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On the Geometry of Newton operators

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Introduction

Constant mean curvature (CMC) hypersurfaces appear as critical points of a natural geometric variational problem: to minimize the area with or without a volume constraint (the unconstrained case corresponds to zero mean curvature, i.e. to minimal hypersurfaces). A fundamental problem of this discipline is the geometric study and classification of CMC surfaces under global hypotheses like compactness, completeness, properness or embeddedness.
Important results in this field are the so-called Jellet-Liebmann theorem and the Alexandrov theorem. In 1853 Jellet in [**43**] showed that a star-shaped compact hypersurface immersed into Euclidean space with constant mean curvature is a round hypersphere. However a related and much weaker result published in 1899 by Liebmann [**45**], was very much recalled. Liebmann's theorem stated that the only compact convex hypersurfaces of Euclidean space with constant mean curvature were the round hyperspheres. Further, in a series of papers between 1956 and 1962, Alexandrov [**5**],[**6**] obtained a fundamental result in this theory. He showed that a compact CMC hypersurface embedded into $\mathbb{R}^3$ must be a hypersphere. Later on, in [**63**], Ros, exploiting an idea of Reilly, was able to obtain an Alexandrov theorem for compact hypersurfaces embedded into the Euclidean space with constant scalar curvature.
The natural generalization of mean and scalar curvature for an n-dimensional hypersurface are the k-mean curvatures H_k, $k = 1, \ldots, n$, that are defined via the elementary symmetric functions of the principal curvatures of the immersion. In fact, H_1 is just the mean curvature and H_2 defines a geometric quantity which is related to the scalar curvature, as we will see later in Section 1.1. Therefore it is natural to try to extend those characterization results to the case of constant higher order mean curvature. A natural question is then to ask if the sphere is the only compact hypersurface (embedded or immersed) in the Euclidean space with constant higher order mean curvature H_k, for some $k = 1, \ldots, n$. In [**40**], Hsiung showed that this is true, provided that the hypersurface is star-shaped. Further, in [**62**] and [**51**], Ros and Montiel proved the validity of the Alexandrov theorem for every compact hypersurface immersed in the Euclidean space with constant higher order mean curvature H_k for some $k = 1, \ldots, n$. Moreover, this method equally works for hypersurfaces in the Hyperbolic space.
The next step is to try to extend this uniqueness results to hypersurfaces immersed in more general ambient spaces with non-constant sectional curvature. Toward this aim we need to consider as ambient spaces manifolds that

have a large number of constant k-mean curvature compact hypersurfaces in order to use them as comparison hypersurfaces. A natural class of ambient manifolds to consider is that of warped products with 1-dimensional basess $I \times_\rho \mathbb{P}^n$ (see [**54**] for more details on warped products). In this case the leaves of the foliation $t \to \mathbb{P}_t := \{t\} \times \mathbb{P}$ (that we will call *slices*) are totally umbilical hypersurfaces of constant mean curvatures. The first attempt to generalize uniqueness theorems to hypersurfaces of constant mean curvature in warped product spaces was made by Montiel in [**49**], that obtained some results for compact hypersurfaces of constant mean curvature, under suitable assumptions on the Ricci curvature of a standard slice. Afterward, more results have been obtained by Alías and Dajczer in [**10**], where they recover some of Montiel's main theorems and, under different assumptions, prove new ones both in the compact and in the complete non-compact case. Our aim is to extend these uniqueness results to hypersurfaces in warped products with constant k-mean curvature, $k = 2, \dots, n$, both in the compact and in the complete non-compact case. We devote Chapter 2 to the study of such a problem, presenting some results we have proved in [**13**], while in Chapter 1 we introduce some basic notions and tools that we will need in the rest of the dissertation. Further, we will use an analytical approach inspired by [**10**], reducing our problem to the study of certain differential equations that arise in this context. For what concern the compact case, the study of these equation is based, firstly, on the well-known property that any C^2 function u on a compact Riemannian manifold attains its maximum and minimum values at some points $p_{\max}$ and $p_{\min}$ respectively. Moreover, the gradient of the function is zero on these values and the Hessian is respectively negative or positive definite. These latter fact implies that if L is any semi-elliptic operator, then it satisfies

$$Lu(p_{\max}) \leq 0, \qquad \text{resp. } Lu(p_{\min}) \geq 0.$$

Further, another important tool is the so-called *classical maximum principle*, that asserts that, given any semi-elliptic operator L on a Riemannian manifold Σ, such that Lu only contains terms involving the second and the first derivatives of the function u, if

$$Lu \leq 0, \qquad \text{resp. } Lu \geq 0$$

and u attains its maximum (resp. minimum) in the interior of Σ, then it has to be constant (see [**33**, Theorem 3.5]).

Using these facts we will be able to prove, for instance, the following theorems. Here $\mathcal{H}$ will denote the mean curvature of the slices and the angle function Θ is defined as the scalar product between the vector field normal to Σ and that normal to the slices.

Theorem (Theorems 2.13,2.14). *Let $f : \Sigma^n \to I \times_\rho \mathbb{P}^n$ be a compact hypersurface such that either*

(i) *the 2-mean curvature H_2 is a positive constant or*
(ii) *the k-mean curvature H_k is constant and there exists an elliptic point on Σ.*

If $\mathcal{H}'(t) \geq 0$ and the angle function Θ does not change sign, then $\mathbb{P}^n$ is necessarily compact and $f(\Sigma^n)$ is a slice.

Theorem (Theorem 2.25)**.** *Let $f : \Sigma^n \to M^{n+1} = I \times_\rho \mathbb{P}^n$ be a compact hypersurface of constant k-mean curvature, $2 \leq k \leq n$ and suppose that $\mathcal{H}$ does not vanish. Assume that*

$$K_{\mathbb{P}} \geq \sup_I \{\rho'^2 - \rho''\rho\},$$

$K_{\mathbb{P}}$ being the sectional curvature of $\mathbb{P}^n$, and that the angle function Θ does not change sign. Then either $f(\Sigma^n)$ is a slice over a compact $\mathbb{P}^n$ or M^{n+1} has constant sectional curvature and Σ^n is a geodesic hypersphere. The latter case cannot occur if the above inequality is strict.

In the complete case, it is not true in general that a continuous function admits a maximum or a minimum. Nevertheless, as explained in Section 1.3, when the *Omori-Yau maximum principle* for semi-elliptic operators holds, it is possible for any function $u \in C^2(\Sigma)$ with $u^* = \sup_\Sigma u < +\infty$ or $u_* = \inf_\Sigma u > -\infty$, to find sequences $\{p_j\}_{j\in\mathbb{N}} \subset \Sigma$ and $\{q_j\}_{j\in\mathbb{N}} \subset \Sigma$ with the properties

$$\lim_{j\to+\infty} u(p_j) = u^*, \ \|\nabla u(p_j)\| < \frac{1}{j}, \ Lu(p_j) < \frac{1}{j},$$

and, respectively,

$$\lim_{j\to+\infty} u(q_j) = u_*, \ \|\nabla u(q_j)\| < \frac{1}{j}, \ Lu(q_j) > -\frac{1}{j}.$$

The general version of the Omori-Yau maximum principle illustrated above allows us to obtain, among others, the next

Theorem (Theorems 2.17, 2.18)**.** *Let $f : \Sigma^n \to I \times_\rho \mathbb{P}^n$ be a complete hypersurface such that either*

(i) *the 2-mean curvature H_2 is a positive constant or*
(ii) *the k-mean curvature H_k is constant and there exists an elliptic point on Σ*

and suppose that

$$K_\Sigma^{\mathrm{rad}} \geq -G(r).$$

Here G is a smooth function on $[0, +\infty)$ which is even at the origin and satisfying conditions (i)–(iv) listed in Theorem 1.12. Assume that $\sup_\Sigma |H_1| < +\infty$ and that Σ is contained in a slab, that is,

$$f(\Sigma^n) \subset [t_1, t_2] \times \mathbb{P}^n,$$

where $t_1, t_2 \in I$ are finite. If $\mathcal{H}'(t) > 0$ almost everywhere and the angle function Θ does not change sign, then $f(\Sigma^n)$ is a slice.

As we will clarify in Section 1.3, the condition on the sectional curvature of the hypersurfaces is assumed in order to guarantee the validity of the general version of the Omori-Yau maximum principle.

A generalization of the classical maximum principle in the complete non-compact case is the property of a manifold to be *parabolic*, that is it does not admit any non-constant C^1 subharmonic (resp. superharmonic) function bounded from above (resp. bounded from below). More generally, we say that a manifold is parabolic with respect to a semi-elliptic operator L if it does not exist any C^1 non-constant bounded above (resp. below) function u satisfying

$$Lu \geq 0, \qquad (\text{resp. } Lu \leq 0).$$

Under an appropriate assumption on the growth of the volumes of geodesic spheres we can guarantee that the hypersurface is parabolic with respect to a suitable semi-elliptic operator and thus we are able to extend the second main theorem of the compact case as follows

Theorem (Theorem 3.20)**.** *Let $M^{n+1} = I \times_\rho \mathbb{P}^n$ be a warped product space and assume that $\mathbb{P}^n$ has constant sectional curvature κ satisfying*

$$\kappa > \sup_I \{\rho'^2 - \rho''\rho\}.$$

Let $f : \Sigma^n \to I \times_\rho \mathbb{P}^n$ be a complete hypersurface with $\sup_\Sigma |H_1| < +\infty$ *and satisfying condition*

$$\big(\sup_{\partial B_t} H_{k-1}\mathrm{vol}(\partial B_t)\big)^{-1} \notin L^1(+\infty),$$

where B_t denotes the geodesic ball of radius t. Suppose that f has constant k-mean curvature, $2 \leq k \leq n$, and

$$f(\Sigma^n) \subset [t_1, t_2] \times \mathbb{P}^n,$$

where t_1, t_2 are finite. Assume that either $k = 2$ and H_2 is positive or $k \geq 3$ and there exists an elliptic point $p \in \Sigma^n$. If $\mathcal{H}(h)$ and the angle function Θ do not change sign, then $f(\Sigma^n)$ is a slice.

In the seventies, with the works of Calabi [**22**], Cheng and Yau [**24**], Brill and Flaherty [**20**], Choquet-Bruhat [**26**], [**27**] and, later on, with the works of some other authors, began the mathematical interest for the study of spacelike CMC hypersurfaces in Lorentzian manifolds. This interest is also motivated by their relevance from a physical point of view (see for instance [**47**] for more details). By then, a lot of problems on hypersurfaces in Riemannian manifolds have been transposed to the Lorentzian setting. In particular, the problem of classifying CMC and, later on, constant higher order mean curvature spacelike hypersurfaces in certain spacetimes has been intensively studied also in this area. Many works have appeared where it is studied the case where the ambient space is a *generalized Robertson-Walker* (GRW) spacetimes, which, as we will see later, are nothing but Lorentzian warped product spaces with one dimensional bases. For instance, in [**16**] Alías, Romero and Sánchez studied the problem of uniqueness of spacelike hypersurfaces in GRW spacetimes in the CMC case. In particular, it is proved that, when the spacetime obeys the so-called *timelike convergence condition*, then every compact CMC spacelike hypersurface must be totally umbilical and, in most of the cases, it must be a spacelike slice. In [**17**], the same authors also observed that these results can be obtained replacing the timelike convergence condition by a weaker one, the so-called *null* *convergence condition*. Later on, the same problem was considered by Montiel [**50**] that classified totally umbilical spacelike CMC hypersurfaces and proved that the only compact CMC spacelike hypersurfaces in a GRW spacetime obeying the null convergence condition are the spacelike slices, unless in the case where the spacetime is a de Sitter space and the hypersurface is a round umbilical hypersphere. Moreover, he also proved a uniqueness result for hypersurfaces of constant scalar curvature. In this circle of ideas, in [**8**] Alías and Colares extended these results to compact spacelike hypersurfaces of constant k-mean curvature, $2 \leq k \leq n$, in proper GRW spacetimes (that is spacetime that cannot be written as trivial products, not even locally) obeying either the null convergence condition or a condition on the warping function. For what concern the complete non-compact case, uniqueness results have been proved by Alías and Montiel in [**15**] for complete CMC spacelike hypersurfaces in GRW spacetimes and, later on, by Romero and Rubio, in [**61**], and by Caballero, Romero and Rubio, in [**21**], for CMC surfaces in GRW spacetimes. In Chapter 3 we face off this problem for complete spacelike hypersurfaces of constant k-mean curvature, $2 \leq k \leq n$. Using the analytical tools developed for the Riemannian case and assuming a condition on the warping function we are able to prove the next

Theorem (Theorem 3.17)**.** *Let $-I \times_\rho \mathbb{P}^n$ be a generalized Robertson-Walker spacetime whose warping function satisfies $(\log\rho)'' \leq 0$, with equality only at isolated points, and suppose that $\mathbb{P}^n$ has sectional curvature bounded from below. Let $f : \Sigma^n \to -I \times_\rho \mathbb{P}^n$ be a complete spacelike hypersurface contained in a slab and assume that either*

(i) *H_2 is a positive constant, or*
(ii) *H_k is constant (with $k \geq 3$) and there exists an elliptic point in Σ.*

If $\sup_\Sigma |H_1| < +\infty$, then Σ is a slice.

Moreover, in the case when the GRW spacetime obeys the null convergence condition, we find that the following result holds.

Theorem (Theorem 3.20)**.** *Let $-I \times_\rho \mathbb{P}^n$ be a GRW spacetime and assume that $\mathbb{P}^n$ has constant sectional curvature κ. Let $f : \Sigma^n \to -I \times_\rho \mathbb{P}^n$ be a complete spacelike hypersurface of constant k-mean curvature, $k \geq 2$, contained in a slab $[t_1, t_2] \times \mathbb{P}^n$ on which ρ' does not change sign and*

$$\kappa > \max_{[t_1,t_2]}((\log\rho)''\rho^2).$$

Suppose that Σ^n satisfies condition

$$\big(\sup_{\partial B_t} H_{k-1}\mathrm{vol}(\partial B_t)\big)^{-1} \notin L^1(+\infty)$$

and either

(i) *$k = 2$ and $H_2 > 0$ or*
(ii) *$k \geq 3$ and there exists an elliptic point $p \in \Sigma^n$.*

If $\sup_\Sigma |H_1| < +\infty$ and $\sup_\Sigma |\Theta| < +\infty$, then $f(\Sigma^n)$ is a slice.

These results and others presented in Chapter 3 are entirely contained in [**14**].

Finally, in the last chapter we will exhibit some results obtained in [**41**], where we study the geometry of spacelike hypersurfaces by means of the analysis of the extrinsic Lorentzian distance function from a point. Under suitable bounds on the sectional curvature of the ambient space we are able to obtain, combining the Hessian and Laplacian comparison theorems proved in Section 1.4, the Omori-Yau maximum principle for the Laplacian and for more general semi-elliptic operators, lower and upper bounds for the mean and the higher order mean curvatures of the immersion in terms of the mean curvatures of the level sets of the distance function. Finally, we restrict ourselves to the case when the ambient space is a space form and we prove a characterization of the round spheres as the unique spacelike hypersurfaces with constant k-mean curvature which are bounded by a level set of the Lorentzian distance function.

CHAPTER 1

Preliminaries

1.1. Geometry of the Newton operators: the Riemannian setting

Let Σ^n be a connected oriented Riemannian n-manifold and let $f : \Sigma^n \to M^{n+1}$ be an isometric immersion of Σ^n into an orientable Riemannian $(n+1)$-manifold M^{n+1}. We will denote by A the linear operator associated to the second fundamental form of the immersion and by N the unit normal vector field globally defined on Σ^n. Let $p \in \Sigma$. Since $A_p : T_p\Sigma \to T_p\Sigma$ is symmetric, there exists an orthonormal basis of eigenvectors $\{E_1, ..., E_n\}$ of $T_p\Sigma$ with real eigenvalues $k_1, ..., k_n$. We call these eigenvectors the *principal directions* and the corresponding eigenvalues the *principal curvatures* of the immersion. Denote by S_k the k-th symmetric function of the principal curvatures, defined as

$$\begin{aligned} S_0 =& 1, \\ S_k =& \sum_{i_1 < ... < i_k} k_{i_1} \cdots k_{i_k}, \ 1 \leq k \leq n, \\ S_k =& 0, \ k > n. \end{aligned}$$

We can then define the k-*mean curvature* of f by

$$S_k = \binom{n}{k} H_k.$$

Thus $H_1 = H$ is the mean curvature, H_n is the Gauss-Kronecker curvature and H_2 is an intrinsic quantity related to the scalar curvature of the hypersurface. Indeed, as a consequence of Gauss equation [1]

$$\mathrm{Ric}(X,Y) = \overline{\mathrm{Ric}}(X,Y) - \left\langle \overline{\mathrm{R}}(N,X)Y, N \right\rangle - \langle AX, AY \rangle + nH_1 \langle AX, Y \rangle$$

and

$$S = \mathrm{Tr}(\mathrm{Ric}) = \overline{S} - 2\overline{\mathrm{Ric}}(N,N) + n(n-1)H_2.$$

Hence, if for instance, M has constant sectional curvature c, then $S = n(n-1)(H_2 + c)$. More generally, if M is an Einstein manifold, then H_2 is a multiple of the scalar curvature modulo a constant.

Associated with the higher order mean curvatures there is a family of operators, the so-called *Newton transformations*, which are related to the second fundamental form A and are inductively defined as

$$\begin{aligned} P_0 =& I, \\ P_k =& S_k I - AP_{k-1}, \ 1 \leq k \leq n. \end{aligned}$$

[1] We are using the convention $\mathrm{R}(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$

Observe that the Newton transformations P_k are self-adjoint operators which commute with the second fundamental form A. Moreover, if $\{E_1, \dots, E_n\}$ is an orthonormal frame in $T_p\Sigma$ that diagonalizes A_p, then

$$(P_k)_p E_i = \mu_{i,k}(p) E_i,$$

where

$$\mu_{i,k} = \sum_{i_1 < \cdots < i_k, i_j \neq i} k_{i_1} \dots k_{i_k} = \frac{\partial S_{k+1}}{\partial k_i}.$$

Performing some simple algebraic computations it is easy to prove the following

Proposition 1.1. *The following properties hold:*

(1) $\mathrm{Tr}(P_k) = c_k H_k$,

(2) $\mathrm{Tr}(AP_k) = c_k H_{k+1}$,

(3) $\mathrm{Tr}(A^2 P_k) = \binom{n}{k+1}(nH_1 H_{k+1} - (n-k-1)H_{k+2})$,

where $c_k = (n-k)\binom{n}{k} = (k+1)\binom{n}{k+1}$.

We refer to [**18**] for a detailed proof.

Associated to each Newton transformation P_k of an immersion $f : \Sigma^n \to M^{n+1}$, there is a second order differential operator $L_k : C^\infty(\Sigma) \to C^\infty(\Sigma)$ defined by

$$L_k u = \mathrm{Tr}(P_k \circ \operatorname{hess} u),$$

$u \in C^\infty(\Sigma)$, where by $\operatorname{hess} u : T\Sigma \to T\Sigma$ we denote the symmetric operator given by $\operatorname{hess} u(X) = \nabla_X \nabla u$ for every $X \in T\Sigma$, and by $\operatorname{Hess} u : T\Sigma \times T\Sigma \to C^\infty(\Sigma)$ the metrically equivalent bilinear form given by

$$\operatorname{Hess} u(X,Y) = \langle \operatorname{hess} u(X), Y \rangle .$$

When the ambient space has constant sectional curvature, as a consequence of the Codazzi equation, the operator L_k can be written in divergence form

$$L_k u = \mathrm{div}(P_k \nabla u).$$

This can be seen as a particular case in the following discussion. In general

$$\mathrm{div}(P_k \nabla u) = L_k u + \langle \mathrm{div} P_k, \nabla u \rangle \tag{1.1}$$

where

$$\langle \mathrm{div} P_k, X \rangle = \sum_{i=1}^n \langle (\nabla_{E_i} P_k) E_i, X \rangle = \sum_{i=1}^n \langle (\nabla_{E_i} P_k) X, E_i \rangle$$

for any $X \in T\Sigma$ and any local orthonormal frame $\{E_1, \dots, E_n\}$ on $T\Sigma$. The last term in (1.1) is strictly related to the curvature of the ambient space, as shown in the following

Proposition 1.2. *Let $\Sigma^n \to M^{n+1}$ be an isometric immersion. Let $E_1, ..., E_n$ be a local orthonormal frame on $T\Sigma$ and N be the (local) unit normal. Then*

$$\sum_{i=1}^{n} \langle (\nabla_{E_i} P_k)X, E_i \rangle = \sum_{j=0}^{k-1}\sum_{i=1}^{n}(-1)^{k-1-j} \left\langle \overline{\mathrm{R}}(E_i, A^{k-1-j}X)N, P_j E_i \right\rangle, \tag{1.2}$$

for every $X \in T\Sigma$.

Proof. We will prove Equation (1.2) by induction on k, $1 \le k \le n-1$. Using Codazzi equation

$$(\overline{\mathrm{R}}(X,Y)N)^T = (\nabla_Y A)X - (\nabla_X A)Y$$

and the definition of P_1 it is not difficult to prove that Equation (1.2) holds for $k = 1$. Assume then that the equation holds for $k-1$. Using again Codazzi equation and the definition of covariant derivative, we get

$$\begin{aligned}\sum_{i=1}^{n} \langle (\nabla_{E_i} P_k)X, E_i \rangle =& -\sum_{i=1}^{n} \langle (\nabla_{E_i} P_{k-1})AX, E_i \rangle + \sum_{i=1}^{n} \left\langle \overline{\mathrm{R}}(E_i, X)N, P_{k-1}E_i \right\rangle \\ &+ X(S_k) - \sum_{i=1}^{n} \langle P_{k-1}(\nabla_X A)E_i, E_i \rangle \\ =&\sum_{j=0}^{k-1}\sum_{i=1}^{n}(-1)^{k-1-j} \left\langle \overline{\mathrm{R}}(E_i, A^{k-1-j}X)N, P_j E_i \right\rangle \\ &+ X(S_k) - \sum_{i=1}^{n} \langle P_{k-1}(\nabla_X A)E_i, E_i \rangle .\end{aligned}$$

We claim that

$$X(S_k) = \sum_{i=1}^{n} \langle P_{k-1}(\nabla_X A)E_i, E_i \rangle .$$

Indeed, assume that the basis $\{E_1, \cdots, E_n\}$ diagonalizes A. Then it diagonalizes simultaneously P_{k-1} and

$$\begin{aligned}\sum_{i=1}^{n} \langle P_{k-1}(\nabla_X A)E_i, E_i \rangle =& \sum_{i=1}^{n} \langle P_{k-1}\nabla_X A E_i, E_i \rangle - \sum_{i=1}^{n} \langle P_{k-1}A(\nabla_X E_i), E_i \rangle \\ =& \sum_{i=1}^{n} \mu_{i,k-1} X(k_i) \\ =& \sum_{i=1}^{n} \frac{\partial S_k}{\partial k_i} X(k_i) \\ =& X(S_k).\end{aligned}$$

For further details see also [**11**], paying attention to the different convention for the sign of the curvature tensor $\overline{\mathrm{R}}$. □

In the next chapters we will prove some uniqueness results for hypersurfaces of constant k-mean curvature, $2 \le k \le n$, immersed in some suitable ambient manifolds. In order to do that we will compute some basic partial differential equation by applying the operators L_k to appropriate functions. As already said in the Introduction, the main tools to prove our theorems will be then either the classical maximum principle, if the hypersurface is compact, or, in the complete non-compact case, a generalized version of the Omori-Yau maximum principle that we will introduce in the next section. Since in both cases we will consider the operators L_k to be elliptic, we state now two propositions where geometric conditions are found in order to guarantee this property.

Proposition 1.3 (Lemma 3.10 in [**28**])**.** *Let $\Sigma^n \to M^{n+1}$ be an isometric immersion. If $H_2 > 0$ on Σ, then L_1 is an elliptic operator (for an appropriate choice of the Gauss map N).*

Proof. Observe that, it follows from the Cauchy-Schwarz inequality and by the assumption on H_2 that

$$H_1^2 \geq H_2 > 0.$$

Hence H_1 never vanishes and we can choose the orientation so that $H_1 > 0$. Moreover, since $\|A\|^2 = n^2 H_1^2 - n(n-1)H_2$,

$$k_1 + \ldots + k_n = nH_1 > \|A\| = \sqrt{k_1^2 + \ldots + k_n^2} \geq |k_i| \geq k_i$$

for all $1 \leq i \leq n$. Hence

$$\mu_{i,1} = \sum_{j, j \neq i} k_j > 0,$$

proving the positive definitness of P_1. □

The next proposition follows directly from the proof of Proposition 3.2 in [**18**], nevertheless we derive it here for the sake of completeness. One of the main ingredients of the proof are the so-called *Garding inequalities* [**32**], that is

$$H_1 \geq H_2^{1/2} \geq \cdots \geq H_n^{1/n} \tag{1.3}$$

with equality only at umbilical points. Garding inequalities hold whenever the functions H_j are strictly positive and can be derived as an application of the well-known Newton inequalities (see [**38**] for a detailed proof)

$$H_{j-1}H_{j+1} \leq H_j^2.$$

Recall that a point $p \in \Sigma$ is said to be *elliptic* if the second fundamental form is positive definite at p.

