \subseteq \mathscr{B}_Z(x,\varepsilon/2)$ and δ_m goes to zero when m goes to infinity. □

2.5 F-norms with the LUR property

In this section we want to prove theorem 1.6. To do that we look for a p-version of [OT09a, Lemma 3.2], but the main ingredient is Deville's master lemma.

Lemma 2.36 (p-version of Deville's lemma) *Let $(\varphi_i)_{i\in I}$ and $(\psi_i)_{i\in I}$ be two families of real valued non-negative functions defined on a Banach space X, which are both uniformly bounded, uniformly continuous on bounded subsets of X and satisfy the p-property. For $i \in I$ and $k \in \mathbb{N}$, let us denote for every $x \in X$*

$$\begin{aligned}
\theta_{i,k}(x) &= \varphi_i^2(x) + \frac{1}{k}\psi_i^2(x);\\
\theta_k(x) &= \sup_{i\in I} \theta_{i,k}(x);\\
\theta(x) &= \|x\|^2 + \sum_{k\in\mathbb{N}} 2^{-k}(\theta_k(x) + \theta_k(-x)),
\end{aligned}$$

where $\|\cdot\|$ is the norm of X. If $q(\cdot)$ denotes the p-Minkowski functional of $B = \{x \in X \,|\, \theta(x) \leq 1\}$, then $q(\cdot)$ is an equivalent p-norm on X with the following property: if $x_n, x \in X$ satisfy $\lim(2q^2(x)+2q^2(x_n)-q^2(x+x_n)) = 0$, then there exists a sequence (i_n) in I such that:

1. $\lim_{n\in\mathbb{N}} \left(\frac{1}{2}\psi^2_{i_n}(x) + \frac{1}{2}\psi^2_{i_n}(x_n) - \psi^2_{i_n}\left(\frac{x+x_n}{2^{1/p}}\right)\right) = 0$;
2. $\lim_{n\in\mathbb{N}} \varphi_{i_n}(x) = \lim_{n\in\mathbb{N}} \varphi_{i_n}(x_n) = \lim_{n\in\mathbb{N}} \varphi_{i_n}\left(\frac{x+x_n}{2^{1/p}}\right) = \sup_{i\in I} \varphi_i(x)$.

Proof The equivalence for p-norm is of the following type $\xi^p\|x\|^p \leq q(x) \leq \zeta^p\|x\|^p$ and follows from boundedness and continuity. Suppose that $x \in X$ and $(x_n)_{n\in\mathbb{N}} \subseteq X$ are such that

$$\lim_{n\in\mathbb{N}}(2q^2(x) + 2q^2(x_n) - q^2(x + x_n)) = 0.$$

From proposition 2.18 we have $\lim_{n\in\mathbb{N}} q(x_n) = q(x)$ and $\lim_{n\in\mathbb{N}} q(x+x_n) = 2q(x)$. Thus $\theta(x) = q(x)$, since θ is uniformly continuous on bounded subsets of X, we have $\lim_{n\in\mathbb{N}} \theta(x_n) = \theta(x)$ and $\lim_{n\in\mathbb{N}} \theta\left(\frac{x+x_n}{2^{1/p}}\right) = \theta(x)$ and consequently

$$\lim_{n\in\mathbb{N}} \left(\frac{1}{2}\theta(x) + \frac{1}{2}\theta(x_n) - \theta\left(\frac{x+x_n}{2^{1/p}}\right)\right) = 0.$$

By proposition 2.18, we have for every $k \in \mathbb{N}$,

$$\lim_{n\in\mathbb{N}} \left(\frac{1}{2}\theta_k(x) + \frac{1}{2}\theta_k(x_n) - \theta_k\left(\frac{x+x_n}{2^{1/p}}\right)\right) = 0.$$

Let $(\alpha_n)_{n\in\mathbb{N}}$ a sequence of real number such that $\alpha_n > 0$ and $\lim_{n\in\mathbb{N}} n\alpha_n = 0$. By a standard convexity argument (see [DGZ93, Fact II.2.3]) there exists $(k_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$

$$\frac{1}{2}\theta_{k_n}(x) + \frac{1}{2}\theta_{k_n}(x_n) - \theta_{k_n}\left(\frac{x+x_n}{2^{1/p}}\right) < \alpha_{k_n}. \tag{2.1}$$

It follow from (2.1) and the very definition of θ_{k_n} that for each $n \in \mathbb{N}$ there exists i_n such that

$$\frac{1}{2}\theta_{k_n}(x) + \frac{1}{2}\theta_{k_n}(x_n) - \theta_{i_n,k_n}\left(\frac{x+x_n}{2^{1/p}}\right) < \alpha_{k_n}.$$

Thus, for every $i \in I$, we have:

$$\begin{aligned}
\alpha_{k_n} &> \frac{1}{2}\theta_{i,k_n}(x) + \frac{1}{2}\theta_{i_n,k_n}(x_n) - \theta_{i_n,k_n}\left(\frac{x+x_n}{2^{1/p}}\right) = \\
&= \frac{1}{2}(\varphi_i^2(x) - \varphi_{i_n}^2(x)) + \frac{1}{2}\varphi_{i_n}^2(x) + 2\varphi_{i_n}^2(x_n) - \varphi_{i_n}^2\left(\frac{x+x_n}{2^{1/p}}\right) + \\
&+ \frac{1}{2k_n}(\psi_i^2(x) - \psi_{i_n}^2(x)) + \frac{1}{k_n}\left(\frac{1}{2}\psi_{i_n}^2(x) + \frac{1}{2}\psi_{i_n}^2(x_n) - \psi_{i_n}^2\left(\frac{x+x_n}{2^{1/p}}\right)\right).
\end{aligned} \tag{2.2}$$

If we choose $i = i_n$ we get by proposition 2.16

$$0 \le \left(\frac{\varphi_{i_n}(x) - \varphi_{i_n}(x_n)}{2}\right)^2 \le \frac{1}{2}\varphi_{i_n}^2(x) + \frac{1}{2}\varphi_{i_n}^2(x_n) - \varphi_{i_n}^2\left(\frac{x+x_n}{2^{1/p}}\right) \le \alpha_{k_n} \tag{2.3}$$

$$0 \le \left(\frac{\psi_{i_n}(x) - \psi_{i_n}(x_n)}{2}\right)^2 \le \frac{1}{2}\psi_{i_n}^2(x) + \frac{1}{2}\psi_{i_n}^2(x_n) - \psi_{i_n}^2\left(\frac{x+x_n}{2^{1/p}}\right) \le k_n\alpha_{k_n} \tag{2.4}$$

Since $\lim_{n\in\mathbb{N}} k_n\alpha_{k_n} = 0$, (2.4) implies 1. Furthermore, (2.3) implies that

$$\lim_{n\in\mathbb{N}}(\varphi_{i_n}(x) - \varphi_{i_n}(x_n)) = 0 \quad \text{and} \quad \lim_{n\in\mathbb{N}}\left(\varphi_{i_n}(x) - \varphi_{i_n}\left(\frac{x+x_n}{2^{1/p}}\right)\right) = 0.$$

On the other hand, if we denote by $M = \sup_{i\in I} \psi_i^2(x)$ then given $n \in \mathbb{N}$, (2.2) yields, for every $i \in I$

$$\varphi_i^2(x) - \varphi_{i_n}^2 < 2\alpha_{k_n} - \frac{1}{k_n}(\psi_i^2(x) - \psi_{i_n}^2(x)) \le 2\alpha_{k_n} - \frac{M}{k_n}.$$

Thus, for $n \in \mathbb{N}$, we have

$$\varphi_{i_n}^2(x) \geq \sup_{i \in I} \varphi_i^2(x) + \frac{M}{k_n} - 2\alpha_{k_n}.$$

Hence $\liminf_{n \in \mathbb{N}} \varphi_{i_n}(x) \geq \sup_{i \in I} \varphi_i(x)$. This concludes the proof of the lemma. □

Observe that if $Z \subseteq X^*$ is a norming subspace of X and if the norm $\|\cdot\|$, the φ_i's and the ψ_i's are all $\sigma(X, Z)$-lower semicontinuous, then the p-norm $q(\cdot)$ is also $\sigma(X, Z)$-lower semicontinuous. Let us state now a p-version of the connection lemma.

Lemma 2.37 (p-connection lemma) *Let $(X, \|\cdot\|)$ a normed space and Z a norming subspace in X^*. Let $\mathcal{B} = \{B_i \,|\, i \in I\}$ an uniformly bounded and p-isolated family of subsets of X, with respect to the $\sigma(X, Z)$-topology. Then there exists an equivalent $\sigma(X, Z)$-lower semicontinuous p-norm $q_{\mathcal{B}}(\cdot)$ on X such that for every $i_0 \in I$, every $x \in B_{i_0}$, and every sequence $(x_n)_{n \in \mathbb{N}}$ in X the condition*

$$\lim_{n \in \mathbb{N}} \left(2q_{\mathcal{B}}^2(x_n) + 2q_{\mathcal{B}}^2(x) - q_{\mathcal{B}}^2(x + x_n)\right) = 0,$$

implies that:

1. *there exists $n_0 \in \mathbb{N}$ such that*

$$x_n, \frac{x_n + x}{2^{1/p}} \notin \overline{\mathrm{conv}_p}^{\sigma(X,Z)} \bigcup \{B_i \,|\, i \neq i_0,\ i \in I\}$$

   for every $n \geq n_0$;

2. *for every $\delta > 0$ there exists $n_\delta \in \mathbb{N}$ such that*

$$x_n \in \overline{\mathrm{conv}(B_{i_0} \cup \{0\}) + \mathscr{B}(0, \delta)}^{\sigma(X,Z)},$$

   whenever $n \geq n_\delta$.

Proof Fix an index $i \in I$ and define the functions φ_i as the p-distance from

$$\overline{\mathrm{conv}_p}^{\sigma(X^{**},X^*)} \bigcup \{B_i \,|\, i \neq i_0,\ i \in I\}$$

Set $D_i = \mathrm{conv}(B_i \cup \{0\})$ and $D_i^\delta = D_i + \mathscr{B}(0, \delta)$, where $\mathscr{B}(0, \delta) = \{x \in X \,|\, \|x\|_Z < \delta\}$, for every $\delta > 0$ and $i \in I$. We denote by $p_{i,\delta}$ the Minkowski functional of the convex body $\overline{D_i^\delta}^{\sigma(X,Z)}$. Then we define the norm p_i by the formula

$$p_i^2(x) = \sum_{q \in \mathbb{N}} \frac{1}{q^2 2^q} p_{i,1/q}^2(x),$$

for every $x \in X$. It is well defined and $\sigma(X,Z)$-lower semicontinuous. Indeed, since $\mathscr{B}(0,\delta) \subseteq \overline{D_i^\delta}^{\sigma(X,Z)}$ we have for every $x \in X$, and $\delta > 0$, that $p_{i,\delta}\left(\frac{\delta x}{\|x\|_Z}\right) \leq 1$, thus $\delta p_{i,\delta}(x) \leq \|x\|_Z$ and hence the above series converges. Finally we define the nonnegative, p-convex and $\sigma(X,Z)$-lower semicontinuous function

$$\psi_i(x) = p_i(x),$$

for every $x \in X$. We are now in position to apply the p-version of Deville's lemma to get an equivalent and $\sigma(X,Z)$-lower semicontinuous p-norm $q_{\mathcal{B}}(\cdot)$ on X. Take $i_0 \in I$, $x \in B_{i_0}$ and a sequence $(x_n)_{n\in\mathbb{N}}$ in X satisfying

$$\lim_{n\in\mathbb{N}} \left(2q_{\mathcal{B}}^2(x_n) + 2q_{\mathcal{B}}^2(x) - q_{\mathcal{B}}^2(x + x_n)\right) = 0.$$

Lemma 2.36 implies the existence of a sequence of indexes $(i_n)_{n\in\mathbb{N}}$ in I such that the conclusion of the previous lemma hold. Our hypothesis on the p-isolated character of the family $\mathcal{B}$ tell us, after lemma 2.21, that since the point x belongs to the set B_{i_0} of the family $\mathcal{B}$, we have $\varphi_{i_0}(x) > 0$, but $\varphi_i(x) = 0$ for all $i \in I \setminus \{i_0\}$. From the assertion 1 of lemma 2.36 follows that there exists $n_0 \in \mathbb{N}$ such that $i_n = i_0$, $\varphi_{i_0}(x_n) > 0$ and $\varphi_{i_0}\left(\frac{x+x_n}{2^{1/p}}\right) > 0$ for all $n \geq n_0$, from where the conclusion 1 of our lemma follows. Moreover, the equation 2, of lemma 2.36 is now of the form

$$\lim_{n\in\mathbb{N}} \left(\frac{1}{2}\psi_{i_0}^2(x_n) + \frac{1}{2}\psi_{i_0}^2(x) - \psi_{i_0}^2\left(\frac{x + x_n}{2^{1/p}}\right)\right) = 0$$

and so by a p-convexity argument (proposition 2.18), for every $q \in \mathbb{N}$, we have

$$\lim_{n\in\mathbb{N}} \left(\frac{1}{2}p_{i_0,1/q}^2(x_n) + \frac{1}{2}p_{i_0,1/q}^2(x) - p_{i_0,1/q}^2\left(\frac{x + x_n}{2^{1/p}}\right)\right) = 0$$

and consequently $\lim_{n\in\mathbb{N}} p_{i_0,1/q}(x_n) = p_{i_0,1/q}(x)$. Fix $\delta > 0$ and $q \in \mathbb{N}$ such that $\frac{1}{q} < \delta$. Since $x \in D_{i_0}^{1/q}$ we have that $p_{i_0,1/q}(x) < 1$ because $D_{i_0}^{1/q}$ is norm open. Therefore, there exists $n_\delta \in \mathbb{N}$ such that for $n \geq n_\delta$ we have that $p_{i_0,1/q}(x_n) < 1$ and thus $x_n \in \overline{D_{i_0}^\delta}^{\sigma(X,Z)}$ that is

$$x_n \in \overline{\mathrm{conv}(B_{i_0} \cup \{0\}) + \mathscr{B}_Z(0,\delta)}^{\sigma(X,Z)}$$

which is 2 for $\|\cdot\|_Z$. Since the proof is valid for every $\delta > 0$ and $\|\cdot\|_Z$ is an equivalent norm the proof is over. □

We are now in position to prove theorem 1.6.

Theorem 2.38 *In a normed space $(X, \|\cdot\|)$ with a norming subspace Z in X^*, if there exist isolated families, with respect to the $\sigma(X,Z)$-topology*

$$\{\mathcal{B}_n \mid n \in \mathbb{N}\}$$

in the unit sphere $\mathcal{S}_X$ such that for every x in $\mathcal{S}_X$ and every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ and $B \in \mathcal{B}_n$ such that $x \in B$ and $\|\cdot\|$-diam$(B) < \varepsilon$, then there exists a LUR F-norm; i.e. there exists an F-norm F such that for every $(x_n)_{n\in\mathbb{N}} \subseteq X$ and $x \in X$ it follows $\|\cdot\|$-$\lim_{n\in\mathbb{N}} x_n = x$, whenever

$$\lim_{n\in\mathbb{N}}(2F^2(x) + 2F^2(x_n) - F^2(x + x_n)) = 0.$$

Proof By the decomposition lemma (lemma 2.22) we may assume that $\{\mathcal{B}_n\}_{n\in\mathbb{N}}$ are p_n-isolated, with respect to the $\sigma(X, Z)$-topology, for some sequence $p_n \searrow 0$. So we can consider the p_n-norms, say $q_{\mathcal{B}_n}(\cdot)$, constructed using the p_n-version of the connection lemma. Define an F-norm in the following way

$$F^2_{\mathcal{B}}(x) := \sum_{n\in\mathbb{N}} c_n q^2_{\mathcal{B}_n}(x),$$

for every $x \in X$, where the sequence $(c_n)_{n\in\mathbb{N}}$ is chosen accordingly for the convergence of the series on bounded set. This is possible because all the p_n-norms $q_{\mathcal{B}_n}(\cdot)$ are equivalent to the original norm and hence there exist numbers ζ_n such that

$$q_{\mathcal{B}_n}(x) \leq \zeta_n^{p_n} \|x\|^{p_n} \leq \zeta_n^{p_n} \max\{1, \|x\|\},$$

so it is enough to take $c_n := \frac{1}{\zeta_n^{2p_n} 2^n}$. Consider $x \in X$ and a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ such that

$$\lim_{n\in\mathbb{N}} \left(2F^2_{\mathcal{B}}(x_n) + 2F^2_{\mathcal{B}}(x) - F^2_{\mathcal{B}}(x + x_n)\right) = 0.$$

Fix $\varepsilon > 0$, we know that there exists $m \in \mathbb{N}$ and $B_0 \in \mathcal{B}_m$ with $\frac{x}{\|x\|} \in B_0 \subseteq x + \frac{\epsilon}{2} B_X$. The condition

$$\lim_{n\in\mathbb{N}} \left(2F^2_{\mathcal{B}}(x_n) + 2F^2_{\mathcal{B}}(x) - F^2_{\mathcal{B}}(x + x_n)\right) = 0$$

implies that

$$\lim_{n\in\mathbb{N}} \left(2q^2_{\mathcal{B}_m}(x_n) + 2q^2_{\mathcal{B}_m}(x) - q^2_{\mathcal{B}_m}(x + x_n)\right) = 0$$

by a p-convexity arguments (proposition 2.18), and then

$$\lim_{n\in\mathbb{N}} \left(2q^2_{\mathcal{B}_m}\left(\frac{x_n}{\|x\|}\right) + 2q^2_{\mathcal{B}_m}\left(\frac{x}{\|x\|}\right) - q^2_{\mathcal{B}_m}\left(\frac{x + x_n}{\|x\|}\right)\right) = 0$$

in particular $\lim_{n\in\mathbb{N}} q_{\mathcal{B}_m}(x_n/\|x\|) = q_{\mathcal{B}_m}(x/\|x\|)$. The p-connection lemma (lemma 2.37) tells us that there exists $n_{\frac{\varepsilon}{2}} \in \mathbb{N}$ such that

$$\frac{x_n}{\|x\|} \in \overline{\text{conv}(B_0 \cup \{0\}) + \mathscr{B}(0, \varepsilon/2)}^{\sigma(X,Z)}$$

whenever $n \geq n_{\frac{\varepsilon}{2}}$. Fix $(y_\beta)_{\beta\in B}$ a subnet of $(x_n/\|x\|)_{n\in\mathbb{N}}$ associated to a subnet map $\sigma : B \to \mathbb{N}$. We know that there exists $\beta_{\frac{\varepsilon}{2}} \in B$ such that for every $\beta \geq \beta_{\frac{\varepsilon}{2}}$ we have $\sigma(\beta) \geq n_{\frac{\varepsilon}{2}}$, and then

$$y_\beta \in \overline{\text{conv}(B_0 \cup \{0\}) + \mathscr{B}(0, \varepsilon/2)}^{\sigma(X,Z)},$$

whenever $\beta \geq \beta_{\frac{\varepsilon}{2}}$. So we have that $\|\cdot\|$- $\operatorname{dist}(y_\beta, I_x) \leq \varepsilon$ for $\beta \geq \beta_{\frac{\varepsilon}{2}}$ where I_x is the segment joining x with the origin, and so there exists a net $r_{(\beta,\varepsilon)} \in [0,1]$ such that

$$\left\| y_\beta - r_{(\beta,\varepsilon)} \frac{x}{\|x\|} \right\| \leq \varepsilon,$$

for $\beta \geq \beta_{\frac{\varepsilon}{2}}$. Consider now the direct set $D := \{(\beta, \varepsilon) \in B \times (0,1] | \beta \geq \beta_{\frac{\varepsilon}{2}}\}$, which is direct with the induced order. Then for the net $\left\{ r_{(\beta,\varepsilon)} \,\middle|\, (\beta,\varepsilon) \in D \right\}$ there exists a subnet map $\tau : C \to D$ for some directed set $(C, \succeq)$ such that the limit

$$r := \lim_{\gamma \in C} r_{\tau(\gamma)}$$

exists by compactness of $[0,1]$. Let us denote with $\overline{\tau}$ the composition of the map τ with the projection from $B \times (0,1]$ onto B, which is a subnet map too, we have

$$\|\cdot\|\text{-} \lim_{\gamma \in C} \frac{x_{\overline{\tau}(\gamma)}}{\|x\|} = r \frac{x}{\|x\|}.$$

In particular by equivalence of $q_{\mathcal{B}_m}(\cdot)$ with $\|\cdot\|$ we have

$$\lim_{\gamma \in C} q_{\mathcal{B}_m} \left(\frac{x_{\overline{\tau}(\gamma)}}{\|x\|} \right) = q_{\mathcal{B}_m} \left(r \frac{x}{\|x\|} \right),$$

and by hypothesis we have $q_{\mathcal{B}_m}(rx/\|x\|) = q_{\mathcal{B}_m}(x/\|x\|) \neq 0$, and so $r = 1$. The proof is over because the former reasoning is valid for every subnet of the given sequence. □

Using this theorem we can construct an F-norm F_1 with the LUR property. By the remark after the proof of theorem 2.26, we also get an F-norm F_2 with the Kadec property. If we define

$$F^2(x) = F_1^2(x) + F_2^2(x) \qquad x \in X,$$

then F is an F-norm which enjoys both the Kadec and the LUR property. This prove theorem 1.6.

