# Continuous groups of transformations: Élie Cartan's structural approach

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Figure 1: Élie Joseph Cartan (1869-1951). Archives of the Academy of Sciences of Paris, Élie Cartan's *dossier biographique*.

# Introduction

*The problems dealt with by Cartan are among the most important, abstract and general problems of mathematics. As we have already said, group theory is, in a certain sense, mathematics itself, deprived of its matter and reduced to its pure form.*[1]

With these praising words Henri Poincaré described Élie Cartan's mathematical works in the report which he wrote for the Faculty of Sciences of Paris in 1912. In the course of his survey, Poincaré singled out two fundamental characteristics of Cartan's mathematical production until that time: a high degree of unity due to his relentless commitment to group theory and a constant concern for issues of structural nature.

In effect, since the beginning of his scientific career, Cartan had, almost exclusively, dealt with the theory of groups by considering different variants of the notion such as finite discontinuous groups, finite continuous groups and infinite continuous groups. Furthermore, in all his researches on the subject, Cartan had emphasized the importance of pursuing an abstract approach which was based on the fundamental notions of structure and isomorphism.

Although it is doubtful that Cartan was willing to completely share Poincaré's view according to whom every mathematical theory was ultimately a branch of group theory, he admitted that the notion of group provided a most precious tool by means of which apparently distinct theories could be reunited under common principles. Already in 1909, he clarified his ideas over this point by making recourse to two examples taken from pure geometry and theoretical physics. Lobachevskian geometry in 3-dimensional space, he asserted, is equivalent to projective geometry of real or imaginary figure upon a straight line. Similarly, the new cinematics of special relativity which is governed by the Lorentz group is equivalent to Laguerre geometry. The reason for such equivalences, Cartan explained, lays in the fact that their corresponding groups are isomorphic, i.e. they exhibit the same structure.

[1] *"On voit que les problèmes traités par M. Cartan sont parmi les plus importants, les plus abstraits et les plus généraux dont s'occupent les Mathématiques; ainsi que nous l'avons dit, la théorie des groupes est, pour ansi dire, la Mathématique entière, dépouille de sa matière et réduite à une forme pure".* [Poincaré 1914, p. 145].

In a certain sense, he concluded, the logical content of many geometrical theories coincides with the structure of their corresponding groups.

In actual fact, geometry will assume a dominant role in Cartan's research priorities only later in his career, namely starting from the 1910's. It is certainly true that such a favourable attitude towards geometrical applications of group theory may have contributed to guide Cartan's interests even in his early researches. Nonetheless, as it will be shown in great detail, applications to integration theory of differential equations seem to have played a by far prevalent role in driving Cartan to conceive a structural theory of continuous groups of transformations.

Until at least 1910, the notion of group structure constituted the main, if not unique, object of Cartan's researches. He defined it to be the law of composition of the transformations of a group when these are considered independently of the nature of the objects upon which they act.

However, as he hastened to remark, depending upon the type of groups under consideration, finite or infinite continuous groups, the study of the structure of groups took on different forms and required quite different techniques.

As for the case of *finite* continuous groups, forefathers of modern Lie groups, Cartan's theory was indeed based upon consideration of infinitesimal transformations. First introduced by Lie, they had been profitably exploited by W. Killing in his monumental classification work in which the grounds for the modern theory of the structure of (complex) Lie algebras were laid. Infinitesimal transformations had proved to be a valuable technical tool essentially in consequence of the fact that they led to the existence of constants which fully characterize the structure of a given group.

On the contrary, in the case of *infinite* continuous groups (which nowadays we would call *Lie pseudogroups*), since still at the beginning of the last century no structural approach was available, it was up to Cartan to build up a brand new theory which introduced, for the first time, structural considerations in the infinite domain. In this respect, Cartan's innovative theory of exterior differential systems turned out to be an essential tool. As a consequence of this, infinitesimal transformations were replaced by invariant exterior forms whose exterior derivatives provided generalization of classical structure constants.

In view of Cartan's constant concern for structural issues, it is not surprising that his figure and work became a kind of benchmark for the Bourbaki group. Dieudonné, for example, saw him as a tutelary deity of incoming generations of mathematicians. By directly addressing Cartan, on the occasion of his seventieth birthday, he said for example: "*vous êtes un jeune, et vous comprenez les jeunes*[2]". Nonetheless, we should not disregard the fact that Cartan's commitment to the structural theory of continuous groups

[2]See [Jubilé, p. 49].

must be situated in the appropriate historical context in which his work saw the light. Indeed, an attentive analysis of the motivations at the basis of his researches in this field reveals, at the same time, a marked inclination towards concrete applications[3].

This emerges quite clearly both from his early works on finite continuous groups and from his subsequent studies on infinite continuous ones. Indeed, one of the main driving forces guiding Cartan's first contributions on the structure of finite continuous groups was represented by the wide variety of applications to the theory of differential equations and, in particular, to Lie's integration theory of complete systems of first order linear PDE's. Similarly, the applications to the theory of general systems of PDE's played a major role in orienting Cartan's research priorities in the realm of infinite continuous groups, too.

Furthermore, his peculiar approach to infinite continuous groups provided him with essential technical tools later on to be profitably employed in differential geometry, namely in his method of moving frames. Indeed, the systematic use of exterior differential forms not only turned out to be indispensable for treating infinite groups, but it also provided a reformulation of Lie's theory of finite continuous groups in terms of the so-called Maurer-Cartan forms which proved to be the most suitable one for geometrical applications.

This constant search for a balance between abstraction and application represented a crucial characteristic of Cartan's entire work, all the more so, since this peculiarity of his mathematical activity frequently reflected his natural tendency to develop general (algorithmic) methods which found application in a large variety of specific problems. His general approach to PDE systems by means of exterior forms only, the theory of equivalence of differential structures and the method of moving frames are the most significant and well known examples of such a tendency.

From a certain point of view, one may even say that the real greatness of Cartan's entire work coincides precisely with the generality and the power of his methods and technical tools rather than with specific achievements in a particular branch of mathematics. In this respect, the ubiquitous recourse to exterior differential forms contribute to bestow to his theory a character of particular exceptionality. Such was the uniqueness of his work that Ugo Amaldi, one of the first to recognize its real significance, could declare to him (Cartan) that "there really is a mathematics of Cartan".
We hope that the following pages will contribute to convey an idea of what such mathematics is.

A few words on the reception of Cartan's work among his contemporaries and subsequent generations of mathematicians are necessary.

However surprising it may appear, Cartan's greatness asserted itself with

[3] In this respect, see also [Hawkins 2000, p. 195].

extreme difficulty and only later in his career. In this respect, Dieudonné very appropriately observed:

> *Cartan's recognition as a first rate mathematician came to him only in his old age; before 1930 Poincaré and Weyl were probably the only prominent mathematicians who correctly assessed his uncommon powers and depth. This was due partly to his extreme modesty and partly to the fact that in France the main trend of mathematical research after 1900 was in the field of function theory, but chiefly to his extraordinary originality.*[4]

We will see that this originality, which sometimes resulted in a real radical break with past tradition, lay mainly in the novelty and audacity of the technical tools which he employed. In particular, the ubiquitous recourse to Pfaffian forms should be indicated as a distinctive feature of his work and, at the same time, as one of the causes for its belated recognition. Needless to say, the proverbial obscurity and objective difficulty which readers encountered when studying Cartan's papers did not foster swift acknowledgments to come.

This dissertation does not provide an analysis of Cartan's mathematical production as a whole; rather, it concentrates its focus on a limited period of Cartan's mathematical activity: from his early works on the structure of complex Lie algebras (1893-1894) to the monumental series of papers laying the basis of his structural theory of infinite continuous groups (1902-1910). The first chapter is devoted to conveying a brief sketch of Cartan's biography. It is mainly based on the first chapter of [Akivis, Rosenfeld 1993] and on Cartan's *Dossier Biographique* which is conserved at the Academy of Sciences in Paris. The second chapter deals with the structural theory of finite continuous groups. The main results of Cartan's doctoral dissertation are presented in some detail. Special care is paid to stressing the application oriented character of his researches on this subject.

The third chapter provides a historical survey of the theory of infinite continuous groups. Starting from Lie's pioneering works dating back to the early 1880's, we describe the subsequent developments up to Vessiot's monumental *Mémoire couronnée*. Beyond preparing the ground for the next chapters by providing a detailed contextualization of Cartan's theory, we hope that it may contribute to a more accurate historical understanding of Lie's theory of transformation groups too.

The fourth chapter deals with the origins of the exterior differential calculus and the integration theory of general Pfaffian systems. Special emphasis is paid to describing the scientific milieu in which Cartan's work saw the light. Relevant papers by F. Engel and E. Ritter von Weber which lay the ground for Cartan's subsequent geometrical methods are, for the first time, analyzed.

[4] See [Dieudonné 2008].

The fifth chapter is devoted to Cartan's structural theory of infinite continuous groups. Its genesis and subsequent development is analyzed thoroughly. Cartan's reinterpretation of Lie's equivalence theory is singled out as the main driving force in the genetical process leading to Cartan's structural theory of infinite groups.

The last chapter is devoted to providing some general ideas of Cartan's method of moving frames. Indeed, as will be explained, Cartan's theory of groups, when applied to finite Lie groups, disclosed new insights which turned out to be most suitable for geometrical applications.

Three appendices can be found at the end of the dissertation. Appendix A conveys a very brief survey of the essential elements of Lie's theory of finite continuous groups. The three fundamental theorems are stated in order to facilitate a comparison with Cartan's innovative approach. Appendices B and C are devoted to providing a more accurate description of the historical context in which Cartan's group theory saw the light; the Picard-Vessiot theory and Drach's subsequent attempts to generalize it to non-linear ODE's are dealt with.

A few remarks on methodological issues are in order here. One major problem frequently encountered by historians of mathematics is represented by the challenge consisting of actualizing the works under examination without distorting their original content. On one hand we should resist the temptation to credit the texts with retrospective meanings which are not inherent in the intention of the authors. On the other, we should also try to take advantage of our present mathematical knowledge in order to clarify, when it is necessary, some obscure and difficult points of the texts themselves. All the more so when, as in the present dissertation, we deal with the history of relatively recent mathematical theories.

In the light of these premises, the following methodological choice will be made. As a rule, we will adhere to the original formalism and language (for this reason, for example, we will not distinguish between the notions of Lie group and Lie algebra[5]). We will make recourse to modern translation only if this is regarded as strictly necessary and it does not produce any misunderstanding or misinterpretation of the texts under examination.

Finally I would like to express my gratitude to Prof. Peter Olver for his precious remarks and comments on some parts of this dissertation. Thanks are also due to Mme. Florence Greffe of the Archives of the Academy of Sciences in Paris for her valuable assistance during archival research.

Last but not least, I thank my Ph.D. advisor Prof. Umberto Bottazzini for teaching me the most important thing: the respect that we owe to the

[5]This seems to be a wise decision especially as far as the history of infinite continuous groups is concerned. Indeed, a translation of Lie's achievements in this field in terms of contemporary mathematical language severely jeopardizes the possibility of grasping the intrinsic unity which characterizes his theory of continuous groups, both finite and infinite. See section (3.1).

masters of the past. Working under his guidance has been a great privilege for me. I hope I deserved it.

# Chapter 1

# Élie Joseph Cartan

Mildness of spirit, modesty, discretion and abnegation marked Cartan's temperament throughout his life. Later in his career, when (too late!) honor and unanimous praise for his imperishable contributions to mathematical science were finally acknowledged, he still never abandoned his almost proverbial humbleness. Such a disposition reflected somehow his humble origins which Cartan constantly recalled with pride and sincere affection.

He was born on 9th April 1869, the second of four children, in Dolomieu (Isère), a small village in south-eastern France. His ancestors were peasants. His father, Joseph was the village blacksmith. As Cartan himself later recalled, Cartan's infancy was spent serenely. Devotion for hard work and a marked sense of dignity were his parents' most precious advice. In 1939, at the occasion of the celebration of his 70th birthday, he wrote:

> *My childhood was cradled under blows of the anvil which resounded every morning from dawn; I can still see my mother working with a spinning-wheel during those rare instants when she was free from taking care of the children and the house.*[1]

Equally valuable for forming his character was the influence exerted by teachers at the municipal primary school in Dolomieu. For all his life, Cartan remembered with gratitude and special affection Monsieur Collomb and, above all, Monsieur Dupuis. Since his childhood, Cartan was an excellent schoolboy. Dupuis described Cartan as a shy student whose eyes shone both with the light of intelligence and with the desire of knowledge. One of a numerous class, his intelligence and almost prodigious memory stood out among all the other pupils. As Cartan himself was later to recall he could, without any hesitation, list all subprefectures in each department of France and remember all the subtlest rules governing the past participle.

[1] *C'est au bruit de l'enclume résonnant chaque matin des l'aube que mon enfance a été bercée, et je vois encore ma mère actionnant le métier du canut, aux instants que lui laissaient libres les soins des ses enfants et les soucis du ménage.* [Jubilé, p. 51].

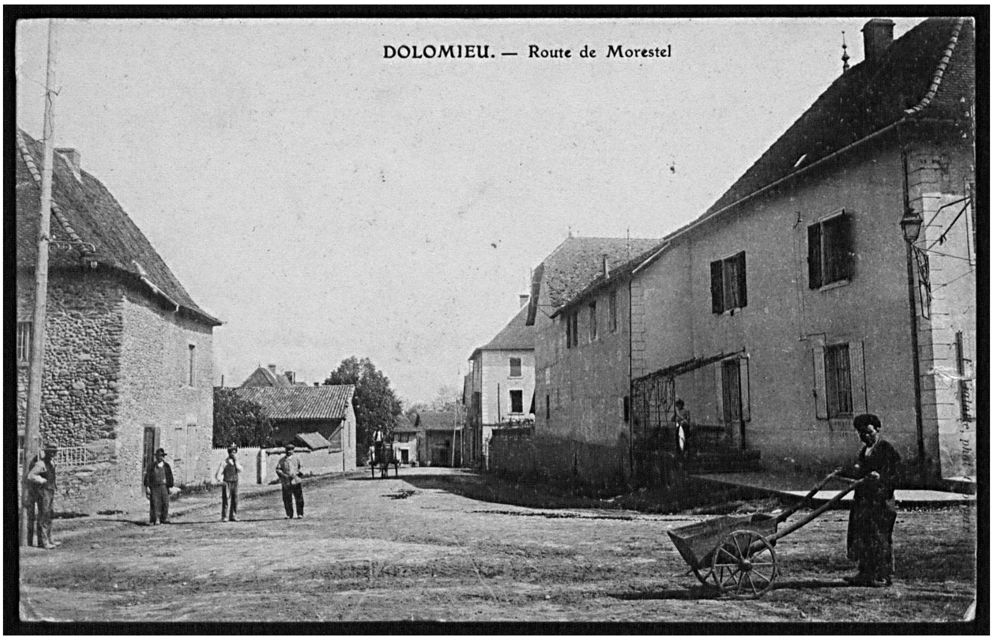


Figure 1.1: Postcard sent by Cartan to F. Engel, 1905, Giessen Archive.

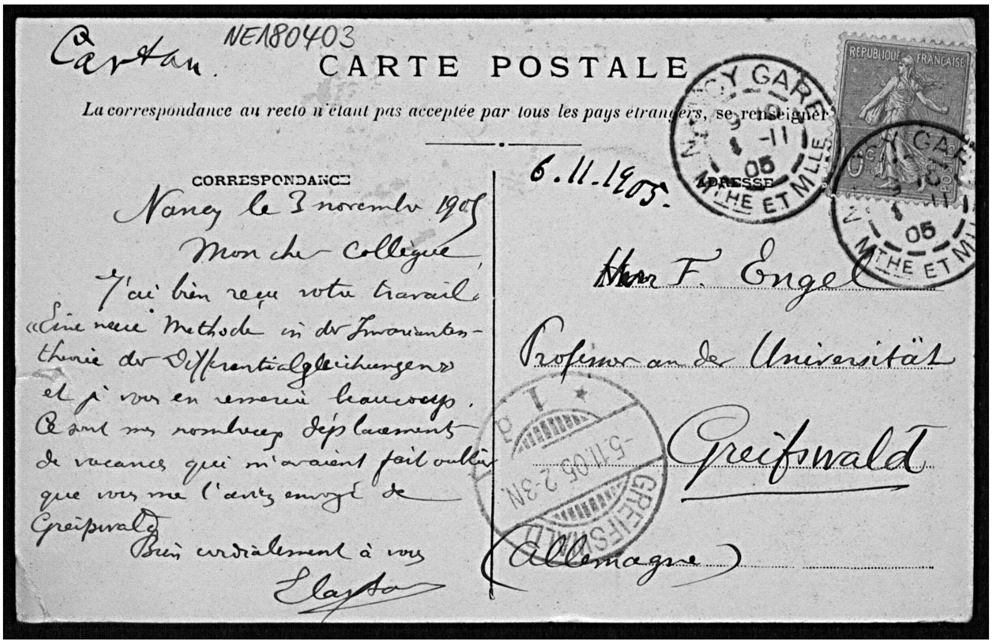


Figure 1.2: Recto of the postcard above, Giessen Archive.

He was equally skilled in orthography, mathematics, sciences, history and geography. The young Cartan, Dupuis concluded, understood everything, even before the teacher had finished his lesson.

Dupuis referred his enthusiastic judgment of Cartan to Antonin Dubost (1842-1921) who came to visit the school of Dolomieu in 1878 in his quality of cantonal inspector. Impressed by such potential, Dubost took Cartan's education to heart and recommended that he apply for a scholarship in a lycée. Monsieur Dupuis took charge of the preparation with unparalleled devotion which Cartan will always recall with gratitude. A brilliant performance in the examination along with the continuing, fatherly concern of Dubost won Cartan a full scholarship in the College de Vienne. Thus, he left Dolomieu at the age of ten. After five years, his scholarship was transferred to the Lycée de Grenoble where he devoted himself to classical studies, namely rhetoric and philosophy. After that, in 1887, Cartan went to the Lycée Janson-de-Sailly in Paris which had been recently inaugurated. Among his comrades, there was Jean Perrin who was destined to become an eminence of French physics.

In 1888, only one year after his entrance in the Lycée, Cartan was admitted to the École Normale Supérieure. Although his rank at admission was not what one might have expected, he soon revealed his real talent. As Tresse, one of his comrades, was later to recall on the occasion of Cartan's scientific jubilee, Cartan rapidly took the place that he deserved: the first in the mathematical section. His opinion in mathematical controversies was the most respected among students; his unremitting diligence became almost proverbial. As Tresse himself recounted, Cartan used to sit in the place nearest to the entrance door of the reading room. The shorter the path he had to walk, the longer he could stay there to study.

During these years, Cartan was able to attend courses from the most prestigious mathematicians in France at that time who had the talent to move in few words from the most elementary notions of Analysis to the most recent problems of mathematics. Amongst others, there were Picard, Tannery, Goursat, Appell, Koenigs, Darboux, Hermite and Poincaré. Cartan remembered with special affection Jules Tannery who seems to have exerted the greatest influence upon his students. Reminiscing, Cartan said:

> *At the École Normale Jules Tannery exerted upon us the most profound influence; as a consequence of a mysterious transposition due to his whole person and possibly to his glance, the respect for rigour, whose necessity in mathematics he constantly emphasized, almost became a moral virtue, frankness, loyalty and respect themselves. Tannery was our conscience.*[2]

[2] *A l'École Normale c'est Jules Tannery qui exçerca sur nous la plus profonde influence; par une sorte de transposition mystérieuse due à l'ensemble de toute sa personne, à son regard peut-être, le respect de la rigueur dont il nous montrait la nécessité en mathématiques*

Lectures by Hermite on elliptic functions and by Darboux on the geometry of curves and surfaces made a big impression on Cartan's education. Of Hermite he once said:

> *Every time I had a chance to listen to Hermite, I had before me an image of quiet and pure joy caused by contemplations about mathematics, joy similar to the one that Beethoven must have felt while feeling his music inside of himself.*[3]

Equally eulogistic were Cartan's recollections of Darboux who exerted a great influence upon Cartan's method of moving frames. Cartan wrote:

> *Gaston Darboux was an analyst and geometer at the same time.* [...] *He surely was not one of the geometers who avoided tarnishing the beauty of geometry by flattering analysis, and neither was he one of the analysts inclined to reduce geometry to calculations without any concern or interest in their geometric meanings. In this respect he followed in Monge's footsteps, connecting fine and well-developed geometric intuition with skilled applications of analysis. All of his method are extraordinarily elegant and perfectly suited for the subject under investigation. While teaching in the department of higher geometry at the Sorbonne, where he succeeded Michel Chasles, he frequently and with reverence spoke about the theory of triple orthogonal systems, with pleasure stressing the importance of Lamé's works.* [...] *Darboux had tremendous influence on the development of geometry.* [...] *Classic in its field, Darboux's work* Théorie des Surfaces *is a splendid monument erected in honor of both analysis and geometry.*[4]

Not less influencing were the lectures by Poincaré on electrodynamics which Cartan attended at the Sorbonne. A giant to whom every field of modern mathematics was somehow indebted, as Cartan described him, Poincaré always showed a great interest in Cartan's career. He seems to have been the first person to recognize Cartan's true greatness and depth, still when Cartan's works were scarcely read and poorly known. Shortly before his premature death in 1912, Poincaré devoted his last contribution to Science to a survey of Cartan's already wide mathematical production (published two years later in *Acta Mathematica*) which finally won Cartan due acknowledgement and fame, by paving the way to Cartan's appointment as full professor in the Sorbonne in 1912.

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*devenait une vertu morale, la franchise, la loyauté, le respect de soi-même. Tannery était notre conscience.* See [Jubilé, p. 53].

[3]See [Akivis, Rosenfeld 1993, p. 297].

[4]*Ibidem.*

Cartan graduated from the École Normale in 1891. Subsequently, he served in the army for one year where he achieved the rank of sergeant. On his return to Paris, Cartan immersed himself in a period (1892-1893) of intense study of the structure of (complex) finite Lie algebras. Such a fortunate choice was thanks to A. Tresse. Indeed, upon his return from Leipzig where he had spent some months to study under the guidance of S. Lie and F. Engel, Tresse informed Cartan of W. Killing's recent researches on the classification of Lie groups and the necessity to emend them from some erroneous statements. It was during these years of hard work, when the most part of his time was spent in the library studying Killing's papers that Cartan had the occasion to meet Lie for the first time. At the invitation of Darboux and Tannery, Lie came to Paris in the Spring of 1893. He had known of Cartan from one of his student, Georg Scheffers who, in March of the same year, had informed him that a friend of Tresse, whose name was Cartan, was working on a thesis on Killing's theory. Although, as is observed in [Hawkins 2000, p. 198], it seems that Lie's direct influence on Cartan's thesis was marginal, his meeting with Lie must have been very inspiring. In this respect Cartan wrote:[5]:

> *In 1892 Sophus Lie came to Paris for six months. With great benevolence he became interested in the researches that young French mathematicians devoted to group theory. One could frequently see him around a table of the Café de la Source, in boulevard Saint-Michel; the top of the white marble table was often covered in hand-written formulas which the illustrious master wrote down in order to illustrate the exposition of his ideas. The indifference with which his early works had been embraced, or better ignored, by most mathematicians, now gave way to admiration. Most academies, with the exception of that of Berlin, held it dear to count him among their members. Over his stay in Paris, on 7th July 1892, the Academy of Sciences of Paris appointed him correspondent member in the section of Geometry.*[6]

[5]Cartan mistakenly put Lie's visit to Paris in 1892 instead of 1893. Contrary to what Cartan said, Lie's stay in Paris was only three weeks long. In this respect see [Stubhaug 2002, p. 373-374].

[6]*En 1892 Sophus Lie vient passer six moins à Paris, s'interessant avec une grande bienveillance aux recherches que de jeuns mathématiciens français consacraient à la théorie des groupes; on pouvait le voir souvent autour d'une table du Café de la Source, boulevard Saint-Michel; il n'était pas rare que le dessus de marbre blanc de la table fût couvert de formules au crayon que le maître écrivait pour illustrer l'exposé des ses idées. L'indifférence avec laquelle ses premiers traveaux avaient été accueillis, on peut même dire ignorés, de la plupart des mathématiciens avait maintenant fait place à l'admiration; la plupart des Académies, sauf toutefois celle de Berlin, avaient tenu à le compter parmi leurs membres. Pendant son séjour à Paris, le 7 juin 1892, l'Académie des Sciences de Paris se l'attachait comme membre correspondant dans la séction de Géométrie.* Fonds É. Cartan, 38J, 8.21, Académie des Sciences.

After more than two years of unremitting work, in March 1894, Cartan successfully defended his doctoral dissertation in front of an examining jury composed of Darboux, Hermite and Picard.

That same year, he was appointed to the University of Montpellier where he lectured from 1894 to 1896. Then, from 1896 to 1903 he was lecturer in the Faculty of Sciences in Lyon. While in Lyon, he married Marie-Louise Bianconi (1880-1903), a young woman of Corsican origin, who was destined to be Cartan's beloved partner for his entire life. In 1903 Cartan moved to Nancy where he became professor in the local university. In 1904, first of four children, Henri (1904-2008) was born. He worked in Nancy until 1909 where he also taught elements of analysis at the Institute of Electrical Engineering and Applied Mechanics. It was in Nancy, as he recalled years later, that he first got used to a large audience.

These years (1894-1909) of scientific isolation, which were spent away from Paris, undisputed center of the French mathematical activity, were nonetheless very fruitful and crucial to the development of Cartan's mathematical thought. Indeed, over these years, he conceived great part of the technical tools which, later on, would be applied by him to different realms of mathematics, namely differential geometry. It was during this period that Cartan developed his theory of exterior differential systems and his peculiar approach to continuous groups of transformation which will be dealt with in this work. In his words:

> [...] *I look at the best memory of the fifteen years I spent in the provinces, first in Montpellier, then in Lyon and afterwards in Nancy. These were years of meditation in the calm, and the germs of all I subsequently did are contained in my works in this period, pondered over at length.*[7]

In 1909 he moved to Paris where he was lecturer at the Faculty of Sciences in the Sorbonne before obtaining a full professor position in 1912, at the age of 43. Cartan also taught at the École Normale Supérieure where he exercised great part of his academic activity. Before teaching higher geometry (in 1924, he was appointed to the Chair which had been Darboux's), he had taught analytical mechanics and potential theory too.

His move to Paris marked a turning point in his career. Supported by the enormous applicative potential both of his integration theory of Pfaffian forms and of his structural theory of continuous Lie groups, Cartan's research interests gradually shifted to the realm of geometry. Still for some years, abstract group theory and the vast field of partial differential equations constituted a driving force for his mathematical activity, nonetheless,

[7] [...] *Je garde le meilleur souvenir des quinze ans que j'ai passés en province, à Montpellier d'abord, à Lyon, et à Nancy ensuite. Ce furent des années de méditation dans la calme, et tout ce que j'ai fait plus tard est contenu en germe dans mes travaux mûrement médités de cette période.* [Jubilé, p. 54].

starting from the late 1910's, the analyst and the algebraist rapidly gave way to the geometer.

It was precisely from his contributions to this branch of mathematics, rather than from his previous, almost disregarded achievements, that due acknowledgment for his work finally came. In particular, it was his groundbreaking approach in dealing with generalizations of the notion of space which won him unanimous distinction as undisputed master of differential geometry.

Though reluctant to build up a mathematical school around his figure (S.S. Chern and Chevalley in [Chern and Chevalley 1952, p. 217] pointed out that Cartan "had too much of sense of humor to organize around himself the kind of enthusiastic fanaticism which helps to form a mathematical school"), it seems that the influence which Cartan's lectures, both at the Sorbonne and at the École Normale, produced was deep. Here follows a vivid picture of his teaching qualities drawn by Gaston Julia on the occasion of Cartan's scientific jubilee in 1939:

> *While your students discerned, at a glance, the eminent algebraist that you are, they discovered the real nature of geometer in you more gradually.* [...] *You endeavored, I dare to say, to foresee* a priori*; and when the result of a calculation was simple, you philosophically taught us to foresee it* a posteriori*, as geometers say, that is to illustrate formulas by means of simple and striking geometrical facts which express their profound truth and confer to them the elegance which provide geometrical question with their veritable beauty.*[8]

Particularly appreciated was Cartan's benevolence towards the new generations of mathematicians who saw him as a source of inspiration for their studies. Willing to share his ideas with young researchers, as is witnessed, for example, by his participation in one of the well known Julia Seminars in 1937, still later in his career Cartan proved to be open-minded and always interested in the most recent changes in mathematics. A young J. A. É. Dieudonné (1906-1992) described this virtue of his character with the following words directly turned to Cartan:

> *What I have been able to appreciate in the most direct way is your spirit which is constantly open and ready to assimilate the*

[8] *Tandis que vos élèves discernaient au premier coup d'œil l'éminent algébriste que vous êtes, ils découvraient plus lentement en vous la vraie nature de géomètre.* [...] *Chez vous on s'éfforçait de prévoir, vraiment* a priori, *si j'ose dire; et lorsque le résultat d'un calcul était simple, vous nous enseigniez philosophiquement à le prévoir* a posteriori, *comme disent les géomètres, c'est-à-dire à l'expliquer, ou encore à illustrer les formules par des faits géométriques simples et frappants qui en expriment la réalité profonde, et par où elles acquièrent une élégance qui confère aux questions géométriques leur veritable beauté.* [Jubilé, p. 41].

> *most recent acquisitions of Science, in order to better them. With you, there has never been the break of contact with new generations. The years have passed, whitening your head. Nonetheless, thanks to your enthusiasm for your work, your ever-awakened curiosity, the unremitting blossoming of your work in all directions, constantly renewed, you are young and you understand the young.*[9]

After more than forty years of academic activity, Cartan retired in 1940. For some years afterwards, he continued his researches, publishing his last mathematical paper in 1949, two year before his death in May 1951.

Over the last years of his long career, numerous acknowledgements for his contributions finally came. He received honorary degrees from Universities all over the world. From the University of Liege in 1934 and from Harvard University in 1936. In 1947 he was awarded three honorary degrees from the Free University of Berlin, the University of Bucharest and the Catholic University of Louvain. In the following year he was awarded an honorary doctorate by the University of Pisa. He was elected member of the Royal Society of London, the Accademia dei Lincei of Rome and the Norwegian Academy. In 1931 he had been elected member of the Academy of Sciences of Paris to which he was appointed president in 1946.

Cartan had erected an enormous mathematical edifice which was destined to exert a deep and long lasting influence on a disparate number of disciplines: Lie group theory, representation theory, PDE's theory, differential geometry, topology, etc..

Ugo Amaldi's remarks, conveyed on the occasion of Cartan's commemoration at the *Accademia dei Lincei* on 14th June 1952, represent a most precious description of Cartan's inheritance:

> *The sovereign talents of His mathematical genius shine* [in Cartan's works]*: his originality in posing problems, his ability to single out the conceptual essence of every analytical method, his exceptional inventiveness and, above all, his dynamical conception of Science which, as already observed, allowed him never to stiffen his contributions into definite doctrines, but, on the contrary, to concretize them in live and viable movements which are destined to exert, in the future too, a powerful influence of ideas and suggestions on different realms of mathematics.*[10]

[9] *Ce que j'ai pu apprécier de la façon plus directe, c'est votre esprit toujours ouvert, toujours prêt à assimiler les plus dernières acquisitions de la Science, pour les dépasser aussitôt. Chez vous il n'y a jamais eu la moindre rupture de contact avec les générations montantes; les années ont pu passer, et blanchir votre tête;* [mais] *par votre ardeur au travail, votre curiosité toujours en éveil, par la floraison ininterrompu de votre œuvre dans des directions toujours renouvelées, vous êtes un "jeune" et vous comprenez les jeunes.* See [p. 49][Jubilé].

[10] *Rifulgono* [in Cartan's works] *le doti sovrane del Suo genio matematico: l'originalità*

*nella posizione dei problemi, l'attitudine a cogliere l'essenza concettuale di ogni metodo analitico, la sorprendente fantasia inventiva e, soprattutto, quella visione dinamica della Scienza, per cui, come già fu rilevato da altri, gli apporti del Cartan non si sono mai irrigiditi negli schemi di dottrine in sè concluse, ma sempre si sono concretati in movimenti d'idee vivi e vitali, destinati ad esercitare, anche in futuro, un potente influsso di spunti e suggestioni in vasti campi della Matematica.* See [Amaldi 1952, p. 773].

# Chapter 2

# Cartan's doctoral dissertation

## 2.1 Finite continuous groups

Cartan entered the mathematical scene in the early 1890's by dealing with Lie's theory of finite (complex) continuous groups. The subject was a very popular one in Paris at that time, especially in view of its fruitful applications to the theory of differential equations. Indeed, Lie's theory of continuous groups had been welcomed in Paris with unparallel favour anywhere else in Europe.

The origins of the strong connection between Lie and the Parisian mathematical community, which very appropriately has been defined as a sort of symbiotic relationship[1], can be traced back to the dawn of Lie's career. Since his first journey to Paris in 1870 together with F. Klein, Lie had found there a stimulating mathematical habitat which he greatly appreciated. During his stay, he had made acquaintance with G. Darboux, who, he said, "understood him best of all"[2], and C. Jordan whose attempts to rigorize and systematize Galois' ideas had undoubtedly exerted upon him a deep influence. On the occasion of another stay in the French capital in 1882, when his researches on continuous groups of transformations were already well under way, Lie had then met other leading figures of the mathematical Parisian *milieu* such as Émile Picard (1856-1941) and Henri Poincaré (1854-1912) who were soon convinced of the enormous importance of his contributions.

For example, in a letter to Lie dating back to 1888, Picard wrote:

> *You have created a theory of major importance that will be counted among the most remarkable works of the second half of this century* [...]. [3]

[1] See [Hawkins 2000, p. 184].

[2] See [Hawkins 2000, p. 184].

[3] *Vous avez créé là une théorie d'une importance capitale, et qui comptera comme une*

Poincaré too was highly generous in his praise of Lie's theory, as is shown, by a letter in January 1888 to Lie where he emphasized its relevance, in relation to the applications to the theory of functions. After all, this favourable attitude should come as no surprise given the well known role that the notion of group plays in Poincaré's entire mathematical production. Indeed, as we learn from a letter of Lie's to Klein, already in 1882 Poincaré was to confess to him (Lie) that all mathematics was a tale about groups (*Gruppengeschichte*).

As a result of such widespread interest towards Lie's works, many graduate students of the *École Normale* were encouraged to devote themselves to the study of Lie's theory and even to spend some time in Leipzig, where Lie had been teaching higher geometry since 1886, in order to hone their knowledge of the subject. Among the *normaliens* directly involved, W. de Tannenberg, É, Vessiot, A. Tresse and, a little later, Jules Drach should be mentioned. In particular, according to Cartan's subsequent recollections, it was Tresse, upon his return to Paris in 1892, to inform him of the interest generated in Leipzig by Killing's works on the structure of finite continuous groups.[4]
Starting from 1888, Wilhelm Killing (1847-1923), an almost unknown professor at the Lyceum Hosianum in Braunsberg, East Prussia, published a series of four papers entitled *Die Zusammensetzung der stetigen endlichen Transformationsgruppen* in which he had provided a large part of the essential elements of what in modern terms we would call "structural theory of complex Lie algebras". Killing's approach to continuous groups of transformations was marked by a sharp distinction with respect to Lie's, both for what pertains the underlying motivations and the technical tools which he employed. Well before 1884 when he first learnt of the existence of Lie's theory from a letter sent by F. Klein, Killing had indeed introduced the notion of *space form* which was later recognized to be analogous to that of continuous group of transformations. The context was represented by Killing's researches on the foundations of non-Euclidean geometries. This was a topic (quite popular in Germany at that time) in which he had been interested since 1872 when, in Berlin, he had the occasion to attend Weierstrass' lectures on the foundations of geometry. Over the following years, the problem of classifying all different space forms (or, continuous groups) had then attracted his attention as a reflection of his general project of classifying all conceivable types of geometry. This led him to develop a structural approach to the classification theory of (complex) continuous groups which, unlike Lie's more rudimentary treatment, took great advantage of the new trend of rigour inaugurated by Weierstrass, among other fields, also in linear

*des oeuvres mathématiques les plus remarquables de la seconde moitié de ce siècle* [...]. Translation taken from [Hawkins 2000, p. 188].

[4]The best and most detailed account of Killing's works is provided by [Hawkins 2000, chap. 4-5].

algebra.

In this respect, the exceptional character of Killing's activity, among those of other mathematicians trained in Berlin, should be stressed. Possibly as a consequence of his distance from the academic routine, Killing was able to exploit all the advantages (commitment for rigour and non-generic reasoning) stemming from his Berliner education without, for this very reason, sharing the censor attitude with which Lie's work was welcomed there. The result was a fortunate marriage between Weierstrassian style and Lie's fertile ideas.

Killing's achievements represent a real landmark in the history of complex Lie algebras. Indeed, not only was he able to introduce all the essential ingredients of the modern theory, but also to convey a first version of the classification of all simple Lie algebras which included the exceptional cases too.

Cartan was soon convinced of the importance of taking up Killing's researches with the aim of providing them with the rigour and exactness in which lacking. Indeed, as had already been pointed out by F. Engel and a student of his in Leipzig, K. Umlauf, en route for his classification enterprise, Killing had relied upon some incorrect results which severely conditioned the tenability of his approach. In this respect, Cartan's remarks contained in [Cartan 1893a] are quite enlightening. After recalling Killing's main result on the classification of all finite simple continuous groups, he wrote:

> *Unfortunately, the considerations which led Mr. Killing to these results are lacking in rigour. It was thus desirable to take up these researches, to single out their incorrect results and to demonstrate their correct theorems.* [...] *In the part of Mr. Killing's memoir where simple groups are dealt with, two major lacunae can be found. Firstly, he does consider the case in which what he calls the characteristic equation of the group admits simple roots only. It is true that he tries to set himself free of this limitation in the third part of his memoir. However, by doing so, he relies upon a theorem which he proves only in particular case and which turns out to be false in general: that, if the group coincides with its derived group, then every general transformation belongs to a subgroup consisting of commuting transformations and the characteristic equation admits identically vanishing roots. Secondly, he traces back the determination of simple groups to the determination of certain systems of integer numbers, but he does not prove at all that all the roots of the characteristic equation depend upon one of these systems only.*[5]

[5] *Malheureusement, les considérations qui conduisent M. Killing à ces résultats manquent de rigueur. Il était, par suite, désiderable de refaire ces recherches, d'indiquer ses théorèmes inexacts et de démontrer ses théorèmes justes.* [...] *Dans la partie du Mémoire*

Actually, Cartan did much more than simply clarify some problematic aspects of Killing's analysis. Especially for the case of semisimple groups, his analysis was far more systematic and simpler with respect to Killing's. It is certainly true that the large majority of the technical tools employed by him had already been introduced by Killing himself (characteristic equation, the so-called Cartan subalgebras and Cartan integers, etc.). However Cartan's approach was characterized by a high degree of originality especially in virtue of the strategic role attributed by him to the quadratic form $\psi_2(e)$ (to be defined in the next section) in order to provide criteria for integrability and semisimplicity.

Furthermore, it seems that one of Cartan's great merit consisted of his capability of finding a match point between two distinct tendencies which, until then, had characterized the theory of finite continuous groups. On one hand, the applicative tendency had been predominant in Lie's attempts to develop a Galois theory of differential equations[6]; on the other, the structural one had been far more prevalent in Killing's researches which laid the algebraic grounds of the theory. Cartan indeed was able to mediate between them by investigating the possibility of applying Killing's results (along with essential improvements) to the integration theory of partial differential equations.[7]

This emerges quite clearly in numerous passages of Cartan's doctoral thesis, for example when he introduced the notion of *maximal, integrable, invariant subgroup* (in modern terms, the radical of a Lie algebra) and when he tackled the study of homogenous, linear, simple groups in $n$ variables. In particular, as will be seen, Cartan was able to provide a formula by whose means the radical of a group could be computed in an easy way. Such an achievement was an important one in view of the applications of group theory to the integration of differential equations. Indeed, it provided a general procedure to compute a normal decomposition series of a given (not semisimple) group $G$.

Cartan's thesis was divided into three parts. The first one consisting of three chapters introduced the basic elements of the theory. After giving

*de M. Killing relative aux groupes simples se trouvent surtout deux lacunes importantes. En premier lieu, il ne considère que le cas où ce qu'il appelle l'équation caractéristique du groupe n'admet que des racines simples; il tente, il est vrai, de s'affranchir de cette restriction dans la troisième partie de son Mémoire, mais il s'appuie pour cela sur un théorème qu'il ne démontre que dans un cas particulier et qui, en général est faux: à savoir que si un groupe est son propre groupe dérivé, chaque transformation générale fait partie d'un sous-groupe formé d'autant de transformations échangeables entre elles, que l'équation caractéristique admet des racines identiquement nulles. En second lieu, il ramène la détermination des groupes simples à la détermination de certains systèmes de nombres entiers, mais il ne prouve pas du tout que toutes les racines de l'équation caractéristique ne dépendent que d'un seul se ces systèmes.* See [Cartan 1893a, p. 785-786].

[6] A concrete example of Lie's project will be provided when Lie's theory of complete systems is dealt with.

[7] For a more detailed discussion over this point, see [Hawkins 2000, §6.1].

a sketchy account of Lie's theory of continuous groups, in view of his own structural approach, Cartan put special emphasis on the notion of adjoint group as well as on the relationships between invariants of the adjoint group and invariant subgroups of a given group $G$. Then, all the fundamental notions of Killing's structural theory were introduced: characteristic equation, roots, Cartan's subalgebras, that which nowadays we would call decomposition of a Lie algebra in generalized eigenspaces, etc.. Finally, in the third chapter, Cartan dealt with the theory of integrable groups by providing his well known criterion in terms of a certain vanishing property of the quadratic form $\psi_2$ (the so-called Cartan-Killing form).
The second part was devoted to semisimple and simple groups. Cartan derived his semisimplicity criterion in terms of the non-degeneracy of $\psi_2$ and, by doing so, he was able to rigorize some of Killing's unprecise statements. He then set out to tackle the problem of determining all simple groups by completing Killing's monumental classification.

The third and last part was finally devoted to the study of groups which are not semisimple. The notion of radical was introduced in a very clear way at the beginning of his analysis. It provided Cartan with the essential tools for developing, in a more rigorous way, Killing's theory of secondary roots which was exploited by him in numerous applications such as the already mentioned derivation of an explicit (algebraic) formula for the radical of a given group $G$.

### 2.1.1 Reduced form of a given group

After providing a sketchy account of Lie's theory of continuous groups[8], Cartan started in the second chapter of [Cartan 1894] by writing down the *characteristic equation* of a group $G$. Although it had been introduced for the first time by Lie[9], it was only with Killing's work that its crucial role for the structural theory of continuous groups was finally acknowledged. Since then, it had been considered as a standard technical tool.

Cartan derived the characteristic equation of $G$ by searching for an infinitesimal transformation $\sum_{k=1}^{r} \lambda_k X_k(f)$ which is left invariant by a given infinitesimal transformation $\sum_{k=1}^{r} e_k X_k(f)$, that is such that

$$\left[\sum_{k=1}^{r} e_k X_k(f), \sum_{k=1}^{r} \lambda_k X_k(f)\right] = \omega \sum_{k=1}^{r} \lambda_k X_k(f).$$

A necessary and sufficient condition for the existence of such an infinitesimal

[8]It is suggested that the reader consult Appendix A in order to gain familiarity with the essential notion of Lie theory.

[9]See [Cartan 1894, p. 23].

transformation is that $\omega$ is a root of the *characteristic equation*:

$$\Delta(\omega) = \begin{vmatrix} \sum_{i=1}^r e_i c_{i11} - \omega & \sum_{i=1}^r e_i c_{i21} & \cdots & \sum_{i=1}^r e_i c_{ir1} \\ \sum_{i=1}^r e_i c_{i12} & \sum_{i=1}^r e_i c_{i22} - \omega & \cdots & \sum_{i=1}^r e_i c_{ir2} \\ \cdots & \cdots & \cdots & \cdots \\ \sum_{i=1}^r e_i c_{i1r} & \sum_{i=1}^r e_i c_{i2r} & \cdots & \sum_{i=1}^r e_i c_{irr} - \omega \end{vmatrix} = 0. \tag{2.1}$$

By developing the determinant above, one obtains:

$$(-1)^r \Delta(\omega) = \omega^r - \psi_1(e)\omega^{r-1} + \psi_2(e)\omega^{r-2} + \cdots + (-1)^{r-1}\psi_{r-1}(e)\omega = 0, \tag{2.2}$$

where the coefficients $\psi_i(e)$ are homogeneous polynomials in $e_i$, $i = 1, \cdots, r$ which are obtained by summing all the principal minors of order $i$ of the matrix $A = [\sum_k e_k c_{kij}]$. The first two of them play a particularly important role in the theory. They have a relatively simple expression: $\psi_1(e) = \sum_{ik} e_i c_{ikk}$, $2\psi_2(e) = \psi_1^2(e) - \sum_{ijkh} e_i e_j c_{ikh} c_{jhk}$.

Killing introduced the characteristic equation in the context of the computation of the differential invariants of the adjoint group. Indeed, as was first proved by Killing himself, it turns out that the coefficients $\psi_i(e)$, $i = 1, \cdots, r-1$ are invariants of the adjoint group. Killing proved such a fundamental property by a step by step procedure which consisted of studying what happens with every coefficient $\psi_i(e)$ separately. Engel proposed a general proof which was preferred by K. Umlauf and Cartan himself. The result was obtained as a consequence of the following chain of equalities

$$E_\mu \Delta = \sum_{i,j} E_\mu \gamma_{ij} \cdot \frac{\partial \Delta}{\partial \gamma_{ij}} = \sum_{\omega,s,i,j} e_\omega c_{\omega\mu s} c_{sij} \frac{\partial \Delta}{\partial \gamma_{ij}},$$

where Cartan posed $\gamma_{ij} \equiv \sum_{s=1}^r e_s c_{sij} - \delta_{ij}\omega$. Infact, from this equation, since:

$$\begin{cases} c_{iks} + c_{kis} = 0 \\ \sum_{\rho=1}^{\rho=r} (c_{ik\rho} c_{h\rho s} + c_{kh\rho} c_{i\rho s} + c_{hi\rho} c_{h\rho s}) = 0 \\ (i,k,h,s = 1, ..., r) \end{cases}$$

one obtains:

$$\begin{aligned} E_\mu \Delta &= \sum_{\omega,s,i,j} e_\omega (c_{\mu is} c_{\omega sj} + c_{\omega is} c_{s\mu j}) \frac{\partial \Delta}{\partial \gamma_{ij}} \\ &= \sum_{s,i,j} c_{\mu is} (\gamma_{sj} + \delta_{sj}\omega) \frac{\partial \Delta}{\partial \gamma_{ij}} - \sum_{s,i,j} c_{\mu sj} (\gamma_{is} + \delta_{is}\omega) \frac{\partial \Delta}{\partial \gamma_{ij}} \\ &= \left( \sum_i c_{\mu ii} - \sum_j c_{\mu jj} \right) \Delta = 0. \end{aligned}$$

The study of the properties of the coefficients of the characteristic equation was of central importance for the development of a structural approach to continuous groups. Indeed, a first classification criterion was provided by the number $l$ of independent coefficients. Killing called it the *rank* of the group. As Killing himself realized, in general, the rank $l$ is not equal to the number of identically null roots of the characteristic polynomial (i.e. the modern definition of rank of Lie algebra). Actually, as is easy to prove, $l \leq k$. Let us look at Cartan's version of the theorem which was very elegant and concise.

By posing $\gamma^0_{\mu i} \equiv \sum_{s=1}^{r} e_s c_{s\mu i}$, Cartan observed that the system of first order linear differential equations:

$$E_\mu(f) = \sum_{i=1}^{r} \gamma^0_{\mu i} \frac{\partial f}{\partial e_i} = 0, \qquad \mu = 1, \cdots, r, \tag{2.3}$$

is complete as a consequence of the fact that the infinitesimal transformations $E_\mu(f)$ generate a continuous group. If one indicates the rank of the matrix $[\gamma^0_{\mu i}]$ with $t$, then Clebsch's theorem[10] guarantees the existence of $r-t$ independent solutions in such a way that every other solution of the system can be expressed as a functions of them. Since the coefficients $\psi(e)$, $i = 1, \cdots, r-1$ are solutions to (2.3) too (indeed, they are invariants of the adjoint group), the rank of the group $l$ can be, at most, equal to $r-t$, i.e. $l \leq r-t$. Furthermore, since $\psi_i$, $i = 1, \cdots, r-1$ are obtained (modulo a sign) as the sum of all the principal minors of order $i$ of the matrix $[\gamma^0_{\mu i}]$, Cartan deduced that

$$\psi_{r-1}(e) = \psi_{r-2}(e) = \cdots = \psi_{r-t+1}(e) = 0.$$

As a consequence of the last set of equations, the number of identically vanishing roots of the characteristic polynomial is $r-t$ and thus the statement of the theorem follows.
On the basis of these premises, Cartan moved to the introduction of what nowadays are known as Cartan subalgebras. Again, despite present wording, the notion was first employed by Killing as a way of simplifying the structure constants of the group. Cartan tackled the matter in the following way. In general, he observed, to every non vanishing root $\omega_0$ of the characteristic equation relative to a given infinitesimal transformation $X_1(f)$, there always corresponds an infinitesimal transformation $X_2(f)$ such that $[X_1, X_2] = \omega_0 X_2(f)$. However, the case of null roots was less trivial. Cartan first proved the following lemma which made recourse to an intuition dating back to Lie.

**Lemma 1 (Cartan 1894)** *Let $F$ indicate a homogenous invariant of degree $m$ in $e_1, \cdots, e_r$ of the adjoint group $\{E_\mu(f)\}_{\mu=1}^r$. Suppose that the rank of the matrix $[\gamma^0_{\mu i}]$ is equal to $r-1$ for some transformation $e^0_1 X_1(f) +$*

[10] See section (4.1) for the statement of the theorem.

$e_2^0 X_2(f) + \cdots + e_r^0 X_r(f)$, *then there exists a constant* $\rho$ *such that*

$$mF = \rho\psi_{r-1}(e).$$

*As a consequence of this, if an infinitesimal transformation* $X = \sum e_i^0 X_i(f)$ *is such that* $\psi_{r-1}(e^0) \neq 0$ *and that* $rk[\gamma_{\mu i}^0(e^0)] = r-1$, *then* $X$ *annihilates* all *the non constant invariants of the adjoint group.*

The lemma was exploited to prove the existence of a transformation, say $X_2$, linearly independent of $X_1$ such that $[X_1, X_2] = 0$. Indeed, if $X_1$ is a transformation whose characteristic polynomial admits $k$ $(1 < k < r)$ vanishing roots, then $X_1$ does not annihilate $\psi_{r-k}$ (otherwise there would be a greater number of null roots). Consequently, in virtue of lemma (1), $X_1$ is such that $rk[\gamma_{\mu i}^0] \leq r-2$ and there exists a transformation $X_2$, linearly independent of $X_1$, such that $[X_1, X_2] = 0$.

This was an essential observation for the construction of "Cartan subalgebras". More explicitly, by supposing that $k$ is the number of identically vanishing roots of the characteristic equation and by considering a *general* (nowadays we would say, regular) transformation $X_1$, i.e. such that $\psi_{r-k}(e) \neq 0$, Cartan was able to construct the generalized eigenspace associated with the null root of the characteristic equation. He showed that to every general transformation $X_1$ one could associate $k-1$ independent transformations $X_2, \cdots, X_k$ such that the following commutation relations hold:

$$\begin{cases} [X_1, X_2] = 0, \\ [X_1, X_3] = c_{131}X_1 + c_{132}X_2, \\ \vdots \\ [X_1, X_k] = c_{1k1}X_1 + c_{1k2}X_2 + \cdots + c_{1kk-1}X_{k-1}. \end{cases} \tag{2.4}$$

As already proved by Killing, it turned out that the set $\{X_j\}_{j=1}^k$ generates a subgroup $\gamma$ (in modern terms, a subalgebra) which is characterized by two outstanding properties. Firstly, the rank of $\gamma$ is equal to zero as was shown by Cartan by computing the characteristic polynomial of $\gamma$ which, in this case turned out to be equal to $(-1)^k\omega^k$. This property can nowadays be expressed by saying that $\gamma$ is a *nilpotent*[11] Lie algebra. Secondly, there exists a family of infinitesimal transformations $X_{k+1}, \cdots, X_r$ independent among themselves and independent of $\gamma$ too, which is left invariant by every transformation of $\gamma$. In modern terms, this means that $\gamma$ is equal to its

[11]The present day definition follows. A Lie algebra $\mathcal{G}$ is said to be nilpotent if the so-called lower central series boils down to zero. Recall that the lower central series $\mathcal{G}, \mathcal{G}^1, \cdots, \mathcal{G}^r$ is inductively defined by: $\mathcal{G}^1 = [\mathcal{G}, \mathcal{G}], \cdots, \mathcal{G}^{r+1} = [\mathcal{G}, \mathcal{G}^r]$. The fact that the rank of $\gamma$ is zero is tantamount to the nilpotency of the operator $adX$, for every $X \in \gamma$. Nilpotency of $\gamma$ can nowadays be interpreted as a consequence of Engel's theorem whose statement reads as follows: *If* $\mathcal{G}$ *is a Lie algebra such that all operators* $adX$, $X \in \mathcal{G}$ *are nilpotent, then* $\mathcal{G}$ *is nilpotent.* See [Samelson 1989, p. 19]. For a thorough historical account of the subject, see [Hawkins 2000, p. 176-177].

own *normalizer*[12]. We thus obtain the modern definition of what a Cartan subalgebra is; indeed, it is defined precisely as a nilpotent subalgebra which coincides with its own normalizer.

Given these premises, Cartan set out to provide a thorough study of the properties of the roots of the characteristic polynomial. If one supposes that the transformation $X_1$ of $\gamma$ is regular, then the corresponding characteristic equation admits $r-k$ not vanishing roots which in general are not all distinct. Let $a_1, \cdots, a_p$ indicate $p$ distinct roots among the $r-k$; besides, let $m_1, \cdots, m_p$ designate their respective multiplicities. By posing,

$$k+m_1=k_1 \qquad k_2=k_1+m_2 \qquad \cdots \qquad r=k_p=k_{p-1}+m_p,$$

Cartan considered the subspaces generated by the $m_i$ transformations

$$X_{k_{i-1}+1}, X_{k_{i-1}+2}, \cdots, X_{k_i},$$

$i=1,\cdots,k$, which could be chosen in such a way as to have:

$$\begin{cases} (X_1, X_{k_{i-1}+1}) = a_i X_{k_{i-1}+1} \\ (X_1, X_{k_{i-1}+2}) = a_i X_{k_{i-1}+2} + c_{1,k_{i-1}+2,k_{i-1}+1} X_{k_{i-1}+1} \\ \cdots \\ (X_1, X_{k_i}) = a_i X_k + c_{1,k_i,k_i-1} X_{k_i-1} + \cdots + c_{1,k_i,k_{i-1}+1} X_{k_{i-1}+1}, \end{cases}$$

and he pointed out that each of them is left invariant by the action of $\gamma$. As a result of this, Cartan observed, the characteristic polynomial relative to $\gamma$ splits into the product of $p+1$ determinants. The first, equal to $(-\omega)^k$, is

$$\begin{vmatrix} \sum_{i=1}^{i=k} e_i c_{i11} - \omega & \cdots & \sum_{i=1}^{i=k} e_i c_{ik1} \\ \cdots & \cdots & \cdots \\ \sum_{i=1}^{i=k} e_i c_{i1k} & \cdots & \sum_{i=1}^{i=k} e_i c_{ikk} - \omega \end{vmatrix}$$

while the remaining $p$ ones could be proven to be equal to:

$$\begin{vmatrix} \sum_{i=1}^{i=k} e_i c_{i,k_{i-1}+1,k_{i-1}+1} - \omega & \cdots & \sum_{i=1}^{i=k} e_i c_{i,k_i,k_{i-1}+1} \\ \cdots & \cdots & \cdots \\ \sum_{i=1}^{i=k} e_i c_{i,k_{i-1}+1,k_i} & \cdots & \sum_{i=1}^{i=k} e_i c_{i,k_i,k_i} - \omega \end{vmatrix} = (\omega_i - \omega)^{m_i}.$$

From the last set of equations, Cartan deduced the fundamental result according to which the roots of the characteristic polynomial are linear functions of the components $e_j$, $j=1,\cdots,k$ of $X \in \gamma$ with respect to the basis $X_1, \cdots, X_k$. Indeed, it was sufficient to observe that the trace of a $m_i \times m_i$ square matrix coincides, modulo a sign, with the coefficient of term $\omega^{m_i-1}$ of its characteristic polynomial. So he obtained the following expressions for

[12]One should remember that the normalizer of a subalgebra $\mathcal{A} \subset \mathcal{G}$ is the set $N(\mathcal{A}) = \{X \in \mathcal{G} : [X, \mathcal{A}] \subset \mathcal{A}\}$.

the roots of the characteristic equation relative to a generic transformation of $\gamma$:

$$\omega_i = \sum_{j=1}^{j=k} e_j \omega_i^{(j)}, \qquad (i = 1, 2, ..., p).$$

Cartan thus succeeded in providing with full rigor what nowadays is known as primary decomposition of a Lie algebra in generalized eigenspaces. Cartan called it *forme réduite du groupe relative au sous-groupe* $\gamma$. From now on, following Cartan, $\gamma$ will designate a "Cartan subalgebra" of $G$. Accordingly, the infinitesimal transformations $X_{k_{i-1}+1}, ...X_{k_i}$ will be said to belong to the root $\omega_i$.

### 2.1.2 Integrability and Semisimplicity Criteria

Since 1874, in [Lie 1874], Lie had introduced a classification criterion in the theory of continuous groups consisting of the distinction between integrable[13] and not integrable groups. As the wording suggests, Lie was led to the notion of integrable group in the context of his theory of differential equations, and in particular of his theory of complete systems of first order linear partial differential equations. Killing himself soon got convinced of the fruitfulness of this notion as a consequence of its close connection with that of group of rank equal to zero. Necessary and sufficient conditions guaranteing the integrability of a given group had already been obtained by Engel and Killing. In particular, Engel had shown that a group $G$ is integrable if and only if its first derived group $G'$ has rank zero.

Cartan took great profit of this result by exploiting it in an essential way to prove his own, well known criterion for integrability:

> *The necessary and sufficient condition that a r-term group be integrable is that the transformations of its derived group annihilate identically the coefficient $\psi_2(e)$ of $\omega^{r-2}$ in its characteristic equation.*[14]

As a consequence of Engel's observation mentioned above, the condition is clearly necessary since if the group is integrable then its first derived group has rank zero and consequently all the coefficients of the characteristic polynomial vanishes identically. To demonstrate that the condition is also sufficient is less trivial. Cartan's proof proceeded by absurdum. He supposed that $\psi_2(e)$ vanishes identically on the first derived group of $G$ and that, in spite of that, the group $G$ is not integrable. Then, there exists an invariant subgroup of $G$, let us indicate it $g$ (be $q$ its order), one of the derived

[13] The notion of integrability coincides with present day definition of solvability.

[14] *La condition nécessaire et suffisante pour qu'un groupe d'ordre r soit intégrable est que les transformations de son groupe dérivé annulent identiquement le coefficient $\psi_2(e)$ de $\omega^{r-2}$ dans son équation caractéristique*, [Cartan 1894, p. 47].

subgroup of $G$, which coincides with its own derived group. Since $g \subseteq G'$, its characteristic equation reads:

$$\omega^q - \psi_3(e)\omega^{q-3} + \psi_4(e) \cdots \pm \psi_{q-1}(e)\omega = 0. \qquad (2.5)$$

Let $\gamma$ indicate a Cartan subalgebra of $g$ and let $X_{01}, \cdots, X_{0k}$ be a system of generators thereof. Since $g' = g$, every transformation of $\gamma$ can be obtained as a linear combination of transformations of type $(X_{0i}, X_{0j})$ and $(X_{\alpha i}, X_{\alpha' j})$. As a consequence of a fundamental lemma already proved by Cartan in 1893 in [Cartan1893c], every root of the characteristic equations relative to $(X_{0i}, X_{0j})$ and $(X_{\alpha i}, X_{\alpha' j})$ is a *real* multiple of every other root. Since $\psi_1 = \psi_2 = 0$, the sum of the squares of all the $q$ roots of (2.5) is equal to zero[15]. Thus, all the roots of the characteristic equations relative to the generators of $\gamma$ are zero. As a consequence of this $k = q$ and the group $g$ has rank zero. In virtue of Engel's criterion, $g$ is integrable and thus it cannot be equal to its first derived group, in contrast with the initial hypothesis. Thus $G$ is integrable.

Cartan's criterion for integrability played a key role in providing a proof of his criterion for semisimplicity too. Cartan's great merit consisted of understanding the centrality of the quadratic form $\psi_2$ and the possibility of characterizing many properties of simple groups purely in terms of the non-degeneracy of $\psi_2$. In order to do so, he had to offer an alternative definition of semisimplicity. Killing had defined a group to be semisimple if it could be decomposed in a (direct) sum of simple groups. Killing's attempts to prove a version of what nowadays is known as *the radical splitting theorem* had then suggested that an equivalent definition could be attained by defining a group to be semisimple if it does not admit any integrable invariant subgroup. Cartan regarded such an alternative as the most suitable one. Indeed, it allowed him to obtain, in an almost straightforward way, a highly manageable criterion for semisimplicity. Cartan's proof relied upon a lemma already stated by him in [Cartan1893c, p. 398] which was generalized in the first chapter of [Cartan 1894]. Essentially, this lemma established a connection between invariant subgroups and the coefficients of the characteristic equation. Such an intuition had already been exploited by Killing, however it was up to Cartan to provide a systematic solution. Cartan's result was the following.

[15]Two clarifications are in order here. The fact that $\psi_1 \equiv 0$ is a consequence of the general property according to which every linear invariant of the adjoint group vanishes identically on the first derived group. As far as the vanishing of the sum of the squares of the roots is concerned, this is a simple consequence of the following well known properties of the coefficients of algebraic equations: given in $\mathbb{C}$ an equation of the form $x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0$, then, indicating with $x_1, \cdots, x_n$ its roots, one has:

$$\sum_{i=1}^{n} x_i = -a_1, \qquad \sum_{1 \leq i < j \leq n} x_i x_j = a_2.$$

**Lemma 2** *Let $F(e)$ be a homogeneous polynomial invariant of degree $m$ in the variables $e_1, \cdots, e_r$ of the adjoint group of $G$; then the linear homogeneous equations obtained by equating all the $(m-1)-th$ order derivatives of $F(e)$ with respect to $e_1, \cdots, e_r$ define (unless they imply $e_1 = \cdots = e_r = 0$) an invariant subgroup of $G$.*[16]

By specializing the lemma to the case $F(e) = \psi_2(e)$ and invoking as well the integrability criterion, Cartan was able to prove that

$$g = \left\{ X = \sum_{j=1}^{r} e_j X_j : \frac{\partial \psi_2}{\partial e_i} = 0, i = 1, \cdots, r \right\}$$

is an integrable invariant subgroup of $G$. On the basis of this observation, it was easy to prove the following criterion:

> *The necessary and sufficient condition that a group be semisimple is that the determinant of the quadratic form $\psi_2(e)$ be not zero.*[17]

Indeed, if $G$ is semisimple, Cartan observed, the equations $\frac{\partial \psi_2}{\partial e_i} = 0$, $i = 1, \cdots, r$ imply $e_i = 0$, $i = 1, \cdots, r$, otherwise there would be a not trivial integrable invariant subgroup. To prove that the condition is sufficient too, let us suppose that $\psi_2$ is not degenerate and at the same time, by absurdum, that $G$ contains an integrable invariant subgroup $g$. By considering the derived groups of $g$, we can find one, say $\tilde{g}$, which is invariant in $G$ and such that $[\tilde{g}, \tilde{g}] = 0$. Let $X_{m+1}, \cdots, X_r$ indicate its generators. Then, the characteristic determinant of $G$ depends only upon $e_1, \cdots, e_m$[18]. Thus the quadratic form $\psi_2(e)$ reduces itself to a sum of $m$ squares, which contradicts the non-degeneracy hypothesis.

Cartan's criterion played a crucial role in his treatment of semisimple and simple groups. The first step consisted in proving the equivalence of Killing's definition of semisimplicity with his own. Indeed, by exploiting the non degeneracy of the quadratic form $\psi_2(e)$, he was able to prove that if a semisimple group $G$ admits an invariant subgroup $g$ of order $m$ then there exists an invariant subgroup of order $r-m$, $g'$ such that every transformation of $G$ can be written in a unique way as a sum of two transformations of $g$ and $g'$. Besides, $g$ and $g'$ were easily proved to be semisimple. On the basis

[16] See [Cartan1893c, p. 398].

[17] *La condition nécessaire et suffisante pour qu'un groupe soit semi-simple est que le discriminant de la forme quadratique $\psi_2(e)$ soit différent de zéro*, [Cartan 1894, p. 52].

[18] Actually, not all the elements of the corresponding matrix are independent of $e_{m+1}, \cdots, e_r$ since in general $c_{ijk} \neq 0$ when $i, k = m+1, \cdots, r$; $j = 1, \cdots, m$. However, as a consequence of the fact that $[\tilde{g}, G] \subset \tilde{g}$, one has $c_{ijk} = 0$ when $i = 1, \cdots, r$ and $j, k = m+1, \cdots, r$. The matrix in question is thus a triangular block matrix. Furthermore, since, in virtue of invariancy and commutativity of the transformations of $\tilde{g}$, the elements of the two diagonal blocks depend only upon $e_1, \cdots, e_m$, so does the determinant.

of this observation, one could guarantee that every semisimple group can be decomposed into the (direct) sum of simple groups.

A major improvement of Cartan's approach with respect to Killing's was then represented by the possibility of providing a fully rigorous proof of some results which turned out to be essential for the structural theory of simple groups. As already thoroughly discussed by Hawkins in [Hawkins 2000, Chap. 5-6], Killing's analysis had relied upon some uncorrect theorems which required a more attentive examination. In particular, Killing had asserted the validity of the theorem according to which a "Cartan subalgebra" of every *perfect*[19] group is abelian. On his part, Cartan set out to vindicate Killing's theorem by specializing it to the case of semisimple groups; in Cartan's own notation, if $G$ is semisimple then $\gamma$ is abelian. Again, essential for Cartan's proof was the non degeneracy of $\psi_2(e)$.

### 2.1.3 Radical and decomposition theorems

Cartan's thesis did not limit to convey a rigorous reformulation of Killing's theory of simple and semisimple complex Lie groups, it also provided a detailed analysis of groups which are neither semisimple nor integrable. In this field too, Cartan's project consisted of providing Killing's researches on the subject with the systematic character they lacked of.

A first crucial contribution provided by Cartan was a clear and rigorous definition of the notion of maximal integrable invariant subgroup of a given group $G$; it is what nowadays we would call the *radical* of an algebra, i.e. a maximal solvable ideal.

Unlike Killing, Cartan attached to such a notion a key role in his analysis. At the very beginning of the third part of his thesis, Cartan provided a proof of the existence of such a maximal, integrable, invariant subgroup $\Gamma$. It was sufficient to observe that given two integrable, invariant subgroups $g, g'$ of a group $G$ (supposed to be neither integrable nor semisimple), the invariant subgroup generated by the transformations of $g$ and $g'$ is still integrable.

A first virtue of this definition was the possibility of associating to $G$, by means of $\Gamma$, a semisimple group which is homomorphic (in the language of that time, meriedrically isomorphic) to $G$ itself. Such a homomorphism was induced by what nowadays we would call canonical projection $\pi : G \to G/\Gamma$, where $G/\Gamma$ indicates the quotient group of $G$ with respect to the invariant subgroup $\Gamma$.[20] The semisimplicity of $G/\Gamma$ was proved by Cartan in a straight-

[19]Such a wording dates back to Lie; a group was said by him to be perfect if it coincides with its first derived group, i.e. $G = [G, G]$.