Proposition 1.4. *Let $\Sigma^n \to M^{n+1}$ be an isometric immersion. If there exists an elliptic point of Σ, with respect to an appropriate choice of the Gauss map N, and $H_k > 0$ on Σ for some $3 \leq k \leq n$, then the operators L_j are elliptic for all $1 \leq j \leq k-1$.*

Proof. Observe that proving that the operators L_j are elliptic for all $1 \leq j \leq k-1$ is equivalent to show that each $\mu_{i,j}$ is positive for all $1 \leq j \leq k-1$, $1 \leq i \leq n$.
The existence of an elliptic point implies that all the principal curvatures are positive at p and hence, by continuity, they are positive in an open

neighbourhood $\mathcal{U}$ centered at p. Moreover, the functions H_j, S_j and $\mu_{i,j}$ are also positive in $\mathcal{U}$ for all $1 \leq i \leq n$, $1 \leq j \leq k-1$. Denote by $\mathcal{K}_j$ the set consisting of the points of Σ where the functions $\mu_{i,j}$ are positive. Then, for each j, $\mathcal{U} \subset \mathcal{K}_j$ and $\mathcal{K}_j$ is an open set. Denote by G_j the connected component of $\mathcal{K}_j$ containing $\mathcal{U}$. It is not difficult to prove that, for each j, $\mathcal{G}_{j+1} \subset \mathcal{G}_j$ (see Lemma 3.3 in [**18**] for a detailed proof).
Let us show that $\mathcal{G}_{k-1}$ is closed. Consider a point $q \in \partial\mathcal{G}_{k-1}$. Then, by continuity, $\mu_{i,k-1}$ is non-negative at q for each $1 \leq i \leq n$. Then, since $\mathcal{G}_{k-1} \subset \mathcal{G}_j$ also $\mu_{i,j}$ is non–negative at q for each $1 \leq j \leq k-2$. Observe that, for each i,

$$S_k = k_i\mu_{i,k-1} + \mu_{i,k}.$$

If $\mu_{i,k-1} = 0$ at q, then $S_k > 0$ implies that $\mu_{i,k} > 0$. Since the $\mu_{i,j}$ can be viewed as symmetric functions of $k_1, \dots, k_{i-1}, k_{i+1}, \dots k_j$, we can apply (1.3) and obtain that

$$0 < \mu_{i,k} \leq \binom{n-1}{k}\binom{n-1}{k-1}^{\frac{k}{k-1}} \mu_{i,k-1}^{\frac{k}{k-1}} = 0,$$

leading to a contradiction. This shows that $\mu_{i,k-1} \neq 0$ at q for each i and hence the point q has to belong to the interior of $\mathcal{G}_{k-1}$. This proves that $\mathcal{G}_{k-1}$ is closed and, since Σ is connected, it must coincide with Σ. Using again the fact that $\mathcal{G}_{k-1} \subset \mathcal{G}_j$ we conclude that $\mathcal{G}_j \equiv M$ and hence $\mu_{i,j} > 0$ for all $1 \leq j \leq k-1$, $1 \leq i \leq n$. □

Remark 1.5. We conclude this section giving a motivation for the introduction of the operators L_k.
It is well known that immersions of constant mean curvature are critical points for the variational problem of minimizing the area functional for compactly supported volume-preserving variations. When the ambient space has constant sectional curvature c, also the higher order mean curvatures come out from a variational problem (see [**59**], [**64**] or [**18**] for more details). Namely, let us consider the functional

$$\mathcal{A}_{k-1} = \int_\Sigma F_{k-1}(S_1, ..., S_{k-1})d\Sigma,$$

for compactly supported volume-preserving variations, where the functions F_k are defined by

$$\begin{aligned} F_0 =& 1, \\ F_1 =& S_1, \\ F_k =& S_k + c\frac{n-k+1}{k-1}F_{k-2},\ 2 \leq k \leq n-1. \end{aligned}$$

It is not difficult to see, calculating the first variation formula, that constant k-mean curvature hypersurfaces are critical points of $\mathcal{A}_{k-1}$. Furthermore, if one is interested in studying when this critical points are also minima of the functional, calculating the second variational formula one sees that they must satisfy

$$(1.4) \quad -\int_\Sigma f(L_{k-1}f + (S_1S_k - (k+1)S_{k+1}f + c(n-k+1)S_{k-1}f)d\Sigma \geq 0,$$

where f is any differentiable function such that

$$\int_\Sigma f d\Sigma = 0.$$

Immersions satisfying condition (1.4) are said to be $(k-1)$-*stable* (see also [**4**], [**18**], [**28**] and [**42**] for more details on $(k-1)$-stable hypersurfaces). Thus, the operators L_k naturally appear in this context.

1.2. Geometry of the Newton operators: the Lorentzian setting

Let $f : \Sigma^n \to M^{n+1}$ be a spacelike hypersurface isometrically immersed into a spacetime M. We recall that a hypersurface is said to be *spacelike* if the induced metric is positive definite (that is, it is a Riemannian manifold with respect to the induced metric). Since M is time-oriented, there exists a unique future-directed timelike unit normal field N globally defined on Σ. We will refer to that normal field as the future-pointing Gauss map of the hypersurface. As in the Riemannian case, we let $A : T\Sigma \to T\Sigma$ denote the second fundamental form of the immersion and we denote by $k_1, ..., k_n$ its eigenvalues, which are the principal curvatures of the hypersurface. Their elementary symmetric functions

$$\begin{aligned} S_0 =& 1, \\ S_k =& \sum_{i_1<...<i_k} k_{i_1}\cdots k_{i_k},\ k = 1, ..., n \\ S_k =& 0,\ k > n \end{aligned}$$

define the k-mean curvatures H_k of the immersion via the formula

$$\binom{n}{k} H_k = (-1)^k S_k.$$

In particular, when $k = 1$,

$$H_1 = -\frac{1}{n}\sum_{i=1}^{n} k_i = -\frac{1}{n}\mathrm{Tr}(A) = H$$

is the mean curvature of Σ. The choice of the sign $(-1)^k$ in our definition of H_k is motivated by the fact that in that case the mean curvature vector is given by $\mathbf{H} = HN$. Therefore $H(p) > 0$ at a point $p \in \Sigma$ if and only if $\mathbf{H}(p)$ is in the same orientation as $N(p)$.

When $k = 2$, H_2 defines an intrinsic geometric quantity which is related to the scalar curvature of the hypersurface. Indeed, it follows by the Gauss equation [2] that

$$\mathrm{Ric}(X,Y) = \overline{\mathrm{Ric}}(X,Y) + \left\langle \overline{\mathrm{R}}(X,N)Y, N\right\rangle - \mathrm{Tr}(A)\left\langle AX, Y\right\rangle + \left\langle AX, AY\right\rangle$$

for $X, Y \in T\Sigma$. Hence the scalar curvature S of of the hypersurface is

$$S = \mathrm{Tr}(\mathrm{Ric}) = \overline{S} + 2\overline{\mathrm{Ric}}(N,N) - n(n-1)H_2$$

[2]We are using the convention $\mathrm{R}(X,Y)Z = -\nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z + \nabla_{[X,Y]}Z$

and, if the ambient space M has constant sectional curvature c we obtain that

$$S = n(n-1)(c - H_2),$$

that is, the scalar curvature is a multiple of H_2 modulo a constant. The same is true under the more general assumption of M being Einsten. Even more, when k is even, it follows from the Gauss equation that H_k is a geometric quantity which is related to the intrinsic curvature of Σ^n.

Notice also that, analogously to the Riemannian case, spacelike hypersurfaces of constant higher order mean curvature in Lorentzian spaceforms are critical points of some area functionals for volume preserving variations (see [**19**] or [**23**] for more details).

We introduce the Newton operators $P_k : T\Sigma \to T\Sigma$ which are inductively defined by

$$\begin{aligned} P_0 =& I, \\ P_k =& \binom{n}{k} H_k I + A P_{k-1}, \ \ k = 1, \dots, n. \end{aligned}$$

We observe that the characteristic polynomial of A can be written in terms of the H_k as

$$\det(tI - A) = \sum_{k=0}^{n} \binom{n}{k} H_k t^{n-k}.$$

By the Cayley-Hamilton Theorem, it follows that $P_n = 0$. We also remark that $P_k = (-1)^k \widetilde{P}_k$, where with $\widetilde{P}_k$ we denote the Riemannian Newton transformations defined in the previous section. Then, as a direct application of this relationship, we can derive most of the properties that we stated for the Riemannian Newton tensors. For instance, the Newton operators commute with the second fundamental form and they are simultaneously diagonalizable. In particular, if $\{E_1, \dots, E_n\}$ is a local orthonormal basis that diagonalizes A, the corresponding eigenvalues of P_k are

$$\mu_{i,k} = (-1)^k \widetilde{\mu}_{i,k} = \frac{\partial S_k}{\partial k_i}.$$

Moreover, using again the relationship $P_k = (-1)^k \widetilde{P}_k$ and applying Proposition 1.1, it is not difficult to obtain the next

Proposition 1.6. *The following properties hold:*

(1) $\mathrm{Tr}(P_k) = c_k H_k$,

(2) $\mathrm{Tr}(AP_k) = -c_k H_{k+1}$,

(3) $\mathrm{Tr}(A^2 P_k) = \binom{n}{k+1}(nH_1 H_{k+1} - (n-k-1)H_{k+2})$,

where $c_k = (n-k)\binom{n}{k} = (k+1)\binom{n}{k+1}$.

Using the Newton operators we can define the second order linear differential operators $L_k : C^\infty(\Sigma) \to C^\infty(\Sigma)$ associated to P_k by

$$L_k f = \mathrm{Tr}(P_k \circ \operatorname{hess} f).$$

Notice that, when $k=0$, $L_0=\Delta$ is in divergence form. Moreover, the next proposition holds

Proposition 1.7 (Lemma 3.1, [**7**]). *Let Σ^n be a spacelike hypersurface isometrically immersed into an $(n+1)$-dimensional spacetime and let $\{E_1,...,E_n\}$ be a local orthonormal frame on $T\Sigma$. Then*

$$\operatorname{div}(P_k\nabla u) = L_k u + \sum_{j=0}^{k-1}\sum_{i=1}^{n}\left\langle \overline{\mathrm{R}}(E_i, A^{k-1-j}\nabla u)N, P_jE_i\right\rangle, \tag{1.5}$$

for every $u\in C^\infty(\Sigma)$.

Proof. Notice that

$$\begin{aligned}\operatorname{div}(P_k\nabla u) =& \sum_{i=1}^{n}\langle \nabla_{E_i}P_k\nabla u, E_i\rangle\\ =&\sum_{i=1}^{n}\langle (\nabla_{E_i}P_k)\nabla u, E_i\rangle + \sum_{i=1}^{n}\langle P_k\nabla_{E_i}\nabla u, E_i\rangle .\end{aligned}$$

Hence we we are done if we prove that

$$\sum_{i=1}^{n}\langle (\nabla_{E_i}P_k)X, E_i\rangle = \sum_{j=0}^{k-1}\sum_{i=1}^{n}\left\langle \overline{\mathrm{R}}(E_i, A^{k-1-j}X)N, P_jE_i\right\rangle, \tag{1.6}$$

for every $X\in T\Sigma$. We will prove the latter equation by induction on k, $1\le k\le n-1$. It is straightforward to prove that this is true for $k=1$. Assume that the equation holds for $k-1$. Then, using Codazzi equation

$$(\nabla_X A)Y - (\nabla_Y A)X = (\overline{R}(X,Y)N)^T,$$

we get

$$\begin{aligned}\sum_{i=1}^{n}\langle (\nabla_{E_i}P_k)X, E_i\rangle =&(-1)^kX(S_k) + \sum_{i=1}^{n}\langle (\nabla_{E_i}P_{k-1})AX, E_i\rangle\\ &+\sum_{i=1}^{n}\langle P_{k-1}(\nabla_X A)E_i, E_i\rangle + \sum_{i=1}^{n}\left\langle \overline{\mathrm{R}}(E_i, X)N, P_{k-1}E_i\right\rangle .\end{aligned}$$

We claim that

$$(-1)^{k-1}X(S_k) = \sum_{i=1}^{n}\langle P_{k-1}(\nabla_X A)E_i, E_i\rangle .$$

Indeed, assume that the basis $\{E_1,...,E_n\}$ diagonalizes A. Let k_i and $\mu_{i,k-1}$ be the eigenvalues of A and P_{k-1} respectively corresponding to the eigenvector E_i. Then

$$\begin{aligned}\sum_{i=1}^{n}\langle P_{k-1}(\nabla_X A)E_i, E_i\rangle =&\sum_{i=1}^{n}\mu_{i,k-1}X(k_i)\\ =&(-1)^{k-1}\sum_{i=1}^{n}\frac{\partial S_k}{\partial k_i}X(k_i)\\ =&(-1)^{k-1}X(S_k).\end{aligned}$$

Thus

$$\begin{aligned}\sum_{i=1}^{n} \langle (\nabla_{E_i} P_k)X, E_i\rangle &= \sum_{i=1}^{n} \langle (\nabla_{E_i} P_{k-1})AX, E_i\rangle + \sum_{i=1}^{n} \left\langle \overline{\mathrm{R}}(E_i, X)N, P_{k-1}E_i\right\rangle \\ &= \sum_{j=0}^{k-2}\sum_{i=1}^{n} \left\langle \overline{\mathrm{R}}(E_i, A^{k-1-j}X)N, P_jE_i\right\rangle \\ &\quad + \sum_{i=1}^{n} \left\langle \overline{\mathrm{R}}(E_i, X)N, P_{k-1}E_i\right\rangle \\ &= \sum_{j=0}^{k-1}\sum_{i=1}^{n} \left\langle \overline{R}(E_i, A^{k-1-j}X)N, P_jE_i\right\rangle .\end{aligned}$$

□

The previous proposition thus shows that the operator L_k can be written in divergence form, for any k, when the ambient spacetime has constant sectional curvature. Moreover, it follows by the definition that L_k is elliptic if and only if P_k is positive definite. As a direct application of the relationship between the definition of the Newton operators in the Riemannian and the Lorentzian case and of Propositions 1.3 and 1.4, we can state the two following propositions in which geometric conditions are given in order to guarantee the ellipticity of L_k when $k \geq 1$ (Note that $L_0 = \Delta$ is always elliptic).

Proposition 1.8. *Let Σ be a spacelike hypersurface isometrically immersed into a spacetime. If $H_2 > 0$ on Σ, then L_1 is an elliptic operator (for an appropriate choice of the Gauss map N).*

We point out that in the Lorentzian case by elliptic point we will mean a point in Σ where the second fundamental form is negative definite.

Proposition 1.9. *Let Σ^n be a spacelike hypersurface isometrically immersed into a $(n+1)$-dimensional spacetime. If there exists an elliptic point of Σ, with respect to an appropriate choice of the Gauss map N, and $H_k > 0$ on Σ, $3 \leq k \leq n$, then for all $1 \leq j \leq k$ the operators L_j are elliptic.*

1.3. The Omori-Yau maximum principle for trace-type semi-elliptic operators

In this section we introduce an analytical tool that will be fundamental for the proofs of our uniqueness results, the Omori-Yau maximum principle.

It is well known that, given a compact Riemannian manifold Σ without boundary, for any $u \in C^2(\Sigma)$, there exists $x_0 \in \Sigma$ such that

$$(i)\ u^* = u(x_0),\ (ii)\ \|\nabla u(x_0)\| = 0,\ (iii)\ \Delta u(x_0) \leq 0.$$

More generally, condition (iii) can be replaced by the stronger condition

$$(i)\ u^* = u(x_0),\ (ii)\ \|\nabla u(x_0)\| = 0,\ (iii)\ \operatorname{Hess} u(x_0) \leq 0,$$

where the above inequality has to be interpreted in the sense of quadratic forms, that is

$$\operatorname{Hess} u(x_0)(v,v) \le 0, \qquad \forall v \in T_{x_0}\Sigma.$$

Furthermore, in [**53**] Omori proved that if Σ is a complete non-compact Riemannian manifold with sectional curvature bounded from below, then there exists a sequence of points $\{x_j\}_{j\in\mathbb{N}} \in \Sigma$ satisfying

$$(i)\ u(p_j) > u^* - \frac{1}{j},\ (ii)\ \|\nabla u(p_j)\| < \frac{1}{j},\ (iii)\ \operatorname{Hess} u(p_j) < \frac{1}{j}.$$

Later on, Yau in [**66**] extended these results to complete non-compact Riemannian manifolds with Ricci curvature bounded from below, replacing condition *(iii)* by the weaker

$$\Delta u(p_j) < \frac{1}{j}.$$

For this reason, following the terminology introduced by Pigola, Rigoli and Setti in [**57**], we introduce the following

Definition 1.10. *The Omori-Yau maximum principle is said to hold on an n-dimensional Riemannian manifold Σ if, for any function $u \in C^2(\Sigma)$ with $u^* = \sup_\Sigma u < +\infty$ there exists a sequence of points $\{p_j\}_{j\in\mathbb{N}}$ in Σ satisfying the properties*

$$(i)\ u(p_j) > u^* - \frac{1}{j},\ (ii)\ \|\nabla u(p_j)\| < \frac{1}{j},\ (iii)\ \Delta u(p_j) < \frac{1}{j}$$

for every $j \in \mathbb{N}$. Equivalently, for any function $u \in C^2(\Sigma)$ with $u_* = \inf_\Sigma u > -\infty$, there exists a sequence $\{q_j\}_{j\in\mathbb{N}} \subset \Sigma$ with the properties

$$(i)\ u(q_j) < u_* + \frac{1}{j},\ (ii)\ \|\nabla u(q_j)\| < \frac{1}{j},\ (iii)\ \Delta u(q_j) > -\frac{1}{j}$$

for every $j \in \mathbb{N}$.

Notice that, as shown in [**57**], despite what have been proved by Omori and Yau, the validity of this principle does not depend on curvature bounds as one would expect. Indeed, it has been proved in [**57**, Theorem 1.9] that the Omori-Yau maximum principle holds on every Riemannian manifold admitting a non-negative C^2 function γ satisfying the following requirements

$$\gamma(x) \to +\infty \qquad \text{as } x \to \infty, \tag{1.7}$$

$$\exists A > 0 \qquad \text{such that } \|\nabla\gamma\| \le A\gamma^{\frac{1}{2}} \qquad \text{off a compact set,} \tag{1.8}$$

$$\exists B > 0 \qquad \text{such that } \Delta\gamma \le B\gamma^{\frac{1}{2}} G(\gamma^{\frac{1}{2}})^{\frac{1}{2}} \qquad \text{off a compact set,} \tag{1.9}$$

where G is a smooth function on $[0,+\infty)$ satisfying:

$$\begin{array}{ll} (i)\ G(0) > 0, & (ii)\ G'(t) \ge 0 \quad \text{on } [0,+\infty), \\ (iii)\ G(t)^{-\frac{1}{2}} \notin L^1(+\infty), & (iv)\ \limsup_{t\to\infty} \frac{tG(t^{\frac{1}{2}})}{G(t)} < +\infty. \end{array} \tag{1.10}$$

For a proof of this result see the proof of Theorem 1.9 in [**57**].

Motivated by these facts, in [**13**] we extended this results to a suitable family of second order semi-elliptic operators, that is, the family of differential operators of the form

$$L = \mathrm{Tr}(P \circ \mathrm{hess}), \tag{1.11}$$

where $P : T\Sigma \to T\Sigma$ is a positive semi-definite symmetric operator satisfying $\sup_\Sigma \mathrm{Tr}P < +\infty$. Using the same terminology of definition 1.10, we state the next

Definition 1.11. Let Σ be a Riemannian manifold and let L be an operator as in (1.11). The Omori-Yau maximum principle is said to hold on Σ for the operator L if, for any function $u \in C^2(\Sigma)$ with $u^* = \sup_\Sigma u < +\infty$, there exists a sequence $\{p_j\}_{j\in\mathbb{N}} \subset \Sigma$ with the properties

$$(i)\ u(p_j) > u^* - \frac{1}{j},\ (ii)\ \|\nabla u(p_j)\| < \frac{1}{j},\ (iii)\ Lu(p_j) < \frac{1}{j}$$

for every $j \in \mathbb{N}$. Equivalently, for any function $u \in C^2(\Sigma)$ with $u_* = \inf_\Sigma u > -\infty$, there exists a sequence $\{q_j\}_{j\in\mathbb{N}} \subset \Sigma$ with the properties

$$(i)\ u(q_j) < u_* + \frac{1}{j},\ (ii)\ \|\nabla u(q_j)\| < \frac{1}{j},\ (iii)\ Lu(q_j) > -\frac{1}{j}$$

for every $j \in \mathbb{N}$.

Moreover, in the spirit of Theorem 1.9 in [**57**], we prove the following

Theorem 1.12 (Theorem 1 in [**13**]). *Let Σ be a Riemannian manifold and let L be as in* (1.11). *Assume that there exists a non-negative C^2 function γ satisfying conditions* (1.7), (1.8) *and such that*

$$\exists B > 0 \quad \text{such that } L\gamma \le B\gamma^{\frac{1}{2}}G(\gamma^{\frac{1}{2}})^{\frac{1}{2}} \quad \text{off a compact set,} \tag{1.12}$$

where G is a smooth function on $[0, +\infty)$ satisfying (1.10). *Then, given any function $u \in C^2(\Sigma)$ with $u^* = \sup_\Sigma u < +\infty$, there exists a sequence $\{p_j\}_{j\in\mathbb{N}} \subset \Sigma$ with the properties*

$$(i)\ u(p_j) > u^* - \frac{1}{j},\ (ii)\ \|\nabla u(p_j)\| < \frac{1}{j},\ (iii)\ Lu(p_j) < \frac{1}{j}. \tag{1.13}$$

Proof. Define the function

$$\varphi(t) = e^{\int_0^t G(s)^{-\frac{1}{2}}\mathrm{d}s}.$$

Note that $\varphi(t)$ is a well defined, smooth, positive function and it satisfies $\varphi(t) \to +\infty$ as $t \to +\infty$. Moreover we record for future use that

$$\varphi'(t) = G(t)^{-\frac{1}{2}}\varphi(t) \quad \text{and} \quad \varphi''(t) = \left(G(t)^{-1} - \frac{1}{2}G(t)^{-\frac{3}{2}}G'(t)\right)\varphi(t),$$

and therefore

$$\left(\frac{\varphi'(t)}{\varphi(t)}\right)^2 - \frac{\varphi''(t)}{\varphi(t)} = \frac{1}{2}G(t)^{-\frac{3}{2}}G'(t) \ge 0. \tag{1.14}$$

Besides, using assumption (1.10),(iv) we also have that

$$\frac{\varphi'(t)}{\varphi(t)} \le c(tG(t^{\frac{1}{2}}))^{-\frac{1}{2}}, \tag{1.15}$$

for some constant $c > 0$.

Fix now a point $p \in \Sigma$ and, for all $j \in \mathbb{N}$ define

$$f_j(x) = \frac{u(x) - u(p) + 1}{\varphi(\gamma(x))^{\frac{1}{j}}}.$$

Then $f_j(p) = 1/\varphi(\gamma(p))^{1/j} > 0$. Moreover, since $u^* < +\infty$ and $\varphi(\gamma(x)) \to +\infty$ as $x \to +\infty$, we have $\limsup_{x\to+\infty} f_j(x) \leq 0$. Thus, f_j attains a positive absolute maximum at $p_j \in \Sigma$. Iterating this procedure, we produce a sequence $\{p_j\}_{j\in\mathbb{N}} \subset \Sigma$. Reasoning as in [**57**], let us show first that

$$\limsup_{j\to+\infty} u(p_j) = u^*. \tag{1.16}$$

Assume by contradiction that there exists $\widehat{p} \in \Sigma$ such that

$$u(\widehat{p}) > u(p_j) + \delta$$

for some $\delta > 0$ and for each $j \geq j_0$ sufficiently large. Consider first the case when p_j remains on a compact set. Then, up to passing a subsequence, one can find $\overline{p} \in \Sigma$ such that $p_j \to \overline{p}$ and

$$u(\widehat{p}) \geq u(\overline{p}) + \delta.$$

Since $f_j(p_j) \geq f_j(\widehat{p})$ for every j we deduce that

$$u(\overline{p}) - u(p) + 1 = \lim_{j\to+\infty} f_j(p_j) \geq \lim_{j\to+\infty} f_j(\widehat{p}) = u(\widehat{p}) - u(p) + 1$$

and hence

$$u(\overline{p}) \geq u(\widehat{p}),$$

which is a contradiction. On the other hand, if $\gamma(p_j) \to +\infty$ as $j \to +\infty$, on a subsequence, for each j such that $\gamma(p_j) > \gamma(\widehat{p})$ we have

$$f_j(\widehat{p}) = \frac{u(\widehat{p}) - u(p) + 1}{\varphi(\gamma(\widehat{p}))^{1/j}} > \frac{u(p_j) - u(p) + 1 + \delta}{\varphi(\gamma(p_j))^{1/j}} > f_j(p_j)$$

contradicting the definition of p_j.