Chapter 3

Characterizing and transferring rotund norms

One of the older results on rotund renorming theory (a part the result of Clarkson [Cla36]) is Day's renorming of $c_0(\Gamma)$ (see [Day55, Theorem 10]):

Theorem 3.1 *If* Γ *is an arbitrary nonempty set, then the* Day's norm *on* $c_0(\Gamma)$ *defined for every* $x = (x(\gamma))_{\gamma\in\Gamma}$ *by*

$$\|x\|_D = \sup\left\{ \left(\sum_{k=1}^{n} \frac{x^2(\gamma_k)}{4^k} \right)^{\frac{1}{2}} \middle| (\gamma_1, \ldots, \gamma_n) \right\},$$

is an equivalent rotund norm.

It is actually possible to prove that Day's norm is a LUR norm on $c_0(\Gamma)$ (see [Rai69] and [DGZ93, Theorem II.7.3]) From that point onward, a lot of theory was develop focusing in particular on LUR renorming theory. Surely one of the next oustanding result was the following theorem of Haydon [Hay99, Theorem 5.1]:

Theorem 3.2 *For a tree* Υ *the following are equivalent:*

1. $\mathscr{C}_0(\Upsilon)$ *admits an equivalent rotund norm;*
2. $\mathscr{C}_0(\Upsilon)$ *admits an equivalent MLUR norm, i.e. for every sequence* $(h_n)_{n\in\mathbb{N}} \subseteq X$ *and* $x \in X$ *we have* $\|\cdot\|\text{-}\lim_{n\in\mathbb{N}} h_n = 0$*, whenever*
$$\lim_{n\in\mathbb{N}}(\|x+h_n\|^2 + \|x-h_n\|^2 - 2\|x\|^2) = 0;$$
3. *there exists a bounded linear injection from* $\mathscr{C}_0(\Upsilon)$ *into some space* $c_0(I)$.

This result has give new life to the study of rotund norms, and in particular the works of Smith (see [Smi06], [Smi09] and [Smi12]), Orihuela, Smith and Troyanski (see [OST12]) and Moltó, Orihuela, Troyanski and Zizler (see [MOTZ07]) give new interesting results in this branch of renorming theory.

Dual rotund norm are important in the theory of Gâteaux smooth space, by the following result of Šmulian (see [Šmu40]):

Theorem 3.3 *Let X a Banach space. If X^* admits an equivalent dual rotund norm, then X admits an equivalent Gâteaux smooth norm.*

As a consequence, characterizing dual spaces which admit equivalent dual rotund has a special interest. Our starting point in this framework is the characterization result in [OST12, Theorem 2.7], which was already stated in theorem 1.8. In order to improve this result in the dual case, we have proved an eating lemma, with the intention of replacing open neighbourhoods with open slices. Let us consider the following derivation process: we fix a family Λ of w^*-open sets of $\mathscr{B}_{X^*}$, $\mathcal{H}$ the family of all w^*-slices of $\mathscr{B}_{X^*}$ and for a set $T \subseteq \mathscr{B}_{X^*}$ we define

$$\mathrm{Sl}^*(T, \Lambda) = \{H \cap T | H \in \mathcal{H} \text{ and } W \in \Lambda \text{ with } \emptyset \neq H \cap T \subseteq W\};$$

now by transfinite induction we set $B^{(0)}(\mathscr{B}_{X^*}, \Lambda) = \mathscr{B}_{X^*}$ and for an ordinal α, such that $B^{(\alpha)}(\mathscr{B}_{X^*}, \Lambda) \neq \emptyset$, we define

$$B^{(\alpha+1)}(\mathscr{B}_{X^*}, \Lambda) = B^{(\alpha)}(\mathscr{B}_{X^*}, \Lambda) \setminus \bigcup \mathrm{Sl}^*(B^{(\alpha)}(\mathscr{B}_{X^*}, \Lambda), \Lambda),$$

and if λ is a limit ordinal then

$$B^{(\lambda)}(\mathscr{B}_{X^*}, \Lambda) = \bigcap_{\alpha<\lambda} B^{(\alpha)}(\mathscr{B}_{X^*}, \Lambda).$$

If we assume that Λ cover a set $D \subseteq \mathscr{B}_{X^*}$, we want to study when there exists a countable ordinal α such that $B^{(\alpha)}(\mathscr{B}_{X^*}, \Lambda) \cap D = \emptyset$. The following two lemmata answer our needs.

Lemma 3.4 (Eating lemma) $B^{(\omega_0)}(\mathscr{B}_{X^*}, \Lambda) = \overline{\mathrm{conv}}^{w^*}(\mathscr{B}_{X^*} \setminus \bigcup \Lambda)$.

Proof We will use the notation $B^{(\alpha)} := B^{(\alpha)}(\mathscr{B}_{X^*}, \Lambda)$ for every ordinal α. It is easy to see that $\mathscr{B}_{X^*} \setminus \bigcup \Lambda \subseteq B^{(\omega_0)}$. By the Krein–Milman theorem the thesis follows if we prove that $\mathrm{ext}(B^{(\omega_0)}) \subseteq \mathscr{B}_{X^*} \setminus \bigcup \Lambda$. Assume by contradiction that there exists $x_0^* \in \mathrm{ext}(B^{(\omega_0)}) \cap \bigcup \Lambda$, then we have

$$x_0^* \notin \bigcup \mathrm{Sl}^*(B^{(n)}, \Lambda) \tag{3.1}$$

for every $n \in \mathbb{N}$. Let us fix $W \in \Lambda$ such that $x_0^* \in W$ and, by Choquet's lemma (see [FHH+11, Lemma 3.69]), there is a w^*-open half-space H such that

$$x_0^* \in H \cap B^{(\omega_0)} \subseteq W. \tag{3.2}$$

Let H_1 a w^*-open half-space such that $x_0^* \in H_1 \subseteq \overline{H_1}^{w^*} \subseteq H$, by (3.1) there exists $x_n^* \in H_1 \cap B^{(n)}$ such that $x_n^* \notin W$ for every $n \in \mathbb{N}$. By the w^*-compactness of the dual unit ball there is x^* a cluster point of the sequence $\{x_n^*\}_{n\in\mathbb{N}}$, thus

$$x^* \in B^{(n)} \cap \overline{H_1}^{w^*} \text{ and } x^* \notin W$$

and so $x^* \in B^{(\omega_0)} \cap \overline{H_1}^{w^*} \subseteq B^{(\omega_0)} \cap H$ which is a contradiction with (3.2). □

The previous lemma says that if Λ is a w^*-open cover of $\mathscr{B}_{X^*}$, then $B^{(\omega_0)}(\mathscr{B}_{X^*}, \Lambda) = \emptyset$. Now we want to prove that $B^{(\omega_0)}(\mathscr{B}_{X^*}, \Lambda) \cap \mathcal{S}_{X^*} = \emptyset$, whenever Λ is an open cover of the unit sphere. To this goal we need the following extreme point lemma of Choquet (see [Cho69b, Lemma 27.8]). We state it here for the sake of completeness.

Lemma 3.5 *Let X a Hausdorff topological vector space, $C \subseteq X$ a convex set and $A \subseteq C$ a convex and linearly compact set (that is, any line intersecting A does so in a closed segment). Suppose that $B = C \setminus A$ is convex. Then if* $\mathrm{ext}(A) \neq \emptyset$, *we have* $\mathrm{ext}(A) \cap \mathrm{ext}(C) \neq \emptyset$.

Proof Let $a \in \mathrm{ext}(A)$. The lemma holds if $a \in \mathrm{ext}(C)$, so suppose that $a \notin \mathrm{ext}(C)$. In this case there exist $x, y \in C$ so that

$$a = \frac{x+y}{2}$$

and $x \neq y$. As A and B are convex, we may suppose $x \in A$ amnd $y \in B$. Let ℓ denote the line containing x and y which meets A in a closed segment $[a, b]$ (one end must be a as $a \in \mathrm{ext}(A)$). By translation we may assume $b = 0$. We now claim that $b \in \mathrm{ext}(C)$. This will conclude the proof since obviously

$$\mathrm{ext}(C) \cap A \subseteq \mathrm{ext}(A)$$

and $b \in A$.

To prove our claim, suppose that $b \notin \mathrm{ext}(C)$. Let $b = (1/2)(b_1 + b_2)$ with $b_1 \neq b_2$. One of this point, say b_1, lies in A. Since

$$A \cap \ell = [a, b],$$

$b_1 \notin \ell$ (otherwise $A \cap \ell = [a, b_2]$). Let ℓ' denote the line through b_1 and b_2 so that ℓ and ℓ' determine a plane π through the origin. It is known that π is isomorphic to $\mathbb{R}^2$ (see for example [Cho69a, Theorem 15.23]). Let $f : \mathbb{R}^2 \to \mathbb{R}$ a continuous linear map such that $\ell = \ker(f)$. Then for

$$c_1, c_2 \in \operatorname{conv}(\{b_1, b_2, y\})$$

and lying in separate open halfspace of f. Define

$$g(c_1, c_2) = \frac{f(c_1)}{f(c_1) - f(c_2)} c_1 - \frac{f(c_2)}{f(c_1) - f(c_2)} c_2,$$

then g is continuous and $g(c_1, c_2) \in [0, y]$, since $f \circ g = 0$. We claim that we may choose $b_2 \in A$. If not, then we may choose $z_n \to 0$ with

$$z_n \in (0, b_2] \cap B.$$

For $z \in [b_1, y)$, z_n and z lie in separate halfspace and $g(z, z_n) \to 0$. Hence either $a = b = 0$ (since $g(z, z_n) \in B$ if $z \in B$) or $z \notin B$. We conclude that

$$[b_1, y) \subseteq A.$$

But since A is linearly compact $y \in A$, a contradiction. Hence we may assume that $b_2 \in A$.

Let c_i denote the endpoint of the segment $[b_i, y]$ in A. Then $c_i \neq y$ and we may choose sequences $d_n^i \to c_i$, such that $d_n^i \in [b_i, y] \cap B$. Let $e_n = g(d_n^i, d_n^2) \cap B$ as $e_n \to g(c_1, c_2) \in A$, it follows that $e = \lim e_n = a$ and hence $a \notin \operatorname{ext}(A)$. This contradiction establishes the result. □

Now we can prove that our derivation process "eats" the whole unit sphere in at most ω_0 steps.

Lemma 3.6 (Eating lemma for the sphere) *If Λ is a w^*-cover of the unit sphere, then $B^{(\omega_0)}(\mathscr{B}_{X^*}, \Lambda) \cap \mathcal{S}_{X^*} = \emptyset$.*

Proof We will use the notation $B^{(\alpha)} := B^{(\alpha)}(\mathscr{B}_{X^*}, \Lambda)$ for every ordinal α. Suppose, by contradiction, that there exists $x^* \in B^{(\omega_0)} \cap \mathcal{S}_{X^*}$. Now consider

$$B_n^{(\omega_0)} = B^{(\omega_0)} \cap \mathscr{B}(0, 1 - 1/(n+1)),$$

and let us observe that, without loss of generality, we can assume $B_1^{(\omega_0)} \neq \emptyset$. We plan to construct a family of slices which press the point x^* in such a way that an extreme point $x_0^* \in \operatorname{ext}(B^{(\omega_0)}) \cap \mathcal{S}_{X^*}$ can be found. Then we can conclude our proof as in lemma 3.4. We start with a preliminary contruction which enable us to apply lemma 3.5. By the Hahn–Banach theorem, for every $n \in \mathbb{N}$ a w^*-open halfspace H_n exists with the following properties:

1. $x^* \in H_n$,

2. $B_1^{(\omega_0)} \subseteq B^{(\omega_0)} \setminus \overline{H_1}^{w^*}$,

3. $\operatorname{conv}((B^{(\omega_0)} \setminus H_n) \cup B_n^{(\omega_0)}) \subseteq B^{(\omega_0)} \setminus \overline{H_{n+1}}^{w^*}$ for every $n \in \mathbb{N}$.

This is possible due to the fact that $x^* \notin \operatorname{conv}((B^{(\omega_0)} \setminus H_n) \cup B_n^{(\omega_0)})$. Let us consider the convex and w^*-compact set

$$H = \bigcap_{n\in\mathbb{N}} (\overline{H_n}^{w^*} \cap B^{(\omega_0)}).$$

It is easy to see that $H \subseteq \mathcal{S}_{X^*}$ and $B^{(\omega_0)} \setminus H$ is convex, then by the Krein–Milman theorem and lemma 3.5 we obtain that there exists $x_0^* \in \operatorname{ext}(H) \cap \operatorname{ext}(B^{(\omega_0)})$. By repeating the proof of lemma 3.4, we get a contradiction, so we are done. □

3.1 Characterizations of rotundity

In this first section we study some topological characterizations of the existence of rotund renormings. In particular we prove that, if the unit sphere in some equivalent norm admits a special separating cover, then we can always put on our space an equivalent rotund norm. Also we prove a version for rotund renorming of [OT09a, Theorem 1.5].

3.1.1 Radial sets with a G_δ-diagonal

First of all we need a well known topological property:

Definition 3.7 *Let (X, τ) a topological space. We say that X has a* G_δ-diagonal *if, and only if, the set $\Delta = \{(x, x) \,|\, x \in X\}$ is a G_δ-set in $X \times X$.*

The following well known theorem (see [Ced61, Lemma 5.4] or [Gru84, Theorem 2.2]) will be usefull for our purposes.

Theorem 3.8 *Let (X, τ) a topological space. X has a G_δ-diagonal if, and only if, there exists a sequence $(\mathcal{G}_n)_{n\in\mathbb{N}}$ of open covers of X such that for each $x, y \in X$ with $x \neq y$, there exists $n \in \mathbb{N}$ with*

$$y \notin \operatorname{st}(x, \mathcal{G}_n) := \bigcup \{U \in \mathcal{G}_n \,|\, x \in U\}.$$

We will call the sequence $(\mathcal{G}_n)_{n\in\mathbb{N}}$ a G_δ-diagonal sequence.

The following proposition is borrowed from [ST10, Proposition 5]. We prove it just for the sake of completeness.

Proposition 3.9 *Let X a normed space and $F \subseteq X^*$ a norming subspace. If X admits an equivalent, $\sigma(X,F)$-lower semicontinuous and rotund norm $\|\cdot\|_R$, then $S_R = \{x \in X \mid \|x\|_R = 1\}$ has a G_δ-diagonal. Furthermore a G_δ-diagonal sequence $(\mathcal{G}_n)_{n\in\mathbb{N}}$ can be obtaines such that for every $n \in \mathbb{N}$ the members of $\mathcal{G}_n$ are open slices of S_R.*

Proof Let S_R^* the dual sphere related with the norm $\|\cdot\|_R$. Given a rational $q \in \mathbb{Q}^+ \cap [0,1)$ consider the families of open slices

$$\mathcal{H}_q = \{\{x \in S_R \mid f(x) > q\} \mid f \in S_R^* \cap F\}.$$

Consider two distinct point $x, y \in S_R$ and $q \in \mathbb{Q}^+ \cap [0,1)$ such that $\left\|\frac{x+y}{2}\right\| < q < 1$, it is obvious that every $H \in \mathcal{H}_q$ cannot contain both x and y. $\square$

By this proposition, the unit sphere under a $\sigma(X,F)$-lower semicontinuous rotund norm has a G_δ-diagonal relative to the $\sigma(X,F)$-topology. Actually we can say more, since all the elements of the families $\mathcal{H}_q$ are $\sigma(X,F)$-open half-spaces. In this situation we say that $\mathcal{S}_X$ has a *G_δ-diagonal with $\sigma(X,F)$-slices.* We are going to prove three theorems. The first one is a converse of proposition 3.9; the second one is an improvement of the first one in case we deal with a dual space; finally the third one is a nonconvex version of the second one.

Theorem 3.10 *Let X a normed space and $F \subseteq X^*$ a norming subspace. X admits an equivalent, $\sigma(X,F)$-lower semicontinuous and rotund norm $\|\cdot\|_R$ if, and only if, X admits an equivalent, $\sigma(X,F)$-lower semicontinuous norm $\|\cdot\|_\delta$ such that $S_\delta = \{x \in X \mid \|x\|_\delta = 1\}$ has a G_δ-diagonal with $\sigma(X,F)$-slices.*

Proof Let $(\mathcal{H}_n)_{n\in\mathbb{N}}$ a G_δ-diagonal sequence for S_δ, such that every memeber of $\mathcal{H}_n$ is a $\sigma(X,F)$-slice. We will apply theorem 1.2 with S_δ and $\mathcal{H}_n$ in order to obtain a countable number of equivalent norms $\|\cdot\|_n$ which satisfy the conclusion of theorem 1.2. Define

$$\|\cdot\|_R^2 = \|\cdot\|_\delta^2 + \sum_{n\in\mathbb{N}} c_n \|\cdot\|_n^2,$$

where the constants c_n are chosen in order to guarantee the uniform convergence of the series on bounded sets. Since there are costants a_n, b_n such that $a_n\|\cdot\| \le \|\cdot\|_n \le b_n\|\cdot\|$, for every $n \in \mathbb{N}$, it is suffices to take $c_n = 1/(2^n b_n)$. We will prove rotundness: fix $x, y \in X$ and consider the condition $2\|x\|_R^2 + 2\|y\|_R^2 - \|x+y\|_R^2 = 0$; suppose by contradiction that $x \neq y$. By a standard convexity argument (see [DGZ93, Fact II.2.3]) we obtain for every $n \in \mathbb{N}$

$$2\|x\|_n^2 + 2\|y\|_n^2 - \|x+y\|_n^2 = 0, \qquad (\dagger_n)$$

as well as $\|x\|_\delta = \|y\|_\delta$. Dividing every equation $(\dagger_n)$ for $\|x\|_\delta^2$ we obtain

$$2\left\|\frac{x}{\|x\|_\delta}\right\|_n^2 + 2\left\|\frac{y}{\|x\|_\delta}\right\|_n^2 - \left\|\frac{x+y}{\|x\|_\delta}\right\|_n^2 = 0, \qquad (\dagger_n^\delta)$$

By the G_δ-diagonal property we know that exists n_0 such that $y/\|x\|_\delta \notin \mathrm{st}(x/\|x\|_\delta, \mathcal{H}_{n_0})$. Considering the thesis of 1.2 and the condition $(\dagger_{n_0}^\delta)$ it follows that and there exists $H \in \mathcal{H}_{n_0}$ such that $\frac{x}{\|x\|_\delta}, \frac{y}{\|x\|_\delta} \in H \cap S_\delta$, but this is a contradiction, so we have that $x = y$. The converse implication is proposition 3.9. □

The previous result appears as an improvement of [MOTZ07, Theorem 1.2]. In the dual case our eating lemma (lemma 3.4) allows us to replace open slices by open neighbourhoods.