[20]In modern terms, remember that it makes sense to speak of the quotient of two Lie algebras $\mathcal{G}$ and $\mathcal{H}$ only if $\mathcal{H}$ is an ideal of $\mathcal{G}$. In this case, the algebra $\mathcal{G}/\mathcal{H}$ consists of the equivalence classes of transformations $X$ with respect to the following equivalence relation: $X, Y \in \mathcal{G}$ are equivalent if, and only if $X - Y \in \mathcal{H}$. By designating with $\bar{X}$ the equivalence class of $X \in \mathcal{G}$, a Lie bracket operation on $\mathcal{G}/\mathcal{H}$ can be defined by: $\left[\bar{X}, \bar{Y}\right] = \overline{[X, Y]}$. This definition is well posed (i.e., it is independent of the particular choice of representatives)

forward way by showing that $G/\Gamma$ cannot contain any integrable invariant subgroup. Indeed, suppose that $g$ is an integrable invariant subgroup of $G/\Gamma$; the corresponding invariant subgroup $g' = \pi^{-1}(g) \subset G$ is integrable as well. Since clearly $\Gamma \subset g'$, $\Gamma$ would not be a maximal integrable invariant subgroup, which is absurd. Thus, $G/\Gamma$ is semisimple.

Thus, somehow, the study of an arbitrary continuous group could be traced back to that of integrable and semisimple ones.

In this respect, particularly useful were the notions of principal and secondary roots. First introduced by Killing in 1889 (he spoke of *Hauptwurzeln* and *Nebenwurzeln*), such a distinction had played a major role in Killing's attempts of attaining a complete classification of perfect groups, i.e. such that $G = [G, G]$. However, his presentation had remained obscure and had been rendered precarious by some unproved suppositions which turn out to be false in general.

On his part, Cartan's treatment was much clearer and easier to follow. The first step consisted of the observation that the characteristic determinant of $G$ with respect to a regular transformation $X$ contains, as a factor, the characteristic determinant of $G/\Gamma$ with respect to the (regular) transformation $\pi(X)$. As a consequence of this, Cartan could operate, among the non vanishing roots of $G$, a distinction between principal roots which belong to the characteristic equation of $G/\Gamma$ and secondary roots which do not. Consequently, every infinitesimal transformation of $G$ which belongs to a secondary root belongs to $\Gamma$; furthermore, since all the non vanishing roots of a semisimple group have multiplicity equal to 1, if $\omega_\alpha$ is a principal root of multiplicity $m_\alpha$, $m_\alpha - 1$ independent infinitesimal transformations belong to $\Gamma$ while there exists one transformation of $G$ belonging to $\omega_\alpha$ which is not in $\Gamma$. In a similar way, if $m_0$ is the multiplicity of 0 regarded as a root of $G$ and $l$ is the rank of $G/\Gamma$, then $m_0 - l$ independent infinitesimal transformations belong to $\gamma \bigcap \Gamma$ ($\gamma$ indicates, as usual, a Cartan subalgebra of $G$).

The employment of such notions allowed Cartan to achieve the following, very important result:

**Theorem 1 (Cartan, 1894)** *Let $\omega_\alpha$ and $-\omega_\alpha$ be two principal roots of the characteristic equation of $G$; furthermore, let $X_\alpha$ be an arbitrary transformation of $G$ belonging to $\omega_\alpha$ but* not *contained in $\Gamma$. Then, one can associate to $X_\alpha$, a transformation $X_{\alpha'}$ which belongs to $-\omega_\alpha$ but is not contained in $\Gamma$, such that, by posing $(X_\alpha, X_{\alpha'}) = Y_\alpha$, one has:*

$$(X_\alpha, Y_\alpha) = 2X_\alpha \qquad (X_{\alpha'}, Y_\alpha) = -2X_{\alpha'}.^{21} \tag{2.6}$$

In other words, the 3-dimensional (simple) subgroups of $G/\Gamma$, whose existence was guaranteed by semisimplicity of $G/\Gamma$ itself, could be realized as

as a result of the fact that $\mathcal{H}$ is an ideal of $\mathcal{G}$. Cartan did not explicitly distinguish between transformations of $G$ and equivalence classes of them. Case by case, the real significance of symbols should be inferred from the contest.

[21] See [Cartan 1894, p. 99-100].

subgroups of $G$.
The theorem was first proved in the case in which the radical $\Gamma$ is abelian and then extended to the general case in which no additional hypothesis on $\Gamma$ is required.

Cartan was able to exploit this result in many fruitful ways. He observed, for instance, that Theorem (1) was fully equivalent with Engel's integrability criterion according to which a group $G$ is integrable if, and only if it does not contain any simple 3-dimensional subgroups.

Interestingly enough, at least for some time, Cartan had also regarded this theorem as the central core of his proof of the radical decomposition theorem which he had stated (without proof) for the first time in a brief note [Cartan 1893b] appeared in the *Comptes Rendus* in 1893:

**Theorem 2** *If one considers the maximal integral invariant subgroup $\Gamma$ of a group $G$, there exists a subgroup $G'$ which completes $G$ with $\Gamma$.* [22]

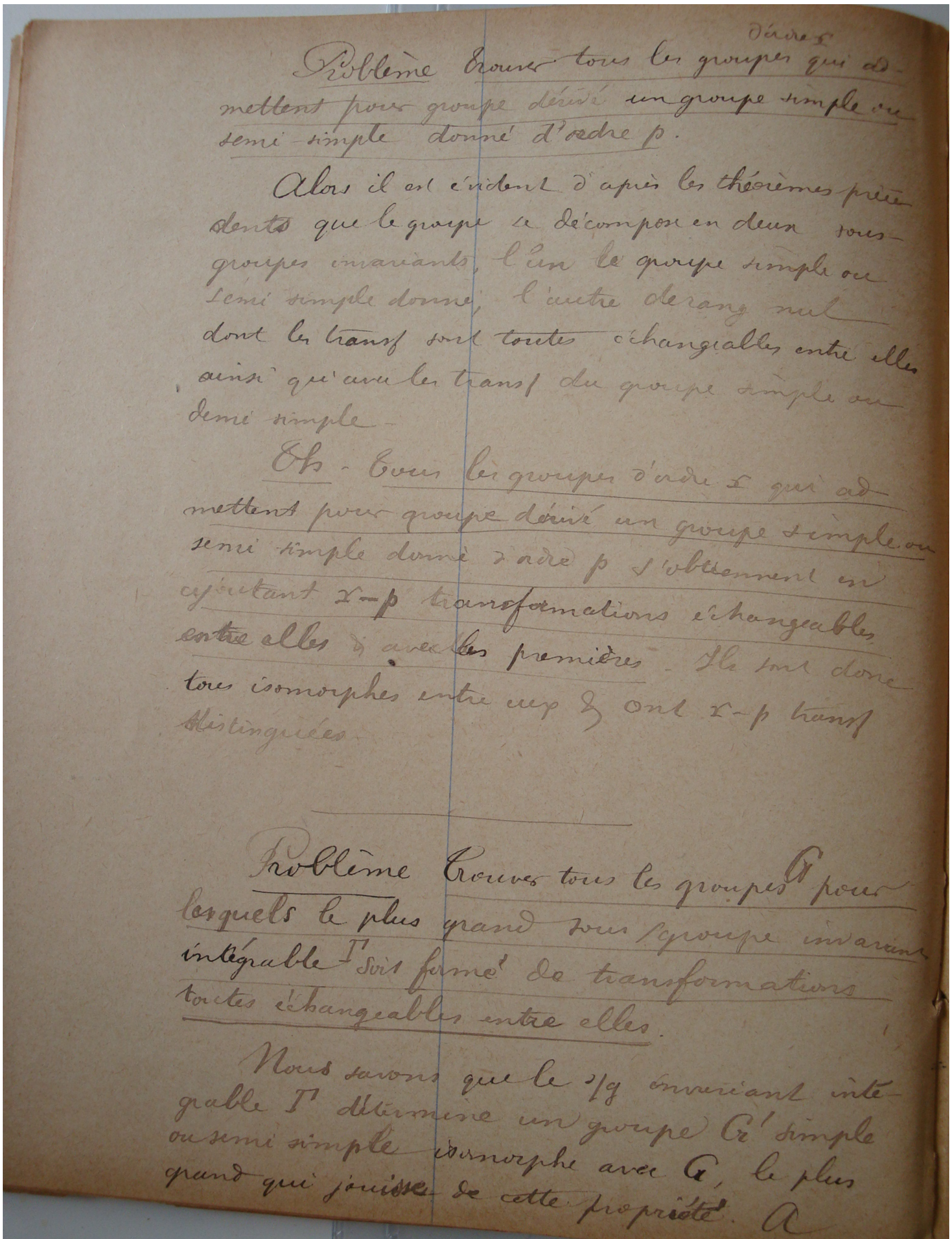
Indeed, in a notebook dating to 1893, Cartan provided a brief proof of this fundamental result which boiled down to a direct application of Theorem (1). The idea for the demonstration which he sketched out was the following. By invoking Theorem (1) to every principal root[23] $\omega_\alpha$ of $G$, he associated transformations $X_\alpha, X_{\alpha'}, Y_\alpha$ such that relations (2.6) hold and he set out to show these transformations generate a subgroup which is isomorphic (holoedrically isomorphic) to $G/H$. He observed that to every transformation $Y_\alpha$ there correspond transformations of 1th genre only, that is such that $(Y_\alpha, X_\beta) = a_{\beta\alpha}X_\beta$, where $X_\beta$ is a transformation which belongs to $\omega_\beta$. However, it is by no means clear how Cartan intended to show that the transformations $X_\alpha, X_{\alpha'}, Y_\alpha, X_\beta, X_{\beta'}, Y_\beta$, etc. actually built up a subgroup of $G$.

It seems likely that at first Cartan thought that he could operate such a verification in a straightforward way, and that for this reason he decided to communicate the theorem in the above mentioned note published in the *Comptes Rendus*. However, when he realized that the relevant computations were far from being trivial, he abandoned the idea of providing in his thesis a complete proof of the general theorem and limited himself to convey a rigorous demonstration[24] of the statement in the particular case in which the rank of $G$ is equal to 1. As for the general statement, he just claimed that it would suffice to prove it in the case in which the group $G$ admits only one integrable invariant subgroup.

[22] *Si l'on considère le plus grand sous-groupe invariant intégrable $\Gamma$ d'un groupe $G$, il existe un sous-groupe $G'$, qui avec $\Gamma$ complète $G$.* See [Cartan 1893b, p. 963].

[23] Cartan actually limited himself to considering a fundamental set of roots of $G$ in terms of which every other roots could be expressed as a linear combination with integer coefficients.

[24] In this respect, see [Hawkins 2000, p. 208-209].


Problème Trouver tous les groupes d'ordre r qui admettent pour groupe dérivé un groupe simple ou semi-simple donné d'ordre p.

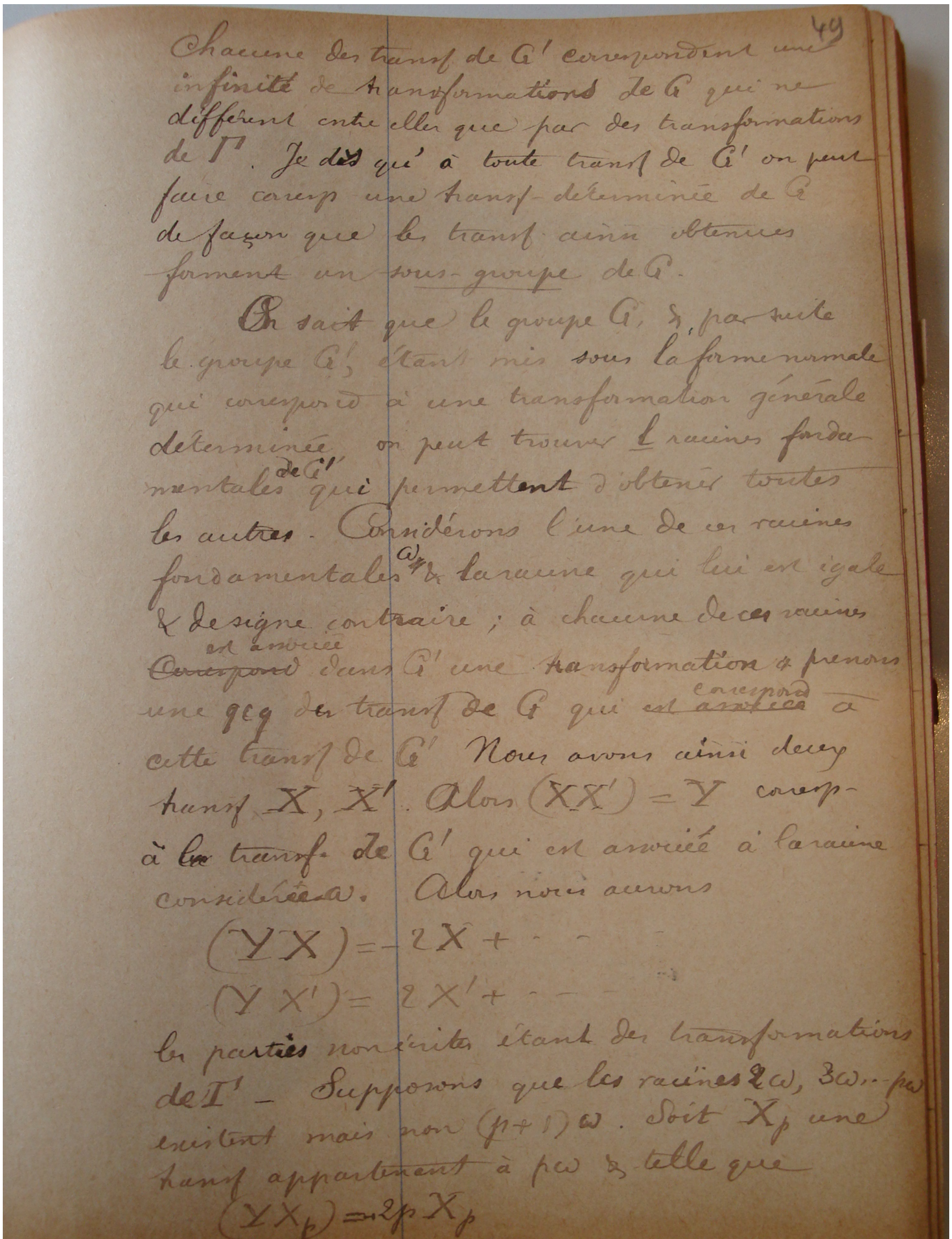
Alors il est évident d'après les théorèmes précédents que le groupe se décompose en deux sous-groupes invariants, l'un le groupe simple ou semi simple donné, l'autre de rang nul dont les transf sont toutes échangeables entre elles ainsi qu'avec les transf du groupe simple ou semi simple.

Obs. Tous les groupes d'ordre r qui admettent pour groupe dérivé un groupe simple ou semi simple donné d'ordre p s'obtiennent en ajoutant r−p transformations échangeables entre elles & avec les premières. Ils sont donc tous isomorphes entre eux & ont r−p transf distinguées.

Problème Trouver tous les groupes G pour lesquels le plus grand sous-groupe invariant intégrable Γ soit formé de transformations toutes échangeables entre elles.

Nous savons que le s/g invariant intégrable Γ détermine un groupe G' simple ou semi simple isomorphe avec G, le plus grand qui jouisse de cette propriété. A

Figure 2.1: Picture taken from Cartan's notebook (1892-1893), 1.04, Fonds É. Cartan, 38J. Archives of the Académie des Sciences de Paris. The frontispiece of the notebook reads: *Cours de Gaston Darboux, Faculté des Sciences de Paris.* However, only the first 14 pages contain notes from Darboux's lectures. The remaining ones consist of personal notes on Lie groups.



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Chacune des transf de G' correspondent une infinité de transformations de G qui ne diffèrent entre elles que par des transformations de Γ. Je dis qu'à toute transf de G' on peut faire corresp une transf déterminée de G de façon que les transf ainsi obtenues forment un sous-groupe de G.

On sait que le groupe G, & par suite le groupe G', étant mis sous la forme normale qui correspond à une transformation générale déterminée, on peut trouver l racines fondamentales de G' qui permettent d'obtenir toutes les autres. Considérons l'une de ces racines fondamentales ω & la racine qui lui est égale & de signe contraire ; à chacune de ces racines est associée dans G' une transformation & prenons une qcq des transf de G qui correspond à cette transf de G'. Nous avons ainsi deux transf $X$, $X'$. Alors $(XX') = Y$ corresp. à la transf. de G' qui est associée à la racine considérée ω. Alors nous aurons

$(YX) = -2X + \cdots$

$(YX') = 2X' + \cdots$

les parties non écrites étant des transformations de Γ' — Supposons que les racines 2ω, 3ω, … pω existent mais non (p+1)ω. Soit $X_p$ une transf appartenant à pω & telle que

$(YX_p) = 2p X_p$

Figure 2.2: Picture taken from Cartan's notebook (1892-1893), 1.04, Fonds É. Cartan, 38J. Archives of the Académie des Sciences de Paris.

A first, fully general proof of the radical decomposition theorem was provided by the Italian mathematician Eugenio Elia Levi (1883-1917) [Levi 1905] which was presented by Luigi Bianchi to the *Accademia delle Scienze di Torino*. It is interesting to note that Levi's proof strategy presented numerous contact points with that employed by Cartan in his notebook. Indeed, as Cartan had done, Levi exploited Theorem (1) to deduce the existence of transformations $X_\alpha, X_{\alpha'}, Y_\alpha, X_\beta, X_{\beta'}, Y_\beta$, etc. satisfying relations (2.6) and then he proved that such transformations build up a subgroup of $G$ isomorphic to $G/\Gamma$. In particular, Levi's treatment made it clear that some of the computations involved were not trivial at all. Some of them even required a careful analysis which consisted of distinguishing among the different cases corresponding to *all* the different types of simple groups.

In spite of the intrinsic importance of the radical decomposition theorem, it is seems that Cartan tended to regard it as a useful technical tool to be employed in further, more application oriented investigations rather than as a crucial theoretical result upon which a fully general structural approach to classification theory of Lie algebras could be built. Indeed, unlike Killing, Cartan's focus of interest was more oriented on issues which were more directly susceptible to applications to Lie's integration theory of differential equations.

In order to obtain an adequate comprehension of this very important historical circumstance, it is first necessary to recall the basic ideas of Lie's approach to differential equations. Later on, we will come back to Cartan's work and we will provide concrete examples which support our assertion. Hawkins was the first person to draw attention on this point in [Hawkins 2000, chap. 6]. On our part, we will try to provide additional evidences in order to strengthen the validity of this position.

## 2.2 Lie's theory of complete systems

As Lie himself often emphasized, his theory of continuous groups found his major motive of inspiration in the fruitful applications to the integration problem of differential equations. His project consisted of developing a general theory of integration which could be regarded as a generalization of Galois' theory of algebraic equations. More precisely, Lie was interested in understanding how the knowledge of certain symmetry properties of a given differential equation could be exploited in order to get information on its solutions. Although such a general research program found application in a wide variety of fields, we will limit ourselves to a very specific case: the integration problem of complete systems of first order linear PDE's which admit a given finite continuous group of transformations.

As already explained in detail by Hawkins, well before Lie's accomplishments, complete systems had played a major role in the theory of first order

PDE's. K. Jacobi's *Nova Methodus*, published posthumously in *Crelle's Journal* in 1862, indeed contained a general procedure which reduced the integration of an arbitrary first order PDE $F\left(x, z, \frac{\partial z}{\partial x}\right) = 0$ to resolution of a system of *linear* first order PDE's of the following form:

$$A_i(f) = \sum_{j=1}^{n} a_j^{(i)}(x) \frac{\partial f}{\partial x_j} = 0, \qquad (i = 1, \cdots, m). \tag{2.7}$$

By exploiting a property of Poisson brackets, Jacobi was led to consider the case in which the operators $A_i(f)$, $i = 1, \cdots, m$ satisfy

$$A_i(A_j(f)) - A_j(A_i(f)) = 0, \qquad (i, j = 1, \cdots, m). \tag{2.8}$$

By developing some of Jacobi's ideas in the context of Pfaffian systems, in 1866 Clebsch introduced the notion of *complete system.* It was a generalization of Jacobi's systems (2.7). Indeed, every system of type (2.7) was said by Clebsch to be complete if operators $A_i(f)$, $i = 1, \cdots, m$, satisfy

$$A_i(A_j(f)) - A_j(A_i(f)) = \sum_{k=1}^{m} d_{ijk}(x) A_k(f), \qquad (i, j = 1, \cdots, m). \tag{2.9}$$

Lie provided the most general and systematic treatment of his integration theory of complete systems in the long and profound memoir [Lie 1885][25]. This memoir was intended to provide, along with more recent achievements, a comprehensive survey of previous results dating back to the early 1870's which Lie had published in Norwegian journals. Indeed, it seems that a dispute with a French mathematician, Georges Halphén (1844-1889), concerning with the priority on some results long known to him, had convinced Lie of the necessity of making his work better known outside Norway. The publication of [Lie 1885] in the *Mathematische Annalen* was part of Lie's attempts of reaching a wider readership.

In accordance with his general project of developing a differential analogous of Galois' theory of algebraic equations, Lie set out to provide a general procedure consisting of exploiting the knowledge of a continuous group admitted (*gestattet*) by the system in order to trace back the integration of the system itself to that of an equation of determined form.[26]

More explicitly, the problem was the following: given a complete system $A_i(f) = 0$, $i = 1, \cdots, m$, which admits a certain finite continuous group $G$

[25] Ugo Amaldi's remarks in a letter to Engel dating back to April 1927 emphasize the importance of this memeoir in Lie's entire mathematical production. While praising Engel's *Anmerkungen* to volume VI of Lie's *Gesammelte Abhandlungen*, he observed: "[...] *la Memoria dei Math. Ann. XXV,* [...] *costituisce indubbiamente il lavoro del Lie più difficile, più profondo e più importante per la comprensione del suo pensiero matematico"*.

[26] For a modern treatment of Lie's integration theory of complete systems, see [Stormark 2000, chap. 10].

generated by $r$ independent infinitesimal transformations

$$B_j(f) = \sum_{k=1}^{n} \xi_{jk}(x) \frac{\partial f}{\partial x_k},$$

to provide a general procedure consisting of tracing back its resolution to that of ordinary differential equations.

Before proceeding further on, it is first necessary to define with precision what it means that a transformation group $G$ is admitted by a complete system. In terms of finite transformations, such a property can be expressed as follows. If $\omega(x_1, \cdots, x_n)$ is an arbitrary solution of $A_i(f)$, $i = 1, \cdots, m$ and $x'_i = f_i(x_1, \cdots, x_n; a_1, a_2, \cdots, a_r)$, $(i = 1, \cdots, n)$ is an arbitrary transformation of $G$ then $\omega(x'_1, \cdots, x'_n)$, where the $x'_i$ are to be considered as functions of $x_i$, is a solution too.

By considering infinitesimal transformations as Lie did, such a symmetry property can be translated by saying that the system in question admits the group $G =< B_1(f), \cdots, B_r(f) >$ if, given an arbitrary solution $\omega(x_1, \cdots, x_n)$, then $B_j(\omega)$ is a solution too, for $j = 1, \cdots, r$.

Lie was able to characterize it purely in terms of commutation relations among the operators $A_i$, $i = 1, \cdots, m$ and the infinitesimal transformations $B_j(f)$, $j = 1, \cdots, r$. Indeed, he proved the following theorem:

**Theorem 3 (Lie, 1885)** *The complete system $A_i(f)$, $i = 1, \cdots, m$ admits the group of infinitesimal transformations $B_j(f)$, $j = 1, \cdots, r$ if, and only if, the following commutation relations hold:*

$$A_i(B_j(f)) - B_j(A_i(f)) = \sum_{k=1}^{m} \alpha_{kij}(x) A_k(f) \qquad i = 1, ..., m; j = 1, \cdots, r.$$[27]

If this is the case, then the integration of the complete system could be further simplified. Indeed, Lie proved, the problem could be traced back to the resolution of a single equation

$$A(f) = \frac{\partial f}{\partial x} + \sum_{k=1}^{r} Y_k(x, y_1, \cdots, y_r) \frac{\partial f}{\partial y_k},$$

admitting a simply transitive group of infinitesimal transformations[28]

$$B_j(f) = \sum_{i=1}^{r} \eta_{ji}(x; y_1, \cdots, y_r) \frac{\partial f}{\partial y_i},$$

[27] See [Lie 1885, p. 78].

[28] In actual fact, the variable $x$ in the coefficients of these transformations should be regarded as a parameter. Lie did not go into detail, however Engel in his *Anmerkungen* provided a thorough analysis of this issue, by showing that one is always entitled to assume that the transformations $B_j(f)$ are independent of $x$.

such that the following relations hold:

$$(A, B_j) = 0, \qquad (B_i, B_j) = \sum_{s=1}^{r} c_{ijs} B_s(f).$$

On the basis of these premises, Lie emphasized the necessity of first providing a thorough algebraic analysis of the properties of the group $G = \{B_j(f)\}_{i=1}^{r}$. Indeed, he observed:

> [...] *I suppose, on one hand, that the finite transformations of the group $B_j(f)$ are known; on the other, that the structure of the group has been determined through a preliminary algebraic discussion.* [29]

In the case of integrable groups, already in [Lie 1874], Lie had shown that the integration of the system could be obtained through quadrature, i.e. by computing integrals.

In the non-integrable case, a first crucial result of Lie's theory was the possibility of reducing the entire procedure to the consideration of simple groups only. In order to do so, Lie first considered a maximal invariant subgroup of $G$, i.e. a maximal ideal of the Lie algebra associated to $G$. He supposed it to be generated by $r - q$ infinitesimal transformations $B_{q+1}(f), \cdots, B_r(f)$ and wrote down the corresponding (complete) system of differential equations:

$$B_{q+1}(f) = 0, \cdots, B_r(f) = 0.$$

By indicating with $x, x_1, \cdots, x_q$ the set of its independent solutions, Lie operated the change of variables consisting of replacing the first $q+1$ variables $x, y_1, \cdots, y_q$ with $x, x_1, \cdots, x_q$. As a consequence of invariance, one has:

$$B_j(B_{q+k}(f)) - B_{q+k}(B_j(f)) = a_{q+1}B_{q+1}(f) + a_{q+2}B_{q+2}(f) + \cdots + a_r B_r(f),$$

for $k = 1, \cdots, r - q$. By posing consecutively $f = x_l$, $l = 1, \cdots, q$, one obtains $B_{q+k}(B_j(x_l)) = 0$, $k = 1, \cdots, r - q$, $l = 1, \cdots, q$, i.e. that $B_j(x_l)$ depend upon $x, x_1, \cdots, x_q$, only. That means that, as a consequence of the above mentioned change of variables, the operator $A(f)$ and the infinitesimal transformations $B_j(f)$, $j = 1, \cdots, r$ take on the following form:

$$A(f) = \frac{\partial f}{\partial x} + \sum_{k=1}^{q} X_k(x, x_1, ..., x_q) \frac{\partial f}{\partial x_q} + \sum_{k=q+1}^{r} Y_k(x, x_1, ..., y_m) \frac{\partial f}{\partial y_k} \quad (2.10)$$

[29] [...] *setze ich jetz einerseits voraus, dass die endlichen Transformationen der Gruppe $B_j(f)$ bekannt sind, andererseits, dass die Zusammensetzung der Gruppe durch eine vorläufige algebraische Discussion bestimmt worden ist* . See [Lie 1885, pag. 136].

$$B_{q+j}(f) = \sum_{i=q+1}^{r} \eta_{q+j,i}(x, ..., x_q, y_{q+1}, ..., y_r)\frac{\partial f}{\partial y_i}, \qquad j = 1, ..., r-q,$$

$$B_k(f) = \sum_{i=1}^{q} \xi_{ki}(x, x_1, ..., x_q)\frac{\partial f}{\partial x_i} + \sum_{i=q+1}^{r} \eta_{ki}(x, x_1, ..., y_r)\frac{\partial f}{\partial y_i}, \qquad k = 1, ...q.$$

At this point, Lie observed, the resolution of (2.10) could be traced back to the resolution of a reduced equation obtained from (2.10) upon cancelation of the terms containing partial derivatives with respect to $y_{q+1}, \cdots, y_r$, i.e.

$$A'(f) = 0 = \frac{\partial f}{\partial x} + \sum_{k=1}^{n} X_k(x, x_1, ..., x_n)\frac{\partial f}{\partial x_k}. \tag{2.11}$$

By construction, the equation $A'(f) = 0$ admits the group generated by:

$$B'_k(f) = \sum_{i=1}^{q} \xi_{ki}(x, x_1, ..., x_q)\frac{\partial f}{\partial x_i}, \qquad k = 1, ...q,$$

which is simple, since it cannot contain any invariant subgroup. By indicating with $\mathtt{x}_1, ..., \mathtt{x}_n$ a set of independent solutions of $A'(f) = 0$ (supposed to be known), the original equation $A(f) = 0$ is transformed into

$$A''(f) = 0 = \frac{\partial f}{\partial x} + \sum_{k=n+1}^{m} Y_k(x, \mathtt{x}_1, ..., \mathtt{x}_n, y_{n+1}, ..., y_m)\frac{\partial f}{\partial y_k},$$

where the quantities $\mathtt{x}_1, ..., \mathtt{x}_n$ are to be regarded as constants. Such an equation, as is easy to see, admits the group of infinitesimal transformations $B_{q+j}(f)$, $j = 1, \cdots, r-q$ acting on the variables $y_{q+1}, \cdots, y_r$.

As a consequence of this Lie was able to prove the following

**Theorem 4 (Lie, 1885)** *The integration of the equation $A(f) = 0$ in $r+1$ variables admitting $r$ infinitesimal transformations $B_j(f)$, $j = 1, \cdots, r$ is split into the resolution of two simpler equations: the first, $A'(f) = 0$, in $q+1$ variables, admits a $q$-parameter simple group; the second, $A''(f) = 0$ in $r-q+1$ variables, admits a $(r-q)$-parameter group.*[30]

In the eventuality in which this last $(r-q)$-parameter group is not simple (*zusammengesetzte*), the procedure described above could be iterated in order to reduce the entire integration of $A(f) = 0$ to equations admitting simple groups. Clearly, in order to do so, as Lie assumed, it was necessary to know a normal decomposition series for the group $G$, i.e. a chain of inclusions of the form

$$G \supset G_1 \supset G_2 \supset G_3 \supset \cdots G_s \supset \{e\},$$

[30]See [Lie 1885, p. 137-138].

where $G_i$ is a maximal invariant subgroup of $G_{i-1}$. With a slight change of notation, the problem was thus the following: to integrate the equation

$$A(f) = \frac{\partial f}{\partial x} + \sum_{k=1}^{r} X_k(x, x_1, \cdots, x_r) \frac{\partial f}{\partial x_k}, \tag{2.12}$$

knowing that it admits a simple, simply transitive continuous group $\{B_i(f)\}_{i=1}^r$.

The principal idea at the basis of Lie's procedure was the notion of reciprocal group. Indeed, as he showed in [Lie 1885, §5], to every simply transitive group $\{B_i(f)\}_{i=1}^r$, a similar, isomorphic, simply transitive group $\{D_j(f)\}_{j=1}^r$ could be associated which was univocally characterized by the property according to which every transformation $D_j(f)$, $j = 1, \cdots, r$ commutes with every single generators $B_i(f)$, $i = 1, \cdots, r$.
The integration of (2.12) was indeed traced back to the following:

**Problem 1** *To integrate the equation $A(f) = 0$ knowing that it admits $r$ independent infinitesimal transformations $B_k(f)$, $k = 1, \cdots, r$ such that $(A, B_i) = 0$, $i = 1, \cdots, r$. Besides, $r$ independent transformations $D_j(f)$, $j = 1, \cdots, r$ (the generators of the reciprocal group) are known such that*

$$(A, D_j) = \sum_{k=1}^{r} \lambda_{jk}(x) D_k(f).$$[31]

Such a formulation of the integration problem, at least in some special cases, allowed Lie to reduce the entire procedure to the integration of an ordinary differential equation of a canonical form.

In order to further simplify the problem under examination, Lie considered a subgroup (necessarily not invariant) of the simple group $\{B_i(f)\}_{i=1}^r$, with a maximal number of parameters $r - q$. Let $B_{q+1}(f), \cdots, B_r(f)$ be its generators. By indicating with $x, z_1, \cdots, z_q$ a set of independent solutions of the associated complete system $B_j(f) = 0$, $j = q + 1, \cdots, r$ and considering the commutation relations $(A, B_j) = 0$, $j = 1, \cdots, r$, and $(B_j, D_k) = 0$, $j, k = 1, \cdots, r$ one concludes that under an appropriate change of coordinates the coefficients of the derivatives $\frac{\partial}{\partial z_k}$, $k = 1, \cdots, q$ in $A(f)$ and in $D_j(f)$, $j = 1, \cdots, r$ depend only upon $x, z_1, \cdots, z_q$. One can then limit oneself to consider a reduced problem where the operators $A(f)$ and $D_k(f)$, $k = 1, \cdots, r$ are replaced by

$$A'(f) = \frac{\partial f}{\partial x} + \sum_{i=1}^{q} Z_i(x, z_1, \cdots, z_q) \frac{\partial f}{\partial z_i}, \tag{2.13}$$

$$D'_k(f) = \sum_{i=1}^{q} Z_{ki}(x, z_1, \cdots, z_q) \frac{\partial f}{\partial z_i}, \qquad k = 1, \cdots, r.$$

[31] See [Lie 1885, p. 139].

Although Lie did not provide any general procedure, he set out to explain how a canonical form could be obtained in some special cases. For example, by choosing $q = 1$, Lie was able to prove that the order of the group $\{B_i(f)\}_{i=1}^r$ has to be equal to 3 and that $B_1(f), B_2(f), B_3(f)$ is isomorphic to the projective group of the straight line. As a consequence of this, the transformations of the (reduced) reciprocal group $\{D'_k(f)\}$ could be chosen so as to coincide with the infinitesimal transformations:

$$\frac{d}{dz}, \quad z\frac{d}{dz}, \quad z^2\frac{d}{dz},$$

and from $(A', D'_j) = \sum_{k=1}^r \lambda_{jk}(x)D'_k(f)$ one could put $A'(f) = 0$ in the following form

$$\frac{\partial f}{\partial x} + X_1(x)\frac{\partial f}{\partial z} + X_2(x)z\frac{\partial f}{\partial z} + X_3(x)z^2\frac{\partial f}{\partial z} = 0,$$

which was easily seen to be equivalent to an ordinary differential equation of Riccati type,

$$\frac{dz}{dx} = X_1 + X_2 z + X_3 z^2.$$

## 2.3 Complete systems and canonical reduction

Beyond its intrinsic theoretical value, the structural theory of finite continuous groups was regarded as highly relevant in Paris at the end of the nineteenth century, especially in view of its fruitful applications in the realm of the theory of differential equations. As it has already been observed, Cartan's interest in the subject did not represent an exception. Indeed, soon after the completion of his doctoral thesis, he devoted himself to an accurate study of the applications of the theoretical achievements accomplished therein. Such a project early materialized in a long memoir [Cartan 1896] in which Lie's theory of complete systems was dealt with in the light of the recent algebraic results of his.

Already in 1894, in a brief note published in the *Comptes Rendus de l'Académie des Sciences*, he had announced his intention of taking up Lie's 1885 memoir with the aim of providing Lie's general procedure with rigor and completeness.

Two years later in [Cartan 1896], Cartan singled out three main problems to which the algebraic aspects of Lie's theory of complete systems could be traced back; (I) starting from the finite transformations of a continuous group to find the equations of its subgroups and deduce from them their differential invariants; (II) from the knowledge of the structure of a group to provide a normal series decomposition thereof; (III) for every simple continuous group, to find a canonical form from which one can deduce a maximal subgroup and a transitive group such that the subgroup leaving a certain point invariant corresponds to the maximal subgroup itself.

In view of Lie's results which were recalled in the preceding section, the relevance of the first two problems for the development of an integration theory of complete systems should be quite clear. On the contrary, the last problem stated by Cartan requires some clarification.
Roughly speaking, one can say that its resolution entails the practical possibility of writing down a canonical form for the equation (2.13) considered above.

The requirement of maximality for the subgroup of $G$ (supposed to be simple, as a result of the procedure described in the preceding section) essentially guarantees that one can rewrite the equation (2.13), modulo the term $\frac{\partial f}{\partial x}$, as a linear combination of the transformations $\{D'_k(f)\}_{k=1}^r$ of the reduced, reciprocal group of $G$ (it is what Cartan indicated with the symbol $\overline{G'}$, see [Cartan 1896, p. 14]).

As for the invariance of a point under the action of corresponding subgroups, it is, again, a technical requirement which assures the actual existence of the change of coordinates which leads to a canonical form for the equation (2.13).

As Cartan himself pointed out[32], in order for the entire procedure to be put in practice, it was necessary to tackle the resolution of the following preliminary problem whose origin could be traced back to a conjecture first stated by Lie in 1885 (this statement will be referred to as *Lie's first conjecture*): given a simple group $G$ with $r$ parameters and a maximal subgroup $g \subset G$ with $r - q$ parameters, to find a $q$-parameters group $H$ which is isomorphic to $G$.

Hawkins [Hawkins 2000, §6.4] has already discussed in detail Cartan's relevant contributions. Namely, he has shown how Cartan in his thesis succeeded in reformulating a special case of Killing's classification problem of Lie groups $G$ such that $[G, G] = G$ in a way that was relevant for a better understanding of Lie's first conjecture. On our part, we will focus on two different examples which will provide further justifications to the above stated assertion about the eminently application-oriented character of Cartan's theory of finite continuous groups. Firstly, a theorem by Cartan on the group of transformations leaving a given semisimple group $G$ invariant will be discussed (it is part of the solution to problem (III) mentioned above). Then, Cartan's general procedure for deducing a normal series decomposition of an arbitrary group (problem (II) mentioned above) will be described.

The origins of the first theorem, once more, can be traced back to Lie's 1885 paper. Just after having stated his first conjecture, Lie had written:

> *My second question is the following: consider a simple (projective) group $G_r$ acting upon a q-dimensional manifold; besides, suppose that $G_r$ does not contain any subgroup with more than*

[32] See [Cartan 1896, p. 15].

> *$r - q$ parameters. Then, is there a group with more than $r$ parameters which contains $G_r$ as an invariant subgroup? It is likely that the question must have a no as an answer.*[33]

Well aware of the relevance of this problem for the integration procedure of complete systems (indeed, a negative answer to the question would guarantee the above mentioned possibility of replacing equation (2.13) with a linear combination of $\{D'_k(f)\}_{k=1}^r$ ), Cartan decided to tackle the question posed by Lie in the sixth chapter of his thesis (p. 112-113). In actual fact, Cartan was able to provide an answer to an even more general problem since he did not limit himself to simple projective groups but he considered arbitrary semisimple ones. Indeed, he succeeded in proving the following:

**Theorem 5 (Cartan, 1894)** *Let $G$ be a semisimple, $r$-parameter group in $q$ variables such that there exists a maximal $(r-q)$-parameter subgroup; then, every infinitesimal transformation which leaves $G$ invariant belongs to $G$.*[34]

At first, Cartan had hoped that he could deduce this result from a direct application of his radical decomposition theorem. Indeed, as we learn from one of his notebooks dating back to 1893, he had provided a proof of Theorem (5) which was based upon the following observation. Given a $r$-parameter (arbitrary) group $G$ which contains a simple or semisimple invariant $q$-parameter subgroup $G$; then $G_r$ can be decomposed into two invariants subgroups, one of them is $G_q$, let the other be $G_{r-q}$, such that $[G_q, G_{r-q}] = 0$. Unfortunately, the proof of this result depended, in an essential way, upon the radical decomposition theorem. As a consequence of this, when Cartan rejected his proof of the latter, he was forced to find an alternative route leading to Theorem (5).

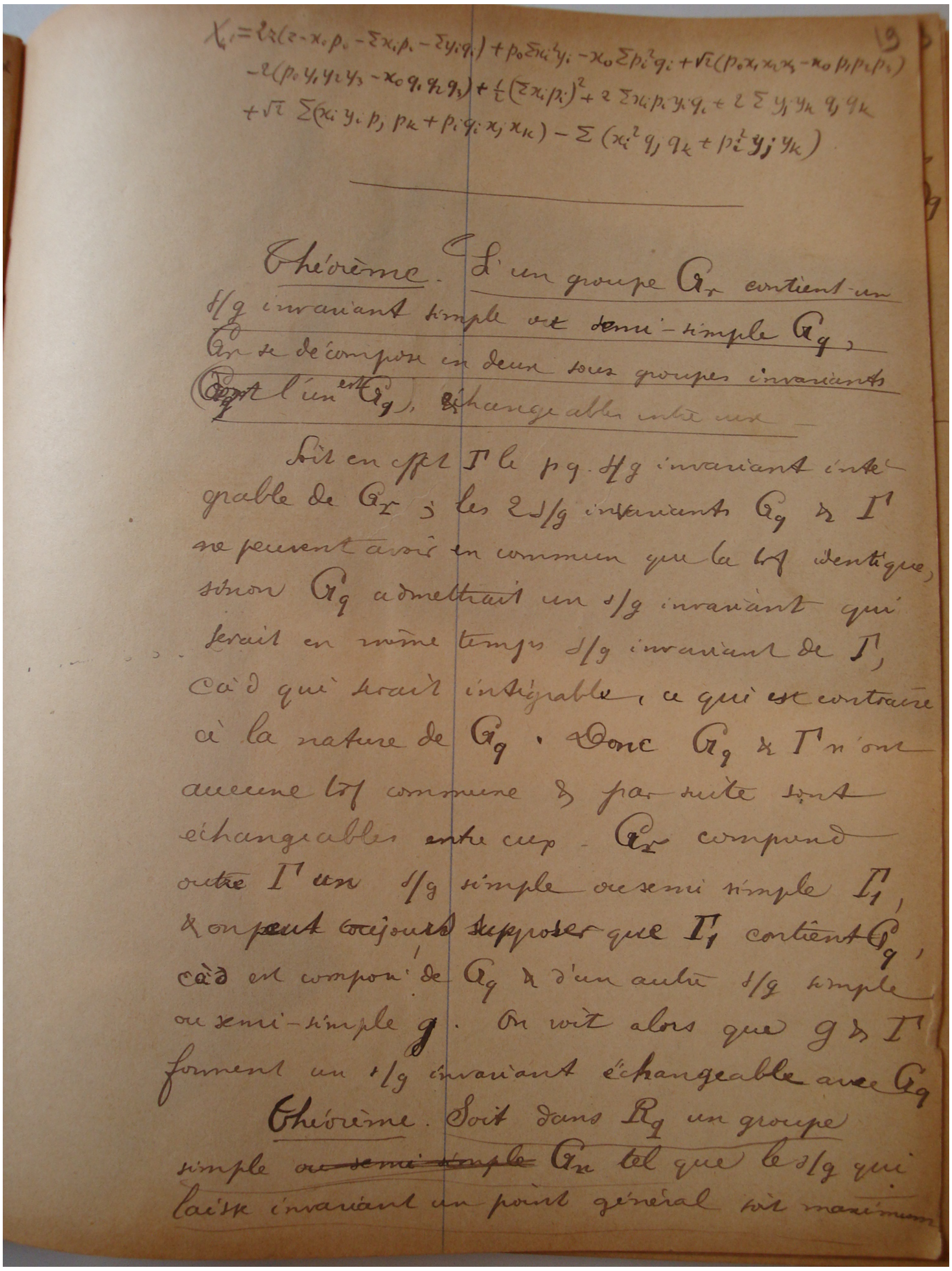
Indeed, in his thesis, he provided a fully satisfying demonstration which relied upon the notion of maximal integrable invariant subgroup (the radical of $G$) and some relevant properties of the quadratic form $\psi_2$. The crucial observation was that the search for transformations leaving a given group $G$ invariant could be reduced to the search of those transformations whose commutators with $G$ belongs to the radical of $G$[35].

---

[33] *Meine zweite Frage ist folgende: Es sei vorgelegte eine einfache (projectivische) Gruppe $G_r$ einer $q$-fach ausgedehnte Mannigfaltigkeit, unter deren Untergruppen keine mehr als $r - q$ Parameter enthält. Giebt es dann eine Gruppe mit mehr als $r$ Parametern, welcher $G_r$ als invariante Untergruppe angehört? Diese Frage muss wahrscheinlich mit Nein beantwortet werden.* See [Lie 1885, p. 135].

[34] See [Cartan 1894, p. 113].

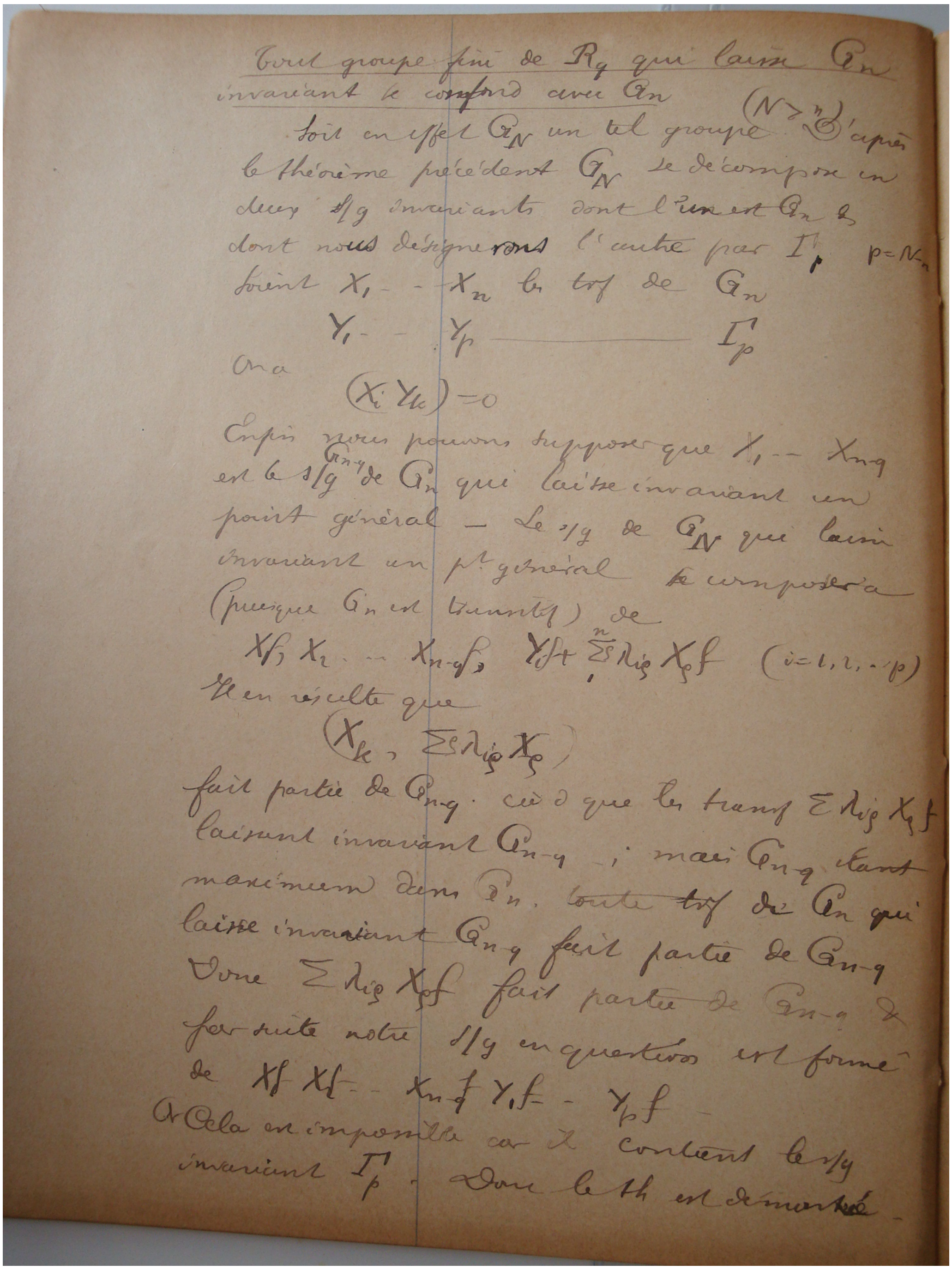
[35] Cartan provided a very simple application of this result in the case of the group of Euclidean movements. Here, $\Gamma$, the radical of $G$, coincides with the subgroup of all translations and it turns out that the transformations which one seeks for are the homotheties of the space. Obviously, it is implicit that every transformation of $G$ leaves $G$ invariant as well.


Théorème. Si un groupe $G_r$ contient un s/g invariant simple ou semi-simple $G_q$, $G_r$ se décompose en deux sous groupes invariants dont l'un est $G_q$, échangeables entre eux.

Soit en effet $\Gamma$ le pg. s/g invariant intégrable de $G_r$; les 2 s/g invariants $G_q$ & $\Gamma$ ne peuvent avoir en commun que la tf identique, sinon $G_q$ admettrait un s/g invariant qui serait en même temps s/g invariant de $\Gamma$, c.à.d qui serait intégrable, ce qui est contraire à la nature de $G_q$. Donc $G_q$ & $\Gamma$ n'ont aucune tf commune & par suite sont échangeables entre eux. $G_r$ comprend outre $\Gamma$ un s/g simple ou semi simple $\Gamma_1$, & on peut toujours supposer que $\Gamma_1$ contient $G_q$, c.à.d est composé de $G_q$ & d'un autre s/g simple ou semi-simple $g$. On voit alors que $g$ & $\Gamma$ forment un s/g invariant échangeable avec $G_q$.

Théorème. Soit dans $R_q$ un groupe simple ~~ou semi simple~~ $G_r$ tel que le s/g qui laisse invariant un point général soit maximum

Figure 2.3: Page taken from Cartan's notebook, 1.17, Fonds É. Cartan, 38J, Archives of the Académie des Sciences de Paris. Here and in the following picture, one can read statement and proof of Theorem (5) mentioned in the text.



Tout groupe fini de $R_q$ qui laisse $G_n$ invariant se confond avec $G_n$ $(N \geq n)$

Soit en effet $G_N$ un tel groupe. D'après le théorème précédent $G_N$ se décompose en deux s/g invariants dont l'un est $G_n$ & dont nous désignerons l'autre par $\Gamma_p$ $p=N-n$

Soient $X_1 \dots X_n$ les trf de $G_n$

$Y_1 \dots Y_p$ —— $\Gamma_p$

On a

$(X_i Y_k) = 0$

Enfin nous pouvons supposer que $X_1 \dots X_{n-q}$ est le s/g $G_{n-q}$ de $G_n$ qui laisse invariant un point général — Le s/g de $G_N$ qui laisse invariant un pt général se composera (puisque $G_n$ est transitif) de

$X_1f, X_2 \dots X_{n-q}f, \; Y_if + \sum_1^n \lambda_{i\varrho} X_\varrho f$ $(i=1,2,\dots p)$

Il en résulte que

$(X_k, \sum \lambda_{i\varrho} X_\varrho)$

fait partie de $G_{n-q}$. c.à.d. que les transf $\sum \lambda_{i\varrho} X_\varrho f$ laissent invariant $G_{n-q}$ ; mais $G_{n-q}$ étant maximum dans $G_n$, toute trf de $G_n$ qui laisse invariant $G_{n-q}$ fait partie de $G_{n-q}$. Donc $\sum \lambda_{i\varrho} X_\varrho f$ fait partie de $G_{n-q}$ & par suite notre s/g en question est formé de $X_1f\, X_2f \dots X_{n-q}f\, Y_1f \dots Y_pf$

Or cela est impossible car il contient le s/g invariant $\Gamma_p$. Donc le th est démontré.

Figure 2.4: Page taken from Cartan's notebook, 1.17, Fonds É. Cartan, 38J, Archives of the Académie des Sciences de Paris.

Again we have an example of how Killing's structural approach was exploited by Cartan in order to solve those problems which were essential for the actual functioning of Lie's integration theory.

Our second and last example deals with the normal series decomposition of an arbitrary group $G$. As we have already seen, Lie was able to reduce the integration problem of a complete system which admits a continuous group of transformations to the integration of a single equation which admits a *simple* group. In order to do that, he had observed, the knowledge of a normal series decomposition of an arbitrary group $G$ was required and to this end he had supposed that a preliminary algebraic investigation of the properties of $G$ could be carried out.[36]

Cartan's thesis allowed to do exactly that. Indeed, it provided, at least in principle, a general procedure for deducing such a normal series. Once more, the notion of maximal integrable invariant subgroup turned out to play a key role.
Recall that a normal series decomposition of a group $G$ consists of finding a chain of invariants subgroups such that $G_i$ is a maximal invariant subgroup of $G_{i-1}$.

Cartan started up by considering the case in which the group $G$ is integrable. In this eventuality, he pointed out, to find such a decomposition did not present any special difficulty. Indeed, it was sufficient to consider the series of derived groups of increasing order $G, G', G^{(2)}$, etc. and to add to $G^{(i+1)}$ $r_i - r_{i+1} - 1$ ($r_i$ is the number of independent infinitesimal transformation which generate $G^{(i)}$) infinitesimal transformations independent of $G^{(i+1)}$.

A little more problematic was the case in which $G$ is not integrable. If so, Cartan explained, one needs to consider the maximal integrable invariant subgroup $\Gamma$. If $\Gamma$ is supposed to be known, then the semisimple group $G/\Gamma$ can be associated to $G$, via the canonical projection $\pi : G \to G/\Gamma$. Moreover, suppose that given an arbitrary semisimple group one knows how to decompose it into simple invariant subgroups. Let us indicate with $g_1, \cdots, g_h$ the (unique) decomposition of $G/\Gamma$ into simple subgroups. Now, define $g' = g_2 \oplus g_3 \oplus \cdots g_h$, $g'' = g_3 \oplus \cdots g_h$, etc.. Then, Cartan concluded, the chain of inclusions

$$G \supset \pi^{-1}\left(g'\right) \supset \pi^{-1}\left(g''\right) \supset \cdots \supset \pi^{-1}\left(g^{(h-1)}\right) \supset \Gamma \supset \Gamma' \cdots,$$

where $\Gamma \supset \Gamma' \cdots$ is a normal series decomposition of $\Gamma$, is a normal series decomposition of $G$. Thus, the entire procedure boiled down to the resolution of the following two problems: i) given a group $G$, to determine its maximal integrable invariant subgroup $\Gamma$; ii) given a semisimple group $G$, to provide its decomposition into invariant simple subgroups.

[36] More recently, in [Vessiot 1892] Vessiot had to face precisely this problem in his integration theory of ordinary *linear* differential equations. See Appendix B.

The first of these two problems was faced and completely solved by Cartan already in [Cartan 1894]. Indeed, by applying his own version of Killing's secondary root theory, he was able to derive a simple formula by means of which the radical of a given group $G$ can be computed once its structure constants $c_{ijk}$ are supposed to be known. Indeed, Cartan proved that the maximal integrable, invariant subgroup of $G$ is generated by all infinitesimal transformations $\sum_k e_k X_k$, such that the components $e_k$, $k = 1, \cdots, r$, satisfy the following linear system of equations:

$$\sum_{\rho=1}^{r} c_{ik\rho} \frac{\partial \psi_2}{\partial e_\rho} = 0, \qquad (i, k = 1, \cdots, r).$$

As for the second of the two problems, a general procedure was indicated in [Cartan 1896, p. 23-28]. To this end the complete classification of all simple structures attained in his thesis proved to be essential; a clever application of the Galois theory of algebraic equations and a careful study of the relations between the roots of the characteristic equation of a given semisimple group were then combined in order to obtain the requested decomposition of the group into invariant, simple subgroups.

In this way, the systematization of Lie's integration theory of complete systems could be regarded as completed.

Before moving to analyze Cartan's subsequent works, namely in the realm of infinite continuous groups, it is necessary to provide a preliminary study of the state of the art of the theory since Lie's first contributions. This is the object of the following chapter.

# Chapter 3

# Infinite continuous groups 1883-1902

Sophus Lie's contributions to the theory of *finite* continuous groups of transformations have been thoroughly examined over the past few decades. Hawkins' authoritative analysis[1] for instance offers a valuable reference for anyone interested in this area of historical research. Nevertheless, it appears that scarce attention has been paid thus far on an other important part of Lie's mathematical production, the theory of infinite continuous groups, which nowadays we would call Lie pseudogroups.

The emphasis on the theory of finite continuous groups and the corresponding neglect towards the theory of infinite continuous ones is justified on one hand by the enormous amount of papers which Lie devoted to the former and, on the other, by the numerous intrinsic difficulties which afflicted, and still afflict, the development of the latter. However such a tendency, already popular in the first half of the past century, runs the risk of restricting, when not to distort, Lie's mathematical thought. In fact, the applicative role played by his theory of groups in the general problem of integration of differential equations induced him to consider all types of continuous groups, finite and infinite ones, without distinction. It is true that the theories of the two types of groups were quite different one from the other and that the theory of infinite continuous groups, as developed by Lie, was little more than rudimentary. Nevertheless, at least in Lie's plans, the two theories should be regarded as different parts of a more general theory of differential invariants in which finite and infinite groups could be treated on the same basis.

In view of the present state of the theory, it may seem surprising that Lie could even conceive the possibility of developing a unified approach for dealing with so different mathematical entities. Indeed, our modern understanding of Lie groups is characterized by a sharp distinction between the

[1] See [Hawkins 2000] and [Hawkins 1991].

two cases. In the finite dimensional case, since C. Chevalley's foundational work [Chevalley 1946], a continuous group is identified with an analytic manifold endowed with a compatible algebraic structure. While in the infinite dimensional case, since no agreement on the existence of an abstract entity underlying them has been found yet, one is compelled to consider only pseudogroups of diffeomorphisms. Nevertheless, it should be observed that in Lie's time and still for some decades later on, finite continuous groups too were not conceived independently of their action on a given manifold and consequently they were regarded more as pseudogroups of diffeomorphisms than as abstract manifolds. Although the notion of an abstract group was already available (the *parametric group*), it seems that no special importance was accorded to it. For this very reason, Lie's project of setting up a unitary conceptual framework in which finite and infinite groups could be studied on the same level, was not only pervious but also greatly desired. The important role attributed by Lie to such an ambitious research program was clarified in the course of the lengthy preface to [Lie 1893] where he wrote:

> *In this respect a general theory of all continuous groups, which are defined by means of differential equations, will be given; a theory according to which finite and infinite groups appear to be as essentially on the same footing.* [2]

Unfortunately Lie did not succeed in fulfilling such an aim; nor he provided a theory of an organic and systematic kind which could be compared to the relative completeness of the theory of finite continuous groups. Ugo Amaldi's effective remarks express very clearly what the state of the theory was soon after Lie's death in 1899.

> *While, upon Lie's death, the theory of finite continuous groups could be considered as essentially completed, as for the infinite groups (that is depending of an infinite number of arbitrary parameters), LIE had limited himself to provide only a short treatment of the foundations of the theory, along with fragmentary mentions of those further general views which also in this field he had achieved (who knows how?), following audacious shortcuts.*[3]

[2] *In Verbindung damit* [ i.e. with the theory of differential invariants ]*wird zugleich eine umfassende Theorie aller continuirlichen Gruppen gegeben werden, die sich durch Differentialgleichungen definiren lassen, eine Theorie, bei der die endlichen und die unendlichen continuirlichen Gruppe als wesentlich gleichberechtigt erscheinen.* See [Lie 1893], *Vorrede XIX.*

[3] [...] *Mentre la teoria dei gruppi continui finiti, alla morte del LIE, si poteva riguardare ormai come sostanzialmente compiuta, pei gruppi infiniti (o dipendenti da più che un numero finito di parametri arbitrari) il LIE non lasciava a noi che una sommaria trattazione dei fondamenti della teoria, accanto a scarsi e frammentari accenni di quelle ulteriori ve-*

Still, his contributions to the subject are worth considering, since they represent the starting point of a long and interesting mathematical enterprise whose first fruitful results would blossom out in the expert hands of E. Vessiot and É. Cartan who, in quite different ways, tried to develop such a theory investigating the possibility to pursue a structural approach.

First, the emergence of the notion of continuous infinite group in Lie's early papers dating back to the 1880's will be described. Then, Engel's *Habilitationsschrift* and Lie's attempts to axiomatize his theory will be discussed. Afterwards, our attention will be focused on Medolaghi's contributions in which, for the first time, the main focus of attention was placed on consideration of finite transformations. Finally, Vessiot's monumental *Mémoire couronnée* will be analyzed. Our emphasis will chiefly be put on abstract as well as structural aspects of his theory; at the same time, Vessiot's attempts to develop applicative implications of the latter, namely differential generalizations of Galois classical theory, will be described.

## 3.1 Lie's first contributions

As Lie himself was once to observe[4], the notion of infinite continuous group (*unendliche Gruppe*) was first introduced by him in [Lie 1883] purely in terms of infinitesimal transformations. He had dealt with infinite groups before; in particular, already in 1872-1873 he had studied the existence of invariants of a partial differential equation with respect to all contact- or point- transformations which are clearly examples of infinite continuous groups. Nevertheless it was only in 1883 that Lie started to offer a systematic treatment of the subject. In a letter to Adolf Mayer (early 1883), he announced his new interests of research with the following optimistic words:

> *Over the last few months I succeeded in taking an important step further in the theory of transformation groups. Before, I essentially limited myself to groups with a finite number of parameters. Nonetheless, there also exist groups with an infinite number of parameters.* [5]

The notion itself of continuity of groups, as is well known, coincided in Lie's view with the possibility of generating every finite transformation belonging to the group (finite or infinite) by the reiterated action of an infinitesimal

*dute generali, che egli pure in questo campo, chissà per quali audaci accorciatoie, aveva saputo conquistare.* See [Amaldi 1908, p. 293].

[4]See [Lie 1891a][p. 300].

[5]*Es ist mir in den letzten Monaten gelungen, sehr wichtige Fortschritte in der Theorie der Transformationsgruppen zu machen. Früher beschränkte ich mich nämlich wesentlich auf Gruppen mit einer endlichen Anzahl Parameter. Es gibt aber auch Gruppen mit unendlich vielen Parametern. Jetzt beherrsche ich auch diese Gruppen. See* Sophus Lie Gesammelte Abhandlungen, Anmerkungen zum sechsten Bande*, p.777.*

transformation. However, in the case of infinite continuous groups the number of such independent infinitesimal transformations was unlimited. For this reason, in 1883 Lie decided to give an alternative and more manageable definition of continuity: a group was said by him to be continuous if the components of a generic infinitesimal transformation belonging to it represent the general solution of a system of (linear) partial differential equations. It is a crucial step in Lie's entire mathematical production. Indeed, still in 1894, he emphasized the importance of this fortunate intuition when recollecting the historical development that had led him to elaborate his theory of infinite groups. He wrote:

> *Finally, in early 1883 I had the fortunate idea of singling out, among all groups, those whose transformations could be defined by means of differential equations.* [6]

Lie started his analysis in [Lie 1883] by taking into consideration finite groups with the hope of extending the validity of his argument to infinite ones. His reasoning was limited to the case in which only two independent variables are involved, that is to the case of continuous transformations of the plane. He first considered $r$ independent infinitesimal transformations of a $r$-parameters finite group:

$$B_i(f) = \xi_i(x,y)\frac{\partial f}{\partial x} + \eta_i(x,y)\frac{\partial f}{\partial y}, \quad (i = 1, \cdots, r). \tag{3.1}$$

Thus, the most general infinitesimal transformation of the group can be written as follows:

$$c_1 B_1(f) + \cdots + c_r B_r(f) = \xi(x,y)\frac{\partial f}{\partial x} + \eta(x,y)\frac{\partial f}{\partial y}, \quad c_i = \text{const.} \tag{3.2}$$

It is clear that, since the differential operators in (3.1) generate a group, the following relations hold:

$$(B_i, B_k)(f) \equiv B_i(B_k(f)) - B_k(B_i(f)) = \sum_{s=1}^{r} c_{iks} B_s(f), \qquad (i, k = 1, \cdots, r).$$

As a consequence of this, Lie observed, the components $\xi$ and $\eta$ of a generic infinitesimal transformation of a given finite group satisfy a certain system of linear partial differential equations of type

$$A\xi + B\eta + C\frac{\partial \xi}{\partial x} + D\frac{\partial \xi}{\partial y} + E\frac{\partial \eta}{\partial x} + \cdots + L\frac{\partial^n \xi}{\partial x^n} + \cdots = 0, \tag{3.3}$$

[6] *Endlich hatte ich im Anfange des Jahres 1883 die glückliche Idee, unter allen Gruppen diejenigen herauszugreifen, deren Transformationen durch Differentialgleichungen definiert werden können.* See [Lie 1895a, p. 43].

where $A, B, C, D, E, \cdots, L, \cdots$ are coefficients which depend on $x, y$ only. According to the structure of the group under consideration, such equations will assume different forms.[7]

For example, in the case of the following 6 parameter finite group whose defining equations (*Definitionsgleichungen*) for finite transformations are given by:

$$x' = \frac{\alpha_1 x + \beta_1}{\gamma_1 x + \delta_1}, \quad y' = \frac{\alpha_2 y + \beta_2}{\gamma_2 y + \delta_2},$$

the equations that the components of a general infinitesimal transformation $B(f) = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y}$ have to satisfy, read as follows:

$$\frac{\partial \xi}{\partial y} = \frac{\partial^3 \xi}{\partial x^3} = \frac{\partial \eta}{\partial x} = \frac{\partial^3 \eta}{\partial y^3} = 0.$$

In general, if $m$ linear partial differential equations of type (3.3) are the defining equations of a given *finite* group then, on the basis of the theory of such groups already developed in detail by Lie, one can ensure that such equations satisfy the following further characteristic properties: i) let $(\xi, \eta)$ and $(X, Y)$ be two pairs of solutions to (3.3), then such is the pair

$$\left(\xi \frac{\partial X}{\partial x} + \eta \frac{\partial X}{\partial y} - X \frac{\partial \xi}{\partial x} - Y \frac{\partial \xi}{\partial y}, \quad \xi \frac{\partial Y}{\partial x} + \eta \frac{\partial Y}{\partial y} - X \frac{\partial \eta}{\partial x} - Y \frac{\partial \eta}{\partial y}\right).$$

ii) the most general solution to the defining equations (3.3) is given by:

$$\xi = c_1 \xi_1 + \cdots + c_r \xi_r$$
$$\eta = c_1 \eta_1 + \cdots + c_r \eta_r,$$

where $(\xi_i, \eta_i)$, $(i = 1, \cdots, r)$ are solutions to (3.3). Viceversa, every system of linear partial differential equations satisfying the above mentioned properties i) and ii) can be regarded as defining a finite continuous group of transformations.

At this point, it was natural for Lie to pose the question about the existence of differential systems of type (3.3) which fulfil condition i) but not condition ii). The answer to such a question, which entailed the question about the existence of infinite continuous groups, was affirmative. A very simple example was given by the following system of differential equations:

$$\frac{\partial \xi}{\partial y} = 0, \quad \frac{\partial \eta}{\partial x} = 0. \tag{3.4}$$

[7] The general procedure for obtaining such a system of equations was provided by Lie, for example, in [Lie 1888, Chap. 11]. It essentially consisted of differentiating an appropriate number of times the equations

$$\xi = \sum_{i=1}^{r} c_i \xi_i, \qquad \eta = \sum_{i=1}^{r} c_i \eta_i,$$

obtained from (3.2), and then of eliminating the parameters $c_i$, $(i = 1, \cdots, r)$.