This proves (1.16) and, up to passing to a subsequence if necessary, we may assume that

$$\lim_{j\to+\infty} u(p_j) = u^*.$$

Let us prove now (1.13),(ii) and (1.13),(iii). Again, if p_j remains in a compact set, then $p_j \to \overline{p} \in \Sigma$ as $j \to +\infty$ and u attains its absolute maximum. Hence we have

$$u(\overline{p}) = u^*, \qquad \|\nabla u(\overline{p})\| = 0, \qquad \operatorname{Hess} u(\overline{p}) \leq 0.$$

In particular, since P is positive semi-definite it holds that $Lu(\overline{p}) \leq 0$. Hence the sequence $y_j = \overline{p}$, for each j, satisfies all the requirements. Consider now the case when p_j diverges off a compact set, so that, according to (1.7), $\gamma(p_j) \to +\infty$. Since f_j attains a positive maximum at p_j we have

$$(i)(\nabla \log f_j)(p_j) = 0, \qquad (ii)\operatorname{Hess}\log f_j(p_j) \leq 0.$$

A simple computation then gives

$$\nabla u(p_j) = \frac{1}{j}(u(p_j) - u(p) + 1)\frac{\varphi'(\gamma(p_j))}{\varphi(\gamma(p_j))}\nabla\gamma(p_j)$$

and

$$\begin{aligned}\operatorname{Hess} u(p_j)(v,v) \leq& \frac{1}{j}(u(p_j)-u(p)+1)\Big\{\frac{\varphi'(\gamma(p_j))}{\varphi(\gamma(p_j))}\operatorname{Hess}\gamma(p_j)(v,v)\\ &+\Big[\Big(\frac{1}{j}-1\Big)\Big(\frac{\varphi'(\gamma(p_j))}{\varphi(\gamma(p_j))}\Big)^2+\frac{\varphi''(\gamma(p_j))}{\varphi(\gamma(p_j))}\Big]\langle\nabla\gamma,v\rangle^2\Big\}\\ \leq& \frac{1}{j}(u(p_j)-u(p)+1)\Big\{\frac{\varphi'(\gamma(p_j))}{\varphi(\gamma(p_j))}\operatorname{Hess}\gamma(p_j)(v,v)\\ &+\frac{1}{j}\Big(\frac{\varphi'(\gamma(p_j))}{\varphi(\gamma(p_j))}\Big)^2\langle\nabla\gamma,v\rangle^2\Big\},\end{aligned}$$

for every $v\in T_{p_j}\Sigma$, where in the last inequality we have used (1.14). Let $\{E_1,\dots,E_n\}\subset T_{p_j}\Sigma$ be an orthonormal basis of eigenvectors of P_{p_j} corresponding to the eigenvalues $\mu_i(p_j)=\langle P_{p_j}E_i,E_i\rangle\geq 0$. Then, for every $1\leq i\leq n$, we have

$$\begin{aligned}\langle P\operatorname{hess} u(p_j)E_i,E_i\rangle =&\mu_i(p_j)\operatorname{Hess} u(p_j)(E_i,E_i)\\ \leq& \frac{1}{j}(u(p_j)-u(p)+1)\Big\{\frac{\varphi'(\gamma(p_j))}{\varphi(\gamma(p_j))}\langle P\operatorname{hess}\gamma(p_j)E_i,E_i\rangle\\ &+\frac{1}{j}\Big(\frac{\varphi'(\gamma(p_j))}{\varphi(\gamma(p_j))}\Big)^2\mu_i(p_j)\langle\nabla\gamma,E_i\rangle^2\Big\}.\end{aligned}$$

Taking traces here and using the fact that

$$\langle P\nabla\gamma,\nabla\gamma\rangle=\sum_{i=1}^n\mu_i\langle\nabla\gamma,E_i\rangle^2\leq \mathrm{Tr}P\|\nabla\gamma\|^2,$$

we obtain that

$$\begin{aligned}Lu(p_j) \leq& \frac{1}{j}(u(p_j)-u(p)+1)\Big\{\frac{\varphi'(\gamma(p_j))}{\varphi(\gamma(p_j))}L\gamma(p_j)\\ &+\frac{1}{j}\Big(\frac{\varphi'(\gamma(p_j))}{\varphi(\gamma(p_j))}\Big)^2\langle P\nabla\gamma(p_j),\nabla\gamma(p_j)\rangle\Big\}\\ \leq& \frac{1}{j}(u(p_j)-u(p)+1)\Big\{\frac{\varphi'(\gamma(p_j))}{\varphi(\gamma(p_j))}L\gamma(p_j)\\ &+\frac{1}{j}\Big(\frac{\varphi'(\gamma(p_j))}{\varphi(\gamma(p_j))}\Big)^2\mathrm{Tr}P\|\nabla\gamma(p_j)\|^2\Big\}.\end{aligned}$$

Since (1.8) and (1.12) hold, they hold at p_j for sufficiently large j and then

$$\begin{aligned}\|\nabla u(p_j)\| =& \frac{1}{j}(u(p_j)-u(p)+1)\frac{\varphi'(\gamma(p_j))}{\varphi(\gamma(p_j))}\|\nabla\gamma(p_j)\|\\ \leq& \frac{c_0A}{j}(u(p_j)-u(p)+1)\frac{\gamma(p_j)^{1/2}}{\gamma(p_j)^{1/2}G\big(\gamma(p_j)^{1/2}\big)^{1/2}}\\ \leq& \frac{c_0A}{j}(u(p_j)-u(p)+1)\frac{1}{G\big(\gamma(p_j)^{1/2}\big)^{1/2}}\end{aligned}$$

for some constant $c_0 > 0$, and the right hand side tends to zero as $j \to +\infty$. Moreover, using (1.15) and letting $\sup_\Sigma \mathrm{Tr} P = C$,

$$\begin{aligned} Lu(p_j) \leq& \frac{1}{j}(u(p_j) - u(p) + 1)\Big\{Bc + \frac{1}{j}c^2A^2CG(\gamma(p_j)^{\frac{1}{2}})^{-1}\Big\} \\ \leq& c_1\frac{u^* - u(p) + 1}{j} \end{aligned}$$

for a positive constant c_1. Since the right hand side tends to zero as $j \to +\infty$, this proves condition (iii) in (1.13). □

Remark 1.13. We observe that in Theorem 1.12 we do not assume the geodesic completeness of the manifold Σ. Actually, as pointed out in [**57**], it is not difficult to prove that assumptions (1.7) and (1.8) imply it. Indeed, reasoning as in [**25**], we consider a divergent path $\sigma : [0, l) \to \Sigma$, that is a path that eventually lies outside any compact subset of Σ parametrized by arc-length. Set $h(t) = \gamma(\sigma(t))$ on $[t_0, l)$, where t_0 is such that $\sigma(t) \notin K$ for all $t_0 \leq t \leq l$, and K is any compact set of Σ. Then, by assumption (1.8)

$$\begin{aligned} 2|\sqrt{h(t)} - \sqrt{h(t_0)}| =& \Big|\int_{t_0}^{t} \frac{h'(u)}{\sqrt{h(u)}}du\Big| = \Big|\int_{t_0}^{t} \frac{\langle\nabla\gamma(\sigma(u)), \sigma'(u)\rangle}{\sqrt{h(u)}}du\Big| \\ \leq& \Big|\int_{t_0}^{t} \frac{|\nabla\gamma(\sigma(u))|}{\sqrt{\gamma(\sigma(u))}}du\Big| \leq A(t - t_0), \end{aligned}$$

for every $t \in [t_0, l)$. Since σ is divergent $\sigma(t) \to +\infty$ as $t \to l^-$ and hence, by assumption (1.7), $h(t) \to +\infty$ as $t \to l^-$. Therefore, letting $t \to l^-$ in the inequality above, we conclude that $l = +\infty$. This shows that divergent paths in Σ have infinite length, that is, Σ is geodesically complete.

Remark 1.14. The proof of the previous theorem shows that one needs the function γ to be C^2 only in a neighbourhood of p_j. When γ is the square of the Riemannian distance from a fixed reference point o (see the examples below), this is the case if p_j does not belong to the cut locus of o. Nevertheless, using a trick of Calabi [**22**], one may assume that γ is always C^2 in a neighbourhood of p_j.

We point out that the function theoretic approach to the generalized Omori-Yau maximum principle given in Theorem 1.12 allows us to apply it in different situations, where the choice of the functions γ and G are suggested by the geometric setting. The next are two significant and useful examples of intrinsic and extrinsic nature, respectively.

Example 1.15. Let Σ be a complete non-compact n–dimensional Riemannian manifold and let $o \in \Sigma$ be a reference point. Denote by $r(x)$ the distance function from o and set $\gamma(x) = r(x)^2$. Then γ satisfies assumptions (1.7) and (1.8) of the previous theorem. Furthermore, γ is smooth within the cut locus of o. Let G be a smooth function on $[0, +\infty)$ even at the origin, i.e. $G^{(2k+1)}(0) = 0$ for each $k = 0, 1, ...$, and satisfying the conditions listed in (1.10). Assume that the radial sectional curvature of Σ, that is, the sectional curvature of the 2-planes containing ∇r, satisfies

$$K_\Sigma^{\mathrm{rad}} \geq -G(r). \tag{1.17}$$

Then assumption (1.12) is satisfied. Namely, assuming that (1.17) holds, by the Hessian comparison theorem (see the next section for further details) within the cut locus of o, one has

$$\operatorname{Hess} r(p)(v,v) \leq \frac{\phi'(r(p))}{\phi(r(p))}(\|v\|^2 - \langle \nabla r(p), v\rangle^2) \tag{1.18}$$

for every $v \in T_p\Sigma$, where $\phi(t)$ is the (positive) solution of the initial value problem

$$\begin{cases} \phi'' - G\phi = 0, \\ \phi(0) = 0, \ \phi'(0) = 1. \end{cases}$$

Now let

$$\psi(t) = \frac{1}{\sqrt{G(0)}}\left(e^{\int_0^t \sqrt{G(s)}ds} - 1\right).$$

Then $\psi(0) = 0$, $\psi'(0) = 1$ and

$$\psi''(t) - G(t)\psi(t) = \frac{1}{\sqrt{G(0)}}\left(G(t) + \frac{G'(t)}{2\sqrt{G(t)}}\, e^{\int_0^t \sqrt{G(s)}ds}\right) \geq 0.$$

Hence, by the Sturm comparison theorem (see Lemma 1.24 in the next section)

$$\frac{\phi'(t)}{\phi(t)} \leq \frac{\psi'(t)}{\psi(t)} = \sqrt{G(t)}\frac{e^{\int_0^t \sqrt{G(s)}ds}}{e^{\int_0^t \sqrt{G(s)}ds} - 1} \leq c\sqrt{G(t)} \tag{1.19}$$

where the last inequality holds for a constant $c > 0$ and t sufficiently large. Therefore, if r is sufficiently large

$$\operatorname{Hess} r \leq c\sqrt{G(r)}(\langle , \rangle - dr \otimes dr).$$

Since $\operatorname{Hess}\gamma = 2r \operatorname{Hess} r + 2dr \otimes dr$, we obtain from here that

$$\operatorname{Hess}\gamma \leq c\sqrt{\gamma G(\sqrt{\gamma})}\,\langle , \rangle \tag{1.20}$$

for a constant c and γ sufficiently large. Then, since P is positive semi-definite

$$L\gamma \leq nc\mathrm{Tr}P\sqrt{\gamma G(\sqrt{\gamma})}.$$

As a consequence, we get the following

Corollary 1.16. *Let $(\Sigma, \langle , \rangle)$ be a complete, non-compact Riemannian manifold whose radial sectional curvature satisfies condition* (1.17)*. Then, the Omori-Yau maximum principle holds on Σ for any semi-elliptic operator L of the form* (1.11)*.*

Example 1.17. Consider $\mathbb{P}^n$ a complete, non-compact, Riemannian manifold, let $o \in \mathbb{P}^n$ be a reference point and denote by $\widehat{r}$ the distance function from o. We will assume that the radial sectional curvature of $\mathbb{P}^n$ satisfies the condition

$$K_{\mathbb{P}}^{\mathrm{rad}} \geq -G(\widehat{r}), \tag{1.21}$$

where G is a smooth function on $[0, +\infty)$ even at the origin and satisfying the conditions listed in (1.10). Let $f : \Sigma^n \to M^{n+1} = I \times_\rho \mathbb{P}^n$ be a hypersurface, where $I \times_\rho \mathbb{P}^n$ is the product $I \times \mathbb{P}^n$ endowed with the metric

$$\langle , \rangle = \pi_I^*(dt^2) + \rho^2(\pi_I)\pi_{\mathbb{P}}^*(\langle , \rangle_{\mathbb{P}}),$$

where $\rho : I \to \mathbb{R}_+$ is a smooth function. Observe that if Σ is compact then every immersion $f : \Sigma^n \to I \times_\rho \mathbb{P}^n$ is proper and contained in a slab, and the Omori-Yau maximum principle trivially holds on Σ for any semi-elliptic operator. Assume then that Σ is non-compact and let $f : \Sigma^n \to I \times_\rho \mathbb{P}^n$ be a properly immersed hypersurface which is contained in a slab, that is, $f(\Sigma) \subset [t_1, t_2] \times \mathbb{P}^n$.

Let $\widehat{\gamma} : \mathbb{P}^n \to \mathbb{R}$ be the function given by $\widehat{\gamma}(x) = \widehat{r}(x)^2$ for every $x \in \mathbb{P}^n$, and set $\gamma : \Sigma \to \mathbb{R}$ for the associated function, defined as

$$\gamma(p) = \widetilde{\gamma}(f(p)) = \widehat{\gamma}(x(p)) = \widehat{r}(x(p))^2$$

for every $p \in \Sigma$, where $\widetilde{\gamma}(t, x) = \widehat{\gamma}(x)$ and $f(p) = (h(p), x(p))$. Since f is proper, if $p \to +\infty$ in Σ then $f(p) \to \infty$ in $I \times_\rho \mathbb{P}^n$, but being f contained in a slab, this means that $x(p) \to \infty$ in $\mathbb{P}^n$. It follows that $\gamma(p) = \widehat{r}(x(p))^2 \to +\infty$ as $p \to +\infty$ in Σ, and γ satisfies condition (1.7) in Theorem 1.12.

Let us denote by $\widetilde{\nabla}$, $\widehat{\nabla}$ and ∇ the Levi-Civita connection (and the gradient operators) in M^{n+1}, $\mathbb{P}^n$ and Σ^n, respectively. Since $\gamma = \widetilde{\gamma} \circ f$, along the immersion f we have

$$\widetilde{\nabla}\widetilde{\gamma} = \nabla\gamma + \left\langle \widetilde{\nabla}\widetilde{\gamma}, N \right\rangle N \tag{1.22}$$

where N is a (local) smooth unit normal field along f. On the other hand, from $\widetilde{\gamma}(t, x) = \widehat{\gamma}(x)$ we have

$$\left\langle \widetilde{\nabla}\widetilde{\gamma}, T \right\rangle = 0,$$

where T stands for the lift of ∂_t to the product $I \times \mathbb{P}^n$, and

$$\left\langle \widetilde{\nabla}\widetilde{\gamma}, V \right\rangle = \left\langle \widehat{\nabla}\widehat{\gamma}, V \right\rangle_{\mathbb{P}}$$

for every V, where V denotes the lift of a vector field $V \in T\mathbb{P}$ to $I \times \mathbb{P}^n$. Since

$$\left\langle \widetilde{\nabla}\widetilde{\gamma}, V \right\rangle = \rho^2 \left\langle \widetilde{\nabla}\widetilde{\gamma}, V \right\rangle_{\mathbb{P}},$$

we conclude from here that

$$\widetilde{\nabla}\widetilde{\gamma} = \frac{1}{\rho^2}\widehat{\nabla}\widehat{\gamma} = \frac{2\widehat{r}}{\rho^2}\widehat{\nabla}\widehat{r}. \tag{1.23}$$

Therefore, since $\left\|\widehat{\nabla}\widehat{r}\right\| = \rho\left\|\widehat{\nabla}\widehat{r}\right\|_{\mathbb{P}} = \rho$ and $\rho(h) \geq \min_{[t_1,t_2]} \rho(t) > 0$, along the immersion we have

$$\|\nabla\gamma\| \leq \left\|\widetilde{\nabla}\widetilde{\gamma}\right\| = \frac{2\sqrt{\gamma}}{\rho(h)} \leq c\sqrt{\gamma} \tag{1.24}$$

for a positive constant c. Thus, γ also satisfies condition (1.8) in Theorem 1.12. In particular Σ is complete (see Remark 1.13).

Now, we will prove that, under appropriate extrinsic restrictions, condition (1.12) in Theorem 1.12 is also satisfied. It follows from (1.22) that

$$\operatorname{Hess}\gamma(X,X) = \operatorname{Hess}\widetilde{\gamma}(X,X) + \left\langle \widetilde{\nabla}\widetilde{\gamma}, N \right\rangle \langle AX, X\rangle$$

for every tangent vector field $X \in T\Sigma$. From (1.23)

$$\widetilde{\nabla}_T \widetilde{\nabla}\widetilde{\gamma} = -\frac{\rho'}{\rho^3}\widehat{\nabla}\widehat{\gamma} = -\mathcal{H}\widetilde{\nabla}\widetilde{\gamma}, \tag{1.25}$$

where $\mathcal{H}(t) = \rho'(t)/\rho(t)$. In particular, $\operatorname{Hess}\widetilde{\gamma}(T,T) = 0$. Then, writing $X = X^* + \langle X, T\rangle T$, where $X^* = \pi_{\mathbb{P}*}X$, we have

$$\operatorname{Hess}\widetilde{\gamma}(X,X) = \operatorname{Hess}\widetilde{\gamma}(X^*,X^*) + 2\langle X, T\rangle \operatorname{Hess}\widetilde{\gamma}(X^*,T).$$

It follows from (1.25) that

$$\operatorname{Hess}\widetilde{\gamma}(X^*,T) = -\mathcal{H}(h)\left\langle \widetilde{\nabla}\widetilde{\gamma}, X\right\rangle = -\mathcal{H}(h)\langle \nabla\gamma, X\rangle .$$

On the other hand, using

$$\widetilde{\nabla}_{X^*}\widetilde{\nabla}\widetilde{\gamma} = \frac{1}{\rho^2}\widehat{\nabla}_{X^*}\widehat{\nabla}\widehat{\gamma} - \frac{\rho'}{\rho^3}\left\langle \widehat{\nabla}\widehat{\gamma}, X^*\right\rangle T$$

we also have

$$\operatorname{Hess}\widetilde{\gamma}(X^*,X^*) = \frac{1}{\rho^2}\left\langle \widehat{\nabla}_{X^*}\widehat{\nabla}\widehat{\gamma}, X^*\right\rangle = \left\langle \widehat{\nabla}_{X^*}\widehat{\nabla}\widehat{\gamma}, X^*\right\rangle_{\mathbb{P}} = \operatorname{Hess}\widehat{\gamma}(X^*,X^*).$$

Summing up,

$$\begin{aligned} \operatorname{Hess}\gamma(X,X) &= \operatorname{Hess}\widehat{\gamma}(X^*,X^*) - 2\mathcal{H}(h)\langle\nabla\gamma, X\rangle\langle T, X\rangle \\ &\quad + \left\langle \widetilde{\nabla}\widetilde{\gamma}, N\right\rangle \langle AX, X\rangle \end{aligned} \tag{1.26}$$

for every tangent vector field $X \in T\Sigma$.

Observe that, using (1.24),

$$|\mathcal{H}(h)\langle\nabla\gamma, X\rangle\langle T, X\rangle| \le |\mathcal{H}(h)|\,\|\nabla\gamma\|\|X\|^2 \le c\sqrt{\gamma}\|X\|^2.$$

for a constant $c > 0$, since $|\mathcal{H}(h)| \le \max_{[t_1,t_2]}|\mathcal{H}(t)|$. On the other hand, reasoning as we did before in deriving (1.20), it follows from condition (1.21) and using the Hessian comparison theorem for $\mathbb{P}^n$ that, if γ is sufficiently large, then

$$\operatorname{Hess}\widehat{\gamma}(X^*,X^*) \le c\sqrt{\gamma G(\sqrt{\gamma})}\|X\|^2$$

for a certain positive constant c, where we are using the fact that

$$\|X^*\|_{\mathbb{P}} \le \frac{1}{\inf_\Sigma \rho(h)}\|X\| \le \frac{1}{\min_{[t_1,t_2]}\rho(t)}\|X\|.$$

Therefore, since $\lim_{t\to+\infty} G(t) = +\infty$ we conclude from (1.26) that

$$\operatorname{Hess}\gamma(X,X) \le c\sqrt{\gamma G(\sqrt{\gamma})}\|X\|^2 + \left\langle \widetilde{\nabla}\widetilde{\gamma}, N\right\rangle\langle AX, X\rangle \tag{1.27}$$

for every tangent vector field $X \in T\Sigma$, outside a compact subset of Σ.

Assume now that $\sup_\Sigma |H_1| < +\infty$. Tracing (1.27) we obtain

$$\Delta\gamma \le nc\sqrt{\gamma G(\sqrt{\gamma})} + nH_1\left\langle \widetilde{\nabla}\widetilde{\gamma}, N\right\rangle$$

outside a compact set, where by (1.24)

$$|H_1 \left\langle \widetilde{\nabla}\widetilde{\gamma}, N \right\rangle| \leq \sup_\Sigma |H_1| \left\| \widetilde{\nabla}\widetilde{\gamma} \right\| \leq c_1\sqrt{\gamma} \leq c_2\sqrt{\gamma G(\sqrt{\gamma})}$$

for some constants $c_1, c_2 > 0$. Thus, we conclude that, outside a compact subset of Σ,

$$\Delta\gamma \leq c\sqrt{\gamma G(\sqrt{\gamma})},$$

for some constant $c > 0$, which means that condition (1.9) is fulfilled for the Laplacian. Therefore, the Omori-Yau maximum principle holds on Σ for the Laplacian.

On the other hand, if we assume instead that $\sup_\Sigma \|A\|^2 < +\infty$ then, using again (1.24), we have

$$|\left\langle \widetilde{\nabla}\widetilde{\gamma}, N \right\rangle \langle AX, X\rangle| \leq \left\| \widetilde{\nabla}\widetilde{\gamma} \right\| \|A\| \|X\|^2 \leq c\sqrt{\gamma G(\sqrt{\gamma})}\|X\|^2$$

for a positive constant c, if γ is sufficiently large. From (1.27) we therefore obtain

$$\operatorname{Hess}\gamma(X, X) \leq c\sqrt{\gamma G(\sqrt{\gamma})}\|X\|^2, \tag{1.28}$$

for every tangent vector field $X \in T\Sigma$, outside a compact subset of Σ. Thus, if L is as in (1.11), we conclude from here that

$$L\gamma \leq nc \sup_\Sigma \operatorname{Tr} P \sqrt{\gamma G(\sqrt{\gamma})}$$

if γ is sufficiently large, which means that condition (1.12) in Theorem 1.12 is fulfilled for the operator L. Therefore the Omori-Yau maximum principle holds on Σ for L. We summarize the above discussion in the following:

Corollary 1.18. *Let $\mathbb{P}^n$ be a complete, non-compact, Riemannian manifold whose radial sectional curvature satisfies condition* (1.21)*. Let $f : \Sigma^n \to I \times_\rho \mathbb{P}^n$ be a properly immersed hypersurface contained in a slab.*

(1) *If $\sup_\Sigma |H_1| < +\infty$, then Σ is complete and the Omori-Yau maximum principle holds on Σ for the Laplacian.*
(2) *If $\sup_\Sigma \|A\| < +\infty$, then Σ is complete and the Omori-Yau maximum principle holds on Σ for any semi-elliptic operator L as in* (1.11)*.*

Remark 1.19. From

$$\|A\|^2 = n^2 H_1^2 - n(n-1)H_2$$

it follows that, under the assumption $\inf_\Sigma H_2 > -\infty$, the condition $\sup_\Sigma \|A\|^2 < +\infty$ is equivalent to $\sup_\Sigma H_1 < +\infty$.

1.4. Hessian and Laplacian comparison theorems in Lorentzian geometry

Riemannian comparison geometry was first developed in the thirties, through the work of Hopf, Morse, Schoenberg, Myers, and Synge and got to the top in the fifties with the pioneering work of Rauch and the foundational work of Alexandrov, Toponogov and Bishop. The core idea is to compare the geometry of an arbitrary Riemannian manifold with that of a model space in order to conclude that the manifold retains particular geometric properties of the model. In particular, comparing the geometry of the manifolds will mean to compare their sectional or sometimes Ricci curvatures. Typically, model spaces are spaces of constant sectional curvature. They can be represented as the products

$$M_c^{n+1} = [0, r_c) \times \mathbb{S}^n,$$

equipped with the rotationally symmetric metric

$$\langle , \rangle = dr^2 + h_c^2(r) d\vartheta^2,$$

where $d\vartheta^2$ is the round metric on the sphere and

$$(i)\ r_c = \begin{cases} +\infty & \text{if } c \leq 0 \\ \pi/\sqrt{c} & \text{if } c > 0 \end{cases}, \quad (ii)\ h_c(r) = \begin{cases} \frac{1}{\sqrt{-c}} \sinh(\sqrt{-c}r) & \text{if } c < 0 \\ r & \text{if } c = 0 \\ \frac{1}{\sqrt{c}} \sin(\sqrt{c}r) & \text{if } c > 0 \end{cases} \tag{1.29}$$

More generally, having fixed a smooth even function G, as observed by Greene and Wu in [**34**], the definition of model spaces can be extended to Riemannian manifolds of sectional radial curvature $-G(r)$ of the form

$$M_{-G}^{n+1} = [0, r_{-G}) \times \mathbb{S}^n,$$

endowed with rotationally symmetric metrics

$$\langle , \rangle = dr^2 + h^2(r) d\vartheta^2,$$

where $h : \mathbb{R} \to \mathbb{R}$ is the unique solution of the Cauchy problem

$$\begin{cases} h'' - Gh = 0 \\ h(0) = 0,\ h'(0) = 1 \end{cases} \tag{1.30}$$

and r_{-G} is the first zero of h in the interval $(0, +\infty)$. It is easy to see that when $G(r)$ is constant, the function h has the explicit expression given in (1.29), (ii).

Among others, we will focus on comparison theorems for the Hessian and the Laplacian of the distance function. In that regard, in [**34**], Hessian and Laplacian comparison theorems are proved using a geometric approach based on the use of Jacobi fields and the second variation of arc length. There the authors compare the radial sectional curvatures (respectively the radial Ricci curvatures) of manifolds that posses a pole and establish a comparison between the Hessians (respectively the Laplacians) of their distance functions. Later on, in [**55**] Petersen recovered some of these results using an analytical approach which is essentially based on the use of comparison results for ODE's. Later on, inspired by Petersen's approach, Pigola, Rigoli and Setti generalized these results in [**58**], where they proved that a lower (resp. upper) bound on the radial sectional curvature of the form

$$K_M(\Pi) \geq -G(r) \qquad (\text{resp. } K_M(\Pi) \leq -G(r))$$

implies an upper (resp. lower) estimate for the Hessian of the distance function r of the type

$$\operatorname{Hess} r \leq \frac{h'(r)}{h(r)}(\langle,\rangle - dr \otimes dr) \qquad (\text{resp. } \operatorname{Hess} r \geq \frac{h'(r)}{h(r)}(\langle,\rangle - dr \otimes dr)),$$

where the function h is a solution of the Cauchy problem (1.30).
It is not difficult to prove (see [**55**, Chapter 3]) that the quantity $h'(r)/h(r)$ is precisely the Hessian of the distance function on the model manifold M_{-G}. Hence the Hessian comparison theorem tells us that comparing the radial sectional curvature of a Riemannian manifold with that of a model allows us to compare their Hessians. Note that, taking traces it is easy to obtain corresponding estimates for the Laplacian under the assumptions on the radial sectional curvature. Nevertheless, an upper estimate for the Laplacian can be also obtained under the weaker condition of a lower bound for the radial Ricci curvature.

Our goal is to obtain similar results for Lorentzian manifolds with sectional curvature of timelike planes bounded by a function of the Lorentzian distance, improving in this way on classical results. We will use these theorems in Chapter 4, where we will give some applications to the study of spacelike hypersurfaces. Before exhibiting the main results of this section let us recall some basic notions.
Let M^{n+1} be an $(n+1)$-dimensional spacetime, that is, an $(n+1)$-dimensional time-oriented Lorentzian manifold, and let $p,\ q \in M$. Using the standard terminology and notation in Lorentzian geometry, we say that q is in the chronological future of p, written $p \ll q$, if there exists a future-directed timelike curve from p to q. Similarly, we say that q is in the causal future of p, written $p \leq q$, if there exists a future-directed causal (that is nonspacelike) curve from p to q. For a subset $S \subset M$, we define the chronological future of S as

$$I^+(S) = \{q \in M | p \ll q \text{ for some } p \in S\},$$

and the causal future of S as

$$J^+(S) = \{q \in M | p \leq q \text{ for some } p \in S\},$$

where $p \leq q$ means that either $p < q$ or $p = q$. In particular, the chronological and the causal future of a point $p \in M$ are, respectively

$$I^+(p) = \{q \in M | p \ll q\}, \qquad J^+(p) = \{q \in M | p \leq q\}.$$

In a dual way, we denote by

$$I^-(S) = \{q \in M | q \ll p \text{ for some } p \in S\},$$

and

$$J^-(S) = \{q \in M | q \leq p \text{ for some } p \in S\},$$

the chronological and the causal past of S.
It is well known that $I^+(p)$ is always open, while $J^+(p)$ is neither open nor closed in general. Given $q \in J^+(p)$, the Lorentzian distance $d(p, q)$ is defined as the supremum of the Lorentzian lengths of all the future-directed causal curves from p to q. If $q \notin J^+(p)$, then $d(p, q) = 0$ by definition. Moreover,

$d(p,q) > 0$ if and only if $q \in J^+(p)$. Given a point $p \in M$ one can define the Lorentzian distance function $d_p : M \to [0, +\infty)$ with respect to p by

$$d_p(q) = d(p,q).$$

Let

$$T_{-1}M_{|p} = \{v \in T_pM | v \text{ is a future-directed timelike unit vector}\}$$

be the fiber of the unit future observer bundle of M^{n+1} at p. Define the function

$$s_p : T_{-1}M_{|p} \to [0, +\infty), \qquad s_p(v) = \sup\{t \geq 0 \mid d_p(\gamma_v(t)) = t\},$$

where $\gamma_v : [0,a) \to M$ is the future timelike geodesic with $\gamma_v(0) = p$, $\gamma_v'(0) = v$. The future timelike cut-locus $\Gamma^+(p)$ of p in T_pM is defined as

$$\Gamma^+(p) = \{s_p(v)v \mid v \in T_{-1}M_{|p} \text{ and } 0 < s_p(v) < +\infty\}$$

and the future timelike cut-locus $C_t^+(p)$ of p in M is $C_t^+(p) = \exp_p(\Gamma^+(p))$ wherever the exponential map $\exp_p$ at p is defined on $\Gamma^+(p)$.
It is well known that the Lorentzian distance function on arbitrary spacetimes may fail in general to be continuous and finite valued. It is also known that this is true for globally hyperbolic spacetimes. Recall that a spacetime M is said to be *globally hyperbolic* if it is strongly causal and it satisfies the condition that $J^+(p) \cap J^-(q)$ is compact for all $p,\ q \in M$. Moreover, a Lorentzian manifold M is said to be *strongly causal* at a point $p \in M$ if for any neighborhood U of p there exists no timelike curve that passes through U more than once. In general, in order to guarantee the smoothness of the distance function we need to restrict it on certain special subsets of M. Let

$$\widetilde{\mathcal{I}}^+(p) = \{tv \mid v \in T_{-1}M_{|p} \text{ and } 0 < t < s_p(v)\},$$

and define

$$\mathcal{I}^+(p) = \exp(\mathrm{int}(\widetilde{\mathcal{I}}^+(\mathrm{p}))) \subset \mathrm{I}^+(\mathrm{p}).$$

Since

$$\exp_p : \mathrm{int}(\widetilde{\mathcal{I}}^+(\mathrm{p})) \to \mathcal{I}^+(\mathrm{p})$$

is a diffeomorphism, $\mathcal{I}^+(p)$ is an open subset of M. In the lemma below we summarize the main properties of the Lorentzian distance function.