Theorem 3.11 *Let X^* a dual Banach space. X^* admits an equivalent dual rotund norm $\|\cdot\|_R$ if, and only if, X^* admits an equivalent dual norm $\|\cdot\|_\delta$ such that $S_\delta = \{x \in X \mid \|x\|_\delta = 1\}$ has a G_δ-diagonal, with respect to the w^*-topology.*

Proof Let $(\mathcal{U}_n)_{n\in\mathbb{N}}$ a countable collection of families of covers of S_δ, such that it gives the G_δ-diagonal property to S_δ and put $B_\delta = \{x \in X \mid \|x\|_\delta \leq 1\}$. We have that for every $n \in \mathbb{N}$

$$B^{(\omega_0)}(B_\delta,\mathcal{U}_n) \cap S_\delta = \emptyset.$$

Now we may apply theorem 1.2 with $B^{(m)}(B_\delta,\mathcal{U}_n)$ and $SL^*(B^{(m)}(B_\delta,\mathcal{U}_n),\mathcal{U}_n)$ to obtain a countable number of norms $\|\cdot\|_{n,m}$ which satisfy the conclusions of theorem 1.2. Define

$$\|\cdot\|_R^2 = \|\cdot\|_\delta^2 + \sum_{n,m\in\mathbb{N}} c_{n,m}\|\cdot\|_{n,m}^2,$$

where the constants $c_{n,m}$ are chosen in order to guarantee the uniform convergence of the series on bounded sets. We will prove rotundness: fix $x, y \in X^*$ and let us consider the condition $2\|x\|_R^2 + 2\|y\|_R^2 - \|x+y\|_R^2 = 0$; suppose, by contradiction that $x \neq y$. By a standard convexity argument (see [DGZ93, Fact II.2.3]) we have, for every $n, m \in \mathbb{N}$,

$$2\|x\|_{n,m}^2 + 2\|y\|_{n,m}^2 - \|x+y\|_{n,m}^2 = 0, \qquad (\dagger_{n,m})$$

as well as $\|x\|_\delta = \|y\|_\delta$. Dividing every equation $(\dagger_n)$ for $\|x\|_\delta^2$ we obtain

$$2\left\|\frac{x}{\|x\|_\delta}\right\|_n^2 + 2\left\|\frac{y}{\|x\|_\delta}\right\|_n^2 - \left\|\frac{x+y}{\|x\|_\delta}\right\|_n^2 = 0, \qquad (\dagger_{n,m}^\delta)$$

Define for every $z \in S_\delta$ and $n \in \mathbb{N}$

$$m_{z,n} = \min\{m \in \mathbb{N} \mid z \in B^{(m)}(B_\delta,\mathcal{U}_n) \text{ and } z \notin B^{(m+1)}(B_\delta,\mathcal{U}_n)\}$$

and consider $n_0 \in \mathbb{N}$, which is provided by the G_δ-diagonal property, such that $y/\|x\|_\delta \notin \mathrm{st}(x/\|x\|_\delta, \mathcal{H}_{n_0})$. Without loss of generality we can consider $m_{x/\|x\|_\delta, n_0} \geq m_{y/\|x\|_\delta, n_0}$, by theorem 1.2 the condition $(\dagger^{\delta}_{n_0, m_{y/\|x\|_\delta, n_0}})$ implies that

$$\begin{aligned}\frac{x}{\|x\|_\delta} \in& B^{(m_{y/\|x\|_\delta,n_0})}(B_\delta, \mathcal{U}_n) \cap S_\delta;\\ \frac{y}{\|x\|_\delta} \in& B^{(m_{y/\|x\|_\delta,n_0})}(B_\delta, \mathcal{U}_n) \cap S_\delta \cap \bigcup SL^*(B^{(m_{y/\|x\|_\delta,n_0})}(B_\delta, \mathcal{U}_{n_0}), \mathcal{U}_{n_0}).\end{aligned}$$

We obtain that there exists $H \cap B^{(m_{y/\|x\|_\delta,n_0})}(B_\delta, \mathcal{U}_n) \in SL^*(B^{(m_{y/\|x\|_\delta,n_0})}(B_\delta, \mathcal{U}_{n_0}), \mathcal{U}_{n_0})$ such that $x/\|x\|_\delta, y/\|x\|_\delta \in H \cap B^{(m_{y/\|x\|_\delta,n_0})}(B_\delta, \mathcal{U}_{n_0})$. By the definition there exists $W \in \mathcal{U}_{n_0}$ such that

$$\frac{x}{\|x\|_\delta}, \frac{y}{\|x\|_\delta} \in H \cap B^{(m_{y/\|x\|_\delta,n_0})}(B_\delta, \mathcal{U}_{n_0}) \subseteq W,$$

but this is a contradiction, then $x = y$. The converse implication is proposition 3.9. □

It is possible to get rid of the convexity assumption with some extra work. Remember that a set A in a vector space is said to be *circled* if αx belongs to A whenever $|\alpha| \leq 1$ and $x \in A$, see [Köt69, Pag. 146].

Theorem 3.12 *Let A a w^*-compact, circled and absorbing subset of a dual space X^*. Let μ_1 and μ_2 the Minkowski functionals of A and $\overline{\mathrm{conv}}^{w^*}(A)$ respectively and consider for $i = 1, 2$*

$$S_i = \{x^* \in X^* \,|\, \mu_i(x^*) = 1\}.$$

If S_1 has a G_δ-diagonal with respect to the w^*-topology, then S_2 has (*) with w^*-slices. In particular X^* admits an equivalent dual rotund norm by [OST12, Theorem 2.7].

Proof Let $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ a G_δ-diagonal sequence for S_1 and $B = \{x^* \in X^* \,|\, \mu_2(x^*) \leq 1\}$. First of all we change our family in order to have a cover of S_2. For every $\alpha \in \mathbb{R}$ and $x \in X$ consider the w^*-open cones

$$C(x, \alpha) = \{x^* \in X^* \,|\, x^*(x) < \alpha \mu_1(x^*)\}.$$

If U is a w^*-open set, we can assume that $U = \bigcup_{i\in I} \bigcap_{j=1}^{n_i} x_j^{-1}(-\infty, \alpha_j)$. Now consider the w^*-open cone

$$U^{\#} = \bigcup_{i\in I} \bigcap_{j=1}^{n_i} C(x_j, \alpha_j).$$

We claim that if $\mathcal{U}$ is an w^*-open cover of S_1, then the family $\mathcal{U}^\# = \{U^\# \,|\, U \in \mathcal{U}\}$ is a w^*-open cover of $X^* \setminus \{0\}$. Indeed let $x^* \in X^* \setminus \{0\}$. We know that $r_{x^*} > 0$ exists such that $r_{x^*}x^* \in S_1$, so there exists $U \in \mathcal{U}$ such that $r_{x^*}x^* \in U$. As before, we can assume that $U = \bigcup_{i \in I} \bigcap_{j=1}^{n_i} x_j^{-1}(-\infty, \alpha_j)$. This means that there exists $i \in I$ such that

$$r_{x^*}x^*(x_j) < \alpha_j = \alpha_j \mu_1(r_{x^*}x^*)$$

for every $j = 1, \dots, n_i$, hence $x^* \in U^\#$. Consider the collection of covers $\{U_n^\#\}_{n \in \mathbb{N}}$ of S_2 and apply lemma 3.6. Let $\mathcal{H}$ the family of all the w^*-open halspace of X^*. We show that the families

$$\mathcal{H}_{n,m} = \{H \cap S_2 | H \in \mathcal{H} \text{ and there exists } W \in \mathcal{U}^\# \text{ such that}$$
$$\emptyset \neq H \cap B^{(m)}(B, \mathcal{U}_n^\#) \subseteq W^\#\}.$$

provide a (*)-sequence for S_2. For every $x^* \in S_2$ consider the natural number

$$m_{x^*,n} = \min\left\{m \in \mathbb{N} \,\middle|\, x \in B^{(m)}(B, \mathcal{U}_n^\#) \text{ and } x \notin B^{(m+1)}(B, \mathcal{U}_n^\#)\right\},$$

well defined by lemma 3.6. For fixed $x^*, y^* \in S_2$, let $r_{x^*}, r_{y^*} \in (0, +\infty)$ such that $r_{x^*}x^*, r_{y^*}y^* \in S_1$ and consider $n_0 \in \mathbb{N}$ such that $\mathcal{U}_{n_0}$ (*)-separates $r_{x^*}x^*$ and $r_{y^*}y^*$. Without loss of generality we can, and do, assume that $m_{x^*,n_0} \geq m_{y^*,n_0}$. We claim that $\mathcal{H}_{n_0, m_{y^*,n_0}}$ (*)-separates x^* and y^*. In fact assume $m_{x^*,n_0} > m_{y^*,n_0}$. Then we know that

$$y^* \in \bigcup \mathcal{H}_{n_0, m_{y^*,n_0}} \cap B^{(m_{y^*,n_0})}(B, \mathcal{U}_{n_0}^\#) \text{ and } x^* \in B^{(m_{y^*,n_0})}(B, \mathcal{U}_{n_0}^\#).$$

Suppose by contradiction that there exists $H \cap S_2 \in \mathcal{H}_{n_0, m_{y^*,n_0}}$ such that $x, y \in H \cap S_2$. This means that there exists $W^\# \in \mathcal{U}_{n_0}^\#$ such that

$$x^*, y^* \in H \cap B^{(m_{y^*,n_0})}(B, \mathcal{U}_{n_0}^\#) \subseteq W^\#$$

which implies $m_{x^*,n_0} = m_{y^*,n_0}$, a contradiction. Suppose now $m_{x^*,n_0} = m_{y^*,n_0}$. In this case

$$x^*, y^* \in \bigcup \mathcal{H}_{n_0, m_{y^*,n_0}} \cap B^{(m_{y^*,n_0})}(B, \mathcal{U}_{n_0}^\#).$$

Suppose by contradiction that there exists $H \cap S_2 \in \mathcal{H}_{n_0, m_{y^*,n_0}}$ such that $x^*, y^* \in H \cap S_2$. This means that there exists $W^\# \in \mathcal{U}_{n_0}^\#$ such that

$$x^*, y^* \in H \cap B^{(m_{y^*,n_0})}(B, \mathcal{U}_{n_0}^\#) \subseteq W^\#$$

which implies that $r_{x^*}x^*, r_{y^*}y^* \in W$, a contradiction. □

Observe that theorems 3.11 and 3.12 give us the following result.

Corollary 3.13 *Let X^* a dual Banach space. X^* admits an equivalent dual rotund norm if, and only if, there exists a w^*-compact, cirlced and absorbent set $A \subseteq X^*$ such that ∂A has a G_δ-diagonal with respect to the w^*-topology, where ∂A is the w^*-boundary of A.*

The theorems of this section and the results of chapter 2 allow us to prove the following fact, which was already proved in [Raj02, Theorem 1.3].

Corollary 3.14 *Let X^* a dual Banach space. X^* admits an equivalent w^*-Kadec norm if, and only if, X^* admits an equivalent dual LUR norm.*

Proof It is a well known fact that a dual LUR norm is w^*-Kadec (see [DGZ93, Proposition 1.4], which can be easily adapted in the w^* case). If X^* admits an equivalent w^*-Kadec norm $\|\cdot\|_*$, then by definition the unit sphere

$$S_* = \{x^* \in X^* \mid \|x^*\|_* = 1\}$$

admits a G_δ-diagonal with respect to the w^*-topology. So by theorem 3.11 we have the thesis. □

Raja had proved in [Raj03b, Theorem 1.3(a)] that X^* admits an equivalent w^*-LUR norm if, and only if, X^* is w^*-descriptive. Using our techniques we can only prove the following partial result.

Corollary 3.15 *Let X^* a dual Banach space. If X^* is w^*-descriptive, then X^* admits an equivalent dual rotund norm.*

Proof By theorem 2.12 there exists an equivalent w^*-lower semicontinuous and w^*-Kadec quasinorm $q(\cdot)$, i.e. a quasinorm such that the w^* and norm topologies agree on the unit "sphere" $\{x \in X \mid q(x) = 1\}$. Observe that the set

$$A = \{x \in X \mid q(x) = 1\}$$

satisfies the condition of theorem 3.12, so we have that X^* admits an equivalent dual rotund norm. □

3.1.2 Characterizations through symmetrics

One of the most well known result in LUR renorming theory is the following Troyanski's characterization (see [Tro79]).

Theorem 3.16 *A Banach space X admits an equivalent LUR norm if, and only if, there exists an equivalent norm $\|\cdot\|_D$ such that every point of $\mathcal{S}_D$ is denting; i.e. for every $\varepsilon > 0$ and $x \in \mathcal{S}_D$ there exists a w-open half-space H such that $x \in H$ and*

$$\|\cdot\|\text{-}\operatorname{diam}(H \cap \mathscr{B}_D) < \varepsilon.$$

In what follows we will generalize the concept of denting point, but in order to do so we need a classical topological concept (see [Gru84]):

Definition 3.17 *Let S a nonempty set. A function $\rho : S \times S \to [0, +\infty)$ is called* symmetric *if for every $x, y \in S$*

- $\rho(x, y) = \rho(y, x)$*;*
- $\rho(x, x) = 0$ *if, and only if,* $x = y$*.*

If a set S has a symmetric ρ, then we can define a topology τ_ρ in the following way: $U \in \tau_\rho$ if, and only if, for every $x \in U$ there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$, where

$$B_\varepsilon(x) = \{y \in S \mid \rho(x, y) < \varepsilon\}.$$

Observe that without additional conditions (such that the triangular inequality of ρ), we cannot assume that $B_\varepsilon(x)$ are neighbourhood of x. Now we make precise what we mean by denting point with respect to ρ.

Definition 3.18 *Let X a normed space, $F \subseteq X^*$ a norming subspace and ρ a symmetric on X. We say that $x \in \mathcal{S}_X$ is a $\sigma(X, F)$*-denting point with respect to ρ, *if for every $\varepsilon > 0$ there exists a $\sigma(X, F)$-open half-space H such that $x \in H$ and*

$$\rho\text{-}\operatorname{diam}(H \cap \mathscr{B}_X) < \varepsilon.$$

In the following lemma we define a natural symmetric on a rotund normed space.

Lemma 3.19 *Let X a normed space and $F \subseteq X^*$ a norming subspace. Consider the function $\rho(x, y) = 2\|x\|^2 + 2\|y\|^2 - \|x + y\|^2$, defined for every $x, y \in X$. The following holds:*

1. *ρ is a non-negative function;*
2. *for every $x \in \mathcal{S}_X$ and $\varepsilon > 0$ there exists a $\sigma(X, F)$-open half-space H such that $x \in H$ and $\rho\text{-}\operatorname{diam}(H \cap \mathscr{B}_X) < \varepsilon$;*
3. *if $\|\cdot\|$ is rotund, then ρ is a symmetric;*
4. *if $\|\cdot\|$ is asymptotically rotund, then ρ is a symmetric such that every sequence has at most an unique limit in the τ_ρ-topology;*
5. *if $\|\cdot\|$ is τ-LUR, where τ is a T_0 topology, then ρ is a symmetric such that τ_ρ is finer than τ;*
6. *if $\|\cdot\|$ is $\sigma(X, F)$-lower semicontinuous and $\sigma(X, F)$-LUR, then ρ is a symmetric such that τ_ρ is finer than $\sigma(X, F)$ and $\tau_\rho = \sigma(X, F)$ when restricted to the sphere.*

Proof 1 and 3 follows from [DGZ93, Fact II.2.3], from this consideration 4 is actually trivial.

2. Let $\mu \in (0,1)$, $x \in \mathcal{S}_X$ and $f \in \mathscr{B}_{X^*} \cap F$ such that $f(x) > 1-\mu$ and define
$$H_\mu = \{y \in X \mid f(y) > 1-\mu\}.$$
Fix $\varepsilon > 0$ and consider μ small enough such that $4(2\mu - \mu^2) < \varepsilon$ and observe that for every $y, z \in H_\mu \cap \mathscr{B}_X$ we have
$$\rho(y,z) = 2\|y\|^2 + 2\|z\|^2 - \|y+z\|^2 \leq 4 - (f(y+z))^2 \leq 4(2\mu - \mu^2) < \varepsilon.$$
So $\rho\text{-}\operatorname{diam}(H_\mu \cap \mathscr{B}_X) < \varepsilon$.

5. Since τ is a T_0 topology, $\|\cdot\|$ is rotund, so ρ is a symmetric. Indeed let $x, y \in X$ satisfy
$$2\|x\|^2 + 2\|y\|^2 - \|x+y\|^2 = 0.$$
For any pair U_x, U_y of τ-neighbourhoods of x and y respectively, we do have $x \in U_y$ and $y \in U_x$, so $x = y$. Let U a τ-open set, $C = X \setminus U$ and assume that for every $\varepsilon > 0$ there exists $y_\varepsilon \in B_\varepsilon(x) \cap C$.For every $n \in \mathbb{N}$ we have
$$2\|x\|^2 + 2\|y_{1/n}\|^2 - \|x + y_{1/n}\| < \frac{1}{n},$$
so $\tau\text{-}\lim y_{1/n} = x$. Since C is τ-closed, we have $x \in C$, a contradiction.

6. The remaining part to be proved follows easily from the equality
$$\{y \in \mathcal{S}_X \mid \rho(x,y) < \varepsilon\} = \left\{y \in \mathcal{S}_X \;\middle|\; \left\|\frac{x+y}{2}\right\| > 1-\varepsilon\right\},$$
that holds for every $x \in \mathcal{S}_X$. □

Before stating our next results, we state a transfer result that will help us later.

Theorem 3.20 *Let X a normed space, $F \subseteq X^*$ a norming subspace and S a nonempty set with a symmetric ρ. Let $\Phi : X \to S$ a map such that for every $x \in \mathcal{S}_X$ and every $\varepsilon > 0$ there exists a $\sigma(X,F)$-open half-space H with $x \in H$ and*
$$\rho\text{-}\operatorname{diam}(\Phi(H \cap \mathscr{B}_X)) < \varepsilon.$$
Then there exists a $\sigma(X,F)$-lower semicontinuous norm $\|\cdot\|_\Phi$ such that
$$\tau_\rho\text{-}\lim \Phi(x_n) = \Phi(x),$$
whenever $x \in \mathcal{S}_X$, $(x_n)_{n\in\mathbb{N}} \subseteq \mathscr{B}_X$ and $2\|x\|_\Phi^2 + 2\|x_n\|_\Phi^2 - \|x + x_n\|_\Phi^2 \to 0$. Furthermore if $2\|x\|_\Phi^2 + 2\|y\|_\Phi^2 - \|x+y\|_\Phi^2 = 0$, for some point $x \in \mathcal{S}_X$ and $y \in \mathscr{B}_X$, then $\Phi(x) = \Phi(y)$.

Proof Consider the families of slices of $\mathscr{B}_X$

$$\mathcal{U}_n = \left\{ H \cap \mathscr{B}_X \,\middle|\, \rho\text{- diam}\,(\Phi(H \cap \mathscr{B}_X)) < \frac{1}{n} \right\}$$

and let $\|\cdot\|_n$ the norms constructed using theorem 1.2; now define

$$\|\cdot\|_\Phi^2 = \sum_{n\in\mathbb{N}} c_n \|\cdot\|_n^2,$$

where the constants c_n are chosen in order to guarantee the uniform convergence of the series on bounded set. Suppose that x and $(x_n)_{n\in\mathbb{N}}$ satisfy the hypothesis of our theorem, by a standard convexity argument (see [DGZ93, Fact II.2.3]) we have that from

$$\lim_{n\in\mathbb{N}} (2\|x\|_\Phi^2 + 2\|x_n\|_\Phi^2 - \|x + x_n\|_\Phi^2) = 0$$

it follows

$$\lim_{n\in\mathbb{N}} (2\|x\|_k^2 + 2\|x_n\|_k^2 - \|x + x_n\|_k^2) = 0,$$

for every $k \in \mathbb{N}$. By theorem 1.2 for every $k \in \mathbb{N}$ we obtain that there exist $n_0 \in \mathbb{N}$ and $(H_n \cap \mathscr{B}_X)_{n\in\mathbb{N}} \subseteq \mathcal{U}_k$ such that $x, x_n \in H_n \cap \mathscr{B}_X$ for every $n \geq n_0$. Then for every $k \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ such that

$$\rho(\Phi(x), \Phi(x_n)) < \frac{1}{k},$$

for every $n \geq n_0$. So by [Gru84, Lemma 9.3] we have that $\tau_\rho\text{-}\lim \Phi(x_n) = \Phi(x)$. The furthermore part follows easily. □

The previous result make it possible to construct equivalent norms that "behave well".