Indeed, if $(X_1(x), Y_1(y))$ and $(X_2(x), Y_2(y))$ are two pairs of solutions to (3.4) then it is clear that $\left(X_1\frac{\partial X_2}{\partial x} - X_2\frac{\partial X_1}{\partial x}, \quad Y_1\frac{\partial Y_2}{\partial y} - Y_2\frac{\partial Y_1}{\partial y}\right)$ is a solution too and, accordingly condition i) above is fulfilled. On the contrary, for what pertains condition ii), every infinitesimal transformation of the form

$$X(x)\frac{\partial}{\partial x} + Y(y)\frac{\partial}{\partial y},$$

for arbitrary functions $X(x)$ and $Y(y)$, is easily recognized as a solution to (3.4); thus ii) is not satisfied. In fact the corresponding group of finite transformations is of an infinite type whose defining equations are:

$$\begin{cases} x' = F(x) \\ y' = \Phi(y) \end{cases},$$

for arbitrary functions $F(x), \Phi(y)$.

This being the general framework, Lie set out to classify all differential systems of type (3.3) which satisfy condition i) but not condition ii). In doing so, he wrote, the problem of finding all continuous groups of transformations acting on the plane could be contemporaneously attained. However, at this point, a delicate technical problem arises. It concerns the connection between the groups of infinitesimal transformations and the groups of finite transformations. In 1883 Lie assumed implicitly that these two notions could be identified via a bijective correspondence according to which a group of infinitesimal transformations is associated to every continuous group of finite transformations and *viceversa*. Only later in 1891 when trying to lay the foundations of the theory of infinite groups, did Lie devote his attention to a more detailed examination of the subject, "demonstrating" the existence of such a correspondence without any exception. Yet later, we will see that Lie's proposed solution to the question remained problematic essentially in consequence of an opaque distinction between what nowadays we would call groups and semigroups of transformations.

Nevertheless, already in 1883 Lie was able to demonstrate a classification theorem for all infinite transformation groups in two variables which could be regarded as an extension of an analogous result he had obtained five years before in [Lie 1878] in the realm of finite groups.

## 3.2 Differential invariants

Despite the difficulties that stemmed from the definition of infinite continuous groups in terms of their infinitesimal transformations, Lie regarded such a definition as a major achievement for his general theory of transformation groups. As he wrote to Klein at the beginning of 1884, the main obstacle to a successful development of his ideas had been posed by the definition itself of what an infinite continuous group is; once this problem was solved, it was relatively simple, Lie asserted, to develop a theory of the invariants of such groups.

Beyond the intrinsic value derived by a deepened study of the subject which provided an important step forward for the whole theory of transformation groups, Lie's interest in infinite continuous groups and namely in the theory of invariants lay in the vast realm of their applications too. Not only, as one may easily expect, the theory of differential equations but also numerous branches of geometry could in principle be subsumed under the general approach offered by his new theory.

This emerges quite clearly by some letters dating back to late 1883 and early 1884. In December 1883, for instance, Lie wrote to Mayer:

> *Gauss' theory of curvature and Mindings' theorem on geodetic curvature, in addition the theories which are connected to them, are special cases of my general theory from which I expect much in many directions.*[8]

and, at the beginning of the following year, along the same lines, he wrote to F. Klein:

> *The theory of invariants under deformation which originated with Gauß and Mindings is clearly a very special case of my general invariant theory which, as a consequence of this, covers many other particular theories. I consider this to be an advantage rather than a shortcoming.*[9]

These remarks and plans soon materialize in a long memoir, [Lie 1884], which Lie concluded in May 1884. Once more, the starting point of Lie's analysis is the concept of infinite groups itself but, again, as in [Lie 1883], no special attention on the distinction between finite and infinitesimal transformations was paid. Let us consider the relevant definition in Lie's words:

> *I say that a continuous family of transformations builds up an infinite continuous group*
>
> - *if the succession of two transformations of the family is equivalent to a unique transformation of the family,*

[8] *Die Gaußische Theorie des Krümmungsmaßes und Mindings Satz über geodätische Krümmung, wie die sich hieran anschließenden Theorien, sind sie specielle Teile meiner allgemeinen Theorie, von der ich mir sehr viel in vielen Richtungen verspreche.* See *Sophus Lie Gesammelte Abhandlungen, Anmerkungen zum sechsten Bande*, p.779.

[9] *Die von Gauß, Minding etc. herrührende Theorie der Invarianten bei Deformation ist natürlich ein sehr spezieller Fall meiner allgemeinen Invariantentheorie, die ebenso mehrere andere partikuläre Theorien umfaßt. Dies betrachte ich indess als Gewinn als wie Verlust.* See *Sophus Lie Gesammelte Abhandlungen, Anmerkungen zum sechsten Bande*, p.788.

- *if every finite transformation of the family can be generated by infinite repetition of a determined infinitesimal transformation of the family,*
- *if the family contains an infinite number of independent infinitesimal transformations.*[10]

Lie limited himself to the case in which, if $B(f) = \xi_1 \frac{\partial f}{\partial x_1} + \cdots + \xi_n \frac{\partial f}{\partial x_n}$ indicates a generic infinitesimal transformation belonging to the group, then its components $\xi_i$, $(i = 1, \cdots, n)$, are determined by a system of partial differential equations of type

$$\Omega_i \left( x_1, \cdots, x_n, \xi_1, \cdots, \xi_n, \frac{\partial \xi_1}{\partial x_1}, \cdots \right) = 0.$$

This system is necessarily linear since it is easy to prove that if $B_1(f)$ and $B_2(f)$ are any two infinitesimal transformations which belong to the group, then also $c_1 B_1(f) + c_2 B_2(f)$, for arbitrarily chosen constants $c_1$ and $c_2$, is an infinitesimal transformation of the same group. Furthermore, Lie demonstrated that if $B_1(f)$ and $B_2(f)$, as before, are any two infinitesimal transformations, the Lie bracket $(B_1, B_2)$ is also an infinitesimal transformation belonging to the group. In this way, as it is easily recognized, Lie recovered the definition of infinite groups in terms of differential systems of a special kind he had given in [Lie 1883].
These preliminary remarks having been stated, the main goal that Lie set out to pursue in [Lie 1884], as the title suggests, was the study of the differential invariants of a given infinite continuous group. The analogous problem for finite groups had already been tackled and worked out by Lie. As he wrote to Klein at the end of 1883, already in 1874 Lie had developed a method for computing the differential invariants in the finite dimensional case. Only more recently, however had he been able to transfer his analysis to a more general setting. Although he had dealt with specific problems connected with the differential invariants of infinite groups, only now was Lie ready to develop the subject in more systematic way. Before providing a demonstration of the existence of differential invariants, Lie offered a detailed analysis of some specific examples which shed some light on the general strategy employed by him.

[10] *Eine kontinuierliche Schaar von Transformationen bildet, sage ich, eine unendliche kontinuierliche Gruppe,*

– *wenn die Sukzession zweier Transformationen der Schaar mit einer einzigen Transformation der Schar äquivalent ist,*

– *wenn jede endliche Transformation der Schar durch unendlichmalige Wiederholung einer bestimmten infinitesimalen Transformation der Schar erzeugt werden kann,*

– *wenn die Schar unendlichviele unabhängige infinitesimale Transformationen enthält.*

See [Lie 1884, p. 553].

The example of the equivalent group of the plane is particulary instructive. This is the group whose transformations are characterized by the property of being volume-preserving[11]. If $B(f) = X(x,y)\frac{\partial f}{\partial x} + Y(x,y)\frac{\partial f}{\partial y}$ indicates a generic infinitesimal transformation of the group then the components $X(x,y)$ and $Y(x,y)$ satisfy the defining equation $\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = 0$. In order to find differential invariants of the group it is first necessary to introduce auxiliary variables, say $u$ and $v$, which are left invariant by any transformations of the group itself. The variables $x$ and $y$ are then considered to be functions of $u$ and $v$. At this point one has to consider the so-called *prolongation* (*Erweiterung*) of the infinitesimal transformation $B(f)$. By putting

$$\frac{\partial x}{\partial u} = x_u, \quad \frac{\partial x}{\partial v} = x_v, \quad \frac{\partial y}{\partial u} = y_u, \quad \frac{\partial y}{\partial v} = y_v,$$

it is easy to calculate the effect of the action of $B(f)$ on the set of variables $x, y, x_u, x_v, y_u, y_v$. Lie determined it by characterizing the prolonged transformations as those that leave the contact forms $dx - x_u du - x_v dv$ and $dy - y_u du - y_v dv$ invariant. Alternatively, such a prolongation can be derived through a direct computation of the following type:

$$\delta x_u = \delta\left(\frac{\partial x}{\partial u}\right) = \frac{\partial(\delta x)}{\partial u} = \frac{\partial X(x,y)}{\partial u}\delta t = \left[\frac{\partial X}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial X}{\partial y}\frac{\partial y}{\partial u}\right]\delta t.$$

In an analogous way one can obtain the expressions for the variations of $x_v, y_u, y_v$ and finally find the first order prolongation of $B(f)$, to be indicated with $B^{(1)}(f)$[12]:

$$B^{(1)}(f) = X\frac{\partial f}{\partial x} + Y\frac{\partial f}{\partial y} + (X_x x_u + X_y y_u)\frac{\partial f}{\partial x_u} + (X_x x_v + X_y y_v)\frac{\partial f}{\partial x_v} + \\ + (Y_x x_u + Y_y y_u)\frac{\partial f}{\partial y_u} + (Y_x x_v + Y_y y_v)\frac{\partial f}{\partial y_v}.$$

Since the only condition that $X$ and $Y$ have to fulfill is that the divergence $X_x + Y_y$ vanishes identically, Lie obtained the following system of linear homogeneous partial differential equations of first order:

$$\begin{cases} \frac{\partial f}{\partial x} = 0, \\ \frac{\partial f}{\partial y} = 0, \\ x_u\frac{\partial f}{\partial x_u} + x_v\frac{\partial f}{\partial x_v} - y_u\frac{\partial f}{\partial y_u} - y_v\frac{\partial f}{\partial y_v} = 0, \\ y_u\frac{\partial f}{\partial x_u} + y_v\frac{\partial f}{\partial x_v} = 0, \\ x_u\frac{\partial f}{\partial y_u} + x_v\frac{\partial f}{\partial y_v} = 0. \end{cases} \tag{3.5}$$

[11] In Lie's words: *Ihre Transformationen sind dadurch charakterisirt, dass sie alle Flächenräume der Cartesischen Ebene x, y invariant lassen.* See [Lie 1884, p. 561].

[12] The symbols $X_x, X_y$, etc. stand for $\frac{\partial X}{\partial x}, \frac{\partial X}{\partial y}$, etc..

It can easily be seen that the system (3.5) is complete in the sense of Clebsch's definition; thus its solution

$$I = x_u y_v - x_v y_u,$$

is the unique, first order invariant of the group. Other differential invariants of higher order can then be derived by observing that since $u, v$ are left invariant by any transformation of the group, all the derivatives of $I$ with respect to $u, v$ are invariants too.
From such a simple example Lie moved on to demonstrate the existence of differential invariants for a generic infinite group. His main result was the following

**Theorem 6 (Lie, 1884)** *Every infinite continuous group determines an infinite sequence of differential invariants which can be defined as solutions of complete systems.* [13]

The rest of [Lie 1884] was devoted to a large amount of examples of the general theory of invariants. To mention only a few, Lie focused his attention on the theory of differential invariants of second order, ordinary differential equations, on the theory of linear ordinary differential equations of any order, on the theory of invariants of first order partial differential equations.

According to the plans communicated in the letters to Mayer and Klein, Lie was also interested in developing the applications of his theory in the realm of pure geometry too. Namely, Lie considered the element of the arc-length of an arbitrary surface; from Gauss' relevant analysis this could be written as:

$$ds^2 = Edx^2 + 2Fdxdy + Gdy^2.$$

By considering the transformations

$$\begin{cases} x_1 = X(x, y) \\ y_1 = Y(x, y) \end{cases},$$

the arc-length element $ds^2$ was transformed into

$$ds^2 = E_1 dx_1^2 + 2F_1 dx_1 dy_1 + G_1 dy_1^2,$$

where $E_1, F_1$ and $G_1$ are appropriate functions of $E, F, G$ and $x, y$. It turned out that such relations build up an infinite continuous group in the five variables $E, F, G, x, y$. By applying the general procedure, Lie observed, the Gaussian curvature of the surface could be obtained as a differential invariant of the continuous group just introduced.

[13] *Jede unendliche continuirliche Gruppe bestimmt eine unendliche Reihe von Differentialinvarianten, die als Lösungen von vollständigen Systemen definirt werden können.* See [Lie 1884, p. 566].

Finally, the last paragraph of [Lie 1884] was devoted to some general remarks where the problem of equivalence of two differential structures was briefly discussed. Indeed, the question whether or not one structure could be brought into the other by a transformation of a given continuous group could be faced, Lie observed, by means of the theory of differential invariants. More precisely, necessary conditions could be obtained for the existence of such transformations in terms of invariant differential relations. Once more Lie explained his point of view by making recourse to Gauss' theory of surfaces. To decide whether a surface with arc-length element

$$ds^2 == Edx^2 + 2Fdxdy + Gdy^2$$

could be transformed into another surface with arc-length element

$$ds_1^2 = E_1 dx_1^2 + 2F_1 dx_1 dy_1 + G_1 dy_1^2$$

one computes certain differential invariants, say $A, B$ and $C$, for both surfaces and then determines the relations among $A, B$ and $C$. If one finds only one relation, say

$$A = \Omega(B, C),$$

then its form is a true picture of all the properties of the surface which are invariant under bending. Indeed, two surfaces can be bent one into the other, i.e. they are equivalent, only if their $A, B$ and $C$ are connected by the same relations.
Lie did not go into details and did not explain how one could obtain not only necessary conditions for the equivalence but also sufficient ones. Almost twenty years later, in the early 1900's, Élie Cartan took up the same question by providing a systematic framework which enabled him to obtain sufficient conditions too. Essentially, the basic idea in Cartan's equivalence theory consisted of the possibility of putting every differential structure into a standard form parameterized by subgroups of the linear general group. A crucial role was played by the development of exterior differential calculus. After all, Lie himself seemed to be well aware of the limits of his equivalence theory which he regarded as highly provisional. Indeed, in a draft of a letter to É. Picard containing his intellectual testimonial he wrote:

> *In volume 24* [of Math. Ann.]*, I treated the important question of how one decides whether given expressions or a given system of differential equations can be brought to given forms through the transformations of a given group. My theory of differential invariants gave me the* necessary *criteria. But I was not able, at that time, to reduce these* sufficient *criteria to their simplest form. Later, I was able to do this for* finite *groups. For finite groups, there exists, most certainly, a full system of differential*

> *invariants,* all *allowing themselves to be derived by differentiation* [...]. *I have now attempted to expand this to infinite groups. For a series of infinite groups, I have* verified *that they have a completely analogous invariant theory. Unfortunately, I have* not *been successful in proving the existence of a complete system of invariants. But, I am convinced that this conjecture is correct.*[Taken from [Fritzsche 1999, p. 8]]

In the same draft Lie turned directly to Picard asking him to think about this problem. He wrote, "if you do not want to deal with it, then make sure that Vessiot does. *Vessiot should be the heir to my integration theories*".

As will be seen, Vessiot actually contributed to Lie's research program in an essential way, especially with his theory of automorphic systems. However, it was Cartan who fully developed a general and systematic approach to the equivalence problem. Interestingly, as will be seen, it was the very need to gain a better understanding of it that led Cartan to conceive a completely different theoretical framework for dealing with infinite continuous groups.

Shortly after Lie had written the above quoted lines[14], the problem of finding sufficient criteria which guarantee the equivalence of two given differential systems was tackled by Arthur Tresse, who had spent some time in Leipzig under the guidaince of Lie and Engel. Upon Lie's direct suggestion, in [Tresse 1893] Tresse decided to undertake the challenge represented by the proof of Lie's conjecture about the existence of a complete system of invariants for (infinite) continuous groups[15]. In accordance with Lie's hopes, Tresse succeeded in proving that every infinite continuous group admits a finite number of independent invariants such that every arbitrary invariant (of any order) can be expressed in terms of them and their derivatives. In this regard, a crucial technical tool was provided by the theorem according to which the study of a general system of PDE's can always be reduced to that of a system consisting of a *finite* number of equations. In Tresse's original words:

**Theorem 7 (Tresse, 1893)** *An arbitrary system of partial differential equations is necessarily limited in the sense that there exists a finite order $s$ such that all the equations of order higher than $s$ which are contained in the system can be deduced from the equations of order less or equal to $s$ only by means of differentiations.*[16]

[14] According to [Fritzsche 1999, p. 19], they can be dated back approximately to 1892.

[15] It appears that Tresse was not completely fair in acknowledging his debt to Lie. Indeed, it is highly probable that most results contained in [Tresse 1893] were Lie's. In this respect, see the first paragraph of [Lie 1895a].

[16] *Un système d'équations aux dérivées partielles étant defini d'une manière quelconque, ce système est nécessairement limité, c'est-à-dire qu'il existe un ordre fini $s$, tel que, toutes les équations d'ordre supérieur à $s$ que comprendent le système, se déduisent par de simples différentiantions des équations d'ordre égal ou inférieur à $s$.* [Tresse 1893, p. 8].

## 3.3 Engel's Habilitationsschrift

Despite the numerous achievements that Lie's theory of transformation groups had attained over the 1870's, still in the early 1880's the diffusion of his ideas was scarce. Lie attributed such difficulty in circulation to his lack of *Redationsfähigkeit*, i.e. the capability of providing his publications with the clarity that was required to reach wider readership. Namely, as he explained to his friend A. Mayer, the main problem was represented by his difficulty in translating his synthetic insights into a more analytical form. Such an openly declared deficiency created concern for himself. Indeed, as he wrote to Klein in 1884, his main fear was that other mathematicians might think that his ideas were proportional to his *Redationsfähigkeit*. In this respect, A. Mayer's and F. Klein's decision to send to Lie one of their best students at the University of Leipzig, turned out to be providential. Their choice was the young mathematician Friedrich Engel (1861-1941). Son of a Lutheran pastor, Engel had studied at Leipzig under the guidance of Mayer, but he also had attended lectures in Berlin where he became familiar with Weierstrass' theory of analytical functions. After the completion (summer 1883) of his thesis "*Zur Theorie der Berührungstransformationen*", he spent one year in Dresden where he served in the army. Having obtained a financial support from Leipzig university as well as from the Saxon Academy of Sciences, in September 1884 F. Engel moved to Christiania (now Oslo) to join Lie. The aim was twofold: on one hand, Engel would support Lie's *Redationsfähigkeit* in the project of editing a compendium of the theory of transformation groups, later on to materialize in the three volumes *Theorie der Transformationsgruppen*; on the other, he would exploit the closeness with Lie to write his *Habilitationsschrift* which he defended after his return in Germany on 15th July 1885. The title of Engel's dissertation was *Ueber die Definitionsgleichungen der continuirlichen Transformationsgruppen*. Its importance for the theory of infinite continuous transformation groups can hardly be overestimated. As the title suggests, it represented the first attempt to develop a general method to build up the defining equations of all continuous groups. At this stage the treatment was rather heuristic and of an unsystematic kind. Nevertheless, it is worth considering especially for the discovery of a remarkable correspondence between infinite continuous groups and certain finite groups of a particular composition. In the introduction to [Engel 1886] Engel wrote:

> *In this dissertation a new method for the determination of the defining equations of groups of infinitesimal transformations will be provided.*
> *In many passages of his already mentioned "Ueber unendliche Gruppe", Lie directly computed several defining equations in the plane. He simply required that if* $\xi_1, \eta_1$ *and* $\xi_2, \eta_2$ *is a solution to*

*the defining system under examination, then also*

$$\xi_1 \frac{\partial \xi_2}{\partial x} + \eta_1 \frac{\xi_2}{\partial y} - \xi_2 \frac{\partial \xi_1}{\partial x} + \eta_2 \frac{\xi_1}{\partial y},$$
$$\xi_1 \frac{\partial \eta_2}{\partial x} + \eta_1 \frac{\eta_2}{\partial y} - \xi_2 \frac{\partial \eta_1}{\partial x} + \eta_2 \frac{\eta_1}{\partial y}$$

*is. In this way, he obtained relations from which the coefficients of the defining equations could be deduced. However, this procedure leads to quite extensive calculations* [$\cdots$]. *The method, which will be expounded here, is more general.*[17]

As Engel explained, he came up with the idea at the basis of his dissertation first by considering the groups of transformations of the straight line, that is the groups in one variable, then by generalizing it, extending his reasoning to transformation groups acting on manifolds of arbitrary dimension $n$. His treatment can be summarized as follows.

He started by considering an infinite continuous group of infinitesimal transformations in the variables $x_1, \cdots, x_n, \alpha_1, \cdots, \alpha_m$ of the following (particular) kind:

$$\sum_{i=1}^{n} \phi_i \frac{\partial f}{\partial x_i} + \sum_{\kappa=1}^{m} \Omega_\kappa \left( \alpha_1, \cdots, \alpha_m, \phi_1, \cdots, \phi_n, \frac{\partial \phi_1}{\partial x_1}, \cdots, \frac{\partial^2 \phi_1}{\partial x_1^2} \cdots, \right) \frac{\partial f}{\partial \alpha_\kappa}, \tag{3.6}$$

where $\phi_i$, $(i = 1, \cdots, n)$ are arbitrary functions of $x_1, \cdots, x_n$ while the $\Omega_\kappa$, $(\kappa = 1, \cdots, m)$ depend on $\alpha_1, \cdots, \alpha_m$ as well as on $\phi_i$ and their derivatives until a certain order, $q$. Furthermore, Engel supposed that the $\Omega_\kappa$ do not contain any arbitrary constant so that they are completely determined once the functions $\phi_i$ are known. He also required that $\Omega_\kappa = 0$, $(\kappa = 1, \cdots, m)$ when all the $\phi_i$ vanish. Lastly, due to the fact that the preceding infinitesimal transformations build up a continuous group, Engel observed, it follows

[17] *Es soll in dieser Abhandlung eine neue Methode entwickelt werden, vermittelst deren Definitionsgleichungen von Gruppen von infinitesimalen Transformationen bestimmt werden können.*
*Lie hat an mehreren Stellen der erwähnte Abhandlung "Ueber unendliche Gruppe" direct einige Definitionsgleichungen in der Ebene berechnet. Er verlangt einfach, dass mit* $\xi_1, \eta_1$ *und* $\xi_2, \eta_2$ *gleichzeitig auch*

$$\xi_1 \frac{\partial \xi_2}{\partial x} + \eta_1 \frac{\xi_2}{\partial y} - \xi_2 \frac{\partial \xi_1}{\partial x} + \eta_2 \frac{\xi_1}{\partial y},$$
$$\xi_1 \frac{\partial \eta_2}{\partial x} + \eta_1 \frac{\eta_2}{\partial y} - \xi_2 \frac{\partial \eta_1}{\partial x} + \eta_2 \frac{\eta_1}{\partial y}$$

*ein Lösungensystem der bettreffenden Differentialgleichungen ist, und erhält auf diese Weise Relationen, aus welchen sich die Coefficienten der Differentialgleichungen ermitteln lassen. Allerdings führt dieses Verfahren auf recht weitläufige Rechnungen* [$\cdots$]. *Die Methode, welche hier auseinandergesetzt werden soll, ist allgemeiner.* See [Engel 1886, p. 10].

that the functions $\Omega_\kappa$, $(\kappa = 1, \cdots, m)$ must be linear in the arguments $\phi_i$ $(i = 1, \cdots, n)$ as well as in their derivatives.

At this point Engel set out to determine among the transformations of type (3.6) those that leave the following system of equations

$$\alpha_i - \alpha_i(x_1, \cdots, x_n) = 0, \qquad (i = 1, \cdots, m),$$

invariant. By posing

$$\Omega_\kappa \left( \alpha_1, \cdots, \alpha_m, \phi_1, \cdots, \phi_n, \frac{\partial \phi_1}{\partial x_1}, \cdots, \frac{\partial^2 \phi_1}{\partial x_1^2} \cdots, \right) = U_\kappa(\alpha, \phi),$$

such transformations can be characterized by the $m$ linear differential equations

$$U_\kappa(\alpha, \phi) - \sum_{i=1}^{n} \frac{\partial \alpha_\kappa}{\partial x_i} \phi_i = 0, \qquad (\kappa = 1, \cdots, m), \tag{3.7}$$

where the functions $\alpha_\kappa$ have to be considered as functions of $x_1, \cdots, x_n$. On one hand, the equations (3.7) determine a group of infinitesimal transformations in the variables $x_1, \cdots, x_n, \alpha_1, \cdots, \alpha_m$[18]; on the other, they can be considered as the defining equations of a group (of infinitesimal transformations) in the variables $x_1, \cdots, x_n$ only. These are the equations of the group one was seeking for. Actually, as Engel explicitly observed, it is necessary to suppose that equations (3.7) possess solutions and to this end Engel required that the (arbitrary) functions $\alpha_1, \cdots, \alpha_m$ are chosen in such a way that the system (3.7) is completely integrable.

After a thorough analysis of the transformation groups of the straight line, Engel set out to develop the details of his method for the groups in $n$ variables by first considering the case in which the coefficients $U_\kappa$ of (3.6) contain first order derivatives of $\phi_i$ only.

As a consequence of this supposition, by indicating with the symbol $A_{i\kappa}(f)$ the infinitesimal transformations in the variables $\alpha$, the general transformation of the infinite group in the variables $x_1, \cdots, x_n, \alpha_1, \cdots, \alpha_m$ reads as follows:

$$X(f) = \sum_{i=1}^{n} \phi_i \frac{\partial f}{\partial x_i} + \sum_{i=1}^{n} \sum_{\kappa=1}^{n} A_{i\kappa}(f) \frac{\partial \phi_i}{\partial x_\kappa} \tag{3.8}$$

The next step in Engel's procedure was to determine the conditions that the transformations $A_{i\kappa}$ have to satisfy in order for the set of transformations (3.6) to actually be a group. To this end, Engel calculated the combination (the so-called Lie bracket) of two infinitesimal transformations

$$X(f) = \sum_{i=1}^{n} \phi_i \frac{\partial f}{\partial x_i} + \sum_{i=1}^{n} \sum_{\kappa=1}^{n} A_{i\kappa}(f) \frac{\partial \phi_i}{\partial x_\kappa},$$

[18] When the general solution to (3.7) is substituted into (3.6). The existence of such a solution was explicitly assumed.

$$Y(f) = \sum_{i=1}^{n} \psi_i \frac{\partial f}{\partial x_i} + \sum_{i=1}^{n}\sum_{\kappa=1}^{n} A_{i\kappa}(f) \frac{\partial \psi_i}{\partial x_\kappa}$$

and imposed that it be equal to:

$$(XY) = \sum_{i=1}^{n}\sum_{\nu=1}^{n} \left( \phi_\nu \frac{\partial \psi_i}{\partial x_\nu} - \psi_\nu \frac{\partial \phi_i}{\partial x_\nu} \right) \frac{\partial f}{\partial x_i} +$$
$$+ \sum_{i=1}^{n}\sum_{\kappa=1}^{n}\sum_{\kappa=1}^{n} A_{i\kappa}(f) \frac{\partial}{\partial x_\kappa} \left( \phi_\nu \frac{\psi_i}{\partial x_\nu} - \psi_\nu \frac{\phi_i}{\partial x_\nu} \right). \tag{3.9}$$

After a certain amount of calculation Engel obtained the following relations which characterize the structure of the (finite) group generated by the $n^2$ (in general not independent) infinitesimal transformations $A_{i\kappa}$, $(i, \kappa = 1, \cdots, n)$:

$$(A_{i\kappa}, A_{\mu\nu}) = \delta_{i\nu} A_{\mu\kappa} - \delta_{\mu\kappa} A_{i\nu}. \tag{3.10}$$

By posing $A_{i\kappa} = -x_i \frac{\partial f}{\partial x_\kappa}$, Engel recognized that the group generated by $A_{i\kappa}$ is isomorphic (in modern terms, we would say homomorphic) to the $n^2$ parameter group generated by $x_i \frac{\partial f}{\partial x_\kappa}$ which coincides with the general linear group in $n$ variables.

Thus Engel concluded that the transformations (3.8) represent the generators of a group if, and only if the finite group generated by $A_{i\kappa}$ is homomorphic to the general linear group in $n$ variables. In this way the problem of determining the groups stemming from the transformations (3.8) was traced back to the problem of determining all the groups homomorphic to the general linear group. Engel did not consider the case in which such a homomorphism is holoedric, i.e. an isomorphism (in modern terms), by saying that its treatment would be rambling (*weitläufig*) and restricted himself to the case in which the isomorphism is simply meriedric. The technique of finding such groups was standard; it essentially consisted of determining all the invariant subgroups (ideals) of the given group (in this case the general linear group in $n$ variables)[19].

For example, by considering the ideal generated by

$$x_i \frac{\partial f}{\partial x_\kappa}, \quad x_i \frac{\partial f}{\partial x_i} - x_\kappa \frac{\partial f}{\partial x_\kappa} \quad (i \neq \kappa),$$

Engel deduced the existence of a group meriedrically isomorphic to the general linear group generated by the infinitesimal transformations

$$A_{11} = A_{22} = \cdots = A_{nn} = \frac{\partial}{\partial \alpha}, \quad A_{i\kappa} = 0, \quad (i \neq \kappa),$$

[19] See [Engel 1886, p. 8-9].

and finally obtained the following expression for the transformation (3.8):

$$\sum_{i=1}^{n} \phi_i \frac{\partial f}{\partial x_i} + \sum_{i=1}^{n} \frac{\partial \phi_i}{\partial x_i} \frac{\partial f}{\partial \alpha},$$

and the corresponding defining equation for a group in $n$ variables:

$$\frac{\partial \xi_1}{\partial x_1} + \cdots + \frac{\partial \xi_\kappa}{\partial x_\kappa} - \frac{\partial \alpha}{\partial x_1}\xi_1 - \cdots - \frac{\partial \alpha}{\partial x_\kappa}\xi_n = 0.$$

Lie had demonstrated that such an equation where $\alpha$ has to be interpreted as an arbitrary function of $x_1, \cdots, x_n$, defines a family of groups which are similar (*ähnlich*) one to the other. In particular, by setting $\alpha = 0$, one obtains:

$$\frac{\partial \xi_1}{\partial x_1} + \cdots + \frac{\partial \xi_n}{\partial x_n} = 0,$$

which is easily recognized to be the defining equation of the equivalent group in $n$ variables.

In the case of transformation groups in one single variable, Engel was able to prove that upon different choices for the composition of the infinite groups (3.6) his method succeeded in recovering all the continuous groups of the straight line. The problem, as he said, was to understand the basis for such a complete determination; in fact, *a priori* it was by no means clear that every group could be obtained by the method proposed by him. In the introduction to [Engel 1886] Engel stated that Lie had provided him with complete proof of the generality of his method. However such a demonstration was too long to be included in his dissertation. In fact complete proof was obtained by him only ten years later, in 1894 when he was able to demonstrate that the defining equations (3.7) were indeed characteristic of *every* group of infinitesimal transformations.

## 3.4 Foundations of infinite continuous groups

After receiving an offer for a professorship at the University of Göttingen, Klein left Leipzig in the summer of 1886. Lie seemed to be his natural successor. Indeed, despite some harsh prejudices expressed by Weierstrass who thought that Lie's mathematical achievements did not justify the choice of preferring a foreigner to other German mathematicians, Lie was appointed with a full professor position at the University of Leipzig. He gave his inaugural lecture in May 1886 which dealt with the influence of geometry in the development of mathematics.

The project of editing a compendium of his group theory could be continued. However, Lie's first years in Leipzig were also a period of intense research on the theory of infinite groups. In this case too, Engel's assistance turned out to be essential.

So far, Lie had limited himself to infinite groups of infinitesimal transformations, assuming at the same time the complete equivalence between groups of infinitesimal transformations and groups of finite transformations. The need for a more attentive analysis of such an asserted equivalence as well as the more general necessity of providing the theory with solid theoretical grounds materialized in a long memoir which was presented in two communications in 1891 to the *Sächsische Akademie der Wissenschaft* in Leipzig. We know from Engel's commentary to Lie's *Gesammelte Abhandlungen* that Lie had worked on them in the spring of 1890 while being treated in a sanatorium for nervous disorders (*Nervenanstalt*) near Hannover. As a consequence of Lie's poor health[20], the task of writing the final draft of the paper was assigned to Engel who was able to rely upon some handwritten notes by Lie.

After a description of the state of the art of the theory in which the relative completion of the theory of finite groups is contrasted with the sketchiness of that of the infinite ones and the relevance of the latter for the theory of partial differential equations is firmly asserted, Lie started his analysis by giving a definition of what an infinite continuous group is. The attempt of providing a rigorous axiomatization is clear. Contrary to his previous treatment, such a definition was given as follows in terms of finite transformations:

> *A family of transformations:*
>
> $$y_i = y_i(x_1, \cdots, x_n) \quad (i = 1, \cdots, n) \tag{3.11}$$
>
> *is said to be an infinite continuous group if (3.13) are the most general solutions to a system of partial differential equations:*
>
> $$W_k\left(x_1, \cdots, x_n, y_1, \cdots, y_n, \cdots, \frac{\partial y_1}{\partial x_1}, \cdots, \frac{\partial y_n}{\partial x_n}, \frac{\partial^2 y_1}{\partial x_1^2}, \cdots\right) = 0, \tag{3.12}$$
>
> $(k = 1, \cdots, l)$, *and if this system possesses the following two properties:*
> *firstly, the most general solutions to the system (3.12) do not depend only upon a finite number of arbitrary constants;*
> *secondly, if*
>
> $$y_i = F_i(x_1, \cdots, x_n) \text{ und } y_i' = \Phi_i(x_1, \cdots, x_n) \quad (i = 1, \cdots, n),$$
>
> *are two systems of solutions of the differential equations (3.12), then also*
>
> $$y_i'' = \Phi_i(F_1(x), \cdots, F_n(x)) \quad (i = 1, \cdots, n)$$

[20]It seems that Lie was affected by a severe metabolic disease called pernicious anaemia, frequent symptoms of which were sleeplessness and depression.

> *is; in other words, the composition of any two arbitrary transformations of the family defined by (3.12) is always a transformation belonging to the family.*[21]

It should be observed that Lie explicitly excluded the case in which the general solution of (3.12) does depend on a finite number of parameters only. Indeed such an eventuality was already covered by his theory of finite continuous groups. Nevertheless, it is important to notice that finite continuous groups too can be defined as the space of solutions of a differential system of type (3.12). In this case, the differential system, known as Lie-Mayer type, is characterized by the property that *all* its derivatives of highest order are principal, which means that they can be expressed in terms of derivatives of lower orders. This condition guarantees that the general solution of (3.12) depends upon a finite number of parameters only.

In 1885, when discussing the degree of indeterminacy of the functions $\alpha_1, \cdots, \alpha_m$, Engel had already required certain integrability conditions. Now Lie decided to include such a requirement in the definition itself of continuous groups by demanding that through iterated differentiation of the system (3.12) no new equation is obtained. In Lie's words:

> [...] *if m is the order of the system (3.12), then all the differential equations of m or lower order which can be derived from differentiation and elimination, follow from the system (3.12) without differentiation.*[22]

---

[21] *Eine Schar von Transformationen:*

$$y_i = y_i(x_1, \cdots, x_n) \quad (i = 1, \cdots, n) \tag{3.13}$$

*soll eine unendlich kontinuierliche Gruppe heißen, wenn 3.13 die allgemeinsten Lösungen eines Systems von partiellen Differentialgleichungen:*

$$W_k\left(x_1, \cdots, x_n, y_1, \cdots, y_n, \cdots, \frac{\partial y_1}{\partial x_1}, \cdots, \frac{\partial y_n}{\partial x_n}, \frac{\partial^2 y_1}{\partial x_1^2}, \cdots\right) = 0, \tag{3.14}$$

$(k = 1, \cdots, l)$ *sind und dieses System die folgenden beiden Eigenschaften besitzt:*
*Erstens sollen die allgemeinsten Lösungen des Systems (3.14) nicht bloss von einer endlichen Anzahl willkürlicher Kostanten abhängen.*
*Zweitens soll stets, wenn:*

$$y_i = F_i(x_1, \cdots, x_n) \text{ und } y_i' = \Phi_i(x_1, \cdots, x_n) \quad (i = 1, \cdots, n),$$

*irgend zwei Lösungensysteme der Differentialgleichungen (3.14) sind, zugleich auch:*

$$y_i'' = \Phi_i(F_1(x), \cdots, F_n(x)) \quad (i = 1, \cdots, n)$$

*eine Lösungensystem dieser Differentialgleichungen sein, mit andern Worten: es sollen zwei beliebige Transformationen der durch (3.14) definierten Schar nach einander ausgeführt stets wieder eine Transformation der Schar ergeben.* See [Lie 1891a, p. 317].

[22] [...] *wenn m die Ordnung des Systems (3.14) ist, so sollen alle Differentialgleichungen m-ter oder niedriger Ordnung, die aus (3.14) durch Differentiationen und Eliminationen*

Furthermore, it should be stressed that no explicit requirement of the existence of the inverse transformation is made. In fact Lie was convinced that such a hypothesis was not independent of those explicitly stated in the quotation above and even provided proof thereof. Unfortunately the demonstration proposed by him was utterly insufficient since it assumed that if one fixes a transformation, say $S$, and $T$ is any arbitrary transformation then the product $ST$ is an *arbitrary* transformation of the group.[23]

Nevertheless Lie emphasized that he wanted to focus his attention on those groups which admit the identity transformation as well as the inverse of any of their transformations[24].

A first important result to be noticed was the possibility to characterize the transformations of the group (3.12) in two different, distinct manners. According to the definition quoted above a transformation $y_i = \Phi_i(x_1, \cdots, x_n)$ belongs to the group if, and only if, it is a solution of the differential system (3.12). Alternatively, Lie demonstrated, the transformations of the group can be identified with those transformations which, in a certain sense to be specified, leave the system of defining equations (3.12) invariant[25]. More precisely, Lie was able to prove the following to be noteworthy:

**Theorem 8 (Lie, 1891)** *If an infinite continuous group with inverse transformations is defined by a system of differential equations of the form:*

$$W_k \left( x_1, \cdots, x_n, y_1, \cdots, y_n, \cdots, \frac{\partial y_1}{\partial x_1}, \cdots, \frac{\partial y_n}{\partial x_n}, \frac{\partial^2 y_1}{\partial x_1^2}, \cdots \right) = 0,$$

*then this defining system of differential equations admits the following transformation*

$$y_i' = F_i(y_1, \cdots, y_n), \quad x_i' = x_i \qquad (i = 1, \cdots, n),$$

*if, and only if the equations:*

$$y_i' = F_i(y_1, \cdots, y_n), \qquad (i = 1, \cdots, n),$$

*hergeleitet werden können, schon ohne Differentiation aus des System (3.14) folgen.* See [Lie 1891a, p. 318]. Such a hypothesis can be understood in the light of what nowadays we would call formal theory of PDE's. In particular, the requirement just stated coincides with the hypothesis of formal integrability. We will see later that Cartan's treatment of the same subject will start from the more restrictive requirement of involution.

[23] For an interesting discussion relating to group axioms in Lie's theory see [Wussing 1894, p. 223-229].

[24] For a modern, rigorous definition of Lie pseudogroup, see [Pommaret 1978, p. 265-266]. The defining equation of an infinite continuous group given by Lie can now be considered as the local expression of a fiber submanifold $\mathcal{R}_q$ of the jet bundle $J_q(X, Y)$, where $X$ and $Y$ indicate the source manifold and the target manifold respectively, i.e. the spaces of untransformed and transformed variables.

[25] See [Pommaret 1978, p. 272], Theorem 1.23 for a modern version of this fundamental result.

*represent a transformation of the given infinite group.*[26]

Following Lie's analysis, let us see what that means in concrete terms. To this end we shall consider the example of the equivalent group acting on a numerical space of dimension $n$. In this case the system (3.12) reduces itself to the following single equation of first order:

$$det\left[\frac{\partial y_i}{\partial x_k}\right] = \frac{\partial(y_1,\cdots,y_n)}{\partial(x_1,\cdots,x_n)} = 1. \tag{3.15}$$

First, it is necessary to verify that every transformation which belongs to the group leaves the equation (3.15) invariant. More precisely, one has to show that the transformation

$$y'_i = \Phi_i(y_1,\cdots,y_n), \quad x'_j = x_j, \qquad (i,j = 1,\cdots,n), \tag{3.16}$$

with $\Phi_i$ equivalent transformation, leaves (3.15) invariant. In order to operate such a change of variables in (3.15), one has to invert the first $n$ equations of (3.16) obtaining:

$$y_i = \bar{\Phi}_i(y'_1,\cdots,y'_n).$$

Since the Jacobian of $\bar{\Phi}\circ\Phi$ is the matrix multiplication of the Jacobians of $\bar{\Phi}$ and $\Phi$, the equation (3.16) takes the form:

$$\frac{\partial(\bar{\Phi}_1,\cdots,\bar{\Phi}_n)}{\partial(y'_1,\cdots,y'_n)}\frac{\partial(y'_1,\cdots,y'_n)}{\partial(x_1,\cdots,x_n)} = 1.$$

Now, since the inverse of every equivalent transformation is an equivalent transformation, $\frac{\partial(\bar{\Phi}_1,\cdots,\bar{\Phi}_n)}{\partial(y'_1,\cdots,y'_n)} = 1$, and the equation (3.15) is consequently transformed into the following

$$\frac{\partial(y'_1,\cdots,y'_n)}{\partial(x_1,\cdots,x_n)} = 1,$$

[26] *Ist eine unendliche kontinuierliche Gruppe mit paarweise inversen Transformationen durch ein System von partiellen Differentialgleichungen:*

$$W_k\left(x_1,\cdots,x_n,y_1,\cdots,y_n,\cdots,\frac{\partial y_1}{\partial x_1},\cdots,\frac{\partial y_n}{\partial x_n},\frac{\partial^2 y_1}{\partial x_1^2},\cdots\right) = 0,$$

$(k = 1,\cdots,l)$, *definirt, so gestattet dieses System von Differentialgleichungen eine Transformation von der Form*

$$y'_i = F_i(y_1,\cdots,y_n), \quad x'_i = x_i \qquad (i = 1,\cdots,n),$$

*stets dann, aber auch nur dann, wenn die Gleichungen:*

$$y'_i = F_i(y_1,\cdots,y_n), \qquad (i = 1,\cdots,n),$$

*eine Transformation der betreffenden unendlichen Gruppe darstellen.* See [Lie 1891a, p. 338].

which proves one part of the asserted invariance property.

We come to the viceversa. If the change of variables (3.16), where now $\Phi$ is to be considered as an arbitrary transformation, leave the equation (3.15) invariant then the Jacobian

$$\frac{\partial(\bar{\Phi}_1, \cdots, \bar{\Phi}_n)}{\partial(y'_1, \cdots, y'_n)}$$

is equal to one, and $\bar{\Phi}$ is an equivalent transformation and consequently also $\Phi$ is.

On the basis of theorem (8) Lie was able to start his analysis on the connection between infinitesimal and finite transformations. The aim was that of demonstrating that his new definition of (infinite) continuous group was indeed tantamount to the previous one given in terms of infinitesimal transformations only. The first step in the study of such a connection was the introduction of the notion of *elementary transformation* (*unendlich kleine Transformation*[27]). This was presented as a generalization of the notion of infinitesimal transformation; indeed, Lie called elementary transformation a transformation, infinitely close to the identity transformation, of the following type:

$$y_i = x_i + \phi_i(x_1, \cdots, x_n)\delta t + \psi_i(x_1, \cdots, x_n)(\delta t)^2 + \cdots$$

where the coefficients of $\delta t, (\delta t)^2$ are arbitrary functions of the variables $x_1, \cdots, x_n$. An infinitesimal transformation constitutes a particularization since in this case the coefficients of all orders are uniquely determined by the first order ones, i.e. $\phi_i(x)$.
Lie's strategy consisted first of demonstrating the existence of elementary transformations belonging to the group and then of deducing from them the existence of an infinite number of infinitesimal transformations. It was clear that a unique infinitesimal transformations could be associated to every elementary transformation, but it was by no means evident *a priori* that this infinitesimal transformation actually belonged to the group. Lie succeeded in proving that this was indeed the case. The relevant theorem was the following:

**Theorem 9** *Let us suppose that the infinite continuous group (3.12) contains a family of transformations of the following form:*

$$y_i = x_i + \epsilon\xi_i(x) + \epsilon^2\theta_i(x) + \cdots, \quad (i = 1, \cdots, n);$$

*then the group contains also the infinitesimal transformation* $\sum_{i=1}^{n} \xi_i(x)\frac{\partial f}{\partial x_i}$ *as well as the one-parameter group generated by it.*

[27] I took the wording *elementary transformation* from [Amaldi 1944]. Literally, we should translate infinitely small transformation.

This theorem laid the basis for the proof of the existence of an infinite number of infinitesimal transformations belonging to the group (3.12). Besides, as a consequence of this, Lie was able to prove that every finite transformation of the group could be obtained as the result of the action of infinitely many infinitesimal transformation. The result was gratifying and in clear accordance with Lie's declared intention to construct a theory of infinite continuous group which was based upon the model of the theory of finite continuous groups; nevertheless, it could not be considered, *stricto sensu*, as a genuine extension of the so-called *first fundamental theorem*. It was by no means clear whether, as for finite groups[28], every finite transformation could be considered as being generated by *one single* infinitesimal transformation or not. In this respect, Lie wrote:

> *In the preceding paragraphs we have proved only that every infinite continuous groups with pairwise inverse transformations contains infinitely many one parameter groups; however, this by no means entails that the group consists of one parameter groups only. Whether this is the case, i.e. that every transformation of the infinite group belongs to a one parameter subgroup of this group and thus is generated by an infinitesimal transformation of the group, it is a functional question whose answer does not appear to be simple.*[29]

On the basis of these premises, Lie set out to face the problem of determining the defining equations of the infinitesimal transformations when the defining equations of the finite transformations (3.12) are supposed to be known[30].

[28] Actually, even in the case of finite groups it is not true that every finite transformation can be generated by an infinitesimal transformation. The assertion is valid only if one restricts himself to an appropriate neighborhood of the identity. What distinguishes infinite groups from finite ones is the fact that in the former case the generability of finite transformations from infinitesimal ones is not guaranteed at all, even upon restriction to a local neighborhood of the identity transformation. In this respect, Singer and Sternberg observed: "[$\cdots$] *the relation between the group and the infinitesimal group (i.e., Lie algebra) is not as simple in the infinite case as it is in the finite case. In fact, one of Lie's conjecture - that a small enough neighborhood of the identity is covered by one parameter subgroups- turns out to be false in the infinite dimensional case"* [Sinberg Sternberg 1965, p. 6].

[29] *Im vorigen Paragraphen haben wir nun allerdings gezeigt, dass jede unendliche kontinuierliche Gruppe mit paarweise inversen Transformationen unbegrentz viele eiglidrigen Gruppen enthält, aber damit ist keineswegs bewiesen, dass die Gruppe aus lauter eiglidrigen Gruppen besteht. Ob das der Fall ist, ob jede Transformation der unendlichen Gruppe einer einglidrigen Untergruppe dieser Gruppe angehört und also von einer infinitesimalen Transformation der Gruppe erzeugt ist - das ist eine funktionentheoretische Frage, deren Beantwortung nicht leicht zu sein scheint.* See [Lie 1891a, p. 344].

[30] Pommaret in [Pommaret 1978, p. 273], called such a procedure *linearisation process*.

In order for an infinitesimal transformation

$$X(f) = \sum_{i=1}^{n} \xi_i(y_1, \cdots, y_n) \frac{\partial f}{\partial y_i} \tag{3.17}$$

to belong to the group (3.12), it is necessary and sufficient that its defining system admits the corresponding prolonged transformation of order $m$ equal to the order of (3.12):

$$X^{(m)} W_k = \sum_{i=1}^{n} \xi_i(y) \frac{\partial W_k}{\partial y_i} + \sum_{ijl} \frac{\partial \xi_i}{\partial y_l} \frac{\partial y_l}{\partial x_j} \frac{\partial W_k}{\partial \frac{\partial y_i}{\partial x_j}} + \cdots,$$

that is, the vanishing of $X^{(m)} W_k$ is a consequence of the vanishing of $W_k$, $k = 1, \cdots, l$. Now, Lie observed, if we consider an *arbitrary* finite transformation of the group

$$y_i = F_i(x_1, \cdots, x_n) \quad (i = 1, \cdots, n), \tag{3.18}$$

then the equations (3.12) are identically satisfied after the following substitutions:

$$y_i = F_i(x_1, \cdots, x_n), \quad , \frac{\partial y_i}{\partial x_j} = \frac{\partial F_i}{\partial x_j}, \quad \frac{\partial^2 y_i}{\partial x_j \partial x_k} = \frac{\partial^2 F_i}{\partial x_j \partial x_k} \quad \cdots \quad . \tag{3.19}$$

Now, if the relations $y_i = F_i(x)$ are solved with respect to $x_i$, one obtains

$$x_i = \Phi_i(y_1, \cdots, y_n) \tag{3.20}$$

and from these also equations of the form:

$$\frac{\partial y_i}{\partial x_j} = \Phi_{ij}(y), \quad \frac{\partial^2 y_i}{\partial x_l \partial x_j} = \Phi_{ilj}(y), \cdots. \tag{3.21}$$

Finally Lie observed that if the last two sets of equations (3.20) and (3.21) were substituted in $X^{(m)} W_k = 0$ (operation to be indicated with square bracket), then the conditions guaranteeing that (3.17) belongs to the group could be written as:

$$\left[X^{(m)} W_k\right] = \sum_i \xi_i(y) \left[\frac{\partial W_k}{\partial y_i}\right] + \sum_{ij} \frac{\partial \xi_i}{\partial y_j} \left[\sum_s \frac{\partial y_j}{\partial x_s} \frac{\partial W_k}{\partial \left(\frac{\partial y_i}{\partial x_s}\right)}\right] + \cdots = 0, \tag{3.22}$$

for $k = 1, \cdots, l$. At first sight it may seem that the equations that the $\xi_i$'s have to obey are infinite in number since infinitely many transformations of type (3.18) should be considered. However, as Lie proved, it turned out that two differential systems corresponding to distinct transformations are completely equivalent in the sense that every solution of one system is automatically a solution of the other. As a consequence of this, one is

entitled to choose, among all possible transformations $y_i = F_i$, the simplest one, that is the identity transformation.
Let us see in some detail how the procedure proposed by Lie works in practice in the case of the equivalent group of the plane which we have already encountered. In this case the system of defining equations consists of the following simple equation:

$$W_1 = \frac{\partial y_1}{\partial x_1}\frac{\partial y_2}{\partial x_2} - \frac{\partial y_1}{\partial x_2}\frac{\partial y_2}{\partial x_1} - 1 = 0. \tag{3.23}$$

The condition for a generic infinitesimal transformation

$$X(f) = \xi_1(y_1, y_2)\frac{\partial f}{\partial y_1} + \xi_2(y_1, y_2)\frac{\partial f}{\partial y_2}$$

to be an infinitesimal transformation of the equivalent group, according to Lie's procedure, is represented by the following relation obtained by applying the first order prolongation of $X(f)$ to $W_1$:

$$\frac{\partial \xi_1}{\partial y_1}\left(\frac{\partial y_1}{\partial x_1}\frac{\partial y_2}{\partial x_2} - \frac{\partial y_1}{\partial x_2}\frac{\partial y_2}{\partial x_1}\right) + \frac{\partial \xi_2}{\partial y_2}\left(-\frac{\partial y_2}{\partial x_1}\frac{\partial y_1}{\partial x_2} + \frac{\partial y_2}{\partial x_2}\frac{\partial y_1}{\partial x_1}\right) = 0.^{31} \tag{3.24}$$

By evaluating the round brackets in correspondence with the identity transformation, $\frac{\partial y_j}{\partial x_i} = \delta_{ij}$, $i, j = 1, 2$, we easily obtain the defining system for the infinitesimal transformations of the equivalent group in two variables:

$$\frac{\partial \xi_1}{\partial y_1} + \frac{\partial \xi_2}{\partial y_2} = 0.$$

Thus, more generally, Lie was able to provide a general procedure for obtaining the defining differential system (3.22) of the infinitesimal transformations of a given group $G$ of finite transformations. He then demonstrated that such a system had the property of being closed with respect bracket-operation in the sense that if $X(f)$ and $Y(f)$ are infinitesimal transformations solutions to (3.22), then also $(X, Y)$ is. In this way, Lie generalized to infinite groups a well-known result corresponding to the first part of the so-called *second fundamental theorem* of his theory of finite groups according to which a $r$-parameter group contains $r$ independent infinitesimal transformations $X_j(f)$, $(j = 1, \cdots, r)$ such that $(X_i, X_j)(f) = \sum_k c_{ijk} X_k(f)$, $(i, j = 1, \cdots, r)$ hold. At the same time he recovered the definition of infinite dimensional continuous group that he had formulated in [Lie 1883] and [Lie 1884].

At this point it was quite natural for Lie to ask the question about the possibility of inverting this result by investigating if, as for the case of finite groups, a group of infinitesimal transformations generates a group of

[31] Actually, one should also take into account other two terms which contain $\frac{\partial \xi_1}{\partial y_2}$ and $\frac{\partial \xi_2}{\partial y_2}$; however, an easy calculation shows that they vanish identically.

finite transformations. This problem was really an important one since an affirmative answer would have meant establishing the complete equivalence between the two definitions of infinite group given by him since then, the one just mentioned stated in terms of infinitesimal transformations and the more recent one proposed in the *Grundlagen*.
Indeed it turned out that this was the case. However, Lie was able to prove only a weaker statement which guaranteed only that an infinite group of infinitesimal transformations generates a semigroup of finite transformations, that is a set that shares all usual group properties with the exception of the existence of inverse elements.

Thus the problem was this: given a continuous group of infinitesimal transformations, to determine an infinite continuous group of *finite* transformations whose infinitesimal transformations (to be determined according to the procedure described above) coincide with the infinitesimal transformations of the group from which one starts with.

The solution proposed by Lie consisted first of a preliminary study of the differential invariants of the group $\Gamma$ of infinitesimal transformations. To this end, he considered the associated defining system of partial differential equations which, when resolved with respect to a certain number of principal derivatives, took on the following form:

$$\frac{\partial^k \xi_j}{\partial y_1^{k_1} \cdots \partial y_n^{k_n}} = \sum_{\lambda=1}^{n} \sum_{\nu_1,\cdots,\nu_n} \theta^{\lambda|\nu_1,\cdots,\nu_n}_{j|k_1,\cdots,k_n}(y) \frac{\partial^\nu \xi_\lambda}{\partial y_1^{k_1} \cdots \partial y_n^{k_n}}. \tag{3.25}$$

He first determined the differential invariants $I$ of order $Q$ of $\Gamma$ by considering the differential system $X^{(Q)}I = 0$ obtained by prolonging a given transformation $X = \sum_j \xi_j(y)\frac{\partial f}{\partial y_j}$ of $\Gamma$, supposed to be acting on the variables $y_1, \cdots, y_n$, until the order $Q$. This differential system gives rise to a complete system of linear first order differential equation. As a result of Clebsch's theorem, such system admits a number of independent solutions of type

$$I(y_1, \cdots, y_n; \cdots, \frac{\partial^k y_j}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}}, \cdots).$$

There are as many of them as the number of principal derivatives, until the order $Q$, which are present in the defining differential system (3.25). If the system (3.25) is of order $q$, Lie observed, the candidate for the requested defining system of finite transformations must be of order $q$ as well. Furthermore, as a consequence of the fact that such system must admit any infinitesimal transformation of (3.25), it could be proven[32] that it takes on

[32]The very same method first proposed by Lie was employed by Pommaret in [Pommaret 1978, p. 279-280].

the following form:

$$I_k(y_1, \cdots, y_n; \cdots, \frac{\partial^k y_j}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}}, \cdots) = \omega_k(x_1, \cdots, x_n), \qquad (k = 1, \cdots, m). \tag{3.26}$$

The number $m$ is equal to the sum of the number of principal derivatives of (3.25) until the order $q$, while the second members $\omega_k$ are equal to $I_k$ upon substitution of $y_k$ with $x_k$, $k = 1, \cdots, n$. This is a direct consequence stemming from having required that the group of finite transformations must contain the identity transformation.

Lie was able to demonstrate that the system (3.26) is completely integrable and that, as requested, it actually defines a a group of finite transformations. Actually Lie limited himself to prove only that the composition of two solutions of (3.26) is again a solution of (3.26). In accordance with the definition of continuous group provided by him, no mention of the existence of inverse transformations was made.

Furthermore, he was able to prove that the infinitesimal transformations associated to the group $G$ defined by (3.26) coincide precisely with those infinitesimal transformations defined by (3.25). By doing so, notwithstanding the limitation imposed by the lack of care in treating inverse transformations, Lie finally achieved a complete generalization of his well known second fundamental theorem. In analogy with the finite dimensional case, for infinite continuous groups as well, a bijective correspondence between groups of finite transformations and group of infinitesimal transformations was thus established.

As a byproduct, Lie had deduced a noteworthy form (3.26) (later on to be referred to as *Lie form*) for the defining system of the finite transformations of an arbitrary infinite group in which independent and dependent variables appeared separated on the two sides of the equations. We will see that this achievement represented a major advance in the theory.

In conclusion, Lie's *Grundlagen* conveyed a theoretical framework which allowed, for the first time, to glimpse the possibility of attaining a systematic theory of infinite continuous groups based upon the model provided by the theory of finite continuous ones. Some of the fundamental properties of the latter such as the existence of differential invariants and the possibility of generating finite transformations by means of infinitesimal ones, though with some precaution, could be preserved. Nonetheless, it was by no means clear how a structural approach to infinite continuous groups could be obtained. The notion of structure itself seemed to lose meaning when one tried to transfer it into the infinite dimensional domain.

## 3.5 On a theorem by Engel

In section 3.3 we saw that Engel in 1885 had succeeded in providing a method for generating continuous groups (finite as well as infinite ones). However it was by no means clear what was the justification at the basis of the choice of the particular infinite group (3.6) upon which the entire procedure depended. Even more opaque was the question about the generality of his method: indeed, no proof of the fact that the structure of *every* conceivable continuous group could be deduced in the proposed way was given. For both these two reasons in 1894 Engel decided to come back to the subject of his *Habilitationsschrift* with the hope to overhaul it. The occasion was offered by Lie's *Grundlagen* which, as we know, had provided a systematic foundation of the theory of infinite continuous groups. Indeed Engel's starting point in [Engel 1894] consisted of the exploitation of the advantages offered by the possibility of writing the defining system (supposed to be of order $q$) of an infinite group in the so-called Lie form:

$$I_k\left(y_1,\cdots,y_n,\frac{\partial y_1}{\partial x_1},\cdots,\frac{\partial y_n}{\partial x_n},\cdots\right)=\omega_k(x_1,\cdots,x_n),\quad (k=1,\cdots,m). \tag{3.27}$$

After the recollection of the main properties of (3.27), Engel considered an infinitesimal transformation which was thought to be acting first on the independent variables $x_1,\cdots,x_n$ and then on the dependent ones, $y_1,\cdots,y_n$. Such a double interpretation produced two distinct infinitesimal transformations:

$$Zf=\sum_{i=1}^{n}\zeta_i(x_1,\cdots,x_n)\frac{\partial f}{\partial x_i}\qquad\text{and}\qquad \mathcal{Z}f=\sum_{i=1}^{n}\zeta_i(y_1,\cdots,y_n)\frac{\partial f}{\partial y_i}.$$

Since they act on two different sets of variables, $Zf$ and $\mathcal{Z}f$ gave rise to two distinct prolongations[33] to be indicated with $Z^{(q)}f$ and $\mathcal{Z}^{(q)}f$ which act on the derivatives of $y_i$ with respect to $x_j$ until the q-th order. It is easily shown, Engel asserted, that the prolongation of the sum $Zf+\mathcal{Z}f$ is equal to the sum of the prolongations $Z^{(q)}f+\mathcal{Z}^{(q)}f$. Then Engel considered an infinitesimal transformation $Xf=\sum_{i=1}^{n}\xi_i(y_1,\cdots,y_n)\frac{\partial f}{\partial y_i}$ which belongs to the group (3.27). As usual, its prolongation was indicated by him with $X^{(q)}(f)$.

These assumptions being stated, Engel's strategy consisted of examining the changes to which the equations $X^{(q)}I_k=0,\quad (k=1,\cdots,s)$, are subject as a consequence of the action of the infinitesimal transformation $Z^{(q)}f+\mathcal{Z}^{(q)}f$.

It is well known that to every infinitesimal transformation corresponds a change of coordinates in the following way. By denoting the infinitesimal

[33] Later on explicit formulas for prolongations will be given. See section 3.6. Nowadays, $Zf$ and $\mathcal{Z}f$ are known as *source* and *target* infinitesimal transformations respectively. In this respect, [Pommaret 1978, §6.5] should be consulted.

transformation with the operator $A(f) = \sum_{j=1}^{n} \alpha_j(x) \frac{\partial f}{\partial x_j}$, one can introduce the associated change of variables:

$$x'_j = x_j + \alpha_j(x)\delta t \qquad (j = 1, \cdots, n).$$

On the basis of this, Engel proceeded with the calculation obtaining that the invariants $I_k$ and the operator $X^{(q)}(f)$ undergo the following transformations[34]:

$$\begin{aligned} &I_k \to I_k - \delta t(Z^{(q)} I_k + \mathcal{Z}^{(q)} I_k), \\ &X^{(q)} f \to X^{(q)} f + \delta t(X^{(q)} f, Z^{(q)} f + \mathcal{Z}^{(q)} f). \end{aligned} \tag{3.28}$$

As the invariance of $I_k$ with respect to the infinitesimal transformation $X^{(q)}(f)$ is preserved under the change of coordinates imposed by $Z^{(q)} f + \mathcal{Z}^{(q)} f$, Engel obtained the following equations:

$$X^{(q)} I_k + \delta t \left\{ X^{(q)} \mathcal{Z}^{(q)} I_k - \mathcal{Z}^{(q)} X^{(q)} I_k - X^{(q)} \mathcal{Z}^{(q)} I_k - X^{(q)} Z^{(q)} I_k \right\} = 0, \tag{3.29}$$

from which Engel deduced the following:

$$X^{(q)} Z^{(q)} I_k \equiv 0, \quad (k = 1, \cdots, m). \tag{3.30}$$

Now, since the functions $I_k$ are a complete set[35] of invariants for the group $G$ defined by (3.27), one finally arrives at the result according to which $Z^{(q)} I_k$

[34]Let us examine in some detail the case of a differential operator. Here is the task to be fulfilled: one want to rewrite a differential operator, $B(f) = \sum_{j=1}^{n} \beta_j(x) \frac{\partial f}{\partial x_j}$ say, with respect to a new set of coordinates defined by the transformations imposed by another differential operator, $A(f) = \sum_{i=1}^{n} \alpha_i(x) \frac{\partial f}{\partial x_i}$ say. In virtue of the transformation $x'_i = x_i + \alpha_i(x)\delta t$, the operator can be written as follows:

$$\begin{aligned} B(f) = \sum_{j=1}^{n} \beta_j(x) \frac{\partial f}{\partial x_j} &= \sum_{j,k=1}^{n} \beta_j(x'_i - \alpha_i(x)\delta t) \frac{\partial x'_k}{\partial x_j} \frac{\partial f}{\partial x'_k} \\ &= \sum_{j,k=1}^{n} \left\{ \beta_j(x'_i) - \sum_{l=1}^{n} \frac{\partial \beta_j}{\partial x'_l} \alpha_l \delta t \right\} \left\{ \delta_{kj} \frac{\partial f}{\partial x'_k} + \frac{\partial \alpha_k}{\partial x'_j} \delta t \frac{\partial f}{\partial x'_k} \right\} \\ &= \sum_{j=1}^{n} \beta_j \frac{\partial f}{\partial x'_j} + \delta t \sum_{j,k=1}^{n} \left\{ \beta_j \frac{\partial \alpha_k}{\partial x'_j} - \alpha_j \frac{\partial \beta_k}{\partial x'_k} \right\} \frac{\partial}{\partial x'_k} \\ &= B(f) + \delta t(B(f), A(f)), \end{aligned}$$

where a first order Taylor's expansion in $\delta t \alpha_i(x)$ has been operated. By posing $A(f) = Z^{(q)} f + \mathcal{Z}^{(q)} f$ and $B(f) = X^{(q)}(f)$, one obtains the wanted result. The transformation rules for the functions $I_k$, $(k = 1, \cdots, m)$, can be deduced in a similar way.

[35]That means that every differential invariant of order $q$ can be expressed as a function of a certain number of independent ones.

can be written as follows:

$$Z^{(q)} I_k = \sum_{i=1}^{n} \sum_{\nu_1,\cdots,\nu_n} \zeta_{i,\nu_1,\cdots,\nu_n}(x) \alpha^{(k)}_{i,\nu_1,\cdots,\nu_n}(I_1,\cdots,I_s), \quad (k=1,\cdots,s)^{36}. \tag{3.31}$$

Consequently, as Engel observed,

$$Z(f) + \sum_{k=1}^{s} \sum_{i=1}^{n} \sum_{\nu_1,\cdots,\nu_n} \zeta_{i,\nu_1,\cdots,\nu_n}(x) \alpha^{(k)}_{i,\nu_1,\cdots,\nu_n}(I_1,\cdots,I_m) \frac{\partial f}{\partial I_k}$$

is the general transformation of a continuous infinite group in the variables $x_1,\cdots,x_n,I_1,\cdots,I_s$. It is noteworthy to observe that this transformation coincides with (3.6) which Engel had given in his *Habilitationsschrift.* According to the procedure proposed therein, a continuous group, in the variables $x_1,\cdots,x_n$ only, could be deduced starting from such transformations by imposing the invariance of the following system of equations:

$$I_k = \omega_k(x_1,\cdots,x_n), \quad (k=1,\cdots,m).$$

By doing so, one obtains the defining equations of the infinitesimal transformations of a continuous group which, once more, coincide with the system of equations already given in [Engel 1886]:

$$\sum_{i=1}^{n} \sum_{\nu_1,\cdots,\nu_n} \zeta_{i,\nu_1,\cdots,\nu_n}(x) \alpha^{(k)}_{i,\nu_1,\cdots,\nu_n}(\omega_1,\cdots,\omega_n) = \sum_{\tau=1}^{n} \zeta_\tau \frac{\partial \omega_k}{\partial x_\tau}, \quad (k=1,\cdots,m). \tag{3.32}$$

By making recourse to Lie's characterization of the infinitesimal transformations of a group, one easily recognizes that such infinitesimal transformations are exactly the infinitesimal transformations of the group defined by (3.27). Since (3.27) was chosen to be a generic continuous group, Engel was finally able to provide his entire procedure with rigorous grounds. In his words:

> *Thus, we have demonstrated that the form proposed by me for the defining equations of the infinitesimal transformations of a continuous group is indeed characteristic and that my method leads to the determination of the defining equations of all finite or infinite continuous groups.*[37]

[36] As usual,

$$\zeta_{i,\nu_1,\cdots,\nu_n}(x) = \frac{\partial \zeta_i}{\partial x_1^{\nu_1} \cdots \partial x_n^{\nu_n}}.$$

[37] *Hiermit ist bewiesen, dass die von mir angegebene Form für die Definitionsgleichungen der infinitesimalen Transformationen einer continuirlichen Gruppen charakteristisch ist und dass meine Methode zu den Definitionsgleichungen aller endlichen oder unendlichen continuirlichen Gruppen führt.*

In the light of this result as well as of the method described in his *Habilitationsschrift*, Engel's entire procedure can be summarized in the following theorem whose statement we take from an important article [Medolaghi 1897] by P. Medolaghi which we will be concerned with in the next paragraph:

**Theorem 10 (Engel, 1885-1894)** *Let*

$$\sum_{i=1}^{n} \xi_i \frac{\partial f}{\partial x_i} + \sum_{\kappa=1}^{m} U_\kappa(\alpha, \xi) \frac{\partial f}{\partial \alpha_\kappa}, \tag{3.33}$$

*be the symbol of an infinitesimal transformation in the variables $x_1, \cdots, x_n$, $\alpha_1, \cdots, \alpha_m$, the functions $U_\kappa(\alpha, \xi)$, homogeneous and linear with respect to the $\xi_i$ and their derivatives, being chosen so that the family of transformations obtained when the $\xi_i$ are free to vary arbitrarily builds up an infinite continuous group. Then let $\alpha_1, \cdots, \alpha_m$ be arbitrary*[38] *functions of $x_1, \cdots, x_n$. The equations:*

$$U_\kappa(\alpha, \xi) - \sum_{\nu=1}^{n} \xi_\nu \frac{\partial \alpha_\kappa}{\partial x_\nu} = 0, \quad (\kappa = 1, \cdots, m), \tag{3.34}$$

*are the defining equations of the infinitesimal transformations of a group (finite or infinite) in the variables $x_1, \cdots, x_n$. By considering all the groups of type (3.33), one obtains the defining equations of* all *the groups in the variables $x_1, \cdots, x_n$.*[39]

## 3.6 Medolaghi's contributions

The Italian mathematician Paolo Medolaghi (1873-1950) is not rated among the leading figures of history of mathematics. Nevertheless this judgement appears to be hasty and not commensurate with the important mathematical contributions carried out by him. Especially in the realm of the theory of infinite continuous groups, his researches were highly respected and often cited as essential reference.