Lemma 1.20 ([**29**], Section 3.1)**.** *Let M be a spacetime and $p \in M$.*

(1) *If M is strongly causal at p, then $s_p(v) > 0$ $\forall v \in T_{-1}M_{|p}$ and $\mathcal{I}^+(p) \neq \emptyset$,*

(2) *If $\mathcal{I}^+(p) \neq \emptyset$, then the Lorentzian distance function d_p is smooth on $\mathcal{I}^+(p)$ and $\overline{\nabla} d_p$ is a past-directed timelike (geodesic) unit vector field on $\mathcal{I}^+(p)$.*

Remark 1.21. If M is a globally hyperbolic spacetime and $\Gamma^+(p) = \emptyset$, then $\mathcal{I}^+(p) = I^+(p)$ and hence the Lorentzian distance function d_p with respect to p is smooth on $I^+(p)$ for each $p \in M$.
We also observe that if M is a Lorentzian space form, then it is globally hyperbolic and geodesically complete. Moreover, every timelike geodesic

realizes the distance between its points. Hence $\Gamma^+(p) = \emptyset$ and we conclude again that the Lorentzian distance function d_p is smooth on $I^+(p)$ for each $p \in M$.

We are now ready to exhibit the estimates for the Hessian and the Laplacian of the Lorentzian distance function in Lorentzian manifolds with sectional or Ricci curvature bounded by a radial function. The case constant bounds on sectional or Ricci curvature was investigated for the first time in [**31**], and later on in [**12**]. In particular, in [**12**] the authors, inspired by the approach of Greene and Wu in [**34**] based on the use of the second variation formula for the arc length and Jacobi fields, proved the comparison described in Propositions 1.22 and 1.23.

Consider, for every $c \in \mathbb{R}$, the function $f_c(s)$ defined as

$$f_c(t) = \begin{cases} \sqrt{c}\coth(\sqrt{c}t) & \text{if } c > 0 \text{ and } t > 0 \\ \frac{1}{t} & \text{if } c = 0 \text{ and } t > 0 \\ \sqrt{-c}\cot(\sqrt{-c}t) & \text{if } c < 0 \text{ and } 0 < t < \pi/\sqrt{-c} \end{cases}$$

and let $\overline{\nabla}$, $\overline{\mathrm{Hess}}$ and $\overline{\Delta}$ respectively the Levi-Civita connection, the Hessian and the Laplacian on the spacetime M. Then the following proposition holds.

Proposition 1.22 (Lemmas 3.1 and 3.2 in [**12**]). *Let M^{n+1} be an $(n+1)$-dimensional spacetime such that that $K_M(\Pi) \leq c$, $c \in \mathbb{R}$, for all timelike planes Π. Assume that there exists a point $p \in M$ such that $\mathcal{I}^+(p) \neq \emptyset$ and let $q \in \mathcal{I}^+(p)$ (with $r(q) < \pi/\sqrt{-c}$ if $c < 0$, $r(\cdot) = d_p(\cdot)$ being the Lorentzian distance function from p). Then*

$$\overline{\mathrm{Hess}} r(X, X) \geq -f_c(r(q)) \langle X, X\rangle$$

for every spacelike $X \in T_qM$. Analogously, if $K_M(\Pi) \geq c$, $c \in \mathbb{R}$, for all timelike planes Π and $r(q) < \pi/\sqrt{-c}$ if $c < 0$, then

$$\overline{\mathrm{Hess}} r(X, X) \leq -f_c(r(q)) \langle X, X\rangle$$

for every spacelike $X \in T_qM$.

Moreover, under the weaker hypothesis

$$\mathrm{Ric}_M(Z, Z) \geq -nc$$

for every unit timelike vector Z, the following Laplacian comparison result holds.

Proposition 1.23 (Lemma 3.3 in [**12**]). *Let M^{n+1} be an $(n+1)$-dimensional spacetime such that such that $\mathrm{Ric}_M(Z, Z) \geq -nc$, $c \in \mathbb{R}$, for every unit timelike vector Z. Assume that there exists a point $p \in M$ such that $\mathcal{I}^+(p) \neq \emptyset$ and let $q \in \mathcal{I}^+(p)$ (with $r(q) < \pi/\sqrt{-c}$ if $c < 0$, $r(\cdot) = d_p(\cdot)$ being the Lorentzian distance function from p). Then*

$$\overline{\Delta} r(q) \geq -n f_c(r(q)).$$

Inspired by these works, we aim at extending the previous comparison results to the more general cases

$$K_M(\Pi) \geq G(r), \qquad K_M(\Pi) \leq G(r),$$

for all timelike planes Π, and

$$\mathrm{Ric}_M(\overline{\nabla}r, \overline{\nabla}r) \geq -nG(r),$$

where G is a smooth even function on $\mathbb{R}$. In order to do that we will use an analytical approach which makes use of a comparison result for solutions of Riccati inequalities. First of all we need the following

Lemma 1.24 (Sturm comparison Theorem)**.** *Let G be a continuous function on $[0,+\infty)$ and let $\phi,\ \psi \in C^1([0,+\infty))$ with $\phi',\ \psi' \in AC([0,+\infty))$ be solutions of the problems*

$$\begin{cases} \phi'' - G\phi \leq 0 & a.e.\ in\ (0,+\infty) \\ \phi(0) = 0 \end{cases},\quad \begin{cases} \psi'' - G\psi \geq 0 & a.e.\ in\ (0,+\infty) \\ \psi(0) = 0,\ \psi'(0) > 0 \end{cases}$$

If $\phi(r) > 0$ for $r \in (0,T)$ and $\psi'(0) \geq \phi'(0)$, then $\psi(r) > 0$ in $(0,T)$ and

$$\frac{\phi'}{\phi} \leq \frac{\psi'}{\psi}\ \text{and}\ \psi \geq \phi \qquad \text{on } (0,T).$$

For a proof of the lemma see [**58**]. Using the above lemma we are able to obtain the desired comparison result for solutions of Riccati inequalities with appropriate asymptotic behaviour.

Corollary 1.25 (Corollary 4 in [**41**])**.** *Let G be a continuous function on $[0,+\infty)$ and let $g_i \in AC((0,T_i))$ be solutions of the Riccati differential inequalities*

$$g_1' - \frac{g_1^2}{\alpha} + \alpha G \geq 0,\ (\text{resp.} \leq 0) \qquad g_2' + \frac{g_2^2}{\alpha} - \alpha G \geq 0,\ (\text{resp.} \leq 0)$$

a.e. in $(0,T_i)$, satisfying the asymptotic conditions

$$g_i(t) = \frac{\alpha}{t} + o(t) \qquad as \qquad t \to 0^+,$$

for some $\alpha > 0$. Then $T_1 \leq T_2$ (resp. $T_1 \geq T_2$) and $-g_1(t) \leq g_2(t)$ in $(0,T_1)$ (resp. $-g_2(t) \leq g_1(t)$ in $(0,T_2)$).

Proof. Since $\widetilde{g_i} = \alpha^{-1}g_i$ satisfies the conditions in the statement with $\alpha = 1$, without loss on generality we may assume that $\alpha = 1$. Notice that $g_i(s) - \frac{1}{s}$ is bounded and integrable in a neighbourhood of $s = 0$. Hence the same is true for the function $-g_1(s) - \frac{1}{s}$. Indeed

$$-\Big(g_1(s) + \frac{1}{s}\Big) < -\Big(g_1(s) - \frac{1}{s}\Big) \leq \Big|g_1(s) - \frac{1}{s}\Big| \leq C,$$

for some constant $C > 0$. Now let $\phi_i \in C^1([0,T_i))$ be the positive functions defined by

$$\phi_1(t) = t\exp\Big(-\int_0^t \big(g_1(s) + \frac{1}{s}\big)ds\Big),\ \phi_2(t) = t\exp\Big(\int_0^t \big(g_2(s) - \frac{1}{s}\big)ds\Big).$$

Then $\phi_i(0) = 0$, $\phi_i' \in AC((0,T_i))$, $\phi_i'(0) = 1$ and

$$\phi_1'(t) = -g_1(t)\phi_1(t),\ \phi_2'(t) = g_2(t)\phi_2(t)$$

Hence

$$\phi_1'' \leq G\phi_1, \quad \phi_2'' \geq G\phi_2 \quad (\text{resp. } \phi_1'' \geq G\phi_1, \quad \phi_2'' \leq G\phi_2).$$

Then, it follows by Lemma 1.24 that $T_1 \leq T_2$ (resp. $T_1 \geq T_2$) and

$$-g_1(t) = \frac{\phi_1'(t)}{\phi_1(t)} \leq \frac{\phi_2'(t)}{\phi_2(t)} = g_2(t) \quad (\text{resp.} - g_2(t) = \frac{\phi_2'(t)}{\phi_2(t)} \leq \frac{\phi_1'(t)}{\phi_1(t)} = g_1(t)).$$

□

Remark 1.26. The proof of the previous Corollary, as well as those of the Hessian and Laplacian comparison theorems, are deeply inspired by the proofs given in [**58**, Chapter 2].

We are now ready to prove the Hessian and Laplacian comparison theorems.

Theorem 1.27 (Hessian Comparison Theorem, Theorem 5 in [**41**])**.** *Let M^{n+1} be an $(n+1)$-dimensional spacetime. Assume that there exists a point $p \in M$ such that $\mathcal{I}^+(p) \neq \emptyset$ and let $r(\cdot) = d_p(\cdot)$ be the Lorentzian distance function from p. Given a smooth even function G on $\mathbb{R}$, let h be a solution of the Cauchy problem*

$$\begin{cases} h'' - Gh = 0 \\ h(0) = 0, \ h'(0) = 1 \end{cases}$$

and let $I = [0, r_G) \subset [0, +\infty)$ be the maximal interval where h is positive and $q \in \mathcal{I}^+(p) \cap B^+(p, r_G)$, where

$$B^+(p, r_G) = \{q \in I^+(p) | d_p(q) < r_G\}.$$

If

$$K_M(\Pi) \leq G(r) \tag{1.31}$$

for all timelike planes Π, then

$$\overline{\text{Hess}} r(X, X) \geq -\frac{h'}{h}(r(q)) \langle X, X \rangle$$

for every spacelike $X \in T_q M$ which is orthogonal to $\overline{\nabla} r(q)$. Analogously, if

$$K_M(\Pi) \geq G(r) \tag{1.32}$$

for all timelike planes Π, then

$$\overline{\text{Hess}} r(X, X) \leq -\frac{h'}{h}(r(q)) \langle X, X \rangle$$

for every spacelike $X \in T_q M$ which is orthogonal to $\overline{\nabla} r(q)$.

Proof. Let $v \in \exp_p^{-1}(q) \in \operatorname{int}(\widetilde{\mathcal{I}}^+(\mathrm{p}))$ and let $\gamma(t) = \exp_p(tv)$, $0 \leq t \leq s_p(v)$, be the radial future directed unit timelike geodesic with $\gamma(0) = p$, $\gamma(s) = q$, $s = r(q)$. Recall that $\gamma'(s) = -\overline{\nabla} r(q)$ and $\overline{\nabla}_{\overline{\nabla} r} \overline{\nabla} r(q) = 0$. Since $\overline{\nabla} r$ satisfies the timelike eikonal inequality, $\overline{\text{Hess}} r$ is diagonalizable (see Chapter 6 in [**31**] or [**30**] for more details) and $T_q M$ has an orthonormal basis consisting of eigenvectors of $\overline{\text{Hess}} r$. Let us denote by $\lambda_{\max}(q)$ and $\lambda_{\min}(q)$ respectively its greatest and smallest eigenvalues in the orthogonal complement of $\overline{\nabla} r(q)$. Notice that the theorem is proved once one shows that

(a) if (1.31) holds, then

$$\lambda_{\min}(q) \geq -\frac{h'}{h}(r(q)).$$

(b) if (1.32) holds, then

$$\lambda_{\max}(q) \leq -\frac{h'}{h}(r(q)).$$

Let us prove claim (a) first. We claim that if (1.31) holds, then $\lambda_{\min}$ satisfies

$$\begin{cases} \frac{d}{dt}(\lambda_{\min} \circ \gamma) - (\lambda_{\min} \circ \gamma)^2 \geq -G & \text{for a.e. } t > 0 \\ \lambda_{\min} \circ \gamma = \frac{1}{t} + o(t) & \text{as } t \to 0^+ \end{cases} \tag{1.33}$$

Namely, by the definition of covariant derivative

$$(\overline{\nabla}_X \overline{\operatorname{hess}} u)(Y) = \overline{\nabla}_X(\overline{\operatorname{hess}} u(Y)) - \overline{\operatorname{hess}} u(\overline{\nabla}_X Y).$$

Hence, recalling the definition of the curvature tensor we find

$$(\overline{\nabla}_Y \overline{\operatorname{hess}} u)(X) - (\overline{\nabla}_X \overline{\operatorname{hess}} u)(Y) = \overline{\mathrm{R}}(X,Y)\overline{\nabla} u.$$

Choose $u = r$, $X = \overline{\nabla} r$. For every spacelike unit vector $Y \in T_qM$, Y is orthogonal to $\gamma'(s)$ and we can define a vector field Y orthogonal to γ' by parallel translation along γ. Then

$$\begin{aligned} \overline{\nabla}_{\gamma'(s)}(\overline{\operatorname{hess}} r(Y)) =& (\overline{\nabla}_{\gamma'(s)} \overline{\operatorname{hess}} r)(Y) + \overline{\operatorname{hess}} r(\overline{\nabla}_{\gamma'(s)} Y) \\ =& -(\overline{\nabla}_{\overline{\nabla} r} \overline{\operatorname{hess}} r)(Y) \\ =& -(\overline{\nabla}_Y \overline{\operatorname{hess}} r)(\overline{\nabla} r) + \overline{\mathrm{R}}(\overline{\nabla} r, Y)\overline{\nabla} r \\ =& \overline{\operatorname{hess}} r(\overline{\nabla}_Y \overline{\nabla} r) + \overline{\mathrm{R}}(\overline{\nabla} r, Y)\overline{\nabla} r. \end{aligned}$$

On the other hand, since Y is parallel

$$\frac{d}{dt}\left\langle \overline{\operatorname{hess}} r(Y), Y \right\rangle \Big|_s = \left\langle \overline{\nabla}_{\gamma'(s)} \overline{\operatorname{hess}} r(Y), Y \right\rangle.$$

Hence

$$\frac{d}{dt}\overline{\operatorname{Hess}} r(\gamma)(Y,Y) - \left\langle \overline{\operatorname{hess}} r(\gamma)(Y), \overline{\operatorname{hess}} r(\gamma)(Y) \right\rangle = -K_M(Y \wedge \gamma')$$

Notice that

$$\overline{\operatorname{Hess}} r(X,X) \geq \lambda_{\min}$$

for every spacelike unit vector field $X \perp \overline{\nabla} r$. Let us choose Y so that at s

$$\overline{\operatorname{Hess}} r(\gamma)(Y,Y) = \lambda_{\min}(\gamma(s)).$$

Then, the function $\operatorname{Hess} r(\gamma)(Y,Y) - \lambda_{\min} \circ \gamma$ attains its minimum at s. Hence

$$\frac{d}{dt}\overline{\operatorname{Hess}} r(\gamma)(Y,Y)\Big|_s = \frac{d}{dt}(\lambda_{\min} \circ \gamma)\Big|_s$$

and we have proved that $\lambda_{\min}$ satisfies the first equation in (1.33), since $K_M(Y \wedge \gamma') \leq G$. The asymptotic behaviour follows from the expression

$$\overline{\operatorname{Hess}} r = \frac{1}{r}(\langle , \rangle + dr \otimes dr) + o(1) \tag{1.34}$$

that can be proved using normal coordinates around p. Now, if we set $\phi = \frac{h'}{h}$, we find that ϕ satisfies

$$\begin{cases} \phi' + \phi^2 = G & \text{on } (0, r_G) \\ \phi = \frac{1}{t} + o(t) & \text{as } t \to 0^+ \end{cases}$$

Then, using Corollary 1.25 with $g_1 = \lambda_{\min}$, $g_2 = \phi$ and $\alpha = 1$ we conclude that

$$\lambda_{\min}(q) \geq -\frac{h'}{h}(r(q))$$

and this concludes the proof of (a).

Finally, for what concerns claim (b), we observe that reasoning as in the proof of claim (a) and choosing Y so that at s

$$\overline{\mathrm{Hess}} r(\gamma)(Y, Y) = \lambda_{\max}(\gamma(s))$$

we can prove that, if (1.32) holds, $\lambda_{\max}$ satisfies

$$\begin{cases} \frac{d}{dt}(\lambda_{\max} \circ \gamma) - (\lambda_{\max} \circ \gamma)^2 \leq -G & \text{for a.e. } t > 0 \\ \lambda_{\max} \circ \gamma = \frac{1}{t} + o(t) & \text{as } t \to 0^+ \end{cases}$$

In this case, setting again $\phi = \frac{h'}{h}$, we find that ϕ satisfies

$$\begin{cases} \phi' + \phi^2 = G & \text{on } (0, r_G) \\ \phi = \frac{1}{t} + o(t) & \text{as } t \to 0^+ \end{cases}$$

Then, we can conclude again using Corollary 1.25 with $g_1 = \lambda_{\max}$, $g_2 = \phi$ and $\alpha = 1$. $\square$

Notice that in case $G(r) \equiv c$, $c \in \mathbb{R}$, then the function h has the expression

$$h(s) = \begin{cases} \frac{1}{\sqrt{c}} \sinh(\sqrt{c}s) & \text{if} \quad c > 0 \\ s & \text{if} \quad c = 0 \\ \frac{1}{\sqrt{-c}} \sin(\sqrt{-c}s) & \text{if} \quad c < 0 \end{cases} \tag{1.35}$$

and

$$r_G = \begin{cases} +\infty & \text{if} \quad c \geq 0 \\ \pi/\sqrt{-c} & \text{if} \quad c < 0 \end{cases}$$

It is then easy to see that $h'/h(s) = f_c(s)$, showing that our theorem extends Proposition 1.22.

Furthermore, observe that, as in the Riemannian case, we can introduce the model manifold

$$M_G = -I \times \mathbb{H}^n, \qquad \langle , \rangle = -dr^2 + h^2(r) d\vartheta^2,$$

where h is a solution of (1.30), $I = (0, r_G)$ and r_G is the first zero of h.

Using the relationship between the curvature tensor of a warped product and the curvature tensor of its base and fibre (see for instance Proposition 42 in [**54**]) one has

$$\begin{aligned} \overline{\mathrm{R}}(U, V)W = & \mathrm{R}_{\mathbb{P}}(U^*, V^*)W^* + ((\log \rho)')^2(\pi_I)(\langle U, W \rangle V - \langle V, W \rangle U) \\ & + (\log \rho)''(\pi_I) \langle W, T \rangle (\langle V, T \rangle U - \langle U, T \rangle V) \\ & - (\log \rho)''(\pi_I)(\langle U, W \rangle \langle V, T \rangle - \langle V, W \rangle \langle U, T \rangle)T, \end{aligned} \tag{1.36}$$

where $X^* = \pi_{\mathbb{P}*}X$ for every $X \in TM$. A straightforward calculation based on the above formula permits to conclude then that the function G is the radial sectional curvature of the model manifold M_G.
Under the weaker assumption of radial Ricci curvature bounded from below we obtain the following

Theorem 1.28 (Laplacian Comparison Theorem, Theorem 6 in [**41**])**.** *Let M^{n+1} be an $(n+1)$- dimensional spacetime. Assume that there exists a point $p \in M$ such that $\mathcal{I}^+(p) \neq \emptyset$ and let $q \in \mathcal{I}^+(p)$. Let $r(\cdot) = d_p(\cdot)$ be the Lorentzian distance function from p. Given a smooth even function G on $\mathbb{R}$, let h be a solution of the Cauchy problem*

$$\begin{cases} h'' - Gh \geq 0 \\ h(0) = 0, \ h'(0) = 1 \end{cases}$$

and let $I = [0, r_G) \subset [0, +\infty)$ be the maximal interval where h is positive. If

$$\mathrm{Ric}_M(\overline{\nabla} r, \overline{\nabla} r) \geq -nG(r), \tag{1.37}$$

then

$$\overline{\Delta} r \geq -n\frac{h'}{h}(r)$$

holds pointwise on $\mathcal{I}^+(p) \cap B^+(p, r_G)$.

Proof. Let $v \in \exp_p^{-1}(q) \in \mathrm{int}(\widetilde{\mathcal{I}}^+(\mathrm{p}))$ and let $\gamma(t) = \exp_p(tv)$, $0 \leq t \leq s_p(v)$, be the radial future directed unit timelike geodesic with $\gamma(0) = p$, $\gamma(s) = q$, $s = r(q)$. Recall that $\gamma'(s) = -\overline{\nabla} r(q)$ and $\overline{\nabla}_{\overline{\nabla} r}\overline{\nabla} r(q) = 0$. Define

$$\varphi(t) = \overline{\Delta} r \circ \gamma(t), \qquad t \in (0, s].$$

Then tracing Equation (1.34)

$$\varphi(t) = \frac{n}{t} + o(t) \qquad \text{as } t \to 0^+.$$

Let $f \in C^\infty(M)$. The following Bochner formula holds

$$\frac{1}{2}\overline{\Delta}\left\langle \overline{\nabla} f, \overline{\nabla} f \right\rangle = \left\|\overline{\mathrm{hess}} f\right\|^2 + \mathrm{Ric}_M(\overline{\nabla} f, \overline{\nabla} f) + \left\langle \overline{\nabla}\overline{\Delta} f, \overline{\nabla} f \right\rangle.$$

See [**31**] for more details. Since $\left\|\overline{\nabla} r\right\|^2 = -1$, it follows that

$$0 = \left\|\overline{\mathrm{hess}} r\right\|^2 + \mathrm{Ric}_M(\overline{\nabla} r, \overline{\nabla} r) + \left\langle \overline{\nabla}\overline{\Delta} r, \overline{\nabla} r \right\rangle.$$

Since $\left\|\overline{\mathrm{hess}} r\right\|^2 \geq \frac{(\overline{\Delta} r)^2}{n}$ and $\mathrm{Ric}_M(\overline{\nabla} r, \overline{\nabla} r) \geq -nG(r)$, we have

$$\frac{1}{n}(\overline{\Delta} r)^2 + \left\langle \overline{\nabla}\overline{\Delta} r, \overline{\nabla} r \right\rangle \leq nG(r).$$

Computing along γ

$$\varphi'(t) = \frac{d}{dt}(\overline{\Delta} r(\gamma(t)))\Big|_s = \left\langle \overline{\nabla}\overline{\Delta} r(\gamma(t)), \gamma'(t) \right\rangle\Big|_s = -\left\langle \overline{\nabla}\overline{\Delta} r, \overline{\nabla} r \right\rangle.$$

Hence the function φ satisfies

$$\begin{cases} \varphi'(t) - \frac{\varphi^2(t)}{n} \geq -nG \\ \varphi(t) = \frac{n}{t} + o(t) \qquad \text{as } t \to 0^+ \end{cases}$$

Set $\phi = n\frac{h'}{h}$. Then ϕ satisfies

$$\begin{cases} \phi'(t) + \frac{\phi^2(t)}{n} \geq nG & \text{on } (0, r_G) \\ \phi(t) = \frac{n}{t} + o(t) & \text{as } t \to 0^+ \end{cases}$$

Hence we conclude again using Corollary 1.25. $\square$

Notice as above that, in case $G(r) \equiv c$, we recover Proposition 1.23.

CHAPTER 2

Hypersurfaces of constant k-mean curvature in warped products

The aim of this chapter is to state and prove uniqueness results for hypersurfaces of constant higher order mean curvature immersed in suitable ambient manifolds. To do that, in the spirit of the Alexandrov Theorem, we will consider manifolds with a large class of embedded umbilical hypersurfaces of constant mean curvatures. We will then look for geometric condition that force an immersed complete hypersurface of constant k-mean curvature, $2 \leq k \leq n$, to be one of those already classified. As pointed out by Montiel in [**49**], a natural class of ambient manifolds to consider is that of warped products $M^{n+1} := \mathbb{R} \times_\rho \mathbb{P}^n$, where $\mathbb{P}^n$ is a complete n-dimensional Riemannian manifold, $\rho : \mathbb{R} \to \mathbb{R}_+$ is a smooth function and the product manifold $\mathbb{R} \times \mathbb{P}^n$ is endowed with the complete Riemannian metric

$$\langle , \rangle = \pi_{\mathbb{R}}^*(dt^2) + \rho^2(\pi_{\mathbb{R}})\pi_{\mathbb{P}}^*(\langle , \rangle_{\mathbb{P}}).$$

Here $\pi_{\mathbb{R}}$ and $\pi_{\mathbb{P}}$ denote the projections onto the corresponding factors and $\langle , \rangle_{\mathbb{P}}$ is the Riemannian metric on $\mathbb{P}^n$. Each leaf $\mathbb{P}_t = \{t\} \times \mathbb{P}^n$ of the foliation $t \to \mathbb{P}_t$ of M^{n+1} by complete hypersurfaces is totally umbilical and has constant k-mean curvature

$$\mathcal{H}_k(t) = \Big(\frac{\rho'(t)}{\rho(t)}\Big)^k, \qquad 0 \leq k \leq n,$$

with respect to $-T = -\partial/\partial t$.
Observe that it is not difficult to check that the vector field $\mathcal{T} = \rho T$ is a closed conformal vector field on M, that is it satisfies

$$\overline{\nabla}_X \mathcal{T} = \varphi X, \qquad \text{for any} X \in TM,\ \varphi \in C^\infty(M),$$

with $\varphi(t) = \rho'(t)$. Actually, it can be proved (see [**49**] for more details) that any complete Riemannian manifold carrying a closed conformal vector field can be constructed from a warped product with one dimensional base.
Space forms are a typical example of manifolds that support a closed conformal vector field.