Theorem 3.21 *Let $(X, \|\cdot\|)$ a normed space, $F \subseteq X^*$ a norming subspace and τ a vector topology on X. X admits an equivalent, $\sigma(X, F)$-lower semicontinuous and rotund norm if, and only if, there exists a symmetric ρ on X and an equivalent norm $\|\cdot\|_\rho$ such that every point of the unit sphere of $\|\cdot\|_\rho$ is a $\sigma(X, F)$-denting point with respect to ρ;*

Proof We apply theorem 3.20 and obtain an equivalent and $\sigma(X, F)$-lower semicontinuous norm $\|\cdot\|_{id}$ such that

$$\tau_\rho\text{- }\lim x_n = x$$

whenever $x \in \mathcal{S}_X$, $(x_n) \subseteq \mathscr{B}_X$ and $2\|x\|_{id}^2 + 2\|x_n\|_{id}^2 - \|x + x_n\|_{id}^2 \to 0$. Furthermore if $2\|x\|_{id}^2 + 2\|y\|_{id}^2 - \|x + y\|_{id}^2 = 0$ for some $x \in \mathcal{S}_X$ and $y \in \mathscr{B}_X$, then $x = y$. Let

$$\|x\|_\rho^2 = \|x\|^2 + \|x\|_{id}^2$$

and take $x, y \in X$ such that

$$2\|x\|_\rho^2 + 2\|y\|_\rho^2 - \|x+y\|_\rho^2 = 0.$$

By a standard convexity argument (see [DGZ93, Fact II.2.3]), we have that $\|x\| = \|y\|$ and $2\|x\|_{id}^2 + 2\|y\|_{id}^2 - \|x+y\|_{id}^2 = 0$. So considering the equation

$$2\left\|\frac{x}{\|x\|}\right\|_{id}^2 + 2\left\|\frac{y}{\|x\|}\right\|_{id}^2 - \left\|\frac{x+y}{\|x\|}\right\|_{id}^2 = 0,$$

by theorem 3.20 we obtain $x = y$. □

Via a suitable use of all the theorems of the previuos section we can prove our result in the dual case.

Theorem 3.22 *Let $(X^*, \|\cdot\|)$ a dual Banach space. X^* admits an equivalent, dual rotund norm if, and only if, there exists a symmetric ρ on X^* and an equivalent dual norm $\|\cdot\|_\rho$ such that every point of the unit sphere of $\|\cdot\|_\rho$ and $\varepsilon > 0$ there exists a w^*-neighbourhood U of x^* such that*

$$\rho\text{-}\operatorname{diam}(U \cap \{x^* \in X^* | \|x^*\|_\rho \le 1\}) < \varepsilon.$$

Proof Define $B = \{x^* \in X^* | \|x^*\|_\rho \le 1\}$ and $S = \{x^* \in X^* | \|x^*\|_\rho = 1\}$. Consider the countable collection of covers of S

$$\mathcal{U}_n = \left\{ U \cap B \,\middle|\, \rho\text{-}\operatorname{diam}(U \cap B) < \frac{1}{n} \right\}.$$

It is easy to see that this family gives us a G_δ-diagonal of the unit sphere of $\|\cdot\|_\rho$, and by theorem 3.11 the thesis follows. □

3.1.3 Characterizations through quasiconvex functions

We state a characterization theorem for rotund renorming using quasiconvex functions in place of half-spaces. Let us remember that a function $\varphi : X \to \mathbb{R}$ is said to be *quasiconvex* (see, for example, [Sio58]) if

$$\varphi((1-\sigma)x + \sigma y) \le \max\{\varphi(x), \varphi(y)\}$$

for $x, y \in X$ and $\sigma \in [0, 1]$. We need the following definition.

Definition 3.23 *Let X a normed space and $F \subseteq X^*$ a norming subspace. We say that a countable collection of families $\mathcal{L}_n = \{\varphi_i^n : X \to [0, +\infty)\}_{i \in I_n}$ of quasiconvex and $\sigma(X, F)$-lower semicontinuous functions is a (*)-sequence for X if the countable collection of families of open sets*

$$\mathcal{V}_{\mathcal{L}_n} = \left\{ (\varphi_i^n)^{-1}(0, +\infty) \,\middle|\, i \in I_n \right\}$$

is a (*)-sequence for X.

We do not loose generality when using non negative functions to separate points. Indeed, assume that a family $\mathcal{I} = \{\varphi_i : X \to \mathbb{R}\}_{i\in I}$ of quasiconvex $\sigma(X,F)$-lower semicontinuous functions makes the family of sets

$$\mathcal{W}_{\mathcal{I}} = \left\{(\varphi_i)^{-1}(\mu_i, +\infty) \,\middle|\, i \in I\right\}$$

(*)-separating points $x, y \in X$. For $\mu_i \in \mathbb{R}$ fixed, we can consider the new family $\mathcal{L} = \{(\varphi_i - \mu_i)^+ : X \to [0, +\infty)\}_{i\in I}$ of positive quasiconvex $\sigma(X,F)$-lower semicontinuous functions which (*)-separates x and y (according to the former definition).

The following theorem is a version for rotund renorming of theorem 1.5 of [OT09a].

Theorem 3.24 *Let X a normed space with a norming subspace $F \subseteq X^*$. They are equivalent:*

1. *There is a countable collection of families $\mathcal{L}_n = \{\varphi_i^n : X \to [0, +\infty)\}_{i\in I_n}$ of $\sigma(X,F)$-lower semicontinuous and quasiconvex functions, which is a (*)-sequence for X.*

2. *X admits an equivalent, $\sigma(X,F)$-lower semicontinuous and rotund norm.*

Proof

(1.⇒ 2.) Since the open sets $(\varphi_i^n)^{-1}(0, +\infty)$ have a convex and $\sigma(X,F)$-closed complementary we can consider the families $\mathcal{H}_n$ of all $\sigma(X,F)$-open half-space H such that

$$H \cap (\varphi_i^n)^{-1}(0) = \emptyset,$$

for some $i \in I_n$ and every $n \in \mathbb{N}$. It follows by the Hahn-Banach theorem that these families of open half-spaces (*)-separates the points of X.

(2.⇒ 1.) For a given family $\mathcal{H}$ of $\sigma(X,F)$-open half-spaces define the F-distance function

$$\varphi_H(x) = \inf\{\|x - c\|_F \,|\, c \in X \setminus H\},$$

for $H \in \mathcal{H}$. It follows that two points $x, y \in X$ are (*)-separated by $\mathcal{H}$ if, and only if, they are (*)-separated by the family of convex functions given by $\{\varphi_H\}_{H\in\mathcal{H}}$. □

It is an easy observation that when we have two normed spaces X and Y together with a map $\Phi : X \to Y$ and

$$\mathcal{L}_n = \{\varphi_i^n : Y \to [0, +\infty) \,|\, i \in I_n\}$$

families of quasiconvex functions such that $\varphi_i^n \circ \Phi$ is quasiconvex for every $i \in I_n$ too, it follow that $x \in \Phi^{-1}(y)$ and $x' \in \Phi^{-1}(y')$ are (*)-separated by the family of quasiconvex functions

$$\mathcal{L}_n \circ \Phi = \{\varphi_i^n \circ \Phi \,|\, i \in I_n\},$$

whenever $y \neq y' \in Y$ are (*)-separates by the family $\mathcal{L}_n$. It follows the following transfer result:

Corollary 3.25 *Let X and Y normed spaces with the norming subspaces $F \subseteq X^*$ and $G \subseteq Y^*$. Let $\Phi : X \to Y$ a one to one map. Assume there exist families*

$$\mathcal{L}_n = \{\varphi_i^n : Y \to [0, +\infty) \,|\, i \in I_n\}$$

of $\sigma(Y, G)$-lower semicontinuous and quasiconvex functions such that for every $y \neq y'$ there exists $p \in \mathbb{N}$ such that $\mathcal{L}_p$ (*)-separates y and y'. If $\varphi_i^n \circ \Phi$ is quasiconvex and $\sigma(X, F)$-lower semicontinuous for every $i \in I_n$ and $n \in \mathbb{N}$, then there exists a $\sigma(X, F)$-lower semicontinuous and equivalent rotund norm on X.

If we involve linear functionals for the range space we obtain:

Theorem 3.26 *Let X and Y normed spaces with norming subspaces $F \subseteq X^*$ and $G \subseteq Y^*$. Let $\Phi : X \to Y$ a one to one map such that for every $g \in G$ we have that $(g \circ \Phi)^+$ is quasiconvex and $\sigma(X, F)$-lower semicontinuous. Assume that Y has a $\sigma(Y, G)$-lower semicontinuous and rotund norm. Then X admits an equivalent rotund and $\sigma(X, F)$-lower semicontinuous norm.*

Proof Consider the families $\mathcal{H}_n$ of $\sigma(Y, G)$-open half-space such that Y has the (*)-separation property with them. If we remind the proof of [OST12, Theorem 2.7], it is not restriction to assume that the $\sigma(Y, G)$-open half-spaces are always of the form

$$H = \{z \in Y \,|\, g(z) > \lambda\}$$

for $g \in \mathscr{B}_{Y^*} \cap G$ and $\lambda > 0$. Take two different points $x, x' \in X$ and choose $p \in \mathbb{N}$ such that $\Phi(x)$ and $\Phi(x')$ are (*)-separated by $\mathcal{H}_p$. Let us write every $H \in \mathcal{H}_n$ as

$$H = \{z \in Y \,|\, g_H^n(z) > \lambda_H\}$$

where $g_H^n \in \mathscr{B}_{Y^*} \cap G$ and $\lambda_n^H > 0$. Denote

$$\varphi_H^n = \max\left\{(g_H^n \circ \Phi)^+, \lambda_H^n\right\} - \lambda_H^n.$$

Our hypothesis imply that φ_H^n is a quasiconvex and $\sigma(X, F)$-lower semicontinuous function. It now follows that the families $\mathcal{L}_p = \left\{\varphi_H^p\right\}_{H \in \mathcal{H}_p}$ (*)-separates the points x and x'. □

3.2 Transfer results

In this section we study some nonlinear transfer results in the spirit of the theorems in [MOTV09]. In particular we study conditions to be put on a function

$$\Phi : X \to (Y, d),$$

where d is a pseudometric, so that we can obtain a rotund norm on X. Later we will generalize [MOTV09, Corollary 4.32] and [OT09a, Theorem 1.5] in the case of rotund renorming.

3.2.1 Pseudometric transfer results

Let us recall that a pseudometric d on a set Y is a function that verifies all the properties of a metric except the condiction $d(x, y) = 0 \Rightarrow x = y$ (for more information see [Eng89]). Let us remark that theorem 1.2 also holds when the subspace $F \subseteq X^*$ is not a norming subspace. Of course, in that case we will obtain a $\sigma(X, F)$-lower semicontinuous and norm-continuous seminorm instead of an equivalent norm.

Theorem 3.27 *Let X a normed space, F a subspace in X^* and (Y, d) a pseudometric space and $\Phi : X \to Y$ a map. Suppose that there exists a sequence (A_n) of subsets of X such that for every $x \in X$ and every $\varepsilon > 0$ we can find $n \in \mathbb{N}$ together with a $\sigma(X, F)$-open half-space H so that $x \in H \cap A_n$ and $d\text{-}\operatorname{diam}(\Phi(\operatorname{conv}(H \cap A_n))) < \varepsilon$, then X admits a $\sigma(X, F)$-lower semicontinuous and norm-continuous seminorm $\|\cdot\|_\Phi$ such that*

$$\lim_{n\in\mathbb{N}}(2\|x\|_\Phi^2 + 2\|y_n\|_\Phi^2 - \|x + y_n\|_\Phi^2) = 0$$

implies that there exists a sequence $\{y_n^{**} \in \overline{D_n}^{w^*} | n \in \mathbb{N}\}$ for a sequence of bounded subsets $D_n \subseteq X$ with $x \in \bigcap_{n\in\mathbb{N}} D_n$, $\lim_{n\in\mathbb{N}} \|y_n^{**} - y_n\|_F = 0$ and

$$\lim_{n\in\mathbb{N}} d\text{-}\operatorname{diam}\Phi(D_n) = 0.$$

Furthermore if F is norming, then $\|\cdot\|_\Phi$ is an equivalent norm.

Proof Without loss of generality we can assume that the terms of the sequence (A_n) are bounded and convex sets. Let $\mathcal{H}$ the family of $\sigma(X, F)$-open half-spaces of X and consider the following families of slices

$$\mathcal{H}_n^\varepsilon = \{H \cap A_n \,|\, H \in \mathcal{H},\ H \cap A_n \neq \emptyset \text{ and } d\text{-}\operatorname{diam}(\Phi(H \cap A_n)) < \varepsilon\}$$

for every $n \in \mathbb{N}$ and $\varepsilon > 0$. Apply theorem 1.2 to every one of the families $\mathcal{H}_n^{1/p}$ and bounded set A_n to obtain a countable number of $\sigma(X,F)$-lower semicontinuous and norm-continuous seminorm $\|\cdot\|_{n,p}$. Consider the series

$$\|\cdot\|_\Phi^2 := \sum_{n,p\in\mathbb{N}} c_{n,p}\|\cdot\|_{n,p}^2$$

where the constants $c_{n,p}$ are chosen for the uniform convergence of the series on bounded sets. If we assume

$$\lim_{n\in\mathbb{N}}(2\|x\|_\Phi^2 + 2\|y_n\|_\Phi^2 - \|x+y_n\|_\Phi^2) = 0,$$

for some sequence $(y_n)_{n\in\mathbb{N}} \subseteq X$ and $x \in X$, by a standard convexity argument (see [DGZ93, Fact II.2.3]), we will have

$$\lim_{n\in\mathbb{N}}(2\|x\|_{m,p}^2 + 2\|y_n\|_{m,p}^2 - \|x+y_n\|_{m,p}^2) = 0,$$

for every $p, m \in \mathbb{N}$. We are going to construct the sequence (y_n^{**}) by induction:

($p = 1$) Take an integer $m_1 \in \mathbb{N}$ such that for some $H \cap A_{m_1} \in \mathcal{H}_{m_1}^1$ we have $x \in H \cap A_{m_1}$ and d-$\operatorname{diam}(\Phi(H \cap A_{m_1})) < 1$. Since

$$\lim_{n\in\mathbb{N}}(2\|x\|_{m_1,1}^2 + 2\|y_n\|_{m_1,1}^2 - \|x+y_n\|_{m_1,1}^2) = 0,$$

theorem 1.2 tells us that there exists a sequence $H_n^1 \cap A_{m_1} \in \mathcal{H}_{m_1}^1$ and $n_1 \in \mathbb{N}$ such that for $n \geq n_1$ we have $x \in H_n^1 \cap A_{m_1}$ and

$$x, y_n \in \overline{(H_n^1 \cap A_{m_1}) + \delta\,\mathscr{B}_X}^{\sigma(X,F)}$$

for some $\delta \in (0,1)$ we fix. Let us observe that for

$$z \in \overline{(H_n^1 \cap A_{m_1}) + \delta\,\mathscr{B}_X}^{\sigma(X,F)}$$

there exists a net $\{z_\alpha = v_\alpha + w_\alpha \mid \alpha \in (D, \preceq)\}$, where $v_\alpha \in H_n^1 \cap A_{m_1}$ and $w_\alpha \in \delta\,\mathscr{B}_X$ for every α in the directed set $(D, \preceq)$ with

$$\lim_{\alpha\in D} |f(z - (v_\alpha + w_\alpha))| = 0$$

for every $f \in F$. By w^*-compactness we can find a w^*-converging subnet to some point in X^{**} that can be written as $v^{**} + w^{**}$ with $v^{**} \in \overline{H_n^1 \cap A_{m_1}}^{w^*}$ and $w^{**} \in \delta\,\mathscr{B}_{X^{**}}$. Then we obtain that z coincides with $v^{**} + w^{**}$ on every element $f \in F$, therefore

$$\|z - v^{**}\|_F = \|w^{**}\|_F \leq \|w^{**}\|_{X^{**}} \leq \delta.$$

In particular we have for some $y_n^{**} \in \overline{H_n^1 \cap A_{m_1}}^{w^*}$ that $\|y_n - y_n^{**}\|_F \leq \delta$, for every $n \geq n_1$.

$(p \rightsquigarrow p+1)$ Take an integer $m_{p+1} \in \mathbb{N}$ such that for some $H \cap A_{m_{p+1}} \in \mathcal{H}^{p+1}_{m_{p+1}}$ we have $x \in H \cap A_{m_{p+1}}$ and $d\text{-}\operatorname{diam}(\Phi(H \cap A_{m_{p+1}})) < \frac{1}{p+1}$. Since

$$\lim_{n\in\mathbb{N}}(2\|x\|^2_{m_{p+1},p+1} + 2\|y_n\|^2_{m_{p+1},p+1} - \|x+y_n\|^2_{m_{p+1},p+1}) = 0,$$

theorem 1.2 tells us that there exists a sequence $H_n^{p+1} \cap A_{m_{p+1}} \in \mathcal{H}^{p+1}_{m_{p+1}}$ and an integer $n_{p+1} > n_p$ such that for every $n \geq n_{p+1}$ we have $x \in H_n^{p+1} \cap A_{m_{p+1}}$ and

$$x, y_n \in \overline{(H_n^{p+1} \cap A_{m_{p+1}}) + \delta^{p+1}\mathscr{B}_X}^{\sigma(X,F)}.$$

Then for every $n > n_{p+1}$ we can take a $y_n^{**} \in \overline{H_n^{p+1} \cap A_{m_p}}^{w^*}$ with $\|y_n - y_n^{**}\|_F \leq \delta^p$.

Now define $D_n = \mathscr{B}(0, 2\|x\|)$ and $y_n^{**} = 0$ for $1 \leq n \leq n_1$, and $D_n = H_n^p \cap A_{m_p}$ for $n_p < n \leq n_{p+1}$. With this definition we have $x \in \bigcap_{n\in\mathbb{N}} D_n$, $y_n^{**} \in \overline{D_n}^{w^*}$ and

$$\lim_{n\in\mathbb{N}} d\text{-}\operatorname{diam}\Phi(D_n) = 0.$$

□

The following corollary is a first step on the way to rotund renorming.

Corollary 3.28 *Let X a normed space, F a subspace in X^*, (Y,d) a pseudometric space and $\Phi : X \to Y$ a map such that the function $d(\Phi(x), \Phi(\cdot))$ is $\sigma(X,F)$-lower semicontinuous on X for every $x \in X$. If there exists a sequence (A_n) of subsets of X such that for every $x \in X$ and every $\varepsilon > 0$ we can find $n \in \mathbb{N}$ and a $\sigma(X,F)$-open half-space H such that $x \in H \cap A_n$ and*

$$d\text{-}\operatorname{diam}(\Phi(\operatorname{conv}(H \cap A_n))) < \varepsilon,$$

then X admits a $\sigma(X,F)$-lower semicontinuous and norm-continuous seminorm $\|\cdot\|_\Phi$ such that the condition $2\|x\|^2_\Phi + 2\|y\|^2_\Phi - \|x+y\|^2_\Phi = 0$, implies that $d(\Phi(x), \Phi(y)) = 0$. Furthermore if F is norming, then $\|\cdot\|_\Phi$ is an equivalent norm.