Medolaghi studied mathematics at the University of Rome where he defended his thesis in June 1895 on some problems of integration of partial differential equations. His thesis advisor, Luigi Cremona, was a fervid admirer of Lie's work and thus it is likely that Cremona played a major role in encouraging the young Medolaghi to deepen the study of Lie's groups. Between 1897 and 1899 he published a great number of papers on the theory of infinite continuous groups. They represented an important contribution to the theoretical development of the subject; at the same time, they provided

[38] Actually one should require certain integrability conditions. Later on, a discussion over this aspect will be given too.

[39] See [Medolaghi 1897, p. 188].

an original and very interesting analysis of the numerous, possible applications of the theory, namely in the realm of the theory of partial differential equations and pure geometry.

Medolaghi's first important paper [Medolaghi 1897] was published in 1897 in the *Annali di Matematica Pura e Applicata*. Its relevance lay mainly in the fact that for the first time therein the classification problem for infinite groups is formulated in terms of finite transformation and not in terms of infinitesimal transformations only. Furthermore, the correspondence between finite and infinite groups and certain finite groups of special composition already discovered by Engel was analyzed by Medolaghi in a more systematic and fruitful way. Indeed he was able to provide the structure of such groups[40] in the general case and thus he succeeded in completing Engel's treatment of the same problem which was limited to the case when the infinitesimal transformations (3.6) contained derivatives of the first and second order only. Finally, the determination of the finite equations of a generic infinite or finite continuous group was traced back to the determination of the finite equations of the corresponding Engel group.

Let us read Medolaghi's relevant synthesis of the main contributions contained in [Medolaghi 1897].

> *Already in 1885, Engel provided a method for constructing these systems of equations* [i.e. the defining systems of infinitesimal transformations.] *for all groups in n variables. Since then, an outstanding correspondence between these groups and certain finite groups of particular composition has emerged. These structures were determined by Engel for those defining equations which are of first and second order. More recent remarks of Engel,* [Medolaghi referred to [Engel 1894].] *in which he set out to prove the validity of his method in full generality, have facilitated my determination of these compositions in the general case too. This determination, along with a new account of Engel's method, can be found in §2 of this paper.*
> *The remaining paragraphs are devoted to showing another aspect of the correspondence between the finite groups so determined (which I call $\gamma_{sn}$) and the groups in n variables. These remarks refer to the equations of finite transformations.*[41]

[40] Later on, such finite groups will be designated as *Engel groups*.

[41] *L'Engel, già fino dal 1885, espose un metodo per formare questi sistemi di equazioni* [i.e. the defining systems of infinitesimal transformations.] *per tutti i gruppi in n variabili, e fino da quel momento apparì una notevole corrispondenza tra questi gruppi e certi gruppi finiti di particolari composizioni. Queste composizioni furono anche determinate dall'Engel pel caso in cui le equazioni di definizione sono dell'ordine primo e secondo. Più recenti considerazioni dell'Engel,* [Medolaghi referred to [Engel 1894].] *intese a dimostrare la generalità del suo metodo, mi hanno reso facile determinare queste composizioni anche nel caso generale. Questa determinazione, insieme ad una nuova esposizione del metodo*

After a sketchy account of Engel's method for generating the defining systems of groups of infinitesimal transformations, Medolaghi set out to solve the problem of determining the structure of *all* Engel groups. The strategy employed by Engel in 1885 was less more than rudimentary: to write down the relations which guarantee that the transformations (3.6) build up a continuous group in the variables $x_1, \cdots, x_n, \alpha_1, \cdots, \alpha_m$. It was rather inefficient since, one had to perform calculations case by case. Hence Medolaghi decided to follow an alternative route. The key idea at the basis of the proposed solution was the exploitation of Engel's 1894 theorem.
Engel had considered an arbitrary infinitesimal transformation

$$Z(f) = \sum \zeta_i(x) \frac{\partial f}{\partial x_i}$$

and then he had prolonged it until a certain order, say $s$, to obtain

$$Z^{(s)} f = \sum_{i=1}^{n} \zeta_i(x) \frac{\partial f}{\partial x_i} + \sum_i \sum_\nu \frac{\partial \zeta_i}{\partial x_\nu} A_{i\nu} f + \sum_i \sum_{\mu\nu} \frac{\partial \zeta_i}{\partial x_\mu \partial x_\nu} A_{i\mu\nu} f + \cdots,$$

where $A_{i\nu}, A_{i\mu\nu}, \cdots$ are infinitesimal transformations in the variables

$$y_{11} = \frac{\partial y_1}{\partial x_1}, \cdots, y_{nn} = \frac{\partial y_n}{\partial x_n}, y_{111} = \frac{\partial^2 y_1}{\partial x_1 \partial x_1}, \cdots$$

Afterwards, he had proved that

$$A_{i,\nu_1,\nu_n}(I_k) = \alpha^k_{i,\nu_1,\nu_n}(I_1, \cdots, I_m),$$

where $I_1, \cdots, I_m$ indicate a complete set of invariants of the group. As a consequence of this, Medolaghi observed, the equations

$$I_1(y_1, \cdots, y_n, y_{11}, \cdots) = c_1, \cdots, I_m(y_1, \cdots, y_n, y_{11}, \cdots) = c_m,$$

where the $c_k$'s are arbitrary constants, determine an invariant foliation of the space $y_1, \cdots, y_n, y_{11}, \cdots$ with respect to the finite group generated by

$$A_{i,\nu_1,\cdots,\nu_n}(f), \quad (i = 1, \cdots, n; \nu_1 + \cdots + \nu_n \leq s).$$

A theorem[42] proved by Lie then guaranteed that the infinitesimal transformations

$$\sum_{k=1}^{n} \alpha^k_{i,\nu_1,\cdots,\nu_n}(I_1, \cdots, I_m) \frac{\partial f}{\partial I_k},$$

*di Engel, si trovano nel §2 di questo lavoro.*
*Gli altri paragrafi sono destinati a mostrare un altro aspetto della corrispondenza tra i gruppi finiti così trovati (che io chiamo gruppi $\gamma_{sn}$) ed i gruppi in n variabili. Le considerazioni si riferiscono alle equazioni delle trasformazioni finite.* See [Medolaghi 1897, p. 179].

[42]The statement of the theorem spoken of is the following:

**Theorem 11** *Let us consider a r-parameter group $G_r$, $X_1(f), \cdots, X_r(f)$ acting on a n-*

builds up a group in the variables $I_1, \cdots, I_m$ which is meriedrically isomorphic[43] to the group $A_{i,\nu_1,\cdots,\nu_n}(f)$.

On the basis of this result Medolaghi was able to characterize the Engel groups as those groups which are homomorphic to the group generated by $A_{i,\nu_1,\cdots,\nu_n}(f)$. In view of Engel's procedure, the first step en route for the determination of all the groups in $n$ variables was consequently identified by Medolaghi with the determination of all groups homomorphic to $A_{i,\nu_1,\cdots,\nu_n}(f)$. Thus Medolaghi's reinterpretation of Engel's theorem opened the possibility of studying the properties of Engel's groups in full generality, without any need to perform explicit calculations in terms of the transformations (3.6). To this end, it is important to recall how the transformations $A_{i,\nu_1,\cdots,\nu_n}(f)$ had been obtained; they had emerged from the prolongation of a generic infinitesimal transformation $Z(f)$. Let us see in some detail the relevant calculations.

In order to prolong the transformation $Z(f)$ it is first necessary to introduce other variables $y_1, \cdots, y_n$, supposed to be invariant, which are regarded as functions of the variables $x_1, \cdots, x_n$. The problem consists now of calculating what is the effect of the transformation $Z(f) = \sum_{i=1}^{n} \zeta_i(x_1, \cdots, x_n)\frac{\partial f}{\partial x_i}$ on the derivatives of $y$ with respect to $x$ until a certain order, say $q$. The infinitesimal transformation which describes such changes is the prolongation of $Z(f)$ of order $q$, to be indicated with the symbol $Z^{(q)}(f)$. For the sake of brevity, we will limit the calculation to the first order prolongation.

As usual the infinitesimal transformation $Z(f)$ may be interpreted as a change of coordinates, $x'_j = x_j + \zeta_j(x)\delta t$; we want to calculate $\frac{\partial y'_k}{\partial x'_j}$ under the hypothesis that $y'_k = y_k$. To this end we observe that

$$\frac{\partial}{\partial x'_j} = \frac{\partial}{\partial x_j} - \delta t \sum_{m=1}^{n} \frac{\partial \zeta_m}{\partial x_j}\frac{\partial}{\partial x_m}, \quad (j = 1, \cdots, n),$$

and thus we obtain:

$$\frac{\partial y'_k}{\partial x'_j} = \frac{\partial y_k}{\partial x'_j} = \frac{\partial y_k}{\partial x_j} - \delta t \sum_{m=1}^{n} \frac{\partial \zeta_m}{\partial x_j}\frac{\partial y_k}{\partial x_m}, \quad (k, j = 1, \cdots, n).$$

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*dimensional space; let it be imprimitive and let*

$$u_1(x_1, \cdots, x_n) = c_1, \cdots, u_{n-q}(x_1, \cdots, x_n) = c_{n-q}$$

*be an invariant foliation (*Zerlegung*) of $G_r$, then the infinitesimal transformations*

$$\sum_{\nu=1}^{n-q} X_k u_\nu \frac{\partial f}{\partial u_\nu} = \sum_{\nu=1}^{n-q} \omega_{k\nu}(u_1, \cdots, u_{n-q})\frac{\partial f}{\partial u_\nu},$$

*generate a finite continuous group in the variables $u_1, \cdots, u_{n-q}$ which is isomorph to $G_r$.*

See [Lie 1888, p. 307].

[43] In modern terms, we would say that the group is homomorphic to $A_{i,\nu_1,\cdots,\nu_n}(f)$.

As a consequence of this the first prolongation of $Z(f)$ assumes the following form:

$$Z^{(1)}(f) = \sum_{i=1}^{n} \zeta_i(x) \frac{\partial f}{\partial x_i} - \sum_j \sum_k \sum_m \frac{\partial \zeta_m}{\partial x_j} \frac{\partial y_k}{\partial x_m} \frac{\partial f}{\partial \left( \frac{\partial y_k}{\partial x_j} \right)}.$$

The linear infinitesimal transformation

$$\sum_k \frac{\partial y_k}{\partial x_m} \frac{\partial f}{\partial \left( \frac{\partial y_k}{\partial x_j} \right)}$$

coincides (modulo a minus sign) with what was previously indicated with $A_{mj}$.

As already observed by Engel in 1894, Medolaghi emphasized the importance of a twofold way to prolong a given infinitesimal transformation. Indeed, besides the prolongation just described, one could think of an infinitesimal transformation acting on the dependent variables $y_1, \cdots, y_n$ only, leaving the independent variables $x_1, \cdots, x_n$ invariant. In such a way, a new transformation is obtained which is the prolongation of an infinitesimal transformation $W(f) = \sum_{i=1}^{n} \xi_i(y) \frac{\partial f}{\partial y_i}$. By performing analogous calculations to those mentioned above, Medolaghi deduced the following expression

$$W^{(1)}(f) = \sum_{i=1}^{n} \xi_i(y) \frac{\partial f}{\partial y_i} + \sum_j \sum_k \sum_m \frac{\partial \xi_m}{\partial y_j} \frac{\partial y_j}{\partial x_k} \frac{\partial f}{\partial \left( \frac{\partial y_m}{\partial x_k} \right)},$$

and introduced the linear infinitesimal transformations

$$B_{mj} = \sum_k \frac{\partial y_j}{\partial x_k} \frac{\partial f}{\partial \left( \frac{\partial y_m}{\partial x_k} \right)}.$$

By extending the same reasoning to prolongations of a general order, say $q$, Medolaghi was then in the position to prove the existence of two finite continuous groups in the variables $\frac{\partial y_i}{\partial x_j}, \frac{\partial^2 y_i}{\partial x_j \partial x_k}$, etc..., the groups generated by $A_{i,\nu_1,\cdots,\nu_n}(f)$ and $B_{i,\nu_1,\cdots,\nu_n}(f)$, $(i = 1, \cdots, n; \sum_i^n \nu_i \leq s)$, in terms of which the twofold prolongation of a given infinitesimal transformation finds its expression. By employing a denomination to appear later on, the group generated by $A_{i,\nu_1,\cdots,\nu_n}(f)$, to be indicated with $\mathcal{A}_s$, will be referred to as the group of right Engel transformations, while the group generated by $B_{i,\nu_1,\cdots,\nu_n}(f)$, indicated by $\mathcal{B}_s$, will be called the group of left Engel transformations.

An adequate comprehension of the role played by these two groups for the whole structural theory was one of Medolaghi's main achievements. Indeed, although the only transformations that explicitly appeared in Engel's procedure were the infinitesimal transformations of Engel right group, it turned out that Engel left group also assumed an outstanding auxiliary role. The reason was a technical one: indeed, the determination of the groups which are meriedrically isomorphic to $\mathcal{A}_s$ could be traced back to the determination of the subgroups of $\mathcal{B}_s$. Finally Medolaghi succeeded in providing the defining equations of the finite transformations of these two groups; they could be easily obtained upon simple application of the formulas for the derivatives of composed functions. As Ugo Amaldi wrote in [Amaldi 1908]:

> [...] *Medolaghi succeeded in identifying Engel characteristic groups as those series of simply transitive, pairwise reciprocal groups whose finite equations can be obtained in the following manner: let us consider the following three series of variables*
>
> $$z_i, y_i, x_i \quad (i = 1, 2, \cdots, n);$$
>
> *firstly, when regarding the $z_i$'s as composed functions of the $x_i$'s by means of the $y_i$'s, express the derivatives of $z$ with respect to $x$ until a certain order as functions of the derivatives of $z$ with respect to $y$ and of the derivatives of $y$ with respect to $x$; secondly, in the equations just obtained, consider the derivatives of $z$ with respect to $x$ as new variables, the derivatives of $z$ with respect to $y$* [or the derivatives of $y$ with respect to $x$ ] *as old variables and the derivatives of $y$ with respect to $x$* [or the derivatives of $z$ with respect to $y$] *as parameters. These are the indicated groups.*[44]

On the basis of these preliminary observations, Medolaghi set out to exhibit a general method for deducing the defining equations of the finite transformations of a group from the knowledge of the equations of the infinitesimal ones (3.32). The idea consisted of exploring the possibilities for obtaining these equations as the result of the integration of one complete system. From Lie's findings, it was well-known that they could be written in the following familiar form:

$$I_k(y_1, \cdots, y_n, y_{11}, \cdots) = \omega_k(x_1, \cdots, x_n), \quad (k = 1, \cdots, m).$$

Thus it was natural to search for functions $I_1, \cdots, I_m$ such that

$$A_{i,\nu_1,\cdots,\nu_n}(I_k) = \alpha^k_{i,\nu_1,\cdots,\nu_n}(I_1, \cdots, I_m), \quad (k = 1, \cdots, m)$$

[44] [...] *il Medolaghi era riuscito a identificare i gruppi caratteristici dell' Engel in quelle due serie di gruppi semplicemente transitivi, a due a due reciproci, le cui equazioni finite si ottengono nel seguente modo: si immaginino tre serie di n variabili*

$$z_i, y_i, x_i \quad (i = 1, 2, \cdots, n);$$

*e, pensate le $z_i$ come funzioni composte delle $x_i$ per mezzo delle $y_i$, si esprimano le derivate, fino ad un certo ordine, delle z rispetto alle x in funzione delle derivate delle z rispetto alle y e delle y rispetto alle x. Infine nelle equazioni così ottenute si considerino le derivate delle z rispetto alle x come variabili nuove, le derivate delle z rispetto alle y [o delle y rispetto alle x] come antiche variabili e le derivate delle y rispetto alle x [o delle z rispetto alle y] come parametri. Sono questi i gruppi indicati.* See [Amaldi 1908, p. 322-323].

and such that $I_k(y_1, \cdots, y_n, y_{11}, \cdots) \equiv \omega_k(x_1, \cdots, x_n)$, when $y_i = x_i$, $y_{i\nu} = \delta_{i\nu}$. Indeed it turned out that these requirements were sufficient for unique determination of them. To this end, Medolaghi explained, one had to consider the following complete system in the variables $y_{11}, \cdots, y_{nn}, \cdots, I_1, \cdots, I_m$:

$$A_{i,\nu_1,\cdots,\nu_n}(f) + \bar{A}_{i,\nu_1,\cdots,\nu_n}(f) = 0, \tag{3.35}$$

where, as usual $A_{i,\nu_1,\cdots,\nu_n}(f)$ designated the generators of the group of Engel's right transformations $\mathcal{A}_s$, while the operators $\bar{A}_{i,\nu_1,\cdots,\nu_n}(f)$ stood for $\sum_{k=1}^{n} \alpha^k_{i,\nu_1,\cdots,\nu_n}(I_1, \cdots, I_m) \frac{\partial f}{\partial I_k}$. By indicating with

$$\Phi_1(I_1, \cdots, I_m, y_{11}, \cdots), \cdots, \Phi_m(I_1, \cdots, I_m, y_{11}, \cdots)$$

the principal solutions of (3.35) with respect to the values $y_{i\nu} = \delta_{i\nu}$, Medolaghi proved that it was sufficient to solve the equations $\Phi_k = \omega_k(y)$ with respect to $I_1, \cdots, I_m$, in order to obtain the functions sought after:

$$I_k = \Omega_k(\omega_1(y), \cdots, \omega_m(y), y_{11}, \cdots), \quad (k = 1, \cdots, m).$$

As a consequence of this, he finally was able to deduce the following general form for the defining equations of the finite transformations of a group:

$$\Omega_k(\omega_1(y), \cdots, \omega_m(y), y_{11}, \cdots) = \omega_k(x), \quad (k = 1, \cdots, m). \tag{3.36}$$

It should be noticed that the dependence of (3.36) upon the variables $x$ and $y$ occurs only through the so-called *characteristic functions*, $\omega_1, \cdots, \omega_m$. Finally, Medolaghi succeeded in making such a characterization of the defining equations of a group even more precise. Indeed, he showed that the functions $\Omega_k$, $(k = 1, \cdots, m)$ could be identified with the finite transformations of Engel group associated to the group of infinitesimal transformations.

As Medolaghi explained in the introductory remarks to [Medolaghi 1897], his research was motivated by wide applications in the realm of partial differential equations. In particular, in 1895 Lie had attempted to generalize his theory of partial differential systems to those systems which admit an infinite continuous group of transformations. He had indicated in the theory of infinite groups the unifying principle of an organic theory of partial differential equations.

Medolaghi followed the route opened up by Lie by putting special emphasis on the role played by the correspondence between infinite groups and finite groups of Engel type. The hope was that of providing a deeper analysis aimed at developing some outstanding intuitions which Lie had exposed in [Lie 1895b].

At the same time, it appears that Medolaghi was interested in studying applications in the realm of pure geometry too. Indeed he was well aware of the geometrical content of his research on the form of the defining equations for the finite transformations of groups. This emerges quite clearly for example from the reading of the letters that he wrote to Engel between 1897 and 1900[45] as well as from some relevant brief papers that Medolaghi devoted to such a geometrical analysis. In regards to this, the contribution *Sopra la forma degli invarianti differenziali* should be mentioned. One finds therein an interesting theorem on the dependence of the differential invariants of a group upon the variables $x$ and $y$; more precisely, Medolaghi succeeded in demonstrating that these variables enter the differential invariants by means of the characteristic functions only. Although the geometrical significance was not made explicitly on this occasion, one can have an idea of Medolaghi's underlying motivation by looking at a letter that he sent to Friedrich Engel on 27th April 1898. After communicating to Engel some important results of his on the possibility that groups belonging to the same Engel type, that is having the same Engel's group, may be not similar[46], Medolaghi emphasized the geometrical significance of the theory whose development he had contributed to. The notion of geometry accepted by Medolaghi is that proposed by Felix Klein in his well-known *Erlanger Programm*. Medolaghi wrote:

> [...] *another example of groups which, not being similar, nevertheless has defining equations of the same form* [i.e. (3.36)] *is that of the group of Euclidean movements and of the group of non-Euclidean movements: now the theorems which I have proved in the recent mémoire of the Accademia dei Lincei suggests that:*
> *among all geometries which are based on groups belonging to the same system (3.36) there exist the same relations as between the Euclidean and the non-Euclidean geometries.*
> *in other words:*
> *One can pass from a geometry with fundamental group (3.36) to the geometry with a fundamental group obtained by (3.36) under a change of the characteristic functions, by changing certain words and expressions into certain other words and expressions.*[47]

[45] The letters to Engel from Medolaghi (in all there are fifteen of them) can be consulted at Engel's *Nachlass* in Giessen. See also [Rogora 2010].

[46] This aspect of the theory will be dealt with in the next section.

[47] [...] *un altro esempio di gruppi non simili e che pure hanno equazioni della stessa forma* [i.e. (3.36)] *è quello del gruppo dei movimenti Euclidei e il gruppo dei movimenti non Euclidei: ora i teoremi sugli invarianti differenziali da me indicati nell'ultima Nota della Accademia dei Lincei mi conducono a credere che*
*fra tutte le geometrie che hanno a fondamento gruppi con uno stesso sistema (3.36) hanno luogo delle relazioni della stessa natura come tra la Geometria Euclidea e quella Non Euclidea.*
*od anche in altre parole.*
*Da una geometria che abbia a fondamento un gruppo (3.36) si passa alla geometria che*

## 3.7 Vessiot and his Mémoire couronnée

In 1900 the Académie des Sciences de Paris published the institution of a price whose aim was that of encouraging researches concerning the application of Lie groups to the study of differential equations[48]. The subject was a very popular one in France at that time. In the 1880's and in the early 1890's Picard and Vessiot had developed a differential Galois theory for linear homogenous ordinary differential equations. More recently, namely in 1898, Jules Drach had tried to generalize it to homogenous linear partial differential equations. Unfortunately, Drach's work was flawed by severe mistakes. Although they did not completely jeopardize his original and very interesting approach, however they rendered his achievements completely ineffective and devoid of any rigorous foundation. It is likely that the very need to remedy Drach's shortcomings may have played a role in influencing the choice of the Parisian mathematical community.

Ernest Vessiot, one of the leading expert in this field and the first person to draw attention to certain incoherences of Drach's thesis, decided to undertake the challenge.

Vessiot was born in Marseille on 8th March 1865. He came from a well-educated family; his father Alexandre was a school teacher. After his secondary studies at the Licée de Marseille, Vessiot was brilliantly admitted to the *École Normale Supérieure* in 1884. Since the beginning of his mathematical activity, Vessiot became interested in Lie's theory of continuous groups, namely in the fruitful applications of the latter to the theory of differential equations. The interest for this subject never abandoned him for the rest of his life. Over his long mathematical career, he devoted himself to different branches of mathematical and physical research such as, conformal geometry, ballistics, relativity theory, celestial mechanics, etc., however, as Cartan was to observe in 1947, Vessiot's whole mathematical production is characterized by a remarkable unity which is to be ascribed to his project of developing and completing the research program started up by Lie in the early 1870's.
The contribution by Vessiot that will be analyzed in this section represent the climax of this enterprize. They are undisputed masterpieces which still today raise admiration and great interest among mathematicians.

In September 1902 Vessiot submitted to the Academy a long memoir divided into three parts which was awarded with the victory of the price. The first part, published in the annals of the *École Normale* in 1903, will be of primary interest for our discussion on the structural theory of infinite

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*ha per fondamento un gruppo diverso da (3.36) soltanto per le funzioni caratteristiche, cambiando certe parole ed espressioni in certe altre parole ed espressioni.*

[48]Here is the announcement text: *"Perfectionner, en un point important, l'application de la théorie des groupes continus à l'étude des équations aux dérivées partielles"*. See *Comptes rendus, Acad. Sci. Paris*, p. 1151, t. 131, 1900.

continuous groups. Vessiot set out to take up the results of Lie, Engel and Medolaghi and to put them in an organic and systematic form that would facilitate the applications of the notion of group to the resolution of partial differential equations. It should be observed that [Vessiot 1903] consisted mainly of a revision of already known results; nonetheless, some noteworthy novelties were introduced too, especially for what concerns the study of similarity between groups as well as a more rigorous analysis of some integrability conditions that have to be introduced at a certain point. Finally, a general definition of isomorphism valid for the case of infinite groups too, which turned out to be equivalent with a definition introduced by Cartan a year before, was proposed. Vessiot started his analysis first by introducing the groups of right and left Engel transformations; in this connection, no new result was obtained but his treatment was more neat and simple with respect to that offered by Medolaghi. The procedure for obtaining such groups from the prolongation of a given infinitesimal transformation was carefully described and then the compositions of $\mathcal{A}_s, \mathcal{B}_s$ and their reciprocal relation were established by providing an explicit expression of their transformations.

The deduction of the defining equations of the finite transformations of group in the form first proposed by Medolaghi in 1897 was particularly clear. As already pointed out by Engel in [Engel 1894] and by Medolaghi himself, it was convenient, Vessiot observed, to start by writing down the defining equations for the finite transformations of a group in Lie's form:

$$I_k(y_1, \cdots, y_n, y_{11}, \cdots) = \omega_k(x_1, \cdots, x_n) \quad (k = 1, \cdots, m). \tag{3.37}$$

Engel and Medolaghi had proved that the infinitesimal transformations of $\mathcal{A}_s$ were such as to leave the foliation (*Zerlegung*) $I_k = c_k$ of the space of variables $y_{11}, \cdots, y_{nn}, \cdots$ invariant, i.e. that there exist appropriate functions $\alpha^k_{i,\nu_1,\cdots,\nu_n}$ such that the following relations hold:

$$A_{i,\nu_1,\cdots,\nu_n}(I_k) = \alpha^k_{i,\nu_1,\cdots,\nu_n}(I_1, \cdots, I_m), \quad (k = 1, \cdots, m). \tag{3.38}$$

From that, one had deduced the existence of a finite group (the Engel group) of infinitesimal transformations univocally associated to the group (3.37). Vessiot decided to concentrate his attention on the equations for the finite transformations of the group $\mathcal{A}_s$ in order to obtain the finite transformations of the Engel group which describe the changes to which the fundamental invariants $I_1, \cdots, I_m$ are subject as a consequence of the transformation $\phi$ acting upon the independent variables $x_1, \cdots, x_n$:

$$\begin{aligned} I'_k &= I_k(y_1, \cdots, y_n, \cdots, y'_{j,\beta_1,\cdots,\beta_n}, \cdots) = \\ &= L_k[\cdots I_h(y_1, \cdots, y_n, \cdots, y_{j,\beta_1\cdots\beta_n}, \cdots) \cdots | \cdots \phi_{i,\delta_1\cdots,\delta_n}, \cdots] \end{aligned} \quad (k = 1, \cdots, m). \tag{3.39}$$

Thus, by introducing the auxiliary variables $u_1, \cdots, u_m$, Vessiot wrote the finite transformations of Engel's group $\mathcal{L}$, as follows:

$$u'_k = L_k(u_1, \cdots, u_m | \cdots, \phi_{j,\delta_1 \cdots \delta_n}), \quad (k = 1, \cdots, m), \tag{3.40}$$

where the $\phi_{j,\delta_1 \cdots \delta_n}$ have to be considered as parameters.

At this point Vessiot considered the general infinitesimal transformation of Engel type (3.6). The finite transformations of $\mathcal{L}$ having been already obtained, he was in the position to write down the corresponding finite transformation[49]:

$$\begin{cases} x'_i = \phi_i(x_1, \cdots, x_n) \\ u'_k = L_k(u_1, \cdots, u_n | \cdots, \phi_{j,\delta_1 \cdots \delta_n}, \cdots) \end{cases} \quad (i = 1, \cdots n; k = 1, \cdots, m). \tag{3.41}$$

He was now ready to deduce the defining equations of the group in Medolaghi's form. It simply rested upon Lie's characterization of the transformations of a group as those transformations which leaves the defining differential system invariant. Explicitly, in order for the transformation $x'_i = \phi_i(x_1, \cdots, x_n)$ to be a transformation of the group, it is necessary and sufficient that the transformed defining system

$$I_k(y_1, \cdots, y_n, \cdots, y'_{j,\beta_1 \cdots \beta_n}, \cdots) = \omega_k(x'_1, \cdots, x'_n), \quad (k = 1, \cdots, m).$$

is a consequence of (3.37) as well as of $x'_i = \phi_i$. Now, Vessiot observed, according to (3.39), this condition could be written as:

$$L_k[\cdots I_h(y_1, \cdots, y_n, \cdots, y_{j,\beta_1 \cdots \beta_n}, \cdots) \cdots | \cdots \phi_{i,\delta_1 \cdots \delta_n}, \cdots] = \omega_k(\phi_1, \cdots, \phi_n),$$

for $k = 1, \cdots, m$. By replacing the letters $\phi$ by $y$ and the $I_h$'s by the corresponding $\omega_h$'s, he finally obtained the announced Medolaghi's equations:

$$L_k[\omega_1(x_1, \cdots, x_n), \cdots, \omega_m(x_1, \cdots, x_n) | \cdots, y_{j,\delta_1 \cdots \delta_n}, \cdots] = \omega_k(y_1, \cdots, y_n), \\ (k = 1, \cdots, m). \tag{3.42}$$

The properties of such a canonical form were then the object of a careful study by Vessiot. The possibility of deducing from them the equations of all the groups similar to a given one was established without special difficulty. Namely, if the defining equations of a group $G$ are given in the just obtained Medolaghi's form, then the equations of the group $G' = T^{-1}GT$, transformed of $G$ under the change of coordinates $x'_i = Tx_i = f_i(x)$, Vessiot

[49] Very interestingly, Pommaret interpreted the equations of this finite transformation as giving the transition function of a fiber bundle $\mathcal{F}$ over $X$ (the space of source variables) which he called *bundle of geometric objects.* Pommaret suggests that, by writing (3.41), Vessiot was laying the grounds of the modern theory of fiber bundles. Though tempting, such an attribution seems to have no historical justification.

proved, assume the following form:

$$L_k[\bar{\omega}_1(x_1, \cdots, x_n), \cdots, \bar{\omega}_m(x_1, \cdots, x_n)| \cdots, y_{j,\delta_1 \cdots \delta_n}, \cdots] = \bar{\omega}_k(y_1, \cdots, y_n), \quad (k = 1, \cdots, m), \tag{3.43}$$

simply deduced upon substitution of $\omega_k$ by

$$\bar{\omega}_k = L_k[\cdots, \omega_h(f_1(x), \cdots, f_n(x)), \cdots | \cdots, f_{j,\delta_1 \cdots \delta_n}, \cdots].$$

Conversely, Vessiot noticed that every group obtained in this way is similar to the group $G$.

Next the procedure for determining all the transformation groups in any number of variables was thoroughly described. It consisted of two steps: first, it was necessary to determine all the groups (3.40) homomorphic to $\mathcal{A}_m$, then one had to choose the characteristic functions

$$\omega_1(x_1, \cdots, x_n), \cdots, \omega_m(x_1, \cdots, x_n)$$

in such a way as to guarantee that the system (3.42) admits solutions $y(x)$ different from the identity transformation.

The first problem did not present any special difficulties. As Medolaghi himself had observed, it could be faced by well-known techniques dating back to Lie. On the contrary, the problem of determining the integrability conditions of (3.42) appeared to be more challenging. In particular, it turned out that such integrability conditions could provide a solution to the problem of distinguishing among groups which, though corresponding to the same Engel's group $\mathcal{L}$, are not similar.

The question had been tackled for the first time and partially solved by Medolaghi some years before[50], but it was left to Vessiot the task of clarifying the point in a more satisfying way.

It should be noticed that Medolaghi was the first one to call attention to the fact that groups corresponding to the same Engel group are not necessarily similar. Actually, still on February 1898 he was convinced of the contrary[51], but he soon discovered that this was by no means the case. Indeed in [Medolaghi 1899] he was able to provide a counterexample. The background was of a geometrical type. Medolaghi considered those transformations $y_i(x_1, \cdots, x_n)$, $i = 1, \cdots, n$ that leave the quadratic differential form

$$\sum_{h,k} f_{hk}(x_1, \cdots, x_n) dx_h dx_k$$

[50] See [Medolaghi 1899].

[51] Indeed, in a letter to Engel dating 21st February 1898 he wrote that *in general* two groups with the same associated Engel group are similar.

invariant. It was easy to show that these transformations which satisfy the following relations:

$$f_{ij}(x_1,\cdots,x_n)=\sum_{h,k} f_{hk}(y_1,\cdots,y_n)\frac{\partial y_h}{\partial x_i}\frac{\partial y_k}{\partial x_j},\quad (i,j=1,\cdots,n)^{52},\quad (3.44)$$

build up a continuous group. However, even if we impose, Medolaghi observed, certain integrability conditions, namely complete integrability, the system (3.44) is insufficient to fully characterize the group under examination. Indeed, the system (3.44) is easily recognized to be a candidate for representing the Medolaghi form of the defining equations of two distinct (i.e. not similar) groups[53]. Indeed, one of these is the group of the Euclidean displacements whose infinitesimal generators are

$$\frac{\partial}{\partial x_k},\qquad x_j\frac{\partial}{\partial x_k}-x_k\frac{\partial}{\partial x_j},\qquad (j,k=1,\cdots,n),$$

the other one is what Medolaghi called the group of non-Euclidean displacements whose generators are

$$\frac{\partial}{\partial x_k}-x_k\sum_{i=1}^{n}x_i\frac{\partial}{\partial x_i},\qquad x_j\frac{\partial}{\partial x_k}-x_k\frac{\partial}{\partial x_j},\qquad (j,k=1,\cdots,n).$$

In such a way, Medolaghi proved the possibility that groups corresponding to the *same* Engel group are not similar. Medolaghi himself in [Medolaghi 1899] set out to provide a method for separating different classes of similarities based on the study of the integrability conditions that have to be imposed on the defining equations of a group. However, as shown by Vessiot, his treatment turned out to be incomplete.

Vessiot completed it by showing that, along with the integrability conditions of the type taken into account by Medolaghi, integrability conditions of a different kind could emerge as well. In particular Medolaghi had shown that the integrability conditions for the system (3.42) could be reduced to systems of the following type:

$$J_i\left(\omega_1,\cdots,\omega_m|\cdots,\frac{\partial\omega_s}{\partial x_l},\cdots\right)=c_i,\quad (i=1,\cdots,r);\qquad (3.45)$$

and from this he had deduced that among groups belonging to the same Engel type it always exists one (a so-called Picard group) which contains the group of all translations.

On his part Vessiot explained that integrability conditions of another type:

$$\Omega_h\left(\omega_1,\cdots,\omega_m|\cdots,\frac{\partial\omega_s}{\partial x_l},\cdots\right)=0,\quad (h=1,\cdots,\rho)\qquad (3.46)$$

[52] In this case the functions $f_{ij},(i,j=1,\cdots,n)$ play the role of characteristic functions.

[53] This is a consequence of the fact that the independent and dependent variables $x,y$ enter (3.44) only through the characteristic functions $f_{ij}$. In other words, by varying $f_{ij}$ one obtains groups corresponding to the same Engel group.

had in fact to be imposed in some cases by providing a counterexample which invalidated Medolaghi's theorem on the existence of Picard's groups belonging to *every* Engel type[54].

At the end of [Vessiot 1903], Vessiot set out to provide a generalization of the notion of isomorphism between groups of transformations which could be apply as well to infinite continuous groups. Vessiot's notion of isomorphism turned out to be equivalent to the one proposed by Cartan one year before in [Cartan 1902a], nevertheless, it seems that it was developed by him completely independently of Cartan's.

From a purely abstract point of view, Vessiot observed, the notion of isomorphism between two infinite continuous groups did not pose particular difficulty. One could generalize to them the relevant definition adopted in the theory of finite continuous groups. As it is well known, this definition relied upon consideration of infinitesimal transformations: two finite continuous groups are said to be isomorphic if their infinitesimal transformations correspond ones to others bijectively in such a way that the Lie brackets of two arbitrary infinitesimal transformations of the first group corresponds to the bracket of the homologue transformations of the second.

However, as Vessiot pointed out, in the infinite dimensional case, the problem consisted of providing a general definition which allowed to translate the correspondence between transformations in a purely analytical manner. Indeed, the fact that an infinite continuous group possesses an infinite number of independent infinitesimal transformations rendered the previous definition utterly inadequate.

In order to remedy this inconvenience Vessiot proceeded as follows. He considered two finite continuous groups $G$, $G_1$ which he supposed to be holoedrically isomorphic. He first constructed simply transitive, imprimitive prolongations of such groups to be indicated with $\bar{G}$ and $\bar{G_1}$; then he proved that these prolongations are necessarily similar and finally he was able to characterize the notion of isomorphism between two groups as the possibility of constructing similar (holoedric) prolongations.

This alternative definition of isomorphism between finite continuous groups was considered by Vessiot to be sufficiently general to be extended to the infinite dimensional case. However, as we will see, it was only with Cartan's fundamental work that this definition would be profitably applied within a systematic structural approach.

Despite his strong interest towards abstract, chiefly structural issues, Vessiot did not disregard to investigate applicative aspects of his theory

[54]The two types of integrability conditions are nowadays known as integrability conditions of the first and second kind. See [Pommaret 1978, §7.3]. In Pommaret's view, the approach pursued by Vessiot represents a genuine generalization of Lie's classical theory of finite groups. Indeed, according to this standpoint, the constants $c$'s emerging from the integrability conditions of the defining equations play the role of structure constants of the pseudogroup under consideration.

of infinite continuous groups. On the contrary, the theory of (partial) differential equations should be regarded as the main, if not unique, source of interest of his researches on this subject. In accordance with a general tendency which was quite popular in France at that time, which we often had the occasion to emphasize, Vessiot devoted conspicuous parts of his *Mémoire couronnée* to developing his own views on the better approach to follow in order to attain a satisfying generalization of Galois theory to the realm of differential equations. As it has already been recalled, some years before (1892) following Picard's work, he had succeeded in achieving such an aim in the limited case of *linear* ordinary differential equations. Some years later, Drach had then tried to develop a general theory for arbitrary ordinary differential equations by providing a Galois theory for equations of type:

$$X(z) = \frac{\partial z}{\partial x} + \sum_{i=1}^{n} P_i(x, x_1, \cdots, x_n) \frac{\partial z}{\partial x_i} = 0.$$

However original and farsighted, Drach's theory was tarnished by mistakes and lacunae which required a careful re-examination of the entire subject. Vessiot decided to tackle the problems disclosed by Drach's thesis in the second part of his *Mémoire couronnée* which was published in 1904 in the *Annales de l'École Normale* under the title *Sur la théorie de Galois et ses diverses généralisations*.

The central idea at the basis of Vessiot's theory consisted of a reformulation of Galois theory which exploited in an essential way the notion of *automorphic system*. Such an innovative approach put Vessiot in the position to provide a fully satisfying differential generalization of Galois classical ideas which was welcomed with the highest favour in the French mathematical community.

Automorphic (differential) systems had already been studied by Lie in 1895 [Lie 1895b]. Three years later, Drach himself had implicitly made recourse to this notion when he supposed that a certain differential system which enter the stage at a certain point of his theory was characterized by the property that its general solution could be obtained as the result of the action of a (generally, infinite) continuous group upon a particular solution[55]. However it was only in 1902 that Vessiot rendered this notion official by defining an automorphic system to be a system of algebraic or differential equations such as its every solution can be obtained from a particular one by means of a (unique) transformation of a group, to be called *fundamental group* of the automorphic system.

It seems interesting to convey some general indications of the route followed by Vessiot in his project of providing an alternative approach to the theory of algebraic equations. Sketchy remarks on his strategy for emending Drach's mistakes will be given too.

[55]See Appendix C.

From the outset, Vessiot emphasized the novelty of his theory, explaining as well the underlying motivations. He wrote:

> *We have abandoned Galois' proof method, which Mr. Picard, as is well-known,* [Vessiot referred to Picard's *Traité d'Analyse*] *has succeeded in extending to the case of linear homogeneous ordinary differential equations. In the present case,* [i.e. that of equations of type $X(z) = 0$] *the employment of this method collides with the following difficulty: the transition from a solution to another cannot be done by means of rational formulas.*
> *The method which we have employed is completely analytic. We have illustrated it first for Galois' classical theory.*[56]

Vessiot's starting point consisted of focusing his attention on the following problem which turns out to be fundamental to his method. Given an algebraic equation $P(x) = x^n - p_1x^{n-1} + p_2x^{n-2} - \cdots + (-1)^n p_n = 0$, Vessiot supposed that one or more *rational*, *entire* relations $G_i(x_1, \cdots, x_n) = 0$, $i = 1, \cdots, q$ among the roots $x_1, \cdots, x_n$ of $P(x) = 0$ exist. Let us indicate them with (A). The problem posed by him was: what information concerning the resolution of $P(x) = 0$ one can draw from the knowledge of (A)?

It is first necessary, Vessiot explained, to replace the equation $P(x)$ with the system (S) obtained from the well known relations between elementary symmetric functions and the coefficients of $P(x)$:

$$\sum x_i = p_1, \qquad \sum x_i x_k = p_2, \qquad \cdots, \qquad x_1 x_2 \cdots x_n = p_n.$$

By construction, the system (S), which admits $n!$ solutions, is an automorphic system with associated group equal to the general substitution group of $n$ objects, $G_n$. Indeed, by denoting with $\sigma$ a particular solution $x_i = \alpha_i$, $i = 1, \cdots, n$, every other solution can be written in the form $Tx_i = \alpha_i$, $i = 1, \cdots, n$, with $T \in G_n$; Vessiot denoted such a solution with the symbol $T\sigma$. By hypothesis, the algebraic system $[S, A]$ consisting of the juxtaposition of the systems $S$ and $A$ admits at least one solution $\sigma$. Therefore, there exists a family of substitutions $F = \bar{T}$ (in general, not a group) such that every solution of $[S, A]$ can be represented in the form $\bar{T}\sigma$, for some $\bar{T} \in F$.

Leaving aside the case in which $F$ reduces to the identity transformation, Vessiot focused his attention upon the general eventuality in which $[S, A]$ has more than one solution.

[56] *Nous avons abandonné la methode de démonstration de Galois, que M. Picard a réussi, comme l'on sait, à étendre au cas des équations différentielles ordinaires, linéaires et homogenes. Dans le cas actuel, en effet, l'emploi de cette méthode se heurte à la difficulté suivante: le passage d'une solution à une autre ne se fait pas, en général, par des formules rationelles.*
*La méthode que nous avons employeée est tout analytique. Nous l'avons exposée d'abord sur la théorie de Galois elle-même* [...]. See [Vessiot 1904a, p. 9-10].

His aim was that of showing that it is possible to derive from $[S, A]$, by means of rational operations only, an automorphic system of the form $[S, A']$, where $A'$ indicates a set of rational relations of the same type as that of $A$. More precisely, Vessiot set out to prove that the system $[S, A]$ could be decomposed in a finite number of automorphic systems associated to groups which are pairwise similar, i.e. which are obtained one from the other by means of an appropriate substitution.

The idea at the basis of this procedure was the following, simple observation. If one operates an arbitrary substitution $T$ upon $[S, A]$, the number of its solution remains unaffected. Besides, since $S$ is automorphic with respect to $G_n$, the result of such a substitution $T$ is a system $[S, TA]$, where $TA$ indicates the set of equations $G_i(Tx_1, \cdots, Tx_n) = 0$, $i = 1, \cdots, q$. It is easy to verify that the general solution of $[S, TA]$ can be written in the form, $\bar{T}T\sigma$, $\bar{T} \in F$.

It is interesting, Vessiot pointed out, to consider the case in which $TA$ admits some solution in common with $[S, A]$; he proved that this was the case when $T = \bar{T}_1^{-1}\bar{T}_2$, for some $\bar{T}_1, \bar{T}_2 \in F$. Furthermore, if $\sigma$ is chosen to be one of these common solutions, then $T$ must be of the form $\bar{T}^{-1}$, for $\bar{T} \in F$. Vessiot was thus led to consider the totality of algebraic systems $\bar{T}^{-1}A$.
It may happen, he observed, that each system $\bar{T}^{-1}A$ admits every solution of $[S, A]$. If so, then the family of substitutions $F$ is a group and consequently $[S, A]$ is automorphic. If this is not the case, nonetheless one can consider the totality of solutions common to $[S, A]$ and all the systems $\bar{T}^{-1}A$. Their general form is of type $\Theta\sigma$. The set of $\Theta$'s builds up a family of transformation to which the identity substitution belongs; indeed, remember that the systems $TA$ have been chosen in such a way as to guarantee that $\sigma$ is a solution thereof.
At this point it was easy for Vessiot to demonstrate that such a family is in fact a group and consequently that the system consisting of $(S)$ and of all the systems $\bar{T}^{-1}A$, for $\bar{T} \in F$, is automorphic.

The (automorphic) system just obtained has been constructed in such a way that $\sigma$ is one of its solutions. If one starts from another solution of $[S, A]$, say $\bar{T}_\alpha\sigma$, then the preceding procedure leads to another automorphic system whose associated group consists of substitutions of type $\bar{T}_\alpha^{-1}\Theta\bar{T}_\alpha$. Furthermore, it is easy to see that either two such automorphic systems have the same solutions or they do not have any in common.

As a result of this, Vessiot could finally deduce the already mentioned assertion according to which the system $[S, A]$ could be decomposed in a finite number of automorphic systems. He also provided a practical procedure to concretely produce automorphic systems starting from relations $A$.

At this point, Vessiot moved on to show how an alternative approach to Galois theory could be developed on the basis of these quite simple premises,. To this end, he needed the following:

**Lemma 3** *Consider a system of rational, entire equations in the unknowns $x_1, \cdots, x_n$:*

$$F_k(x_1, \cdots, x_n) = 0, \qquad k = 1, \cdots, p; \tag{3.47}$$

*if an equation of type $Q(x_1, \cdots, x_n) = g$, where $Q$ is a rational function of its arguments, admits all the solutions of (3.47), then the constant $g$ can be rationally expressed in term of the coefficients of $F_k$, $k = 1, \cdots, n$ and $Q$.*[57]

As Vessiot explained in details, this result played an important, technical role; namely it provided a means to obtain canonical forms of rational automorphic systems. Indeed, every automorphic system $(B)$ of rational and entire equations in the unknowns $x_1, \cdots, x_n$ with associated group $G$, Vessiot proved, is equivalent to the system obtained by adding to $(S)$ a single equation of type:

$$\Omega(x_1, \cdots, x_n) = \omega. \tag{3.48}$$

Here $\Omega$ indicates what Vessiot called a characteristic invariant of $G$, with non-vanishing discriminant. By this expression, he meant that $\Omega$ is a rational function of its arguments marked by the following properties: i) $\Omega$ admits all the substitutions which belongs to $G$ and no others, ii) if two arbitrary substitutions of the general group transform $\Omega$ into two distinct functions, then such functions assume different (numerical) values for every solution of $(S)$. Now, let $\omega$ be the value which $\Omega$ assumes in correspondence with a certain solution of the system $(B)$. By construction, the equation (3.48) admits every solution of $(B)$. As a consequence of the above mentioned lemma, $\omega$ is rationally known, i.e. it can be computed in terms of the coefficients of $(B)$ and $\Omega$. Thus, the equation (3.48) is the canonical form of the automorphic system $(B)$.

At this point, Vessiot was finally able to provide the answer to the problem he had posed at the very beginning of [Vessiot 1904a]. Indeed, in view of the above described decomposition of the system [S,A] in automorphic subsystems as well as of the canonical form just obtained, he concluded that the knowledge of rational relations (A) implies the knowledge of the value $\omega$ which a characteristic invariant $\Omega(x_1, \cdots, x_n)$, with non-vanishing discriminant, associated to a group $G$, assumes when its arguments $x_1, \cdots, x_n$ are replaced by a given solution $x_i = \alpha_i$, $i = 1, \cdots, n$ of $(S)$.

How this result could find application in Vessiot's intent to convey a new treatment of Galois theory? In order to gain an idea thereof, following Vessiot, let us consider a polynomial equation $P(x) = 0$. By employing denominations first introduced by Drach in his thesis, Vessiot defined the equation $P(x) = 0$ to be *special* if there exist rational systems of type (A) which admit some solutions of (S) without admitting all of them. In the contrary case, the equation $P(x) = 0$ was said to be *general*.

[57] See [Vessiot 1904a, p. 19].

If $P(x) = 0$ is special, one can pose the problem of classifying all conceivable rational relations such as (A). As a consequence of the preceding remarks, Vessiot observed, one can limit himself to consider all rational functions of type $\Omega(x_1, \cdots, x_n)$ which assume a rational value $\omega$ when $x_1, \cdots, x_n$ are replaced by a certain solution $\sigma$ of $(S)$. Vessiot proved that the subgroup $\Gamma$ which is obtained from the intersection of the groups corresponding to all such functions possesses an invariant which is part of the same family, i.e. it assumes a rational value $j$ in correspondence with $x_i = \alpha_i$, $i = 1, \cdots, n$. Vessiot indicated it with $J(x_1, \cdots, x_n)$.

Conversely, if $\Omega(x_1, \cdots, x_n)$ is an invariant of a group $G \supset \Gamma$, then the equation $\Omega(x_1, \cdots, x_n) = \Omega(\alpha_1, \cdots, \alpha_n)$ admits all the solutions of the systems obtained from $(S)$ by adding the equation $J(x_1, \cdots, x_n) = j$. Thus, as a consequence of Lemma (3), $\Omega(\alpha_1, \cdots, \alpha_n)$ is a rational number. Vessiot could finally deduce his own version of Galois fundamental theorem from which Galois classical statement could be derived in a straightforward way.

**Theorem 12 (Vessiot 1902)** *Consider the set of all rational, with non-vanishing discriminant, functions $\Omega(x_1, \cdots, x_n)$ whose coefficients belongs to a certain rationality domain $[R]$. The subset consisting of those functions which take on a rational value in correspondence with a given solution of the system (S) coincides with the set of those functions which are left invariant under the action of a given substitution group $\Gamma$.*[58]

Vessiot's contributions to the theory of algebraic equations did not provide any substantial new achievement with respect to the state of the art of the discipline. Nonetheless, his new theoretical framework had the merit of clarifying the connection between two distinct approaches which had characterized the work of mathematicians over the 19th century in this realm of research. On one hand, Galois' standpoint which, as Vessiot saw it, consisted of examining all simplifications which can be operated in the resolution of a given equation; on the other, Abel's point of view (later on shared by Lie as well) consisting of drawing information for the resolution of an arbitrary equation from the knowledge of particular given relations such as the existence of transformations mapping solutions into solutions. As Drach himself had pointed out in a brief note [Drach 1893] published in 1893, it was in the search for a clear understanding of the differences between Galois' and Abel's original approaches to the theory of algebraic equations that a generalization to the theory of differential equations could hopefully be found.
Vessiot succeeded in doing precisely that. With appropriate modifications, his theory of algebraic equations could be extended to linear ordinary differential equations as well as to linear partial differential equations of first order.

[58]See [Vessiot 1904a, p. 24-25].

In full analogy with the algebraic case, the generic ordinary differential equation of type:

$$P(x) = \frac{d^n x}{dt^n} + p_1(t)\frac{d^{n-1}x}{dt^{n-1}} + \cdots + p_n(t)x = 0, \tag{3.49}$$

was replaced by the corresponding automorphic system $(S)$ whose associated group coincides with the general linear group in $n$ variables. Again, Vessiot posed the question about what information one can drawn from the knowledge of rational relations $A$ subsisting among a fundamental system of solutions of (3.49), $x_1, \cdots, x_n$. He proved that every system of type $[S, A]$ could be decomposed into a series of automorphic systems of the same type $[S, A']$. He then provided a canonical form for them and finally he showed how the fundamental theorem of Picard-Vessiot theory could be deduced in a simple and direct way. By doing so, Vessiot pointed out, the researches contained in his thesis [Vessiot 1892] could be endowed with the rigour they lacked.

Much more articulate was the extension of the theory to the case of equations of type:

$$P(x) = \frac{\partial x}{\partial t} + \sum_{i=1}^{n} p_i(t, t_1, \cdots, t_n)\frac{\partial f}{\partial t_i} = 0. \tag{3.50}$$

As it had already been observed by Drach, it was convenient to replace this equation with an automorphic system $(S)$ whose associated group coincides with the (infinite) group of all diffeomorphisms in $n$ variables. Again, Vessiot posed the question about what information can be drawn from the knowledge of one or more differential relations $A$ among a set of solutions $x_1, \cdots, x_n$ of $(S)$. As in the case of algebraic and linear ordinary differential equations, Vessiot proved that the system $[S, A]$ can be decomposed into a family of infinitely many automorphic systems.
Then, a canonical form for such automorphic systems was provided. In this respect, his researches on infinite continuous groups turned out to be essential. In particular, Vessiot took great profit from the flexibility of Medolaghi's form of the defining equations of a group with respect to change of coordinates[59]. Unlike the analysis of the two preceding cases, it was by no means obvious that there exists an automorphic system consisting of rational relations only. As Vessiot himself emphasized, such a novel difficulty was intrinsically connected with the type of equation one was dealing with. Indeed, the most general solution of $P(x) = 0$ can be obtained from a particular one $x_i$, $i = 1, \cdots, n$ by means of the action of an arbitrary diffeomorphism $\phi$ which in general (contrary to what happens for algebraic and ordinary linear differential equations) is not a rational function of its

[59]See [Vessiot 1904a, p. 49-51].

arguments. Vessiot succeeded in overcoming such an obstacle by providing a general procedure for obtaining automorphic systems of rational type. Finally, in the last chapter of [Vessiot 1904a], Vessiot was able to provide a rigorous emendation of Drach's results which allowed him to state precisely which assumptions should be introduced in order to guarantee the existence of a differential generalization of the Galois group for the equation (3.50).

Interestingly enough, in one of his rare publications on this subject [Cartan 1938], Cartan proved to be highly interested in Drach's and Vessiot's theories. He agreed with Vessiot on the central role played by the notion of automorphic system, so much so as to declare that all the different variants of Galois theory ultimately reduced to the study of automorphic systems. Furthermore, by resting upon the notion of reducibility first introduced by Drach, he was able to provide a very synthetic proof of the existence of the rationality group which enabled him to treat the algebraic and the differential cases on the same ground. To this end, Cartan singled out two main hypotheses which, according to his opinion, turned out to be very important. First, the existence of rational, irreducible systems which are compatible with the given automorphic system; secondly, the property of the transformations of the fundamental group of being rational. At first sight, Cartan observed, it seemed that this second hypothesis was to be regarded as a requirement which could not be renounced. Nevertheless, as he pointed out, the works by Drach and Vessiot had the great merit of indicating an ingenious way to sidestep such an obstacle.

As it will be seen in the next chapters, unlike Vessiot and Drach, Cartan would never get actively involved in the project of erecting a differential extension of Galois theory. Certainly, as [Cartan 1894] and [Cartan 1896] show, he was well aware of the enormous potentiality of applications of his theory of continuous groups, especially in the vast field of partial differential equations. Nonetheless, his attention was for large part concentrated on the ambitious project of investigating chiefly structural properties. Besides, when in the 1910's his interest of research became more application oriented, geometry had assumed in his mind a privileged position upon any other branch of mathematical science.

# Chapter 4

# Exterior differential systems

Historians and mathematicians are unanimous in considering Cartan's work on Pfaffian systems (what nowadays we would call exterior differential systems) as a landmark both for what concerns his entire mathematical production and the development itself of 20th century mathematics. The strategic role played by such systems in so many realms of mathematical research such as general theory of partial differential equations, continuous infinite Lie groups, theory of equivalence and differential geometry, to mention only a few, is unitedly acknowledged.

Nevertheless, it appears that scarce attention has been paid to this area of historical research thus far. Authoritative scholars[1] dealt with Pfaff's problem and the foundation of exterior differential calculus in Cartan's early papers; however no specific analysis of his subsequent works laying the foundations of what nowadays is known as the Cartan-Kähler theory has been provided so far.

In this chapter, we will try to remedy this unsatisfactory state of affairs. Our specific aim is that of providing a historical account of the genesis of the integration theory of general Pfaffian systems which will allow us, in the next chapter, to discuss Cartan's theory of infinite continuous groups. A full understanding of the necessity for such a preliminary *excursus* will be attained only later. For the time being, we will limit to recall an anecdote told by André Weil which provides a vivid idea of the role played by exterior differential forms in Cartan's mathematics. Over his infancy, Henri, Cartan's son, developed a most peculiar idea of what doing mathematics meant. The ubiquitous presence of exterior differential forms in his father's notes was such that, as a child, Henri thought that it was all about drawing $\omega$'s (remember that $\omega$ is the standard symbol employed by Cartan to designate an exterior differential form).

We will try first to discuss some works by F. Engel and E. Ritter von Weber in which a first geometrical approach to the integration theory of

[1] See [Hawkins 2005] and [Katz 1985].

general Pfaffian systems was developed. Afterward, Cartan's fundamental papers [Cartan 1901a] and [Cartan 1901b] will be analyzed.

## 4.1 Some technical preliminaries

This section is devoted to some general remarks on the mathematics which we are about to deal with; historical accuracy will not be main focus of attention here, we will limit ourselves to give the necessary information that will be helpful in understanding the discussion that is going to follow. The interested reader can find a detailed historical account of this material in the paper [Hawkins 2005] by T. Hawkins.

The main topic of our discussion will be the problem of integration of differential systems of Pfaffian equations; thus, it seems appropriate to describe briefly what a Pfaffian equation is and what it means to integrate such equations. Moreover, it is useful to emphasize a crucial separation that has to be operated in the theory and that will be of primary importance for our purposes, that is the distinction between the completely integrable (*unbeschränkt integrable*) case and the not completely integrable one.

In modern terms, what nineteenth century mathematicians meant by a Pfaffian form in $n$ variables can be identified with the local expression of a differential 1-form defined on a $n$-dimensional manifold. However, until 1899, when Cartan gave a symbolic definition of what he named differential expression (*expression différentielle*), it appears that no autonomous status was attributed to it. Rather, what was considered to be meaningful was the problem of its vanishing on suitable regions of the space. This was interpreted as the manifestation of certain finite relations (to be determined) among the independent variables.

Thus, a Pfaffian equation in $n$ variables is a differential relation of the following type:

$$\omega = A_1(x_1, \cdots, x_n)dx_1 + \cdots + A_n(x_1, \cdots, x_n)dx_n = 0. \tag{4.1}$$

To find integrals[2] of (4.1) means to determine, functionally independent, finite relations among the variables $x_1, \cdots, x_n$, $f_j(x_1, \cdots, x_n) = 0$, $(j = 1, \cdots, m)$ such that the vanishing of (4.1) is a consequence of the $2m$ relations $f_j = 0, df_j = \sum_{k=1}^{n} \frac{\partial f_j}{\partial x_k} dx_k = 0\,(j = 1, \cdots, m)$. These integrals can be thought of geometrically as defining an integral submanifold of dimension $n - m$ given by the intersection of $m$ hypersurfaces $f_j = 0$[3].

During the nineteenth century, one of the main problem in the theory of Pfaffian equations was that of finding a canonical form for $\omega$, that is, the

[2] In the classical literature one often finds the wording *integral equivalents*.

[3] The present-day definition of what an integral variety of a 1-form is, is quite the same, only rephrased in different language: $i : S \hookrightarrow M$ is an integral submanifold of the equation $\omega = 0$ if, and only if, the pullback of $\omega$, $i^*(\omega)$, vanishes identically.

problem of finding a suitable change of variables, $y_i = (x_1, \cdots, x_n)$ so that the Pfaffian expression $\omega$ could be written in such a way as to contain the minimal number of variables. Clearly, the determination of such a canonical form coincides with the determination of the minimal number of integral equivalents of $\omega = 0$ and consequently with the individuation of the integral varieties of maximal dimension.
The main results in this field were obtained by Frobenius in 1877, [Frobenius 1877], with the introduction of two notions: the bilinear covariant (*bilineare Covariante*) and the class (*Classe*) of a Pfaffian expression.

The bilinear covariant[4] of $\omega$ was defined by Frobenius as the following expression:

$$\sum_{i,j=1}^{n} a_{ij} dx_i \delta x_j, \qquad \text{with} \qquad a_{ij} = \frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i}, \tag{4.2}$$

where $d$ and $\delta$ are differentials in different directions; the word *covariant* indicates the crucial property according to which, if, under a change of coordinates $x'_i = \phi_i(x_k)$, $\sum A_j dx_j = \sum A'_j dx'_j$, then

$$\sum_{i,j}^{n} a_{ij} dx_i \delta x_j = \sum_{i,j}^{n} a'_{ij} dx'_i \delta x'_j.$$

A first application of this notion was Frobenius' analytical classification theorem for Pfaffian forms[5]. Indeed, he considered the matrix

$$M = [a_{ij}] \qquad \text{and} \qquad M' = \begin{bmatrix} a_{11} & \cdots & a_{nn} & A_1 \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & A_n \\ -A_1 & \cdots & -A_n & 0 \end{bmatrix} \tag{4.3}$$

and defined the class of a Pfaffian form $\omega$ as the number (invariant under arbitrary change of coordinates) $p = \frac{rk(M)+rk(M')}{2}$. He then demonstrated that $p$ is the minimal number of independent variables in term of which $\omega$ can be expressed. In other words, $p$ individuates the canonical form to which $\omega$ belongs: if $p = 2r$, then $\omega = y_{r+1}dy_1 + \cdots + y_{2r}dy_r$, if $p = 2r+1$, then $\omega = dy_0 + y_{r+1}dy_1 + \cdots + y_{2r}dy_r$, under appropriate changes of coordinates.

A second application of the bilinear covariant of which Frobenius took great advantage was the so-called integrability theorem for systems of Pfaffian equations. A Pfaffian system of type

$$\omega_\mu = a_{\mu 1}dx_1 + \cdots + a_{\mu n}dx_n, \qquad (\mu = 1, \cdots, m). \tag{4.4}$$

[4] For a detailed historical account see [Hawkins 2005, §6].

[5] Important results in this field were obtained by G. Darboux almost at the same time. However Darboux did not submit them for publication immediately. A paper [Darboux 1882] by him on Pfaff's problem appeared in 1882. For an analysis of Darboux' contribution and a comparison with Frobenius' approach, see [Hawkins 2005, p. 420-424].

was said to be completely integrable if it admits $m$ independent integrals, that is if it admits an integral variety of dimension $n-m$. Frobenius dealt with this special kind of Pfaffian systems en route for the proof of the analytical classification theorem of single Pfaffian equations. His main result was a characterization of complete integrability in terms of the properties of the bilinear covariants of (4.4):

**Theorem 13 (Frobenius, 1877)** *Given the system (4.4) of $m$ linearly independent Pfaffian equations, it is completely integrable if, and only if, the vanishing of all its bilinear covariants is an algebraic consequence of the system itself.*

Frobenius' demonstration relied upon a result due to Clebsch which can now be interpreted as the dual counterpart of Frobenius' theorem. Indeed, in [Clebsch 1866], Clebsch had devoted his attention to a generalization of Jacobi's theory of linear partial differential equations by introducing the notion of complete (*vollständig*) integrability. A system of linear partial differential equations of type

$$A_i(f) = X_{i1}\frac{\partial f}{\partial x_1} + \cdots + X_{in}\frac{\partial f}{\partial x_n} = 0, \qquad (i = 1, \cdots, r) \tag{4.5}$$

was said by Clebsch to be complete if all expressions $(A_i, A_j)(f) = A_i(A_j(f)) - A_j(A_i(f))$ are linear combinations (in general with non-constant coefficients) of (4.5). He was able to demonstrate the following:

**Theorem 14 (Clebsch, 1866)** *If the system (4.5) is complete, then it admits a system of $n-r$ functionally independent solutions $f_1, \cdots, f_{n-r}$.*

It turns out that requiring complete integrability of (4.4) is equivalent to the supposition that an appropriate system (actually, its dual[6]) of linear differential equations of type (4.5) is complete in the sense of Clebsch's definition; moreover, it should be observed that a system of integrals of (4.4) is also a system of solutions for (4.5) and viceversa.

To conclude the present introductory section, we recall that if the integrability conditions of Theorem (13) are not satisfied, then, in general, $\omega_\mu$ cannot be expressed as a linear combination of the total differentials of $m$ appropriate functions $f_j$, $(j = 1, \cdots, m)$. If this is the case, then the system (4.4) is said to be not completely integrable. To be precise, one should distinguish further the case in which some (although not all) of the integrability conditions are satisfied from the case in which *none* of them is; in the

[6]It appears that Mayer was the first one to call attention over this dual connection in [Mayer 1872]; see also [Hawkins 2005, p. 408-410]. In this regard, Frobenius spoke of *adjungirt* or *zugehörig* system. His characterization of duality was purely algebraic as one can see by consulting [Frobenius 1877, §13] or [Hawkins 2005, p. 411-415]. We will see later Engel's interpretation in terms of infinitesimal transformations.

former case, one speaks of incompletely integrable systems; in the latter, of non-integrable systems. Since the study of incompletely integrable systems can be traced back to the study of non-integrable ones, we will often ignore such a distinction in the following discussion.

## 4.2 The state of the art in the early 1890's

Well after the publication in 1877 of the seminal work [Frobenius 1877] by Frobenius, the problem of finding solutions of not completely integrable Pfaffian systems remained open and almost untouched. As we have just seen, Frobenius was able to give necessary and sufficient conditions that guarantee the complete integrability of a given differential system. However, except some brief remarks[7], no specific attention was paid by him to the more general problem of finding integral equivalents of not completely integrable systems of Pfaffian equations.

A common feeling of inadequacy in relation to the state of the theory of Pfaffian systems of this more general kind was frequently expressed by mathematicians in the early 1890's. For instance, Forsyth in [Forsyth 1890] complained about the lack of new results in this realm of the theory and tried to indicate a path to be followed in order to achieve a satisfying generalization of the study of a single non-exact Pfaffian equation to systems of many equations. As in the case of a single equation, he said, it is desirable to have the integral equivalent of the system as general as possible and, in order to fulfill this aim, he individuated three different steps: i) the determination of the number of equations in the integral equivalent of a non-integrable system; ii) the deduction of some simple integral equivalent of such a system and finally, iii) the generalization of such an integral equivalent once it has been obtained. According to Forsyth, some advances had only been achieved in relation to step i) by the work [Biermann 1885] of Otto Biermann, who had demonstrated that the maximal dimension of the integral varieties of an unconditioned[8] Pfaffian system is given by the integer part of the ratio between the number of variables and the number of equations augmented by one and that the rest of this division gives information about the degree of indeterminacy of the integral solutions. As far as the remaining two steps were concerned, Biermann's analysis had made it clear that the methods of integration at that time known (in particular the so-called Clebsch's second method) did not permit a general solution to be obtained. Forsyth's effective synthesis of the state of the art of the theory is worth quoting.