Example 2.1. The Euclidean space $\mathbb{R}^{n+1}$ carries many nontrivial closed (in fact exacts) conformal vector fields. One of them is the costant vector field

$$\mathcal{T}(p) = c, \qquad \forall c \in \mathbb{R}^{n+1}, \qquad p \in \mathbb{R}^{n+1}.$$

This vector field generates a foliation of the space by means of parallel hyperplanes and can be represented as the product $\mathbb{R} \times \mathbb{R}^n$ (i.e. the warping function ρ is constantly equal to 1).

Further, let us fix an origin $a \in \mathbb{R}^{n+1}$. Then it is not difficult to check that the position vector field

$$\mathcal{T}(p) = p - a, \qquad p \in \mathbb{R}^{n+1},$$

is again a closed conformal vector field that generates a foliation of the space by means of concentric round hyperspheres centered at a. This corresponds to the representation of $\mathbb{R}^{n+1}$ as warped product with base $I = \mathbb{R}_+$, fiber $\mathbb{P}^n = \mathbb{S}^n$ and warping function $\rho(t) = t$.

Example 2.2. Round spheres $\mathbb{S}^{n+1}$ admit also exact conformal vector fields. Indeed, once we fix an origin $a \in \mathbb{S}^{n+1}$, we can verify that the vector field

$$X(p) = a - \langle a, p\rangle\, p, \qquad p \in \mathbb{S}^{n+1}$$

is closed and conformal and foliates the hypersphere by means of umbilical hyperspheres $\mathbb{S}^n$ parallel to the equator orthogonal to a. The corresponding representation as warped product space is then $(0, \pi) \times_{\sin t} \mathbb{S}^n$.

Example 2.3. Finally, let us consider the Hyperbolic space $\mathbb{H}^{n+1}$ viewed as a hypersphere in the Lorentz-Minkowski space, that is a connected component of the hyperquadric

$$\{p \in \mathbb{R}^{n+2}_1 | \, \langle p, p\rangle = -1\}.$$

Fix $a \in \mathbb{R}^{n+2}_1$ and consider the closed conformal vector field

$$\mathcal{T}(p) = a + \langle a, p\rangle\, p, \qquad p \in \mathbb{H}^{n+1}.$$

Depending on the causal character of a we have different foliations of $\mathbb{H}^{n+1}$ and hence different descriptions of it as warped product space. Namely, if a is timelike, $\mathbb{H}^{n+1}$ is foliated by spheres and can be described as the warped product $\mathbb{R}_+ \times_{\sinh t} \mathbb{S}^n$; if a is lightlike the foliation is by means of horospheres and the space can be viewed as $\mathbb{R} \times_{e^t} \mathbb{R}^n$; finally, if a is spacelike, the vector field $\mathcal{T}$ generates a foliation of $\mathbb{H}^{n+1}$ by means of totally geodesic hyperbolic hyperplanes and it can be represented as the warped product with base $\mathbb{R} \times_{\cosh t} \mathbb{H}^n$.

Consider now $f : \Sigma^n \to I \times_\rho \mathbb{P}^n$, $I \subset \mathbb{R}$, being an isometric immersion of an n-dimensional Riemannian manifold Σ^n. Our aim is to show that, under suitable geometric conditions, every hypersurface Σ of constant k-mean curvature, $2 \le k \le n$, has to be a slice. The case of constant mean curvature has already been studied in [**49**] and [**9**].
Given $f : \Sigma^n \to I \times_\rho \mathbb{P}^n$, we define the *height function* $h \in C^\infty(\Sigma)$ as $h = \pi_\mathbb{R} \circ f$. In this context and following the terminology introduced in [**9**], we will say that the hypersurface is *contained in a slab* if $f(\Sigma)$ lies between two leaves of the foliation, $\mathbb{P}_{t_1}, \mathbb{P}_{t_2}$ with $t_1 < t_2$.
In what follows, we will assume that the immersion is *two-sided*, which holds always true at least locally. Recall that a submanifold $f : \Sigma^n \to I \times_\rho \mathbb{P}^n$ is called two-sided if its normal bundle is trivial, i.e. there exists a globally defined unit normal vector field. For instance, every hypersurface

with nonzero constant mean curvature is trivially two sided.
We can then define the *angle function* $\Theta : \Sigma^n \to [-1, 1]$ by

$$\Theta(p) = \langle N(p), T(f(p)\rangle ,$$

where N denotes the global normal unit vector field and $T = \partial/\partial t$.

2.1. Curvature estimates for hypersurfaces in warped products

In this section we will derive some estimates for the mean and the higher order mean curvatures of a hypersurface in a slab of a warped product space. The idea is that, under suitable geometric conditions, it is possible to compare the (higher order) mean curvature of the hypersurface with that of a slice.

Similar results have been proved for the mean curvature in [**9**] for surfaces in warped product spaces $\mathbb{R} \times_\rho \mathbb{P}^2$, where $\mathbb{P}^2$ is a complete surface of non-negative Gaussian curvature. The technique there used exploits the parabolicity of the surface, which is guaranteed by the hypothesis on the curvature of $\mathbb{P}^2$. Unfortunately, the fact that the latter assumption imply the parabolicity of the immersed surface holds only in case $n = 2$. Thus it is necessary to work out a different proof to extend these results to the case of arbitrary n. It is not surprising that the right tool turns out to be the Omori-Yau maximum principle.

To prove our estimates we will need the following computational result

Proposition 2.4. *Let $f : \Sigma^n \to I \times_\rho \mathbb{P}^n$ be an isometric immersion. If*

$$\sigma(t) = \int_{t_0}^{t} \rho(u)du,$$

then

$$L_{k-1}h = \mathcal{H}(h)(c_{k-1}H_{k-1} - \langle P_{k-1}\nabla h, \nabla h\rangle) + c_{k-1}\Theta H_k \tag{2.1}$$

and

$$L_{k-1}\sigma(h) = c_{k-1}\rho(h)(\mathcal{H}(h)H_{k-1} + \Theta H_k), \tag{2.2}$$

where $c_{k-1} = (n-k+1)\binom{n}{k-1} = k\binom{n}{k}$.

Proof. The gradient of $\pi_{\mathbb{R}} \in C^\infty(M)$ is $\overline{\nabla}\pi_{\mathbb{R}} = T$, hence

$$\nabla h = (\overline{\nabla}\pi_{\mathbb{R}})^T = T - \langle T, N\rangle N = T - \Theta N.$$

Recall that the Levi-Civita connection of a warped product satisfies (see [**54**] for more details)

$$\overline{\nabla}_X T = \mathcal{H}(X - \langle X, T\rangle T), \qquad \text{for any } X \in TM.$$

Thus

$$\overline{\nabla}_X \nabla h = \mathcal{H}(h)(X - \langle X, T\rangle T) - X(\Theta)N + \Theta AX,$$

for any $X \in T\Sigma$. Then

$$\mathrm{hess}(h)(X) = \nabla_X \nabla h = \mathcal{H}(h)(X - \langle X, \nabla h\rangle \nabla h) + \Theta AX. \tag{2.3}$$

Composing with P_{k-1} and taking the trace

$$L_{k-1}h = \mathcal{H}(h)(\mathrm{Tr}(P_{k-1}) - \langle P_{k-1}\nabla h, \nabla h\rangle) + \Theta\mathrm{Tr}(P_{k-1}A).$$

On the other hand, since $\nabla\sigma(h) = \rho(h)\nabla h$, we have

$$\mathrm{hess}(\sigma(h))(X) = \rho'(h)\langle\nabla h, X\rangle\nabla h + \rho(h)\,\mathrm{hess}(h)(X).$$

Again, composing with P_{k-1} and taking the trace

$$\begin{aligned}L_{k-1}\sigma(h) =& \rho'(h)\langle P_{k-1}\nabla h, \nabla h\rangle + \rho(h)\mathcal{H}(h)\big(\mathrm{Tr}P_{k-1} - \langle P_{k-1}\nabla h, \nabla h\rangle\big)\\ &+ \rho(h)\Theta\mathrm{Tr}(P_{k-1}A)\end{aligned}$$

and we conclude using the expressions of the traces of P_{k-1} and $P_{k-1}A$. $\square$

As a first application of the computations above, we derive the following:

Theorem 2.5 (Theorem 7 in [**13**]). *Let $f : \Sigma^n \to I \times_\rho \mathbb{P}^n$ be an immersed hypersurface. If the Omori-Yau maximum principle holds on Σ for the Laplacian and if $h^* = \sup_\Sigma h < +\infty$, then*

$$\sup_\Sigma |H_1| \geq \inf_\Sigma \mathcal{H}(h).$$

In other words, there is no properly immersed hypersurface contained in a slab $[t_1, t_2] \times \mathbb{P}^n$ with

$$\sup_\Sigma |H_1| < \inf_\Sigma \mathcal{H}(h).$$

In particular, as an application of Corollary 1.18, we deduce the following result, which generalizes Theorem 2 in [**9**].

Corollary 2.6. *Let $\mathbb{P}^n$ be a complete, non-compact, Riemannian manifold whose radial sectional curvature satisfies condition* (1.21)*, that is*

$$K_{\mathbb{P}}^{rad} \geq -G(\widehat{r}),$$

where $\widehat{r}$ is the distance from a reference point in $\mathbb{P} = n$ and G is a smooth function on $[0, +\infty)$, even at the origin and satisfying the conditions listed in (1.10)*. If $f : \Sigma^n \to I \times_\rho \mathbb{P}^n$ is a properly immersed hypersurface contained in a slab, then*

$$\sup_\Sigma |H_1| \geq \inf_\Sigma \mathcal{H}(h). \tag{2.4}$$

In other words, there is no properly immersed hypersurface contained in a slab $[t_1, t_2] \times \mathbb{P}^n$ with

$$\sup_\Sigma |H_1| < \inf_{[t_1,t_2]} \mathcal{H}(t).$$

For the proof of Corollary 2.6, observe that, if $\sup_\Sigma |H_1| = +\infty$, then the inequality (2.8) trivially holds. On the other hand, if $\sup_\Sigma |H_1| < +\infty$ then by Corollary 1.18 we know that the Omori-Yau maximum principle holds on Σ and the result follows from Theorem 2.5.

Proof of Theorem 2.5. Since h is bounded from above, we may find a sequence $\{p_j\} \subset \Sigma^n$ such that

$$\begin{aligned}&\lim_{j\to+\infty} h(p_j) = h^* := \sup h,\\ &\|\nabla h(p_j)\|^2 = 1 - \Theta^2(p_j) < \left(\frac{1}{j}\right)^2,\\ &\Delta h(p_j) = \mathcal{H}(h(p_j))(n - \|\nabla h(p_j)\|^2) + nH_1(p_j)\Theta(p_j) < \frac{1}{j}.\end{aligned}$$

Then

$$\frac{1}{j} > \Delta h(p_j) \geq \mathcal{H}(h(p_j))(n - \|\nabla h(p_j)\|^2) - n \sup_\Sigma |H_1|.$$

Making $j \to +\infty$ we get

$$0 \geq \mathcal{H}(h^*) - \sup_\Sigma |H_1|,$$

so that

$$\sup_\Sigma |H_1| \geq \mathcal{H}(h^*) \geq \inf_\Sigma \mathcal{H}(h).$$

□

Using the Omori-Yau maximum principle for elliptic operators of the form (1.11) and assuming some extra assumption it is not difficult to extend the previous estimate to the k-mean curvature, $2 \leq k \leq n$.

Assume that $H_k > 0$, $2 \leq k \leq n$. Then we normalize the operators P_{k-1} to the following

$$\widehat{P}_k = \frac{1}{H_k} P_k.$$

Observe that

$$\mathrm{Tr}(\widehat{P}_k) = c_k,$$

so that the operators $\widehat{P}_k$ always have trace bounded from above. Moreover, we will denote by $\widehat{L}_k$ the corresponding differential operator, that is

$$\widehat{L}_k = \mathrm{Tr}(\widehat{P}_k \circ \mathrm{hess}).$$

In case $k = 2$ we obtain the next

Theorem 2.7 (Theorem 10 in [**13**])**.** *Let $f : \Sigma^n \to I \times_\rho \mathbb{P}^n$ be an immersed hypersurface with $H_2 > 0$. If the Omori-Yau maximum principle holds on Σ for $\widehat{L}_1$ and $h^* = \sup_\Sigma h < +\infty$, then*

$$\sup_\Sigma H_2^{1/2} \geq \inf_\Sigma \mathcal{H}(h).$$

Proof. We may assume without loss of generality that $\sup_\Sigma H_2 < +\infty$ and $\inf_\Sigma \mathcal{H}(h) \geq 0$. Otherwise the inequality trivially holds. Since $|H_1| \geq \sqrt{H_2}$, the mean curvature H_1 never vanishes and we can choose the orientation on Σ so that $H_1 > 0$. Then $\widehat{L}_1$ is a well defined elliptic operator. Moreover, since h is bounded from above and $\sigma(t)$ is an increasing function, $\sup_\Sigma \sigma(h) = \sigma(h^*) < +\infty$ and we may find a sequence $\{p_j\} \subset \Sigma^n$ such that

$$\lim_{j\to+\infty} h(p_j) = h^*,$$
$$\|\nabla h(p_j)\|^2 = 1 - \Theta^2(p_j) < \left(\frac{1}{j}\right)^2,$$
$$\widehat{L}_1 h(p_j) < \frac{1}{j}.$$

Then,

$$\begin{aligned}\frac{1}{j} >&\widehat{L}_1(\sigma\circ h)(p_j) = n(n-1)\rho(h(p_j))\left(\mathcal{H}(h(p_j)) + \Theta(p_j)\frac{H_2}{H_1}(p_j)\right)\\ \geq& n(n-1)\rho(h(p_j))\left(\mathcal{H}(h(p_j)) - \frac{H_2}{H_1}(p_j)\right)\\ \geq& n(n-1)\rho(h(p_j))\left(\mathcal{H}(h(p_j)) - \sqrt{H_2}(p_j)\right).\end{aligned}$$

Taking into account that $\sup_\Sigma \mathrm{Tr}(\widehat{P}_1) < +\infty$ and taking $j \to +\infty$, up to passing to a subsequence we get

$$0 \geq \mathcal{H}(h^*) - \sup_\Sigma \sqrt{H_2}$$

and this concludes the proof. □

As a consequence of the previous theorem and Corollary 1.18 we also have

Corollary 2.8. *Let $\mathbb{P}^n$ be a complete, non-compact, Riemannian manifold whose radial sectional curvature satisfies condition* (1.21)*. If $f : \Sigma^n \to I \times_\rho \mathbb{P}^n$ is a properly immersed hypersurface with $\sup_\Sigma |H_1| < +\infty$ which is contained in a slab, then*

$$\sup_\Sigma H_2^{1/2} \geq \inf_\Sigma \mathcal{H}(h).$$

In other words, there is no properly immersed hypersurface with $H_2 > 0$ and $\sup_\Sigma |H_1| < +\infty$ contained in a slab $[t_1, t_2] \times \mathbb{P}^n$ with

$$\sup_\Sigma H_2^{1/2} < \inf_{[t_1,t_2]} \mathcal{H}(t).$$

The previous results can be extended to the case $3 \leq k \leq n$ once we guarantee the ellipticity of the operator $\widehat{L}_{k-1}$. Using Proposition 1.4 one obtains the next

Theorem 2.9 (Theorem 12 in [**13**])**.** *Let $f : \Sigma^n \to I \times_\rho \mathbb{P}^n$ be an immersed hypersurface having an elliptic point and satisfying $H_k > 0$. If the Omori-Yau maximum principle holds on Σ for $\widehat{L}_{k-1}$, with $3 \leq k \leq n$, and $h^* < +\infty$ then*

$$\sup_\Sigma H_k^{1/k} \geq \inf_\Sigma \mathcal{H}(h).$$

The proof proceeds as in the previous theorem taking into account that by Garding inequalities

$$H_{k-1} \geq H_k^{(k-1)/k}.$$

Observe that Proposition 1.4 also implies that each H_j is positive for any $1 \leq j \leq k-1$. In particular H_2 is positive and hence, keeping in mind Remark 1.19, the condition $\sup_\Sigma |H_1| < +\infty$, together with the assumption of Σ being properly immersed, suffice to guarantee the validity of the Omori-Yau maximum principle for $\widehat{L}_{k-1}$. Hence the next corollary is straightforward.

Corollary 2.10. *Let $\mathbb{P}^n$ be a complete, non-compact, Riemannian manifold whose radial sectional curvature satisfies condition* (1.21)*. Assume that $f : \Sigma^n \to I \times_\rho \mathbb{P}^n$ is a properly immersed hypersurface having an elliptic point satisfying $H_k > 0$ and $\sup_\Sigma |H_1| < +\infty$. If $f(\Sigma)$ is contained in a slab, then*

$$\sup_\Sigma H_k^{1/k} \geq \inf_\Sigma \mathcal{H}(h)$$

for every $3 \leq k \leq n$. In other words, there is no properly immersed hypersurface having an elliptic point and contained in a slab $[t_1, t_2] \times \mathbb{P}^n$ with

$$\sup_\Sigma H_k^{1/k} < \inf_{[t_1,t_2]} \mathcal{H}(t).$$

2.2. Uniqueness of hypersurfaces: compact case

The computational results found in Proposition 2.4 can be used also to prove uniqueness results for hypersurfaces with either constant mean curvature H_1 or constant k-mean curvature H_k, $2 \leq k \leq n$ immersed in warped product spaces. The idea is that, since the geometry of these spaces is completely determined by the warping function and by the geometry of the fiber, appropriate conditions on them force the hypersurface to be a slice. In this section we will concentrate on uniqueness results obtained by imposing conditions on the warping function. Similar results with conditions on the geometry of a standard slice are obtained in Section 2.4.

Let us consider first the case of constant mean curvature hypersurfaces. In [**10**] it has been obtained the following result.

Theorem 2.11 (Theorem 2.4 in [**10**])**.** *Let $f : \Sigma^n \to I \times_\rho \mathbb{P}^n$ be a compact two-sided hypersurface of constant mean curvature H_1. Assume that $\mathcal{H}' \geq 0$ and that the angle function Θ does not change sign. Then $\mathbb{P}^n$ is compact and $f(\Sigma^n)$ is a slice.*

Observe that, since $\mathcal{H}(t) = \frac{\rho'}{\rho}(t) = (\log \rho)'(t)$, the assumption on $\mathcal{H}(h)$ being non-decreasing is equivalent to $(\log \rho)''(h)$ being convex. Concerning the assumption that the angle function does not change sign, notice that, if the immersion f is locally a graph, then either $\Theta < 0$ or $\Theta > 0$ along Σ. Hence, by requiring that Θ does not change on Σ we are relaxing the hypothesis of f being a local graph.

The proof of the previous theorem is essentially based on the use of the classical maximum principle applied to some basic partial differential equations (more precisely equations (2.1) and (2.2) in case $k = 0$). Our aim is to extend this result to compact hypersurfaces of constant higher order mean curvature. The idea is to find the right partial differential equation and to use again the classical maximum principle that holds for every semi-elliptic operator on a compact Riemannian manifold. Looking at Proposition 2.4 it seems natural, once we assume that the k-mean curvature H_k is constant, $2 \leq k \leq n$, to try to perform the same computations as in Theorem 2.11 simply replacing the Laplacian operator by the operator L_{k-1}. Unfortunately, Equations (2.1) and (2.2) involve both the $k-1$ and k mean curvatures and, since in general we don't know much on the first one, it is not possible to

reach the desired conclusion. The idea is then to find a suitable family of semi-elliptic operators, which will turn out to be a combination of the L_k's, that make things work.
Let us start with the simpler case $k = 2$. In this case we will assume H_2 a positive constant, so that the operator L_1 is elliptic if we choose the normal unit vector N on Σ such that $H_1 > 0$.

Let $\sigma(t) = \int_{t_0}^{t} \rho(u)du$. By Proposition 2.4 we know that

$$\Delta\sigma(h) = n\rho(h)(\mathcal{H}(h) + \Theta H_1),$$

$$L_1\sigma(h) = n(n-1)\rho(h)(\mathcal{H}(h)H_1 + \Theta H_2). \tag{2.5}$$

Therefore, if we define

$$\mathcal{L}_1 = (n-1)\mathcal{H}(h)\Delta - \Theta L_1 = \mathrm{Tr}(\mathcal{P}_1 \circ \mathrm{hess}),$$

with

$$\mathcal{P}_1 = (n-1)\mathcal{H}(h)I - \Theta P_1.$$

a straightforward computation shows that

$$\mathcal{L}_1\sigma(h) = n(n-1)\rho(h)(\mathcal{H}(h)^2 - \Theta^2 H_2), \tag{2.6}$$

Hence, if we assume that the angle function Θ does not change sign, since we need the operator $\mathcal{L}_1$ (respectively the operator $-\mathcal{L}_1$) to be semi-elliptc, we would like to have

$$\mathcal{H}(h) \geq 0 \text{ when } \Theta \leq 0 \qquad (\text{resp. } \mathcal{H}(h) \leq 0 \text{ when } \Theta \geq 0).$$

This is guaranteed by the following

Lemma 2.12. *Let $f : \Sigma^n \to I \times_\rho \mathbb{P}^n$ be a compact hypersurface with non-vanishing mean curvature. Assume that the angle function Θ does not change sign and that $\mathcal{H}' \geq 0$. Then*

$$(i) \text{ if } \Theta \leq 0, \qquad \text{then } \mathcal{H}(h) \geq 0,$$

$$(ii) \text{ if } \Theta \geq 0, \qquad \text{then } \mathcal{H}(h) \leq 0.$$

PROOF. Choose the orientation so that $H_1 > 0$. Since Σ is compact, there exist points $p_{\min}, p_{\max}$ where the height function h attains its minimum and maximum values respectively. Moreover, set

$$\underline{h} := \min_\Sigma h = h(p_{\min}),$$

$$\overline{h} := \max_\Sigma h = h(p_{\max}).$$

Then

$$\nabla h(p_{\min}) = \nabla h(p_{\max}) = 0$$

and

$$\Theta(p_{\min}) = \Theta(p_{\max}) = \pm 1.$$

Moreover

$$0 \leq \Delta h(p_{\min}) = n\mathcal{H}(\underline{h}) + n\Theta(p_{\min})H_1(p_{\min})$$

$$0 \geq \Delta h(p_{\max}) = n\mathcal{H}(\overline{h}) + n\Theta(p_{\max})H_1(p_{\max})$$

Assume $\Theta \leq 0$. Then $\Theta(p_{\min}) = -1$ and

$$0 \leq n\mathcal{H}(\underline{h}) - nH_1(p_{\min}) \leq \mathcal{H}(\underline{h}).$$

Hence, since $\mathcal{H}' \geq 0$

$$\mathcal{H}(h) \geq \mathcal{H}(\underline{h}) \geq 0.$$

On the other hand, if $\Theta \geq 0$, $\Theta(p_{\max}) = 1$ and

$$0 \geq n\mathcal{H}(\overline{h}) + nH_1(p_{\max}) \geq \mathcal{H}(\overline{h})$$

and we conclude as above. □

We can now state the first main result of this section, which extends Theorem 2.11 to the case of constant 2-mean curvature H_2.

Theorem 2.13 (Theorem 15 in [**13**])**.** *Let $f : \Sigma^n \to I \times_\rho \mathbb{P}^n$ be a compact hypersurface of constant positive 2-mean curvature H_2. If $\mathcal{H}'(h) \geq 0$ and the angle function Θ does not change sign, then $\mathbb{P}^n$ is necessarily compact and $f(\Sigma^n)$ is a slice.*

Proof. As above, we choose the orientation of Σ so that $H_1 > 0$. We may apply Lemma 2.12 and consider first the case $\Theta \leq 0$, for which $\mathcal{H}(h) \geq 0$. In this case the operator $\mathcal{P}_1$ is positive semi-definite or, equivalently, $\mathcal{L}_1$ is semi-elliptic.

Since Σ is compact, there exist points $p_{\max}, p_{\min} \in \Sigma$ such that

$$h(p_{\max}) = \overline{h} = \max_\Sigma h \quad \text{and} \quad h(p_{\min}) = \underline{h} = \min_\Sigma h.$$

Therefore, $\|\nabla h(p_{\max})\| = \|\nabla h(p_{\min})\| = 0$, which yields

$$\Theta(p_{\max}) = \Theta(p_{\min}) = -1.$$

Observe that

$$\max_\Sigma(\sigma \circ h) = \sigma(\overline{h}) = \sigma(h(p_{\max}))$$

and

$$\min_\Sigma(\sigma \circ h) = \sigma(\underline{h}) = \sigma(h(p_{\min})),$$

because $\sigma(t)$ is strictly increasing. In particular,

$$\operatorname{Hess}\sigma(h)(p_{\max}) \leq 0 \quad \text{and} \quad \operatorname{Hess}\sigma(h)(p_{\min}) \geq 0.$$

Taking into account that $\mathcal{P}_1$ is positive semi-definite, this yields

$$\mathcal{L}_1\sigma(h)(p_{\max}) = n(n-1)\rho(\overline{h})(\mathcal{H}(\overline{h})^2 - H_2) \leq 0$$

and

$$\mathcal{L}_1\sigma(h)(p_{\min}) = n(n-1)\rho(\underline{h})(\mathcal{H}(\underline{h})^2 - H_2) \geq 0.$$

Then, since $\mathcal{H}(h) \geq 0$ on Σ, we obtain

$$\mathcal{H}(\underline{h}) \geq H_2^{1/2} \geq \mathcal{H}(\overline{h}).$$

On the other hand, by $\mathcal{H}' \geq 0$ we also have $\mathcal{H}(\underline{h}) \leq \mathcal{H}(\overline{h})$. Thus, we deduce the validity of the equality $\mathcal{H}(\underline{h}) = \mathcal{H}(\overline{h})$ and $\mathcal{H}(h) = H_2^{1/2}$ is constant on Σ. By (2.5), using the basic inequality $H_1 \geq H_2^{1/2}$ and the fact that $\Theta \geq -1$, we obtain that

$$\begin{aligned} L_1\sigma(h) &= n(n-1)\rho(h)H_2^{1/2}(H_1 + \Theta H_2^{1/2}) \\ &\geq n(n-1)\rho(h)H_2^{1/2}(H_1 - H_2^{1/2}) \\ &\geq 0. \end{aligned}$$

That is, $L_1\sigma(h) \geq 0$ on the compact manifold Σ. Thus, by the maximum principle applied to the elliptic operator L_1 we conclude that $\sigma(h)$, and hence h, is constant.