Proof Theorem 3.27, applied to x and the constant sequence $y_n = y$ for every $n \in \mathbb{N}$, gives us $y_n^{**} \in \overline{D_n}^{w^*}$ such that $\lim_{n\in\mathbb{N}} \|y - y_n^{**}\|_F = 0$. Then we have that for every $n \in \mathbb{N}$

$$y \in \overline{\bigcup_{m\geq n} D_m}^{\sigma(X,F)}.$$

Indeed, take any $f \in F \cap \mathscr{B}_{X^*}$ and $\mu > 0$, and choose $m \in \mathbb{N}$ big enough such that $m \geq n$ and $\|y - y_n^{**}\|_F \leq \frac{\mu}{2}$, then we have $|f(y - y_n^{**})| \leq \frac{\mu}{2}$. If we select now $z_m \in D_m$ so that $|f(y_m^{**} - z_m)| \leq \frac{\mu}{2}$, we finally have

$$|f(y - z_m)| \leq |f(y - y_m^{**})| + |f(y_m^{**} - z_m)| \leq \mu.$$

Since $x \in \bigcap_{n\in\mathbb{N}} D_n$ and $\lim_{n\in\mathbb{N}} d\text{-diam}\,\Phi(D_n) = 0$ we have, for every $\varepsilon > 0$

$$d(\Phi(x), \Phi(z_n)) \leq \varepsilon,$$

for $n \geq n_\varepsilon$ and $z_n \in D_n$. If we take a net $\{z_\alpha \mid \alpha \in (D, \preceq)\} \subseteq \bigcup_{n\geq n_\varepsilon} D_n$ with

$$y = \sigma(X, F)\text{-}\lim_{\alpha\in D} z_\alpha$$

we will have, by $\sigma(X, F)$-lower semicontinuity of the function $d(\Phi(x), \Phi(\cdot))$,

$$d(\Phi(x), \Phi(y)) \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary we finally have $d(\Phi(x), \Phi(y)) = 0$, as we wanted. □

As a consequence we have the following nonlinear version of [DGZ93, Theorem 2.4].

Corollary 3.29 *Let X, Y two normed spaces, F a subspace in X^* and G a norming subspace in Y^*. Let $\Phi : X \to Y$ a one to one, $\sigma(X, F)$ to $\sigma(Y, G)$ continuous map. If there exists a sequence of subsets (A_n) of X such that for every $x \in X$ and $\varepsilon > 0$ there exists $p \in \mathbb{N}$ and a $\sigma(X, F)$-open half-space H with $x \in H \cap A_p$ and*

$$\|\cdot\|_G\text{-}\operatorname{diam}(\Phi(\operatorname{conv}(A_p \cap H))) \leq \varepsilon,$$

then X admits an equivalent $\sigma(X, F)$-lower semicontinuous, norm-continuous and rotund seminorm $\|\cdot\|_\Phi$. Furthermore if F is norming, then $\|\cdot\|_\Phi$ is an equivalent norm.

Proof Apply the former corollary with the pseudometric $d(x, y) = \|x - y\|_G$. □

3.2.2 Subdifferential transferring results

In this section we extend some results of [MOTV09, Chapter 4] to the case of rotund renorming. Remember that, if A is a subset of a topological vector space, $\varphi : A \to \mathbb{R}$, $x \in U \subseteq A$ and $\varepsilon > 0$, then we call *ε-subdifferential of φ as a function on U at the point x*, the set

$$\partial_\varepsilon\varphi(x|U) = \{x^* \in X^* \mid \varphi(y) \geq \varphi(x) + x^*(y - x) - \varepsilon, \text{ for every } y \in U\}.$$

For a deeper analysis of this concept see [Phe93]. The next theorem deals with the local convexity condition for the map $\varphi_x(u) = \|\Phi(u) + \Phi(x)\|_Y$ at x.

Theorem 3.30 *Let* $(X, \|\cdot\|)$ *and* $(Y, \|\cdot\|_Y)$ *normed spaces such that* $\|\cdot\|_Y$ *is asymptotically rotund. Let* $\Phi : X \to Y$ *a map, and* $\varphi_x(u) := \|\Phi(u) + \Phi(x)\|_Y$ *for all* $u, x \in X$. *If* $\partial_\varepsilon \varphi_x(x| \mathscr{B}_r(0)) \neq \emptyset$ *for every* $x \in \mathscr{B}_r(0)$, $\varepsilon > 0$ *and* $r \in (0, +\infty)$, *then* X *admits an equivalent norm* $\|\cdot\|_\Phi$ *such that*

$$2\|x\|_\Phi^2 + 2\|y\|_\Phi^2 - \|x+y\|_\Phi^2 = 0$$

implies $\Phi(x) = \Phi(y)$, *whenever* $\|\Phi(x)\|_Y = \|\Phi(y)\|_Y$. *Furthermore if* $\|\Phi(x)\|_Y = \|\Phi(y)\|_Y = \left\|\Phi\left(\frac{x+y}{2}\right)\right\|_Y$ *and*

$$2\|x\|_\Phi^2 + 2\|y\|_\Phi^2 - \|x+y\|_\Phi^2 = 0,$$

then $\Phi(x) = \Phi(y) = \Phi\left(\frac{x+y}{2}\right)$.

Proof Let us select $f_\varepsilon^{x,r} \in \partial_\varepsilon \varphi_x(x| \mathscr{B}_r(0))$ for every $x \in \mathscr{B}_r(0)$ and define

$$H(f_\varepsilon^{x,r}) = \{z \in X \mid f_\varepsilon^{x,r}(z) > f_\varepsilon^{x,r}(x) - \varepsilon\}.$$

Consider the families of slices

$$\mathcal{H}_\varepsilon^{x,r,\rho,\mu} = \{H(f_\varepsilon^{x,r}) \cap \mathscr{B}_r(0) \mid \rho < \|\Phi(x)\|_Y < \mu\}$$

where $\varepsilon > 0$, $r, \rho, \mu \in \mathbb{Q}^+$ and $x \in \mathscr{B}_r(0)$. Observe that when $y \in H \cap \mathscr{B}_r(0) \in \mathcal{H}_\varepsilon^{x,r,\rho,\mu}$ and $y \in \mathscr{B}_r(0)$ we have

$$\|\Phi(x) + \Phi(y)\|_Y \geq 2\|\Phi(x)\|_Y - 2\varepsilon$$

If we have two points $x, y \in \mathscr{B}_r(0)$ with $\xi := \|\Phi(x)\|_Y = \|\Phi(y)\|_Y$, and we assume that for every $m \in \mathbb{N}$, $\rho \leq \xi \leq \mu$, $\rho, \mu \in \mathbb{Q}^+$, there is some $u(1/m, \rho, \mu) \in \mathscr{B}_r(0)$ such that

$$x, y, \frac{x+y}{2} \in H(f_{1/m}^{u(1/m,\rho,\mu),r}) \cap \mathscr{B}_r(0) \in \mathcal{H}_{1/m}^{u(1/m,\rho,\mu),r,\rho,\mu}$$

we will have

$$\|\Phi(x) + \Phi(u(1/m,\rho,\mu))\|_Y \geq 2\|\Phi(u(1/m,\rho,\mu))\|_Y - 2/m,$$
$$\|\Phi(y) + \Phi(u(1/m,\rho,\mu))\|_Y \geq 2\|\Phi(u(1/m,\rho,\mu))\|_Y - 2/m$$

and $\rho < \|\Phi(u(1/m,\rho,\mu))\|_Y < \mu$. If we take two sequences $\rho_n \leq \|\Phi(x)\|_Y = \|\Phi(y)\|_Y \leq \mu_n$ and ε_n such that

$$\lim_{n\in\mathbb{N}} \varepsilon_n = 0 \qquad \lim_{n\in\mathbb{N}} \rho_n = \|\Phi(x)\|_Y = \|\Phi(y)\|_Y = \lim_{n\in\mathbb{N}} \mu_n$$

we will have, by asymptotically rotundness, that $\Phi(x) = \Phi(y)$. Furthermore if $\left\|\Phi\left(\frac{x+y}{2}\right)\right\|_Y = \|\Phi(x)\|_Y = \|\Phi(y)\|_Y$, then repeating the same argument we obtain that $\Phi\left(\frac{x+y}{2}\right) = \Phi(x) = \Phi(y)$. To get the equivalent norm we are looking for let us consider the families

$$\left\{\mathcal{H}_{1/n}^{u(1/n,\rho,\mu),r,\rho,\mu} \,\middle|\, n \in \mathbb{N},\ r,\rho,\mu \in \mathbb{Q}^+,\ \rho < \mu\right\}$$

and the sets $\{\mathscr{B}_r(0) \,|\, r \in \mathbb{Q}^+\}$ and apply theorem 1.2 to obtain the equivalent norms $\|\cdot\|_{r,n,\rho,\mu}$ that localizes slices through the LUR condition. In particular we have that

$$2\|x\|^2_{r,n,\rho,\mu} + 2\|y\|^2_{r,n,\rho,\mu} - \|x+y\|^2_{r,n,\rho,\mu} = 0$$

should imply that for some $H \cap \mathscr{B}_r(0) \in \mathcal{H}_{1/n}^{u(1/n,\rho,\mu),r,\rho,\mu}$ we have $x, y, \frac{x+y}{2} \in H \cap \mathscr{B}_r(0)$ when either:

- $x \in \mathscr{B}_r(0) \cap \bigcup \mathcal{H}_{1/n}^{u(1/n,\rho,\mu),r,\rho,\mu}$ and $y \in \mathscr{B}_r(0)$ or
- $y \in \mathscr{B}_r(0) \cap \bigcup \mathcal{H}_{1/n}^{u(1/n,\rho,\mu),r,\rho,\mu}$ and $x \in \mathscr{B}_r(0)$.

Since we always are able to ensure this condition for every $n \in \mathbb{N}$, r big enough and suitable ρ, μ, we will have the norm we are looking for adding all this information. Thus we write

$$\|\cdot\|^2_\Phi = \sum_{\substack{n\in\mathbb{N},\ r\in\mathbb{Q}^+ \\ \rho,\mu\in\mathbb{Q}^+,\ \rho<\mu}} c_{r,n,\rho,\mu}\|\cdot\|^2_{r,n,\rho,\mu}$$

where the constant $c_{r,n,\rho,\mu}$ are chosen for the uniform convergence of the series on bounded sets. Take two points $x, y \in X$ with $\|\Phi(x)\|_Y = \|\Phi(y)\|_Y$ and such that

$$2\|x\|^2_\Phi + 2\|y\|^2_\Phi - \|x+y\|^2_\Phi = 0.$$

Fix $r \in \mathbb{Q}^+$ such that $\max\{\|x\|_Y, \|y\|_Y\} < r$ together with sequences $\rho_n \leq \|\Phi(x)\|_Y = \|\Phi(y)\|_Y \leq \mu_n$ such that

$$\lim_{n\in\mathbb{N}} \rho_n = \|\Phi(x)\|_Y = \|\Phi(y)\|_Y = \lim_{n\in\mathbb{N}} \mu_n.$$

By a standard convexity argument (see [DGZ93, Fact II.2.3]) we have

$$2\|x\|^2_{r,n,\rho_n,\mu_n} + 2\|y\|^2_{r,n,\rho_n,\mu_n} - \|x+y\|^2_{r,n,\rho_n,\mu_n} = 0$$

for every $n \in \mathbb{N}$ and theorem 1.2 gives us $u(1/n, \rho_n, \mu_n) \in \mathscr{B}_r(0)$ such that

$$x, y, \frac{x+y}{2} \in H(f_{1/n}^{u(1/n,\rho_n,\mu_n),r}) \cap \mathscr{B}_r(0) \in \mathcal{H}_{1/n}^{u(1/n,\rho_n,\mu_n),r,\rho_n,\mu_n}$$

thus we will have, as above, $\Phi(x) = \Phi(y)$. Furthermore if $\left\|\Phi\left(\frac{x+y}{2}\right)\right\|_Y = \|\Phi(x)\|_Y = \|\Phi(y)\|_Y$, then we obtain that $\Phi\left(\frac{x+y}{2}\right) = \Phi(x) = \Phi(y)$. □

This result enables us to prove the following version of [MOTV09, Corollary 4.32], for rotund renorming.

Corollary 3.31 *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_Y)$ normed spaces such that $\|\cdot\|_Y$ is asymptotically rotund. Let $\Phi : X \to Y$ a locally bounded map such that*

$$2\|\Phi(x)\|_Y \leq \sum_{i=1}^{n} \lambda_i \|\Phi(x) + \Phi(x_i)\|_Y$$

whenever $n \in \mathbb{N}$, $x, x_i \in X$, $\sum_{i=1}^{n} \lambda_i = 1$, $x = \sum_{i=1}^{n} \lambda_i x_i$ and $\lambda_i \geq 0$ for $i = 1, \ldots, n$. Then X admits an equivalent norm $\|\cdot\|_\Phi$ such that

$$2\|x\|_\Phi^2 + 2\|y\|_\Phi^2 - \|x+y\|_\Phi^2 = 0$$

implies $\Phi(x) = \Phi(y) = \Phi\left(\frac{x+y}{2}\right)$. In particular $\|\cdot\|_\Phi$ is an equivalent rotund norm, whenever Φ is one to one.

Proof The fact that our conditions on Φ implies that $\partial_\varepsilon \varphi_x(x|\mathscr{B}_r(0)) \neq \emptyset$ for every $x \in \mathscr{B}_r(0)$, $\varepsilon > 0$ and $r > 0$, where

$$\varphi_x(u) = \|\Phi(u) + \Phi(x)\|_Y \qquad u, x \in X,$$

follows from the proof done in [MOTV09, Corollary 4.32]. Theorem 3.30 give us an equivalent norm $\|\cdot\|_1$ such that $2\|x\|_1^2 + 2\|y\|_1^2 - \|x+y\|_1^2 = 0$ implies that $\Phi(x) = \Phi(y) = \Phi\left(\frac{x+y}{2}\right)$, whenever $\|\Phi(x)\|_Y = \|\Phi(y)\|_Y = \left\|\Phi\left(\frac{x+y}{2}\right)\right\|_Y$. Let us consider the sets

$$A_r = \{x \in X \mid \|\Phi(x)\|_Y \leq r\},$$

for every $r \in \mathbb{Q}^+$. Without loss of generality we can assume that $\Phi(0) = 0$. Consider $\tau(x) := \|\Phi(x)\|_Y$ and observe that τ is convex and continuous (since Φ is locally bounded), and let p_{A_r} the Minkowski functionals of the sets A_r. We have that

$$\|\cdot\|_\Phi^2 = \|\cdot\|_1^2 + \sum_{r \in \mathbb{Q}^+} p_{A_r}^2(\cdot)$$

is the norm we were seeking. Indeed if $x, y \in X$ are such that $2\|x\|_\Phi^2 + 2\|y\|_\Phi^2 - \|x+y\|_\Phi^2 = 0$, then by a standard convex argument (see [DGZ93, Fact II.2.3]) we have that $\|x\|_1 = \|y\|_1 = \left\|\frac{x+y}{2}\right\|_1$ and $p_{A_r}(x) = p_{A_r}(y) = p_{A_r}\left(\frac{x+y}{2}\right)$ for all $r \in \mathbb{Q}^+$. This last condition implies that $\tau(x) = \tau(y) = \tau\left(\frac{x+y}{2}\right)$ and then by theorem 3.30 the proof follows. □

A straightforward application of the last corollary gives the following result (in fact, turns out to be a particular case of [OST12, Theorem 2.9]).

Corollary 3.32 *Let $(X, \|\cdot\|)$ a normed space and $\Phi : X \to c_0(\Gamma)$ a locally bounded and one-to-one map. If $\delta_\gamma \circ \Phi$ is convex and non-negative for every $\gamma \in \Gamma$, then X admits an equivalent rotund norm.*

One more application is the following generalization of a result of Zizler in [Ziz84], which has been already proved, in a different way, in [DGZ93, Proposition VII.1.5].

Corollary 3.33 *Let $(X, \|\cdot\|)$ a normed space and $\{T_\alpha\}_{\alpha\in\Gamma}$ a family of bounded linear operators $T_\alpha : X \to X$. If the following conditions hold:*

1. *the function T defined on X by $T(x) = (\|T_\alpha(x)\|)_{\alpha\in\Gamma}$ maps X into $c_0(\Gamma)$;*
2. *$\bigcap_{\alpha\in\Gamma} \{x \in X \mid T_\alpha(x) = 0\} = \{0\}$;*
3. *for every $\alpha \in \Gamma$, $T_\alpha(X)$ admits an equivalent rotund norm $\|\cdot\|_\alpha$,*

then X admits an equivalent rotund norm.

Proof Let c_α and C_α such that

$$c_\alpha\|y\|_\alpha \le \|y\| \le C_\alpha\|y\|_\alpha$$

for every $y \in T_\alpha(X)$. Consider the function $\widetilde{T}(x) = (c_\alpha\|T_\alpha(x)\|_\alpha)_{\alpha\in\Gamma}$: it is easy to prove that this function satisfies the conditions of corollary 3.31, which gives us an equivalent notm $\|\cdot\|_{\widetilde{T}}$ such that

$$2\|x\|^2_{\widetilde{T}} + 2\|y\|^2_{\widetilde{T}} - \|x+y\|^2_{\widetilde{T}} = 0$$

implies $\widetilde{T}(x) = \widetilde{T}(y) = \widetilde{T}\big(\frac{x+y}{2}\big)$. Then for every $\alpha \in \Gamma$

$$c_\alpha\|T_\alpha(x)\|_\alpha = c_\alpha\|T_\alpha(y)\|_\alpha = c_\alpha\left\|T_\alpha\left(\frac{x+y}{2}\right)\right\|_\alpha,$$

and by the linearity of T_α and the rotundity of $\|\cdot\|_\alpha$ we have that $T_\alpha(x) = T_\alpha(y)$. By condition 2 we obtain that $x = y$. □

A straightforward application of this result gives us the following corollary.

Corollary 3.34 *Let $(X, \|\cdot\|)$ a normed space and $\{P_\alpha\}_{\alpha\in[\omega_0,\mu]}$ a projectional resolution of the identity (see [FHH+11, Definition 11.5]). Suppose that for every $\alpha \in [\omega_0, \mu)$ the space*

$$T_\alpha(X) := (P_{\alpha+1} - P_\alpha)(X)$$

admits an equivalent rotund norm, then X admits an equivalent rotund norm.

Via corollary 3.31, a result on vector valued sequence spaces can be obtained.

Corollary 3.35 *Let X a normed space and Γ a nonempty set. $c_0(\Gamma, X)$ admits an equivalent rotund norm if, and only if, X admits an equivalent rotund norm. The same result is valid for $\ell_1(\Gamma, X)$.*

Proof Observe that $c_0(\Gamma, X)$ admits an equivalent lattice and LUR norms $\|\cdot\|_L$ and define $\|\cdot\|_X$ an equivalent rotund norm on X. Consider the function $T(x) = (\|x(\gamma)\|_X)_{\gamma\in\Gamma}$, defined for every $x = (x(\gamma))_{\gamma\in\Gamma} \in c_0(\Gamma, X)$, and observe that satisfies the condition of corollary 3.31. So there exists an equivalent norm $\|\cdot\|_T$ such that if $2\|x\|_T^2 + 2\|y\|_T^2 - \|x+y\|_T^2 = 0$, then for every $\gamma \in \Gamma$ we have

$$2\|x(\gamma)\|_X^2 + 2\|y(\gamma)\|_X^2 - \|x(\gamma) + y(\gamma)\|_X^2 = 0.$$

Since $\|\cdot\|_X$ is rotund, we have $x(\gamma) = y(\gamma)$ for every $\gamma \in \Gamma$. The proof in the $\ell_1(\Gamma, X)$ case is similar. □

Another application is a known three space result.