*And so the solution of the problem of obtaining the integral*

[7] See §20 of [Frobenius 1877].

[8] That means that no specification of the coefficients of the Pfaffian system has been made.

> *equivalent of a simultaneous system of unconditioned Pfaffians does not appear possible by any methods at present known which are effective for the case of a single Pfaffian. It is, in fact, one of the most general problems of the integral calculus; the discovery of its solution lies in the future.*[9]

## 4.3 Engel's invariants theory of Pfaffian systems

Quite similar remarks of dissatisfaction for the state of the art of the theory were expressed by F. Engel at the beginning of the first of two memoirs [Engel 1890] that were dedicated to the invariants theory of Pfaffian systems and were communicated by M. A. Mayer in 1889 and in 1890 to the *Sächsische Akademie der Wissenschaft* in Leipzig. Engel wrote:

> *The invariant theory of a single Pfaffian equation has been completed for some time; on the contrary, as far as systems of Pfaffian equations are concerned, almost everything remains to be done.*[10]

Engel's approach was deeply influenced by the work of his highly respected master, S. Lie. Moreover, it appears that the main concern that led him to deal with such systems of total differentials equations was their application to the theory of continuous groups of transformations. Nonetheless, beyond their applicative character, Engel's contributions are of considerable historical interest since they represented a source of inspiration for the forthcoming papers by E. von Weber and E. Cartan himself.

### 4.3.1 Invariant correspondences

Engel's strategy was dominated by the persistent recourse to structures invariantly connected to the given Pfaffian system. The very first example of such connected structures had been the so-called bilinear covariant of a Pfaffian expression upon which Frobenius had constantly relied in his work. Engel took up this fertile idea and generalized it, proposing the following definition: two differential systems (depending on the circumstances, a differential system can be a system of partial differential equations, a system of Pfaffian equations or a set of infinitesimal transformations) are said to be invariantly associated (*invariant verknüpft*) if a bijective correspondence exists between them that is preserved under arbitrary changes of coordinates. The knowledge of these connected structures, as in the case of a single Pfaffian equation or in the case of a complete system of Pfaffian equations examined by Frobenius, was considered by Engel quite useful since

[9] See [Forsyth 1890, §185].

[10] *Die Invariantentheorie einer einzelnen Pfaff'sche Gleichung ist schon lange erledigt, dagegen bleibt für die Systeme von Pfaff'schen Gleichungen fast noch Alles zu thun.*

the study of their properties allowed him to get information, for example, regarding the normal form of the original Pfaffian system.

The starting point of his analysis was the observation that a reciprocal connection (*Zusammenhang*) exists between Pfaffian systems and systems of linear homogeneous partial differential equations of first order. According to Engel, the origin of this connection stemmed from two distinct interpretations one could ascribe to a given system of $m$ Pfaffian equations of the following form:

$$\omega_\mu = \sum_{i=1}^{n} a_{\mu i}(x_1 ..., x_n) dx_i = 0, \qquad (\mu = 1, ..., m). \tag{4.6}$$

One can interpret (4.6), in the usual way, as a system of differential equations and, correspondingly, one can undertake the task to determine all its integral equivalent equations, that is, to determine all the equations

$$\Phi_1(x_1 ... x_n) = 0, ..., \Phi_q(x_1 ... x_n) = 0,$$

such that the $2q$ relations

$$\Phi_1 = 0, ..., \Phi_q = 0, d\Phi_1 = 0, ..., d\Phi_q = 0,$$

imply, identically, $\omega_\mu \equiv 0$, $\mu = 1, ..., m$. On the other hand, Engel explained, one can regard the quantities $dx_1, ..., dx_n$ in (4.6) as the infinitesimal increments to which the variables $x_1, ..., x_n$ are subject as a consequence of the action of an infinitesimal transformation,

$$X(f) = \sum_{j=1}^{n} \xi_j(x_1 ... x_n) \frac{\partial f}{\partial x_j}.$$

According to this interpretation, equations (4.6) define a family (*Schaar*) of infinitesimal transformations, namely the set of all infinitesimal transformations $X(f)$ that satisfy the following $m$ relations:

$$\sum_{j=1}^{n} a_{\mu j} \xi_j = 0, \qquad (\mu = 1, ..., m). \tag{4.7}$$

Since the rank of the matrix $A = [a_{\mu j}]$ is supposed to be maximal (and so equal to $m < n$), equation (4.7) admits $n - m$ linearly independent solutions $\xi_i^{(k)}$, for $i = 1, ..., n$ and $k = 1, ..., n - m$. Therefore one obtains $n - m$ linearly independent infinitesimal transformations:

$$X_k(f) = \sum_{i=1}^{n} \xi_i^{(k)}(x_1, ..., x_n) \frac{\partial f}{\partial x_i}, \qquad (k = 1, ..., n - m).$$

which are the generators of the set of infinitesimal transformations that is associated to the given Pfaffian system (4.6). An arbitrary transformation of this set takes on the following expression:

$$W(f) = \chi_1(x_1, ..., x_n)X_1(f) + ... + \chi_{n-m}(x_1, ..., x_n)X_{n-m}(f). \qquad (4.8)$$

where the $\chi_i$, $(i = 1, ..., n - m)$ are arbitrary functions of $n$ variables.

Engel observed that this reciprocal correspondence between Pfaffian systems and sets of infinitesimal transformations is not only bijective but it is also preserved under arbitrary transformations of coordinates, so that it is, in fact, an example of invariant association. Finally, by setting all these transformations equal to zero, one obtains the following system of independent differential equations:

$$X_1(f) = 0, \quad \cdots, \quad X_{n-m}(f) = 0.$$

which is also invariantly connected with (4.6).

This dual connection was used by Engel to build up new auxiliary Pfaffian systems which introduce remarkable simplification in the theory. The first one of these auxiliary systems is obtained as a consequence of the action of a generic infinitesimal transformation of type (4.8) on the Pfaffian system (4.6), now supposed to be rewritten in the following (*aufgelöst*) form[11]:

$$\Delta_\mu = dx_\mu - \sum_{k=1}^{n-m} a_{m+k,\mu} dx_{m+k} = 0, \qquad (\mu = 1, \cdots, m). \qquad (4.9)$$

Correspondingly, the infinitesimal transformations which are associated with it are now written as:

$$A_{m+k}(f) = \frac{\partial f}{\partial x_{m+k}} + \sum_{\mu=1}^{m} a_{m+k,\mu} \frac{\partial f}{\partial x_\mu}, \qquad (k = 1, ..., n - m).$$

If we define $W(f)$ to be a generic transformation of type:

$$W(f) = \sum_{k=1}^{n-m} \chi_{m+k} A_{m+k}(f),$$

its action on (4.9) transforms the latter in the system:

$$\Delta_\mu + \delta t(W\Delta_\mu) = 0, \qquad (\mu = 1, ..., m)^{12}$$

[11] Here and in what follows I adhere to the original notation employed by Engel.

[12] The expression $W\Delta_\mu$ is what today we would call Lie derivative of $\Delta_\mu$ with respect to the vector field $W$. Such a denomination is very appropriate from a historical point of view. Indeed, Lie was the first one to introduce it. See for example [Lie 1888, p. 529-530].

which is easily demonstrated to be equivalent and invariantly connected to the following system of Pfaffian equations:

$$\Delta_1 = 0, ..., \Delta_m = 0, A_{m+k}\Delta_1 = 0, ..., A_{m+k}\Delta_m = 0, \qquad (k = 1, ..., n-m). \tag{4.10}$$

Finally, few manipulations give the following equivalent and simplified form written in terms of the coefficients of Frobenius' bilinear covariants:

$$\begin{cases} dx_\mu - \sum_{k=1}^{n-m} a_{m+k,\mu} dx_{m+k} = 0, \\ \sum_{k=1}^{n-m} \{A_{m+k} a_{m+j,\mu} - A_{m+j} a_{m+k,\mu}\} dx_{m+k} = 0 \\ (\mu = 1, ..., m; \quad j = 1, ..., n-m). \end{cases} \tag{4.11}$$

Engel observed that it may happen that the system (4.11) coincides with (4.9); if this is the case, then (4.9) is completely integrable and it admits every infinitesimal transformation (4.8), that is, for a generic transformation $W(f)$, $W\Delta_\mu = 0$, $(\mu = 1, ..., m)$, are a consequence of $\Delta_\mu = 0$, $(\mu = 1, ..., m)$.

In virtue of the dual correspondence between Pfaffian systems and sets of infinitesimal transformations, the system (4.11) can be considered as defining a set (*Schaar*) of infinitesimal transformations. It turns out that these transformations are precisely those transformations that leave the original Pfaffian system (4.9) invariant and, besides, as Engel demonstrated, that the Pfaffian system (4.11) is completely integrable. By using anachronistic terminology, such transformations can be called *characteristic transformations* and, correspondingly, the Pfaffian system (4.11) defining them, *characteristic system.*

A second differential system invariantly connected to (4.9) was obtained by Engel by making recourse to the following simple remark. If one considers a system of $n - m$ linear homogeneous partial differential equations of the following form:

$$C_k(f) = \sum_{i=1}^{n} \beta_{ik}(x_1, \cdots, x_n) \frac{\partial f}{\partial x_i} = 0 \qquad (k = 1, \cdots, n-m), \tag{4.12}$$

and the system of equations

$$\begin{cases} (C_k, C_j)(f) = C_k(C_j(f)) - C_j(C_k(f)) = 0, \\ C_k(f) = 0, \quad (k, j = 1, \cdots, n-m), \end{cases} \tag{4.13}$$

then, they are invariantly associated. Since there is an adjoint Pfaffian system associated to every system of linear homogeneous partial differential equations, it is clear that two systems of Pfaffian equations which are invariantly connected correspond to (4.12) and (4.13). As a result of this, Engel stated the following:

**Theorem 15** *The system of Pfaffian equations which is dual to the system of partial differential equations*

$$\left\{ \begin{array}{c} A_{m+k}f = \frac{\partial f}{\partial x_{m+k}} + \sum_{\mu=1}^{m} a_{m+k,\mu}\frac{\partial f}{\partial x_\mu} = 0 \\ (A_{m+k}, A_{m+j})f = \sum_{\mu=1}^{m}(A_{m+k}a_{m+j,\mu} - A_{m+j}a_{m+k,\mu})\frac{\partial f}{\partial x_\mu} = 0 \\ (k,j = 1,\cdots,n-m), \end{array} \right. \quad (4.14)$$

*is invariantly connected to the Pfaffian system (4.9).*

Other differential systems invariantly connected to the given Pfaffian system were obtained by Engel in the course of his researches. However, the function they fulfilled was in any case the same: to deduce from them the normal form of the considered Pfaffian system and to develop applications in the realm of the theory of continuous groups and of the theory of contact transformations as well. To name just a few concrete examples, Engel succeeded in giving a complete invariant theory of Pfaffian systems of two equations in four independent variables; furthermore, he utilized some of his results to give a simpler treatment of the problem, already faced by M. Page in [Page 1888], of the classification of all imprimitive continuous transformation groups in space in four dimensions[13] and, finally, he was able to present a very clear demonstration of a theorem originally due to A. V. Bäcklund[14] which gave a complete characterization of all contact transformations.

## 4.4 von Weber's contributions: 1898-1900

As E. Goursat[15] was once to observe, before Cartan's seminal papers [Cartan 1901a] and [Cartan 1901b], first rigorous results in the field of the theory of general Pfaffian systems were obtained, along with Engel, by the young mathematician Eduard Ritter von Weber (1870-1934) in a series of articles which laid the ground for the subsequent geometrical developments of Cartan's theory of exterior differential systems. von Weber's approach was profoundly inspired to Engel's researches. Wide use of invariantly associated differential systems, frequent application of infinitesimal characteristic transformations and consistent reference to geometrical visualization were for von Weber, as

[13] A group of $r$ independent infinitesimal transformations in $n$ variables is said to be *imprimitive* if it leaves a family of $\infty^{n-q}$ $q-$dimensional subvarieties $M_q$:

$$\phi_1(x_1,...,x_n) = c_1, \qquad \cdots \qquad \phi_{n-q}(x_1,...,x_n) = c_{n-q},$$

invariant; that is, if

$$X_i(\phi_k) = \Omega_{ki}(\phi_1,\cdots,\phi_{n-q}), \qquad i = 1,\cdots,r \quad k = 1,\cdots,n-q$$

where the $\Omega$ are some functions of $\phi_1,\cdots,\phi_{n-q}$; see [Page 1888, p. 297-300].

[14] See [Bäcklund 1876].

[15] See [Goursat 1922, p. 259].

for Engel, the main technical and conceptual tools to which he had recourse to tackle the resolution of generalized Pfaffian systems.

Nonetheless, it appears that a specific motivation guided von Weber's interest in his attempt to classify the large variety of Pfaffian systems, that is the hope of applying Pfaffian systems to a systematic study of general system of partial differential equations already started up by C. Méray and C. Riquier. Moreover, von Weber took advantage of some of the main results of the general theory of systems of partial differential equations, namely existence theorems for the so-called passive systems[16] which von Weber used to demonstrate the existence of integral varieties of the given Pfaffian system and the consequent possibility of writing it in a simple normal form containing a reduced number of differentials. Finally a regular application of the theory of linear complexes and congruences in projective space has to be indicated as one of the most original technical innovation introduced by von Weber into the theory[17].

von Weber's contributions, in which we are interested, are spread over a certain number of memoirs which he published between 1898 and 1900. Our attention will be mainly concentrated on [von Weber 1898], in which the notion of character of a Pfaffian system and that of derived system were introduced for the first time. Nevertheless, since it appears that some of his later developments may have played a role in influencing Cartan's geometrical approach to a generalization of the problem of Pfaff, a brief survey of [von Weber 1900a], [von Weber 1900b] and [von Weber 1900c] will be given too.

### 4.4.1 Character and characteristic transformations

As von Weber himself observed in the final historical remarks of [von Weber 1900c, p. 609], since the introduction of the bilinear covariant by Frobenius and Darboux, invariantly associated structures had played a major role in the

[16] An explanation of this intricate notion will be given later.

[17] The following remarks taken from the introduction to [von Weber 1901] are quite enlightening. von Weber wrote:
*"Unsere Aufgabe lässt sich als Specialfall der allgemeinen Theorie der Differentialsysteme auffassen, wie sich auch umgekehrt die letztere, von einem andern Standpunkt aus betrachtet, der ersteren als Specialfall einordnet. Das neue Hilfsmittel jedoch, das wir bei unseren Untersuchungen verwenden und mit der Theorie der Differentialsysteme in mannigfache Beziehung setzen werden ist die Theorie der Liniencomplexe und -Congruenzen in $m-1$-dimensionalen Raum, also der Schaaren von alternirenden Bilinearformen mit $m$ Variabelnpaaren."*
(*"If our task can be considered as a special case of the general theory of differential systems, it is also true that, from another point of view, the latter can be regarded as a special case of the former. Yet, the new auxiliary means that we will utilize in our analysis and that will be connected in many ways to the theory of differential systems, is represented by the theory of linear complexes and congruences in $(m-1)$-dimensional space as well as by the theory of families of antisymmetric bilinear forms in $2m$ variables"*).

theory of Pfaffian systems. As we have seen, Engel had taken great advantage of them and had succeeded in providing some new applications of Pfaffian equations especially in the classification problem of continuous groups of transformations. von Weber acknowledged the fruitfulness of this approach and tried to give it systematic basis within the context of an invariants theory of systems of Pfaffian equations.

von Weber started his analysis in [von Weber 1898] by considering a system of $n-m$ Pfaffian equations in the following, resolved form[18]:

$$\nabla_s = dx_{m+s} - \sum_{i=1}^{m} a_{si} dx_i = 0, \qquad (s = 1, \cdots, n-m). \tag{4.15}$$

It was proved by Engel that the differential system for two independent variations of the $n$ variables $x_1, \cdots, x_n$, $dx_i$ and $\delta x_i$, $(i = 1, \cdots, n)$,

$$\begin{cases} dx_{m+s} = \sum a_{si} dx_i; \quad \delta x_{m+s} = \sum a_{si} \delta x_i, \\ \sum_{k=1}^{m} \sum_{i=1}^{m} a_{iks} dx_i \delta x_k = 0, \end{cases} \quad (s = 1, ..., n-m). \tag{4.16}$$

is invariantly associated with (4.15)[19]. As a consequence of this, von Weber observed, the study of invariant quantities attached to (4.15) could be transferred to that of the invariants of the system (4.16). He defined the first of these invariants, the *character* $K$ of the system (4.15), as the rank of the matrix

$$\left[ \sum_{k=1}^{m} a_{iks} \lambda_k \right] \qquad (i = 1, \cdots, m, s = 1, \cdots, n-m),$$

when $x_1, \cdots, x_n$ and $\lambda_1, \cdots, \lambda_m$ assume arbitrary values.

The importance of the notion of character lay in the fact that it offered a first classification criterion for the large variety of Pfaffian systems and a measure of the difficulty degree of the problem one has to face, as it were: the greater the character, the harder is the task to undertake. If, for example, $K = 0$ then system (4.15) is completely integrable, since clearly the coefficients of the bilinear covariants vanish identically.

von Weber's attention in [von Weber 1898] was almost exclusively concentrated on Pfaffian systems of character *one*, but some important results concerning characteristic transformations were obtained for the general case too. Since characteristic transformations will also play a key role in Cartan's analysis, it seems appropriate to describe their frequent use in von Weber's theory in some detail.

Already introduced by Engel, characteristic transformations are defined as the infinitesimal transformations which are dually associated with (4.15)

[18]Though it may appear bizarre, the $\nabla$ notation was that employed by von Weber.

[19]As usual, it is supposed that the following relations hold: $A_i(f) = \frac{\partial f}{\partial x_i} + \sum_{s=1}^{n-m} a_{si} \frac{\partial f}{\partial x_{m+s}}$ and $a_{iks} = -a_{kis} = A_i(a_{sk}) - A_k(a_{si})$.

and, at the same time, leave these equations invariant; that is, if we consider a generic infinitesimal transformation associated with (4.15), $X(f) = \sum_{i=1}^{m} \xi_i A_i(f)$, this transformation is characteristic if the following identities are satisfied in virtue of equations (4.15):

$$X(\nabla_s) = 0, \qquad (s = 1, \cdots, n-m). \tag{4.17}$$

In a more explicit form, this means:

$$X(\nabla_s) = d\xi_{m+s} - \sum_{i=1}^{m} X(a_{si})dx_i - \sum_{i=1}^{m} a_{si} d\xi_i = 0,$$

and, consequently, we have

$$d\left(\xi_{m+s} - \sum_{i=1}^{m} a_{si}\xi_i\right) + \sum_{i=1}^{m} (da_{si}\xi_i - X(a_{si})dx_i) = 0^{20},$$

and finally, since $\xi_{m+s} = \sum_{i=1}^{m} a_{si}\xi_i$ and since, as a consequence of (4.15), for an arbitrary function of $n$ variables, $df = \sum_{k=1}^{m} A_k(f)dx_k$:

$$\sum_{k=1}^{m} \xi_k a_{iks} = 0, \qquad (i = 1, \cdots, m; \quad s = 1, \cdots, n-m). \tag{4.18}$$

von Weber supposed that there are $h$ independent solutions $\vec{\xi}^{(i)}$, $(i = 1, \cdots, h)$ of equations (4.18), so that the set of infinitesimal transformations leaving the Pfaffian system (4.15) invariant is generated by the following differentials operators:

$$X_i(f) = \sum_{j=1}^{m} \xi_j^{(i)} \frac{\partial f}{\partial x_j}, \qquad (i = 1, \cdots, h). \tag{4.19}$$

As already pointed out by Engel and demonstrated by von Weber through a direct computation, the system of $h$ differential equations $X_i(f) = 0$ is complete in the sense of Clebsch's definition[21]. von Weber acknowledged the importance of such transformations and explained how they could be usefully employed to simplify the integration of the Pfaffian system under consideration. In particular, he proved the following:

**Theorem 16** *For the Pfaffian system (4.15) to be reducible, through a change of coordinates, to a system of equations in $n-h$ variables, it is necessary and sufficient that it admits $h$ (independent) characteristic infinitesimal transformations.*

[20] A modern version of this formula would read as follows: $X(\nabla_s) = d\nabla_s(X) + d(\nabla_s(X))$. Cartan is usually acknowledged as its first discoverer, see [Ivey, Landsberg 2003, p. 339]. However, this attribution appears to be not very accurate from a historical point of view. Indeed, it can be found already in [Engel 1896, p. 415].

[21] In modern terms, that means that the operators $X_i$, $(i = 1, \cdots, h)$ define an involutive distribution of tangent vector fields. See section 4.1 of this paper.

Indeed, if one introduces a change of coordinates in which $n-h$ variables are identified with the $n-h$ independent solutions of the complete system (4.19), it is easy to show that the Pfaffian system so obtained only depends upon these $n-h$ variables[22].

### 4.4.2 Pfaffian systems of character one, I

The remaining part of the memoir [von Weber 1898] was devoted to a thorough analysis of a very special type of Pfaffian systems, namely those whose character is equal to one. Since Cartan would take up the same topic in 1901 by reinterpreting von Weber's result in the light of his new geometrical methods based on the brand new exterior differential calculus, it appears appropriate to discuss von Weber's accomplishments in order to facilitate a comparison between Cartan's and von Weber's approaches.

Pfaffian systems of character one represent the simplest eventuality one can conceive, after the case of completely integrable systems. Frobenius had showed that if the system is completely integrable, then the vanishing of all its bilinear covariants is an algebraic consequence of the equations of the system itself. Instead, in the case of systems of character one the bilinear covariants reduce to a single bilinear form whose vanishing is not implied by the equations of the system itself. In other words, the following relations among the coefficients of the bilinear covariants hold: $a_{iks} = \mu_s a_{ik1}$, $(s = 2, \cdots, n-m; i, k = 1, \cdots, m)$, where $\mu_s$ are functions of the $n$ variables $x_1, \cdots, x_n$. von Weber supposed that the matrix $[a_{ik1}]$, $(i, k = 1, \cdots, m)$ has rank equal to $2\nu$, so that he could deduce the existence of $m - 2\nu$ linearly independent characteristic transformations

$$X^{(k)}f = \sum_{i=1}^{n} \xi_i^{(k)} \frac{\partial f}{\partial x_i}, \quad k = 1, \cdots, m - 2\nu.$$

As already explained, the existence of such transformations was exploited to obtain a reduced form of the Pfaffian system under examination; indeed, by appropriate definition of new variables $y_1, \cdots, y_{\nu+1}, \cdots, y_{\nu+n-m}$, the number of differentials can be lowered to $\nu + n - m$ to give the following reduced form of (4.15):

$$dy_{\nu+s} = \sum_{i=1}^{\nu} \eta_{si} dy_i, \quad (s = 1, \cdots, n-m)^{23}.$$

von Weber's treatment of this special type of systems was marked by the definite and profitable distinction between the case in which $2\nu = 2$ and

[22] For details, see [von Weber 1898][p. 210-211]. A modern statement of this theorem can be found in [Olver 1995][p. 430].

[23] It should be observed that, in general, the $\eta_{si}$ are functions of the $n$ variables $x_1, \cdots, x_n$.

the case in which $2\nu > 2$. Let us consider in some detail the case $2\nu > 2$. The study of such systems was carried out by exploiting the existence of the so-called derived system (*"das abgeleitete System von* (4.15)*"*) which in this case turns out to be completely integrable. Indeed, von Weber considered the following system of partial differential equations (the dual counterpart of the derived system):

$$A_i(f) = 0, \quad B(f) = \frac{\partial f}{\partial x_{m+1}} + \sum_{s=2}^{n-m} \mu_s \frac{\partial f}{\partial x_{m+s}} \quad (i = 1, \cdots, m)^{24} \qquad (4.20)$$

and demonstrated that it is complete in the sense of Clebsch. In fact, as $(A_i A_k)(f) = a_{ik1} B(f) \quad (i, k = 1, \cdots, m)$, all he had to show was that $(A_i B)(f)$ could be expressed as a linear combination (in general with non-constant coefficients) of $A_i(f)$'s and $B(f)$. Supposing[25] $m \geq 3$, from Jacobi's identity and from $((A_i A_k) A_l) = \left( \sum_{s=1}^{n-m} a_{iks} \frac{\partial f}{\partial x_{m+s}}, A_l \right)$, it follows that

$$\Phi_{0i} a_{kl1} + \Phi_{0k} a_{li1} + \Phi_{0l} a_{ik1} = 0 \quad (i, k, l = 1, \cdots, m), \qquad (4.21)$$

where $\Phi_{0l} = -\Phi_{l0} = (BA_l) - B(f) \cdot B(a_{1l})$. From this one deduces that in the $(m+1) \times (m+1)$ antisymmetric matrix

$$\begin{bmatrix} 0 & \Phi_{01} & \Phi_{02} & \cdots & \Phi_{0m} \\ \Phi_{10} & 0 & a_{121} & \cdots & a_{1m1} \\ \Phi_{20} & a_{211} & 0 & \cdots & a_{2m1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Phi_{m0} & a_{m11} & a_{m21} & \cdots & 0 \end{bmatrix} \qquad (4.22)$$

all the principal minors (*Hauptunterdeterminanten*) of order four containing elements from the first column and the first row vanish. As a consequence of antisymmetry, *all* principal minors of order four vanish and consequently[26], either the rank of $[a_{ik1}]$ is two or all $\Phi_{0l}$ vanish. Since $rk\,[a_{ik1}] > 2$, the only possibility is that $\Phi_{0l} = -\Phi_{l0} = (BA_l) - B(f) \cdot B(a_{1l}) \equiv 0$ and so the system (4.20) is complete. From complete integrability of the system (4.20), von Weber straightforwardly derived the complete integrability of what he called the derived Pfaffian system of (4.15):

$$\nabla_s - \mu_s \nabla_1 = 0 \quad (s = 2, 3, \cdots, n-m). \qquad (4.23)$$

[24] Rember the definition of $A_i(f) = \frac{\partial f}{\partial x_i} + \sum_{s=1}^{n-m} a_{si} \frac{\partial f}{\partial x_{m+s}} \quad (i = 1, \cdots, m)$.

[25] The case $m = 2$ is indeed trivial.

[26] This implication holds in virtue of the antisymmetry. Remember that the rank of an antisymmetric matrix is always even and that it is equal to $r$ if, and only if all the principal minors of order $r+2$ vanish and a non vanishing principal minor of order $r$ exists.

By indicating with

$$\begin{cases} z_{2\nu+2}(x_1, \cdots, x_n) = c_1, \\ z_{2\nu+3}(x_1, \cdots, x_n) = c_2, \\ \vdots \\ z_{2\nu+n-m}(x_1, \cdots, x_n) = c_{n-m-1}, \end{cases}$$

its integral equivalents, he was finally able to provide a normal form for (4.15) given by the system

$$\begin{cases} dz_{2\nu+1} = z_{\nu+1}dz_1 + z_{\nu+2}dz_2 + \cdots + z_{2\nu}dz_\nu \\ dz_{2\nu+2} = 0, dz_{2\nu+3} = 0, \cdots, dz_{2\nu+n-m} = 0, \end{cases} \tag{4.24}$$

where $z_1, \cdots, z_{2\nu+1}$ are appropriate functions of $x_1, \cdots, x_n$. We will see later Cartan's reinterpretation of the notion of derived system. For the time being, it should be observed, as von Weber did, that the derived system (4.23) represents an example of differential structure invariantly connected to (4.15) in the sense of Engel's definition. Indeed Weber's derived system (4.23) does coincide with the Pfaffian system introduced by Engel in Theorem 15.

### 4.4.3 Reducibility of a Pfaffian system to its normal form

Although von Weber's analysis in [von Weber 1898] can certainly be considered as remarkable progress with respect to years of relative stagnation, the results therein contained were of an unsystematic kind and often limited to very particular cases (e. g. character equal to one, as we have seen in the preceding paragraph). Over the following years, von Weber tried to remedy this inconvenience and developed a more organic theory which, in principle, could be applied to Pfaffian systems of a general type. A crucial role, as we will see, was played by geometrical insight and by frequent reliance upon the theory of linear complexes and linear congruences in projective spaces.

As for von Weber's results in this period, the following account is mainly based upon [von Weber 1900a] and [von Weber 1900b]. I will linger on some details, since in von Weber's papers for the first time we encounter problems, results and technical tools of great importance for the development of Cartan's geometrical theory of exterior differential systems. When discussing Cartan's papers I will endeavour to indicate limits and relative importance of his debt to von Weber.

At the beginning of [von Weber 1900a], von Weber singled out the main problem in the theory of general Pfaffian equations as the answer to the following question (indeed, a genuine generalization of the problem of Pfaff for a single total differentials equation):

**Problem 2** *What are the necessary and sufficient conditions for the system (4.15) to be reducible to the following (normal) form*

$$\sum_{s=1}^{\tau} F_{sh} df_s = 0, \qquad (h = 1, ..., n-m), \tag{4.25}$$

*containing only $\tau$ differentials, where $f_1, \cdots, f_\tau$ are independent functions of $x_1, \cdots, x_n$ and $\tau$ indicates an integer not smaller than $n-m$ and not greater than $n-2$?*[27]

As already observed by Frobenius[28], if the Pfaffian system (4.15) admits a normal form of type (4.25), then the system of $\tau$ equations

$$dx_{m+h} = \sum_{i=1}^{m} a_{ih} dx_i; \quad df_1 = 0, \cdots, df_\rho = 0 \quad (h = 1, \cdots, n-m; \rho = \tau - n + m) \tag{4.26}$$

is completely integrable. Thus, a necessary and sufficient condition for the existence of the normal form (4.25) is that the $n-m$ bilinear forms

$$\Omega(dx, \delta x) = \sum_{i=1}^{m} \sum_{k=1}^{m} a_{iks} dx_i \delta_k x, \quad (s = 1, \cdots, n-m) \tag{4.27}$$

vanish as a consequence of the following relations:

$$\sum_{i=1}^{m} A_i(f_k) dx_i = 0, \quad \sum_{i=1}^{m} A_i(f_k) \delta x_i = 0 \quad (k = 1, \cdots, \rho)^{29}. \tag{4.28}$$

von Weber observed that the same problem can be considered from a different and more geometrical perspective whose usefulness Cartan would thoroughly examine in his work.

If the Pfaffian system (4.15) can be rewritten in the normal form (4.25), then the equations

$$f_1(x_1, \cdots, x_n) = c_1, f_2(x_1, \cdots, x_n) = c_2, \cdots, f_\tau(x_1, \cdots, x_n) = c_\tau \tag{4.29}$$

[27] The reason for these limitations is easily explained: if $\tau = n-1$, then one is brought back to the problem of determining 1-dimensional integral manifolds of (4.15); if, on the other hand, $\tau = n-m$, this means that the system (4.15) is completely integrable and so Frobenius' theory can be applied.

[28] See [Frobenius 1877, §20].

[29] This is a good point at which the following important observation should be made: in the classical literature no conceptual and notational distinctions were made between base elements of what we would nowadays call cotangent space and the components of tangent vectors. If one cannot resist the temptation to restore such a distinction, it should be observed that in formulas (4.27) and (4.28) the $dx_i$'s and the $\delta x_i$'s have to be regarded as components of tangent vectors whereas in formula (4.26) the $dx_i$'s and $df_k$'s are indeed to be considered as elements of a cotangent space. Clearly, this comes as no surprise as far as the theory lacks a formal definition of what a differential form is. A little bit more surprising will be the discovery that even Cartan's theory was affected by this "flaw".

represent[30] an integral equivalent of (4.15), that is an $(n-\tau)$-dimensional integral variety, which von Weber indicated with $M_{n-\tau}$. If equations (4.29) are replaced by their equivalent parametric expression

$$x_i = \phi_i(u_1, \cdots, u_n) \qquad (i = 1, \cdots, n; \nu = n - \tau), \tag{4.30}$$

then these functions are solutions of a first order differential system, to be indicated with the symbol $S_\nu$, which takes on the following form[31]:

$$\begin{cases} \frac{\partial x_{m+h}}{\partial u_r} = \sum_{i=1}^m a_{ih} \frac{\partial x_i}{\partial u_r} & (r = 1, \cdots, \nu; h = 1, \cdots, n-m), \\ \sum_{i=1}^m \sum_{k=1}^m a_{ikh} \frac{\partial x_i}{\partial u_r} \frac{\partial x_k}{\partial u_s} = 0 & (r, s = 1, \cdots, \nu; h = 1, \cdots, n-m). \end{cases} \tag{4.31}$$

Thus, if the differential system $S_\nu$ is such that there is a $M_{n-\tau}$ integral variety through every point $(x_1^0, x_2^0, \cdots, x_n^0)$ of a certain domain of the whole variety $M_n$, then the Pfaffian system admits the normal form (4.25). In this way, problem (2) was traced back to the analysis of the conditions guaranteeing the existence of solutions of the differential system $S_\nu$.

It was at this very point that von Weber made recourse to the theory of the so-called general systems of partial differential equations developed, among others, by C. Méray and C. Riquier[32]. As we will see, a crucial role was played by the notion of passivity (or involution) which assures, under regularity conditions, the existence of integrals of the system itself.

If one adds to the system $S_\nu$ all the equations that can be obtained from $S_\nu$ through repeated derivations (finite in numbers), either of the following two eventualities must occur: either a contradiction, that is a relation among the variables $x_1, \cdots, x_n$ only is produced or a differential system is obtained such that, by solving it with respect to certain partial derivatives, it can be put in the so-called *canonical passive form*, $\Sigma$.[33] If this is the case and if $\nu$

[30] For arbitrary values of the $c_i$'s.

[31] The second group of equations can be easily deduced from the first one simply by differentiating with respect to $u_1, \cdots, u_\nu$ and remembering that $\frac{\partial^2 x_i}{\partial u_r \partial u_s} = \frac{\partial^2 x_i}{\partial u_s \partial u_r}$. For a detailed verification of this very simple statement, see e. g. [Amaldi 1942, p. 110-112]

[32] For a historical account of the theory, Riquier's remarks in the preface to [Riquier 1910] can be consulted.

[33] I will not insist on detail. For further details, see Riquier's papers [Riquier 1893] and his treatise [Riquier 1910] or von Weber's encyclopaedia article [von Weber 1900d] on partial differential equations. I will give just few remarks. The expression *canonical form* means a system (to be indicated with $\Sigma$) of equations of type:

$$\frac{\partial^{\alpha_1+\alpha_2+\cdots+\alpha_\nu} x_i}{\partial u_1^{\alpha_1} \partial u_2^{\alpha_2} \cdots \partial u_\nu^{\alpha_\nu}} = \phi_{i,\alpha_1,\cdots,\alpha_\nu}\left(x_1, \cdots, x_\nu, \cdots, \frac{\partial^{\beta_1+\cdots+\beta_\nu} x_k}{\partial u_1^{\beta_1} \cdots \partial u_\nu^{\beta_\nu}}\right) \tag{4.32}$$

which satisfies the following requirements: i) No derivative making its appearance in the left side of (4.32) is contained in the right side. ii) for every derivative $\frac{\partial^{\beta_1+\cdots+\beta_\nu} x_k}{\partial u_1^{\beta_1} \cdots \partial u_\nu^{\beta_\nu}}$ contained in the expression for $\phi_{i,\alpha_1,\cdots,\alpha_\nu}$, is $\sum_{j=1}^\nu \beta_j \leq \sum_{j=1}^\nu \alpha_j$; if, in particular, $\sum_{j=1}^\nu \beta_j = \sum_{j=1}^\nu \alpha_j$ then $k \leq i$; if even $k = i$ then the first non-vanishing number in constants $u_1^0, \cdots, u_\nu^0$ are arbitrarily chosen, the system $S_\nu$ admits a unique solution $x_k(u_1, \cdots, u_\nu), (k = 1, \cdots, n)$ with the property that the parametric quantities valued in $u_1^0, \cdots, u_\nu^0$ assume the initial values

$$x_1^0, \cdots, x_n^0, \cdots, \left( \frac{\partial^{i+k+\cdots+l} x_h}{\partial u_1^i \partial u_2^k \cdots \partial u_\nu^l} \right)_0 \cdots \quad ,$$

provided that the right-hand sides of equations composing $\Sigma$ are sufficiently regular and also provided that the $n$ power series (the sum being extended to all parametric derivatives):

$$\sum \left( \frac{\partial^{i+k+\cdots+l} x_k}{\partial u_1^i \cdots u_\nu^l} \right)_{u_i = u_i^0} (u_1 - u_1^0)^i \cdots (u_\nu - u_\nu^0)^l \qquad (4.33)$$

converge in a certain region of $\mathbb{C}^n$. In this way, the admissibility of a normal form of type (4.25) and the consequent existence of $n-\tau$-dimensional integral varieties of the Pfaffian system was characterized by von Weber in terms of the possibility to write the differential system $S_\nu$ in a canonical passive (involutive) form. Moreover, as he observed, to establish whether such an eventuality was fulfilled or not, reduced, at least in principle, to a simple procedure consisting of differentiations and eliminations to be operated upon $S_\nu$.

At this point, von Weber made an interesting observation that was destined to assume a role of outstanding importance in Cartan's theory. Motivated by the possibility of developing fruitful applications in the realm of the theory of general partial differential equations, he introduced further hypotheses which guarantee the existence of integral varieties of increasing dimensions. Indeed, he explained, if one supposes that the system $S_\nu$ can be put in a canonical passive form simply by solving it with respect to certain derivatives $\frac{\partial x_i}{\partial u_\nu}$ and furthermore, one supposes that from $S_\nu$ and from equations obtained from it by differentiation and elimination, no relation among the variables $x_i, \frac{\partial x_i}{\partial u_1}, \cdots, \frac{\partial x_i}{\partial u_\nu}, \quad (i = 1, \cdots, n)$, already contained in $S_{\nu-1}$, can be deduced, and so on for the systems $S_{\nu-1}, \cdots, S_1$, then every 1-dimensional integral variety of (4.15) $M_1$ belongs at least to one 2-dimensional integral variety $M_2$, etc. and, finally, every $\nu - 1$-dimensional integral variety $M_{\nu-1}$ belongs at least to one $\nu$-dimensional integral variety $M_\nu$.

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the series of differences $\beta_1 - \alpha_1, \beta_2 - \alpha_2, \cdots$ is required to be positive.

On the other hand, passivity coincides with the following hypothesis: if the partial derivatives contained in the left side of the equations in $\Sigma$ and all the other deduced from them through repeated derivations with respect to $u_1, \cdots, u_\nu$ are called *principal*, and the remaining ones, along with the dependent variables $x_1, \cdots, x_n$, are called *parametric* quantities, then it is required that from $\Sigma$ and from equations deduced by $\Sigma$ through differentiation, every principal derivative is required to be expressed in terms of the parametric quantities in a unique way.

Thus, von Weber arrived at the statement of what I will refer to as the second problem of his theory of Pfaffian systems.

**Problem 3** *What are the necessary and sufficient conditions that have to be satisfied so that every* 1*-dimensional integral variety* $M_1$ *of (4.15) belongs at least to one* 2*-dimensional integral variety* $M_2$ *etc., and so that every* $\nu-1$*-dimensional integral variety* $M_{\nu-1}$ *belongs at least to one* $\nu$*-dimensional integral variety* $M_\nu$*?*

Clearly, both for Problem 1 and Problem 2, the aim was that of expressing such conditions by means of algebraic and differential relations among the coefficients $a_{si}, (s = 1, \cdots, n-m;\ i = 1, \cdots, m)$ only.
For instance, it turns out that a necessary condition to be fulfilled for the system (4.15) to possess $\nu$-dimensional integral varieties, $M_\nu$ (Problem 2), is the existence of a system of linearly independent functions $\eta_i^{(s)}$, $(i = 1, \cdots, m; s = 1, \cdots, \nu)$ satisfying the following bilinear equations:

$$\sum_{i=1}^{m}\sum_{k=1}^{m} a_{ikh}\eta_i^{(r)}\eta_k^{(s)} = 0, \quad (r, s = 1, \cdots, \nu; h = 1, \cdots, n-m). \tag{4.34}$$

A detailed account of von Weber's achievements over this point would go beyond our present purposes; we will include just a few observations.
His analysis was mainly based on the theory of bilinear forms and, more precisely, on the classification of linear complexes and linear congruences in $(m-1)$-dimensional[34] projective spaces. This should come as no surprise, since the coefficients of the bilinear covariants, $a_{iks}$, $(i, k = 1, \cdots, m; s = 1, \cdots, n-m)$ were interpreted by von Weber geometrically as defining a system of linear complexes, equal in number to the character $K$ of (4.15):

$$\sum_{i=1}^{m}\sum_{k=1}^{m} a_{iks}\eta_i\xi_k = 0, \quad (s = 1, \cdots, K). \tag{4.35}$$

In this geometrical context, von Weber identified the characteristic transformations, $X_i(f) = \sum_{j=1}^{m} \xi_j^{(i)} \frac{\partial f}{\partial x_j}$, $(i = 1, \cdots, h)$, as those for which the components $\xi_j$ represent the singular points of the congruence consisting of all the straight lines belonging to complexes (4.35).
von Weber was able to give a detailed analysis of Pfaffian systems with $m = 3, 4, 5, 6$ but, ultimately, despite his hopes[35], his approach did not succeed in providing a general theory of unlimited validity. Nevertheless, he

[34] The number $m$, that is the difference between the number of variables and the number of equations of which the system (4.15) consists, was called by von Weber *die Stufe* and it was considered by him, along with the *character*, as a measure of the difficulties one has to face in tackling the study of a given system of Pfaffian equations. See [von Weber 1901, p. 387].

[35] See, [von Weber 1901, p. 388].

should be acknowledged for opening a new phase in the studies of general Pfaffian systems, assessing some of the problems to be considered relevant and also for introducing useful technical tools which were destined to outlive his theory itself and to be reinterpreted in the light of Cartan's exterior differential calculus.

## 4.5 The foundations of the exterior differential calculus

After the composition of his doctoral thesis [Cartan 1894] where he gave a rigorous and complete treatment of the classification problem of finite-dimensional, semisimple, complex Lie algebras already started up and developed by Killing, for some years Cartan devoted himself to applications of the theoretical results contained therein. The theory of partial differential equations appears to be one of the main fields of his interest. This emerges quite clearly, for example, from the reading of a dense memoir, [Cartan 1896], dedicated to the theory of those systems of partial differential equations whose solutions depend only upon arbitrary constants and such that they admit a continuous group of transformations[36]. Cartan's work [Cartan 1899] on Pfaffian forms and more specifically on the problem of Pfaff was part of this interest. Indeed, as Lie had demonstrated, the integration of partial differential equations and the integration of Pfaffian forms were considered as equivalent formulations of the same problem.

V. Katz [Katz 1985] and T. Hawkins [Hawkins 2005] have already given a full and authoritative account of the large part of the material contained in [Cartan 1899]. For this reason we will limit ourselves to recall the main notions which will be useful for the rest of our discussion.

Cartan organized his treatment in a deductive way by first presenting a full set of definitions and conventions. He started up by giving a *symbolic definition* of what a differential expression in $n$ variables is; this was defined as a homogenous expression built up by means of a finite number of additions and multiplications of the $n$ differentials $dx_1, \cdots, dx_n$ as well as of certain coefficients which are functions of $x_1, \cdots, x_n$. In such a way, a Pfaffian expression was defined as a differential expression of degree one of type: $A_1dx_1 + \cdots + A_ndx_n$; a differential form of degree two was given, for example, by $A_1dx_2 \wedge dx_1 + A_2dx_3 \wedge dx_2$[37].

A very important notion of Cartan's new calculus was the exterior multiplication between two differential expressions[38]. Cartan himself observed

[36]See Section 2.3.

[37]Cartan did not employ the wedge product symbol $\wedge$.

[38]One can find the germs of this crucial notion already in some works on integral invariants by Poincaré. See [Katz 1985, p. 322] and [Olver 2000, p. 69] for the relevant bibliography.

that already in 1896 he had realized that the variables change formulas in multiple integrals could be easily derived by submitting the differentials under the integration sign to appropriate laws of calculation which coincide with Grassmann's exterior calculus. By developing such an intuition, in 1899 he was able to present convincing arguments to justify such rules, which relied upon the idea of the *value* of a differential form.

To this end, Cartan considered a differential expression $\omega$ of degree $h$ and then supposed that the $n$ involved variables are functions of $h$ arbitrary parameters $(\alpha_1, \cdots, \alpha_h)$. By indicating with $(\beta_1, \cdots, \beta_h)$ one of the $h!$ permutations of the parameters $\alpha_1, \cdots, \alpha_h$, Cartan associate to it the value that $\omega$ assumes when the differentials occupying the $i^{th}$ $(i = 1, \cdots, h)$ position are replaced by the corresponding derivative of $x$ with respect to $\beta_i$. By attributing to such a quantity the sign + or - depending on the parity of the permutation considered and then by summing over all $h!$ permutations, Cartan finally obtained what he called the *value* of the differential expression. For example, the value of the differential form $A_1 dx_2 \wedge dx_1 + A_2 dx_3 \wedge dx_2$ is:

$$A_1 \frac{\partial x_2}{\partial \alpha_1}\frac{\partial x_1}{\partial \alpha_2} + A_2 \frac{\partial x_3}{\partial \alpha_1}\frac{\partial x_3}{\partial \alpha_2} - A_1 \frac{\partial x_2}{\partial \alpha_2}\frac{\partial x_1}{\partial \alpha_1} - A_2 \frac{\partial x_3}{\partial \alpha_2}\frac{\partial x_2}{\partial \alpha_1}.$$

At this point, Cartan defined two differential expressions of degree $h$ to be equivalent if their value is the same independently from the choice of parameters $\alpha_1, \cdots, \alpha_h$. In this a way he was able to establish Grassmann's well-known rules for the multiplication to be interpreted, in Cartan's view, as equalities between equivalence classes of exterior differential forms. For example, one has $dx_1 \wedge dx_2 = -dx_2 \wedge dx_1$ or $dx_4 \wedge dx_4 = 0$, as it is easy to see by calculating the values of the differential expressions appearing in the equations.

A second crucial novelty of Cartan's theory was the exterior derivative of a given Pfaffian expression[39] which he explicitly connected with Frobenius' and Darboux's notion of bilinear covariant. Cartan's definition reads as follows. Given a Pfaffian form of type $A_1 dx_1 + \cdots + A_n dx_n$, its derived expression was the form of degree two:

$$\omega' = dA_1 \wedge dx_1 + \cdots + dA_n \wedge dx_n{}^{40}.$$

The invariant character of such derivative was then established upon reliance of the notion of *value* by observing that if $\bar{\omega}$ indicates the expression of $\omega$ with respect to a new set of coordinates $y_i(\vec{x})$ then the differential forms of degree two $\overline{\omega'}$ and $(\bar{\omega})'$ are equivalent in the sense specified above.

On the basis of such a new calculus, Cartan not only was able to reformulate all the known results of the theory of Pfaffian equations, including

[39]The definition was generalized to enclose derivatives of differential forms of degree greater than one in [Cartan 1901b, p. 243].

[40]The notation $d\omega$ was introduced by Kähler in [Kähler 1934, p. 6].

Frobenius' analytical classification theorem[41], but he also succeeded in obtaining new remarkable achievements concerning the resolution of systems (particularly relevant for the theory of partial differential equations of first order ) consisting of a single Pfaffian equation and a certain number of finite relations[42].

## 4.6 Cartan's theory of general Pfaffian systems

After laying the foundations of his new exterior differential calculus, Cartan devoted himself to the study of not completely integrable systems of Pfaffian equations. As for Weber, in accordance with the motivations laying at the basis of [Cartan 1899], it appears that the main reason for this was to be found in the wide applications of Pfaffian expressions to the theory of partial differential equations.

Cartan started his analysis in [Cartan 1901a] by recalling Biermann's[43] efforts to determine the maximal dimension and the degree of indeterminacy of integral varieties of unconditioned Pfaffian systems. In this respect, Cartan complained about the lack of rigorous (not generic) and systematic results, and emphasized the urgency to provide the theory with solid theoretical grounds in order to remedy this unsatisfactory state of affairs.

As already observed by Hawkins[44], a key role was played by the notion of the bilinear covariant which, as we have seen, Cartan interpreted as the first exterior derivative of a Pfaffian form. However, one should not forget that the use of such a notion was not Cartan's prerogative since, as we have seen, its employment was quite frequent among other mathematicians too. Instead, what characterizes Cartan's approach with respect to his contemporaries was the ubiquitous recourse to geometrical insight, and the foundational role of his exterior differential calculus which introduced considerable simplifications in the theory.

### 4.6.1 Geometrical representation

Apparently, Lie had been the first one to attribute a geometrical interpretation to the system of Pfaffian equations (4.15) in the context of his synthetic approach to differential equations. Engel took it from Lie and profitably applied it to his researches in [Engel 1890] by writing:

> *We can attribute to the system (4.15) also an illustrative representation. Indeed, through the equations (4.15), to every point*

[41] See section 4.1 above.
[42] See [Cartan 1899, Chapter V].
[43] See section 4.2 above.
[44] See [Hawkins 2005, p. 430]

*of the space $x_1, \cdots, x_n$ the corresponding plane bundle of directions: $dx_1 : dx_2 : \cdots : dx_n$, is associated. If in every point of the space the corresponding bundle of directions is considered then one obtains a figure which is the exact image of the system (4.15).*[45].

von Weber himself took great advantage of such a geometrical representation, and even widened it by introducing for the first time (September 1900, [von Weber 1900b]) the notion of *element* of a Pfaffian system as the set of directions tangent to the integral variety of the system itself.
Clearly inspired by this longstanding tradition, Cartan opened his analysis in [Cartan 1901a] by emphasizing the importance of geometrical representation in the problems he was about to deal with. Let us consider a system of Pfaffian equations of the following type:

$$\begin{cases} \omega_1 = a_{11}dx_1 + \cdots + a_{1r}dx_r = 0 \\ \omega_2 = a_{21}dx_1 + \cdots + a_{2r}dx_r = 0 \\ \qquad\qquad \vdots \\ \omega_s = a_{s1}dx_1 + \cdots + a_{sr}dx_r = 0. \end{cases} \tag{4.36}$$

Cartan supposed that $n$ out of the $r$ variables should be regarded as independent, so that the remaining $r-n$ could be expressed as functions of them. In this way, a $n$-dimensional variety $M_n$ of the $r$-dimensional total manifold[46] was defined. Then, Cartan observed, the system (4.36) can be thought of geometrically as prescribing those conditions that have to be satisfied by the differentials $dx_1, \cdots, dx_r$ when one considers an arbitrary displacement on $M_n$. Furthermore, since the differentials $dx_i$ can be assimilated[47] to direction parameters of the tangent lines to the variety $M_n$, the system (4.36) can be interpreted by saying that, as a consequence of (4.36), these tangent lines belong to a certain linear variety (*multiplicité plane*) which depends upon the point considered. Hence, the problem of finding $n$-dimensional integral varieties of (4.36) was traced back to the following:

*We associate to every point of the space a linear variety passing through this point; then we determine a $n$-dimensional variety*

[45] *Man kann mit dem Systeme (4.15) auch eine anschauliche Vorstellung verbinden. Durch die Gleichungen (4.15) wird nämlich jedem Punkte des Raumes $x_1, \cdots, x_n$ ein ebenes Bündel von $\infty^{m-1}$ Fortschreitungsrichtungen: $dx_1 : dx_2 : \cdots : dx_n$ zugeordnet. Denkt man sich in jedem Punkte des Raumes das zugehörige Bündel von Fortschreitungsrichtungen, so erhält man eine Figur, welche das genaue geometrische Bild des Systems (4.15) ist*

[46] I will designate with the expression *total manifold* the set of all $r$-uple of type $(x_1, \cdots, x_r)$.

[47] Again, no distinction between base elements of cotangent spaces and components of tangent vectors was made.

> *$M_n$, such that in its every point all the tangents to the variety belong to the linear variety corresponding to this point.* [48]

After posing the problem in precise geometrical terms, Cartan defined a linear element (*élément linéaire*) to be the set consisting of a point and a straight line passing through this point which can be denoted with $(\vec{x}, d\vec{x})$. Then, he specified a linear element to be integral if the differentials $dx_1, \cdots, dx_n$, regarded as direction parameters of the tangent line of the linear element, satisfy the systems of linear equations obtained from (4.36) after evaluating the coefficients $a_{ij}$, $(i = 1, \cdots, s; j = 1, \cdots, r)$, in $\vec{x}$. He finally arrived at the the following proposition which characterized the integral varieties of (4.36).

**Proposition 1** *For a variety to be integral is necessary and sufficient that every linear element of it be integral.*

The notion of linear element was soon generalized to a greater number of dimensions. Indeed, a $p$-dimensional element was defined to be the couple consisting of a point and a linear variety passing through this point. Cartan indicated it with the symbol $E_p$. It was clear that every $p$-dimensional element $E_p$ of an integral variety $M_n$ (necessarily, $p \leq n$) consists of *integral* linear elements; however, as already observed by von Weber[49], $E_p$ has to satisfy further conditions which, in general[50], are not algebraically implied by (4.36); they are the relations obtained by requiring the vanishing of all bilinear expressions

$$\left\{ \begin{array}{c} \omega_1' = \sum_{i,k=1}^{r} \left( \frac{\partial a_{1i}}{\partial x_k} - \frac{\partial a_{1k}}{\partial x_i} \right) (dx_i \delta x_k - dx_k \delta x_i), \\ \vdots \\ \omega_s' = \sum_{i,k=1}^{r} \left( \frac{\partial a_{si}}{\partial x_k} - \frac{\partial a_{sk}}{\partial x_i} \right) (dx_i \delta x_k - dx_k \delta x_i), \end{array} \right. \tag{4.37}$$

where $d\vec{x}$ and $\delta\vec{x}$ are arbitrary integral linear elements belonging to $E_p$. Thus, Cartan arrived at the crucial definition of integral element of more than one dimension:

[48] *A chaque point de l'espace on fait correspondre une multiplicité plane passant per ce point; déterminer une multiplicité à n dimensions $M_n$, telle qu'en chacun des ses points toutes les tangentes à cette multiplicité soient situées dans la multiplicité plane correspondente à ce point.* Cartan's formulation of the integration problem can be translated as follows. Consider the exterior ideal $\mathcal{I}$, simply generated by the 1-forms $\omega_1, \cdots, \omega_s$. To find integral submanifolds of $\mathcal{I}$ is equivalent to the problem of determining integral submanifolds of the distribution of vector fields which is dual to $\mathcal{I}$. If we indicate this distribution with $\mathcal{V} = \{v_1, \cdots, v_{r-s}\}$, the vector space $\mathcal{V} \mid_x$ is precisely the linear variety to which Cartan referred. Indeed, for a submanifold $M_n$ to be an integral variety of $\mathcal{V}$ is necessary and sufficient that $TM_n \mid_x \subset \mathcal{V} \mid_x, \forall x \in M_n$.

[49] Here we are referring to von Weber's remarks according to which one has to consider in the differential system $S_\nu$, together with equations of type $\frac{\partial x_{m+h}}{\partial u_r} = \sum a_{ih} \frac{\partial x_i}{\partial u_r}$, also the equations obtained by the them through differentiation.

[50] Unless the system (4.36) is completely integrable.

> *An element consisting of linear integral elements is said to be an integral element of dimension* $2, 3, \ldots$ *if every two linear integral elements satisfy (4.37).* [51]

Then he defined two integral linear elements $d\vec{x}, \delta\vec{x}$ to be *associated* or in *involution* if all bilinear expressions (4.37) vanish. As a consequence of this, the definition of integral element could be rephrased as follows:

> *A linear integral element of dimension* $2, 3, \cdots$ *is an element consisting of linear integral elements which are associated pairwise.*[52]

Once more, Cartan found it useful to emphasize his geometrical approach by observing that if the quantities $dx_i\delta x_k - dx_k\delta x_i$ are regarded as Plücker's coordinates of a straight line then the bilinear relations (4.37) can be interpreted as defining linear complexes in projective spaces; in this respect, he often insisted upon the possibility of yielding a thorough classification of all Pfaffian systems on the basis of the classification of linear complexes themselves.

It is somehow surprising that Cartan never mentioned von Weber's valuable researches[53] in this field, namely Weber's frequent recourse to the theory of linear complexes and bilinear forms. It is true that they had been published only a year before the publication of Cartan's epoch-making article [Cartan 1901a] and he thus might not have had the opportunity of studying them carefully or even of reading them at all. Nevertheless, it is of considerable historical interest that many ideas fully developed by Cartan could already be found in von Weber's work.

We will see later that for the crucial notion of a Pfaffian system in involution (closely related to Problem 2 of von Weber's theory) too, no mention of von Weber's analysis was made by Cartan. Once more, it is probable that Cartan was not aware of von Weber's work, and that he introduced involutive systems borrowing them directly from the theory of general systems of partial differential equations as developed by Méray, Riquier and É. Delassus.

[51] *Appelons élément intégral à* $2, 3, \ldots$ *dimensions un élément formé d'éléments linéaires intégraux et tel, de plus, que deux quelconques d'entre eux satisfassent au system (4.37).* In modern terms, Cartan's definition of integral element of (4.36) can be translated as follows: a $p$-dimensional integral element of (4.36) is a subspace $E_p \subset TM_r\mid_x$ such that:

$$< \omega_j; v_i >= 0 \quad \wedge \quad < d\omega_j; v_i, v_k >= 0 \quad \forall v_i, v_k \in E_p, \quad (j = 1, \cdots, s).$$

[52] *Un élément integral à* $2, 3, \cdots$ *dimensions est un élément formé d'éléments linéaires intégraux associés deux à deux.*

[53] The only work by von Weber, among those here considered, explicitly cited by Cartan both in [Cartan 1901a] and [Cartan 1901b] was [von Weber 1898].

### 4.6.2 Cauchy's first theorem

What Cartan called *Cauchy's first problem* was the following:

> *Given a p-dimensional integral variety $M_p$ of a system of total differentials equations, to determine a $(p+1)$-dimensional integral variety $M_{p+1}$ passing through $M_p$.*[54]

Its relevance can hardly be overestimated. Indeed, it turns out that the problem of integration of (4.36) can be solved by a step by step procedure consisting of determining integral varieties of increasing dimensions. Clearly, a necessary condition for the existence of a $(p+1)$-dimensional integral variety $M_{p+1}$ passing through $M_p$ is that every $p$-dimensional integral element $E_p$ of $M_p$ is contained at least in one $(p+1)$-dimensional integral element $E_{p+1}$. However this condition is not sufficient. To guarantee the existence of such integral varieties one has to require more, i.e. that at least a $(p+1)$-dimensional $E_{p+1}$ integral element passes through *every* $p$-dimensional integral element of the space.

Before moving on to the resolution of Cauchy's first problem, Cartan observed, it is useful to state a few geometrical remarks on the structure of the integral elements $E_{p+1}$. If $E_p$ is supposed to be generated by $p$ linearly independent vectors, i.e. $E_p =< e_1, \cdots, e_p >$ then an integral element $E_{p+1} \supset E_p$ can be defined by adding a linear element $e$ to $E_p$, which is linearly independent from $E_p$; clearly, one requires both that $e$ is integral and that $e$ is in involution with $e_i$, $\forall i = 1, \cdots, p$.

The $(p+1)$-dimensional element $E_{p+1}$ containing $E_p$ depends on $r-p$ homogeneous parameters[55], and the equations expressing that $E_{p+1}$ is integral are linear with respect to these parameters. Let us suppose that these equations reduce in number to $r-p-u-1$ (with $s \geq 0$), then at least one integral element $E_{p+1}$ passes through every integral element $E_p$. In particular if $u = 0$ then such a $E_{p+1}$ is unique. In general, these integral elements build up an infinite family depending on $u$ arbitrary constants. It may happen that in particular cases the degree of indeterminacy is greater than $u$; in this eventuality, Cartan said that $E_p$ is a *singular* element, otherwise $E_p$ was said to be regular[56]. Besides, the notion of singularity was easily transferred to integral varieties by defining them to be singular when all their integral elements are singular in the specified sense.

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[54] *Étant donnée une multiplicité intégrale à p dimensions $M_p$ d'un système d'equations aux différentielles totales, faire passer par $M_p$ une multiplicité intégrale à $p+1$ dimensions $M_{p+1}$.*

[55] The numbers of effective parameters is $r-p-1$; in general, every $q$-dimensional element $E_q$ containing a $p$-dimensional element $E_p$ depends on $(q-p)(r-q)$ effective parameters. See Goursat's proof in [Goursat 1922, §81].

[56] A more clear explanation of Cartan's notion of regularity will be given in the next section in terms of the so-called characteristic integers.

At this point Cartan was ready to face the solution of Cauchy's first problem proving what he called Cauchy's first theorem. This is a crucial point in the whole theory of exterior differential systems; indeed, it can be considered as the gist of what is nowadays known as Cartan-Kähler theorem, since one can find in it the proof of the inductive step that is crucial for assessing the existence of varieties "integrating" integral elements. As is well-known, the validity of the theorem is limited to the class of analytic Pfaffian system. For this reason we will suppose, as Cartan explicitly did, that the coefficients of (4.36) are analytic functions of $x$.

**Theorem 17 (Cauchy's first theorem)** *Suppose that every $p$-dimensional, regular, integral element passes through at least one $(p+1)$-dimensional integral elements $E_{p+1}$; then, given a non-singular $p$-dimensional integral variety of (4.36), $M_p$, a $(p+1)$-dimensional integral variety $M_{p+1}$ exists which passes through $M_p$. More precisely, if $\infty^u$ $(p+1)$-dimensional integral elements passes through every regular, integral element $E_p$, then many integral varieties $M_{p+1}$ exist which depend upon $u$ arbitrary functions.*

In order to offer a general idea of how it works it will be enough to limit ourselves to the following remarks.
Cartan's starting point was to translate the geometrical content of the statement into an analytic form. To this end he considered on $M_p$ a $p$-dimensional regular integral element $E_p^0$ that he supposed to have its centre in a fixed point $P^0 = (x_1^0, \cdots, x_r^0)$, of $M_p$. In an appropriate open subset containing $P^0$, $M_p$ can be represented by $r-p$ analytic functions expressing, for instance, the variables $x_{p+1}, \cdots, x_r$ in terms of $x_1, \cdots, x_p$. Furthermore, the linear equations (with constant coefficients) defining the integral element $E_p^0$ are solvable with respect to the differentials $dx_{p+1}, \cdots, dx_r$. Then if a $(p+1)$-dimensional integral element $E_{p+1}^0$ passing through $E_p^0$ is considered, the $r-p-1$ equations defining it are solvable with respect to the $r-p-1$ differentials $dx_{p+2}, \cdots, dx_r$, say. Thus, a $(p+1)$-dimensional integral variety $M_{p+1}$ admitting $E_{p+1}^0$ will be expressed, in an appropriate neighbourhood of $P^0$, by $r-p-1$ analytic functions, $x_{p+2}, \cdots, x_r$ of $x_1, \cdots, x_{p+1}$.

At this point Cartan introduced some simplification in the notation: $x_{p+1}$ was replaced by $x$ and $x_{p+2}, \cdots, x_r$ by $z_1, \cdots, z_m$ (clearly, $m = r-p-1$). As a consequence of this, the equations for $M_p$ (in a neighbourhood of $P^0$) can be written as follows:

$$\begin{cases} x = \phi(x_1, \cdots, x_p), \\ z_1 = \phi_1(x_1, \cdots, x_p) \\ \vdots \\ z_m = \phi_m(x_1, \cdots, x_p), \end{cases} \tag{4.38}$$

while the variety $M_{p+1}$ can be expressed (in a neighborhood of $P^0$) by:

$$z_j = z_j(x, x_1, \cdots, x_{p+1}), \qquad (j = 1, \cdots, m = r-p-1). \tag{4.39}$$

If $x$ is replaced by $x - \phi(x_1, \cdots, x_p)$ then (4.38) can be rewritten as

$$\begin{cases} x = 0, \\ z_j = \phi_j(x_1, \cdots, x_p), \qquad (j = 1, \cdots, m). \end{cases}$$

Consequently, the condition that $M_p$ passes through $M_{p+1}$ requires that:

$$z_j(0, x_1, \cdots, x_p) = \phi_j(x_1, \cdots, x_p) \quad (j = 1, \cdots, m).$$

Now, the problem consists of writing down the differential equations for the $r - p - 1$ unknown functions that have to be satisfied for $M_{p+1}$ to be an integral variety of (4.36).To this end, it is first necessary to consider the generic $(p+1)$-dimensional element (having its centre in a neighborhood of $P^0$) consisting of the linear elements that one obtains when infinitesimal increments are attributed to every single variable $x, x_1, \cdots, x_p$. Their direction parameters are given by the following table:

$$\begin{array}{ll} e: & \frac{dx}{1} = \frac{dx_1}{0} = \cdots = \frac{dx_p}{0} = \frac{dz_1}{\frac{\partial z_1}{\partial x}} = \cdots = \frac{dz_m}{\frac{\partial z_m}{\partial x}}, \\ e_1: & \frac{dx}{0} = \frac{dx_1}{1} = \cdots = \frac{dx_p}{0} = \frac{dz_1}{\frac{\partial z_1}{\partial x_1}} = \cdots = \frac{dz_m}{\frac{\partial z_m}{\partial x_1}}, \\ \vdots & \qquad\qquad \vdots \\ e_p: & \frac{dx}{0} = \frac{dx_1}{0} = \cdots = \frac{dx_p}{1} = \frac{dz_1}{\frac{\partial z_1}{\partial x_p}} = \cdots = \frac{dz_m}{\frac{\partial z_m}{\partial x_p}}. \end{array} \tag{4.40}$$

Then one requires that this $(p+1)$-dimensional element is indeed an integral element; that is, one requires i) that every linear element is integral and ii) that the linear integral elements are pairwise in involution. Cartan observed that it is useful to separate such a system of differential equations into two groups: the first one (I) assures that the element $E_p =< e_1, \cdots, e_p >$ is integral, the second one (II) that the linear element $e$ is integral and in involution with $E_p$. It is easily seen that (I) does not contain the derivatives $\frac{\partial z_1}{\partial x}, \cdots, \frac{\partial z_m}{\partial x}$ while the second system of equations (II) is linear with respect to such derivatives. At this point the regularity assumptions come into play: essentially, the fact that the indeterminacy degree of $E_{p+1}$ is minimal guarantees the possibility to rewrite the system (II) in a Cauchy-Kovalevskaya form where $m - u$ derivatives are expressed in terms of an equal number of analytic functions:

$$\begin{cases} \frac{\partial z_1}{\partial x} = \Phi_1 \left( x, x_i, z_k, \frac{\partial z_k}{\partial x_j}, \frac{\partial z_{m-u+1}}{\partial x}, \cdots, \frac{\partial z_m}{\partial x} \right) \\ \qquad\vdots \\ \frac{\partial z_{m-u}}{\partial x} = \Phi_{m-u} \left( x, x_i, z_k, \frac{\partial z_k}{\partial x_j}, \frac{\partial z_{m-u+1}}{\partial x}, \cdots, \frac{\partial z_m}{\partial x} \right). \end{cases} \tag{4.41}$$

The theory of the systems of partial differential equations of this particular kind guarantees the existence of holomorphic solutions in a neighbourhood of $P^0$ depending on $u$ arbitrary[57] functions $z_{m-u+1}, \cdots, z_m$, such that for $x = 0$ they reduce to $\phi_1, \cdots, \phi_m$.

Cartan's final step consisted of proving that the solution so obtained indeed represents an $(p+1)$-dimensional integral variety of the system (4.36), that is, it satisfies both system (I) and (II). First, he proved that every solution of the Cauchy-Kovalevskaya system satisfies systems (I) and (II) for $x = 0$. Then he demonstrated that every solution satisfying (I) and (II) for a generic $x$ also satisfies (I) and (II) for every infitesimally close point $x + \delta x$, showing, in such a way, that the theorem is true for every point within an appropriate neighbourhood of $P^0$.