Finally, in the case where $\Theta \geq 0$ we know from Lemma 2.12 that $\mathcal{H}(h) \leq 0$ on Σ, so that the operator $-\mathcal{L}_1$ is semi-elliptic. The proof then follows as in the case $\Theta \leq 0$, working with $-\mathcal{L}_1$ instead. $\square$

In order to extend our previous results to the case of higher order mean curvatures, we introduce a family of operators that generalize $\mathcal{L}_1$. For $1 \leq k \leq n$, let us consider the operator

$$\mathcal{L}_{k-1} = \mathrm{Tr}\Big(\Big[\sum_{j=0}^{k-1}(-1)^j\frac{c_{k-1}}{c_j}\mathcal{H}(h)^{k-1-j}\Theta^j P_j\Big] \circ \mathrm{hess}\Big) = \mathrm{Tr}(\mathcal{P}_{k-1} \circ \mathrm{hess}),$$

where

$$\mathcal{P}_{k-1} = \sum_{j=0}^{k-1}(-1)^j\frac{c_{k-1}}{c_j}\mathcal{H}(h)^{k-1-j}\Theta^j P_j. \tag{2.7}$$

We claim that

$$\mathcal{L}_{k-1}\sigma(h) = c_{k-1}\rho(h)(\mathcal{H}(h)^k + (-1)^{k-1}\Theta^k H_k).$$

We can prove the claim by induction. We have already seen in (2.6) that the equation above holds when $k = 2$. For $k \geq 3$, observe that

$$\mathcal{P}_{k-1} = \frac{c_{k-1}}{c_{k-2}}\mathcal{H}(h)\mathcal{P}_{k-2} + (-1)^{k-1}\Theta^{k-1}P_{k-1}$$

and

$$\mathcal{L}_{k-1} = \frac{c_{k-1}}{c_{k-2}}\mathcal{H}(h)\mathcal{L}_{k-2} + (-1)^{k-1}\Theta^{k-1}L_{k-1}$$

Therefore, if $k \geq 3$ and we assume that the claim is true for $\mathcal{L}_{k-2}$, then using (2.2) we conclude that

$$\begin{aligned}\mathcal{L}_{k-1}\sigma(h) =& \frac{c_{k-1}}{c_{k-2}}\mathcal{H}(h)\mathcal{L}_{k-2}\sigma(h) + (-1)^{k-1}\Theta^{k-1}L_{k-1}\sigma(h)\\ =& c_{k-1}\rho(h)(\mathcal{H}(h)^k + (-1)^{k-2}\mathcal{H}(h)\Theta^{k-1}H_{k-1}\\ &+ (-1)^{k-1}\mathcal{H}(h)\Theta^{k-1}H_{k-1} + (-1)^{k-1}\Theta^k H_k)\\ =& c_{k-1}\rho(h)(\mathcal{H}(h)^k + (-1)^{k-1}\Theta^k H_k).\end{aligned}$$

Using the operators $\mathcal{L}_{k-1}$ and the the maximum principle on the compact manifold Σ we are able to give the following extension of Theorem 2.13.

Theorem 2.14 (Theorem 20 in [**13**]). *Let $f : \Sigma^n \to I \times_\rho \mathbb{P}^n$ be a compact hypersurface with constant k-mean curvature H_k, with $3 \leq k \leq n$. Assume that there exists an elliptic point in Σ. If $\mathcal{H}'(t) \geq 0$ and the angle function Θ does not change sign, then $\mathbb{P}^n$ is necessarily compact and $f(\Sigma^n)$ is a slice.*

Proof. Choose the orientation of Σ so that $H_1 > 0$. Let us apply Lemma 2.12 considering first the case $\Theta \leq 0$. In this case $\mathcal{H}(h) \geq 0$ and, since by Lemma 1.4 all the P_j are positive definite, $1 \leq j \leq k-1$, the operator $\mathcal{P}_{k-1}$ is positive semi-definite or, equivalently, $\mathcal{L}_{k-1}$ is semi-elliptic. Reasoning now as in the proof of Theorem 2.13, we obtain

$$\mathcal{L}_{k-1}\sigma(h)(p_{\max}) = c_{k-1}\rho(\overline{h})(\mathcal{H}(\overline{h})^k - H_k) \leq 0$$

and

$$\mathcal{L}_{k-1}\sigma(h)(p_{\min}) = c_{k-1}\rho(\underline{h})(\mathcal{H}(\underline{h})^k - H_k) \geq 0.$$

Then, since $\mathcal{H}(h) \geq 0$ on Σ,

$$\mathcal{H}(\underline{h}) \geq H_k^{1/k} \geq \mathcal{H}(\overline{h}).$$

Again, the assumption on $\mathcal{H}'(h)$ implies that $\mathcal{H}(h) = H_k^{1/k}$ is constant on Σ. Therefore, by the Garding inequality $H_{k-1} \geq H_k^{(k-1)/k}$ and the fact that $\Theta \geq -1$, we obtain that

$$\begin{aligned} L_{k-1}\sigma(h) &= c_{k-1}\rho(h)H_k^{1/k}(H_{k-1} + \Theta H_k^{(k-1)/k}) \\ &\geq c_{k-1}\rho(h)H_k^{1/k}(H_{k-1} - H_k^{(k-1)/k}) \geq 0. \end{aligned}$$

That is, $L_{k-1}\sigma(h) \geq 0$ on the compact manifold Σ. Therefore, by the maximum principle applied to the elliptic operator L_{k-1} we conclude that $\sigma(h)$, and hence h, must be constant.

Finally, when $\Theta \geq 0$ we can apply again Lemma 2.12 in order to conclude that $\mathcal{H}(h) \leq 0$ on Σ. Consider then the operator

$$\mathcal{L}_{k-1} = \mathrm{Tr}\Big(\Big[\sum_{j=0}^{k-1}(-1)^{k-1-j}\frac{c_{k-1}}{c_j}\mathcal{H}(h)^{k-1-j}\Theta^j P_j\Big] \circ \mathrm{hess}\Big) = \mathrm{Tr}(\mathcal{P}_{k-1} \circ \mathrm{hess}),$$

where

$$\mathcal{P}_{k-1} = \sum_{j=0}^{k-1}(-1)^{k-1-j}\frac{c_{k-1}}{c_j}\mathcal{H}(h)^{k-1-j}\Theta^j P_j. \tag{2.8}$$

which is semi-elliptic. Furthermore, it is not difficult to prove using induction on k that

$$\mathcal{L}_{k-1}\sigma(h) = c_{k-1}\rho(h)((-1)^{k-1}\mathcal{H}(h))^k + \Theta^k H_k).$$

The proof then follows as in the case $\Theta \leq 0$. □

2.3. Uniqueness of hypersurfaces: complete non-compact case

We consider now the case when the immersed hypersurface is complete non-compact and we try to extend in this context the results proved in the previous section. Again, for what concern the mean curvature case, the desired uniqueness result has already been proved in [**10**] and it reads as follows

Theorem 2.15 (Theorem 2.9 in [**10**]). *Let $f : \Sigma^n \to I \times_\rho \mathbb{P}^n$ be a complete two-sided hypersurface of constant mean curvature H_1, with Ricci curvature bounded from below and*

$$f(\Sigma^n) \subset [t_1, t_2] \times \mathbb{P}^n,$$

where t_1, $t_2 \in \mathbb{R}$ are finite. Assume that $\mathcal{H}' \geq 0$ and that the angle function Θ does not change sign. Then $f(\Sigma^n)$ is a slice.

For the proof, one proceeds just as in the compact case, using the Omori-Yau maximum principle for the Laplacian rather than the classical maximum principle. This result can be extended to the case of complete constant k-mean curvature hypersurfaces, $2 \leq k \leq n$, by means of the family of operators $\mathcal{L}_{k-1}$. Since, as we will see, these operators belong to the family (1.11) introduced in Section 1.3, the proof of the uniqueness results is straightforward once one guarantees the conditions for the validity of the Omori-Yau maximum principle for semi-elliptic operators of the form (1.11). In this regard, let us introduce the following

Lemma 2.16. *Let $f : \Sigma^n \to I \times_\rho \mathbb{P}^n$ be a hypersurface with non-vanishing mean curvature which is contained in a slab. Assume that $\mathcal{H}' > 0$ and that the angle function Θ does not change sign. If the Omori-Yau maximum principle holds on Σ, then*

$$\begin{aligned} &(i)\ \textit{if } \Theta \leq 0, && \textit{then } \mathcal{H}(h) \geq 0,\\ &(ii)\ \textit{if } \Theta \geq 0, && \textit{then } \mathcal{H}(h) \leq 0. \end{aligned}$$

Proof. Choose on Σ the orientation so that $H_1 > 0$. Since h is bounded from below and the Omori-Yau maximum principle holds on Σ, we can find a sequence $\{q_j\}_{j\in\mathbb{N}} \subset \Sigma^n$ such that

$$\begin{aligned} &\lim_{j\to+\infty} h(q_j) = h_*,\\ &\|\nabla h(q_j)\|^2 = 1 - \Theta^2(q_j) < \left(\frac{1}{j}\right)^2,\\ &\qquad \Delta h(q_j) = \mathcal{H}(h(q_j))(n - \|\nabla h(q_j)\|^2) + nH_1(q_j)\Theta(q_j) > -\frac{1}{j}. \end{aligned}$$

Then

$$-nH_1(q_j)\Theta(q_j) < \frac{1}{j} + \mathcal{H}(h(q_j))(n - \|\nabla h(q_j)\|^2). \tag{2.9}$$

Similarly, since h is bounded from above, we can also find a sequence $\{p_j\} \subset \Sigma^n$ such that

$$\begin{aligned} &\lim_{j\to+\infty} h(p_j) = h^*,\\ &\|\nabla h(p_j)\|^2 = 1 - \Theta^2(p_j) < \left(\frac{1}{j}\right)^2,\\ &\qquad \Delta h(p_j) = \mathcal{H}(h(p_j))(n - \|\nabla h(p_j)\|^2) + nH_1(p_j)\Theta(p_j) < \frac{1}{j}. \end{aligned}$$

Then

$$-nH_1(p_j)\Theta(p_j) > -\frac{1}{j} + \mathcal{H}(h(p_j))(n - \|\nabla h(p_j)\|^2). \tag{2.10}$$

Assume first that $\Theta \leq 0$. Since $\lim_{j\to+\infty} -\Theta(q_j) = -\operatorname{sgn}(\Theta) = 1 > 0$, then $-\Theta(q_j) > 0$ for j sufficiently large. Since $H_1(q_j) > 0$, it then follows from (2.9) that

$$0 \leq \liminf_{j\to+\infty} H_1(q_j) \leq \liminf_{j\to+\infty} \Big(-H_1(q_j)\Theta(q_j)\Big) \leq \mathcal{H}(h_*).$$

Therefore $\mathcal{H}(h_*) \geq 0$ and, by $\mathcal{H}' \geq 0$, we conclude that

$$\mathcal{H}(h) \geq \mathcal{H}(h_*) \geq 0.$$

On the other hand, if $\Theta \geq 0$ then $\lim_{j\to+\infty} \Theta(p_j) = \operatorname{sgn}(\Theta) = 1 > 0$, so that $\Theta(p_j) > 0$ for j sufficiently large. Therefore, since $H_1(p_j) > 0$, from (2.10) we have that

$$0 \leq \liminf_{j\to+\infty} H_1(p_j) \leq \liminf_{j\to+\infty} \Big(H_1(p_j)\Theta(p_j)\Big) \leq -\mathcal{H}(h^*).$$

Therefore $\mathcal{H}(h^*) \leq 0$ and

$$\mathcal{H}(h) \leq \mathcal{H}(h^*) \leq 0.$$

This concludes the proof. □

Using the previous lemma and under some additional assumption guaranteeing the validity the Omori-Yau maximum principle for semi-elliptic trace-type differential operators (see Corollary 1.16), we are able to extend Theorem 2.13 to the complete case. The uniqueness result we obtain reads as follows.

Theorem 2.17 (Theorem 16 in [**13**])**.** *Let $f : \Sigma^n \to I \times_\rho \mathbb{P}^n$ be a complete hypersurface of constant positive 2-mean curvature H_2 such that*

$$K_\Sigma^{\mathrm{rad}} \geq -G(r). \tag{2.11}$$

Here G is a smooth function on $[0, +\infty)$ which is even at the origin and satisfying conditions (i)–(iv) listed in Theorem 1.12. Assume that $\sup_\Sigma |H_1| < +\infty$ and that Σ is contained in a slab. If $\mathcal{H}'(h) > 0$ almost everywhere and the angle function Θ does not change sign, then $f(\Sigma^n)$ is a slice.

Proof. Choose the orientation of Σ so that $H_1 > 0$. By Corollary 1.16 we know that the Omori-Yau maximum principle holds on Σ for the Laplacian operator, so that we may apply Lemma 2.16.

In the case where $\Theta \leq 0$, we have $\mathcal{H}(h) \geq 0$, and therefore the operator $\mathcal{P}_1$ is positive semi-definite. Furthermore

$$\mathrm{Tr}\mathcal{P}_1 = n(n-1)\mathcal{H}(h) - n(n-1)H_1\Theta \leq n(n-1)(\mathcal{H}(h^*) + H_1^*),$$

where $h^* = \sup_\Sigma h < +\infty$ and $H_1^* = \sup_\Sigma H_1 < +\infty$. Hence, since Corollary 1.16 implies the validity of the Omori-Yau for any semi-elliptic operator as in (1.11), we conclude that we can apply it for the operator $\mathcal{L}_1$.

Since $\sup_\Sigma \sigma(h) = \sigma(h^*) < +\infty$, there exists a sequence $\{p_j\}_{j\in\mathbb{N}} \subset \Sigma$ such that

$$\begin{aligned}
&(i) \quad \lim_{i\to+\infty} \sigma(h(p_i)) = \sup_\Sigma \sigma(h) = \sigma(h^*),\\
&(ii) \quad \|\nabla(\sigma\circ h)(p_i)\| = \rho(h(p_i))\|\nabla h(p_i)\| < \frac{1}{i},\\
&(iii) \quad \mathcal{L}_1(\sigma\circ h)(p_i) < \frac{1}{i}.
\end{aligned}$$

Observe that condition (i) implies that $\lim_{i\to+\infty} h(p_i) = h^*$, because $\sigma(t)$ is strictly increasing. By condition (ii) we also have $\lim_{i\to+\infty} \|\nabla h(p_i)\| = 0$. Moreover

$$\mathcal{L}_1\sigma(h)(p_i) = n(n-1)\rho(h(p_i))(\mathcal{H}(h(p_i))^2 - \Theta^2(p_i)H_2) < \frac{1}{i},$$

and taking the limit for $i \to +\infty$ and observing that $\Theta^2(p_i) = 1 - \|\nabla h(p_i)\|^2 \to 1$ as $i \to +\infty$, we find

$$\mathcal{H}(h^*)^2 - H_2 \le 0.$$

On the other hand, since $h_* = \inf_\Sigma h > -\infty$, then $\inf_\Sigma \sigma(h) = \sigma(h_*) > -\infty$. Thus, we can find a sequence $\{q_i\}_{i\in\mathbb{N}} \subset \Sigma$ such that

$$\begin{aligned}
&(i) \quad \lim_{i\to+\infty} \sigma(h(q_i)) = \inf_\Sigma \sigma(h) = \sigma(h_*),\\
&(ii) \quad \|\nabla(\sigma\circ h)(q_i)\| = \rho(h(q_i))\|\nabla h(q_i)\| < \frac{1}{i},\\
&(iii) \quad \mathcal{L}_1(\sigma\circ h)(q_i) > -\frac{1}{i}.
\end{aligned}$$

Proceeding as above and using

$$\mathcal{L}_1\sigma(h)(q_i) = n(n-1)\rho(h(q_i))(\mathcal{H}(h(q_i))^2 - \Theta^2(q_i)H_2) > -\frac{1}{i},$$

we obtain

$$\mathcal{H}(h_*)^2 - H_2 \ge 0.$$

Thus $\mathcal{H}(h_*)^2 \ge \mathcal{H}(h^*)^2$ and, taking into account that $\mathcal{H}(h_*), \mathcal{H}(h^*) \ge 0$, this gives $\mathcal{H}(h_*) \ge \mathcal{H}(h^*)$. Therefore, since $\mathcal{H}(h)$ is an increasing function we conclude that $h^* = h_*$.

Finally, let us consider the case where $\Theta \ge 0$. By Lemma 2.16 $\mathcal{H}(h) \le 0$ and then the operator $-\mathcal{L}_1$ is semi-elliptic. Moreover

$$\mathrm{Tr}(-\mathcal{P}_1) = -n(n-1)\mathcal{H}(h) + n(n-1)H_1\Theta \le n(n-1)(-\mathcal{H}(h_*) + H_1^*).$$

Hence the trace of $-\mathcal{P}_1$ is bounded from above and by Corollary 1.16 the Omori-Yau maximum principle holds for the operator $-\mathcal{L}_1$. Proceeding as above we get the two inequalities

$$H_2 - \mathcal{H}(h_*)^2 \ge 0 \quad \text{and} \quad H_2 - \mathcal{H}(h^*)^2 \le 0.$$

Thus $\mathcal{H}(h_*)^2 \le \mathcal{H}(h^*)^2$. Since $\mathcal{H}(h_*), \mathcal{H}(h^*) \le 0$, this implies $\mathcal{H}(h_*) \ge \mathcal{H}(h^*)$. But $\mathcal{H}(t)$ being increasing, this gives $h_* = h^*$ and this concludes the proof. □

Analogously to the compact case, Theorem 2.17 can be generalized to higher order mean curvatures as follows.

Theorem 2.18 (Theorem 23 in [**13**]). *Let $f : \Sigma^n \to \mathbb{R}\times_\rho\mathbb{P}^n$, $n \ge 3$, be a two-sided complete hypersurface of constant k-mean curvature H_k, $3 \le k \le n$, satisfying* (2.11) *and such that $\sup_\Sigma H_1 < +\infty$. Assume that there exists an elliptic point $p \in \Sigma^n$, that $\mathcal{H}' > 0$ almost everywhere and that the angle function Θ does not change sign. Then $f(\Sigma^n)$ is a slice.*

PROOF. By Corollary 1.16 the Omori-Yau maximum principle holds on Σ for any semi-elliptic operator as in (1.11) and hence, in particular, it holds for the Laplacian. We may then apply Lemma 2.16. Thus, in the case $\Theta \leq 0$ we have $\mathcal{H}(h) \geq 0$ and therefore

$$(-1)^j \mathcal{H}(h)^{k-1-j} \Theta^j \geq 0$$

for all $0 \leq j \leq k-1$. Since the operators P_j are positive definite, $\mathcal{P}_{k-1}$ is positive semi-definite or, in other words, the differential operator $\mathcal{L}_{k-1}$ is semi-elliptic. Furthermore, since $0 \leq -\Theta \leq 1$,

$$\operatorname{Tr}(\mathcal{P}_{k-1}) = c_{k-1} \sum_{j=0}^{k-1} (-1)^j \mathcal{H}(h)^{k-1-j} \Theta^j H_j \leq c_{k-1} \sum_{j=0}^{k-1} \mathcal{H}(h^*)^{k-1-j} H_j^*,$$

where $H_j^* = \sup_\Sigma H_j \leq (\sup_\Sigma H_1)^j < +\infty$ by (1.3). Hence the trace of $\mathcal{P}_{k-1}$ is bounded from above and we can apply the Omori-Yau maximum principle for the operator $\mathcal{L}_{k-1}$. Proceeding as in the proof of Theorem 2.17, we may find two sequences $\{p_i\}_{i\in\mathbb{N}} \subset \Sigma$ and $\{q_i\}_{i\in\mathbb{N}} \subset \Sigma$ satisfying

$$\lim_{i\to+\infty} h(p_i) = h^*, \quad \text{and} \quad \lim_{i\to+\infty} h(q_i) = h_*,$$

$$\lim_{i\to+\infty} \Theta(p_i) = \lim_{i\to+\infty} \Theta(q_i) = -1,$$

$$\mathcal{L}_{k-1}\sigma(h)(p_i) = c_{k-1}\rho(h(p_i))(\mathcal{H}(h(p_i))^k + (-1)^{k-1}\Theta^k(p_i)H_k) < \frac{1}{i},$$

and

$$\mathcal{L}_{k-1}\sigma(h)(q_i) = c_{k-1}\rho(h(q_i))(\mathcal{H}(h(q_i))^k + (-1)^{k-1}\Theta^k(q_i)H_k) < \frac{1}{i}.$$

Letting $i \to +\infty$ in the inequalities above, we obtain that

$$\mathcal{H}(h^*)^k \leq H_k \leq \mathcal{H}(h_*)^k,$$

which implies that $h_* = h^*$, as in the proof of Theorem 2.17.

Finally, in the case where $\Theta \geq 0$ we proceed as above with

$$\mathcal{L}_{k-1} = \operatorname{Tr}\Big(\Big[\sum_{j=0}^{k-1} (-1)^{k-1-j} \frac{c_{k-1}}{c_j} \mathcal{H}(h)^{k-1-j} \Theta^j P_j\Big] \circ \operatorname{hess}\Big) = \operatorname{Tr}(\mathcal{P}_{k-1} \circ \operatorname{hess}),$$

which is a semi-elliptic trace-type operator with $\operatorname{Tr}(\mathcal{P}_{k-1})$ bounded from above. □

Observe that Theorems 2.17 and 2.18 remain true if we replace condition (2.11) by the stronger condition of Σ^n having (radial) sectional curvature bounded from below by a constant. This happens, for instance, when the sectional curvature of $\mathbb{P}^n$ is bounded from below, as proved in the following

Lemma 2.19. *Let $\mathbb{P}^n$ be a Riemannian manifold with sectional curvature bounded from below and let $f : \Sigma^n \to I \times_\rho \mathbb{P}^n$ be an immersed hypersurface. Assume that $\sup_\Sigma \|A\|^2 < +\infty$ and that Σ is contained in a slab. Then the sectional curvature of Σ is bounded from below.*

Proof of Lemma 2.19. Recall that the Gauss equation for a hypersurface $f:\Sigma^n\to M^{n+1}$ is given by

$$\langle \mathrm{R}(X,Y)Z,V\rangle = \left\langle \overline{\mathrm{R}}(X,Y)Z,V\right\rangle - \langle AY,V\rangle\langle AX,Z\rangle + \langle AX,V\rangle\langle AY,Z\rangle,$$

for $X,Y,Z,V\in T\Sigma$, where R and $\overline{\mathrm{R}}$ are the curvature tensors of Σ^n and M^{n+1}, respectively. Then, if $\{X,Y\}$ is an orthonormal basis for an arbitrary 2-plane tangent to Σ, we have

$$\begin{aligned} K_\Sigma(X,Y) =& \overline{K}(X,Y) + \langle AX,X\rangle\langle AY,Y\rangle - \langle AX,Y\rangle^2 \\ \geq& \overline{K}(X,Y) - \|AX\|\|AY\| - \|AX\|^2 \\ \geq& \overline{K}(X,Y) - 2\|A\|^2, \end{aligned} \tag{2.12}$$

where the last inequality follows from the fact that

$$\|AX\|^2 \leq \mathrm{Tr}(A^2)\|X\|^2 = \|A\|^2$$

for every unit vector X tangent to Σ. Since we are assuming that $\sup_\Sigma\|A\|^2 < +\infty$, it suffices to have $\overline{K}(X,Y)$ bounded from below in order to conclude.

The curvature tensor of M^{n+1} expressed in terms of the curvature tensor of $\mathbb{P}^n$ is

$$\begin{aligned} \overline{\mathrm{R}}(U,V)W =& \mathrm{R}_{\mathbb{P}}(U^*,V^*)W^* - \mathcal{H}^2(\pi_I)(\langle V,W\rangle U - \langle U,W\rangle V) \\ &+ \mathcal{H}'(\pi_I)\langle W,T\rangle(\langle U,T\rangle V - \langle V,T\rangle U) \\ &- \mathcal{H}'(\pi_I)(\langle V,W\rangle\langle U,T\rangle - \langle U,W\rangle\langle V,T\rangle)T, \end{aligned} \tag{2.13}$$

for every $U,V,W\in TM$, where $T=\partial_t$ and we are using the notation U^* to denote $\pi_{\mathbb{P}*}U$ for an arbitrary $U\in TM$. Then, since $\{X,Y\}$ is orthonormal, we find that

$$\begin{aligned} \overline{K}(X,Y) &= \frac{1}{\rho^2(h)}K_{\mathbb{P}}(X^*,Y^*)\|X^*\wedge Y^*\|^2 \\ &\quad -\mathcal{H}^2(h) - \mathcal{H}'(h)(\langle X,\nabla h\rangle^2 + \langle Y,\nabla h\rangle^2) \\ &\geq \frac{1}{\rho^2(h)}K_{\mathbb{P}}(X^*,Y^*)\|X^*\wedge Y^*\|^2 - \mathcal{H}^2(h) - 2|\mathcal{H}'(h)|, \end{aligned} \tag{2.14}$$

since

$$\langle X,\nabla h\rangle^2 + \langle Y,\nabla h\rangle^2 \leq 2\|\nabla h\|^2 \leq 2.$$

On the other hand, observe that

$$\begin{aligned} \|X^*\wedge Y^*\|^2 &= \|X^*\|^2\|Y^*\|^2 - \langle X^*,Y^*\rangle^2 \\ &= 1 - \langle X,T\rangle^2 - \langle Y,T\rangle^2 \leq 1. \end{aligned}$$

Therefore, if $K_{\mathbb{P}}\geq c$ for some constant c, we deduce

$$\frac{1}{\rho^2(h)}K_{\mathbb{P}}(X^*,Y^*)\|X^*\wedge Y^*\|^2 \geq -\frac{|c|}{\rho^2(h)}. \tag{2.15}$$

Finally, since the hypersurface is contained in a slab, h is a bounded function and we conclude from (2.12), (2.14) and (2.15) that the sectional curvature $K_\Sigma(X,Y)$ is bounded from below by a constant. □

Taking into account the equality

$$\|A\|^2 = \mathrm{Tr}(A^2) = n^2 H_1^2 - n(n-1)H_2$$

it follows by Proposition 1.4 that, if H_k is positive and, in case $k \geq 3$, we assume also that there exists an elliptic point on Σ, then each H_j is positive, $1 \leq j \leq k-1$ and

$$\sup_{\Sigma} \|A\|^2 \leq n^2 (\sup_{\Sigma} H_1)^2 - n(n-1)H_2 < n^2 (\sup_{\Sigma} H_1)^2.$$

Thus, if $\sup_{\Sigma} |H_1| < +\infty$, the previous lemma implies that the sectional curvature of Σ is bounded from below by a constant.