Corollary 3.36 *Let X a normed space and $Y \subseteq X$ a closed subspace. If X/Y admits an equivalent LUR norm and Y admits an equivalent rotund norm, then X admits an equivalent rotund norm.*

Proof Consider the continuous projection $Q : X \to X/Y$ and let $B : X/Y \to X$ the continuous and positively homogeneous Bartle–Graves selection of the inverse Q^{-1} (see [FHH+11, Corollary 7.56]). By corollary 3.31 there exists an equivalent norm $\|\cdot\|_Q$ on X such that for every $x, y \in X$ such that

$$2\|x\|_Q^2 + 2\|y\|_Q^2 - \|x+y\|_Q^2 = 0$$

implies $Q(x) = Q(y)$. Let $\|\cdot\|_1$ an equivalent LUR norm on X/Y, $\|\cdot\|_2$ an equivalent norm on X such that its restriction on Y is rotund (using [DGZ93, Lemma II.8.1] for rotund norm) and

$$S_1 = \{\widehat{x} \in X/Y \mid \|\widehat{x}\|_1 = 1\}.$$

For every $\widehat{a} \in S_1$ consider $f_{\widehat{a}} \in (X/Y)^*$ such that $f_{\widehat{a}}(\widehat{a}) = 1$ and $\|f_{\widehat{a}}\|_2^* \le M$, and define

$$\begin{aligned} P_{\widehat{a}}(x) &= f_{\widehat{a}}(Qx)B\widehat{a}; \\ \varphi_{\widehat{a}}(x) &= \inf\{r > 0 \mid \left\|r^{-1}Qx + \widehat{a}\right\|_1 \le 2\}; \\ \psi_{\widehat{a}}(x) &= \|x - P_{\widehat{a}}x\|_2. \end{aligned}$$

We can apply Deville's master lemma (lemma 1.1) with the families $(\varphi_{\widehat{a}})_{\widehat{a}\in S_1}$ and $(\psi_{\widehat{a}})_{\widehat{a}\in S_1}$, and obtains the norm $\|\cdot\|$ which satisfies for every $x \in X$ and $(x_n)_{n\in\mathbb{N}} \subseteq X$ such that if

$$\lim_{n\in\mathbb{N}} (2\|x\|^2 + 2\|x_n\|^2 - \|x + x_n\|^2) = 0,$$

then there exists $(\widehat{a}_n)_{n\in\mathbb{N}} \subseteq S_1$ such that

1. $\lim_{n\in\mathbb{N}}\left(\frac{1}{2}\psi^2_{\widehat{a}_n}(x)+\frac{1}{2}\psi^2_{\widehat{a}_n}(x_n)-\psi^2_{\widehat{a}_n}\left(\frac{x+x_n}{2}\right)\right)=0$;
2. $\lim_{n\in\mathbb{N}}\varphi_{\widehat{a}_n}(x)=\lim_{n\in\mathbb{N}}\varphi_{\widehat{a}_n}(x_n)=\lim_{n\in\mathbb{N}}\varphi_{\widehat{a}_n}\left(\frac{x+x_n}{2}\right)=\sup_{\widehat{a}\in S_1}\varphi_{\widehat{a}}(x)$.

Let $x,y\in X$ such that $2\|x\|^2+2\|y\|^2-\|x+y\|^2=0$, observe the following fact:

- $\sup_{\widehat{a}\in S_1}\varphi_{\widehat{a}}(x)=\|Qx\|_1$. Indeed
$$\left\|\frac{Qx}{\|Qx\|_1}+\widehat{a}\right\|_1\leq 2$$
for every $\widehat{a}\in S_1$ and this implies $\sup_{\widehat{a}\in S_1}\varphi_{\widehat{a}}(x)\leq\|Qx\|_1$. Furthermore
$$\left\|\frac{Qx}{r}+\frac{Qx}{\|Qx\|_1}\right\|_1=\frac{\|Qx\|_1}{r}+1\leq 2,$$
thus $\|Qx\|_1\leq r$, then $\|Qx\|_1\leq\sup_{\widehat{a}\in S_1}\varphi_{\widehat{a}}(x)$.

- $\lim_{n\in\mathbb{N}}\widehat{a}_n=\frac{Qx}{\|Qx\|_1}$. Indeed for every $\varepsilon\in(0,\|Qx\|_1)$ there exists $n_\varepsilon\in\mathbb{N}$ such that
$$2\leq\left\|\frac{Qx}{r}+\widehat{a}_n\right\|_1,$$
whenever $r\leq\|Qx\|_1-\varepsilon$ and $n\geq n_\varepsilon$. Now let $r=\|Qx\|_1-\varepsilon$ and observe that
$$\begin{aligned}2\leq\left\|\frac{Qx}{\|Qx\|_1}+\widehat{a}_n+\left(\frac{1}{r}-\frac{1}{\|Qx\|_1}\right)Qx\right\|_1\leq\\ \leq\left\|\frac{Qx}{\|Qx\|_1}+\widehat{a}_n\right\|+\frac{\varepsilon}{\|Qx\|_1-\varepsilon}.\end{aligned}$$
It follows $\liminf_{n\in\mathbb{N}}\|Qx\|Qx\|_1^{-1}+\widehat{a}_n\|_1\geq 2$ and by the LUR property we have our claim.

- $\lim_{n\in\mathbb{N}}P_{\widehat{a}_n}(x)=BQx$. Indeed
$$\begin{aligned}\frac{1}{\|Qx\|_1}\|P_{\widehat{a}_n}(x)-BQx\|_2=\left\|f_{\widehat{a}_n}\left(\frac{Qx}{\|Qx\|_1}\right)B\widehat{a}_n-f_{\widehat{a}_n}(\widehat{a}_n)B\frac{Qx}{\|Qx\|_1}\right\|_2\leq\\ \leq\left\|f_{\widehat{a}_n}\left(\frac{Qx}{\|Qx\|_1}-\widehat{a}_n\right)B\frac{Qx}{\|Qx\|_1}\right\|_2+\left\|f_{\widehat{a}_n}\left(\frac{Qx}{\|Qx\|_1}\right)\left(B\widehat{a}_n-B\frac{Qx}{\|Qx\|_1}\right)\right\|_2\leq\\ \leq M\left\|\frac{Qx}{\|Qx\|_1}-\widehat{a}_n\right\|_1\left\|B\frac{Qx}{\|Qx\|_1}\right\|_2+M\left\|B\widehat{a}_n-B\frac{Qx}{\|Qx\|_1}\right\|_2,\end{aligned}$$
by continuity of B we have our claim.

Now observe that these facts give us

$$2\|x - BQx\|_2^2 + 2\|y - BQy\|_2^2 - \|x + y - BQx - BQy\|_2^2 = 0,$$

since $\|\cdot\|_2$ is rotund on Y and $x - BQx, y - BQy \in Y$ we have that $x - BQx = y - BQy$. Consider now the equivalent norm

$$\|\cdot\|_3^2 = \|\cdot\|_Q^2 + \|\cdot\|^2,$$

it is easy to see that this norm satisfies the thesis of our corollary. □

3.2.3 Nonlinear (*)-transference

The following results provide versions of [MOTV09, Theorem 1.1] for the rotund renorming case. The first theorem is a nonlinear transfer of the (*)-property.

Theorem 3.37 *Let X, Y normed spaces, $F \subseteq X^*$ and $G \subseteq Y^*$ norming subspaces. Suppose that Y admits an equivalent $\sigma(Y, G)$-lower semicontinuous rotund norm. If $\Phi : X \to Y$ is a $\sigma(X, F)$-$\sigma(Y, G)$-continuous and one-to-one function, such that there exists a family $\{A_p\}_{p\in\mathbb{N}}$ of convex sets such that for every $x \in X$ and K $\sigma(Y, G)$-open half-space with $\Phi(x) \in K$, there exists $p \in \mathbb{N}$ and a $\sigma(X, F)$-open halfspace H such that $x \in A_p \cap H$ and $\Phi(A_p \cap H) \subseteq K$. Then X admits an equivalent, $\sigma(X, F)$-lower semicontinuous and rotund norm.*

The theorem above can be obtained as a corollary from the following more general result.

Theorem 3.38 *Let X a normed space, $F \subseteq X^*$ a norming subspace and (Y, τ) a topological space which has (*) with closed subsets. Suppose that $(\mathcal{K}_n)_{n\in\mathbb{N}}$ is a (*)-sequence of families of closed subsets for Y and let*

$$\mathcal{K} = \bigcup_{n\in\mathbb{N}} \mathcal{K}_n.$$

If $\Phi : X \to Y$ is a $\sigma(X, F)$-τ-continuous and one-to-one function, such that there exists a family $\{A_p\}_{p\in\mathbb{N}}$ of convex sets such that for every $x \in X$ and $K \in \mathcal{K}$ such that $\Phi(x) \in K$, there exists $p \in \mathbb{N}$ and a $\sigma(X, F)$-open halfspace H such that $x \in A_p \cap H$ and $\Phi(A_p \cap H) \subseteq K$. Then X admits an equivalent, $\sigma(X, F)$-lower semicontinuous and rotund norm.

Proof Without loss of generality we may assume that $X \subseteq F^*$, in such a way that the $\sigma(X,F)$-topology is induced by the w^*-topology on F^*, also we can assume that the family $\{A_p\}_{p\in\mathbb{N}}$ is composed of bounded sets, indeed the sets

$$A_{p,q} = A_p \cap \mathscr{B}(0,q),$$

for $p \in \mathbb{N}$ and $q \in \mathbb{Q}$, satisfy the same property. Let $\mathcal{H}$ the family of $\sigma(X,F)$-open halfspaces of X. Consider the families of slices

$$\mathcal{H}_{n,p} = \left\{ \overline{A_p}^{\sigma(X,F)} \cap H \,\middle|\, H \in \mathcal{H} \text{ and there exists } K \in \mathcal{K}_n \text{ such that } \Phi(A_p \cap H) \subseteq K \right\}$$

and let us apply theorem 1.2 to every set $\overline{A_p}^{\sigma(X,F)}$ and each family $\mathcal{H}_{n,p}$ in order to obtain the norms $\|\cdot\|_{n,p}$ which satisfy for every $(x_k)_{k\in\mathbb{N}} \subseteq X$ and $x \in \overline{A_p}^{\sigma(X,F)} \cap \bigcup \mathcal{H}_{n,p}$ such that if

$$\lim_{k\in\mathbb{N}} (2\|x\|_{n,p}^2 + 2\|x_k\|_{n,p}^2 - \|x + x_k\|_{n,p}^2) = 0,$$

then there exists a sequence of slices $(\overline{A_p}^{\sigma(X,F)} \cap H_k)_{k\in\mathbb{N}} \subseteq \mathcal{H}_{n,p}$ such that

1. there exists $k_0 \in \mathbb{N}$ such that $x, x_k \in \overline{A_p}^{\sigma(X,F)} \cap H_k$, whenever $k \geq k_0$ and $x_k \in \overline{A_p}^{\sigma(X,F)}$;
2. for every $\delta > 0$ there exists $k_\delta \in \mathbb{N}$ such that
$$x, x_k \in \overline{(\overline{A_p}^{\sigma(X,F)} \cap H_k) + \mathscr{B}(0,\delta)}^{\sigma(X,F)},$$
for every $k \geq k_\delta$.

Consider the equivalent norm

$$\|x\|^2 = \sum_{n,p\in\mathbb{N}} c_{n,p} \|x\|_{n,p}^2$$

where the constants $c_{n,p}$ are chosen in order to guarantee the convergence of the series on bounded subsets. Let $x, y \in X$ such that

$$2\|x\|^2 + 2\|y\|^2 - \|x + y\|^2 = 0$$

and suppose, by contradiction, that $x \neq y$. By a standard convexity argument (see [DGZ93, Fact II.2.3]) we obtain for every $n, p \in \mathbb{N}$

$$2\|x\|_{n,p}^2 + 2\|y\|_{n,p}^2 - \|x + y\|_{n,p}^2 = 0.$$

Consider $n_0 \in \mathbb{N}$ such that $\mathcal{K}_{n_0}$ (*)-separates $\Phi(x)$ and $\Phi(y)$. Without loss of generality we can assume that $\Phi(x) \in \bigcup \mathcal{K}_{n_0}$ and, by hypothesis, we know that there exists $p_0 \in \mathbb{N}$ and a $\sigma(X,F)$-open halfspace H such that $x \in A_{p_0} \cap H$ and

$$\Phi(A_{p_0} \cap H) \subseteq K \in \mathcal{K}_{n_0}$$

for a fixed K. Consider the condition

$$2\|x\|^2_{n_0,p_0} + 2\|y\|^2_{n_0,p_0} - \|x+y\|^2_{n_0,p_0} = 0,$$

from $x \in A_{p_0} \cap \bigcup \mathcal{H}_{n_0,p_0}$ we obtain that there exists a sequence of slices $(\overline{A_{p_0}}^{\sigma(X,F)} \cap H_k)_{k\in\mathbb{N}} \subseteq \mathcal{H}_{n_0,p_0}$ such that

1. there exists $k_0 \in \mathbb{N}$ such that $x, y \in \overline{A_{p_0}}^{\sigma(X,F)} \cap H_k$, whenever $k \geq k_0$ and $y \in \overline{A_{p_0}}^{\sigma(X,F)}$;
2. for every $\delta > 0$ there exists $k_\delta \in \mathbb{N}$ such that

$$x, y \in \overline{(\overline{A_{p_0}}^{\sigma(X,F)} \cap H_k) + \mathscr{B}(0,\delta)}^{\sigma(X,F)},$$

for every $k \geq k_\delta$.

Observe that if

$$z \in \overline{(\overline{A_{p_0}}^{\sigma(X,F)} \cap H_k) + \mathscr{B}(0,\delta)}^{\sigma(X,F)} \subseteq \overline{(\overline{A_{p_0}}^{\sigma(X,F)} \cap H_k) + \mathscr{B}(0,\delta)}^{w^*}$$

then there exists a net $\{z_\alpha = v_\alpha + w_\alpha \,|\, \alpha \in (D, \preceq)\}$, where $v_\alpha \in \overline{A_{p_0}}^{\sigma(X,F)} \cap H_k$ and $w_\alpha \in \mathscr{B}(0,\delta)$ for every $\alpha \in D$, such that

$$\lim_{\alpha\in D} |f(z - (v_\alpha + w_\alpha))| = 0 \tag{†}$$

for every $f \in F$. By w^*-compactness we can find a w^*-covergent subnet to some point of F^* which we can write as $v^* + w^*$ where $v^* \in \overline{\overline{A_{p_0}}^{\sigma(X,F)} \cap H}^{w^*} \subseteq \overline{A_{p_0} \cap H}^{w^*}$ and $w^* \in \overline{\mathscr{B}(0,\delta)}^{w^*}$. We obtain that z coincide with $v^* + w^*$ on every element $f \in F$ by condition (†), so

$$\|z - v^*\|_{F^*} = \|w^*\|_{F^*} \leq \delta.$$

So in our case we obtain that for every $\delta > 0$ there exists $y_k^\delta \in \overline{A_{p_0} \cap H_k}^{w^*}$ such that $\|y - y_k^\delta\|\delta$. So there exists a sequence $y_n \in \overline{A_p}^{w^*}$ such that $\lim_{n\in\mathbb{N}} \|y - y_n\|_{F^*} = 0$ and so $y \in \overline{A_p}^{w^*}$. By hypothesis we have that $y \in \overline{A_{p_0}}^{\sigma(X,F)} = \overline{A_{p_0}}^{w^*} \cap X$. By condition 1. we obtain that there exists $H \in \mathcal{H}_{n_0,p_0}$ such that $x, y \in \overline{A_{p_0}}^{\sigma(X,F)} \cap H$, by the very definition of $H \in \mathcal{H}_{n_0,p_0}$, there exists $K \in \mathcal{K}_{n_0}$ such that

$$\Phi(A_{p_0} \cap H) \subseteq K,$$

but this is a contradiction since

$$\Phi(x), \Phi(y) \in \Phi(\overline{A_{p_0}}^{\sigma(X,F)} \cap H) \subseteq \Phi(\overline{A_{p_0} \cap H}^{\sigma(X,F)}) \subseteq \overline{\Phi(A_{p_0} \cap H)}^{\tau} \subseteq K$$

by continuity, but $\mathcal{K}_{n_0}$ (*)-separates $\Phi(x)$ and $\Phi(y)$. □

Observe that if Y is a normed space for which a norimg space $G \subseteq Y^*$ exists such that Y admits an equivalent, $\sigma(Y,G)$-lower semicontinuous and rotund norm then Y satisfies the assumption of our theorem. Furthermore in the same way (without any substantial change) the following theorem can be proved.

Theorem 3.39 *Let X a normed space, $F \subseteq X^*$ a norming subspace and (Y,τ) a topological space which has (*). Suppose that $(\mathcal{K}_n)_{n\in\mathbb{N}}$ is a (*)-sequence for Y and let*

$$\mathcal{K} = \bigcup_{n\in\mathbb{N}} \mathcal{K}_n.$$

If $\Phi : X \to Y$ is a one-to-one function, such that there exists a family $\{A_p\}_{p\in\mathbb{N}}$ of convex and $\sigma(X,F)$-closed sets such that for every $x \in X$ and $K \in \mathcal{K}$ such that $\Phi(x) \in K$, there exists $p \in \mathbb{N}$ and a $\sigma(X,F)$-open halfspace H such that $x \in A_p \cap H$ and $\Phi(A_p \cap H) \subseteq K$. Then X admits an equivalent, $\sigma(X,F)$-lower semicontinuous and rotund norm.

Now we state a "symmetric" characterization result:

Theorem 3.40 *Let X a normed space and $F \subseteq X^*$ a norming subspace. X admits an equivalent, $\sigma(X,F)$-lower semicontinuous an rotund norm if, and only if, there exists a symmetric $\rho : X \times X \to [0,+\infty)$ such that for every $\varepsilon > 0$ it is possible to write*

$$X = \bigcup_{n\in\mathbb{N}} X_{n,\varepsilon},$$

such that every one of $X_{n,\varepsilon}$ is a convex and $\sigma(X,F)$-closed set, and for every $x \in X$ there exists $n \in \mathbb{N}$ and a $\sigma(X,F)$-open halfspace H such that $x \in X_{n,\varepsilon} \cap H$ and $\rho\text{-diam}(X_{n,\varepsilon} \cap H) < \varepsilon$.

Proof We use the same idea of the previous theorem. Without loss of generality we may assume that $X \subseteq F^*$, in such a way that the $\sigma(X,F)$-topology is induced by the w^*-topology on F^*, also we can assume that the sets $X_{n,\varepsilon}$ are bounded sets, indeed the sets

$$X_{n,q,\varepsilon} = X_{n,\varepsilon} \cap \mathscr{B}(0,q),$$

for $n \in \mathbb{N}$, $q \in \mathbb{Q}$ and $\varepsilon > 0$, satisfy the same property. Consider the families of slices

$$\mathcal{H}_{n,m} = \left\{ X_{n,1/m} \cap H \,\middle|\, \rho\text{- diam}(X_{n,1/m} \cap H) < \frac{1}{m} \right\}$$

and let us apply theorem 1.2 to every set $X_{n,1/m}$ and each family $\mathcal{H}_{n,m}$ in order to obtain the norms $\|\cdot\|_{n,m}$ which satisfy for every $(x_k)_{k\in\mathbb{N}} \subseteq X$ and $x \in X_{n,1/m} \cap \bigcup \mathcal{H}_{n,m}$ such that if

$$\lim_{k\in\mathbb{N}} (2\|x\|^2_{n,m} + 2\|x_k\|^2_{n,m} - \|x + x_k\|^2_{n,m}) = 0,$$

then there exists a sequence of slices $(X_{n,1/m} \cap H_k)_{k\in\mathbb{N}} \subseteq \mathcal{H}_{n,m}$ such that

1. there exists $k_0 \in \mathbb{N}$ such that $x, x_k \in X_{n,1/m} \cap H_k$, whenever $k \geq k_0$ and $x_k \in X_{n,1/m}$;
2. for every $\delta > 0$ there exists $k_\delta \in \mathbb{N}$ such that

$$x, x_k \in \overline{(X_{n,1/m} \cap H_k) + \mathscr{B}(0,\delta)}^{\sigma(X,F)},$$

for every $k \geq k_\delta$.