As we have seen, von Weber had been the first one to set the problem of the integration of general Pfaffian systems in terms of the existence of integral varieties of increasing dimension. Yet he was unable to yield a systematic analysis of the conditions whose determination was the object of Problem 2 of his theory. For his part, Cartan considered the determination of chains of integral varieties of increasing dimension as the gist of the integration procedure of general Pfaffian systems. To this end, he introduced the following definition of Pfaffian system in involution. A Pfaffian system (4.36) was said by him to be in involution if at least one 2-dimensional integral variety $M_2$ passes through each integral curve $M_1$, at least one 3-dimensional integral variety $M_3$ passes through each 2-dimensional integral variety $M_2$, etc., and finally at least one $g$-dimensional integral variety $M_g$ passes through each $(g-1)$-dimensional integral variety $M_{g-1}$[58]. Now, from the first Cauchy's theorem it follows that a sufficient and necessary condition for the system (4.36) to be in involution is that every regular 1-dimensional integral element $E_1$ belongs at least to one 2-dimensional integral element $E_2$, every regular 2-dimensional integral element $E_2$ belongs at least to one 3-dimensional integral element, etc., and finally, every regular $(g-1)$-dimensional integral element $E_{g-1}$ belongs at least to one $g$-dimensional integral element $E_g$.

In a such a way Cartan was able to provide a complete answer to Weber's Problem 2 by obtaining at the same time an answer to Problem 1 since the determination of the number $g$ coincides with the determination of integral (regular) varieties of maximal dimension.

Some remarks on the historical origin of the notion of involutive systems of Pfaffian equations are in order here. Weber was the first, to my knowledge, to draw attention on the relation between the general theory of canonical passive (involutive) systems of partial differential equations and the existence of chains of integral varieties of (4.36) of increasing dimensions.

[57] Clearly, for $x = 0$ they have to reduce to $\phi_{m-u+1}, \cdots, \phi_m$.

[58] The number $g$ was called by Cartan the *genre* of the system (4.36); it is defined as the maximal dimension of regular integral varieties of (4.36). In the next section, we will see how the existence of such an integer can be deduced from the so-called characteristic integers of (4.36).

Thus, it would be natural to trace back Cartan's notion of involution to von Weber's works on general Pfaffian systems, namely to [von Weber 1900a]. However, I cannot produce any evidence testifying that Cartan derived from von Weber the inspiration for his researches over this point. On the contrary, Cartan's recollection contained in [Cartan 1939, p. 28-29] seems to support the possibility that he developed such a notion independently of von Weber. In fact he recognized that his own notion of involution was analogous to the one introduced, among others, by Méray and Riquier in the context of the theory of general systems of partial differential equations.
Lastly, it seems that an important influence on Cartan was exerted by the work of Étienne Delassus, namely by [Delassus 1896], in which, for the first time, the study of solutions of general systems of partial differential equations had been traced back to the study of Cauchy-Kovalevskaya systems. In particular, Delassus had shown that the whole integration procedure could be reduced to successive integrations of Cauchy-Kovalevskaya systems in an increasing number of independent variables. From this point of view, Cartan's achievements can be seen as a translation and a development of Delassus' results in the geometric, coordinates-independent language of exterior differential forms.

### 4.6.3 *Genre* and characters

Cauchy's first theorem highlights the need to proceed toward a detailed geometrical analysis of the properties of integral elements. In particular, as we have just seen, Cartan was interested in studying the conditions guaranteeing the existence of integral elements of increasing dimensions. His procedure can be summarized as follows: let us consider a $p$-dimensional integral element $E_p$ of a certain point $x$; we can think of it as being generated by $p$ linearly independent linear integral elements reciprocally in involution: $E_p =< e_1, \cdots, e_p >$. To construct an integral element $E_{p+1}$ containing $E_p$, as we already know, one has to add to $E_p$ an integral linear element $e$ independent from $E_p$ as well as in involution with $e_i$, $(i = 1, \cdots, p)$[59]. If, as Cartan would do in [Cartan 1901b], we introduce the linear variety[60] $H(E_p)$, the *polar element*[61] of $E_p$, of all integral linear elements in involution with $E_p$, we can rephrase the preceding sentence by saying that $E_{p+1} =< E_p, e >$, with $e \in H(E_p)$ and $e \notin E_p$.

[59]Remember that $e$ is in involution with $E_p =< e_1, \cdots, e_p >$ if, and only if $d\omega_k(e, e_i) = 0$, $(k = 1, \cdots, s; i = 1, \cdots, p)$.

[60]The fact that $H(E_p)$ is a linear variety is a consequence of the bilinearity of (4.37). It should be observed that, although consisting of linear integral elements, in general, $H(E_p)$ is *not* an integral element. The reason for this is due to the fact that two linear integral elements in involution with a third are not necessarily in involution between themselves. See [Cartan 1901a, p. 250].

[61]To my knowledge, this denomination was first introduced by Cartan when discussing Pfaffian systems of character one in [Cartan 1901b, §18].

To characterize the structure of such polar elements as well as to obtain information on the degree of indeterminacy of integral elements, Cartan introduced a sequence of integers, which we will indicate with $\tilde{r}_i$, to be defined as follows:

$$\dim H(E_p) = \tilde{r}_{p+1} + p + 1.$$

Geometrically, this means that the polar element $H(E_p)$ is generated by the $p$ base vectors of $E_p$ and by $\tilde{r}_{p+1} + 1$ linear integral elements, $e_0, e_1, \cdots, e_{\tilde{r}_{p+1}}$. As a consequence of this, the $(p+1)$-dimensional integral elements $E_{p+1}$ passing through $E_p$ depend on $\tilde{r}_{p+1}$ parameters or, as Cartan expressed himself, $\infty^{\tilde{r}_{p+1}}$ integral elements $E_{p+1}$ pass through $E_p$.

Although Cartan was not very explicit, it is clear that the coefficients $\tilde{r}_p$ depend not only on the point $x$ to which the $E_p$'s belong but also on the choice of the basis of the tangent space in $x$ to the integral variety (indeed, even on the ordering of such a basis)[62]. Thus, to be rigorous, he should have written $\tilde{r}_p(x; e_1, \cdots, e_{p-1})$ instead of $\tilde{r}_p$. In fact, it appears that the coefficients $r_p$ effectively introduced by Cartan, the so-called *characteristic integers*, should be interpreted as the minimum values of $\tilde{r}_p(x; e_1, \cdots, e_{p-1})$ when $x$ varies over $M_r$ and the $e_i$'s $(i = 1, \cdots, p-1)$ vary over $TM_r \mid_x$:

$$r_p = Min\{\tilde{r}_p(x, e_1, \cdots, e_{p-1}) \mid x \in M_r,\ e_i \in TM_r \mid_x\}.$$

This lack of notational precision was justified, in Cartan's view, by the necessity to focus his analysis[63] on the so-called *regular integral elements*, *i.e.* on those integral elements for which $\tilde{r}_p = r_p$. More precisely, according to Cartan's definition, a $p$-dimensional integral element $E_p$ is regular if, and only if, $\tilde{r}_{p+1} = r_{p+1}$[64], or, in another words, when its polar element has minimal dimension. As a consequence of this, in that which follows, as Cartan did, we will limit our attention to *non-singular* (*i.e.* regular) integral elements and for this reason we will ignore the distinction between $r_p$ and $\tilde{r}_p$.

From these preliminary remarks, Cartan moved on to demonstrate certain arithmetical relations among the characteristic integers which turn out to be very useful for the following discussion.

A first result is that the integers $r_p$ decrease when the index $p$ increases. Indeed let us consider a regular integral element $E_p$ and a regular integral element $E_{p-1}$ contained therein; since every linear integral element in involution with $E_p$ is, *a fortiori*, in involution with $E_{p-1} \subset E_p$, we have $H(E_p) \subset H(E_{p-1})$ and thus, $r_p \geq r_{p+1} + 1$.

From this, it follows that the succession of integers $\{r_p\}$ is decreasing and that an integer $g$ exists such that $r_{g+1} = -1$. Therefore, the polar

62 See [Olver 1995, p. 450-454].

63 As we will see later, characteristic elements are a major exception.

64 It should be observed that this is different from the notion of regularity given for example by Olver in [Olver 1995, p. 456].

space $H(E_g)$ does coincide with $E_g$ (supposed to be regular) and no $(g+1)$-dimensional integral element $E_{g+1}$ passes through $E_g$. The integer $g$ was called by Cartan the *genre* of the differential system (4.36).

Another chain of inequalities gives information on the differences among three consecutive characteristic integers:

$$r_p - r_{p+1} \geq r_{p+1} - r_{p+2} \qquad (p \leq g-2).$$

As for the preceding inequality, its demonstration relies on geometric considerations concerning the polar spaces, $H(E_{p-1}), H(E_p)$ and $H(E_{p+1})$.

From this and from $r_{g-1} - r_g - 1 \geq r_g$ Cartan finally deduced the following fundamental chain of inequalities:

$$r - r_1 - 1 \geq r_1 - r_2 - 1 \geq \cdots \geq r_{g-1} - r_g - 1 \geq r_g. \tag{4.42}$$

The numbers present in such inequalities assume a great importance in the theory. As the characteristic integers $r_p$, they are invariants of the system (4.36) with respect to arbitrary changes of coordinates[65] and provide a useful tool for the classification of general Pfaffian systems. Cartan indicated them with $s_1, \cdots, s_g$, namely[66]:

$$\begin{cases} s_1 = r_1 - r_2 - 1 \\ \vdots \\ s_{g-1} = r_{g-1} - r_g - 1 \\ s_g = r_g. \end{cases} \tag{4.43}$$

Cartan observed that the first of these integers, $s_1$, had already been introduced by Weber under the denomination of *character* of the system (4.36). By generalizing such a notion to the subsequent integers, Cartan spoke of second, third, etc. character, respectively. As we have seen, von Weber had introduced the character $s_1$ in a purely algebraic manner (except for the subsequent interpretation in terms of linear complexes) as the number of linearly independent relations built up with the bilinear covariants of the Pfaffian system. It is easy to demonstrate that Cartan's definition coincides with that of Weber. Indeed, it is sufficient to observe that, according to Cartan's definition, $s_1$ is the number of linearly independent equations which one has to add to (4.36) in order to obtain the polar element of a (regular) linear integral element, $E_1$.

The relevance of such integers was clarified by the possibility of determining the most general (regular) integral variety $M_g$ of (4.36), by repeated

[65]Such an invariance property is essentially due to the covariance of the exterior derivative. See [Cartan 1901b, p. 236-237].

[66]The first integer $r - r_1 - 1$ is ignored since it is easily demonstrated to be equal to $s$, the number of linear independent Pfaffian equations of the system. $s$ was sometimes called the zero-th character of (4.36), for example in [Amaldi 1942].

application of the so-called Cauchy's first theorem. More precisely, Cartan was able to yield a full characterization of the indeterminacy degree of the solutions of (4.36) by demonstrating the following

**Theorem 18 (Cauchy's second theorem)** *Given a Pfaffian system of $s$ linearly independent equations in $r$ variables, let us indicate with $g$ its* genre *and with $s_1, \cdots, s_g$ its characters. Then, the $r$ variables can be divided into $g+2$ groups:*

$$\begin{array}{cccc} x_1, & x_2, & \cdots, & x_g; \\ z_1, & z_2, & \cdots, & z_s; \\ z_1^{(1)}, & z_2^{(1)}, & \cdots, & z_{s_1}^{(1)} \\ \vdots & \vdots & \vdots & \vdots \\ z_1^{(g)}, & z_2^{(g)}, & \cdots, & z_{s_g}^{(g)}. \end{array}$$

*such that on the most general integral variety $M_g$ the variables $x_1, x_2, \cdots, x_g$ can be regarded as independent and, in a neighborhood of a regular point $(x_1^0, \cdots, x_g^0)$, $M_g$ is determined by the following specification of initial conditions: on $M_g$, $z_1^{(g)}, z_2^{(g)}, \cdots, z_{s_g}^{(g)}$ reduce to $s_g$ arbitrary functions of $x_1, \cdots, x_g$; for $x_g = x_g^0$, the $z_1^{(g-1)}, \cdots, z_{s_{g-1}}^{(g-1)}$ reduce to $s_{g-1}$ arbitrary functions of $x_1, \cdots, x_{g-1}$; etc.; for $x_g = x_g^0, \cdots, x_2 = x_2^0$ the $z_1^{(1)}, z_2^{(1)}, \cdots, z_{s_1}^{(1)}$ reduce to $s_1$ functions of $x_1$; finally, for $x_g = x_g^0, \cdots, x_2 = x_2^0, x_1 = x_1^0$, the $z_1, \cdots, z_g$ reduce to $s$ arbitrary constants.*

As Cartan observed, the theorem includes the results already obtained by Biermann[67] in the realm of unconditioned Pfaffian systems. Clearly, Cartan's achievements were far more general, rigorous and complete. Furthermore, the characterization of the indeterminacy of the integral solutions agreed with analogous results obtained by Delassus in [Delassus 1896] in his researches on general systems of partial differential equations. Nevertheless, Cartan claimed the superiority of his new approach through exterior differential forms since it had the advantage of being independent of a particular choice of coordinates.

### 4.6.4 Characteristic elements

Thus far, we have dealt with regular integral elements. However, singular integral elements play an important role too. Following Engel, von Weber had introduced characteristic transformations as those infinitesimal transformations which are dual to the Pfaffian equations of the system and leave the system invariant. For his part, Cartan introduced what he called *characteristic elements* by observing that in some cases the differential equations of the Cauchy-Kovalevskaya system determining the integral variety $M_{p+1}$ passing through a given integral variety $M_p$ assume a particularly simple

[67] See section 4.2 of this paper.

form which greatly simplifies their integration. By using the notation of section (4.6.2), Cartan considered the eventuality in which such equations do not depend upon the derivatives $\frac{\partial z_i}{\partial x_k}$, $(i = 1, \cdots, r-p-1; k = 1, \cdots, p)$, that is the case in which the Cauchy-Kovalevskaya system is independent of the linear integral elements $e_1, \cdots, e_p$ generating the integral element $E_p$. As a consequence of this, the partial derivatives $\frac{\partial z_1}{\partial x}, \cdots, \frac{\partial z_{r-p-1}}{\partial x}$ define a linear integral element $e$ which depends only on the point considered and is as well in involution with all the integral elements $E_p$ passing through this point. If this is the case, in every point of the space[68] a linear integral element exists that is in involution with *every* linear integral element drawn from this point. Cartan called such linear integral elements *characteristic*[69]. Clearly, a linear integral characteristic element is also *singular*, since $\infty^{r_1-1}$ integral elements $E_2$ pass through it.

As von Weber had already observed, the importance of characteristic elements[70] lay in the possibility to exploit their existence to simplify the integration of the Pfaffian system under examination. Indeed, after defining the characteristic Pfaffian system as the system of total differentials equation determining characteristic elements, he was able to reformulate von Weber's Theorem 16 in the following way:

**Theorem 19** *The minimal number of variables upon which, by means of a change of variables, the coefficients and the differentials of a given Pfaffian system can depend is equal to the number of linear independent equations of the characteristic system; these variables are given by the integration of such a system.*

As for von Weber's analysis, a crucial point in Cartan's treatment was the fact that the characteristic system is completely integrable. However, whereas von Weber had established such a property by making recourse to what we called characteristic transformations and then relying upon Clebsch's theorem on complete systems of linear partial differential equations, Cartan deliberately avoided such expedients and managed to demonstrate the complete integrability of the characteristic system by using differential forms only. Within few months, he was able to propose two different demonstrations. The first one contained in [Cartan 1901a, p. 302-305] consisted of a step by step procedure which relied on the basic property according to which a Pfaffian system of $r-1$ equations in $r$ variables is necessarily completely integrable. The second, contained in [Cartan 1901b, p. 248-249],

[68] We actually should limit ourselves to some open subset of the space.

[69] The denomination stemmed from the theory of partial differential equations, namely from the theory of Cauchy's characteristics. The connection between Cartan's characteristic elements and Cauchy's characteristics is explained very clearly in [Amaldi 1942, p. 180-182].

[70] Actually, as Engel did, he spoke of transformations leaving the Pfaffian system invariant.

was instead presented as a more direct application of the symbolic calculus with exterior differential forms that Cartan had developed in [Cartan 1899].

It should be noticed that, as we will observe in the next section too, Cartan's refusal to utilize infinitesimal transformations was by no means casual. We suggest that his need to avoid any recourse to them was due to the project of developing an approach to continuous Lie's groups purely in terms of Pfaffian forms without any use of infinitesimal transformations which, according to Cartan's view[71], did not represent an appropriate technical tool to deal with the structural theory of infinite dimensional continuous groups of transformations.

### 4.6.5 Pfaffian systems of character one, II

The present section will be devoted to a discussion of Cartan's analysis contained in [Cartan 1901b] of this special type of Pfaffian systems with the aim to draw a comparison between Weber's and Cartan's approaches. Whereas von Weber's treatment was almost entirely based on analytical and algebraic considerations, Cartan carried out his analysis in geometrical terms, heavily relying, as in his approach to characteristic elements, on the new properties of his exterior differential calculus.

Since the beginning of his discussion, Cartan affirmed that his analysis of Pfaffian systems of character one does not bring to any new results with respect to von Weber's paper [von Weber 1898]. Nevertheless, he set out to reinterpret von Weber's achievements in order to yield a concrete application of the principles of his theory.

Let us begin with Cartan's deduction of the so-called *derived* system of (4.36). Contrary to Weber's analysis which had had recourse to infinitesimal transformations (indeed, the dual counterpart of differential expressions) to deduce, via Theorem 15, the existence of the system (4.23) invariantly connected to the Pfaffian system under examination, Cartan first defined the notion of congruence between two differential forms[72], and then he introduced the derived system in the following way.

He considered a Pfaffian system of $s$ independent equations, $\omega_i = 0$, $(i = 1, \cdots, s)$, in $r$ variables and he introduced $r - s$ Pfaffian forms $\bar{\omega}_j$, $(j = 1, \cdots, r - s)$ such that $\{\omega_i, \bar{\omega}_j\}\,(i = 1, \cdots, s; j = 1, \cdots, r - s)$ are $n$ independent Pfaffian forms[73]. As a consequence of this, the $s$ bilinear covariants of

[71] For a detail discussion over this point, see the following chapter.

[72] If $\Omega$ and $\Pi$ designate two differential forms with the same degree and $\omega_1, \cdots, \omega_p$ designate $p$ homogeneous differential forms with degree less or, at most, equal to that of $\Omega$ (and $\Pi$), then Cartan defined $\Omega$ and $\Pi$ to be congruent module $\omega_1, \cdots, \omega_p$ if $p$ differential forms $\chi_1, \cdots, \chi_p$ exist such that: $\Omega = \Pi + \omega_1 \wedge \chi_1 + \cdots + \omega_p \wedge \chi_p$.

[73] In modern terms, one can say that $\{\omega_i, \bar{\omega}_j\}\,(i = 1, \cdots, s; j = 1, \cdots, r - s)$ define a *coframe*.

the system can be written as

$$\omega_i' \equiv \sum_{j,k=1}^{n-s} A_{ijk}\bar{\omega}_j \wedge \bar{\omega}_k = \Omega_i, \quad (\operatorname{mod} \omega_1, \omega_2, \cdots, \omega_s), \quad (i = 1, \cdots, s). \tag{4.44}$$

Now, in general, the differential forms $\Omega_i$ are not independent; if, for example:

$$l_1\Omega_1 + l_2\Omega_2 + \cdots + l_s\Omega_s = 0^{74},$$

then one has:

$$(l_1\omega_1 + l_2\omega_2 + \cdots + l_s\omega_s)' \equiv 0, \quad (\operatorname{mod} \omega_1, \omega_2, \cdots, \omega_s). \tag{4.45}$$

In such a way, Cartan demonstrated that appropriate linear combinations of the Pfaffian equations of the system (4.36) exist such that every couple of integral elements of (4.36) is in involution with respect to them. He then considered all the equations of type $l_1\omega_1 + l_2\omega_2 + \cdots + l_s\omega_s = 0$, and built up what he called the *derived system* of (4.36). It is clear that Cartan's definition was a generalization of that given by von Weber which was limited to systems of character one. Furthermore, it is important to emphasize the fact that the introduction of derived systems was brought about by Cartan purely in terms of exterior differential forms without any recourse to operations with vector fields (infinitesimal transformations, in his wording). To this end, an important role may have been played by the remark, already implicit in Engel's and Weber's work, that exterior differentiation could be considered in a certain sense as the dual counterpart of the Lie-bracketing operation between two infinitesimal transformations[75].

Specializing his discussion to systems of character one, in accordance with Weber's results, Cartan was able to show that in this case the derived system of (4.36) consists of the $s-1$ equations:

$$\omega_i - l_i\omega_1 = 0 \qquad (i = 2, \cdots, s).$$

and, consequently, that the equations of (4.36) could be chosen in such a way that:

$$\omega_2' \equiv \omega_3' \equiv \cdots \equiv \omega_s' \equiv 0 \qquad (\operatorname{mod} \omega_1, \cdots, \omega_s).$$

Before turning to a detailed study of the derived system and exploiting its properties to integrate the Pfaffian system under examination, it is necessary, Cartan observed, to examine carefully the geometric properties of the (unique) linear complex associated to (4.36). A first problem to be solved is the determination of the maximal dimension of (regular) integral elements and consequently the maximal dimension of (regular) integral varieties. In

[74] Here and in what follows, $l_i$, $(i = 1, \cdots, s)$ indicate $s$ arbitrary functions of $x_1 . \cdots, x_r$.

[75] For a detailed discussion of the notion of derived system with special emphasis on duality, see [Stormark 2000, p. 24-26]

the light of Weber's results, one may expect that characteristic elements play a role of strategic importance and, indeed, it turns out that this is the case also for Cartan's treatment of the subject.

Cartan started by considering the linear variety of *all* linear integral elements of (4.36); he indicated it with $H_\rho$, where $\rho$ designates its dimension that is, $\rho = r - s$. He supposed that $\sigma$ is the dimension of the greatest characteristic element $\epsilon_\sigma$ and then considered a linear integral element $E_1 \notin \epsilon_\sigma$; since $E_1$ is supposed to be regular and the character of (4.36) is assumed to be equal to 1, its polar element $H_{\rho-1}$, is a linear variety of dimension $\rho - 1$. Now, with respect to the linear integral elements of $H_{\rho-1}$, Cartan observed, a characteristic element $\epsilon_{\sigma+1}$ of dimension $\sigma + 1$ exists such that $\epsilon_{\sigma+1} =< \epsilon_\sigma, E_1 >$[76]. It turns out, as Cartan demonstrated in full detail, that $\epsilon_{\sigma+1}$ is the greatest characteristic element with respect to $H_{\rho-1}$. At this point, he considered another linear integral element $E_1'$ not belonging to $\epsilon_{\sigma+1}$; thus, the linear elements of $H_{\rho-1}$ in involution with $E_1'$ generate a linear variety $H_{\rho-2}$ whose greatest characteristic element $\epsilon_{\sigma+2}$, Cartan demonstrated, can be described as the linear variety $< \epsilon_\sigma, E_1, E_1' >$. Iterating the same process an appropriate number of times, one finally arrives at a (necessarily integral) element $H_{\rho-\nu}$[77] which coincides with its characteristic element $\epsilon_{\sigma+\nu}$. The number $\nu$ is obtained by equating the dimension of $H_{r-\nu}$ with that of $\epsilon_{\sigma+\nu}$; thus:

$$\rho - \sigma = 2\nu.$$

Consequently, since $H_{\rho-\nu}$ is one of the integral elements of (4.36) of maximal dimension, the *genre* of the Pfaffian system is $\rho - \nu$[78].
Now, from the previous geometrical construction of maximal integral elements, it follows that the bilinear covariant $\omega_1'$ must be expressed in terms of $2\nu$ independent Pfaffian forms, so that:

$$\omega_1' = \bar{\omega}_1 \wedge \bar{\omega}_{\nu+1} + \cdots \bar{\omega}_\nu \wedge \bar{\omega}_{2\nu} \quad (\text{mod } \omega_1, \cdots, \omega_s). \tag{4.46}$$

This formula is the starting point for the subsequent analytical study of integral varieties of systems of character one.

To emphasize the novelty of Cartan's technical tools with respect to those utilized by von Weber, let us consider his demonstration of the theorem according to which if $\nu > 1$ then the derived system of (4.36) is completely integrable. Cartan supposed that the Pfaffian forms of (4.36) are chosen in such a way that its derived system can be written as:

$$\omega_2 = \omega_3 = \cdots = \omega_s = 0.$$

[76]The fact that $\epsilon_{\sigma+1}$ is characteristic with respect to $H_{\rho-1}$ means that $e \in \epsilon_{\sigma+1}$ if, and only if $\omega_i(e) = 0$, $(i = 1, \cdots, s)$ and $\omega_1'(e, e') = 0 \ \forall e' \in H_{\rho-1}$. The statement that $< \epsilon_\sigma, E_1 >$ is characteristic with respect to $H_{\rho-1}$ should now be clear if one recalls the definition of polar element.

[77]Cartan indicated with $h$ what here is indicated with $\nu$. The change in notation is aimed at facilitating the comparison with [von Weber 1898].

[78]This is in accordance with von Weber's normal form (4.24) for the case in which $\nu > 1$.

From Cartan's definition of derived system it follows that:

$$\omega_2' \equiv \omega_3' \equiv \cdots = \omega_s' \equiv 0 \qquad (\mathrm{mod}\ \omega_1, \cdots, \omega_s). \tag{4.47}$$

As a consequence of $\omega_2' \equiv 0$, $(\mathrm{mod}\ \omega_1, \cdots, \omega_s)$, one has:

$$\omega_2' \equiv \omega_1 \wedge \chi,\ (\mathrm{mod}\ \omega_2, \cdots, \omega_s), \tag{4.48}$$

where $\chi$ is a form of degree one which depends upon $\omega_1, \bar{\omega}_1, \cdots, \bar{\omega}_{2\nu}$[79]. By calculating the derivative of the last congruence, one obtains that

$$\omega_1' \wedge \chi - \chi' \wedge \omega_1 \equiv 0 \quad (\mathrm{mod}\ \omega_2, \cdots, \omega_s; \omega_1', \cdots, \omega_s')$$

and, consequently, that $\omega_1' \wedge \chi \equiv 0$, $(\mathrm{mod}\ \omega_1, \cdots, \omega_s)$. This is equivalent to:

$$\omega_1' = \chi \wedge \pi + \omega_1 \wedge \pi_1 + \cdots + \omega_s \wedge \pi_s,$$

for appropriate forms of degree one, $\pi, \pi_1, \cdots, \pi_s$. Now, unless $\chi \equiv 0$ $(\mathrm{mod}\ \omega_1)$, the integral elements of (4.36) that satisfy $\chi = 0$ and $\pi = 0$ would be characteristic elements and consequently the number of equations of the characteristic system would be $s+2$. However, in such an eventuality, we would have $\nu = 1$ which contradicts the hypothesis of the theorem to be demonstrated. Thus, the only possibility is that $\chi \equiv 0$ $(\mathrm{mod}\ \omega_1)$. As result of this, from (4.48) we obtain: $\omega_2' \equiv 0$ $(\mathrm{mod}\ \omega_2, \cdots, \omega_s)$, and, after repeating the same reasoning for $\omega_3, \cdots, \omega_s$, we finally deduce that :

$$\omega_2' \equiv \omega_2' \equiv \cdots \equiv \omega_s' \equiv 0 \quad (\mathrm{mod}\ \omega_2, \cdots, \omega_s),$$

which, according to Cartan's reformulation of Frobenius' theorem[80], implies the complete integrability of the derived system of (4.36).

* * * * * * * * * * * * * * * * * * * *

Contrary to prevalent opinion, far from being the result of the work of an isolated mathematical genius, Cartan's theory of exterior differential systems (later on generalized by Kähler in [Kähler 1934] to differential systems of any degree), was deeply rooted in the historical context of the late nineteenth century theory of partial differential equations. Indeed, as we have seen, his achievements were situated at the intersection of two closely related strands of research: the theory of not completely integrable systems of Pfaffian equations as developed by Engel and von Weber, and the theory of general systems of partial differential equations that was the main focus of attention of Méray, Riquier and Delassus, among others. Cartan's great

[79] This is due to the fact that, according to (4.46), the characteristic element of maximal dimension is individuated by: $\bar{\omega}_i = 0$ $(i = 1, \cdots, 2\nu)$.

[80] See [Cartan 1901b, p. 247] and [Hawkins 2005, p. 429].

merit was to reinterpret them systematically in a new and powerful geometrical language whose central core was represented by his exterior differential calculus. At the same time, the very emphasis given by him on the language of exterior differential forms may be indicated as the main cause for the undeserved scarcity of attention that characterized for some years the response of the mathematical community towards his achievements in this field. In this connection it is interesting that still in 1924 Vessiot in [Vessiot 1924], while praising the beauty of Cartan's integration theory, felt the necessity to translate it into its dual counterpart by replacing exterior Pfaffian forms with the notion of *faisceau* of infinitesimal transformations.
As Kähler suggested in the introduction to his masterpiece [Kähler 1934] in a really effective and historically accurate way, such a double historical origin was reflected in the twofold virtue of the theory: on one hand, with its emphasis on exterior forms, it yielded to Cartan the necessary tools for the subsequent applications to geometry (namely, the method of moving frames) as well as to the theory of infinite continuous Lie groups[81]. On the other hand, it offered a deeper insight into the machinery (*Mechanik*) of partial differential equations.

[81] The next chapter is devoted precisely to a detailed discussion over this point.

# Chapter 5

# Cartan's theory (1902-1909)

## 5.1 On the genesis of the theory

It rarely happens that history of mathematics develops through radical changes; gradualness seems to be the general rule according to which the mathematical thought evolves. When historians think that they have singled out some sudden and unexpected breakthrough, it is usually the case that their impression is dictated by a poor comprehension of the historical context in which some events took place.

On the base of this premise, it might seem hazardous to argue in favour of the radicalness of Cartan's innovations in the realm of infinite continuous groups. Nevertheless, Cartan's theory appears as one of those rare cases in history of mathematics in which continuity gives way to radical change.

Cartan's approach to infinite continuous groups was indeed characterized by a sharp break with the past tradition dating back to Lie, Engel, Medolaghi and Vessiot. Such a discontinuity involved at least two aspects of the theory: the technical tools employed and the priorities of the theory itself. On the technical side, Cartan took great profit of his theory of exterior differential systems, namely of his existence and uniqueness results for not integrable Pfaffian systems. As far as priorities were concerned, Cartan's theory was marked by a radical change of perspective: no more the emphasis was put on the problem of determining, modulo similarities, all infinite continuous groups of transformations in a given number of variables; rather, what Cartan considered to be essential was to develop a structural theory of such groups in which the notion of isomorphism played a central role. Clearly, the problem of the classification of infinite groups was still considered of great importance by him, nevertheless it was regarded more as a particular application of his theory than as a starting point.

Cartan himself was well aware of the place that his theory occupied in the history of infinite continuous groups. The remarks contained in his *Notice sur les travaux scientifiques* emphasized its character of radicalness. After providing a sketchy historical account of the development of the theory from Lie's foundational work to Vessiot's more recent contributions, he wrote:

> *These, with the exception of Picard's researches of 1901 on certain infinite groups which generalize the general group of analytic transformations in one complex variable, were the works that had been published on the general theory of infinite groups when I started my researches on the same subject. My researches are completely independent of the preceding ones both for what pertains to methods and aim sought for. Instead of dealing with the determination of infinite groups in a given number of variables, I set out to erect a general theory of* isomorphism *of continuous groups; a theory which had to lead, as particular application, to the effective determination of all groups in n variables.*[1]

The reason for the necessity of such a radical change of conceptual framework was indicated by Cartan in the impossibility of extending Lie's structural approach in terms of infinitesimal transformations to infinite groups. Indeed, in the case of finite continuous groups, their structure could be deduced from the consideration of the infinitesimal transformations whose Lie brackets provide the so-called structure constants $c_{ijk}$. The entire theory of the classification of groups was essentially nothing else than a classification of all different algebraic systems of constants $c_{ijk}$, which are subjected to the constraints imposed by antisymmetry and by Jacobi's condition. On the contrary, in the infinite dimensional case, since, as Lie had demonstrated in his *Grundlagen*, groups are generated[2] by an infinite number of independent infinitesimal transformations, the corresponding Lie algebra is infinite dimensional and thus the possibility of exploiting its properties for the task of an algebraic classification is heavily jeopardized.

These preliminary remarks being stated, one can try to understand the historical development which led Cartan to conceive his theory in the terms that will be thoroughly described later on. Indeed, despite the break with the past theoretical framework, it is possible to describe the genesis process of his theory by taking into consideration Cartan's interests of research in the early 1900's. Again, his recollections in [Cartan 1939] turn out to be particularly useful as therein Cartan went back over the conceptual course

[1] *Tels étaient, abstraction faite des recherches faites par M. Picard en 1901 sur certains groupes infinis qui généralisent le groupe général des transformations analytiques d'une variable complexe, les travaux qui avaient paru sur la théorie générale des groupes infinis, lorsque j'ai commencé mes recherches sur la même sujet. Mes travaux sont tout à fait indépendants des travaux précédents, quant aux méthodes et même quant au but poursuivi. Au lieu de m'occuper de la détermination des groupes infinis à un nombre donné de variables, je me suis proposé de fonder une théorie générale de l'*isomorphisme *des groupes continus, théorie qui devait conduire,* comme application particulière*, à la détermination effective de tous les groupes à n variables.* [Cartan 1939, p. 52].

[2] On the real significance of the word *generate*, see the detailed discussion in chapter 3.

which led him to conceive his treatment of infinite continuous groups. It seems that this recollection may be considered reliable since it is supported by attentive analysis of the relevant papers.

Cartan traced back the origin of his theory to two, at first sight unrelated, strands of research: the integration theory of completely integrable (in the sense of Clebsch's definition) systems of first order linear homogeneous differential equations which admit infinitesimal transformations and the theory of equivalence of what nowadays we would call differential structures.

Let us begin with the theory of completely integrable systems. In 1896 Cartan had succeeded in completely solving the integration problem which Lie had singled out in [Lie 1885]. To this end the classification of all simple, complex Lie algebra had turned out to be essential[3]. In 1902, he decided to go back over the same subject with the aim of investigating the possibility of using the new technical tools provided by his theory of exterior differential forms. This plan materialized in a couple of very brief and sketchy notes, [Cartan 1902b] and [Cartan 1902c], communicated by Picard to the *Académie des Sciences* in 1902.

Cartan started in [Cartan 1902b] by considering a completely integrable Pfaffian system $\mathcal{P}$ consisting of $r$ independent 1-forms in $n$ variables:

$$\begin{cases} \omega_1 = a_{11}dx_1 + \cdots + a_{1n}dx_n = 0, \\ \qquad\cdots \\ \omega_r = a_{r1}dx_1 + \cdots + a_{rn}dx_n = 0. \end{cases} \tag{5.1}$$

If $u_1, \cdots, u_r$ indicate $r$ independent integral of $\mathcal{P}$, then Cartan defined a Pfaffian form $\varpi = \alpha_1 dx_1 + \cdots + \alpha_n dx_n$ to be an integral differential expression[4] (*expression différentielle intégrale*) if, under an appropriate change of coordinates, $\varpi$ can be rewritten in the following form:

$$\varpi = A_1 du_1 + \cdots + A_r du_r,$$

where the coefficients $A_1, \cdots, A_r$ are functions of the integral $u_1, \cdots, u_r$ only. The knowledge of such Pfaffian forms, Cartan observed, could profitably be applied in order to simplify the integration procedure of the system. Indeed, he claimed, if the system $\mathcal{P}$ admits the infinitesimal transformation

$$X(f) = \xi_1 \frac{\partial f}{\partial u_1} + \cdots + \xi_n \frac{\partial f}{\partial u_n},$$

then the function $(X, \varpi) = \sum_{i=1}^{n} \xi_i \alpha_i$ is an integral of $\mathcal{P}$.[5] Furthermore, if $r$ linearly independent integral differential expressions $\varpi_1, \cdots, \varpi_r$ are known

[3]See Section 2.3.

[4]On the importance of the notion of integral differential expression for the development of Cartan's theory of groups, see also the introduction to [Cartan 1922, p. V].

[5]Cartan did not provide any explicit proof of the statement, however it is not difficult to provide one. Let $u_1, \cdots, u_r$ indicate $r$ independent integrals of $\mathcal{P}$. By definition $\mathcal{P}$ admits

and if $U$ is an integral of $\mathcal{P}$, then new integrals $U_1, \cdots, U_r$ which are defined by the following relation:

$$dU = U_1\varpi_1 + U_2\varpi_2 + \cdots + U_r\varpi_r$$

can be obtained.

More generally, Cartan observed, once $r$ independent integral differential expressions are known, there exists a general procedure leading to the integration of $\mathcal{P}$. Such a method relied upon the following theorem which Cartan formulated without providing any proof thereof:

**Theorem 20 (Cartan 1902)** *The necessary and sufficient condition in order for the $\omega_1, \cdots, \omega_r$ to be integral differential expressions of the system $\mathcal{P}$ is that their bilinear covariants $d\omega_1, \cdots, d\omega_r$ can be expressed by the following formulas:*

$$d\omega_i = \sum_{\lambda,\mu=1}^{r} \frac{1}{2} c_{i\lambda\mu}\omega_\lambda \wedge \omega_\mu. \tag{5.2}$$

*Furthermore, if this is the case, then the functions $c_{i\lambda\mu}$ are integrals of $\mathcal{P}$.*[6]

the infinitesimal transformation $X = \sum_{i=1}^n \xi_i(x)\frac{\partial}{\partial x_i}$ if and only if $X(u_j) = f_j(u_1, \cdots, u_r)$, $j = 1, \cdots, r$, for appropriate functions $f_j$, $j = 1, \cdots, r$. Under the following change of coordinates:

$$(x_1, \cdots, x_n) \mapsto (\bar{x}_1, \cdots, \bar{x}_n) = (u_1(x), \cdots, u_r(x), x_{r+1}, \cdots, x_n)$$

$X$ is transformed into $X = \sum_{i=1}^r f_i(u_1, \cdots, u_r)\frac{\partial}{\partial u_i} + \sum_{i=r+1}^n \xi_i \frac{\partial}{\partial x_i}$. Now, by hypothesis, under the same change of coordinates $\varpi$ is transformed into $\varpi = \sum_{i=1}^r A_j(u_1, \cdots, u_r)du_j$; thus, $(X, \varpi) = \sum_{j=1}^r A_j(u_1, \cdots, u_r)f_j(u_1, \cdots, u_r)$, i.e. $(X, \varpi)$ is an integral of $\mathcal{P}$. Note that $(X, \varpi)$ is independent of the choice of coordinates employed to compute it.

[6]Let us see in some detail how one can prove the theorem. As it is implicitly assumed by Cartan, we suppose that the Pfaffian forms $\omega_1, \cdots, \omega_r$ are linear combinations of the differentials of the integral of the system $\mathcal{P}$, i.e. $\omega_j = \sum_{i=1}^r f_{ji}du_i$, $j = 1, \cdots, r$, where $f_{ji}$ are functions of $x_1, \cdots, x_n$. The condition stated in the theorem is clearly necessary. One has to prove that it is also sufficient. To this end, let us compute the bilinear covariants of $\omega_1, \cdots, \omega_r$.

$$\begin{aligned} d\omega_j &= \sum_{i=1}^r df_{ji} \wedge du_i = \sum_{\lambda,\mu=1}^r \frac{1}{2}c_{j\lambda\mu}\omega_\lambda \wedge \omega_\mu = \\ &= \sum_{\lambda,\mu=1}^r \sum_{i,k=1}^r \frac{1}{2}c_{j\lambda\mu}f_{\lambda i}f_{\mu k}du_i \wedge du_k, \end{aligned} \tag{5.3}$$

for $j = 1, \cdots, r$. From this, it follows that

$$\sum_{k=1}^r \left( df_{jk} - \sum_{i=1}^r \sum_{\lambda,\mu=1}^r \frac{1}{2}c_{j\lambda\mu}f_{\lambda i}f_{\mu k}du_i \right) \wedge du_k = 0.$$

As a consequence of this

$$df_{jk} - \sum_{i=1}^r \sum_{\lambda,\mu=1}^r \frac{1}{2}c_{j\lambda\mu}f_{\lambda i}f_{\mu k}du_i \equiv 0, \qquad mod(du_1, \cdots, du_r), \qquad j, k = 1, \cdots, r.$$

By applying the above mentioned remarks, from the integrals $c_{i\lambda\mu}$ one can deduce other integrals of the system. After iteration of the method, one finally is led to the case in which $r - s$ independent integrals of $\mathcal{P}$ have been derived and no other integral can be obtained by further application of the described procedure. By equating these $r - s$ integrals of the system to an equal number of arbitrary constants, one is finally brought back to the case in which there exist $s$ independent integral differential expressions whose corresponding coefficients $c_{i\lambda\mu}$ are constants. It turned out, Cartan observed, that such constants define the structure of a finite continuous group $G$. Depending on the structure of $G$, the integration of $\mathcal{P}$ could then be traced back to the integration of a series of systems of canonical form which are associated to simple continuous groups.

Cartan regarded his theory as a generalization of Lie's theory of integration of complete systems which admits a symmetry group of transformations. Indeed, he observed, if the system $\mathcal{P}$ admits a $r$-parameters, transitive group $G = \{X_1, \cdots, X_r\}$ such that $det[X_j(\omega_i)] \neq 0$, then $r$ independent Pfaffian forms, $\omega_1, \cdots, \omega_r$ could be chosen in such a way that $(X_i, \omega_j) = \delta_{ij}$. As a consequence of theorem (20), $\omega_1, \cdots, \omega_r$ are integral differential forms and the coefficients $c_{\lambda\mu i}$ of formula (5.2) coincide with the structure constants of the group $G$.[7]

However, in the general case studied by Cartan, no integral Pfaffian forms for the system $\mathcal{P}$ are known and one has to provide a method in order to exploit the existence of symmetries of the system $\mathcal{P}$ to simplify the integration procedure. To this end, Cartan pointed out that, in any case, the differential forms $\omega_1, \cdots, \omega_r$ could be obtained as linear combinations with non constant coefficients of $r$ *unknown* integral differential forms $\Omega_1, \cdots, \Omega_r$, that is as the result of the action of a transformation of the linear general group. Besides, he pointed out that the knowledge of some infinitesimal transformation which is admitted by $\mathcal{P}$ as well as that of some integral of $\mathcal{P}$, could be used to reduce this group to some subgroup $\Gamma$ of the linear general group.

On the basis of these observations, Cartan formulated his own generalization of Lie's integration problem in the following way:

*Consider a completely integrable Pfaffian system*

$$\omega_1 = \omega_2 = \cdots = \omega_r = 0,$$

Thus, $df_{jk} \equiv 0$, $mod\ (du_1, \cdots, du_r)$, $j, k = 1, \cdots, r$. We conclude that $f_{jk}$, $j, k = 1, \cdots, r$ are functions of $u_1, \cdots, u_r$ only, that is $\omega_j$, $j = 1, \cdots, r$ are differential integral Pfaffian forms. That $c_{i\lambda\mu}$ are integrals of $\mathcal{P}$ is a simple consequence of the fact that the Pfaffian forms $\omega_i$, $i = 1, \cdots, r$ depend only upon $u_1, \cdots, u_r$.

[7] In modern terms, this is a simple consequence of the fact that the Pfaffian forms $\omega_1, \cdots, \omega_r$ have been chosen in such a way to be the differential forms dual to the infinitesimal generators of the group $G$.

> *to integrate such a system, knowing that the first members $\omega_1, \cdots, \omega_r$ are deduced from (unknown) integral expressions $\Omega_1, \cdots, \Omega_r$ by means of a linear substitution of a known linear group $\Gamma$, the coefficients of such transformation possibly depending upon functions of independent and dependent variables.* [8]

It was the very necessity to develop a general method to handle with this problem that lead Cartan to pursue a detailed study of the equivalence problem of Pfaffian forms with respect to some subgroup of the general linear group. Indeed, as he explained in [Cartan 1939, p. 49], his theory of equivalence offered a systematic procedure to tackle this task. For example, it provided a method to compute the invariants of $\omega_i$, $i = 1, \cdots, r$ with respect to the group $\Gamma$ from which integrals of the Pfaffian system under examination could be deduced.

Cartan was thus led to the second strand of research mentioned above: the theory of equivalence. We will see how this theory was intimately connected to his structural theory of infinite continuous groups. From the brief note published in the *Comptes Rendus* we know that already in 1902 Cartan had at his disposal all the essential elements of the theory. A more complete (though sometimes hard reading and obscure) treatment of the subject was provided in 1908, in the first chapter of [Cartan 1908]. Finally, a more accessible account of the theory was offered in 1937 in the course of a conference for the *Séminaire de Mathématiques* whose printed version was published in Cartan's *Œuvres* (second volume of the second part, pp. 1311-1334). Here is the formulation of the problem which Cartan's theory of equivalence deals with:

> *Given, on one hand, a system of $n$ linearly independent Pfaffian expressions in $x_1, x_2, \cdots, x_n$ and $m$ independent functions of $m$; on the other, a system of $n$ linearly independent Pfaffian expressions $\Omega_1, \Omega_2, \cdots, \Omega_n$ in $X_1, X_2, \cdots, X_n$ and $m$ independent functions $Y_1, Y_2, \cdots, Y_m$ of these variables $X$. Determine if there exists a change of variables which transforms the functions $y_1, y_2, \cdots, y_m$ in the functions $Y_1, Y_2, \cdots, Y_m$, respectively, also such that, by this change of variables, $\Omega_1, \Omega_2, \cdots, \Omega_n$ can be deduced from $\omega_1, \omega_2, \cdots, \omega_n$ by means of a linear substitution which belongs to a given linear group $\Gamma$, the coefficients of the finite*

[8] *Étant donné un système de Pfaff complétament intégrable*

$$\omega_1 = \omega_2 = \cdots = \omega_r = 0,$$

*intégrer ce système, sachant que les premiers membres $\omega_1, \cdots, \omega_r$ se déduisent d'expressions intégrales (inconnues) $\Omega_1, \cdots, \Omega_r$ par une substitution linéare appartenant à un groupe linéaire connu $\Gamma$, les coefficients de cette substitution pouvant être des fonctions des variables, dépendantes et indépendentes.*. See [Cartan 1939, p. 49].

*equations of this group possibly depending upon $y_1, y_2, \cdots, y_m$.*[9]

Cartan was able to provide a complete solution of the problem which consisted of giving necessary and sufficient conditions which guarantee the existence of diffeomorphisms exhibiting the equivalence. The main result of the theory, in view of Cartan's approach of infinite groups, was the following: every transformation mapping $x$ in $X$ which has the required property is the combination of a local equivalence followed by a transformation (a self-equivalence) which maps the $X$'s into themselves. Now, it turns out that these latter transformations form a continuous group; furthermore (and this is a crucial point) such a group is defined as the group of those transformations that leave invariant certain functions of $X$ and certain Pfaffian forms[10]. It was then natural for Cartan to ask the question about the possibility of characterizing *every* continuous group as the set of those transformations characterized by invariance properties of functions and Pfaffian forms.

Let us see in some detail how Cartan's theory of equivalence works in the simplest conceivable case in which there are no functions $y_1, \cdots, y_m$ and the linear group $\Gamma$ reduces to the identity transformation[11]. Our account follows the treatment provided by Cartan in [Cartan 1908, p. 60-63].

Cartan considered two systems of $n$ Pfaffian systems in $n$ independent variables. The first one is:

$$\begin{cases} \omega_1 = a_{11}dx_1 + a_{12}dx_2 + \cdots + a_{1n}dx_n, \\ \omega_2 = a_{21}dx_1 + a_{22}dx_2 + \cdots + a_{2n}dx_n, \\ \qquad \cdots, \\ \omega_n = a_{n1}dx_1 + a_{n2}dx_2 + \cdots + a_{nn}dx_n. \end{cases} \tag{5.4}$$

The other one in the variables $X_1, X_2, \cdots, X_n$ is:

$$\begin{cases} \Omega_1 = A_{11}dX_1 + A_{12}dX_2 + \cdots + A_{1n}dX_n, \\ \Omega_2 = A_{21}dX_1 + A_{22}dX_2 + \cdots + A_{2n}dX_n, \\ \qquad \cdots, \\ \Omega_n = A_{n1}dX_1 + A_{n2}dX_2 + \cdots + A_{nn}dX_n. \end{cases} \tag{5.5}$$

He set out to determine if there exist functions $X_1, \cdots, X_n$ of $x_1, \cdots, x_n$

[9] *Étant donnés d'une part un système de $n$ expressions de Pfaff linéairement indépendentes $\omega_1, \omega_2, \cdots, \omega_n$ en $x_1, x_2, \cdots, x_n$ et $m$ fonctions indépendent $y_1, y_2, \cdots, y_m$ des $x$; d'autre part un système de $n$ expressions de Pfaff linéairement indépendentes $\Omega_1, \Omega_2, \cdots, \Omega_n$ en $X_1, X_2, \cdots, X_n$, et $m$ fonctions indépendentes $Y_1, Y_2, \cdots, Y_m$ de ces $n$ variables $X$; recconaître s'il existe un changement de variables transformant respectivement les fonctions $y_1, y_2, \cdots, y_m$ dans les fonctions $Y_1, Y_2, \cdots, Y_m$, et tel de plus que, par ce changement de variables, $\Omega_1, \Omega_2, \cdots, \Omega_n$ se déduisent de $\omega_1, \omega_2, \cdots, \omega_n$ par une substitution linéare appartenant à une groupe linéaire donné $\Gamma$, les coefficients des équations finies de ce groupe pouvant dépendre de $y_1, y_2, \cdots, y_m$.*

[10] See [Cartan 1939, p. 45].

[11] Nowadays, this case is called equivalence of $e$-structures.

such that the $\Omega_j$, $j = 1, \cdots, n$ reduce to $\omega_j$, $j = 1, \cdots, n$.[12] To this end, he observed, a key role was played by the employment of the exterior derivatives (*covariants bilinéaires*) of $\omega_i$ and $\Omega_i$, $i = 1, \cdots, n$. He indicated with $d\omega_i = \frac{1}{2}\sum_{j,k=1}^{n} c_{jki}(x)\omega_j \wedge \omega_k$ and with $d\Omega_i = \frac{1}{2}\sum_{j,k=1}^{n} C_{jki}(X)\Omega_j \wedge \Omega_k$ the exterior derivatives of $\omega_i$ and $\Omega_i$, $i = 1, \cdots, n$, respectively. From the fact that $\Omega_i = \omega_i$, $i = 1, \cdots, n$, Cartan deduced that $C_{jki}(X) = c_{jki}(x)$, $j, k, i = 1, \cdots, n$, obtaining in such a way a first set of necessary conditions for the existence of the equivalence. He supposed that among the $n^3$ functions $c_{jki}(x)$ there exist $r$ of them which are independent and such that all the remaining ones can be expressed in terms of them. Indicating these functions with $y_1, \cdots, y_r$, Cartan computed their differentials.

If the basis of the differential forms is chosen to coincide with the coframe $\{\omega_j\}_{j=1}^n$, then these differentials can be put in the following form:

$$\begin{cases} dy_1 = h_{11}\omega_1 + h_{12}\omega_2 + \cdots + h_{1n}\omega_n, \\ dy_2 = h_{21}\omega_1 + h_{22}\omega_2 + \cdots + h_{2n}\omega_n, \\ \qquad \cdots, \\ dy_r = h_{r1}\omega_1 + h_{r2}\omega_2 + \cdots + h_{rn}\omega_n. \end{cases} \tag{5.6}$$

Cartan supposed that among the $nr$ functions there exists $r_1 - r$ of them which are independent. By repeating the same reasoning, he then deduced the existence of $p \leq n$ independent functions of $x$, which satisfy the following properties: i) the coefficients $c_{jki}$ are functions of $y_1, \cdots, y_p$; ii) the coefficients of the differentials $dy$ expressed in terms of $\{\omega_i\}_{i=1}^n$ are functions of $y_1, \cdots, y_p$ only, as well. He then applied the same procedure to the Pfaffian forms $\Omega_1, \Omega_2, \cdots, \Omega_n$. If there exists a transformation mapping the $\Omega_i$'s into the $\omega_i$, he claimed, there exist $p$ functions $Y_1, \cdots, Y_p$ of the variables $X_1, \cdots, X_n$ such that the functions $C_{jki}$ and the coefficients $H_{ik}$ of $dY$ with respect to the basis $\{\Omega_i\}_{i=1}^n$ can be expressed in terms of $Y_1, \cdots, Y_p$, only. Furthermore, these functions of $Y$ coincide with those functions of $y$ associated to the coframe $\{\omega_i\}_{i=1}^n$.
This condition is clearly necessary for the equivalence of the two coframes. Cartan proved that it is also sufficient. The proof provided by Cartan relied on a technique, later on to be called *Cartan's graph technique*, which consisted of constructing the diffeomorphism $X_i = X_i(x)$ in a purely geometrical way, starting from integral varieties of appropriate Pfaffian systems which, under certain regularity conditions, could be interpreted as graphs of the diffeomorphisms sought after.

In a more general setting Cartan's integration theory of involutive Pfaffian system must be invoked. However, in the simple case considered herein,

[12] In modern terms, the problem can be formulated as follows. Given a coframe $\{\omega_j\}_{j=1}^n$ on a $n$-dimensional differential manifold $M$ and a coframe $\{\Omega_j\}_{j=1}^n$ on a $n$-dimensional differential manifold $\bar{M}$, one wants to determine, if it exists, a diffeomorphism $\Phi : M \to \bar{M}$ such that $\Phi^*\Omega_i = \omega_i$, $i = 1, \cdots, n$.

Frobenius' theorem finds application. Indeed Cartan considered the mixed Pfaffian system:

$$\begin{cases} Y_1 - y_1 = 0, \\ Y_2 - y_2 = 0, \\ \cdots, \\ Y_p - y_p = 0, \\ \Omega_1 - \omega_1 = 0, \\ \cdots, \\ \Omega_n - \omega_n = 0, \end{cases} \tag{5.7}$$

and he observed that it is completely integrable in the sense that its bilinear covariants vanish as a consequence of the system itself. In virtue of of Frobenius' theorem, thus there exists a $(n-p)$-parameter family of equivalences. Furthermore, Cartan pointed out, the most general transformation mapping $\{\omega_i\}_{i=1}^n$ into $\{\Omega_i\}_{i=1}^n$ could be represented as the composition of an arbitrary transformation joining of this property followed by a transformation of the continuous group generated by those transformations that leaves the Pfaffian forms $\omega_1, \cdots, \omega_n$ invariants. As it will be explained, Cartan's structural theory allowed to interpret this result as a particular case of a far more general situation.

Indeed this special circumstance emerging from Cartan's original reinterpretation of Lie's equivalence theory was promoted by him to the rank of a heuristic principle. In this way, a key technical ingredient of his theory, that is the possibility of characterizing the transformations of an arbitrary (infinite) continuous group by means of the invariance properties of a system of Pfaffian equations, made its appearance.

This noteworthy aspect of the genesis of the theory was emphasized by Ugo Amaldi on the occasion of the commemorative ceremony of Cartan's death at the *Accademia dei Lincei* on 12th June 1952 as follows:

> *By taking a step further along the way thus disclosed, Cartan tackled, for Pfaffian systems, "the general problem of equivalence" with respect to an arbitrary continuous group of transformations, which Lie, with completely different devices, had already discussed for systems of partial differential equations written in traditional form. However, Cartan's deductions and conclusions, in view of their geometrical applications, are more expressive and full of consequences. In particular, Cartan deduced from them, we would say almost in an experimental way, the observation - so unexpected as to constitute a fundamental discovery - that every conceivable continuous group in Lie's sense (i.e. definable by means of differential equations) can be characterized as the family of all those transformations which admit a certain finite number of invariant Pfaffian expressions, or what is the same, of invariant curvilinear integrals, which represent the basis of a*

*vector space which is close or open with respect to exterior differentiation, depending on the group being finite or infinite.*[13]

Few words on the historical origin of Cartan's equivalence theory are in order here. It hass sometimes been stated[14] that Klein's well known *Erlanger Programm* played a role in the historical development of Cartan's ideas over the subject. However, there is no evidence that Cartan in his researches during this period. referred to Klein. On the other hand, the actual impact of the *Erlanger Programm* has been convincingly questioned by [Hawkins 1984]. Indeed, it should by now be clear that the influence of Lie's views was by far more decisive. Furthermore, it should be stressed that the context in which Cartan's ideas first emerged was not that of geometry, but that of Lie's theory of first order linear PDE's.

## 5.2 Cartan's test for involutivity

Before moving to discuss Cartan's approach to the theory of infinite dimensional continuous groups, it is necessary to provide some technical details of his theory of exterior differential systems which have not been dealt with so far, namely the notion of involution with respect to a given independence condition and the numerical test which Cartan devised in order to establish if such an involution property does take places or not.
In principle the problem is quite simple: consider a Pfaffian system $\mathcal{P}$ which consists of $s$ independent 1-forms $\theta_1, \cdots, \theta_s$ in $n$ variables $x_1, \cdots, x_n$. We want to find integral varieties of dimension $p$ such as they do not produce any dependence among $p$ variables chosen in advance; in other words, we want to determine $p$ dimensional integral varieties such as they do not establish any linear relation among $p$ given linear independent Pfaffian forms $\omega_1, \cdots, \omega_p$, which are independent of $\theta_i$, $(i = 1, \cdots, s)$ as well. The relevance of this problem is easily (anachronistically) explained if one recalls that the

[13] *Con un passo ulteriore sulla via così aperta* [Amaldi referred to Cartan's theory of Pfaffian systems] *affrontò per i sistemi Pfaffiani "il problema generale dell'equivalenza" rispetto ad un qualsiasi gruppo continuo di trasformazioni, che già il Lie, con tutt'altri mezzi, aveva discusso per i sistemi a derivate parziali sotto la loro forma tradizionale; ma le deduzioni e le conclusioni del Cartan, soprattutto ai fini delle applicazioni geometriche, sono più espressive e più feconde di conseguenze. In particolare il Cartan ne trasse, quasi si direbbe, per via sperimentale, la constatazione - tanto inattesa da costituire una vera e fondamentale scoperta - che ogni possibile gruppo continuo nel senso del Lie (cioè definibile per mezzo di equazioni differenziali) si può caratterizzare come la famiglia di tutte e sole le trasformazioni, che ammettono un certo numero finito di pfaffiani invarianti o - se si vuole - di invarianti integrali curvilinei, costituenti la base di un insieme lineare, che, rispetto alla differenziazione esterna, risulta chiuso od aperto, secondo che il gruppo considerato è finito o infinito* [...]. See [Amaldi 1952, p. 769-770].

[14] See, for instance, [Gardner 1989, p. V].

formulation of a system of partial differential equations in terms of the contact Pfaffian system and the corresponding submanifold $\mathcal{R}$ of the jet bundle $J^k(\mathbb{C}^p, \mathbb{C}^r)$ requires the imposition of an independence condition of the type above described, namely, if $x_1, \cdots, x_p$ indicate the set of independent variables,

$$\Omega|_{\mathcal{R}} = dx_1 \wedge \cdots \wedge dx_p|_{\mathcal{R}} \neq 0.$$

Beyond the notation employed, it is likely that Cartan was referring exactly to applications of this type when he stated that his theory of exterior differential system needed to be completed with an adequate treatment of the problem just described whose practical importance, he said, was *evident*. Cartan tackled the resolution of the problem in the first chapter of a long mémoire published in two parts between 1904 and 1905 which, with the exception of a brief communication dating back to 1902, represents Cartan's first contribution to the theory of infinite groups. It is of no surprise that the just stated problem of involution with independence condition was carried out by Cartan on this very occasion; indeed, such a collocation must have appeared quite natural since, as we already know, according to Lie's approach, the transformations of an infinite continuous group are defined precisely as solutions to system of PDE's.

Our account will follow as close as possible Cartan's original analysis; at the same time, hopefully without distorting the letter of Cartan's work, we will take great profit from Stormark's recent treatment[15] of the subject where many technical points find adequate clarification. Cartan's treatment can be translated, as Stormark did, in modern terms by means of the language of differential geometry. However, we decided not to do so. The persistent ambiguity which afflicts Cartan's papers about the meaning to attribute to Pfaffian forms, somewhere regarded as differential forms, elsewhere regarded as components of vector fields, has been conserved. The necessity of historical adherence is considered to be prevalent with respect to clarity. The only modification of Cartan's notation which we have adopted will be the employment of the wedge product symbol.

Cartan's starting point consists of considering the following coframe[16] of $n$ independent Pfaffian forms obtained by adding to $\theta_i$ and $\omega_j$, for $k = 1, \cdots, s$ and $j = 1, \cdots, p$, $q = n - s - p$ linearly independent 1-forms (linearly independent of the preceding $p + s$ ones, as well), $\varpi_1, \cdots, \varpi_q$.
As a consequence of this, the bilinear covariants of the original Pfaffian

[15] See [Stormark 2000].

[16] Cartan did not employed this wording, he simply spoke of a set of $n$ independent Pfaffian forms in an $n$-dimensional space.

system can be written as follows:

$$d\theta_k = \sum_{i<j} c_{ijk}\omega_i \wedge \omega_j + \sum_{i=1}^{p}\sum_{\rho=1}^{q} a_{i\rho k}\omega_i \wedge \varpi_\rho + \sum_{\rho<\sigma} b_{\rho\sigma k}\varpi_\rho \wedge \varpi_\sigma,$$

$$(mod \quad \theta_1, \cdots, \theta_s), \quad k = 1, \cdots, s. \tag{5.8}$$

The first step in Cartan's reasoning was the demonstration of the possibility of getting rid of the terms containing $\varpi_\rho \wedge \varpi_\sigma$ by prolonging the system under examination to a Pfaffian system with a higher number of variables. Indeed, Cartan observed, every $p$-dimensional integral element $E_p$ not only satisfies the equations $\theta_1, \cdots, \theta_s$ but also equations of the form:

$$\varpi_k = \sum_{j=1}^{p} l_{kj}\omega_j, \quad k = 1, \cdots, q, \tag{5.9}$$

where $l_{kj}$ are unknown functions of the variables $x_1, \cdots, x_n$. Since, by definition, two linear integral elements belonging to $E_p$ are in involution, then $l_{kj}$ have to be such as to satisfy certain compatibility conditions $F(x; l_{kj}) = 0$ obtained upon substitution of (5.9) in $d\theta_k = 0$, $(k = 1, \cdots, s)$. At this point, different eventualities may present:
i) such equations impose certain finite relations among the variables $x_1, \cdots, x_n$, which define a submanifold $M$ of the $n$ dimensional numerical space. If $\Omega|_M = 0$, then no integral element of the type looked for exists, instead if $\Omega|_M \neq 0$, then one has to start again with a Pfaffian system for which the number $q$ is reduced,
ii) the equations $F(x; l_{kj}) = 0$ are compatible and all the coefficient $l_{kj}$ can be expressed in terms of the variables $x_1, \cdots, x_n$ and other auxiliary quantities, $y_1, \cdots, y_t$, say.
If this is the case and the prolonged Pfaffian system $\mathcal{P}^{(1)}$ in the variables $x_1, \cdots, x_n, y_1, \cdots, y_t$, obtained by adding to $\theta_1, \cdots, \theta_s$ the Pfaffian forms $\theta_{s+k} = \varpi_k - \sum_{j=1}^{p} l_{kj}(x, y)\omega_j, k = 1, \cdots, q$, is considered, then it is not difficult to see that its bilinear covariants do not contain terms in $dy_i \wedge dy_j$. Indeed, by construction $d\theta_k \equiv 0$, $k = 1, \cdots, s$, mod $\mathcal{P}^{(1)}$; furthermore: $d\theta_{s+k} = d\varpi_k - \sum_{j=1}^{p} dl_{kj} \wedge \omega_j + \sum_{j=1}^{p} l_{kj}d\omega_j$, $(k = 1, \cdots, q)$. As $d\varpi_k \equiv 0$ and $d\omega_j \equiv 0$, (mod $\theta_1, \cdots, \theta_s, \omega_1, \cdots, \omega_p, \varpi_1, \cdots, \varpi_q$) for $k = 1, \cdots, q$ and $j = 1, \cdots, p$, it is also true that the same equivalences hold modulo $(\theta_1, \cdots, \theta_s, \theta_{s+1}, \cdots, \theta_{s+q}, \omega_1, \cdots, \omega_p)$. We finally obtain:

$$d\theta_{s+j} \equiv 0, \quad mod(\theta_1, \cdots, \theta_s, \theta_{s+1}, \cdots, \theta_{s+q}, \omega_1, \cdots, \omega_p),$$

i.e. the bilinear covariants of $\mathcal{P}^{(1)}$ do not contain terms in $dy_i \wedge dy_j$, $(i, j = 1, \cdots, t)$. Thus we conclude that, if necessary upon substitution of the original Pfaffian system with an appropriate prolongation, that the coefficients $b_{\rho\sigma k}$ vanish identically.

A second major simplification introduced by Cartan exploited the possibility of choosing the 1-forms $\varpi_1, \cdots, \varpi_q$ in such a way that the coefficients $c_{ijk}$ in (5.8) disappear. Indeed, it turned out that this is the case when integral elements of dimension $p$ exist; one simply had to add to $\varpi_k$, $k = 1, \cdots, q$ appropriate linear combination of $\omega_j$, $j = 1, \cdots, p$.

As a result of this Cartan was able to prove that if a $p$-dimensional integral element $E_p$ exists such that $\Omega|_{E_p} \neq 0$, then the covariants of $\mathcal{P}$ find expression in the following relation:

$$d\theta_k = \sum_{i=1}^{p} \sum_{\rho=1}^{q} a_{i\rho k} \omega_i \wedge \varpi_\rho, \quad k = 1, \cdots, s. \tag{5.10}$$

On the basis of these premises, Cartan was now ready to tackle the problem of individuating the necessary and sufficient conditions for the existence of $p$ dimensional integral elements subjected to the independence condition $\Omega \neq 0$.