Then the following corollaries are straightforward.

Corollary 2.20. *Let $\mathbb{P}^n$ be a complete Riemannian manifold with sectional curvature bounded from below and let $f : \Sigma^n \to I \times_\rho \mathbb{P}^n$ be a complete hypersurface of constant positive 2-mean curvature H_2. Assume that $\sup_{\Sigma} |H_1| < +\infty$ and that Σ is contained in a slab. If $\mathcal{H}'(t) > 0$ almost everywhere and the angle function Θ does not change sign, then $f(\Sigma^n)$ is a slice.*

Corollary 2.21. *Let $\mathbb{P}^n$ be a complete Riemannian manifold with sectional curvature bounded from below and let $f : \Sigma^n \to I \times_\rho \mathbb{P}^n$ be a complete hypersurface with constant k-mean curvature H_k, $3 \leq k \leq n$. Assume that there exists an elliptic point in Σ, $\sup_{\Sigma} |H_1| < +\infty$ and Σ is contained in a slab. If $\mathcal{H}'(t) > 0$ almost everywhere and the angle function Θ does not change sign, then $f(\Sigma^n)$ is a slice.*

Finally, we observe that condition (2.11) has been used in the proof of Theorems 2.17 and 2.18 only to guarantee that the Omori-Yau maximum principle holds on Σ for the Laplacian and for the semi-elliptic operator $\mathcal{L}_k$ (or $-\mathcal{L}_k$), $1 \leq k \leq n-1$. Therefore, the theorems remain true under any other hypothesis guaranteeing this latter fact. Thus, and as a consequence of Corollary 1.18, we can also state the following:

Theorem 2.22. *Let $\mathbb{P}^n$ be a complete, non-compact, Riemannian manifold whose radial sectional curvature satisfies condition (1.21). Let $f : \Sigma^n \to I \times_\rho \mathbb{P}^n$ be a properly immersed hypersurface of constant positive 2-mean curvature H_2. Assume that $\sup_{\Sigma} |H_1| < +\infty$ and that Σ is contained in a slab. If $\mathcal{H}'(t) > 0$ almost everywhere and the angle function Θ does not change sign, then $f(\Sigma^n)$ is a slice.*

Theorem 2.23. *Let $\mathbb{P}^n$ be a complete, non-compact, Riemannian manifold whose radial sectional curvature satisfies condition (1.21). Let $f : \Sigma^n \to I \times_\rho \mathbb{P}^n$ be a properly immersed hypersurface of constant k-mean curvature, $3 \leq k \leq n$. Assume that there exists an elliptic point in Σ, $\sup_{\Sigma} |H_1| < +\infty$ and that Σ is contained in a slab. If $\mathcal{H}'(t) > 0$ almost everywhere and the angle function Θ does not change sign, then $f(\Sigma^n)$ is a slice.*

2.4. Further results for hypersurfaces of constant higher order mean curvatures

We are now going to prove further uniqueness results for constant k-mean curvature hypersurfaces both in the compact and in the complete non-compact case under assumptions on the geometry of the fiber $\mathbb{P}^n$, whichtake the form of a control on its Ricci or sectional curvature. As we will see, this conditions allow to prove the desired results exploiting the subharmonicity of a certain function involving the height function h. The technique we will use to conclude is no longer the classical or the Omori-Yau maximum principle, but the parabolicity (in some sense that we will clarify later on) of the hypersurface, which is authomatic in case the last one is compact but need to be guaranteed by some extra assumption in the complete non-compact case (see Remark 2.29 for more details).

Again, the case Σ compact of constant mean curvature as already been treated (see [**10**] or [**49**]) and the following uniqueness result holds.

Theorem 2.24 (Theorem 2.4 in[**10**], Corollary 7 in [**49**])**.** *Let $f : \Sigma^n \to I \times_\rho \mathbb{P}^n$ be a compact two sided hypersurface of constant mean curvature. Assume that*

$$\mathrm{Ric}_{\mathbb{P}} \geq \sup_I \{\rho'^2 - \rho''\rho\}, \tag{2.16}$$

$\mathrm{Ric}_{\mathbb{P}}$ *being the Ricci curvature of $\mathbb{P}^n$, and that the angle function Θ does not change sign. Then either $f(\Sigma^n)$ is a slice over a compact $\mathbb{P}^n$ or M^{n+1} has constant sectional curvature and Σ^n is a geodesic hypersphere. The latter case cannot occur if the inequality* (2.16) *is strict.*

There are several way to prove the latter theorem. Among the others, we will illustrate the proof given in [**10**] since, as we will see, it is easier to extend it to the complete non-compact constant mean curvature case and also to the constant higher order mean curvature case, both in the compact and complete non-compact settings. Namely, in [**10**], the authors introduce the function

$$\phi = \sigma(h)H + \rho(h)\Theta$$

which, assuming the validity of (2.16), ends up to be a subharmonic function. Then, using the classical maximum principle, one concludes that the function ϕ has to be constant and hence it is trivially a harmonic function. An analysis of the terms that appear in the equation $\Delta\phi = 0$ leads then to the desired conclusion.

Following this approach it is not difficult to extend Theorem 2.24 to the case of hypersurfaces with constant higher order mean curvature. The result we obtain is the following.

Theorem 2.25 (Theorem 24 in [**13**])**.** *Let $f : \Sigma^n \to I \times_\rho \mathbb{P}^n$ be a compact hypersurface of constant k-mean curvature, $2 \leq k \leq n$ and suppose that $\mathcal{H}$ does not vanish. Assume that*

$$K_{\mathbb{P}} \geq \sup_I \{\rho'^2 - \rho''\rho\}, \tag{2.17}$$

$K_{\mathbb{P}}$ being the sectional curvature of $\mathbb{P}^n$, and that the angle function Θ does not change sign. Then either $f(\Sigma^n)$ is a slice over a compact $\mathbb{P}^n$ or $I \times_\rho \mathbb{P}^n$ has constant sectional curvature and Σ^n is a geodesic hypersphere. The latter case cannot occur if the inequality (2.17) *is strict.*

To the proof we need the next computational result.

Lemma 2.26. *Let Σ^n be a hypersurface immersed into a warped product space $I \times_\rho \mathbb{P}^n$, with angle function Θ and height function h. Let $\widehat{\Theta} = \rho(h)\Theta$. Then, for every $1 \leq k \leq n$ we have*

$$\begin{aligned} L_{k-1}\widehat{\Theta} = & - \binom{n}{k}\rho(h)\left\langle \nabla h, \nabla H_k \right\rangle - \rho'(h)c_{k-1}H_k \\ & - \widehat{\Theta}\mathcal{H}'(h)(\|\nabla h\|^2 c_{k-1}H_{k-1} - \langle P_{k-1}\nabla h, \nabla h\rangle) - \frac{\widehat{\Theta}}{\rho(h)^2}\beta_{k-1} \\ & - \widehat{\Theta}\binom{n}{k}(nH_1H_k - (n-k)H_{k+1}), \end{aligned}$$

where

$$\beta_{k-1} = \sum_{i=1}^{n} \mu_{i,k-1}K_{\mathbb{P}}(E_i^* \wedge N^*)\|E_i^* \wedge N^*\|^2.$$

Here $\mu_{i,k-1}$ stand for the eigenvalues of P_{k-1} and $\{E_1, \dots, E_n\}$ is a local orthonormal frame on Σ diagonalizing A.

Proof. Since $\rho(t)T$ is a conformal vector field

$$\nabla\widehat{\Theta} = -\rho(h)A\nabla h.$$

Therefore, using Equation (2.3) we find

$$\nabla_X\nabla\widehat{\Theta} = -\rho(h)(\nabla_X A)\nabla h - \rho'(h)AX - \widehat{\Theta}A^2X.$$

Hence

$$\begin{aligned} L_{k-1}\widehat{\Theta} = & - \rho(h)\sum_{i=1}^{n}\langle P_{k-1}(\nabla_{E_i}A)\nabla h, E_i\rangle \\ & \qquad - \rho'(h)c_{k-1}H_k - \binom{n}{k}\widehat{\Theta}(H_1H_k - (n-k)H_{k+1}). \end{aligned}$$

Using the expression of the covariant derivative of a tensor field we get

$$\begin{aligned} -P_{k-1}(\nabla_{E_i}A)\nabla h = & (\nabla_{E_i}P_{k-1})A\nabla h - (\nabla_{E_i}P_{k-1}A)\nabla h \\ = & (\nabla_{E_i}P_{k-1})A\nabla h + (\nabla_{E_i}P_k)\nabla h - E_i(S_k)\nabla h. \end{aligned}$$

Equation (1.2) implies that

$$\begin{aligned}
-\sum_{i=1}^{n}\langle P_{k-1}(\nabla_{E_i}A)\nabla h, E_i\rangle =&\sum_{i=1}^{n}\langle(\nabla_{E_i}P_{k-1})A\nabla h, E_i\rangle\\
&+\sum_{i=1}^{n}\langle(\nabla_{E_i}P_k)\nabla h, E_i\rangle-\nabla h(S_k)\\
=&\sum_{i=1}^{n}\left\langle\overline{\mathrm{R}}(E_i,\nabla h)N, P_{k-1}E_i\right\rangle-\nabla h(S_k).
\end{aligned}$$

Since $\nabla h = T-\Theta N$, we can write

$$\overline{\mathrm{R}}(E_i,\nabla h)N=\overline{\mathrm{R}}(E_i,T)N-\Theta\overline{\mathrm{R}}(E_i,N)N.$$

Using Equation (2.13) and observing that $T^*=0$ we get

$$\overline{\mathrm{R}}(E_i,T)N=-(\mathcal{H}(h)^2+\mathcal{H}'(h))\Theta E_i=-\frac{\rho''(h)}{\rho(h)}\Theta E_i,$$

which implies

$$\sum_{i=1}^{n}\left\langle\overline{\mathrm{R}}(E_i,T)N, P_{k-1}E_i\right\rangle=-\frac{\rho''(h)}{\rho(h)}\Theta c_{k-1}H_{k-1}.$$

Again by (2.13)

$$\begin{aligned}
\overline{\mathrm{R}}(E_i,N)N=&\mathrm{R}_{\mathbb{P}}(E_i^*,N^*)N^*-\mathcal{H}(h)^2E_i\\
&+\mathcal{H}'(h)\Theta(\langle E_i,\nabla h\rangle N-\Theta E_i)-\mathcal{H}'(h)\langle E_i,\nabla h\rangle T.
\end{aligned}$$

Assume that the orthonormal basis $\{E_i\}_1^n$ diagonalizes A and hence P_{k-1}, that is $P_{k-1}E_i=\mu_{i,k-1}E_i$. Then

$$\begin{aligned}
\sum_{i=1}^{n}\left\langle\overline{\mathrm{R}}(E_i,N)N, P_{k-1}E_i\right\rangle=&\frac{1}{\rho(h)^2}\sum_{i=1}^{n}\mu_{i,k-1}K_{\mathbb{P}}(E_i^*\wedge N^*)\|E_i^*\wedge N^*\|^2\\
&-\frac{\rho''(h)}{\rho(h)}c_{k-1}H_{k-1}\\
&+\mathcal{H}'(h)(\|\nabla h\|^2c_{k-1}H_{k-1}-\langle P_{k-1}\nabla h,\nabla h\rangle).
\end{aligned}$$

Then

$$\begin{aligned}
\sum_{i=1}^{n}\left\langle\overline{\mathrm{R}}(E_i,\nabla h)N, P_{k-1}E_i\right\rangle=&\sum_{i=1}^{n}\left\langle\overline{\mathrm{R}}(E_i,T)N, P_{k-1}E_i\right\rangle\\
&-\Theta\sum_{i=1}^{n}\left\langle\overline{\mathrm{R}}(E_i,N)N, P_{k-1}E_i\right\rangle\\
=&-\frac{\Theta}{\rho(h)^2}\sum_{i=1}^{n}\mu_{i,k-1}K_{\mathbb{P}}(E_i^*\wedge N^*)\|E_i^*\wedge N^*\|^2\\
&-\Theta\mathcal{H}'(h)(\|\nabla h\|^2c_{k-1}H_{k-1}-\langle P_{k-1}\nabla h,\nabla h\rangle)
\end{aligned}$$

and this concludes the proof of the lemma. □

In case $\mathbb{P}^n$ has constant sectional curvature we obtain the following

Corollary 2.27. *Let Σ^n be a hypersurface immersed into a warped product space $I \times_\rho \mathbb{P}^n$, with angle function Θ and height function h. Assume that $\mathbb{P}^n$ has constant sectional curvature κ and let $\widehat{\Theta} = \rho(h)\Theta$. Then, for every $1 \leq k \leq n$ we have*

$$\begin{aligned}L_{k-1}\widehat{\Theta} = &-\binom{n}{k}\rho(h)\left\langle \nabla h, \nabla H_k\right\rangle - \rho'(h)c_{k-1}H_k\\ &-\widehat{\Theta}\Big(\frac{\kappa}{\rho^2(h)} + \mathcal{H}'(h)\Big)(\|\nabla h\|^2 c_{k-1}H_{k-1} - \left\langle P_{k-1}\nabla h, \nabla h\right\rangle)\\ &-\widehat{\Theta}\binom{n}{k}(nH_1H_k - (n-k)H_{k+1}).\end{aligned}$$

Proof. The corollary follows immediately by Lemma 2.26 once one writes explicitly the term

$$\frac{\widehat{\Theta}}{\rho(h)^2}\beta_{k-1},$$

where

$$\beta_{k-1} = \sum_{i=1}^n \mu_{i,k-1}K_{\mathbb{P}}(E_i^* \wedge N^*)\|E_i^* \wedge N^*\|^2.$$

Recall that

$$\|E_i^* \wedge N^*\|^2 = \|E_i^*\|^2\|N^*\|^2 - \left\langle E_i^*, N^*\right\rangle^2.$$

Moreover

$$E_i^* = E_i - \left\langle E_i, \nabla h\right\rangle T, \quad N^* = N - \Theta T,$$

hence

$$\begin{aligned}\|E_i^*\|^2 &= 1 - \left\langle E_1, \nabla h\right\rangle^2,\\ \|N^*\|^2 &= 1 - \Theta^2 = \|\nabla h\|^2,\\ \left\langle E_i^*, N^*\right\rangle &= -\Theta\left\langle E_i, \nabla h\right\rangle,\end{aligned}$$

and

$$\|E_i^* \wedge N^*\|^2 = \|\nabla h\|^2 - \left\langle E_1, \nabla h\right\rangle^2\|\nabla h\|^2 - \Theta^2\left\langle E_1, \nabla h\right\rangle^2 = \|\nabla h\|^2 - \left\langle E_1, \nabla h\right\rangle^2.$$

Then

$$\begin{aligned}\sum_{i=1}^n \mu_{i,k-1}\|E_i^* \wedge N^*\|^2 &= \sum_{i=1}^n \mu_{i,k-1}\|\nabla h\|^2 - \sum_{i=1}^n \mu_{i,k-1}\left\langle E_1, \nabla h\right\rangle^2\\ &= \mathrm{Tr}(P_{k-1})\|\nabla h\|^2 - \left\langle P_{k-1}\nabla h, \nabla h\right\rangle.\end{aligned}$$

Using the expression of the trace of P_{k-1} we obtain

$$\sum_{i=1}^n \mu_{i,k-1}\|E_i^* \wedge N^*\|^2 = c_{k-1}H_{k-1}\|\nabla h\|^2 - \left\langle P_{k-1}\nabla h, \nabla h\right\rangle$$

and this concludes the proof. □

We are now ready to give the

PROOF OF THEOREM 2.25. We may assume without loss of generality that $\mathcal{H}(h) > 0$ on Σ. Since Σ^n is compact, there exists a point $p_{\max} \in \Sigma$ where the height function attains its maximum. Then $\nabla h(p_{\max}) = 0$, $\Theta(p_{\max}) = \pm 1$ and, by (2.3),

$$\operatorname{Hess} h(p_{\max})(v,v) = \mathcal{H}(h^*)\langle v,v\rangle + \Theta(p_{\max})\langle Av,v\rangle(p_{\max}) \leq 0.$$

If $\Theta(p_{\max}) = -1$, then

$$\langle Av,v\rangle(p_{\max}) \geq \mathcal{H}(h^*)\langle v,v\rangle > 0,$$

for any $v \neq 0$. Thus $p_{\max}$ is an elliptic point, H_k is a positive constant and by Garding inequalities

$$H_1 \geq H_2^{\frac{1}{2}} \geq \cdots \geq H_k^{\frac{1}{k}} > 0$$

with equality only at umbilical points. In particular Σ is two-sided and $\Theta \leq 0$. If $\Theta(p_{\max}) = 1$, changing the orientation of Σ we reach the same conclusion.

Consider the function

$$\phi = \sigma(h)H_k^{\frac{1}{k}} + \rho(h)\Theta.$$

Let us prove that $L_{k-1}\phi \geq 0$. Since H_k is constant we have

$$\begin{aligned}
L_{k-1}\phi =& H_k^{\frac{1}{k}} L_{k-1}\sigma(h) + L_{k-1}\widehat{\Theta}\\
=& c_{k-1}H_k^{\frac{1}{k}}(\rho'(h)H_{k-1} + \widehat{\Theta}H_k)\\
&- \widehat{\Theta}\mathcal{H}'(h)\Big(\|\nabla h\|^2 c_{k-1}H_{k-1} - \langle P_{k-1}\nabla h, \nabla h\rangle\Big)\\
&- \widehat{\Theta}\binom{n}{k}(nH_1H_k - (n-k)H_{k+1}) - \rho'(h)c_{k-1}H_k\\
&- \frac{\widehat{\Theta}}{\rho(h)^2}\sum_{i=1}^{n}\mu_{i,k-1}K_{\mathbb{P}}(E_i^*\wedge N^*)\|E_i^*\wedge N^*\|^2\\
=& A + B + C,
\end{aligned}$$

where

$$A = -\widehat{\Theta}\binom{n}{k}(nH_1H_k - (n-k)H_{k+1} - kH_k^{\frac{k+1}{k}}),$$

$$B = c_{k-1}\rho'(h)(H_{k-1}H_k^{\frac{1}{k}} - H_k)$$

and

$$\begin{aligned}
C =& -\widehat{\Theta}\mathcal{H}'(h)(\|\nabla h\|^2 c_{k-1}H_{k-1} - \langle P_{k-1}\nabla h, \nabla h\rangle)\\
&- \frac{\widehat{\Theta}}{\rho(h)^2}\sum_{i=1}^{n}\mu_{i,k-1}K_{\mathbb{P}}(E_i^*\wedge N^*)\|E_i^*\wedge N^*\|^2.
\end{aligned}$$

Then by Garding inequalities

$$H_{k-1}H_k^{\frac{1}{k}} - H_k = H_k^{\frac{1}{k}}(H_{k-1} - H_k^{\frac{k-1}{k}}) \geq 0.$$

Moreover

$$nH_1H_k - kH_k^{\frac{k+1}{k}} \geq nH_k^{\frac{k+1}{k}} - kH_k^{\frac{k+1}{k}} = (n-k)H_k^{\frac{k+1}{k}},$$

hence

$$nH_1H_k - kH_k^{\frac{k+1}{k}} - (n-k)H_{k+1} \geq (n-k)(H_k^{\frac{k+1}{k}} - H_{k+1}) \geq 0.$$

Finally, let $\alpha := \sup_I\{\rho'^2 - \rho''\rho\}$. Since

$$\|E_i^* \wedge N^*\|^2 = \|\nabla h\|^2 - \langle E_i, \nabla h\rangle^2 \,,$$

and taking into account that the $\mu_{i,k-1}$ are positive and the proof of Corollary 2.27, we have

$$\begin{aligned}
&\sum_{i=1}^n \mu_{i,k-1} K_{\mathbb{P}}(E_i^* \wedge N^*)\|E_i^* \wedge N^*\|^2 \\
&\geq \alpha \sum_{i=1}^n \mu_{i,k-1}\|E_i^* \wedge N^*\|^2 \\
&= \alpha(c_{k-1}H_{k-1}\|\nabla h\|^2 - \langle P_{k-1}\nabla h, \nabla h\rangle).
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{1}{\rho(h)^2}&\sum_{i=1}^n \mu_{i,k-1} K_{\mathbb{P}}(E_i^* \wedge N^*)\|E_i^* \wedge N^*\|^2 \\
&+ \mathcal{H}'(h)(\|\nabla h\|^2 c_{k-1}H_{k-1} - \langle P_{k-1}\nabla h, \nabla h\rangle) \\
\geq&\Big(\frac{\alpha}{(\rho)^2(h)} + \mathcal{H}'(h)\Big)(\|\nabla h\|^2 c_{k-1}H_{k-1} - \langle P_{k-1}\nabla h, \nabla h\rangle) \geq 0,
\end{aligned}$$

where the last inequality follows from $\alpha = \sup_I\{-\rho^2\mathcal{H}'\}$ and from the positive definitness of P_{k-1}. Hence $L_{k-1}\phi \geq 0$ and we conclude by the maximum principle that ϕ must be constant. In particular, $L_{k-1}\phi = 0$ and the three terms A, B and C must vanish on Σ. Notice that $B = 0$ implies that Σ is a totally umbilical hypersurface. Moreover, since H_k is a positive constant and Σ is totally umbilical, all the higher order mean curvatures are constant. Hence H_1 is constant and the conclusion follows by Theorem 2.24. □

Let us focus now on the complete non-compact case. Observe first that Theorem 2.24 can be extended to the complete non-compact case once one guarantees the parabolicity of Σ. In that case it is straightforward to prove the next

Theorem 2.28. *Let $I \times_\rho \mathbb{P}^n$ be a warped product space and assume that the Ricci curvature of $\mathbb{P}^n$ satisfies*

$$\mathrm{Ric}_{\mathbb{P}} > \sup_I\{\rho'^2 - \rho''\rho\}.$$

Let $f : \Sigma^n \to I \times_\rho \mathbb{P}^n$ be a complete parabolic two-sided hypersurface with constant mean curvature contained in a slab. If the angle function Θ does not change sign, then $f(\Sigma^n)$ is a slice.

For the proof, observe that, since the function $\phi = H\sigma(h) + \widehat{\Theta}$ is subharmonic and bounded from above and Σ is parabolic, then ϕ must be constant. In particular, $\Delta\phi = 0$ and by

$$\Delta\phi = -\widehat{\Theta}\{\|A\|^2 - nH_1^2 + (n-1)\,\mathrm{Ric}_{\mathbb{P}}(N^*, N^*) + \mathcal{H}'(h)\|\nabla h\|^2\}$$

we conclude that h has to be constant, because of the strict inequality in (2.16).

Remark 2.29. The problem of establishing whether a Riemannian manifold is parabolic or not has been intensively studied. This is due in part by the connection of this property with the recurrence properties of the Brownian motion in a Riemannian manifold (see the survey paper [**37**] for a background on this subject). Indeed, this connection encouraged the understanding of parabolicity from a potential-theoretic point of view and the attempt to find geometric conditions on the manifold guaranteeing this property. In particular, it turned out that parabolicity is strictly related to the volume growth of geodesic balls. Namely, it was proved in [**24**] that in a complete Riemannian manifold, if the volume of the geodesic balls grows at most like a quadratic polynomial, then there are no non-constant positive superharmonic functions defined on it. Thereafter, Karp [**44**] and, later on, Varopoulos [**65**] and Grigor'yan [**35**], [**36**] proved that a sufficient condition for parabolicity is the following

$$\frac{r}{\mathrm{vol}B_r} \notin L^1(+\infty), \tag{2.18}$$

where B_r denotes the geodesic ball of radius r. Notice that condition (2.18) is satisfied if the the geodesic balls have at most quadratic polinomial growth. However, condition (2.18) is far away from being necessary, as shown by a counterexample of Greene quoted in [**65**]. On the other hand, another sufficient condition is the following

$$\big(\mathrm{vol}\partial B_r\big)^{-1} \notin L^1(+\infty). \tag{2.19}$$

This was proved by Ahlfors [**1**] and Nevanlinna [**52**] for Riemannian surfaces and later on by Lyons and Sullivan [**46**] and by Grygor'yan [**35**], [**36**]. Moreover, this condition turns out to be also necessary when the Riemannian manifold is a geodesically complete model manifold. We also observe that (2.19) is always implied by (2.18). Further, as observed in [**60**], it is easy to construct examples of manifolds of exponential volume growth where (2.19) holds while (2.18) obviously does not. One shall therefore concentrate on conditions involving $\mathrm{vol}\partial B_r$ as in (2.19) rather than $\mathrm{vol}B_r$ itself.

Finally, using condition (2.19), it is easy to see that, between the space forms, the Euclidean space $\mathbb{R}^n$ is parabolic only when $n \leq 2$ since the boundary of a geodesic ball ∂B_r grows as r^{n-1}, while the Hyperbolic space is not parabolic since the geodesic spheres have exponential growth.

From the above circle of ideas one obtains the following Corollary to Theorem 2.28.

Corollary 2.30. *Let $I \times_\rho \mathbb{P}^n$ be a warped product space and assume that the Ricci curvature of $\mathbb{P}^n$ satisfies* (2.16). *Let $f : \Sigma^n \to I \times_\rho \mathbb{P}^n$ be a complete two-sided hypersurface with constant mean curvature satisfying condition* (2.19) *and contained in a slab. If the angle function Θ does not change sign, then $f(\Sigma^n)$ is a slice.*

Concerning the case when Σ has constant higher order mean curvature, as one can see from the proof of Theorem 2.25, the same result can be extended to the complete non-compact case if the hypersurface is parabolic in an appropriate sense. Our goal is then to find geometric conditions that guarantee this parabolicity. This is done for a general class of divergence form operators in [**56**] but for simplicity we will focus only on the following divergence form operator, that belongs to that family. More precisely, we will consider

$$\mathfrak{L}_{k-1}f = \operatorname{div}(P_{k-1}\nabla f),$$

where $f \in C^{\infty}(\Sigma)$. Notice that

$$\mathfrak{L}_{k-1}f = \langle \operatorname{div}P_{k-1}, \nabla f\rangle + L_{k-1}f.$$

As a consequence of Proposition 1.2, we obtain the following lemma, in which the expression of the divergence of P_{k-1} is given in the case when the fiber $\mathbb{P}^n$ has constant sectional curvature.