Consider the equivalent norm

$$\|x\|^2 = \sum_{n,m\in\mathbb{N}} c_{n,m}\|x\|_{n,m}^2$$

where the constants $c_{n,m}$ are chosen in order to guarantee the convergence of the series on bounded subsets. Let $x, y \in X$ such that

$$2\|x\|^2 + 2\|y\|^2 - \|x+y\|^2 = 0$$

and suppose, by contradiction, that $x \neq y$. By a standard convexity argument (see [DGZ93, Fact II.2.3]) we obtain for every $n, m \in \mathbb{N}$

$$2\|x\|_{n,m}^2 + 2\|y\|_{n,m}^2 - \|x+y\|_{n,m}^2 = 0.$$

Consider $\varepsilon_0 > 0$ such that $\rho(x,y) > \varepsilon_0$ and $m_0 \in \mathbb{N}$ such that $1/m_0 < \varepsilon_0$. By hypothesis, we know that there exists $n_0 \in \mathbb{N}$ and a $\sigma(X,F)$-open halfspace H such that $x \in X_{n_0,1/m_0} \cap H$ and

$$\rho - \operatorname{diam}(X_{n_0,1/m_0} \cap H) < \frac{1}{m_0}.$$

Consider the condition

$$2\|x\|_{n_0,p_0}^2 + 2\|y\|_{n_0,p_0}^2 - \|x+y\|_{n_0,p_0}^2 = 0,$$

from $x \in X_{n_0,1/m_0} \cap \bigcup \mathcal{H}_{n_0,m_0}$ we obtain that there exists a sequence of slices $(X_{n_0,1/m_0} \cap H_k)_{k\in\mathbb{N}} \subseteq \mathcal{H}_{n_0,m_0}$ such that

1. there exists $k_0 \in \mathbb{N}$ such that $x, y \in X_{n_0,1/m_0} \cap H_k$, whenever $k \geq k_0$ and $y \in X_{n_0,1/m_0}$;
2. for every $\delta > 0$ there exists $k_\delta \in \mathbb{N}$ such that

$$x, y \in \overline{(X_{n_0,1/m_0} \cap H_k) + \mathscr{B}(0,\delta)}^{\sigma(X,F)},$$

for every $k \geq k_\delta$.

Observe that if

$$z \in \overline{(X_{n_0,1/m_0} \cap H_k) + \mathscr{B}(0,\delta)}^{\sigma(X,F)} \subseteq \overline{(X_{n_0,1/m_0} \cap H_k) + \mathscr{B}(0,\delta)}^{w^*}$$

then there exists a net $\{z_\alpha = v_\alpha + w_\alpha \,|\, \alpha \in (D, \preceq)\}$, where $v_\alpha \in X_{n_0,1/m_0} \cap H_k$ and $w_\alpha \in \mathscr{B}(0,\delta)$ for every $\alpha \in D$, such that

$$\lim_{\alpha \in D} |f(z - (v_\alpha + w_\alpha))| = 0 \tag{†}$$

for every $f \in F$. By w^*-compactness we can find a w^*-covergent subnet to some point of F^* which we can write as $v^* + w^*$ where $v^* \in \overline{X_{n_0,1/m_0} \cap H}^{w^*}$ and $w^* \in \overline{\mathscr{B}(0,\delta)}^{w^*}$. We obtain that z coincide with $v^* + w^*$ on every element $f \in F$ by condition (†), so

$$\|z - v^*\|_F = \|w^*\|_F \leq \|w^*\|_{F^*} \leq \delta.$$

So in our case we obtain that for every $\delta > 0$ there exists $y_k^\delta \in \overline{X_{n_0,1/m_0} \cap H_k}^{w^*}$ such that $\left\|y - y_k^\delta\right\| \leq \delta$. So there exists a sequence $y_n \in \overline{X_{n_0,1/m_0}}^{w^*}$ such that $\lim_{n\in\mathbb{N}} \|y - y_n\|_{F^*} = 0$ and so $y \in \overline{X_{n_0,1/m_0}}^{w^*}$. By hypothesis we have that $y \in X_{n_0,1/m_0} = \overline{X_{n_0,1/m_0}}^{w^*} \cap X$. By condition 1 we obtain that there exists $H \in \mathcal{H}_{n_0,m_0}$ such that $x, y \in X_{n_0,1/m_0} \cap H$

$$\rho\text{-}\operatorname{diam}(X_{n_0,1/m_0} \cap H) < \frac{1}{m_0}$$

but this is a contradiction. □

It is possible to prove (without substantial changes) the following result.

Theorem 3.41 *Let X a normed space and $F \subseteq X^*$ a norming subspace. If there exists a $\sigma(X,F)$-lower semicontinuous symmetric $\rho : X \times X \to [0,+\infty)$ such that for every $\varepsilon > 0$ it is possible to write*

$$X = \bigcup_{n\in\mathbb{N}} X_{n,\varepsilon},$$

such that every one of $X_{n,\varepsilon}$ is a convex and $\sigma(X,F)$-closed set, and for every $x \in X$ there exists $n \in \mathbb{N}$ and a $\sigma(X,F)$-open halfspace H such that $x \in X_{n,\varepsilon} \cap H$ and ρ-diam$(X_{n,\varepsilon} \cap H) < \varepsilon$; then X admits an equivalent, $\sigma(X,F)$-lower semicontinuous an rotund norm.

Chapter 4

Uniformly rotund renorming

In this chapter we start studying uniformly rotund renorming in the same spirit as the previous part of our work.

Lemma 4.1 (uniformly version of Deville's lemma) *Let $(\varphi_i)_{i\in I}$, $(\psi_i)_{i\in I}$ be two families of real valued, convex and nonnegative functions defined on a normed space X, which are both uniformly bounded on bounded subsets of X. For every $i \in I$ and $k \in \mathbb{N}$, let us denote*

$$\begin{aligned} \theta_{i,k}(x) &= \varphi_i^2(x) + \frac{1}{k}\psi_i^2(x); \\ \theta_k(x) &= \sup_{i\in I} \theta_{i,k}(x); \\ \theta(x) &= \|x\|^2 + \sum_{k\in\mathbb{N}} 2^{-k}(\theta_k(x) + \theta_k(-x)), \end{aligned}$$

where $\|\cdot\|$ is the norm of X. If $\|\cdot\|_\theta$ denotes the Minkowski functional of the set $B = \{x \in X \mid \theta(x) \le 1\}$, then $\|\cdot\|_\theta$ is an equivalent norm on X with the following property: if $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \subseteq X$ satisfy

$$\lim_{n\in\mathbb{N}} \left(2\|x_n\|_\theta^2 + 2\|y_n\|_\theta^2 - \|x_n + y_n\|_\theta^2\right) = 0,$$

then there exists a sequence $(i_n) \subseteq I$ such that

1. $\lim_{n\in\mathbb{N}} \left(\frac{1}{2}\psi_{i_n}^2(x_n) + \frac{1}{2}\psi_{i_n}^2(y_n) - \psi_{i_n}^2\left(\frac{x_n+y_n}{2}\right)\right) = 0$;
2. $\lim_{n\in\mathbb{N}}(\varphi_{i_n}(x_n) - \varphi_{i_n}(y_n)) = 0$;
3. $\lim_{n\in\mathbb{N}} \left(\varphi_{i_n}(y_n) - \varphi_{i_n}\left(\frac{x_n+y_n}{2}\right)\right) = 0$;
4. $\liminf_{n\in\mathbb{N}} \varphi_{i_n}(x_n) = \liminf_{n\in\mathbb{N}} \sup_{i\in I} \varphi_i(x_n)$;

5. $\liminf_{n\in\mathbb{N}} \varphi_{i_n}(y_n) = \liminf_{n\in\mathbb{N}} \sup_{i\in I} \varphi_i(y_n)$;

6. $\liminf_{n\in\mathbb{N}}(\varphi_{i_n}(x_n) + \varphi_{i_n}(y_n)) = \liminf_{n\in\mathbb{N}} \sup_{i\in I}(\varphi_i(x_n) + \varphi_i(y_n))$.

Proof Following the proof, and with the same notations as in [DGZ93, Lemma VII.1.1], we shall arrive to the following facts:

$$\lim_{n\in\mathbb{N}}(\varphi_{i_n}(x_n) - \varphi_{i_n}(y_n)) = \lim_{n\in\mathbb{N}}\left(\varphi_{i_n}(y_n) - \varphi_{i_n}\left(\frac{x_n + y_n}{2}\right)\right) = 0.$$

On the other hand, if we denote by

$$M_n = \sup_{i\in I} \psi_i^2(y_n) \quad \text{and} \quad M = \sup_{n\in\mathbb{N}} M_n,$$

then the equation (8) of [DGZ93, Lemma VII.1.1] for this case tell us that, given $n \in \mathbb{N}$ we have for all $i \in I$

$$\varphi_i^2(y_n) - \varphi_{i_n}^2(y_n) \leq -\frac{1}{2k_n}(\psi_i^2(y_n) - \psi_{i_n}^2(y_n)) + 2\alpha_{k_n} \leq \frac{M_n}{2k_n} + 2\alpha_{k_n}.$$

Thus, for every $n \in \mathbb{N}$, we have

$$\varphi_{i_n}^2(y_n) \geq \sup_{i\in I} \varphi_i^2(y_n) - \frac{M_n}{2k_n} - 2\alpha_{k_n},$$

hence $\liminf_{n\in\mathbb{N}} \varphi_{i_n}(y_n) = \liminf_{n\in\mathbb{N}} \sup_{i\in I} \varphi_i(y_n)$. By the simmetry of our reasoning we also have $\liminf_{n\in\mathbb{N}} \varphi_{i_n}(x_n) = \liminf_{n\in\mathbb{N}} \sup_{i\in I} \varphi_i(x_n)$ and

$$\liminf_{n\in\mathbb{N}}(\varphi_{i_n}(x_n) + \varphi_{i_n}(y_n)) = \liminf_{n\in\mathbb{N}} \sup_{i\in I}(\varphi_i(x_n) + \varphi_i(y_n)).$$

□

We also need a generalization of the connection lemma [OT09b, Theorem 3]. We start by giving the following definition:

Definition 4.2 *Given an half-space $H = \{x \in X \mid f(x) > \mu\}$ for a $f \in X^*$ we define the ε-enlargement of H as the half-space*

$$H - \varepsilon := \{x \in X \mid f(x) > \mu - \varepsilon\},$$

and the ε-shrinkage of H as the half-space

$$H + \varepsilon := \{x \in X \mid f(x) > \mu + \varepsilon\}.$$

Theorem 4.3 (Uniform slice localization theorem) *Let X a normed space with a norming subspace F in X^*. Let $A \subseteq B$ a bounded subsets in X and $\mathcal{H}$ a family of $\sigma(X, F)$-open half-spaces such that for every $H \in \mathcal{H}$ the set $A \cap H$*

is nonempty. Then there exists an equivalent $\sigma(X,F)$-lower semicontinuous norm $\|\cdot\|_{A,B,\mathcal{H}}$ such that for every sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \subseteq B$, if

$$\lim_{n\in\mathbb{N}}(2\|x_n\|^2_{A,\mathcal{H}} + \|y_n\|^2_{A,\mathcal{H}} - \|x_n + y_n\|^2_{A,\mathcal{H}}) = 0,$$

then for every $\varepsilon > 0$ there exists a sequence of $\sigma(X,F)$-open half-spaces $\{H^\varepsilon_n\}_{n\in\mathbb{N}} \subseteq \mathcal{H}$ such that

1. *if $(x_n)_{n\in\mathbb{N}}$ or $(y_n)_{n\in\mathbb{N}}$ are eventually in $A\cap\bigcup\mathcal{H}$, then there exists $n_\varepsilon \in \mathbb{N}$ such that both $x_n, y_n \in H^\varepsilon_n - \varepsilon$ for every $n \geq n_\varepsilon$. Moreover, for every $\delta > \varepsilon$ there exists some $n_\delta \in \mathbb{N}$ such that for every $n \geq n_\delta$*

$$x_n, y_n \in \operatorname{conv}(A \cap (H^\varepsilon_n - \varepsilon)) + \mathscr{B}(0,\delta).$$

2. *if $(x_n)_{n\in\mathbb{N}}$ or $(y_n)_{n\in\mathbb{N}}$ are frequently in $A \cap \bigcup\mathcal{H}$, then both x_n and y_n are frequently in $H^\varepsilon_n - \varepsilon$. Moreover, for every $\delta > \varepsilon$ frequently we have*

$$x_n, y_n \in \operatorname{conv}(A \cap (H^\varepsilon_n - \varepsilon)) + \mathscr{B}(0,\delta).$$

Proof We shall consider the $\sigma(X,F)$-lower semicontinuous and convex functions (φ^ε_H) and (ψ^ε_H) for every $H \in \mathcal{H}$ defined as follows:

$$\varphi^\varepsilon_H(x) := \inf\left\{\|x-c\|_F \,\middle|\, c \in \overline{X \setminus (H-\varepsilon) \cap \operatorname{conv}(B)}^{\sigma(X^{**},X^*)}\right\}$$

for every $x \in X$. Let us choose a point $a_H \in A \cap H$ and set $D^\varepsilon_H = \operatorname{conv}(A \cap (H-\varepsilon))$ for every $H \in \mathcal{H}$, and $D^{\varepsilon,\delta}_H = D^\varepsilon_H + \mathscr{B}(0,\delta)$, for every $\delta > 0$ and $H \in \mathcal{H}$. We are going to denote with $p_{H,\varepsilon,\delta}$ the Minkowski functional of the convex body $\overline{D^{\varepsilon,\delta}_H}^{\sigma(X,F)} - a_H$ and $p_{H,1}$ will be $\|\cdot\|_F$. Then we define the $\sigma(X,F)$-lower semicontinuous norm $p_{H,\varepsilon}$ by the formula

$$p^2_{H,\varepsilon}(x) = \sum_{n\in\mathbb{N}} \frac{1}{n^2 2^n} p^2_{H,\varepsilon,1/n}(x),$$

for every $x \in X$. Indeed, since $\mathscr{B}(0,\delta) + a_H \subseteq \overline{D^{\varepsilon,\delta}_H}^{\sigma(X,F)}$ we have for every $x \in X$, and $\delta > 0$, that $p_{H,\varepsilon,\delta}(\delta x/\|x\|_F) \leq 1$, thus $\delta p_{H,\varepsilon,\delta}(x) \leq \|x\|_F$ and the above series converges. Finally we define the nonnegative, convex and $\sigma(X,F)$-lower semicontinuous function ψ^ε_H as $\psi^\varepsilon_H(x) := (p^2_{H,\varepsilon}(x - a_H))^{1/2}$ for every $x \in X$. We are now in position to apply lemma 4.1 for a fixed $\varepsilon > 0$ to get an equivalent norm $\|\cdot\|_{A,\mathcal{H},\varepsilon}$ on X such that the condition

$$\lim_{n\in\mathbb{N}}(2\|x_n\|^2_{A,\mathcal{H},\varepsilon} + \|y_n\|^2_{A,\mathcal{H},\varepsilon} - \|x_n + y_n\|^2_{A,\mathcal{H},\varepsilon}) = 0,$$

for sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in X, implies that there exists a sequence of indexes (H^ε_n) in $\mathcal{H}$ such that:

1. $\lim_{n\in\mathbb{N}}\Big(\frac{1}{2}(\psi^{\varepsilon}_{H^{\varepsilon}_n})^2(x_n)+\frac{1}{2}(\psi^{\varepsilon}_{H^{\varepsilon}_n})^2(y_n)-(\psi^{\varepsilon}_{H^{\varepsilon}_n})^2\big(\frac{x_n+y_n}{2}\big)\Big)=0$;
2. $\lim_{n\in\mathbb{N}}(\varphi^{\varepsilon}_{H^{\varepsilon}_n}(x_n)-\varphi^{\varepsilon}_{H^{\varepsilon}_n}(y_n))=0$;
3. $\lim_{n\in\mathbb{N}}\Big(\varphi^{\varepsilon}_{H^{\varepsilon}_n}(y_n)-\varphi^{\varepsilon}_{H^{\varepsilon}_n}\big(\frac{x_n+y_n}{2}\big)\Big)=0$;
4. $\liminf_{n\in\mathbb{N}}\varphi^{\varepsilon}_{H^{\varepsilon}_n}(x_n)=\liminf_{n\in\mathbb{N}}\sup_{H\in\mathcal{H}}\varphi^{\varepsilon}_{H}(x_n)$;
5. $\liminf_{n\in\mathbb{N}}\varphi^{\varepsilon}_{H^{\varepsilon}_n}(y_n)=\liminf_{n\in\mathbb{N}}\sup_{H\in\mathcal{H}}\varphi^{\varepsilon}_{H}(y_n)$;
6. $\liminf_{n\in\mathbb{N}}(\varphi^{\varepsilon}_{H^{\varepsilon}_n}(x_n)+\varphi^{\varepsilon}_{H^{\varepsilon}_n}(y_n))=\liminf_{n\in\mathbb{N}}\sup_{H\in\mathcal{H}}(\varphi^{\varepsilon}_{H}(x_n)+\varphi^{\varepsilon}_{H}(y_n))$.