A linear integral element is defined by assigning $p+q$ parameters:

$$\frac{\omega_1}{u_1} = \cdots = \frac{\omega_p}{u_p} = \frac{\varpi_1}{v_1} = \cdots = \frac{\varpi_q}{v_q}.$$

If this element is regular, Cartan's theory of total differential equations tells us that the number of linearly independent equations:

$$\sum_{i\rho} a_{i\rho k}(u_i \varpi_\rho - v_\rho \omega_i) = 0, \quad (k = 1, \cdots, s).$$

is equal to the first character of $\mathcal{P}$. If, as the independence condition requires, these relations do not establish any linear dependence among $\omega_i$, then, since it must be possible to solve these equations with respect to $s_1$ of the $\varpi_\rho$, the rank of the matrix:

$$\begin{bmatrix} \sum a_{i11} u_i & \sum a_{i21} u_i & \cdots & \sum a_{iq1} u_i \\ \cdots & \cdots & \cdots & \cdots \\ \sum a_{i1s} u_i & \sum a_{i2s} u_i & \cdots & \sum a_{iqs} u_i \end{bmatrix} \tag{5.11}$$

is equal to $s_1$. In an analogous way, Cartan showed that the rank of the matrix obtained from:

$$\begin{bmatrix} \sum a_{i11} u_i & \sum a_{i21} u_i & \cdots & \sum a_{iq1} u_i \\ \cdots & \cdots & \cdots & \cdots \\ \sum a_{i1s} u_i & \sum a_{i2s} u_i & \cdots & \sum a_{iqs} u_i \\ \sum a_{i11} u'_i & \sum a_{i21} u'_i & \cdots & \sum a_{iq1} u'_i \\ \cdots & \cdots & \cdots & \cdots \\ \sum a_{i1s} u'_i & \sum a_{i2s} u'_i & \cdots & \sum a_{iqs} u'_i \\ \cdots & \cdots & \cdots & \cdots \\ \sum a_{i1s} u_i^{(p-1)} & \sum a_{i2s} u_i^{(p-1)} & \cdots & \sum a_{iqs} u_i^{(p-1)} \end{bmatrix}, \tag{5.12}$$

where $u_i, u'_i, \cdots, u_i^{(p-1)}$ are $p^2$ unknowns, by considering the first $ks$ rows, $k = 2, \cdots, p$, must be equal to $s_1 + \cdots + s_k$. As a consequence of this the most general $p$ dimensional integral element depends of $pq - (p-1)s_1 - (p-2)s_2 \cdots - s_{p-1}$.
Conversely, given a Pfaffian system whose bilinear covariants can be expressed in the form (5.10), by introducing the so-called *reduced characters* $\sigma_k$, $k = 1, \cdots, p$, defined to be the rank of the matrix obtained from (5.12) by considering the first $k$ rows, Cartan was able to prove the following:

**Theorem 21 (Cartan 1904)** *The number of parameters upon which the most general p-dimensional integral element that does not establish any linear relations among the $\omega_i$, $i = 1, \cdots, p$ depends, is always less than or equal to*

$$pq - (p-1)\sigma_1 - \cdots - \sigma_{p-1};$$

*if this number is attained then the Pfaffian system is in involution and its p-dimensional integral varieties do not establish any linear relations among the $\omega_i$, $i = 1, \cdots, p$. In this case the quantities $\{a_{i\rho k}\}$ are said to constitute an involutive system.*

Let us see how Cartan demonstrated the first part of the theorem. To this end, he introduced the Pfaffian forms $\varpi_{ik} = \sum_{\rho=1}^{q} a_{i\rho k}\varpi_\rho$, so that the covariants of the system could be written as:

$$d\theta_k = \sum_{i=1}^{p} \omega_i \wedge \varpi_{ik}, \quad k = 1, \cdots, s.$$

Since, Cartan observed, the parameters $u_i, \cdots, u_i^{(p-1)}$, $i = 1, \cdots, p$ can be chosen such as $u_i^{(j)} = \delta_{j+1,i}$ (it is assumed that $u_i^{(0)} = u_i$), without affecting the rank of the relevant matrices, it follows that among the $\varpi_{1k}$ there are precisely $\sigma_1$ independent 1-forms. Cartan chose them to be $\varpi_{11}, \varpi_{12}, \cdots, \varpi_{1\sigma_1}$. In an analogous way, he pointed out that among the $\varpi_{2k}$ there are $\sigma_2$ independent 1-forms to be indicated with $\varpi_{21}, \varpi_{22}, \cdots, \varpi_{2\sigma_2}$. By continuing in this way the existence of $\sigma_1 + \sigma_2 + \cdots + \sigma_p$ independent 1-forms, those $\varpi_{ik}$ for which $k \leq \sigma_i$, was asserted. Cartan named them as the *principal* ones. However, it may happen, he observed, that not all $\varpi_k$, $k = 1, \cdots, q$ can be expressed in terms of these principal 1-forms. If this is the case (it turns out that this eventuality is actually attained when the system admits Cauchy characteristics), then it is easy to see that there are $q - (\sigma_1 + \sigma_2 + \cdots + \sigma_p)$ of them. In this way we obtain already

$$p(q - \sigma_1 - \sigma_2 - \cdots - \sigma_p)$$

arbitrary parameters upon which the general $p$-dimensional integral element depends.

In order to determine the number of remaining parameters Cartan introduced new coefficients $l_{ijk}$, $i,j = 1,\cdots,p$, $k = 1,\cdots,s$, defined by the following relations which express the fact that no more than $p$ independent 1-forms can exist over an integral element of dimension $p$:

$$\varpi_{ik} = \sum_{j=1}^{p} l_{ijk}\omega_j, \quad i = 1,\cdots,p; \quad k = 1,\cdots,s.$$

These coefficients are not all independent. Since $\omega_j$, $j = 1,\cdots,p$ are linearly independent, every linear relations among the $\varpi_{ik}$ produce for each $j$ a similar linear relations among $l_{ijk}$. Besides, as the bilinear covariants $d\theta_k$, $k = 1,\cdots,s$ must vanish identically on general $p$-dimensional integral elements, it follows that $l_{ijk}$ are symmetric under permutation of the indices $i$ and $j$, i.e. $l_{ijk} = l_{jik}$. Such relations allowed Cartan to determine the number of arbitrary parameters upon which an integral element $E_p$, with $\Omega|_{E_p} \neq 0$, might depend. It is given by the sum of $p(q - \sigma_1 - \sigma_2 - \cdots - \sigma_p)$ plus the number of arbitrary coefficients $l_{ijk}$ which turns out to be equal to $\sigma_1 + 2\sigma_2 + \cdots + p\sigma_p$[17]. In this way the first part of the theorem was proved. Let us move to the second part. To this end, following Cartan, we suppose that this number is actually attained and we demonstrate that the system $\mathcal{P}$ is in involution and that its reduced characters are equal to the ordinary ones, $s_i$. Cartan considered a linear integral element $E_1$ which, for the sake of simplicity, he supposed to be defined by the following set of equalities:

$$\frac{\omega_1}{1} = \frac{\omega_2}{0} = \cdots = \frac{\omega_p}{0} = \frac{\varpi_{ik}}{v_{i1k}}, \quad i = 1,\cdots,p; k = 1,\cdots,s.$$

The quantities $v_{i1k}$ are arbitrary being subjected only to the restriction that they satisfy the same linear relations which subsist among the $\varpi_{ik}$. Now, the equations expressing the fact that a linear integral element is in involution with $E_1$ are of the following type:

$$\varpi_{1k} - v_{11k}\omega_1 - v_{21k}\omega_2 - \cdots - v_{p1k}\omega_p = 0, \quad k = 1,\cdots,s. \tag{5.13}$$

These equations, Cartan observed, cannot establish any linear relations among the $\omega_i$, $i = 1,\cdots,p$. Indeed, if we suppose that such a relation exists, there is a relation among the $\varpi_{1k}$ which is not verified by $v_{i1k}$, for at least one $i$. However, since $v_{i1k} = v_{1ik}$, this produces a contradiction with respect to the hypothesis that $v_{1ik}$ have to satisfy the same relations to which the $\varpi_{1k}$ are subjected. As a consequence of this $s_1 = \sigma_1$.
By iterating similar arguments for integral elements of increasing dimension (until order $p$), Cartan was able to prove also the second part of the stated theorem. In this way the problem of establishing whether a given Pfaffian system $\mathcal{P}$ with independent condition is in involution was traced back to an, at least in principle, straightforward numerical test.

[17] Indeed, the arbitrary $l_{ijk}$ are those for which $k \leq \sigma_i$ and $k \leq \sigma_j$.

## 5.3 Cartan's theory of infinite continuous groups

As we have already observed, Cartan's approach to infinite continuous groups was characterized by the centrality of the notion of isomorphism. Independently of Vessiot, already in 1902 Cartan had provided a generalization of such a notion which allowed him to develop a structural approach to the theory.

Cartan's definition, which was equivalent to that provided by Vessiot in [Vessiot 1903], reads as follows. He considered $m + n$ variables

$$x_1, x_2, \cdots, x_m; \qquad y_1, y_2, \cdots, y_n,$$

and two (infinite) groups $G$ and $G'$. $G$ was supposed to be acting upon the variables $x_1, \cdots, x_m$ only, while the group $G'$ transforms all the variables $x_1, \cdots, x_m, y_1, \cdots, y_n$. Cartan defined $G'$ to be a prolongation of $G$, if $G'$ transforms the variables $x_1, \cdots, x_m$ among themselves in the same way as $G$ does. Furthermore, $G'$ was said to be a holoedric prolongation of $G$ if every transformation of $G'$ which leaves the variables $x_1, \cdots, x_m$ invariant reduces to the identity transformation of $G'$ itself. If this is not the case, then $G'$ was said to be a meriedric (Cartan named it *hémiédrique*) prolongation of $G$.

On the basis of these definitions, Cartan was finally able to provide his own definition of isomorphism: two groups $G_1$ and $G_2$ were said to be holoedrically isomorphic, or isomorphic *tout court*, if they admit holoedric prolongations $G'_1$ and $G'_2$ which are similar one with respect to the other. In this case, the groups $G_1, G_2$ were also said to have the same *structure*. As Cartan explicitly observed, when the groups $G_1, G_2$ are finite dimensional, his new definition of isomorphism coincided with the usual definition introduced by Lie in his theory of finite continuous groups.

A generalization of the notion of meriedric isomorphism was introduced too. $G_1$ was said to be meriedrically isomorphic to $G_2$ if there exists a meriedric prolongation of $G_1$ which is similar to a holoedric prolongation of $G_2$.

With the exception of [Cartan 1902a], the first papers in which Cartan provided a general account of his theory of infinite continuous groups were [Cartan 1904] and [Cartan 1905]. He started in the second chapter of [Cartan 1904] by providing what he considered to be the genuine generalization of the three fundamental theorems of Lie's theory of finite continuous groups.

### 5.3.1 First fundamental theorem

Cartan's starting point consisted of considering the defining system of the *finite* transformations of an infinite group $G$.[18] This choice had already been operated some years before by Medolaghi and Vessiot, nevertheless Cartan adapted it to the technical tools he had developed over the years 1899-1901 to deal with systems of PDE's. To this end, he wrote down the involutive Pfaffian system obtained by restricting the contact Pfaffian system to the submanifold of the jet space corresponding to the PDE's system of the group G. In addition, he supposed the group $G$ to be acting on the variables $x_1, \cdots, x_n$. He considered also the case of an intransitive group possessing $n-m$ independent invariants of order zero. In this eventuality, Cartan assumed that such invariants could be chosen as $n-m$ coordinates, $x_{m+1}, \cdots, x_n$, which are left unchanged under the action of $G$. As a consequence of this, by denoting the transformed variables with $X_1, \cdots, X_n$, the defining equations of $G$ contain relations of zero order of the form:

$$X_k = x_k, \qquad (k = m+1, \cdots, n).$$

The remaining transformed variables $X_1, \cdots, X_m$ regarded as functions of $x_1, \cdots, x_n$ are defined by a system of Pfaffian equations which is involutive with respect to the independence condition $dx_1 \wedge \cdots \wedge dx_n \neq 0$. If the defining system of the group is of order $h$, then the corresponding Pfaffian system obtained by restricting the contact Pfaffian system to the submanifold corresponding to the defining system itself can be subdivided into $h$ families of equations. The first one of this family consists of $m$ equations which express the differential $dX_j$, $j = 1, \cdots, m$ in terms of the differentials of $x_1, \cdots, x_n$:

$$\begin{cases} dX_1 - \alpha_{11} dx_1 + \cdots + \alpha_{1n} dx_n = 0, \\ \cdots, \\ dX_m - \alpha_{m1} dx_1 + \cdots + \alpha_{mn} dx_n = 0. \end{cases} \tag{5.14}$$

[18] The remarks provided in [Akivis, Rosenfeld 1993, p. 125] seem to be misleading, if not completely uncorrect. Akivis and Rosenfeld wrote:

> [...] *Cartan considered manifolds whose points are defined by complex coordinates and assumed that the transformations which he studied were given by analytic functions of these coordinates. As in the case of the finite-dimensional Lie groups, Cartan considered only "infinitesimal transformations". This explains why he did not encounter the cases when for two transformations the result of their successive realization cannot be found. Because of this, Cartan used the term "groups" for sets of such transformations.*

Now, it is true that at the time no special care was paid to specifying the domains of definition of the transformations of (finite and infinite) groups, but this did not prevent Cartan and other mathematicians from considering finite transformations. On the contrary, as it has already been emphasized, since Medolaghi's contributions, the shift of attention towards finite transformations to the detriment of infinitesimal ones was regarded as a major advance in the theory. Akivis' and Rosenfeld's remarks should be emended by replacing the expression *infinitesimal transformations* with *finite transformations defined in a sufficiently small neighborhood of the identity transformation.*

The coefficients $\alpha_{ij}$, $i = 1, \cdots, m$, $j = 1, \cdots, m$ are functions of the variables $x, X$ and of $p_1$ additional variables $y_1, \cdots, y_{p_1}$ which coincide with the first order parametric variables of the defining system.
The second family of Pfaffian equations consists of $p_1$ equations which give the expressions for the differentials of $y_1, \cdots, y_{p_1}$ in terms of $dx_1, \cdots, dx_n$:

$$\begin{cases} dy_1 - \beta_{11} dx_1 + \cdots + \beta_{1n} dx_n = 0, \\ \cdots, \\ dy_{p_1} - \beta_{p_1 1} dx_1 + \cdots + \beta_{p_1 n} dx_n = 0. \end{cases} \tag{5.15}$$

Here the coefficients $\beta_{ij}$, $i = 1, \cdots, p_1$, $j = 1, \cdots, n$ are functions of the variables $x$, $X$, $y$ and $p_2$ additional variables $z_1, \cdots, z_{p_2}$ that are the second order parametric variables of the defining system. Iterating the same reasoning until the $h$-th order, Cartan finally wrote the last system of equation expressing the differentials of the parametric variables of $(h-1)$-th order in terms of $dx_1, \cdots, dx_n$:

$$\begin{cases} du_1 - \lambda_{11} dx_1 + \cdots + \lambda_{1n} dx_n = 0, \\ \cdots, \\ du_{p_h - 1} - \lambda_{p_{h-1} 1} dx_1 + \cdots + \lambda_{p_{h-1} n} dx_n = 0. \end{cases} \tag{5.16}$$

The coefficients $\lambda_{ij}$, $i = 1, \cdots, p_{h-1}$, $j = 1, \cdots, n$ are functions of $x$, $X$, $y$, $z$ and $p_h$ additional variables $v_1, \cdots, v_{p_h}$.
The system of $h$ equations so obtained (5.14-5.16) is involutive but in general not completely integrable (in the sense of Frobenius theorem). However, Cartan observed, it reduces to a completely integrable system in the case in which no parametric variables of $h$-th order exist (or equivalently, in the case in which all the derivatives of highest order are principal). In such an eventuality, all the bilinear covariants of the system (5.14-5.16) vanish as a consequence of the system itself; thus, Frobenius' theorem applies and the general solution depends upon a finite number of parameters only. It is the well known finite dimensional case.

At this point, the central idea of Cartan's theory enters the stage. Instead of regarding the system (5.14-5.16) as defining the group $G$ in the sense that its solutions coincide with the transformations of the group itself, he introduced a holoedric prolongation of $G$, $G'$ which he defined as the set of transformations

$$\begin{cases} X_i' = X_i & (i = 1, \cdots, m), \\ x_i' = f_i(x_1, \cdots, x_n) & (i = 1, \cdots, m), \\ x_{m+j}' = x_{m+j} & (j = 1, \cdots, n-m), \\ y_i' = \phi_i(x, X, y) & (i = 1, \cdots, p_1), \\ \cdots, \\ v_i' = \Phi_i(x, X, y, \cdots, v) & (i = 1, \cdots, p_h). \end{cases} \tag{5.17}$$

which leave invariant the system (5.14-5.16). This means that a transformation of type (5.17) transforms the first members of the equations of (5.14-5.16) into a homogeneous linear combination thereof. The fact that the system of equations (5.17) actually defines a holoedric prolongation of $G$ is a consequence of the property, first proved by Lie in his *Grundlagen*, according to which every transformation of $G$ is such as to leave invariant its defining system and, conversely, every transformation which leaves this system invariant is a transformation of $G$.

A first result of Cartan's theory was the possibility of characterizing this holoedric prolongation, and thus indirectly the group $G$ too, as the group of those transformations which leave a certain number of Pfaffian forms invariant. This was the essential content of what Cartan later called the first fundamental theorem of his theory.

From the equations (5.17) of $G'$, it follows in a straightforward way, as a consequence of the fact that $G'$ leaves $X_i$, $i = 1, \cdots, m$, and $x_{m+j}$, $j = 1, \cdots, n-m$, invariant, that $G'$ admits a first set of invariant Pfaffian expressions:

$$\begin{cases} \omega_i = \alpha_{i1}(x, X, y)dx_1 + \cdots + \alpha_{in}(x, X, y)dx_n & (i = 1, \cdots, m), \\ \omega_{m+j} = dx_{m+j} & (j = 1, \cdots, n-m). \end{cases} \tag{5.18}$$

From this set of $n$ independent Pfaffian expressions, Cartan deduced the existence of other invariant Pfaffian forms. To this end, he observed that the system $\omega_i = 0$, $i = 1, \cdots, n$ is completely integrable and thus Frobenius theorem guarantees that the vanishing of its bilinear covariants is a consequence of the system itself. In particular one has that

$$d\omega_i = \sum_{k=1}^{n} \omega_k \wedge \varpi_{ik}, \qquad i = 1, \cdots, m,$$

where $\varpi_{ik}$ are $mn$ Pfaffian forms in the variables $x, X, y$ and their differentials. Since the Pfaffian forms $\omega_i$, $i = 1, \cdots, m$ are left invariant by every transformation of $G'$, so are their bilinear covariants. As a consequence of this, one obtains:

$$\sum_k \omega_k \wedge [\varpi_{ik}(x, X, y; dx, dX, dy) - \varpi_{ik}(x', X', y'; dx', dX', dy')] = 0.$$

which implies that

$$\varpi_{ik}(x, X, y; dx, dX, dy) \equiv \varpi_{ik}(x', X', y'; dx', dX', dy') \quad mod(\omega_1, \cdots, \omega_n). \tag{5.19}$$

Now, among these $mn$ Pfaffian expressions $\varpi_{ik}$, there exist $p_1$ of them which are linearly independent with respect to $dy_1, \cdots, dy_p$. Let us indicate them with $\varpi_i(x, X, y; dx, dX, dy)$, $i = 1, \cdots, p_1$. By stressing their dependence upon $dy_\tau$, $\tau = 1, \cdots, p_1$ and $dX_j$, $j = 1, \cdots, n$, one can write:

$$\varpi_i(x, X, y; dx, dX, dy) \equiv \sum_{\tau=1}^{p_1} a_{i\tau} dy_\tau + \sum_{j=1}^{m} b_{ij} dX_j, \quad (i = 1, \cdots, p_1), \quad mod(\omega_1, \cdots, \omega_n). \tag{5.20}$$

By adding suitable linear combinations of $\omega_i$, $i = 1, \cdots, n$, Cartan observed, one could make $\varpi_i$ to be linear combinations ($mod(\omega_i, \cdots, \omega_n)$) of the first members of (5.14-5.15)[19]. As a consequence of this, one has that, under the change of coordinates imposed by a transformation of $G'$, the $\varpi_i$'s transform into a linear combination of the first members of (5.14-5.15). In view of relations (5.19), the only possibility is that

$$\varpi_i(x', X', y', z'; dx', dX', dy') = \varpi_i(x, X, y, z; dx, dX, dy), \qquad (i = 1, \cdots, p_1),$$

for every transformation of $G'$.

Upon iteration of the preceding procedure, Cartan was thus able to build a set of independent Pfaffian forms equal in number to $s = n+p_1+\cdots+p_{h-1}$, which are left invariant by the action of $G'$. This set could be divided into $h$ subsets. The first subset is constituted by the Pfaffian forms $\omega_1, \cdots, \omega_n$; note that whereas their differentials depend on the variables $x_1, \cdots, x_n$ and $X_1, \cdots, X_n$ only, their coefficients depend in general upon the variables $x, X, y$. The second one consists of the Pfaffian forms $\varpi_i$, $i = 1, \cdots, p_1$, later on to be indicated with $\omega_{n+1}, \cdots, \omega_{n+p_1}$; they contain differentials of $x, X, y$ while their coefficients may depend upon the variables $z_1, \cdots, z_{p_2}$ as well. And so on, until one arrives at $p_{h-1}$ invariant Pfaffian forms in the differentials $dx, dX, dy, dz, \cdots, du$, whose coefficients depend in general upon $p_h$ auxiliary variables $v_1, \cdots, v_{p_h}$, as well.

The final result thus consisted of the possibility of characterizing the finite transformations of a holoedric prolongation (actually, the one obtained from $G'$ by assigning to $X_1, \cdots, X_m$ arbitrary constant values) of $G$ by means of the invariance properties of a system of Pfaffian equations. More precisely, Cartan arrived at the following *first fundamental theorem*:

**Theorem 22 (First fundamental theorem)** *Every infinite continuous group in $n$ variables, defined by an involutive $h$-order system of partial differential equations admits a holoedric prolongation $G'$, which is defined by the following conditions: first, $x'_k = x_k$, $k = m+1, \cdots, n$; secondly, $\omega'_i = \omega_i$, $i = 1, \cdots, s$, (here, $\omega'_i$ designates the Pfaffian form $\omega_i$ after the substitution of $x, y, \cdots$ with $x', y', \cdots$).*

Beyond representing the starting point of Cartan's entire theory, as it is was pointed out by Engel in one of the rare documental reactions to Cartan's work on infinite groups[20], this result provided a very simple and elegant expression for the defining equations of the group since it has the advantage of being fully symmetric with respect to both the untransformed and the transformed variables.

[19] Note that this operation will in general have the affect of introducing additional variables of the second order: $z_1, \cdots, z_{p_2}$.

[20] See *Jahrbuch über die Fortschritte der Mathematik* 35.0176.04.

### 5.3.2 Second and third fundamental theorems

The introduction of the $s$ invariant Pfaffian forms $\omega_1, \cdots, \omega_s$ enabled Cartan to introduce something very similar to the structure constants of Lie's theory of finite continuous groups. As the preceding theorem could be interpreted as a generalization of Lie's first fundamental theorem, in an analogous way Cartan set out to attain a generalization of Lie's second fundamental theorem by characterizing the expressions of the exterior derivatives of these $s$ Pfaffian forms. In this regard, Amaldi's remarks are worth quoting.

> *If one wants to reconnect Cartan's theory to Lie's classical theory of finite continuous groups, one could say that the result proved in the last paragraph - that is the possibility to characterize any arbitrary continuous group, or at least a holoedric prolongation thereof, by means of the invariance property of a certain number of Pfaffian expressions (and possibly of a certain number of functions) - constitutes somehow the same as Lie first fundamental theorem which, in its conceptual content, asserts the possibility to generate a group by means of infinitesimal transformations. In actual fact, the essential role accorded by Lie to infinitesimal transformations is attributed by Cartan to invariant Pfaffian forms.*
> *Thus, to Lie's second (or principal) fundamental theorem, which rendered precise the necessary and sufficient conditions for infinitesimal transformations to generate a finite continuous group, there corresponds, in Cartan's theory, a theorem which provides the characteristic form of the exterior derivatives of the invariant Pfaffian forms of a group (indifferently, finite or infinite).*[21]

Indeed, by exploiting in an essential way the possibility of replacing the Pfaffian forms $\omega_1, \cdots, \omega_s$ with linear combinations thereof (whose coefficients are invariant of the group $G'$), Cartan was able to show the following, very important:

[21] *Se si vuole riavvicinare la teoria del Cartan alla classica teoria dei gruppi continui finiti del Lie, si può dire che il risultato stabilito al paragrafo precedente - cioè la possibilità di caratterizzare un qualsiasi gruppo continuo, o quanto meno un suo prolungamento oloedrico, mediante l'invarianza di un certo numero di pfaffiani (ed eventualmente di un certo numero di funzioni) - costituisca, in qualche modo, l'analogo del primo teorema fondamentale del Lie, che, nel suo contenuto concettuale, esprime la generabilità del gruppo per mezzo di trasformazioni infinitesime; ed effettivamente l'ufficio essenziale, che il Lie conferisce alle trasformazioni infinitesime, viene invece assegnato dal Cartan ai pfaffiani invarianti.*
*Così al secondo teorema fondamentale o principale del Lie, che precisa le condizioni necessarie e sufficienti affinché più trasformazioni infinitesime siano atte a generare un gruppo continuo finito, fa riscontro, nella teoria del Cartan, un teorema che assegna la forma caratteristica dei differenziali esterni dei pfaffiani invarianti di un gruppo (indifferentemente finito o infinito).* See [Amaldi 1944, 267-268].

**Theorem 23 (Second fundamental theorem)** *Let $\omega_i$, $i = 1, \cdots, s$ be the invariant Pfaffian forms associated to a continuous group $G$. Let $\varpi_1, \cdots, \varpi_{p_h}$ be Pfaffian forms involving the differentials of the variables $v_1, \cdots, v_{p_h}$ which coincide with the $h$-order parametric derivatives of the defining system of $G$ ($h$ is supposed to be the order of such a system). Then, $\varpi_1, \cdots, \varpi_{p_h}$ can be chosen in such a way that the following relations hold:*

$$d\omega_i = \frac{1}{2} \sum_{j,k=1}^{s} c_{jki} \omega_j \wedge \omega_k + \sum_{\rho=1}^{p_h} \sum_{l=1}^{s} a_{l\rho k} \omega_l \wedge \varpi_\rho, \quad (i = 1, \cdots, s), \tag{5.21}$$

*where the coefficients $c_{jki}$ and $a_{l\rho k}$ are invariant of the group $G$.*

As Cartan was quick to observe, it should be noticed that in the case in which the group $G$ is transitive, that is it does not admit any zero-th order invariant, the coefficients $c_{jki}$ and $a_{l\rho k}$ reduce to constant numerical values. Furthermore, when $G$ is a finite continuous group, since, in this case, $p_h = 0$, there is no variable of type $v_i$ and the structure equations take on the following simple expressions: $d\omega_i = \frac{1}{2} \sum_{j,k=1}^{s} c_{jki} \omega_j \wedge \omega_k$, $i = 1, \cdots, s$.

Now, as we have already observed, the (mixed) Pfaffian system

$$\begin{cases} x'_{m+k} = x_{m+k}, & k = 1, \cdots, n-m, \\ \omega'_i = \omega_i, & i = 1, \cdots, s, \end{cases} \tag{5.22}$$

defines the finite transformations of the holoedric prolongation $G'$, in the sense that its integral varieties coincide with the graphs of the transformations of $G'$. The system (5.22) is thus involutive with respect to the independence condition $\omega_1 \wedge \cdots \wedge \omega_s \neq 0$. As a consequence of this, Cartan observed, the coefficients $\{a_{l\rho k}\}$ are involutive according to the definition provided in 5.2.

In this way, a first characterization of the coefficients $c_{jki}$ and $a_{l\rho k}$ appearing in the structure equations (5.21) was obtained. A detailed study of the properties of these coefficients led Cartan to a generalization of Lie's third fundamental theorem which, as it is well known, provides necessary and sufficient conditions to be imposed on the structure constants $c_{jki}$ in order to guarantee the existence of a corresponding finite continuous group. However, whereas in the finite dimensional case all these conditions are of an algebraic nature, in the case of infinite continuous groups, it turns out, differential conditions emerge as well.

Cartan was able to show this in a straightforward way, simply by imposing that the exterior derivatives, or as he called them, the *trilinear covariants* of $d\omega_i$, $i = 1, \cdots, s$ vanish identically. The computations involved being quite long and tedious, we will limit ourselves to providing the final result.

**Theorem 24 (Third fundamental theorem)** *The coefficients $c_{jki}$ and $a_{l\rho k}$ appearing in the structure equations (5.21) are invariants of the group $G$ and satisfy the following four conditions.*

- $c_{jki} + c_{kji} = 0$,
- *The system* $\{a_{l\rho k}\}$ *is involutive,*
- *The infinitesimal transformations* $U_\rho = \sum_{i,k=1}^{p_h} a_{i\rho k} u_i \frac{\partial f}{\partial u_k}$, $\rho = 1, \cdots, p_h$, *generate a* $p_h$ *parameter finite, linear continuous group, to be indicated with* $\Gamma$. *Cartan called it the* adjoint group *of* $G$.
- *The linear system for the unknowns* $z_{\lambda\rho\tau}$, $\lambda = 1, \cdots, s; \rho, \tau = 1, \cdots, p_h$:

$$\sum_{\tau=1}^{p_h} (a_{\lambda\tau k} z_{\mu\rho\tau} - a_{\mu\tau k} z_{\lambda\rho\tau}) =$$

$$= \sum_{i=1}^{s} (c_{\lambda\mu i} a_{i\rho k} + c_{i\lambda k} a_{\mu\rho i} - c_{i\mu k} a_{\lambda\rho i}) + \frac{\partial a_{\mu\rho k}}{\partial x_\lambda} - \frac{\partial a_{\lambda\rho k}}{\partial x_\mu},$$

*is compatible, i.e. it admits solutions.*

- *The linear system for the unknowns* $y_{\lambda\mu\tau}$, $\lambda, \mu = 1, \cdots, s$, $\tau = 1, \cdots, p_h$:

$$\sum_{\tau=1}^{p_h} (a_{\lambda\tau k} y_{\mu\nu\tau} + a_{\mu\tau k} y_{\nu\lambda\tau} + a_{\nu\tau k} y_{\mu\lambda\tau}) =$$

$$= \sum_{i=1}^{s} (c_{\lambda\mu i} c_{i\nu k} + c_{\mu\nu i} c_{i\lambda k} + c_{\nu\lambda i} c_{i\mu k}) + \frac{\partial c_{\lambda\mu k}}{\partial x_\nu} + \frac{\partial c_{\mu\nu k}}{\partial x_\lambda} + \frac{\partial c_{\nu\lambda k}}{\partial x_\mu},$$

*is compatible, i.e. it admits solutions.*

At this point, Cartan posed the problem of inverting this result by investigating the possibility of constructing an infinite continuous group starting from given sets of functions $c_{jki}$ and $a_{l\rho k}$ which satisfy the conditions imposed by the preceding theorem. In effect, he was able to do that by making recourse to his involutivity test for establishing the existence of appropriate integral varieties. More explicitly, Cartan strategy consisted of proving the existence of $s + p_h$ Pfaffian expressions $\omega_i$, $i = 1, \cdots, s$ and $\varpi_j$, $j = 1, \cdots, p_h$ such as to satisfy structure equations of type (5.21).

Thus, the generalization of Lie's classical theorems could be regarded as completed.

## 5.4 Subgroups of a given continuous group

In the finite dimensional case, the problem of determining all the subgroups of a given continuous group had already been tackled and solved by Lie. He had succeeded in tracing back such a task to a series of algebraic operations which exploited, in an essential way, the possibility of characterizing the structure of a given finite continuous group in terms of its infinitesimal transformations only.

As Cartan himself emphasized in the introduction to [Cartan 1908], the generalization of this approach to infinite groups was considered by him to be impracticable. The reason for this, he claimed, was the lack, in Lie's original approach, of an adequate structural theory which was valid in the infinite dimensional case too.

The researches which Cartan had developed during the years 1902-1905 put him in the position to provide a general solution which, at least in principle, offered a uniform method for the determination of all subgroups of infinite and finite continuous groups, without any distinction. Beyond its intrinsic theoretical value, the solution to this problem was regarded as highly relevant in view of its applications. Indeed, as Cartan did in 1908, it could be employed in the general classification problem of all infinite continuous groups in $n$ variables. In chapters III and IV of [Cartan 1908], the method was applied to the determination of *all* continuous groups in two variables which were considered as subgroups of the group of the diffeomorphisms of the plane. Furthermore, Cartan observed, the procedure could be applied also in the case of the transformation groups of the 3-dimensional space. He claimed that, as far as transitive infinite groups were concerned, there exist 137 types[22] of degree one[23] which could be obtained by exploiting the knowledge of all the subgroups of the general linear groups in three variables. To provide a general idea of the enormous difficulty to be faced, he observed that to each of these groups other groups of degree higher than one are associated; for example, to one of them there correspond 98 different types. However, he pointed out, such an enumeration was not worth the effort since no new simple transitive group could be obtained in this way; nevertheless, Cartan had to admit, it seemed that this method could exhibit other important classes of groups which he called *improperly simple* and which were relevant for the problem of determining normal series of subgroups.

Some years before the publication of Cartan's paper, in 1902, Vessiot had addressed the same problem[24] by applying Engel's theory and the more recent results by Medolaghi. Vessiot had studied in some details the case of transitive subgroups of a given transitive group $G$, limiting himself to some sketchy remarks concerning the intransitive case. He had set out to determine the subgroups of order $m$ (this means that their defining equations contain derivatives of order $m$). In this case, by prolonging, if necessary, to order $m$ the defining equations of $G$, we know that a finite group $\mathcal{L}$ (meriedrically isomorph to the group of Engel right transformations $\mathcal{A}_m$), the corresponding Engel group, can be associated to $G$. It is also known that $\mathcal{L}$ describes how the invariants (actually, the invariants of the group $G$ itself) of a certain subgroup $\mathcal{K}$ of the group of left Engel transformations $\mathcal{B}_m$ are transformed among themselves as a consequence of the action of a (finite)

[22] As we will see, two subgroups are said to be homologue or to belong to the same *type* if there exists a transformation of the group which transforms one into the other.

[23] For the definition of *degree*, see [Cartan 1908, p. 85].

[24] See [Vessiot 1903, §§18-22].

transformation of the group $\mathcal{A}_m$. Vessiot's idea for determining all the $m$ order subgroups of $G$ had consisted of determining all the subgroups of $\mathcal{K}$ (in principle, this does not produce any difficulty since $\mathcal{K}$ is a finite group) and of deducing from them the Engel groups corresponding to the subgroups sought after. The complete determination of subgroups had relied then upon a thorough (and sometimes hard) discussion of the integrability conditions that have to be required in order to guarantee complete integrability of the differential system of their defining equations.

Cartan shared with Vessiot the necessity of reasoning in terms of finite transformations only. However, in accordance with his general strategy, Cartan's treatment of the problem was marked on one hand by the centrality of the notion of holoedric prolongation and on the other by the ubiquitous use of his exterior differential calculus.

Cartan's starting point in the second chapter of [Cartan 1908] consisted of formulating with precision the objects of his investigation which he divided into two steps: i) To determine all the types of subgroups of a given group $G$ and to exhibit a representant for each of them. ii) for every subgroups so obtained, to determine the maximal subgroup in which it is invariant.

We will follow the general account of this method which was provided by him in [Cartan 1937a]. Cartan specified that the only subgroups considered by him was those of Lie type, that is those which are defined by systems of PDE's.

In principle, the study of the subgroups $g$ of a given group $G$ reduces to the problem of adding new equations to the defining equations of $G$ in such a way that the subgroups so obtained are of Lie type. Clearly, the difficulty consists of singling out which are the possible candidates for these equations. Cartan supposed that the group $G$ is normal in such a way that its defining equations are of first order. This hypothesis was not very restrictive since every group admits a normal holoedric prolongation; correspondingly, the subgroups $g$ are prolonged as well. Cartan thus was traced back to the problem of determining all the subgroups of a given group $G$ of first order. However, he pointed out, the defining equations of $g$ could still be of any order, zero, one, two, etc..

We will see how Cartan succeeded in proving that one could always limit himself to consider subgroup of zero order. He first supposed that among the defining equations of $g$ there certain equations of zero order, that is involving the primitive and the transformed variables $x, X$, only. This means that the group $g$ admits not only the invariant of $G$ but also new invariant equal in number to the independent equations added to the defining equations of $G$. It may happen that these equations are sufficient to define $g$. If this is the case then $g$ can be characterized as the set of those transformations of $G$ which leave invariant a certain number of independent functions of the variables $x$. These functions were called by Cartan *characteristic invariants* of $g$.

Nevertheless, in general, in order to obtain $g$ one has to add to the defining equations of $G$ also equations involving first derivatives of $X$ with respect to $x$. Remember that, according to Cartan's general theory, the transformations of $G$ can be characterized as those transformations which leave invariant a set of $n$ independent Pfaffian forms $\omega_i = \sum a_{ik}(x,u)dx_k$, $i = 1, \cdots, n$. The action of $G$ on the variables $u$, is obtained from

$$\sum_{m=1}^{n} a_{im}(X,U)\frac{\partial X^m}{\partial x_k} = a_{ik}(x,u),$$

and from these equations by calculating $X$ as functions of $x$ one can derived the defining equations of $G$. If, however, a transformation of $g$ is considered, then other relations among the variables $x, u, X, U$ have to be considered. By introducing the first normal prolongation of $G$, $G^{(1)}$ which acts upon the variables $x, u$ then one is traced back to the preceding case since a first order equation with respect to $G$ has been transformed into a zero order relation among the variables $x, u$ upon which the first normal prolongation of $G$ acts.

In general, by iteration of the preceding reasoning Cartan was able to prove the following result which he called *théorème fondamental*:

**Theorem 25** *Given a subgroup $g$ of a normal group $G$, there exists a normal prolongation $G^{(h)}$ of $G$, such that the corresponding prolonged subgroup $g^{(h)}$ of $g$ is defined by the set of those transformations of $G^{(h)}$ which leave invariant a certain number of functions of the variables upon which $G^{(h)}$ acts.*[25]

A first important problem to face was the determination of the characteristic invariants corresponding to the subgroups of $G$. Cartan pointed out that these invariants cannot be arbitrary functions of the transformed variables of the group. For example, a necessary condition that they have to fulfill is that the transformation of $G$ which leave invariant the characteristic functions associated to $g$, do not leave invariant any other function independent of them.

Let us see how Cartan solved the question of determining the characteristic invariants of zero order. For the sake of simplicity, let us suppose that the group $G$ is transitive so that it does not admit any invariant. Now, suppose that a subgroup $g$ of $G$ admits independent characteristic invariants $J_1, \cdots, J_p$. The central idea of Cartan's procedure was that of regarding

[25] *Étant donné un sous-groupe $g$ d'un groupe normal $G$, il existe un prolongement normal $G^{(h)}$ de $G$, tel quel le sous-groupe correspondant $g^{(h)}$ prolongé de $g$ soit défini par l'ensemble des transformations de $G^{(h)}$ qui laissent invariantes un certain nombre de fonctions des variables transformées par $G^{(h)}$. Ces fonctions sont les invariants caractéristiques du sous-groupe $g$.*

them as independent integrals of a completely integrable Pfaffian system in the variables $x_1, \cdots, x_n$, which could be written in the following form:

$$\theta_i \equiv \omega_i + \sum_{k=1}^{n-p} \alpha_{ik}\omega_{p+k} = 0, \qquad (i = 1, \cdots, p). \tag{5.23}$$

The fact that system (5.23) defines the characteristic invariants of $g$ implies some important consequences which have to be carefully analyzed. Every transformation of $G$ and then of $g$ itself leaves invariant the Pfaffian forms $\omega_1, \cdots, \omega_n$. Furthermore, the transformations of $g$ leaves the system (5.23) invariant since they have to transform every integral $J_i$ into itself. As a result of this, the coefficients $\alpha_{ik}$ are invariants of $g$ as well and thus integrals of the system (5.23). As a consequence of complete integrability, the Pfaffian forms $\theta_i$, $i = 1, \cdots, p$ are linear combinations of $dJ_1, \cdots, dJ_p$. Furthermore, since $g$ transforms into itself every $\theta$ as well as every $dJ$, the coefficients of these linear combinations are functions of $J_1, \cdots, J_p$ too. Cartan finally concluded that the exterior derivatives of $\theta_i$, $i = 1, \cdots, p$ could be expressed as quadratic forms of the differential expression $\theta_i$, $i = 1, \cdots, p$ only. In the case of a single Pfaffian equation, $\theta = 0$, that means that $d\theta = 0$, identically, since $\theta \wedge \theta = 0$.

By recalling the structure equations of the group $G$, the exterior derivatives of $\theta_i$, $i = 1, \cdots, p$ could then be written as a linear combination of the 2-forms:

$$\theta_k \wedge \theta_h, \quad \theta_k \wedge \omega_{p+l}, \quad \omega_{p+l} \wedge \omega_{p+m}, \quad d\alpha_{ik} \wedge \omega_{p+l}, \quad \theta_k \wedge \varpi_j, \quad \omega_{p+l} \wedge \varpi_j.$$

Cartan pointed out that, as a consequence of complete integrability of the system (5.23), the differentials $d\alpha_{ik}$ could be expressed as linear combinations $d\alpha_{ik} = \sum_{h=1}^{p} \alpha_{ikh}\theta_k$ where $\alpha_{ikh}$ are functions of the integrals $J_1, \cdots, J_p$ as well. It is then necessary to choose the Pfaffian forms $\varpi_j$ in such a way that $d\theta_i$ are quadratic forms in $\theta_k \wedge \theta_h$. The conditions that have to be imposed to the invariants $\alpha_{ik}$ thus consist of a finite number of algebraic relations. Such relations, Cartan observed, can be thought of, geometrically as defining irreducible algebraic varieties in the $p(n-p)$-dimensional space of the coefficients $\alpha_{ik}$. In this way, the problem of determining all the subgroups of a given group was traced back by Cartan to a purely algebraic task consisting in classifying all the different algebraic varieties arising from the described procedure.

Simple applications of his procedure were given by Cartan in [Cartan 1937a] where he determined all the subgroups of the translation group of the plane and of the projective groups of the straight line[26]. Cartan also applied his method to the determination of all continuous groups in one variables[27]

[26] See [Cartan 1937a, p. 40-44]
[27] See [Cartan 1937a, p. 44-49]

when these are considered as subgroups of the general groups of the diffeomorphisms of the straight line.

## 5.5 Simple infinite continuous groups

In the preceding sections, we saw how Cartan's approach to infinite continuous groups was characterized by his constant concern for structural issues rather than by the need of achieving a complete classification of their different similarity types. Nonetheless, the great number of applications in the field of the theory of differential equations made such a classification a highly promising task. Though not an absolute research priority of his, Cartan too devoted conspicuous efforts to it.

On numerous occasions, Lie himself was very explicit over the necessity to pursue this challenge. For example in [Lie 1895b, p. 291-292], he stressed the importance of carrying out a thorough classification of all simple infinite continuous groups. The motivation was provided by his integration theory of general systems of PDE's which admit (infinite) continuous groups. In this context, Lie was lead to consider special types of such systems (later on to be called by Vessiot *automorphic systems*[28]) characterized by the property that every solution thereof can be obtained from one single fixed solution as the result of the action of an infinite continuous group. In particular, as Lie proved, the integration of the systems of this special type, Lie proved, could be traced back to the resolution of "automorphic systems" whose groups are *simple*.

That same year, Lie himself had devoted large part of [Lie 1895a] to the classification problem of simple groups. However, the methods developed therein were suitable to treat only the case of primitive (and thus transitive) infinite continuous groups. Lie was able to single out four different types: i) the group of all transformations in $n$ variables; ii) the equivalent group, i.e. the group of volume-preserving transformations in $n$ variables; iii) the group of all transformations in $2n \geq 4$ variables which leave invariant the exterior quadratic form $dx_1 \wedge dx_{n+1} + \cdots + dx_n \wedge dx_{2n}$; the group of all transformations in $2n+1$ variables which one obtains when regarding the contact transformations of an $n$-dimensional space as point transformations.

On the occasion of the *Grand Prix des sciences mathématiques* in 1902, Vessiot had taken up the problems raised by Lie's integration methods with the hope of providing them with systematic character[29]. By generalizing two noteworthy examples which Lie had studied in [Lie 1895c, chap. IV], Vessiot had provided a technique for integrating systems of PDE's which admit a (infinite) continuous group of transformations. Following Lie, Vessiot

[28] See chapter 3.

[29] Vessiot's theory of automorphic systems was published two years later in [Vessiot 1904b]

indicated a procedure for decomposing the integration of the given system in two steps: the integration of a *resolvent* system which does not admit any group of transformations; the integration of an automorphic system.[30]

Vessiot agreed with Lie in according great importance to the classification problem of simple infinite groups. In particular, he posed the question about the possible existence of simple groups which do not belong to any of the four classes indicated by Lie, however, it seems that he did not expect that other simple groups, beyond those mentioned above, could exist as well.

On the basis of this conjecture, Vessiot was able to trace back the integration of an arbitrary automorphic system to the resolution of ordinary differential equations only.

Greatly unexpected, in 1907 Cartan proved that new types of *simple* infinite groups do exist. More than that, Cartan succeeded in attaining a complete classification of all types of simple infinite groups. In particular he was able to show that whereas, among transitive groups, the only types of simple ones coincided with those already enumerated by Lie, *intransitive*, simple, infinite continuous groups exist as well.

These results were first announced in a brief note [Cartan 1907] and then explained in more detail in [Cartan 1909]. Cartan's introductory remarks to [Cartan 1909] are worth quoting. He wrote:

> *The important role played by simple groups in the different applications of transformation group theory has been known for long. In particular, for what pertains the integration of differential systems which admit a continuous group of transformations whose structure is known* [This structure is known when the defining equations of the finite transformations of the group are known], *such a prominence is witnessed by the now antique researches of S. Lie as well as by the more recent ones due to Mr. Vessiot* [Cartan referred to [Vessiot 1904b]]. *The integration of a given differential system which admits a group G is traced back to that of a* resolvent *system whose nature may be arbitrary and to that of a series of particular systems (Mr. Vessiot's* automorphic *systems) each one of which corresponds to one of the simple groups appearing in the normal series decomposition of G. The nature of these systems depends upon the structure of the considered simple group.*
>
> *As far as* finite *simple groups are considered, their complete determination was attained by Mr. Killing's researches* [Cartan referred to [Killing 1888-1890]], *later on confirmed by mine* [He referred to [Cartan 1894]]*; with the exception of a limited number of simple groups of a special type, every other simple groups belongs to one of four classes which have been known for long.*

[30]For a modern treatment of Vessiot's splitting technique, see [Ovsiannikov 1982, §26].

> *On the contrary, in the case of infinite continuous groups, S. Lie had indicated four classes as well but it was by no means clear whether other classes could exist; given the lack of a complete structural theory of infinite continuous groups, the problem appeared to be extremely difficult. It turned out to be even more difficult than* a priori *expected, since, unlike to what happens in the finite dimensional case, there exist* intransitive *infinite group which are not isomorphic to any transitive group and among these groups, there may exist simple ones.*[31]

Cartan's classification enterprize consisted of a laborious computational effort which rested upon an ingenuous combination of different techniques stemming on one hand from his structural theory of infinite continuous groups and on the other from some results already contained in [Cartan 1894] concerning the determination of linear (complex) groups which leave nothing planar invariant (i.e. leave no vector subspace invariant).

A crucial step in this direction was represented by the complete determination of all primitive infinite continuous groups from which, he observed, all transitive infinite groups could be deduced. In this respect, Cartan's classification procedure depended in an essential way upon consideration of what he called the *adjoint group* of $G$. Already introduced in [Cartan 1904] and [Cartan 1905], this group, not to be confused with the adjoint group which we dealt with in section 2.1.1, is a linear, finite group which essentially describes the effect that the transformations of $G$ produce upon the

[31] *On sait quelle est l'importance des groupes simples dans les différentes applications qu'on peut faire de la théorie des groupes de transformations. En particulier, en ce qui concerne l'intégration des systèmes différentiels qui admettent un groupe de transformations continu $G$ de structure connue* [*Cette structure est connue si l'on connait les équations des transformations finies du groupe*], *cette importance résulte des recherches déjà anciennes de S. Lie, et de celles, plus récentes, dues à M. Vessiot* [Cartan referred to [Vessiot 1904b]]. *L'intégration d'un système différentiel donné admettant le groupe $G$ est, en effet, ramenée à celle d'un système* résolvant *dont la nature peut être quelconque, et à celle d'une suite de systèmes particuliers (systèmes* automorphes *de M. Vessiot) dont chacun correspond à l'un des groupes simples qui se présentent dans la décomposition du groupe $G$ en une série normale de sous-groupes, et dont la nature ne dépend que de la structure du groupe simple considéré.*
*En ce qui concerne les groupes simple* finis, *leur détermination complète résulte des recherches de M. Killing* [Cartan referred to [Killing 1888-1890]], *confirmées par les miennes* [He referred to [Cartan 1894]]*; en dehors d'un nombre très restreint de groupes simples particuliers, tout les autres se partagent en quatre grandes classes, connues d'ailleurs depuis longstemps.*
*En ce qui concerne au contraire les groupes simples infinis, S. Lie en avait indiqué également quatre grandes classes, mais on ne savait s'il en existait d'autres; le probème paraissait difficile à aborder en l'absence de toute théorie précise sur la structure des groupes infinis. Il était même plus compliqué qu'on ne pouvait* a priori *se le figurer, car, à l'inverse de ce qui se passe pour les groupes finis, il existe des groupes infinis* intransitifs *qui ne sont isomorphes à aucun groupe transitif, et, parmi ces groupes intransitifs, il peut en exister de simples.* See [Cartan 1909, 93-94].

linear elements having a fixed center, say $P$.

The notion was not a new one. Lie had profitably employed it already in 1888 en route towards the classification of finite transitive continuous groups in $n$ variables[32]. Indeed, in [Lie 1888, chap. 28] he proved that every $r$-term transitive continuous group has associated with it a linear group of infinitesimal transformations defined as follows. Let

$$X_k(f) = \sum_{i=1}^{n} \xi_{ik}(x) \frac{\partial f}{\partial x_i}, \quad k = 1, \cdots, r,$$

indicate $r$ independent infinitesimal transformations of $G$. Lie considered then the first prolongation of $G$ by regarding the variables $x_1, \cdots, x_n$ as functions of a single auxiliary variable $t$ (not transformed by $G$) and introducing the derivatives $x'_i = \frac{dx_i}{dt}$. It is the group generated by the following transformations:

$$X_k^{(1)}(f) = \sum_{i=1}^{n} \xi_{ki}(x) \frac{\partial f}{\partial x_i} + \sum_{i=1}^{n} \left( \sum_{\nu=1}^{n} \frac{\partial \xi_{ki}}{\partial x_\nu} x'_\nu \right) \frac{\partial f}{\partial x'_i}, \quad k = 1, \cdots, r.$$

He then observed that this prolonged group indicates precisely how the $\infty^{2n-1}$ line elements (*Linienelemente*) $x_1, \cdots, x_n; x'_1 : \cdots : x'_n$ are transformed by the action of $G$.

At this point, Lie introduced what today we would call the isotropy algebra associated to a given point $P = (x_1^0, \cdots, x_n^0)$ of "general position", i.e. the set $H$ of all those infinitesimal transformations of $G$ such that $X(x_0) = 0$. He then supposed that the dimension of the linear isotropy algebra is equal to $r - q$ and designated its generators with $X_1^0, \cdots, X_{r-q}^0$. As a consequence of the fact that $x_0$ is left invariant by $X_1^0, \cdots, X_{r-q}^0$, Lie could write the following power series expansion in $(x_i - x_i^0)$ in the form:

$$X_k^0(f) = \sum_{\nu=1}^{n} \left\{ \sum_{i=1}^{n} \alpha_{ki\nu}(x_1^0, \cdots, x_n^0)(x_i - x_i^0) + \cdots \right\} \frac{\partial f}{\partial x_\nu}, \quad k = 1, \cdots, r - q.$$

Correspondingly, the infinitesimal transformations of the prolonged group associated to $H$ could be expanded in the following way:

$$\begin{aligned} X_k^{(1),0}(f) &= \sum_{\nu=1}^{n} \left\{ \sum_{i=1}^{n} \alpha_{ki\nu}(x_1^0, \cdots, x_n^0)(x_i - x_i^0) + \cdots \right\} \frac{\partial f}{\partial x_\nu} \\ &+ \sum_{\nu=1}^{n} \left\{ \sum_{i=1}^{n} \alpha_{ki\nu}(x_1^0, \cdots, x_n^0) x'_\nu + \cdots \right\} \frac{\partial f}{\partial x'_\nu} \quad k = 1, \cdots, r - q. \end{aligned}$$

By restricting himself to the line elements passing through the point $P = (x_1^0, \cdots, x_n^0)$, Lie finally obtained the linear homogeneous group $L$ in the

[32] In this respect, see also [Hawkins 2000, §7.1].

variables $x_i'$ (what today we would call the linear isotropy algebra) which describes how the linear elements in $P$ (nowadays we would say the tangent vectors in $P$) are transformed by the action of $G$:

$$L_k(f) = \sum_{i,\nu=1}^{n} \alpha_{ki\nu}(x_1^0, \cdots, x_n^0) x_i' \frac{\partial f}{\partial x_\nu'}, \quad k = 1, \cdots, r-q. \tag{5.24}$$

In the case of transitive groups, Lie observed, the linear group $\mathcal{L}$ is independent of the choice of the fixed point $P$ in the sense that if $y_0$ is another point in general position then the corresponding linear homogeneous group, say $\mathcal{M}$, is conjugate to $\mathcal{L}$, i.e. there exists a linear transformation $T$ such that $\mathcal{M} = T^{-1}\mathcal{L}T$.

As a consequence of this, Lie explained, the linear stability group could be regarded as a useful classification criterion for transitive groups in $n$ variables. He even provided a general procedure consisting of four steps[33], the first one being the determination of all types of linear homogeneous groups in $n$ variables, which could, at least in principle, lead to a complete determination of all (finite) transitive groups. However, Lie himself expressed doubts on the practical possibility of carrying out such a determination in full generality. For this reason in [Lie 1888, chap. 29] he decided to limit himself to addressing the long way more accessible problem of determining all (transitive) finite continuous groups $G$ acting on $n$-dimensional manifolds whose linear stability group $\mathcal{L}$ coincides either with the general linear group, $\mathbf{GL}(n, \mathbb{C})$ or with the special linear group, $\mathbf{SL}(n, \mathbb{C})$. The noteworthy result that Lie was able to prove was that in this case there exist three possibilities only: either $G$ is similar (*ähnlich*) to the general projective group $\mathbf{PGL}(n, \mathbb{C})$, or to the general linear group $\mathbf{GL}(n, \mathbb{C})$ or to $\mathbf{SL}(n, \mathbb{C})$.

Some years later, similar procedures and techniques were applied by Lie in [Lie 1895a, chap. 2] to determine those infinite groups which exhibit the maximum degree of transitivity in the infinitesimal (*die im Infinitesimalen die größtmöglichst Transitivität besitzen*), that is are such that their linear stability group coincide either with $\mathbf{GL}(n, \mathbb{C})$ or to $\mathbf{SL}(n, \mathbb{C})$. In analogy with the finite dimensional case, it turned out that, under such hypotheses, there are only three possible types of infinite continuous group: the group of all point transformations of a $n$-dimensional space, the group of all volume-preserving transformations (the equivalent group) in $n$-variables and the group of all transformations which preserve volume elements module a constant factor (the proportional group).

In [Cartan 1909] Cartan was able to take great profit of some of the techniques already employed by Lie and, at the same time, to provide them with the systematic character they lacked of. In this respect, Cartan's structural theory as developed in [Cartan 1904] and [Cartan 1905] proved to play an essential role in the classification undertaking.

[33] See [Lie 1888, §151].

In particular, Cartan's approach had the advantage of providing directly the infinitesimal transformations of the linear stability algebra (the adjoint group, in Cartan's language[34]) of an infinite transitive group $G$. Indeed, as he had observed in [Cartan 1905, §35], the linear stability algebra of an infinite group coincides with the linear group $\Gamma$ whose existence was guaranteed by his Third Fundamental Theorem.

Clearly inspired by Lie's results, Cartan showed that the classification of all primitive infinite continuous groups could be carried out by starting from a complete determination of all linear homogeneous groups $\Gamma$ which fulfill the following requirements: i) it leaves no planar variety invariant; ii) it is *semi-involutive*[35]. By exploiting results already obtained by him in [Cartan 1894], Cartan was able to attain a classification of such groups from which he deduced the complete determination of all primitive infinite continuous groups in $n$ variables. On the basis of these results, Cartan was finally able to provide a classification of all infinite simple groups[36].

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[Cartan 1909] marked the end of an era in Cartan's mathematical production. After more than ten years over which his focus of attention had been almost exclusively concentrated on developing a structural theory of infinite continuous groups, Cartan turned to more application oriented researches. With the exception of three important papers[37] on representation theory and the classification of real semisimple Lie algebras, the theory of partial differential equations, and above all, differential geometry assumed a predominant position among his research priorities.

Despite the strategic and almost propaedeutical role which his approach to group theory by means of exterior differential forms played in his subsequent works, Cartan's theory of infinite continuous groups did not attract much interest among his contemporaries. On the contrary, it could even been argued that they were almost ignored.

In this respect, Ugo Amaldi's remarks in the preface to his monumental memoir[38] on the classification of all point transformation groups in three variables are quite enlightening. Although Amaldi recognized the importance and the fruitfulness of Cartan's approach, he preferred to rely upon Lie's classical theory. Still in 1909, Cartan's techniques were considered by him to be too audacious, if not extravagant[39]. He wrote:

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[34] Cartan would shift to the wording *stability group* only in [Cartan 1937a].

[35] For the relevant definition, see [Cartan 1909, p. 99].

[36] Attempts to rigorize Cartan's classification results were pursued only recently in [Sinberg Sternberg 1965].

[37] In this respect, see [Hawkins 2000, chap. 8].

[38] See [Amaldi 1912] and [Amaldi 1913].

[39] In actual fact, over the following years, Amaldi became an enthusiastic admirer (one of the few) of Cartan's work. It seems that the publication of [Cartan 1909] can be indicated

> *As for Cartan's researches, we have to acknowledge the exceptional importance of the results which they have produced. Nonetheless, a discerning critic could perhaps argue that Cartan has gone too far away from the principles of the theory founded by Lie and that, in particular, he has avoided explicit use of Lie's wonderful theory of differential invariants. He has replaced Lie's ingenious synthetic views, which nothing for now allows us to consider as depleted in their native fecundity, with a complex and sometimes involuted system of analytical tools.*[40]

as the main motive for such a radical change of attitude. As for Amaldi's interest in Cartan's mathematical activity, see [Nastasi and Rogora Ed.s 2007].

[40] *E per quanto riguarda le ricerche del Cartan, è bensì doveroso riconoscere la eccezionale importanza dei risultati a cui esse hanno già approdato. Ma un critico esigente potrebbe forse rammaricare che il Cartan si sia troppo allontanato dai principi stabiliti per la teoria dal Lie e abbia in particolare evitato, almeno esplicitamente la maravigliosa teoria degli invarianti differenziali, per sostituire un complesso e talvolta involuto sistemi di procedimenti analitici alle geniali vedute sintetiche del Lie, che nulla per ora ci permette di considerare esaurite nella loro nativa fecondità.* [Amaldi 1912, p. 277-278].

# Chapter 6

# Cartan's method of moving frames

When we think of Cartan's contributions to differential geometry, our mind turns spontaneously to the method of moving frames. Indeed, a conspicuous amount of Cartan's achievements in this field can be described as an application or a further development of those fruitful views disclosed by this general technique. Roughly speaking, the method of moving frames consists of an algorithmic approach to the computation of the differential invariants of varieties immersed in some homogeneous space. Its generality lies mainly in the possibility of being applied in various different settings independently of the fundamental group which defines the geometry (Euclidean, affine, projective, etc.) of the homogeneous space under examination.

Retrospectively, it would be quite natural to traced back the origin of Cartan's techniques to the research program inaugurated by F. Klein in 1872 in his well known *Erlanger Programm*; nevertheless, it appears that the early development of Cartan's geometrical methods should rather be described in the light of his structural theory of continuous (finite and infinite) groups as well as of the longstanding French tradition dating back to G. Darboux consisting of a cinematical approach to differential geometry of curves and surfaces. Cartan himself was very explicit in acknowledging such a twofold origin of his theory when, for example in the introduction to [Cartan 1910], he claimed that a geometrical (cinematical) interpretation of his structural theory of finite continuous groups quite naturally leads to a generalization of Darboux's classical theory of moving trihedrons. After all, attempts of generalization of Darboux' technique to group action other than that of the Euclidean group had already been tempted some years before by other mathematicians such A. Demoulin and É. Cotton[1].

At the same time, Cartan observed, the application of group theory to differential geometry was for him the occasion to provide some clarification

[1] In this respect, see [Akivis, Rosenfeld 1993, p. 147-148].

on the connection between his own approach to finite continuous groups and Lie's. Finally, such a geometrical reflection allowed him to remarkably simplify his theory in the finite dimensional case. Indeed, he admitted, since his 1904-1905 theory was designed mainly to handle with infinite groups, the general procedure so far developed appeared scarcely adapted to deal with the finite dimensional case. Small adjustments had to be operated especially for what concerned the method for obtaining the invariant Pfaffian expressions (the Maurer-Cartan forms) which define the structure of a group.

Cartan's starting point in [Cartan 1910] was the specialization of the first fundamental theorem of his structural theory of continuous groups to the case of a finite group. To this aim, he considered a finite $r$ parameters continuous group $G_r$ acting on a $n$ dimensional numerical space. Furthermore, he supposed that $G_r$ is transitive (this requires that $n \leq r$). On the basis of these premises, the first fundamental theorem reads as follows:

**Theorem 26 (Cartan 1904)** *One can add to the $n$ variables $x_1, \cdots, x_n$ of the space upon which $G_r$ acts $r-n$ auxiliary variables $u_1, \cdots, u_{r-n}$ such that its holoedric prolongment $G'_r$ that one obtains is characterized by the property of leaving invariant $r$ linearly independent Pfaffian forms $\omega_1, \cdots, \omega_r$.*

In 1904 Cartan had deduced the existence of these invariant Pfaffian forms from the defining differential system of the group. However, in the case of finite dimensional group, as he said, the procedure that was adopted appeared to be artificial. For this reason, in 1910 he set out to provide an alternative, more direct deduction which relied upon what Cartan later would call the abstract group associated to $G_r$, i.e. the group of parameters of $G_r$[2].

The construction of the parameter group, already introduced by Lie[3], was addressed by Cartan in the following way. He considered a $r$-parameter finite groups of transformations:

$$x'_i = f_i(x_1, \cdots, x_n; a_1, \cdots, a_r), \qquad (i = 1, \cdots, n).$$

By indicating with $S_a$ the transformation which corresponds to the set of parameters $a = (a_1, \cdots, a_r)$, the fundamental group property of closeness under composition, reads $S_c = S_b S_a$, where $a$ and $b$ are arbitrary parameters and $c$ are determined analytical functions of $a$ and $b$, $c_i = \phi_i(a, b)$, $i = 1, \cdots, r$. The parameter group $\Gamma$ associated to $G_r$ is defined by the formula $S_{\xi'} = S_a S_\xi$, where $a$ are to be considered as parameters of $\Gamma$, and $\xi$ and $\xi'$ are the original and the transformed variables, respectively. It is easy to see that $\Gamma$ is simply transitive since given two arbitrary transformations $S_{\xi'}$ and $S_\xi$ there exists one and only one transformation $S_a$ that maps one into the

[2]Since $G'_r$ is a simply transitive group, it can be identified with the parameter group of $G_r$. For this reason, there was no need to resort to the notion of holoedric prolongation.

[3]See for example [Lie 1888], in particular chapter 21. See also appendix A.

other.
Suppose that $a_1^0, \cdots, a_r^0$ indicate the parameters of $G_r$ corresponding to the identity transformation, then the parameters of $S_\xi^{-1} S_{\xi+d\xi}$ considered as a transformation of $G_r$ are infinitely close to $a_1^0, \cdots, a_r^0$ and consequently can be written as

$$a_1^0 + \omega_1, a_2^0 + \omega_2, \cdots, a_r^0 + \omega_r,$$

where $\omega_i$, $i = 1, \cdots, r$ are Pfaffian forms in the variables $\xi$,

$$\omega_i = \alpha_{i1}(\xi_1, \cdots, \xi_r) d\xi_1 + \cdots + \alpha_{ir}(\xi_1, \cdots, \xi_r) d\xi_r, \qquad (i = 1, \cdots, r). \quad (6.1)$$

A first fundamental result obtained by Cartan was the discovery that these Pfaffian forms are precisely those forms which are left invariant by the action of the parameter group $\Gamma$ (or, what is the same modulo similarity, by the action of $G'_r$). Indeed, if one replaces in (6.1) the variables $\xi$ by their transformed $\xi'$ under the action of an arbitrary transformation of $\Gamma$, one recovers the same expressions. This is a simple consequence of the following chain of equalities:

$$S_{\xi'}^{-1} S_{\xi'+d\xi'} = (S_a S_\xi)^{-1} (S_a S_{\xi+d\xi}) = S_\xi^{-1} S_a^{-1} S_a S_{\xi+d\xi} = S_\xi^{-1} S_{\xi+d\xi}.$$

The converse is also true, that is every transformation acting upon the variables $\xi$ which leaves the Pfaffian forms (6.1) invariant is a transformation belonging to the parameter group.

Cartan provided a very simple example in which a direct computation of the Pfaffian expressions (6.1) can be operated in a straightforward way. He considered the affine group of the straight line: $x' = ax + b$. It is easy to compute the defining equations of the parameter group:

$$\begin{cases} \xi' = a\xi, \\ \eta' = a\eta + b. \end{cases} \qquad (6.2)$$

The parameters of $S_\xi^{-1} S_{\xi+d\xi}$ are $1 + \frac{d\xi}{\xi}, \frac{d\eta}{\xi}$ and consequently, since the parameters of the identity transformation are 1 and 0, the forms $\omega_1, \omega_2$ can be chosen to be:

$$\omega_1 = \frac{d\xi}{\xi}, \quad \omega_2 = \frac{d\eta}{\xi},$$

which are immediately recognized to be invariant under the action of (6.2).

On the basis of these premises, Cartan set out to provide a geometrical interpretation of the invariant Pfaffian forms which allowed him to regard them as the components of the relative displacement of a set of moving references (*références mobiles*). The first problem that Cartan had to solve was an adequate, general definition of what a moving reference is. In the study of curves and surfaces in Euclidean geometry, such a definition was imposed by intuitive geometrical insight. The geometry of curves, for example, according to Darboux's approach could be profitably tackled by introducing a family of trirectangular trihedra, the so-called Frenet's trihedra, whose axes coincide with the tangent, the principal normal and the binormal to the curve under examination. In an analogous way, the study of surfaces was carried out by Darboux by introducing a set of moving trirectangular trihedra intrinsically associated to the given surface in such a way that their axes coincided with the two principal directions and with the normal to the surface.

The terms of the question were quite clear: how a generalization of Darboux's moving frames could be obtained when the group of Euclidean movements is replaced by an arbitrary transitive $r$-parameter group?

Cartan's starting point consisted in singling out what he considered to be the essential property upon which the success of Darboux's method relied: any two trirectangular trihedra could be transformed one into the other by the action of one, and only one, transformation of the Euclidean group. According to this observation, families of general moving references should be defined in such a way that this property continues to be valid.

To this end, Cartan considered a $n$ dimensional manifold $M$ upon which a transitive group $G_r$ acts. He claimed that a geometric figure $F_0$ in $M$ consisting of points, or lines, etc. could be chosen as to ensure that the following requirement is satisfied. By indicating with $F_1$ the geometric figure deduced from $F_0$ by the action of the transformation $S_a$, there does not exist another transformation belonging to the group which maps $F_0$ into $F_1$.

In every conceivable case, Cartan observed, one is lead to consider a family of geometrical figures in the $n$ dimensional space which depend upon $r$ parameters. Once a choice of these parameters has been operated, one can indicate with $F_\xi$ the geometrical figure which corresponds to the set of parameters $\xi = (\xi_1, \cdots, \xi_r)$. The set of $F_\xi$ could then be considered as a system of reference figures (*figures de référence*) which could be employed to introduce the notion of *relative coordinates.*

Cartan defined them in the following way. The relative coordinates of a generic point $M$ with respect to the reference $F_a$ were defined to be the coordinates of the point $P$ such that $S_a P = M$. In particular, the relative coordinates of a point $M$ with respect to the original figure $F_0$ coincide with ordinary coordinates which Cartan called *absolute coordinates.*

On the basis of these definitions, Cartan could attribute to the transformation $S_a$ of $G_r$,

$$x'_i = f_i(x_1, \cdots, x_n; a_1, \cdots, a_r), \qquad (i = 1, \cdots, n), \tag{6.3}$$

the following geometrical interpretation: (6.3) establishes the relations among the absolute coordinates of a point $M$, $x'_1, \cdots, x'_n$, the relative coordinates $x_1, \cdots, x_n$ and the parameters $a_1, \cdots, a_r$ of the reference figure $F_a$ to which the relative coordinates are referred.

In an analogous way, Cartan introduced the notion of relative and absolute coordinates of a transformation of a group. He actually employed the cinematical expression *displacement* instead of transformation when such a transformation was considered to be acting on geometrical figures. To this end, he preliminarily defined the relative position of $F_a$ and $F_b$ to be equal to the relative position of $F_{a'}$ and $F_{b'}$ when there exist a (necessarily unique) transformation $S_c$ such that $F_{a'} = S_c F_a$ and $F_{b'} = S_c F_b$.