If a point y_{n_0} belongs to one of the open half-spaces $H\in\mathcal{H}$ then we have $\varphi_{H,\varepsilon}(y_{n_0})>\varepsilon$ and so we have that:

$$\sup_{H\in\mathcal{H}}\varphi_{H,\varepsilon}(y_{n_0})\geq\varphi_{H,\varepsilon}(y_{n_0})>\varepsilon.$$

Under the assumption that the sequence $(y_n)_{n\in\mathbb{N}}$ is eventually in $A\cap\bigcup\mathcal{H}$, for instance, the condition 5 above forces that

$$\liminf_{n\in\mathbb{N}}\varphi^{\varepsilon}_{H^{\varepsilon}_n}(y_n)=\liminf_{n\in\mathbb{N}}\sup_{H\in\mathcal{H}}\varphi^{\varepsilon}_{H}(y_n)\geq\varepsilon,$$

there exists $m_0\in\mathbb{N}$ such that $\varphi^{\varepsilon}_{H^{\varepsilon}_n}(y_n)\geq\varepsilon/2$, whenever $n\geq m_0$, thus condition 2 and 3 tell us that there exists $n_0\in\mathbb{N}$ such that both $\varphi^{\varepsilon}_{H^{\varepsilon}_n}(x_n)\geq\varepsilon/2>0$ and $\varphi^{\varepsilon}_{H^{\varepsilon}_n}(y_n)\geq\varepsilon/2>0$, therefore $x_n,y_n\in H^{\varepsilon}_n-\varepsilon$. Moreover, condition 1 above and a standard convexity argument (see [DGZ93, Fact II.2.3]) imply now that for every $q\in\mathbb{N}$ we have that

$$\lim_{n\in\mathbb{N}}\Bigg(\frac{1}{2}p^2_{H^{\varepsilon}_n,\varepsilon,1/q}(x_n-a_{H^{\varepsilon}_n})+\frac{1}{2}p^2_{H^{\varepsilon}_n,\varepsilon,1/q}(y_n-a_{H^{\varepsilon}_n})+ -p^2_{H^{\varepsilon}_n,\varepsilon,1/q}\left(\frac{x_n+y_n}{2}-a_{H^{\varepsilon}_n}\right)\Bigg)=0,$$

and

$$\lim_{n\in\mathbb{N}}\left(\frac{1}{2}\|x_n-a_{H^{\varepsilon}_n}\|^2_F+\frac{1}{2}\|y_n-a_{H^{\varepsilon}_n}\|^2_F-\left\|\frac{x_n+y_n}{2}-a_{H^{\varepsilon}_n}\right\|^2_F\right)=0.$$

Consequently

$$\lim_{n\in\mathbb{N}}\Big(p^2_{H^{\varepsilon}_n,\varepsilon,1/q}(x_n-a_{H^{\varepsilon}_n})+p_{H^{\varepsilon}_n,\varepsilon,1/q}(y_n-a_{H^{\varepsilon}_n})\Big)=0.$$

If we fix $\delta>0$, an open half-space $H\in\mathcal{H}$ and any $y\in A\cap(H-\varepsilon)$ we have that

$$y-a_H+(y-a_H)\frac{\delta}{\|y-a_H\|_F}\in\mathscr{B}(0,\delta)+(y-a_H)\subseteq D^{\varepsilon,\delta}_H-a_H,$$

and therefore

$$p_{H,\varepsilon,\delta}(y-a_H) < \frac{\|y-a_H\|_F}{\delta+\|y-a_H\|_F},$$

because $D_H^{\varepsilon,\delta} - a_H$ is a norm open set. Let us choose now an integer $q \in \mathbb{N}$ such that $1/q < \delta$ and assume that the sequence $(y_n)_{n\in\mathbb{N}}$ is such that $y_n \in A \cap \bigcup \mathcal{H}$ for every $n \geq N_0$ (respectively there is a subsequence $(y_{n_k})_{k\in\mathbb{N}}$ with $y_{n_k} \in A \cap \bigcup \mathcal{H}$ for every $k \in \mathbb{N}$). We know that $x_n \in H_n^\varepsilon - \varepsilon$, whenever $\varphi^\varepsilon_{H_n^\varepsilon}(x) > 0$, therefore

$$p_{H,\varepsilon,1/q}(x_n-a_H) < \frac{\|x_n-a_H\|_F}{1/q+\|y-a_H\|_F},$$

and we can find a real number $\xi \in (0,1)$ such that $p_{H_n^\varepsilon,\varepsilon,1/q}(x_n - a_{H_n^\varepsilon}) < 1-\xi$, for every $n \in \mathbb{N}$ by boundness of A (respectively $p_{H_{n_k}^\varepsilon,\varepsilon,1/q}(x_{n_k} - a_{H_{n_k}^\varepsilon}) < 1-\xi$, for every $k \in \mathbb{N}$ big enough). If we now take the integer $n \in \mathbb{N}$ big enough to have $p_{H_n^\varepsilon,\varepsilon,1/q}(y_n - a_{H_n^\varepsilon}) < 1-\xi$, too (respectively $p_{H_{n_k}^\varepsilon,\varepsilon,1/q}(y_{n_k} - a_{H_{n_k}^\varepsilon}) < 1-\xi$, for every $k \in \mathbb{N}$ big enough) we arrive to the fact that for both sequences we have: $x_n - a_{H_n^\varepsilon}, y_n - a_{H_n^\varepsilon} \in D^{\varepsilon,\delta}_{H_n^\varepsilon} - a_{H_n^\varepsilon}$ and indeed

$$x_n, y_n \in \operatorname{conv}(A \cap (H_n^\varepsilon - \varepsilon)) + \mathscr{B}(0,\delta),$$

(respectively $x_{n_k}, y_{n_k} \in \operatorname{conv}(A \cap (H_{n_k}^\varepsilon - \varepsilon)) + \mathscr{B}(0,\delta)$, for $k \in \mathbb{N}$ big enough). In order to have the norm valid for every $\varepsilon > 0$ we simply define

$$\|x\|^2_{A,\mathcal{H}} := \sum_{n\in\mathbb{N}} c_n \|x\|^2_{A,\mathcal{H},1/n}$$

with a choice of $(c_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ according with the convergence of the series on bounded subsets. □

The main result of this chapter is the following generalization of [MOTV09, Theorem 1.1] and [Lan95, Proposition 2.1].

Theorem 4.4 *Let X a normed space. X admits an equivalent uniformly rotund norm if, and only if, for every $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that it is possible to write the unit ball as*

$$\mathscr{B}_X = \bigcup_{n=1}^{N_\varepsilon} B_i^\varepsilon,$$

and for every $n = 1, \ldots, N_\varepsilon$ there exist $\delta \in (0,\varepsilon)$ and a family of open half-spaces $\mathcal{H}_{n,\varepsilon}$ which cover B_n such that for every $H \in \mathcal{H}_{n,\varepsilon}$

$$\operatorname{diam}(B_n \cap (H-\delta)) < \varepsilon.$$

Note that sets B_i^ε in the previous decomposition need not to be convex. Moreover they may not be obtained inductively by derivation with prescribed order like in Lancien's construction (see [Lan95]).

Proof We begin by proving the if part. Let $\|\cdot\|$ an equivalent uniformly rotund norm on X and observe that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x, y \in X$ with $2\|x\|^2 + 2\|y\|^2 - \|x+y\|^2 < \delta$ it follows that $\|x - y\| < \varepsilon$. Indeed, if there exists $\varepsilon_0 > 0$ such that for every $\delta > 0$ exists $x_\delta, y_\delta \in X$ with $2\|x_\delta\|^2 + 2\|y_\delta\|^2 - \|x_\delta + y_\delta\|^2 < \delta$, but $\|x_\delta - y_\delta\| \geq \varepsilon_0$, then observe that

$$\lim_{n\in\mathbb{N}}(2\|x_{1/n}\|^2 + 2\|y_{1/n}\|^2 - \|x_{1/n} + y_{1/n}\|^2) = 0$$

and for every $n \in \mathbb{N}$ we have $\|x_{1/n} - y_{1/n}\| \geq \varepsilon_0$, which is a contradiction. Fix $\varepsilon > 0$ and consider $\eta > 0$ such that whenever $2\|x\|^2 + 2\|y\|^2 - \|x+y\|^2 < \eta$ it follows that $\|x - y\| < \varepsilon$. Fix $n \in \mathbb{N}$ big enough such that $\frac{8}{n} < \eta$ and consider the following partition of the unit ball

$$B_k^n = \left\{ x \in \mathscr{B}_X \,\middle|\, \frac{k}{2n} < \|x\| \leq \frac{k+1}{2n} \right\}$$

for $k = 0, \ldots, 2n-1$ and $B_{2n}^n = \{0\}$. For every point $x \in \mathscr{B}_X \setminus \{0\}$ consider $f_x \in \mathscr{B}_{X^*}$ such that $f_x(x) = \|x\|$ and define

$$H(f_x, \|x\|, n) = \left\{ y \in \mathscr{B}_X \,\middle|\, f_x(y) > \|x\| - \frac{1}{4n} \right\},$$

and $\mathcal{H}_{n,k} = \{H(f_x, \|x\|, n) \mid x \in B_k^n\}$. Consider $x, y \in B_k^n \cap (H - 1/4n)$ for $H \in \mathcal{H}_{n,k}$ and $k = 0, \ldots, 2n-1$ and observe that if $H = H(f_z, \|z\|, n)$ then

$$\begin{aligned} 2\|x\|^2 + 2\|y\|^2 - \|x+y\|^2 &\leq 4\left(\frac{k+1}{2n}\right)^2 - \left(2\|z\| - \frac{1}{n}\right)^2 \leq \\ &\leq \frac{(k+1)^2}{n^2} - \frac{(k-1)^2}{n^2} + \frac{4}{n^2} = 4\frac{k+1}{n^2} \leq \frac{8}{n} < \eta. \end{aligned}$$

Thus $\|x - y\| < \varepsilon$ and we have our thesis.

Let now prove the only if part. Consider the family of half-spaces $\mathcal{H}_{k,1/m}$ and the sets $B_k^{1/m} \subseteq \mathscr{B}_X$, applying theorem 4.3 we obtain a countable family of norms $\|\cdot\|_{k,m}$ such that for every $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \subseteq X$ such that if $x_n \in B_k \cap \bigcup \mathcal{H}_{k,m}$ frequently and

$$\lim_{n\in\mathbb{N}}(2\|x_n\|_{k,m}^2 + 2\|y_n\|_{k,m}^2 - \|x_n + y_n\|_{k,m}^2) = 0,$$

then for every $\varepsilon > 0$ there exists a sequence of open half-spaces $\{H_n^\varepsilon\}_{n\in\mathbb{N}} \subseteq \mathcal{H}$ such that for every $\eta > \varepsilon$ frequently we have

$$x_n, y_n \in \operatorname{conv}(A \cap (H_n^\varepsilon - \varepsilon)) + \mathscr{B}(0, \eta).$$

Consider the equivalent norm

$$\|x\|^2 = \sum_{k,m\in\mathbb{N}} c_{k,m}\|x\|_{k,m}^2$$

where $c_{k,m}$ are positive real number chose in order to guarantee the convergence of the series on bounded subsets. Consider $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \subseteq \mathscr{B}_X$ such that

$$\lim_{n\in\mathbb{N}}(2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n + y_n\|^2) = 0,$$

by a standard convexity argument (see [DGZ93, Fact II.2.3]) we have for every $k, m \in \mathbb{N}$

$$\lim_{n\in\mathbb{N}}(2\|x_n\|_{k,m}^2 + 2\|y_n\|_{k,m}^2 - \|x_n + y_n\|_{k,m}^2) = 0.$$

Consider $(n_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$ such that $(x_{n_k})_{k\in\mathbb{N}}$ and $(y_{n_k})_{k\in\mathbb{N}}$ are subsequence, and put $z_k^1 = x_{n_k}$ and $w_k^1 = y_{n_k}$. We will construct a converging subsequence proceeding by induction:

$(m = 1)$ We know that there exists h_1 such that $z_k^1 \in B_{h_1}^{1/4} \cap \bigcup \mathcal{H}_{h_1,1/4}$ frequently. From the condition

$$\lim_{k\in\mathbb{N}}(2\left\|z_k^1\right\|_{h_1,1/4}^2 + 2\left\|w_k^1\right\|_{h_1,1/4}^2 - \left\|z_k^1 + w_k^1\right\|_{h_1,1/4}^2) = 0,$$

follows that there exists a sequence of open half-spaces $\{H_k^{\delta_{1/4}}\}_{n\in\mathbb{N}} \subseteq \mathcal{H}_{h_1,1/4}$ such that frequently we have

$$z_k^1, w_k^1 \in \operatorname{conv}(A \cap (H_k^{\delta_{1/4}} - \delta_{1/4})) + \mathscr{B}(0, 1/2);$$

this means that there exists $(k_l)_{l\in\mathbb{N}} \subseteq \mathbb{N}$ such that

$$\left\|z_{k_l}^1 - w_{k_l}^1\right\| < \frac{1}{4} + \frac{1}{2} = \frac{3}{4}.$$

In order to proceed by induction define $z_{l-1}^2 = z_{k_l}^1$ and $w_{l-1}^2 = w_{k_l}^1$ for every $l \geq 2$.

$(m \rightsquigarrow (m+1))$ Using induction we obtain the sequences $(z_k^m)_{k\in\mathbb{N}}$ and $(w_k^m)_{k\in\mathbb{N}}$. We know that there exists h_m such that $z_k^m \in B_{h_1}^{1/(m+1)^2} \cap \bigcup \mathcal{H}_{h_m,1/(m+1)^2}$ frequently. From the condition

$$\lim_{k\in\mathbb{N}}(2\|z_k^m\|_{h_m,1/(m+1)^2}^2 + 2\|w_k^m\|_{h_m,1/(m+1)^2}^2 - \|z_k^m + w_k^m\|_{h_1,1/(m+1)^2}^2) = 0,$$

follows that there exists a sequence of open half-spaces $\{H_k^{\delta_{1/(m+1)^2}}\}_{n\in\mathbb{N}} \subseteq \mathcal{H}_{h_m,1/(m+1)^2}$ such that frequently we have

$$z_k^m, w_k^m \in \operatorname{conv}(A \cap (H_k^{\delta_{1/(m+1)^2}} - \delta_{1/(m+1)^2})) + \mathscr{B}(0, 1/(m+1));$$

this means that there exists $(k_l)_{l\in\mathbb{N}} \subseteq \mathbb{N}$ such that

$$\|z^m_{k_l} - w^m_{k_l}\| < \frac{1}{(m+1)^2} + \frac{1}{m+1} = \frac{m+2}{(m+1)^2}.$$

In order to proceed by induction define $z^{m+1}_{l-m} = z^m_{k_l}$ and $w^{m+1}_{l-m} = w^m_{k_l}$ for every $l \geq m+1$.

Consider now the diagonal subsequences $(z^m_m)_{m\in\mathbb{N}}$ and $(w^m_m)_{m\in\mathbb{N}}$ and observe that

$$\lim_{m\in\mathbb{N}} \|z^m_m - w^m_m\| = 0.$$

Thus every subsequence of $(x_n - y_n)_{n\in\mathbb{N}}$ admits a subsequence which converge in norm to zero, then

$$\lim_{n\in\mathbb{N}} \|x_n - y_n\| = 0.$$

□

Since the beginning we have worked a lot with cover of the unit sphere in order to obtain equivalent norms; the following result goes in this direction.

Lemma 4.5 *Let X a normed space and $\varepsilon > 0$. If there exists a family $\mathcal{H}$ of open half-spaces which cover the unit sphere such that there exists $\theta \in (0, \varepsilon)$ with*

$$\operatorname{diam}(\mathscr{B}_X \cap (H - \theta)) < \varepsilon,$$

then there exists an equivalent norm $\|\cdot\|_{\mathcal{H}}$ such that the condition

$$\lim_{n\in\mathbb{N}} (2\|x_n\|^2_{\mathcal{H}} + 2\|y_n\|^2_{\mathcal{H}} - \|x_n + y_n\|^2_{\mathcal{H}}) = 0$$

implies that $\|x_n - y_n\| \leq 2\varepsilon$ for $n \in \mathbb{N}$ big enough, $(x_n)_{n\in\mathbb{N}} \subseteq \mathscr{B}_X$ and $(y_n)_{n\in\mathbb{N}} \subseteq \mathcal{S}_X$.

Proof If we apply theorem 4.3 with $\mathcal{S}_X \subseteq \mathscr{B}_X$ and the family $\mathcal{H}$, we obtain an equivalent norm $\|\cdot\|_{\mathcal{H}}$ such that the condition

$$\lim_{n\in\mathbb{N}} (2\|x_n\|^2_{\mathcal{H}} + 2\|y_n\|^2_{\mathcal{H}} - \|x_n + y_n\|^2_{\mathcal{H}}) = 0$$

implies that for every $\eta > 0$ if $(y_n)_{n\in\mathbb{N}} \subseteq \mathcal{S}_X$ definitively then there exists $\{H^\eta_n\} \subseteq \mathcal{H}$ such that for every $\delta > \eta$

$$x_n, y_n \in \operatorname{conv}(\mathcal{S}_X \cap (H^\eta_n - \eta)) + \mathscr{B}(0, \delta),$$

definitively. Fix $\varepsilon > 0$ and consider the θ given from the hypothesis, we know that

$$\lim_{n\in\mathbb{N}} (2\|x_n\|^2_{\mathcal{H}} + 2\|y_n\|^2_{\mathcal{H}} - \|x_n + y_n\|^2_{\mathcal{H}}) = 0$$

implies that there exists $\{H^\theta_n\} \subseteq \mathcal{H}$ such that

$$x_n, y_n \in \operatorname{conv}(\mathcal{S}_X \cap (H^\theta_n - \theta)) + \mathscr{B}(0, \varepsilon),$$

definitively. So we have that $\|x_n - y_n\| \leq 2\varepsilon$ definitively. □

We obtain the following characterization.

Theorem 4.6 *Let X a normed space. X admits an equivalent uniformly rotund norm if, and only if, there exists an equivalent norm $\|\cdot\|_U$ such that for every $\varepsilon > 0$ there exists a family $\mathcal{H}_\varepsilon$ of open half-spaces which covers the sphere $S_U = \{x \in X \mid \|x\|_U = 1\}$, such that there exists $\theta_\varepsilon \in (0, \varepsilon)$ with*

$$\operatorname{diam}((H - \theta_\varepsilon) \cap \{x \in X \mid \|x\|_U \leq 1\}) < \varepsilon$$

for every $H \in \mathcal{H}_\varepsilon$.

Proof For every $k \in \mathbb{N}$ we apply the preceding lemma for the family $\mathcal{H}_{1/k}$, obtaining a norm $\|\cdot\|_k$ such that the condition

$$\lim_{n\in\mathbb{N}}(2\|x_n\|_k^2 + 2\|y_n\|_k^2 - \|x_n + y_n\|_k^2) = 0$$

implies $\|x_n - y_n\| \leq 2/k$ for $n \in \mathbb{N}$ big enough, $(x_n)_{n\in\mathbb{N}} \subseteq \mathscr{B}_X$ and $(y_n)_{n\in\mathbb{N}} \subseteq \mathcal{S}_X$. So by a standard convexity argument (see [DGZ93, Fact II.2.3]) the norm

$$\|x\|^2 = \sum_{k\in\mathbb{N}} c_k \|x\|_k^2,$$

where the constant c_k are chosen in order to guarantee the convergence of the series on bounded subset, is the norm we were looking for. □

Chapter 5

Conclusion and open problems

In connection with descriptive properties, let us remind that, for a descriptive Banach space with respect to the w-topology, the weak Borel sets coincide with the norm Borel sets (see [Han01] and [Onc00]). Based on a sophisticated construction of Todorčević [Tod05], Marciszewki and Pol have proved that it is consistent the existence of a compact scattered space K such that in the function space $\mathscr{C}(K)$ each norm open set is an $\mathcal{F}_\sigma$-set with respect to the w-topology, while the identity map

$$Id : (\mathscr{C}(K), w) \longrightarrow (\mathscr{C}(K), \|\cdot\|_\infty)$$

is not σ-continuous, see [MP09]. Descriptive Banach spaces are Souslin sets built by $\sigma(X^{**}, X^*)$ open or closed subsets of their biduals. Spaces in this class are called weakly Čech analytic and are exactly the ones that can be represented with a Souslin scheme of Borel subsets in their $\sigma(X^{**}, X^*)$ biduals. The fact that every weakly Čech analytic Banach space is σ-fragmented is the main result in [JNR93]. The reverse implications is an open question, that is considered in [JNR92a] and [JNR92b]. We recall here the following.

Problem 5.1 Is there any gap between the class of descriptive Banach spaces, with respect to the w-topology, and the class of the σ-fragmented Banach spaces?

After the seminal paper of R. Hansell [Han01] we know that a covering property on the w-topology of a Banach space, known as hereditarely weakly θ-refinability, is a necessary and sufficient condition for that classes to coincide. Indeed, all known normed spaces which are not weakly θ-refinable are not σ-fragmentable by the norm (see [DJP97] and [DJP06]). For spaces of continuous functions on trees, Haydon has proved that there is no gap between σ-fragmented and the pointwise Kadec renormability property of the space (see [Hay99]). We can consider a particular case of the former question as follows. For this question Hansell conjectures a positive answer:

Problem 5.2 Let X a weakly Čech analytic Banach space where every norm open set is a countable union of sets which are differences of w-closed sets. Does it follow that the identity map $Id : (X, w) \to (X, \|\cdot\|)$ is σ-continuous?

In the special case when the Banach space X enjoys the Radon-Nikodým property, i.e every bounded closed convex subset of X has slices of arbitrarily small diameter, it is still an open problem to decide whether X has even an equivalent rotund norm or not. In that case, by the results in [MOTV00], the LUR renormability reduces to the question of Kadec renormability. So we summarize here.

Problem 5.3 If the Banach space X enjoys the Radon-Nikodým property, does it follow that X has an equivalent Kadec norm? Does it have an equivalent rotund norm?

Let us remark here that Yost and Plicko in [PY01] proved that the Radon-Nikodým property does not imply the separable complementation property. Thus it is not possible any approach to the former question based on the projectional resolution of the identity; at the contrary, such an approch works for the dual case, as in [FG88].

Our previous results, as well as those in [OST12], suggest the following open question.

Problem 5.4 Is it true that a dual Banach space X^* admits an equivalent dual norm if, and only if, $\mathscr{B}_{X^*}$ enjoys (*) with respect to the w^*-topology?

Following our construnction, such a question can be answered in the affermative if it possible to adapt lemma 3.4, in order to work without covers. Finally a positive answer to the following question would be in interesting.

Problem 5.5 Let X a normed space with the separable complementation property. Is it true that, if $\text{dens}(X) \leq \aleph_1$, then X admits an equivalent rotund norm?

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