The absolute coordinates of the displacement which maps $F_a$ into $F_b$ were defined to be the parameters of the transformation $S$ which gives $F_b$ when applied to $F_a$, i.e. the parameters of $S_a S_b^{-1}$. Instead, the relative coordinates of the displacement mapping $F_a$ into $F_b$ were defined to be the parameters of the transformation mapping $F_0$ into the geometrical figure which occupies, with respect to $F_0$, the same relative position of $F_b$ with respect to $F_a$. In view of the preceding, it is easy to see that the relative coordinates coincide with the parameters of the transformation $S_a^{-1} S_b$.

By relying upon such definitions, Cartan was finally able to provide a geometrical interpretation of the Maurer-Cartan forms associated to the parameter group $\Gamma$. They coincide, he concluded, with the relative coordinates of the infinitesimal displacement $S_\xi^{-1} S_{\xi+d\xi}$ of the moving reference figure.

In the light of such a geometrical insight, Cartan set out to examine the connection between his structural theory of continuous groups and Lie's. To this end, he considered a moving reference figure $F_\xi$ depending on $r$ parameters $\xi = (\xi_1, \cdots, \xi_r)$ and a fixed point $M$ whose relative and absolute coordinates are given by $x = (x_1, \cdots, x_n)$ and $x^0 = (x_1^0, \cdots, x_n^0)$, respectively. On the basis of the preceding geometrical interpretation, if one indicates with $z' = f(z, \xi)$ the transformation of the group $G_r$ corresponding to $\xi$, then these coordinates are connected by the following set of $n$ relations:

$$x_i^0 = f_i(x_1, \cdots, x_n; \xi_1, \cdots, \xi_r), \qquad (i = 1, \cdots, n). \tag{6.4}$$

As a consequence of this, when $x_i$, $i = 1, \cdots, n$ are regarded as functions of the variables $\xi_1, \cdots, \xi_r$, they can be considered as the integrals of a completely integrable[4] Pfaffian system which, by introducing the invariants Pfaffian forms $\omega_1, \cdots, \omega_r$, can be written in the following form:

$$\begin{cases} dx_1 + X_{11}(x,\xi)\omega_1 + \cdots + X_{r1}(x,\xi)\omega_r = 0, \\ \cdots \\ dx_n + X_{1n}(x,\xi)\omega_1 + \cdots + X_{rn}(x,\xi)\omega_r = 0. \end{cases} \tag{6.5}$$

The above mentioned connection with Lie's classical theory could be established, Cartan observed, as a result of the fact that the coefficients $X_{ki}$, $(k = 1, \cdots, r; i = 1, \cdots, n)$ do not depend upon the variables $\xi$. In order to prove this, Cartan supposed that a transformation $S_a$ acts upon the reference $F_\xi$, producing a new reference $F_{\xi'}$. The relative coordinates of the point $M' = S_a M$ with respect to $F_{\xi'}$ are still identified with the n-uple

[4] Complete integrability is a consequence of the fact that $x_i$, $i = 1, \cdots, n$ depend only upon $n$ arbitrary constants, $x_i^0$, $(i = 1, \cdots, n)$.

$(x_1, \cdots, x_n)$. Consequently, the system (6.5) remains invariant if, without affecting $x_i$'s, one replaces $\xi$ with $\xi'$ defined by $S_{\xi'} = S_a S_\xi$. Since the linearly independent Pfaffian forms $\omega_1, \cdots, \omega_r$ are invariant under the action of the parameter group $\Gamma$, Cartan deduced that

$$X_{ki}(x, \xi') = X_{ki}(x, \xi), \qquad (k = 1, \cdots, r; i = 1, \cdots, n).$$

Finally, since $\Gamma$ is a simply transitive group, such relations hold for arbitrary values of $\xi$ and $\xi'$, and the only possibility is that $X_{ki}$ do not depend upon $\xi$ at all.

As Cartan emphasized, the consequences of this theorem were highly significant, since, as it was easy to prove, the coefficients $X_{ki}$ could be interpreted as the components of $r$ independent infinitesimal transformations generating $G_r$. Indeed by introducing the symbols:

$$X_k(f) = \sum_{i=1}^{n} X_{ki}(x) \frac{\partial f}{\partial x_i}, \qquad (k = 1, \cdots, r),$$

equations 6.5 could be written in the compact form:

$$df + X_1(f)\omega_1 + X_2(f)\omega_2 + \cdots + X_r(f)\omega_r = 0; \tag{6.6}$$

furthermore, since the coefficients $X_{ki}$ can be computed in correspondence with any arbitrary set of parameters $\xi$, one can differentiates 6.4 and then impose $\xi = \xi^0$. In this way, one obtains that

$$X_{ki} = \left( \frac{\partial f_i}{\partial \xi_k} \right)_{\xi = \xi^0},$$

which is easily seen to be just the expression for the components of infinitesimal transformations in Lie's sense.

By discussing a concrete, simple example Cartan explained how the just prescribed procedure could be profitably apply in order to deduce from the defining equations of a group its infinitesimal transformations as well as its invariant Pfaffian forms. To this end, he considered the projective group of the straight line whose defining equations can be written as

$$x' = \frac{\xi_1 x + \xi_2}{\xi_2 x + 1}.$$

The Pfaffian system describing the relative (with respect to a moving reference) coordinate of a fixed point on the straight line read as follows:

$$d \frac{\xi_1 x + \xi_2}{\xi_3 x + 1} = 0.$$

By developing this equation Cartan deduced:

$$(\xi_1 - \xi_2 \xi_3)dx + d\xi_2 + x(d\xi_1 - \xi_2 d\xi_3 + \xi_3 d\xi_2) + x^2(\xi_3 d\xi_1 - \xi_1 d\xi_3) = 0,$$

from which, in view of (6.5), the infinitesimal transformations and the invariant Pfaffian forms of the projective group of the straight line could be derived:

$$X_1(f) = \frac{\partial f}{\partial x}, \quad X_2(f) = x\frac{\partial f}{\partial x}, \quad X_3(f) = x^2\frac{\partial f}{\partial x},$$

$$\omega_1 = \frac{d\xi_2}{\xi_1 - \xi_2\xi_3}, \quad \omega_2 = \frac{d\xi_1 - \xi_2 d\xi_3 + \xi_3 d\xi_2}{\xi_1 - \xi_2\xi_3}, \quad \omega_3 = \frac{\xi_3 d\xi_1 - \xi_1 d\xi_3}{\xi_1 - \xi_2\xi_3}.$$

As far as the structure of the group $G_r$ was concerned, Cartan emphasized the importance of the formula (6.6) which he regarded as a direct connection between his own approach to the theory of continuous groups and Lie's[5]. Indeed, Cartan was able to prove that, as a consequence of equation (6.6), the structure equations of his theory

$$d\omega_s = \frac{1}{2}\sum_{i,j} c_{ijs}\omega_i \wedge \omega_j, \qquad s = 1, \cdots, r, \tag{6.7}$$

are fully equivalent to Lie's classical equations

$$[X_i, X_j] = \sum_{k=1} c_{ijk}X_k, \qquad i, j = 1, \cdots, r. \tag{6.8}$$

[5]Poincaré himself attributed to this formula an essential role. In his report of Cartan's work [Poincaré 1914], he wrote:

> *The structure theory, as illustrated by Lie in his study of finite groups, cannot be generalized in a straightforward way to infinite groups. Mr. Cartan replaced it with another structure theory which, being equivalent to the first as far as finite groups are concerned, is nonetheless susceptible to generalizations. If $f$ indicates an arbitrary function of the variables $x$ and $X_i(f)$ represent the symbols of Lie, one has, identically:*
>
> $$df + \sum X_i(f)\omega_i = 0,$$
>
> *$\omega_i$ being Pfaffian expressions which depend upon the parameters of the group and their differentials.*
>
> *La théorie de la structure, telle que Lie l'expose dans l'étude des groupes finis, n'est pas susceptible d'être immédiatement généralisée et étendue aux groupes infinis. M. Cartan lui substitue donc une autre théorie de la structure, équivalente à la première en ce qui concerne les groupes finis, mais susceptible de généralisation. Si $f$ est une fonction quelconque des variables $x$, et si les $X_i(f)$ représentent les symboles de Lie, on aura identiquement:*
>
> $$df + \sum X_i(f)\omega_i = 0,$$
>
> *les $\omega_i$ étant des expressions de Pfaff dépendent des paramètres du groupe et de leurs différentielles.*

By calculating the exterior derivative of the equation (6.6), Cartan obtained:

$$\sum_{j=1}^{r} X_j(f)d\omega_j + \sum_{j=1}^{r} d(X_j(f)) \wedge \omega_j = 0. \tag{6.9}$$

By replacing in (6.6) $f$ with $X_j(f)$ (for fixed $j$), one has:

$$d(X_j(f)) = -\sum_{k=1}^{r} X_k(X_j(f))\omega_k, \qquad j = 1, \cdots, r.$$

Recalling the structure equations, finally Cartan arrived at the following equations:

$$\frac{1}{2}\sum_{i,j=1}^{r} \left\{ \sum_{k=1}^{r} c_{ijk} X_k(f) - (X_i X_j(f) - X_j X_i(f)) \right\} \omega_i \wedge \omega_j = 0,$$

from which, in virtue of the linear independence of the forms $\omega_i \wedge \omega_j$, Cartan deduced (6.8). In an analogous way, Cartan was able to prove that (6.8) implied (6.7).

Cartan's structure equations (6.7) played an essential role in the application of group theory to geometry. Indeed, a crucial achievement of [Cartan 1910] was the identification of the equations of Darboux' theory of surfaces treated with the method of moving frames with Cartan's structure equations of the Euclidean group. Darboux appears to have been the first person to observe that the components of the relative displacements of a moving frame (he employed the wording *système mobile*) depending upon two parameters (this is the relevant case for the study of surfaces) are not arbitrary; as he had shown[6], they had to fulfill a certain set of partial differential equations of first order which restricts the set of possible moving frames introduced to describe the surface under examination.

Cartan considered the defining equations of the Euclidean group and deduced from them the structure equations which in this case read as follows:

$$\begin{cases} d\omega_1 = \omega_3 \wedge \varpi_2 - \omega_2 \wedge \varpi_3, \\ d\omega_2 = \omega_1 \wedge \varpi_3 - \omega_3 \wedge \varpi_1, \\ d\omega_3 = \omega_2 \wedge \varpi_1 - \omega_1 \wedge \varpi_2, \\ d\varpi_1 = \varpi_3 \wedge \varpi_2, \\ d\varpi_2 = \varpi_1 \wedge \varpi_3, \\ d\varpi_3 = \varpi_2 \wedge \varpi_1. \end{cases} \tag{6.10}$$

Since, in this case, the equations (6.5) can be written as:

$$\begin{cases} dx + \omega_1 + z\varpi_2 - y\varpi_3 = 0, \\ dy + \omega_2 + x\varpi_3 - z\varpi_1 = 0, \\ dz + \omega_3 + y\varpi_1 - x\varpi_2 = 0, \end{cases} \tag{6.11}$$

[6]See [Darboux 1887]; namely, chapters V and VII of the *Première Livre*.

the forms $\omega_1, \omega_2, \omega_3$ and $\varpi_1, \varpi_2, \varpi_3$ could be interpreted as the components of the instantaneous translation and the instantaneous rotation of the moving trihedra respectively. In this way, it was easy to recognize that equations (6.10) are equivalent to Darboux' equations mentioned above.

Over the following years, Cartan applied his method of moving frames to a large variety of cases. However, first attempts of systematization of the theory came only quite later, namely in [Cartan 1935] and in his book [Cartan 1937b] which contains a modified version of a set of lectures delivered during the winter semester 1931-1932 at the Sorbonne University.

This last work, though obscure in certain points, is particularly interesting since not only Cartan developed a general procedure for the specialization of the moving frames associated to an immersed variety in a homogenous space, but he also investigated the possibility of developing a purely geometrical approach to the theory of finite continuous groups which allowed him to avoid any resort to his structural theory of infinite continuous groups, evidently regarded as unsuitable for being digested by a wider audience. Indeed, as Cartan himself was well aware of, his theory of infinite continuous group was far from being the object of the interest of the mathematical community. As he once wrote in October 1928 to Ugo Amaldi, who was one of the very few who read and appreciated Cartan's work on infinite groups, the attention of large part of mathematicians was instead focused on the theory of finite groups, especially in view of the fruitful applications to theoretical physics, namely quantum theory[7].

To overcome such inconvenience, Cartan decided to apply his method of moving frames to deduce all the essential theorems of Lie's theory in a more direct and intuitive way without any mention whatsoever of the theory of infinite groups. The historical connection between the method of moving frames and the infinite group theory was thus obscured in favour of didactic clarity. In this respect, it is interesting to observe that in Cartan's hopes, the treatment of finite group theory so provided in terms of differential forms only could be considered as a first, accessible introduction to his theory of infinite continuous groups. Again, a letter to Amaldi[8] turns out to be very

[7] Cartan wrote:

> *For the moment the interest of mathematicians is more concentrated upon finite continuous groups, given that Weyl has just published a grand volume on the theory of groups and quantum mechanics* [Cartan referred to *Gruppentheorie und Quantenmechanik*, Leipzig, 1928.]. *Infinite groups may become fashionable one day.*
> *Pour le moment l'intérêt des mathématiciens est plûtot concentré sur les groupes finis et continus, puisque H. Weyl vient de faire paraître un gros volume sur la théorie des groupes et la mécanique des quanta. Les groupes infinis viendront peut-être aussi à la mode un jour.* See [Nastasi and Rogora Ed.s 2007, p.184].

[8] See [Nastasi and Rogora Ed.s 2007, p. 262-264].

enlightening. He wrote:

> *Je viens de corriger les dernières épreuves de mon cours de 1931-1932; il paraîtra dans deux mois sous le titre "La théorie des groupes finis et continus et la Géométrie différentielle traités par la méthode du repère mobile". En un certain sens ce livre prépare à l'intelligence de la structure de groupes infinis parce que les groupes finis y sont envisagés suivant la méthode qui convient aussi aux groupes infinis.*[9]

To conclude, Cartan's contributions to differential geometry *via* moving frames can be regarded as a direct outgrowth of his theory of continuous groups which was developed by him in order to handle the infinite dimensional case. As Cartan's subsequent works would reveal, his own structural approach turned out to be the most suitable one for geometrical applications.

[9] *"I have just finished correcting the last drafts of my course lessons of 1931-1932; they will appear in two months under the title "La théorie des groupes finis et continus et la Géométrie différentielle traités par la méthode du repère mobile". In a certain sense this book prepares for the understanding of the structure of infinite groups since therein finite continuous groups are treated according to the method which suits infinite groups as well."*

# Appendix A

# Finite continuous groups

The present appendix is devoted to providing a general account of Lie's theory of finite continuous groups with special attention on the three fundamental theorems. It is intended to fix fundamental notations and technical words[1]. For a thorough historical account on the genesis of the theory, [Hawkins 2000] and [Hawkins 1991] should be consulted.

A finite continuous group of transformations $G$ consists of a family of analytic complex transformations defined on $\mathbb{C}^n$ and parameterized by a finite number of complex parameters:

$$y = f_a(x) = f(x_1, \cdots, x_n; a_1, \cdots, a_r), \tag{A.1}$$

such that if $y = f(x, a)$ and $y' = f(y, b)$ are transformations belonging to this family then also $y' = f(x, c)$ is, for appropriate analytic functions $c_j = \phi_j(a, b)$, $j = 1, \cdots, r$, of $a$ and $b$.
Lie called this family of transformations a $r$-term finite continuous group if the parameters $a_1, \cdots, a_r$ are essential, i.e. ' if it is impossible to introduce as new parameters independent functions of $a_1, \cdots, a_r$ so that as a result the equations (A.1) contain fewer than $r$ parameters" [Lie 1888, p. 12].

It should be noticed that continuous groups entered the scene as pseudogroups of local diffeomorphisms acting on a given $n$-dimensional space rather than as abstract manifolds endowed with a given group structure. The closest Lie and his followers came to the modern notion of Lie group is maybe represented by the so-called *parameter group* associated to the group $G$. It consists of the transformations of type $c_j = \phi_j(a, b)$, $j = 1, \cdots, r$ which are associated to the composition law of the transformations of $G$ itself, $f_b \circ f_a = f_c$. However, in this case too, the treatment provided by Lie remained at a purely local level without any consideration of topological questions. Although there is little doubt that Lie was well aware of the local

[1] The best guide to Lie's theory in the spirit of nineteenth century mathematics is, in our opinion, [Bianchi 1918].

nature of his theory, nonetheless the limitations imposed by such a point of view were rarely considered by him in an explicit way.

A crucial technical tool of Lie's theory was represented by the notion of infinitesimal transformation. It can be introduced in the following way: let $a^0 = (a_1^0, \cdots, a_r^0)$ correspond to the identity transformation, i.e. $x = f(x, a^0)$. Besides, consider the set of increments $da = (\lambda_1 dt, \cdots, \lambda_r dt)$, for constant $\lambda_i$, $i = 1, \cdots, r$. Then, by Taylor expansion in the $a$-variables, one has

$$x_i' = f_i(x, a^0 + da) = f_i(x, a^0) + \sum_{k=1}^{r} \left[\frac{\partial f_i}{\partial a_k}\right]_{a=a^0} \lambda_k dt + \cdots .$$

By posing

$$\xi_{ki}(x) = \left[\frac{\partial f_i}{\partial a_k}\right]_{a=a^0}$$

and introducing the operator symbols $X_k(f) = \sum_{i=1}^{n} \xi_{ki}(x) \frac{\partial f}{\partial x_i}$, we obtain the general infinitesimal transformation $\sum_k \lambda_k X_k(f)$ belonging to the group $G$.

Nowadays, we can interpret these operators as the generators of the (complex) Lie algebra associated to $G$. In Lie's time, the wording usual employed was that of "group of infinitesimal transformations".

### A.0.1 The three fundamental theorems

In order for the family of transformations $y = f_a(x)$ to build up a continuous group it is necessary that certain conditions are satisfied. What Lie called *the first fundamental theorem* of his theory allows precisely to state what these conditions are. At the same time, this theorem guarantees the possibility of generating the finite transformations of $G$ (in a sufficiently small neighborhood of the identity transformation) by means of $r$ independent infinitesimal transformations.

**Theorem 27 (First fundamental theorem)** *Suppose that the family of transformations $y = f_a(x)$ builds up a $r$-term continuous group $G$, then there exists functions $\alpha_{ih}(a_1, \cdots, a_r)$ and $\xi_{ij}(y_1, \cdots, y_n)$ such that the following equations:*

$$\frac{\partial y_j}{\partial a_h} = \sum_{i=1}^{r} \alpha_{ih}(a_1, \cdots, a_r) \xi_{ij}(y_1, \cdots, y_n), \quad j = 1, \cdots, n; h = 1, \cdots, r, \quad \text{(A.2)}$$

*where the determinant $|\alpha_{ih}|$ does not vanish identically and the $\xi_{ij}$ are functions of their arguments such that the $r$ expressions*

$$X_i(f) = \sum_{j=1}^{n} \xi_{ij}(x) \frac{\partial f}{\partial x_j}, \quad i = 1, \cdots, r$$

*represent r independent infinitesimal transformations of G.*
*Viceversa, if a family of transformations $y = f_a(x)$ satisfies the equations (A.2) and besides, it contains the identity transformation in correspondence of a r-uple of parameters $(\bar{a}_1, \cdots, \bar{a}_n)$ such that $|\alpha_{ih}(\bar{a}_1, \cdots, \bar{a}_n)| \neq 0$, then the family of transformations $y = f_a(x)$ is a r-term continuous group which coincides with the totality of all transformations belonging to the 1-parameter subgroups generated by the infinitesimal transformations $\sum_{k=1}^{r} \lambda_k X_k(f)$.*

A second, essential result of Lie's theory, the so-called second fundamental theorem, establishes a bijective correspondence between continuous groups of finite transformations and Lie algebras. From a modern, global standpoint, such a correspondence is clearly defective, however, in view of the fact that the continuous groups considered by Lie had only a local nature, this identification must have appeared to be quite natural and not particularly problematic[2]. Present day terminological distinction between groups of infinitesimal transformations and Lie algebras date back to 1930's when Hermann Weyl in [Weyl 1935] suggested that the wording "group of infinitesimal transformations" be replaced by "Lie algebras". Here is the statement of the theorem which was provided by Lie in [Lie 1893, p. 590]:

**Theorem 28 (Second fundamental theorem)** *Every r-term continuous group $y = f_a(x)$ contains r independent infinitesimal transformations*

$$X_k(f) = \sum_{\nu=1}^{r} \xi_{k\nu}(x) \frac{\partial f}{\partial x_\nu}, \quad (k = 1, \cdots, r),$$

*which obey the following commutation relations:*

$$X_i(X_k(f)) - X_k(X_i(f)) = [X_i, X_k] = \sum_{s=1}^{r} c_{iks} X_s(f), \quad (i, k = 1, \cdots, r), \tag{A.3}$$

*where the $c_{iks}$ are constants. Conversely, r independent infinitesimal transformations which satisfy (A.3) generate a r-term continuous group.*

A third pillar of Lie's theory was represented by the following result which is of essential importance for the classification of finite continuous groups.

**Theorem 29 (Third fundamental theorem)** *Let $X_k(f) = \sum_{\nu=1}^{r} \xi_{k\nu}(x) \frac{\partial f}{\partial x_\nu}$, $(k = 1, \cdots, r)$, be r independent infinitesimal transformations of a r-term finite continuous group G. If $c_{iks}$ indicate the corresponding structure constants, then the following set of (algebraic) relations hold:*

$$\begin{cases} c_{iks} + c_{kis} = 0, \\ \sum_{\tau=1}^{r} \{c_{ik\tau} c_{\tau js} + c_{kj\tau} c_{\tau is} + c_{ji\tau} c_{\tau ks}\} = 0. \end{cases} \tag{A.4}$$

[2]Not unproblematic though. See [Hawkins 2000, p. 86-87] for an interesting discussion of the discovery made by F. Engel around 1890 that not every finite transformation of $\mathbf{SL}(2, \mathbb{C})$ is generated by an infinitesimal transformation.

*Conversely, if $r^3$ constants satisfy the preceding relations, then there exist $r$ infinitesimal transformations $X_k(f) = \sum_{\nu=1}^{r} \xi_{k\nu}(x) \frac{\partial f}{\partial x_\nu}$, $(k = 1, \cdots, r)$, (acting on a sufficiently high dimensional manifold) which generate a $r$-term continuous group.*

### A.0.2 The adjoint group

The adjoint group associated to a given $r$-term continuous group plays a crucial role in the classification theory of complex Lie algebras. Our account will closely follow that provided in [Hawkins 2000, §3.3].

First introduced in 1876, such a notion was extensively dealt with in [Lie 1885, p. 91-95]. The basic idea consisted of the observation that a generic infinitesimal transformation could be considered as acting upon other infinitesimal transformations in the following way. Lie considered two infinitesimal transformations

$$B(f) = \sum_{k=1}^{n} X_k(x) \frac{\partial f}{\partial x_k} \text{ and } C(f) = \sum_{j=1}^{n} \xi_k(x) \frac{\partial f}{\partial x_j}.$$

One of them, say $C(f)$, can be regarded as defining the infinitesimal variable change $x'_i = x_i + \xi_i(x)\delta t$, $i = 1, \cdots, n$. When the new variables are substituted in the expression for $B(f)$, one obtains:

$$B(f) = \sum_{k=1}^{n} X_k(x') \frac{\partial f}{\partial x'_k} + \delta t \, [B, C] \,. \tag{A.5}$$

As a consequence of this, Lie observed, upon the action of the infinitesimal transformation $C(f)$, the transformation $B(f)$ is mapped into the transformation $B(f) + \delta t[B, C]$.

Now, if $X = \sum e_i X_i$ indicates the general transformation of a $r$-term group $G$ generated by $X_1, X_2, \cdots, X_r$, then, according to the preceding observation, the transformation $X_k$ maps $X$ into $X + [X, X_k] \, \delta t = \sum_{i=1}^{r} (e_i + \delta e_i) \, X_j$, where $\delta e_i = \sum_{s=1}^{r} c_{ski} e_s$. Thus, in general, to every transformation $X_k \in G$ there is an associated linear infinitesimal transformation

$$E_k = \sum_{s,i=1}^{r} e_s c_{ski} \frac{\partial f}{\partial e_i}, \qquad k = 1, \cdots, r.$$

The set of infinitesimal transformations $E_1, \cdots, E_r$, which are not necessarily independent, builds up, Lie proved, a finite continuous group which is homomorphic to $G$ and which is called the *adjoint group* of $G$.

# Appendix B

# Picard-Vessiot theory

When describing the context in which Cartan's work saw the light, we often insisted on the widespread interest of the French mathematical community towards Lie's theory of groups in view of the foreseen fruitful applications to the realm of integration theory of differential equations. What follows is intended to give a general account of one of the most important manifestations of such an interest: the so-called Picard-Vessiot theory. It is well known that Lie was led to most of his researches of continuous groups by the project of developing a differential analogous of Galois' theory of algebraic equations. Nevertheless, possibly as a consequence of the extreme wideness and generality of his proposal, he was not able to completely turn such an analogy into an outright differential Galois' theory.

On the contrary, by restricting its investigation to a special class of differential equations, namely linear homogenous ordinary differential equations of any degree, the Picard-Vessiot theory succeeded in obtaining such an aim by providing a general theoretical framework which exhibits numerous contact points with its algebraic counterpart.

The genesis of the theory can be traced back to 1883 when É. Picard published a short memoir [Picard 1883] in the *Comptes Rendus de l'Académie des Sciences*, to be followed four years later by a more elaborate paper [Picard 1887] which appeared in the *Annales de la Faculté des Sciences de Toulouse* where he had been teaching for some years (1879-1881) before moving to Paris.
Picard's starting point in [Picard 1887] consisted of underlining the analogies between the theory of algebraic equations and that of ordinary linear differential equations. These analogies had been known for long. It seems that Lagrange had been the first person to draw attention upon them when he proved that if an integral of a linear ordinary differential equation of order $m$ is known then the integration of the equation reduces to the resolution of a linear ordinary equation of order $m-1$, just as in the case of a polynomial equation. Later on, other aspects of this tight connection (e.g. great

common divisor and elimination theory, notion of irreducibility, etc.) were investigated by numerous mathematicians such as Frobenius[1] and Fuchs. However, it was only in the early 1880's that a systematic study of such analogy was inaugurated.
Let us now recall the main facts about this parallelism. If one considers a differential equation of the form

$$\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_n y = 0, \tag{B.1}$$

where $p_j$, $j = 1, \cdots, n$ are rational functions of the independent variable $x$, it is easy to show that the coefficients $p_j$, $j = 1, \cdots, n$ can be expressed in terms of a set of fundamental (i.e. independent) solutions of (B.1), $y_1, y_2, \cdots, y_n$, as follows:

$$p_j = -\frac{\Delta_j}{\Delta}, \qquad (j = 1, \cdots, n),$$

where

$$\Delta = \begin{vmatrix} y_1 & \frac{dy_1}{dx} & \cdots & \frac{d^{n-1}y_1}{dx^{n-1}} \\ y_2 & \frac{dy_2}{dx} & \cdots & \frac{d^{n-1}y_2}{dx^{n-1}} \\ \cdots & \cdots & \cdots & \cdots \\ y_n & \frac{dy_n}{dx} & \cdots & \frac{d^{n-1}y_n}{dx^{n-1}} \end{vmatrix},$$

$$\Delta_j = \begin{vmatrix} y_1 & \frac{dy_1}{dx} & \cdots & \frac{d^{n-j-1}y_1}{dx^{n-j-1}} & \frac{d^n y_1}{dx^n} & \frac{d^{n-j+1}y_1}{dx^{n-j+1}} & \cdots & \frac{d^{n-1}y_1}{dx^{n-1}} \\ y_2 & \frac{dy_2}{dx} & \cdots & \frac{d^{n-j-1}y_2}{dx^{n-j-1}} & \frac{d^n y_2}{dx^n} & \frac{d^{n-j+1}y_2}{dx^{n-j+1}} & \cdots & \frac{d^{n-1}y_2}{dx^{n-1}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ y_n & \frac{dy_n}{dx} & \cdots & \frac{d^{n-j-1}y_n}{dx^{n-j-1}} & \frac{d^n y_n}{dx^n} & \frac{d^{n-j+1}y_n}{dx^{n-j+1}} & \cdots & \frac{d^{n-1}y_n}{dx^{n-1}} \end{vmatrix}.$$

As a consequence of their expression in terms of $\Delta, \Delta_j$ the functions $p, p_j$, $j = 1, \cdots, n$ are (differential) invariants of the general linear group in $n$ variables regarded as acting on a fundamental set of solutions, $Y_i = \sum_j \alpha_{ij} y_j$, $i = 1, \cdots, n$.

Thus, the coefficients of the differential equation (B.1) can be expressed in terms of appropriate functions of its solutions which are invariant under the action of a group. This is in full analogy with the algebraic case in which the coefficients of a given equation can be written in terms of the symmetric functions. The role played by the general linear group in the differential setting is replaced, in the algebraic counterpart, by that of the group of substitutions. Furthermore, as P. Appell had proved in [Appell 1881], every *invariant* (with respect to the linear general group) rational function of the integrals $y_1, \cdots, y_n$ (and of their derivatives until the $n^{th}$ order) can be written as a function of the fundamental invariants $\frac{\Delta}{\Delta_j}$, $j = 1, \cdots, n$, and of the independent variable $x$. This result was regarded to be in complete

[1] Among others, the paper [Frobenius 1873] should be recalled as the one introducing a generalization of the notion of irreducibility to ordinary differential equations.

agreement with the so-called fundamental theorem of symmetric polynomials according to which every symmetric polynomial can be expressed as a polynomial in the elementary symmetric functions.

Motivated by these well-established results as well as by Lie's relatively recent researches on continuous groups of transformations, Picard set out to provide an extension of Galois' algebraic theory according to which the existence of a group (analogous to the Galois group) to be associated to a given differential equation (B.1) could be guaranteed. The construction of this group, which Picard named *groupe de transformations linéaires* of the equation under examination, closely followed Galois' original procedure which, as is well known, exploited in an essential way the notion of *resolvent* of a given algebraic equation.

Picard considered the following linear homogeneous expression in the fundamental integrals $y_1, \cdots, y_n$,

$$V = A_{11}y_1 + A_{12}y_2 + \cdots + A_{1n}y_n + A_{21}\frac{dy_1}{dx} + \cdots + A_{2n}\frac{dy_n}{dx} + \cdots + A_{nn}\frac{d^{n-1}y_n}{dx^{n-1}}, \tag{B.2}$$

where the coefficients $A_{11}, \cdots, A_{nn}$ are arbitrary rational functions of the independent variable $x$. The function $V$, Picard claimed, satisfies a linear differential equation, later to be called Picard's resolvent, of order $n^2$ of the following type:

$$\frac{d^{n^2}V}{dx^{n^2}} + P_1\frac{d^{n^2-1}V}{dx^{n^2-1}} + \cdots + P_{n^2}V = 0, \tag{B.3}$$

where, again, the coefficients $P_j$, $j = 1, \cdots, n^2$ are rational functions of $x$. By differentiating equation (B.2) an appropriate number of times, Picard observed that, once more in analogy with the algebraic case, a system of integrals of (B.1) could be obtained from a given solution of the resolvent equation (B.3):

$$\begin{cases} y_1 = \alpha_1 V + \alpha_2 \frac{dV}{dx} + \cdots + \alpha_{n^2}\frac{d^{n^2-1}V}{dx^{n^2-1}}, \\ y_2 = \beta_1 V + \beta_2 \frac{dV}{dx} + \cdots + \beta_{n^2}\frac{d^{n^2-1}V}{dx^{n^2-1}}, \\ \cdots, \\ y_n = \beta_1 V + \beta_2 \frac{dV}{dx} + \cdots + \beta_{n^2}\frac{d^{n^2-1}V}{dx^{n^2-1}}. \end{cases} \tag{B.4}$$

It may happen, Picard observed, that the set of functions $(y_1, \cdots, y_n)$ provided by this system does not represent a fundamental set of solutions of (B.1). This is the case when the determinant of the $y_i$, $i = 1, \cdots, n$ and their derivatives until the $(n-1)$-th order vanishes. By exploiting eq.s (B.4), this condition takes on the following the form:

$$\phi\left(V, \frac{dV}{dx}, \cdots, \frac{d^kV}{dx^k}\right) = 0, \tag{B.5}$$

where the order $k$ is, at most, equal to $n^2 - 1$. Thus, to every solution $V$ of the resolvent (B.3) which is not a solution to (B.5), there corresponds, by means of (B.4), a fundamental system of solutions of (B.1).
In full analogy with the algebraic case, when the integrals $y_1, \cdots, y_n$ are left undetermined, no solution of (B.3) which does not satisfy (B.5) can be solution of an algebraic differential equation of order less than $n^2$.[2] On the contrary, a particular choice of the integrals $y_1, \cdots, y_n$, has the effect of reducing the resolvent equation; that is, there exists a solution of (B.3) which does not satisfy (B.5) and is a solution of a rational differential equation of order less than $n$. Let us indicate with

$$f\left(x, V, \frac{dV}{dx}, \cdots, \frac{d^pV}{dx^p}\right) = 0. \tag{B.6}$$

an equation which fulfill such an hypothesis; let $p$ be its order.

At this point, Picard had to make explicit recourse to the notion of irreducibility of algebraic differential equations which he took from [Kœnigsberger 1882]. He required $f = 0$ to be *irreducible* in the sense that it is algebraically irreducible with respect to $\frac{d^pV}{dx^p}$ and it does not have any solution in common with another equation of the same type but with order less than $p$.

Under these conditions, Picard observed, every solution of (B.6) is a solution of (B.3); furthermore, (B.6) cannot have any solution in common with (B.5).[3] Thus, to every solution of (B.6), there corresponds, by means of (B.4), a fundamental systems of solutions. Let $y_1, \cdots, y_n$ be the fundamental system corresponding to a particular solution $V$ and let $Y_1, \cdots, Y_n$ be the fundamental system corresponding to the general solution of (B.6). Clearly, one has:

$$Y_j = \sum_{i=1}^{n} a_{ji} y_i, \qquad (j = 1, \cdots, n). \tag{B.7}$$

Picard was able to prove that such coefficients depend algebraically upon $p$ parameters and that the set of transformations (B.7) builds up a group (in accordance with the common usage of the time, he actually limited himself to demonstrating that the product of any two transformations of type (B.7) is again a transformation of the same type). In such a way, Picard arrived at establishing the existence of an algebraic linear group associated to every equation (B.1) which consists of all the transformations of type (B.7) which connect all the fundamental systems of solutions of (B.1) stemming from solutions of (B.6).
This group $G$, later on to be named by F. Klein *rationality group* (*Ra-*

[2]This is the case which corresponds to the case in which the algebraic Galois resolvent is irreducible. In this respect, it may be helpful to consult [Schlesinger 1895,1897,1898, §144 and §149].

[3]This is a consequence of what Schlesinger called Kœnigsberger's theorem. See [Schlesinger 1895,1897,1898, §146].

*tionalitätsgruppe*), turned out to be a perfect candidate for the sought for generalization of Galois group of algebraic equations.

Indeed, Picard was able to prove the following:

**Theorem 30 (Picard, 1887)** *Every rational function of $x$, $y_1, \cdots, y_n$ and their derivatives which is a rational function of $x$ remains invariant (as a function of $x$) under the action (upon $y_1, \cdots, y_n$) of the group $G$.*[4]

Picard's proof can be summarized as follows. Let $R(x, y_1, \cdots, y_n)$ be such a function. When $y_1, \cdots, y_n$ and their derivatives are replaced by their expressions in terms of a particular solution $V$ of (B.6), the hypothesis of the theorem reads:

$$F(x, V, \frac{dV}{dx}, \cdots, \frac{d^pV}{dx^p}) = R(x). \tag{B.8}$$

Picard has thus obtained an equation which has a solution $V$ in common with (B.6). As a consequence of the irreducibility of (B.6), (B.8) admits *any* solution of (B.6). Thus, the function $F$ remains invariant (i.e. remains the same function of $x$) when $V$ is replaced by any other solution of (B.6). But, as Picard observed, this is equivalent to saying that $R(x, y_1, \cdots, y_n)$ remains invariant if a transformation of $G$ is operated upon $y_1, \cdots, y_n$ and so the theorem is proved.

More difficult for Picard was to prove the reciprocal statement according to which every function of $x$, $y_1, \cdots, y_n$ which is invariant under the action of $G$ is a rational function of $x$. Indeed, Picard did not succeed in achieving such a result. He was able to prove only a weaker statement: every invariant function of $y_1, \cdots, y_n$ is a uniform function of the independent variable $x$.

It was only five years later, in 1892, that such a disturbing asymmetry could be removed. The advancement was provided by Vessiot's doctoral dissertation in which Picard's results were taken up from a different perspective in which the influence of S. Lie's researches on continuous group became even more evident. Vessiot's strategy for establishing the existence of the rationality group of a given equation (B.1) consisted first of the study of the differential rational functions associated to (B.1). By these, Vessiot referred to rational functions of the integrals $y_1, \cdots, y_n$ (forming a fundamental set) and their derivatives to be indicated with $R(y_1, \cdots, y_n)$. To every function $R(y_1, \cdots, y_n)$ of this type, there corresponds a group belonging to the general linear group which is defined as the family of those linear transformations $\bar{y}_j = \sum a_{ij} y_j$ which leave $R$ formally invariant, i.e. such that the equation $R(\bar{y}_1, \cdots, \bar{y}_n) = R(y_1, \cdots, y_n)$ reduces to an identity. The equations of the group $\Gamma$ obtained in this way, Vessiot observed, depend algebraically upon a certain number of parameters. Consequently, as he called it by employing a wording already to be found in Picard, $\Gamma$ is a linear algebraic group. The

[4] See [Picard 1887, p. A.5].

study of the invariance of such functions was then carried out by Vessiot by introducing the notion of *transform* of a given rational function $R$. Indeed, the function $R$ considered as a function of $x$ satisfies, Vessiot proved, an algebraic differential equation (the transform of $R$) whose order is connected to the number of parameters of the group $\Gamma$.
This notion enabled Vessiot to prove a theorem which was regarded by him as the differential counterpart of Lagrange's theorem on rational functions of the roots of an algebraic equation. More explicitly, Vessiot's result was the following:

**Theorem 31 (Vessiot 1892)** *Consider a rational function $S(y_1, \cdots, y_n)$. Suppose that $S$ admits every linear transformation which is admitted by a rational function $R(y_1, \cdots, y_n)$. Then, $S$ can be rationally expressed in terms of $R$, of the fundamental invariants $p_j$, $j = 1, \cdots, n$, their derivatives and $x$.*[5]

This achievement played a crucial role in Vessiot's analysis. Indeed, by means of it, Vessiot could deduced in a straightforward way the existence of the rationality group of a given linear differential equation. No explicit recourse to the notion of resolvent was made while the two fundamental properties of Galois group were somehow elevated to the rank of definition[6].

**Theorem 32 (Vessiot 1892)** *To every linear ordinary differential equation, there corresponds a group $G$ of linear homogeneous transformations which is characterized by the following properties: (I) Every rational function of the integrals which admits a rational expression is left invariant by the action of $G$; (II) conversely, every rational function of the integrals which is left invariant by $G$ admits a rational expression.*[7]

In order to show this, Vessiot considered the totality of all rational functions of $y_1, \cdots, y_n$ which have a rational expression in term of $x$ and, among them, he singled out one function which admits (formally) a group with the minimal number of parameters. Let $\Phi$ and $G$ indicate this function and this group, respectively. Vessiot's claim consisted of the assertion that $G$ does coincide with the rationality group of the particular linear ordinary equation under examination. Indeed, he explained, if $R$ is *any* rational function which has a rational expression, then it admits the group $G$ since otherwise the function $\Phi + uR$, where $u$ is an arbitrary rational function of $x$, would admit a group with a number of parameter less than that of $G$, which is absurd in view of the minimality of $G$. Conversely, as a consequence of his own generalization of Lagrange's theorem, *every* rational function $R$ which admits the transformation of $G$ can be expressed in term of $x, p_j$ and $\Phi$ and, thus, it has a rational expression.

[5]See [Vessiot 1892, p. 223].
[6]In this respect, see the final remarks in [Schlesinger 1895,1897,1898, §151].
[7]See [Vessiot 1892, p. 231].

By these results, Vessiot was quick to observe, the theory first sketched by Picard in 1883 and, later on, in 1887 could be regarded as completed since all the characteristic properties of the classical Galois' group were established, without restriction, for the so-called rationality group of a linear ordinary differential equations.

Although they were soon regarded as a major advancement of the theory, Vessiot's results were not beyond censure. As was first pointed out by Klein in 1894 in his *Vorlesungen über die hypergeometrische Funktion*, the approach employed by Vessiot to establish the existence of the rationality group suffered from a lack of discrimination between the notions of formal and numerical invariance. Indeed, as it happens in the algebraic Galois theory, the two types of invariance cannot be regarded as equivalent since it may be the case that a given function $R(y_1, \cdots, y_n)$, when regarded as a function of the independent variable $x$ (i.e. when $y_1, \cdots, y_n$ are replaced by a determined solution of the given equation), admits linear transformations which are not admitted by the same $R(y_1, \cdots, y_n)$ when the latter is regarded as a function of the undetermined quantities $y_1, \cdots, y_n$[8]. As a consequence of this, the identification of the rationality group with the set of linear transformations which leave *formally* invariant all the rational functions $R$ admitting a rational expression in terms of $x$ remained highly questionable.[9]

Picard himself seems to have shared Klein's perplexities, when in 1894, in a brief note published in the *Comptes Rendus*, while praising the important achievements contained in Vessiot's thesis, he still considered his own approach to be the more appropriate one in virtue of its close adherence to Galois' original standpoint. Picard was able to plug the gap of his preceding papers in a way utterly independent of Vessiot's method which rested upon the employment of the so-called *Picard resolvent.*

The problems posed by Vessiot's thesis reflected the enormous difficulties which one encountered in the attempt of providing a reconciliation (intensely longed for in France at that time[10]) between Galois' methods and Lie's more recent techniques. The contrast between numerical and formal invariance

[8]It may be useful to clarify this important point by means of an example taken from Vessiot's recollections [Vessiot 1932, p. 57] dating back to 1932. On this occasion, Vessiot emphasized the fact that even in the algebraic case the two notions must not be confused. Indeed, as is easy to see through direct computation, the expression $3x_1^2+2x_1x_2+3x_2^2+x_3^2$ is numerically invariant when the variables $x_1, x_2, x_3$ are replaced by the solutions (no matter what the order chosen is) of the equation $x^3 + px + q = 0$, $p, q$ being arbitrary but given constants. On the contrary, when $x_1, x_2, x_3$ are considered as undetermined quantities, this expression is not left invariant by *every* substitution of 3 objects.

[9]Schlesinger spoke of *formelle Unveränderlichkeit* and of *Unveränderlichkeit als Funktion von* $x$, [Schlesinger 1895,1897,1898, §152]. Unlike Vessiot, Schlesinger proved to be very attentive to distinguishing between the two types of invariance.

[10]In this regard and more particularly on the necessity of extending Galois' original point of view to the realm of differential equations, [Drach 1893] should be consulted.

was a most patent manifestation of the intrinsic differences between the approaches of the two illustrious mathematicians. Indeed, it was the very recourse to Lie's theory of differential invariants, from which Vessiot took such a great profit, to somehow force him to privilege (unlike Galois' classical treatment) the notion of formal invariance. A function $R$ could be considered to be a differential invariant in Lie's sense only if invariance was intended in its formal variant. All the more so, since it does not even make sense to speak of the group of transformations which leave *numerically* invariant a given differential function $R$ given that, in general, the set of such transformations is not a group.

It was only ten years later, in 1902, on the occasion of the *Grand Prix de Mathematique* that Vessiot was able to attain a fully satisfying treatment of the subject. A decisive role was played by the introduction of the notion of *automorphic system.* The relevant result consisted of the following emendation of Theorem (32): the rationality group turned out to be identified with the group of all (linear) transformations which are *formally* admitted by all the functions $R(y_1, \cdots, y_n)$ having a rational expression in term of $x$, with the exclusion, this is the crucial point, of those $R$'s which numerically admit transformations which are not formally admitted.

In spite of all these severe difficulties, Vessiot's thesis truly represented a landmark in the development of differential Galois theory. Indeed, he was able to develop a reducibility theory for linear ODE's where the structural theory of continuous groups played a decisive role. In analogy with Galois classical theory and Lie's more recent theory of complete systems, Vessiot clarified in a very effective way the intimate connection between the integration procedure of a given equation and the structure of its rationality group.

In order to do this, Vessiot had first to introduce the notion of *domain of rationality* (Vessiot actually did not employ this expression in 1892) which enabled him to regard as rational every function which can be expressed as a rational function of certain, assigned *a priori* functions and their derivatives.
The validity of Theorem (32) was clearly independent of the particular domain of rationality which was considered. Thus, therein the word "rational" could be interpreted in the light of such an extended meaning. On the contrary, the rationality group of a given differential equation essentially depended upon it. Indeed, a first fundamental result of Vessiot's reducibility theory was that the adjunction of some element to the domain of rationality has the effect of reducing the rational group to one of its subgroup.

More specifically, Vessiot was able to demonstrate that upon appropriate choice of the adjoined elements (to be chosen among a fundamental set of solution of certain auxiliary equations) the group $G$ reduces to an invariant subgroup thereof.

On the basis of these results, Vessiot provided a necessary and sufficient condition for the linear equation (B.1) to be solvable by quadratures (i.e. integrations of 1st order equations) only: (B.1) is solvable by means of quadratures if, and only if its rationality group is integrable in the sense of Lie's theory. As a corollary of this theorem, he finally showed that the general linear ordinary differential of order greater than 1 is not integrable by quadratures.

In conclusion, Picard's pioneering works and Vessiot's later developments succeeded in giving concreteness to Lie's *idée fixe* of elaborating a theory of integration for differential equations which was analogous to Galois algebraic theory. All the more so, since their researches made it clear that Lie's generic project was susceptible of being turned into a systematic theoretical framework in which rather ephemeral analogies with Galois classical theory gave way to precise correspondences.

Although some details still needed to be perfected, in the short space of a few years the so-called Picard-Vessiot theory became a classical topic to be included in general treaties on differential equations. At the same time, it inaugurated a fruitful period of intense researches, especially in France, in which wider extensions of Galois classical ideas to general systems of differential equations were sought for.

# Appendix C

# Jules Drach, the Galois of his generation

With the exception of few authoritative scholars[1], it seems that the work of Jules Drach (1871-1949) did not obtain the acknowledgement which it would have deserved. Mathematicians such as Jules Pommaret[2] emphasized the outstanding relevance of Drach's achievements in the development of modern differential Galois theory, nevertheless, to our knowledge, no specific historical analysis of Drach's mathematical contributions has been provided thus far. This appendix is intended to yield some indications of the general ideas at the basis of his work. Special attention will be paid to the intimate connection between Drach's logical theory of integration and the theory of infinite continuous groups; by doing so, we will be able to obtain a better understanding of the mathematical context in which Vessiot's and Cartan's papers on infinite continuous groups saw the light.

Jules Drach was born in 1871 in Sainte-Marie-aux-Mines, a small village of Alsace, from a humble family of farmers. Admitted at the École Normale Supérieure in 1889 along with Émile Borel with whom he soon became good friend, Drach rapidly gained the admiration of his teachers. Jules Tannery was particularly lavish in his judgements. In the introduction to [Tannery 1895], he employed prophetical words of praise which give a vivid and effective description of Drach's personality and talent. After speaking of É. Borel and of his already numerous achievements, presenting the reader with Drach's contributions, Tannery wrote:

> *I do believe that Drach too will soon take a similar place: he is one of those people who are concerned, above all, with the nature of things, one of those people who remain fidgety and unsatisfied*

[1] See, for instance, [Dugac, Taton 1981]. Therein the reader can find a rich survey of documental sources on J. Drach, also including Drach's correspondence with E. Vessiot which we will be concerned with.

[2] See [Pommaret 1988, chap. I].

> *until they reach the core. This philosophical attitude of the spirit may be dangerous when it works in vain and is not accompanied with the knowledge of facts or despises particular truths which are essential components of science. Nevertheless, it is such an attitude which presides the following material.*[3]

As it was observed years later by A. Chatelet, these early observations by Tannery described in a precise way the twofold nature of Drach's mathematical thought: on one hand, they indicated Drach's interest in metamathematical issues, on the other, they emphasized his firm conviction that specific applications too should not be disregarded.

Since his first works, Drach's concern with foundational problems was manifest. For example, in the second part of [Tannery 1895] which contained a general introduction to the theory of algebraic numbers, he provided an original reinterpretation of Galois theory of algebraic equations in which the main focus of attention was devoted to giving a proof of the possibility of extending the notion of rational number by adding to a certain known domain (namely, the absolute domain of rational numbers) new symbols (the roots of a given algebraic equation) which obey the same fundamental properties of the usual composition laws of addition and multiplication. In modern terms, Drach's treatment of the foundations of Galois theory can be described as an attempt to provide a constructive proof of the existence of a splitting field for a given irreducible polynomial with rational coefficients[4].

In his doctoral dissertation, which he defended in 1898, Drach tried to extend his researches contained in [Tannery 1895] to the realm of analysis by providing a logical foundation of the theory of functions which are defined by systems of differential equations. In view of such an aim, he first convey an axiomatic, algebraic theory of differentiation from which any recourse to the notion of continuity and to that of limit was excluded. By doing so, however rudimentarily, he attained the abstract notion of differential field.

As Drach explained in the introduction to [Drach 1898] and in his later recollections [Drach 1909], this ambitious research program made it necessary to have a generalization of Galois theory of algebraic equations to general differential systems.

He acknowledged the enormous importance of Lie's contribution to such an attempt, however, he regarded Lie's efforts as essentially pointing in the

[3] *Je crois bien que M. J. Drach ne tardera pas, lui aussi, à prendre une pareille place: il est de ceux qui se préoccupent avant tout du fond des choses, qui restent mécontents et inquiets tant qu'ils n'ont pas atteint le roc. Cette tendence philosophique de l'esprit est un danger quand elle travaille à vide, qu'elle n'est pas accompagnée de la connaisance des faits, et qu'elle engendre le mépris des vérités particulières, matérieux essentiels de la science: c'est elle seule, malgré tout, qui préside à l'arrangement des ces materieux.* See [Tannery 1895], *Préface*, II.

[4] In this respect, [Edwards 1993, §§49-61] should be consulted, although no mention of Drach's contribution is made there.

*wrong* direction. As he explained in 1893 in [Drach 1893], in contrast with Galois' point of view, Lie's strategy was more similar to the approach followed by Abel which consisted of drawing attention to the equation itself rather than on the corresponding system of equations deduced from equating the symmetric functions of the roots to the coefficients of the given equation[5] Indeed, he observed, the success of the Picard-Vessiot theory had to be ascribed, for large part, to the restoration of Galois' point of view, consisting of the substitution of the linear ordinary differential equation under examination with the corresponding invariant system built up by means of the fundamental invariant functions (invariant with respect to the general linear group) as well as of the (rational) coefficients of the equation itself.

Drach declared that in pursuing this aim he was profoundly inspired by some remarks made by S. F. Lacroix (1765-1843) in his *Traité de Calcul différentiel et intégral* where, after noticing the impressive variety of elements which satisfy differential relations and the corresponding slenderness of cases in which such relations can be solved in terms of elementary transcendents, he had written:

> *These remarks induce me to believe that what may contribute the most to the development of the integral calculus is both the classification of completely irreducible and consequently essentially distinct transcendents and the search for the properties of each one of these types.*[6]

This very challenge proposed almost a century before represented Drach's main motivation. Indeed, as the classical Galois theory and the more recent Picard-Vessiot theory provided a classification of the transcendents defined by algebraic and linear ordinary differential equations respectively, his aim was that of generalizing this classification in such a way as to cover transcendents defined by more general differential systems. More precisely, Drach tackled the problem of developing a theory of integration for systems of ordinary differential equations of the following type:

$$\tfrac{dx_1}{dx} = A_1(x, x_1, \cdots, x_n), \quad \cdots \quad \tfrac{dx_n}{dx} = A_n(x, x_1, \cdots, x_n), \qquad \text{(C.1)}$$

[5] More explicitly, a given algebraic equation $x^n - p_1x^{n-1} + p_2x^{n-2} - \cdots (-1)^n p_n = 0$, is replaced by the following system in $n$ variables:

$$\begin{cases} \sum_{i=1}^{n} x_i = p_1 \\ \vdots \\ x_1x_2\cdots x_n = p_n \end{cases}.$$

[6] *Ces considérations me portent à croire que ce qui peut le plus contribuer aux progrès du Calcul intégral, c'est la classification des transcendantes absolument irréductibles, et par là essentiellement distincts, et la recherche des propriétés particulières à chacun de ces genres.*

whose first integrals $z(x, x_1, \cdots, x_n)$ are provided by the following linear, first order differential equation

$$X(z) = \frac{\partial z}{\partial x} + A_1 \frac{\partial z}{\partial x_1} + \cdots + A_n \frac{\partial z}{\partial x_n} = 0. \tag{C.2}$$

His strategy consisted of providing a proof of the existence of a *rationality group* $\Gamma$ acting on a fundamental system of solutions to (C.2), to be indicated with $z_1, \cdots, z_n$[7], which, in analogy with the Galois theory as well as with the theory of Picard and Vessiot, was characterized by the following two fundamental properties: i) every rational invariant of the group $\Gamma$ is a rational function of the independent variables $x, x_1, \cdots, x_n$; ii) every rational function of $z_1, \cdots, z_n$ and their derivatives which is equal to a rational function of $x, x_1, \cdots, x_n$ is an invariant of the group $\Gamma$.

Whereas in the algebraic case the rationality group is a subgroup of the permutation group and in the Picard-Vessiot theory it is identified with a subgroup of the general linear group, in the present case, it turns out that the group $\Gamma$ is an infinite continuous group in the sense of Lie.

First, Drach's construction will be summarized; later on we will try to give an account of the criticisms brought up by Vessiot (and Cartan) who disputed with Drach on the legitimacy of the procedure employed by him.

Following the fruitful approach already employed by Galois, Picard and Vessiot, which was mentioned above, Drach's starting point in the third chapter of his thesis consisted of replacing the equation $X(z) = 0$ with a system of equations for a fundamental set of solution thereof:

$$\frac{D}{1} = \frac{D_1}{A_1} = \cdots = \frac{D_n}{A_n}, \tag{C.3}$$

where $D, D_1, \cdots, D_n$ indicate the coefficient of the development of the determinant

$$\Delta = det \begin{bmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial x_1} & \cdots & \frac{\partial z}{\partial x_n} \\ \frac{\partial z_1}{\partial x} & \frac{\partial z_1}{\partial x_1} & \cdots & \frac{\partial z_1}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial z_n}{\partial x} & \frac{\partial z_n}{\partial x_1} & \cdots & \frac{\partial z_n}{\partial x_n} \end{bmatrix}$$

with respect to the first row of the corresponding matrix[8]. From the theory of first order, linear PDE's, it was known that if the $n$-uple $(z_1, \cdots, z_n)$ denotes a fundamental set of solutions to (C.3), then also $(F_1(z), \cdots, F_n(z))$, where $F_j$, $j = 1, \cdots, n$ are a set of *independent* functions of $n$ variables, is. Viceversa, it was also known that any two sets of fundamental solutions to (C.3), $(z_1, \cdots, z_n)$ and $(Z_1, \cdots, Z_n)$ say, are necessarily linked by

[7]It should be recalled that a first order linear PDE of type $X(z) = 0$ has exactly $n$ independent solutions $z_j(x, x_1, \cdots, x_n)$, $j = 1, \cdots, n$. In this context, independence means functional independence.

[8]This system of equation is analogous to the invariant differential system which represents the starting point of the Picard-Vessiot theory.

functional relations of the form: $Z_j = F_j(z_1, \cdots, z_n)$, where the set $\{F_j\}$, $j = 1, \cdots, n$ can be regarded as a transformation of the general group of point-transformations in $n$ variables, $\Gamma_n$. In other words, the system (C.3) defines a set of fundamental solutions modulo transformations of $\Gamma_n$.

At this point, Drach posed the problem of establishing whether it is possible or not to operate a classification among different fundamental systems of a given equation $X(z) = 0$.

To this end, he introduced the crucial notion of irreducibility (and consequently of reducibility) of a given differential system. The system (C.3) was defined by him to be *irreducible* if every rational relation among the variables $x, x_1, \cdots, x_n$, the functions $z_1, \cdots, z_n$ and their derivatives which is compatible with (C.3) is a *necessary* consequence thereof. In the opposite case, i.e. when rational relations exist which are not necessary consequences of (C.3), the system was said to be *reducible*.

In the first case, Drach observed, there is no possibility of drawing any distinction among different fundamental systems of solutions and borrowing a terminology from algebraic Galois theory, he called the equation (C.2) *general*. On the contrary, if the system (C.3) is reducible, one can provide a classification of the different elements satisfying $X(z) = 0$ by introducing new rational relations which are not implied by (C.3). In this case, the equation $X(z) = 0$ was said to be *special*.

In the case in which $X(z) = 0$ is general, Drach easily proved that the rationality group coincides with $\Gamma_n$, since every rational relation among the elements $z_1, \cdots, z_n$ of a fixed fundamental system is left invariant by any transformation of $\Gamma_n$ and besides, every differential invariant of $\Gamma$ which is a rational expression of its arguments can be expressed as a rational function of $x, \cdots, x_n$.

Much more delicate was the proof of the existence of the rationality group in the reducible case. If $X(z) = 0$ is special, then by definition one can add to (C.3) a certain number of *rational* relations which together with (C.3) can be supposed to build up a new irreducible system, to be indicated with (S). If $(z_1, \cdots, z_n)$ indicates a set of solution to (S), then it is clear that the equations of (S) are *not* satisfied by the functions $Z_i = F_i(z_1, \cdots, z_n)$, $i = 1, \cdots, n$ for an arbitrary transformation $\{F_i\}$ of the group $\Gamma_n$; in order for $Z_i$, $i = 1, \cdots, n$ to be a fundamental system of solutions of (S), it is necessary to impose to $\{F_i\}$ additional conditions which, as Drach observed, give a measure of the transcendency degree of the elements $z_1, \cdots, z_n$, as it were. These conditions, he claimed, can be expressed by a limited number of rational relations among the variables $z_1, \cdots, z_n$, the functions $F_1, \cdots, F_n$ and their derivatives. From them one can deduced a completely integrable differential system which he indicated with $(\Sigma)$. The transformations which fulfill $\Sigma$, he asserted, are such as to leave the system (S) invariant and build up a continuous (generally infinite) group of transformations which is a subgroup of $\Gamma_n$[9]. This group, which Drach indicated with $\Gamma$, was called the rationality group of the equation $X(z) = 0$. Indeed, as Drach showed, it was characterized by the two properties mentioned above: every rational function is an invariant of the group and, conversely, every invariant of the group is a rational function of the independent variables $x, x_1, \cdots, x_n$.

As Drach hastened to explain, although at first it may seem ambiguous, the notion of rationality group is well-posed since it actually depends upon the equation $X(z)$ only and not on a particular choice of the irreducible system (S). Indeed, different irreducible systems stemming from a unique equation give rise to rationality groups which belong to the same similarity type. On the basis of these premises, Drach singled out three different (independent) steps in his study of the solutions to $X(z) = 0$: i) determination of all different systems $(\Sigma)$, i.e. determination of the defining equations of a representative $\Gamma$ for every type of group contained in $\Gamma_n$; ii) determination of the system $(\Sigma)$ which corresponds to the rationality group of $X(z) = 0$; iii) thorough investigation of the properties of $\Gamma$ in order to specify the nature of the transcendents $z_1, \cdots, z_n$.

Connections and interrelations between Drach's theory of integration and the theory of infinite continuous groups should now be clear. Indeed, not only Drach's theory required a full classification of the different type of continuous infinite groups (in particular, Drach emphasized the importance of obtaining a complete classification of all simple infinite types); general techniques for solving algebraic issues such as that of finding a normal decomposition series of a given group were also needed.

Far from being an organic and completed theory of integration, Drach thesis should rather be considered as a wide and ambitious research program aimed at attaining a full generalization of Galois theory. Quite audaciously, Drach considered his work as a genuine attempt of concretizing Galois' dream of building up a theory of ambiguity. By this, Drach referred to some well known remarks (which he quoted *in extenso* at the end of his thesis) which Galois had expressed in a letter to his friend Auguste Chevalier the very night before his tragic death in a duel. Galois had written (May 1832):

> *My main thoughts have been directed, for some time, towards the application of the theory of ambiguity to transcendent Analysis. One needs to understand, a propri, which changes can be operated, which quantities can be substituted to given ones, in a certain relation among transcendent quantities or functions in such a way that this relation remains verified. This allows us to recognize the impossibility of many expression which one may look for. Unfortunately, I do not have enough time and my ideas*

[9] More explicitly, the system $(\Sigma)$ coincides with the system of PDE's whose solutions are the transformations of $\Gamma$.

*in this field, which is immense, are not sufficiently developed yet.*[10]

Drach's thesis was welcomed at first with the highest favour. The jury chosen to judge his work consisted of three leading mathematicians: É. Picard, G. Darboux and H. Poincaré. Picard's comments were particularly commendatory. He wrote in the report on the thesis:

> *From a philosophical and logical standpoint, the results achieved by Drach are of considerable importance. They are expression of a profound and original spirit which has meditated for long upon the principles of science. They open a brand new way to the classification theory of transcendents.*
> *This is not only an excellent dissertation; it is a contribution full of original insights, a real landmark which, without any doubt, will represent the starting point of many researches.*[11]

Despite such praising tones, few months later, first serious criticisms to Drach's theory were moved by E. Vessiot who, at that time was acknowledged as one of the leading expert in the vast field of applications of continuous groups to the theory of differential equations. Vessiot's concerns about the validity of Drach's reasoning found expression in a long letter which was sent on 3rd October 1898. Vessiot explained to Drach that a careful reading of his thesis had cast grave doubts on essential parts of the theory developed therein, namely on theorems and proofs contained in the the third chapter where Drach had set out to provide a differential Galois theory for linear first order PDE's. After conferring with Cartan who had declared to share completely his perplexities on the subject, Vessiot had decided to ask directly Drach for necessary clarification. Vessiot's criticisms involved various aspects of the theory, however the essentials can be summarized as follows.

In the course of his reasoning, Vessiot explained, Drach implicitly assumed that the irreducible system $(S)$ associated to a *special* equation $X(z) = 0$ has the property that its general solution can be obtained from a

[10] *Mes principales méditations, depuis quelque temps, étaient dirigées sur l'application à l'Analyse transcendente de la théorie de l'ambiguité. Il s'agissait de voir, a priori, dans une relation entre des quantités ou fonctions transcendantes, quels échanges on pouvait faire, quelles quantités on pouvait substituer aux quantités données, sans que la relation pût cesser d'avoir lieu. Cela fait reconnaître de suite l'impossibilité de beaucoup d'expressions que l'on pourrait chercher. Mais je n'ai pas le temps, et mes idées ne sont pas éncore bien développées sur ce terrain qui est immense.*

[11] *Les résultats obtenus par M. Drach sont, au point de vue philosophique et logique, d'une importance considérable; ils témoignent d'un esprit profond et original qui a beaucoup réfléchi sur les principes de la science, et ils ouvrent une voie toute nouvelle pour la classification des transcendantes.*[...]
*Ce n'est pas seulement une thèse excellente; c'est un travail rempli d'aperçus originaux, qui marquera dans la science, et sera sans doute l'origine de bien des recherches.* See [Dugac, Taton 1981, p. 39].

particular set of integrals $z_1, \cdots, z_n$ by means of formulas of type $F_i(z_1, \cdots, z_n)$, where the functions $\{F_i\}$ are defined by a differential system $(\Sigma)$ which is independent of the choice of the particular system $(z_1, \cdots, z_n)$. Vessiot communicated a counterexample due to Cartan which invalidated Drach's assumption. As a consequence of this, Drach's procedure for defining the rationality group $\Gamma$ became untenable and the existence itself of the rationality group characterized by the two properties mentioned above remained doubtful.

Few days later, Drach answered Vessiot's criticisms by denying that the doubts which he had expressed could invalidate the essential content of his thesis. Rather, he imputed Vessiot's doubts to a misunderstanding of some notions introduced by him, which, he conceded, might sometimes looked ambiguous. Drach insisted in particular on the necessity of focusing on a special type of irreducible systems $(S)$ which he defined to be of *minimal order* and which later on he would have called *regular irreducible systems*. By operating such a restriction, he wrote to Vessiot, one could dissipate any misunderstanding and doubt.

Drach's attempts of explanation were not successful and did not mitigate Vessiot's censorious attitude. Indeed, Vessiot regarded them to be too hasty and unprecise. For this reason, he pointed out, conspicuous parts of Drach's thesis needed to be heavily revised.

We do not know how, or even if, Drach reacted to Vessiot further remarks. However, it is certain that Drach never publicly recognized the severity of mistakes and lacunae which afflicted his thesis. Still some years later, despite his various efforts of founding his theory on more solid grounds, he tended to minimize them; he was at most disposed to admit a certain degree of ambiguity in some of the notions that he had introduced therein. On the contrary, the Parisian mathematical society quite rapidly agreed on the importance of taking up the subject of Drach's thesis again in order to provide requested emendations. Already in October 1898, for example, P. Painlevé and É. Picard embraced Vessiot's point of view openly speaking of a severe mistake in Drach's work.

Less than two years later, in 1900, the *Académie des Sciences* announced the topic of the *Grand Prix des Sciences Mathematiques* for the year 1902: applications of continuous groups to the theory of partial differential equations. It is highly probable that such a choice was dictated by the largely shared urgency of revising Drach's results as well as of providing them with the rigour they lacked. This intention was made explicit one year later in the motivations for the assignment of the price (consisting of 3000 francs) to Vessiot himself who had submitted a long paper under the title: *Essai sur la nature des integrations auxquelles conduit l'application de la théorie des groupes aux sistèmes différentielles quelconques.* Referring to the second part of Vessiot's memoir which was published in the *Annales de l'École Normale* in 1904, the jury, which consisted of H. Poincaré, C. Jordan, P.

Painlevé, É. Picard and P. Appel, declared that, thanks to Vessiot, it was now possible to render precise and rigorous the theory outlined by Drach in his thesis. They concluded:

> [Vessiot's paper] *fully relieves the lacunae which still afflicted the theory inaugurated by Drach on linear partial differential equations* [and] *the jury is unanimous in awarding it the grand prix de mathematiques.*[12]

Indeed, Vessiot succeeded in providing a coherent and systematic theoretical framework in which Drach's research program finally found a satisfying concretization; his solution consisted of a fruitful combination of Lie's theory of infinite continuous groups as developed by Engel and Medolaghi with an original reinterpretation of classical Galois theory which exploited in an essential way the notion of *automorphic* system.

Drach himself had tried to refine his previous approach in an extended paper which he had submitted for the 1902 competition as well. However, his work was rejected since, this was the explanation provided by the jury, the theory contained therein was still incomplete and in need of further reflection.

In conclusion, although it was afflicted by severe flaws, Drach's thesis not only had the great merit of stimulating new researches in the realm of applications of (infinite) Lie groups to the theory of PDE's, it also contained some technical insights which turned out to be of great importance for Vessiot's subsequent work.

[12] [Vessiot's paper] *comble entièrement les lacunes qui subsistaient dans l'importante question ouverte par M. Drach pour les équations linéaires aux dérivées partielles,* [and] *la Commision est unanime à lui accorder le grand prix des Sciences mathématiques.*[13]

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