Lemma 2.31. *Let $\Sigma^n \to I \times_\rho \mathbb{P}^n$ be an isometric immersion and assume that $\mathbb{P}^n$ has constant sectional curvature κ. Then*

$$\operatorname{div}P_{k-1} = -(n-k+1)\Theta\Big(\frac{\kappa}{\rho^2(h)} + \mathcal{H}'(h)\Big)P_{k-2}\nabla h. \tag{2.20}$$

Proof. Let $E_1, ..., E_n$ be a local orthonormal frame on Σ^n and recall that

$$\langle \operatorname{div}P_{k-1}, X\rangle = \sum_{i=1}^{n} \langle (\nabla_{E_i}P_{k-1})X, E_i\rangle$$

for every vector field $X \in T\Sigma$. For any $0 \le j \le k-2$ we have that

$$\begin{aligned}\sum_{i=1}^{n}\Big\langle \overline{\mathrm{R}}(E_i, A^{k-2-j}X)N, P_jE_i\Big\rangle =& \sum_{i=1}^{n}\Big\langle \mathrm{R}_{\mathbb{P}}(E_i^*, (A^{k-2-j}X)^*)N^*, P_jE_i\Big\rangle \\ &+ \Theta\mathcal{H}'(h)\big(\Big\langle P_j\nabla h, A^{k-2-j}X\Big\rangle \\ &- c_jH_j\Big\langle \nabla h, A^{k-2-j}X\Big\rangle\big).\end{aligned}$$

Since $\mathbb{P}^n$ has constant sectional curvature κ it follows that

$$\mathrm{R}_{\mathbb{P}}(Y,Z)W = \kappa(\langle Z,W\rangle_{\mathbb{P}}\, Y - \langle Y,W\rangle_{\mathbb{P}}\, Z).$$

Hence a direct calculation shows that

$$\begin{aligned}\sum_{i=1}^{n}\Big\langle R_{\mathbb{P}}(E_i^*, (A^{k-2-j}X)^*)N^*, P_jE_i\Big\rangle =& \frac{\kappa}{\rho^2(h)}\Theta\big(\Big\langle P_j\nabla h, A^{k-2-j}X\Big\rangle \\ &- c_jH_j\Big\langle \nabla h, A^{k-2-j}X\Big\rangle\big).\end{aligned}$$

We claim that

$$B_{k-1} := \sum_{j=0}^{k-2}(-1)^{k-j-2}(P_jA^{k-2-j} - c_jH_jA^{k-2-j}) = -(n-k+1)P_{k-2}.$$

In this case the conclusion of the corollary follows immediately by Proposition 1.2. Let us prove the claim by induction on k, $k = 1, ..., n$. The case $k = 1$ is trivial, so assume that we the equation holds for $k-2$. Then

$$\begin{aligned} B_{k-1} =& P_{k-2} - c_{k-2}H_{k-2}I - B_{k-2} \circ A \\ =& P_{k-2} - c_{k-2}H_{k-2}I + (n-k+2)P_{k-3}A \\ =& -(n-k+1)P_{k-2}. \end{aligned}$$

□

Since, in general, the divergence of P_{k-1} has an expression that can not be dealt with unless, as we have just seen, in the case when $\mathbb{P}^n$ has constant sectional curvature, we will restrict to this case.

Following the terminology introduced in [**56**], we present the next

Definition 2.32. *We will say that the manifold $\Sigma^n \hookrightarrow I \times_\rho \mathbb{P}^n$ is $\mathfrak{L}_{k-1}$-parabolic if the only bounded above C^1 solutions of the inequality*

$$\mathfrak{L}_{k-1} f \geq 0$$

are constant.

The next theorem gives some geometric conditions that guarantee the $\mathfrak{L}_{k-1}$-parabolicity. We present here a direct proof of the theorem in order to make the exposition self-contained, although the result it can be obtained as a Corollary of the more general Theorem 2.6 in [**57**], .

Theorem 2.33. *Let $\Sigma^n \hookrightarrow I \times_\rho \mathbb{P}^n$ be a complete Riemannian manifold. If*

$$\left(\sup_{\partial B_t} H_{k-1} \mathrm{vol}(\partial \mathrm{B_t})\right)^{-1} \notin \mathrm{L}^1(+\infty), \tag{2.21}$$

where ∂B_t is a geodesic sphere of radius t, then Σ^n is $\mathfrak{L}_{k-1}$-parabolic.

Proof. Let f be a bounded above C^1 solution of the inequality

$$\mathfrak{L}_{k-1} f \geq 0.$$

We claim that

$$\left(\sup_{\partial B_t} H_{k-1} \int_{\partial B_t} \mathrm{e}^{\sigma f}\right)^{-1} \notin L^1(+\infty), \tag{2.22}$$

for some $\sigma > 0$. Indeed, since $f^* = \sup_\Sigma f < +\infty$, then $\mathrm{e}^{\sigma f} \leq \mathrm{e}^{\sigma f^*}$ and

$$\frac{1}{\sup_{\partial B_t} H_{k-1} \int_{\partial B_t} e^{\sigma f}} \geq \frac{1}{\sup_{\partial B_t} H_{k-1} \int_{\partial B_t} e^{\sigma f^*}} \geq \frac{C}{\sup_{\partial B_t} H_{k-1} \mathrm{vol}(\partial B_t)}$$

and hence, condition (2.21) implies (2.22).

Assume now by contradiction that f is not constant and consider the vector field

$$Z = \mathrm{e}^{\sigma f} P_{k-1} \nabla f.$$

An easy calculation gives

$$\mathrm{div} Z = \sigma \mathrm{e}^{\sigma f} \langle P_{k-1} \nabla f, \nabla f \rangle + \mathrm{e}^{\sigma f} \mathfrak{L}_{k-1} f \geq \sigma \mathrm{e}^{\sigma f} \mu_{\min,k-1} \|\nabla f\|^2.$$

Applying the divergence theorem we get

$$\int_{\partial B_t} \langle Z, \nabla r\rangle = \int_{B_t} \mathrm{div} Z \geq \sigma \inf_{B_t} \mu_{\min,k-1} \int_{B_t} \mathrm{e}^{\sigma f} \|\nabla f\|^2.$$

On the other heand, using the Cauchy-Schwarz inequality and the Hölder inequality with conjugate exponents $p = q = 2$, we have that

$$\begin{aligned}\int_{\partial B_t} \langle Z, \nabla r\rangle \leq \int_{\partial B_t} \|Z\| &= \int_{\partial B_t} \mathrm{e}^{\sigma f} \|P_{k-1}\| \|\nabla f\| \\ &\leq \Big(\int_{\partial B_t} \mathrm{e}^{\sigma f} \|P_{k-1}\|^2\Big)^{1/2} \Big(\int_{\partial B_t} \mathrm{e}^{\sigma f} \|\nabla f\|^2\Big)^{1/2} \\ &\leq \Big(c_{k-1} \sup_{\partial B_t} H_{k-1} \int_{\partial B_t} \mathrm{e}^{\sigma f}\Big)^{1/2} \Big(\int_{\partial B_t} \mathrm{e}^{\sigma f} \|\nabla f\|^2\Big)^{1/2}.\end{aligned}$$

Set

$$G(t) = \int_{B_t} \mathrm{e}^{\sigma f} \|\nabla f\|^2.$$

Then

$$G'(t) = \int_{\partial B_t} \mathrm{e}^{\sigma f} \|\nabla f\|^2$$

and, combining the previous inequalities,

$$G^{-2}(t) G'(t) \geq C \Big(\sup_{\partial B_t} H_{k-1} \int_{\partial B_t} \mathrm{e}^{\sigma f}\Big)^{-1}.$$

Integrating for $t \in (R, r)$,

$$G^{-1}(R) \geq G^{-1}(R) - G^{-1}(r) \geq C \int_R^r \Big(\sup_{\partial B_t} H_{k-1} \int_{\partial B_t} \mathrm{e}^{\sigma f}\Big)^{-1} dt,$$

that is

$$\int_{B_R} \mathrm{e}^{\sigma f} \|\nabla f\|^2 \leq \frac{C'}{\int_R^r (\sup_{\partial B_t} H_{k-1} \int_{\partial B_t} \mathrm{e}^{\sigma f})^{-1} dt}.$$

Letting $r \to +\infty$ and using assumption (2.22), we conclude that f must be constant, reaching in this way a contradiction. $\square$

We are then ready to state our last result of this section, that extends Theorem 2.25 to complete non-compact hypersurfaces in warped product spaces $I \times_\rho \mathbb{P}^n$, at least when the fiber $\mathbb{P}^n$ has constant sectional curvature.

Theorem 2.34 (Theorem 32 in [**13**])**.** *Let $I \times_\rho \mathbb{P}^n$ be a warped product space and assume that $\mathbb{P}^n$ has constant sectional curvature κ satisfying*

$$\kappa > \sup_I \{\rho'^2 - \rho''\rho\}. \tag{2.23}$$

Let $f : \Sigma^n \to I \times_\rho \mathbb{P}^n$ be a complete hypersurface with $\sup_\Sigma |H_1| < +\infty$ and satisfying condition (2.21). Suppose that f has constant k-mean curvature, $2 \leq k \leq n$, and that $f(\Sigma^n)$ is contained in a slab. Assume that either $k = 2$ and H_2 is positive or $k \geq 3$ and there exists an elliptic point $p \in \Sigma^n$. If $\mathcal{H}(h)$ and the angle function Θ do not change sign, then $f(\Sigma^n)$ is a slice.

Remark 2.35. Comparing with Theorem 2.25 we have relaxed the condition on $\mathcal{H}$ but we are requiring, as it will be clear from the proof, the existence of an elliptic point. That, on a compact manifold was guaranteed by the assumption $\mathcal{H}(h) \neq 0$. Moreover, we observe that the angle function is indeed well defined because Σ is two-sided. For $k = 2$, this follows from the positivity of H_2 since $H_1^2 \geq H_2 > 0$. In the remaining cases we obtain this from the Garding inequalities, as in the compact case. In any case we choose the orientation so that $H_1 > 0$.

PROOF. It follows from the hypotheses that $\sup_\Sigma \|A\| < +\infty$ and therefore, by Lemma 2.19, the sectional curvature of Σ^n is bounded from below. We deduce then the validity of the Omori-Yau maximum principle for any semi-elliptic operator as in (1.11) and hence, in particular, for the Laplacian. Assume that $\mathcal{H}(h) \geq 0$. Applying the Omori-Yau maximum principle for the Laplacian and using equation (2.1) we find that

$$-\operatorname{sgn}(\Theta) \liminf_{j \to +\infty} H_1(q_j) \geq \mathcal{H}(h^*) \geq 0.$$

Therefore for the chosen orientation, $\operatorname{sgn}\Theta = -1$ and $\Theta \leq 0$ on Σ. Consider the operator

$$\mathfrak{L}_{k-1} u = \operatorname{div}(P_{k-1}\nabla u)$$

and the function

$$\phi = H_k^{\frac{1}{k}} \sigma(h) + \widehat{\Theta},$$

where, we recall, $\widehat{\Theta} = \rho(h)\Theta$. Since $\mathbb{P}^n$ has constant sectional curvature κ, it follows by Equation (2.20) that

$$\begin{aligned}
\mathfrak{L}_{k-1}\phi =& -(n-k+1)\Theta\Big(\frac{\kappa}{\rho^2(h)} + \mathcal{H}'(h)\Big)\langle P_{k-2}\nabla h, \nabla \phi\rangle + L_{k-1}\phi \\
=& -(n-k+1)\widehat{\Theta}\Big(\frac{\kappa}{\rho^2(h)} + \mathcal{H}'(h)\Big)\langle P_{k-2}\nabla h, \nabla h\rangle H_k^{\frac{1}{k}} \\
&+ (n-k+1)\widehat{\Theta}\Big(\frac{\kappa}{\rho^2(h)} + \mathcal{H}'(h)\Big)\langle P_{k-2}A\nabla h, \nabla h\rangle \\
&+ H_k^{\frac{1}{k}} L_{k-1}\sigma(h) + L_{k-1}\widehat{\Theta}.
\end{aligned}$$

Using Equation (2.2) and Corollary 2.27 we find

$$\begin{aligned}
\mathfrak{L}_{k-1}\phi =& c_{k-1}\rho'(h) H_k^{\frac{1}{k}}(H_{k-1} - H_k^{\frac{k-1}{k}}) \\
&- \binom{n}{k}\widehat{\Theta}\left(nH_1H_k - (n-k)H_{k+1} - kH_k^{\frac{k+1}{k}}\right) \\
&- (n-k)\widehat{\Theta}\Big(\frac{\kappa}{\rho^2(h)} + \mathcal{H}'(h)\Big)\langle P_{k-1}\nabla h, \nabla h\rangle \\
&- (n-k+1)\widehat{\Theta} H_k^{\frac{1}{k}}\Big(\frac{\kappa}{\rho^2(h)} + \mathcal{H}'(h)\Big)\langle P_{k-2}\nabla h, \nabla h\rangle .
\end{aligned} \tag{2.24}$$

Then using Garding inequalities as in Theorem 2.25, it is easy to prove that the first and the second terms are non-negative. By the fact that each P_j is an elliptic operator, $j = 0, ..., k-1$, and by Equation (2.17) it follows that also all the remaining terms in the previous equation are non-negative and hence $\mathfrak{L}_{k-1}\phi \geq 0$. Since, by assumption (2.21), Σ^n is $\mathfrak{L}_{k-1}$-parabolic, we conclude that ϕ has to be constant. In particular, $\mathfrak{L}_{k-1}\phi = 0$ and the four terms on the right-hand side of Equation (2.24) vanish. Let us prove that $\mathcal{U} = \{p \in \Sigma^n : \Theta(p) = 0\}$ has empty interior. Indeed, assume the contrary and let $\mathcal{V} \neq \emptyset$ be an open subset of $\mathcal{U}$. On $\mathcal{V}$ the function $\phi = \sigma(h)H_k^{1/k}$ is constant. Hence, since $H_k \neq 0$, then $\sigma(h)$ and, consequently h, is constant on $\mathcal{V}$. But this is not possible since $\|\nabla h\|^2 = 1 - \Theta^2 = 1$ on $\mathcal{V}$. Therefore, since the third term on the right-hand of (2.24) vanishes identically, Σ^n is totally umbilical and

$$\langle P_j \nabla h, \nabla h \rangle = 0, \quad j = k-2, k-1$$

Since the P_j's are positive definite, this means that h has to be constant and this concludes the proof. $\square$

CHAPTER 3

Spacelike hypersurfaces of constant k-mean curvature in generalized Robertson-Walker spacetimes

In this chapter we want to study uniqueness of spacelike hypersurfaces of constant higher order mean curvature in Lorentzian manifolds, in the case when the ambient space has a large number of totally umbilical hypersurfaces of constant mean curvature (and hence of constant higher order mean curvatures). For what concerns hypersurfaces of constant mean curvature, as already said in the Introduction, several papers have appeared where the same problem has been studied and it turned out that a natural class of ambient spaces to consider is that of *conformally stationary-closed spacetimes*. A conformally stationary-closed spacetime M is a time-orientable spacetime equipped with a globally defined timelike closed conformal vector field $\mathcal{T}$. It is a well-known fact that the distribution on the spacetime orthogonal to $\mathcal{T}$ provides a foliation of M by means of umbilical leaves so that the geometry of a standard leaf, toghether with the conformal function, determines the geometry of the whole M. A special family of conformally stationary-closed spacetimes is that of generalized Robertson-Walker spacetimes, which is the one we will focus on. Following the terminology used in [**16**], by a generalized Robertson-Walker spacetime (GRW) we mean a Lorentzian warped product $-I \times_\rho \mathbb{P}^n$ with 1-dimensional base $I \subseteq \mathbb{R}$ and Riemannian fibre $\mathbb{P}^n$, endowed with the Lorentzian metric

$$\langle , \rangle = -\pi_I(dt^2) + \rho^2(\pi_I)\pi_{\mathbb{P}}(\langle , \rangle_{\mathbb{P}}).$$

In a GRW spacetime, the vector field given by $\mathcal{T}(t,x) = \rho(t)(\partial/\partial t)_{(t,x)}$ is a globally defined timelike closed conformal field. In fact, a kind of converse also holds. Indeed, as observed by Montiel in [**50**], every conformally stationary spacetime admitting a timelike closed conformal vector field is locally isometric to a GRW spacetime. For a global analogue of this assertion, under the assumption of timelike geodesic completeness, see [**50**], Proposition 2.

Here are some significant examples.

Example 3.1. The first and easiest example of a spacetime admitting a timelike closed conformal vector field, and hence admitting a warped product representation, is the Lorentz-Minkowski space $\mathbb{R}^{n+1}_1$. In fact, $\mathbb{R}^{n+1}_1$ admits many timelike closed (actually exact) conformal vector fields and hence many representations as warped product space. A first (trivial) example is, for each timelike vector $a \in \mathbb{R}^{n+1}_1$ the constant field

$$\mathcal{T}(p) = a \qquad \text{for all } p \in \mathbb{R}^{n+1}_1.$$

The vector field $\mathcal{T}$ generates a foliation of $\mathbb{R}^{n+1}_1$ by means of parallel spacelike hyperplanes. This corresponds to the representation of the Lorentz-Minkowski spacetime as the (trivial) warped product $-\mathbb{R}\times_\rho\mathbb{R}^n$ with $\rho(t)=1$.

Another timelike closed conformal vector field is the following

$$\mathcal{T}(p)=p \qquad \text{for } p\in C^+,$$

C^+ denoting the future directed lightcone of the origin, which generates a foliation of $\mathbb{R}^{n+1}_1$ by hyperquadrics of equation $\|p\|^2=-\tau^2$, $\tau>0$, which are umbilical spacelike hypersurfaces isometric to the hyperbolic spaces $\mathbb{H}^n$ of negative constant sectional curvature $-1/\tau$. This corresponds to the representation $-\mathbb{R}_+\times_\rho\mathbb{H}^n$ with $\rho(t)=t$.

Example 3.2. Another important example of spacetime admitting a timelike closed conformal vector field is the de Sitter space $\mathbb{S}^{n+1}_1$. We recall that the de Sitter space can be viewed as the hyperquadric

$$\mathbb{S}^{n+1}_1=\{p\in\mathbb{R}^{n+2}_1|\|p\|^2=1\}$$

and is a model of spacetime of positive constant sectional curvature. For any $a\in\mathbb{R}^{n+1}_1$ let us consider on $\mathbb{S}^{n+1}_1$ the closed conformal vector field

$$\mathcal{T}(p)=a-\langle a,p\rangle\,p, \qquad p\in\mathbb{S}^{n+1}_1.$$

It is not difficult to see that $\mathcal{T}$ is a timelike vector field if we restrict on certain subsets of $\mathbb{S}^{n+1}$ and it generates different foliations depending on the causal character of a.

If a is a unit timelike vector, then $\mathcal{T}$ is timelike on the whole $\mathbb{S}^{n+1}_1$ and foliates the de Sitter space by means of umbilical round spheres of radii $\sqrt{1+\tau^2}$ and described by the equation $\langle p,a\rangle=\tau$, $\tau\in\mathbb{R}$ (see [**48**, Example 1] for more details). On the other hand, if a is a null vector, the closed conformal vector field $\mathcal{T}$ is timelike on the open set

$$\{p\in\mathbb{S}^{n+1}_1|\,\langle p,a\rangle\neq 0\}.$$

Consider the connected component of this set characterized by $\langle p,a\rangle>0$. In this case the corresponding foliation has as leaves the spacelike hypersurfaces described by the equation $\langle a,p\rangle=\tau$, $\tau\in\mathbb{R}$, which are isometric to Euclidean spaces $\mathbb{R}^n$ for any $\tau\in\mathbb{R}$. Finally, if a is a unit spacelike vector, the vector field $\mathcal{T}$ has a timelike character only on the open set

$$\{p\in\mathbb{S}^{n+1}_1|\,\langle p,a\rangle^2>1\}.$$

If we restrict ourselves to the component with $\langle p,a\rangle>1$ it can be seen that the leaves of the corresponding foliation are the spacelike hypersurfaces of equation $\langle p,a\rangle=\tau$, $\tau>1$ which are isometric to Hyperbolic spaces $\mathbb{H}^n$ of negative constant sectional curvature $-1/(\tau^2-1)$. Moreover, the case $\|a\|^2=-1$ corresponds to the representation of the de Sitter space as $-\mathbb{R}\times_\rho\mathbb{S}^n$, where $\rho(t)=\cosh t$, the case $\|a\|=0$ corresponds to the representation $-\mathbb{R}\times_\rho\mathbb{R}^n$, where $\rho(t)=\mathrm{e}^t$ (the so called *steady state space*) and, finally, the case $\|a\|^2=1$ corresponds to $-\mathbb{R}_+\times_\rho\mathbb{H}^n$, where $\rho(t)=\sinh t$.

Example 3.3. The last significative example we want to illustrate is that of the anti-de Sitter spacetime $\mathbb{H}_1^{n+1}$. Similarly to the de Sitter space, it can be viewed as the hyperquadric

$$\mathbb{H}_1^{n+1} = \{p \in \mathbb{R}_1^{n+2} | \|p\|^2 = -1\}$$

and is a model of spacetime of negative constant sectional curvature. Fixed any $a \in \mathbb{R}_2^{n+1}$, its causal character determines a closed conformal vector field

$$\mathcal{T}(p) = a + \langle a, p \rangle \, p, \qquad p \in \mathbb{H}_1^{n+1}$$

Since

$$\|\mathcal{T}(p)\|^2 = \|a\|^2 + \langle a, p \rangle^2 \, ,$$

it is easy to see that $\mathcal{T}$ is timelike only when a is itself timelike. Moreover, if this is the case, then $\mathcal{T}$ is timelike on the two components of the open set consisting of the points $p \in \mathbb{H}_1^{n+1}$ satisfying $\langle a, p \rangle^2 < 1$, it generates a foliation by means of umbilical spacelike hypersurfaces isometric to two copies of $\mathbb{H}^n$ and each of the two components can be described as the warped product $-(-\pi/2, \pi/2) \times_\rho \mathbb{H}^n$, $\rho(t) = \cos t$.

Consider now a spacelike hypersurface $f : \Sigma^n \to M^{n+1} := -I \times_\rho \mathbb{P}^n$. In this case, since $T := \frac{\partial}{\partial t}$ is a timelike unit vector field globally defined on $-I \times_\rho \mathbb{P}^n$, there exists a unique timelike unit normal field N globally defined on Σ with the same orientation as T. We will refer to that normal field N as the future-pointing Gauss map of the hypersurface. Moreover, the angle function

$$\Theta(p) := \langle N(p), T(f(p)) \rangle$$

is globally defined and it satisfies

$$\Theta(p) \leq -1 < 0.$$

3.1. The operator L_k acting on the height and the angle functions

We devote this section to some computational results that will be fundamental to recover the main theorems of the chapter. Let us start with the following

Proposition 3.4. *Let $f : \Sigma^n \to -I \times_\rho \mathbb{P}^n$ be a spacelike hypersurface and let*

$$\sigma(t) = \int_{t_0}^{t} \rho(u) du.$$

Then

$$L_{k-1} h = -(\log \rho)'(h)(c_{k-1} H_{k-1} + \langle P_{k-1} \nabla h, \nabla h \rangle) - \Theta c_{k-1} H_k, , \tag{3.1}$$

$$L_{k-1} \sigma(h) = -c_{k-1}(\rho'(h) H_{k-1} + \Theta \rho(h) H_k)., \tag{3.2}$$

where $c_k = (n-k+1)\binom{n}{k-1} = k\binom{n}{k}$

Proof. Observe that the gradient of $\pi_{\mathbb{R}} \in C^\infty(M)$ is $\overline{\nabla} \pi_{\mathbb{R}} = -T$. Hence

$$\nabla h = (\overline{\nabla} \pi_{\mathbb{R}})^T = -T - \Theta N.$$

Moreover

$$\nabla \sigma(h) = \rho(h) \nabla h = -\rho(h) T - \rho(h) \Theta N.$$

Since $\rho(t)T$ is a non-vanishing closed conformal vector field on $-I\times_\rho\mathbb{P}^n$ with conformal function $\rho'(t)$, we have

$$\overline{\nabla}_Z(\rho(t)T)=\rho'(t)Z,$$

for every vector Z tangent to $-I\times_\rho\mathbb{P}^n$. Hence

$$\overline{\nabla}_X\nabla\sigma(h)=-\rho'(h)X+\rho(h)\Theta AX-X(\rho(h)\Theta)N$$

and

$$\nabla_X\nabla\sigma(h)=(\overline{\nabla}_X\sigma(h))^T=-\rho'(h)X+\rho(h)\Theta AX. \tag{3.3}$$

Then

$$\begin{aligned}L_{k-1}\sigma(h)=&\mathrm{Tr}(P_{k-1}\circ\mathrm{hess}(\sigma(h)))\\ =&-c_{k-1}\rho(h)\mathrm{Tr}(P_{k-1})+\rho(h)\Theta\mathrm{Tr}(P_{k-1}A)\\ =&-c_{k-1}(\rho'(h)H_{k-1}+\rho(h)\Theta H_k).\end{aligned}$$

Moreover

$$\nabla_X\nabla h=-\frac{\rho'(h)}{\rho(h)}\langle X,\nabla h\rangle\nabla h+\frac{1}{\rho(h)}\nabla_X\sigma(h)$$

and therefore

$$\begin{aligned}L_{k-1}h=&-(\log\rho)'(h)\langle P_{k-1}\nabla h,\nabla h\rangle+\frac{1}{\rho(h)}L_{k-1}\sigma(h)\\ =&-(\log\rho)'(h)(\langle P_{k-1}\nabla h,\nabla h\rangle-c_{k-1}H_{k-1})-c_{k-1}\rho(h)\Theta H_k).\end{aligned}$$

□

Before proving the second useful computational result, we need the following

Lemma 3.5. *Let Σ^n be a spacelike hypersurface immersed into a GRW spacetime $-I\times_\rho\mathbb{P}^n$, with angle function Θ and height function h. Let $\widehat{\Theta}=\rho(h)\Theta$ and let $\{E_1,...,E_n\}$ be an arbitrary local orthonormal frame on Σ. Then, for every $k=1\dots n$ we have*

$$\begin{aligned}\rho(h)\sum_{i=1}^n\left\langle\overline{\mathrm{R}}(E_i,\nabla h)N,P_{k-1}E_i\right\rangle=&\frac{\widehat{\Theta}}{\rho^2(h)}\sum_{i=1}^n\mu_{i,k-1}K_{\mathbb{P}}(N^*\wedge E_i^*)\|N^*\wedge E_i^*\|^2\\ &-\widehat{\Theta}(\log\rho)''(h)(c_{k-1}H_{k-1}\|\nabla h\|^2\\ &-\langle P_{k-1}\nabla h,\nabla h\rangle),\end{aligned}$$

where we set $N^*=\pi_{\mathbb{P}*}N$, $E_i^*=\pi_{\mathbb{P}*}E_i$.

Proof. Observe that, using the decomposition

$$\nabla h=-T-\Theta N,$$

we can write

$$\overline{\mathrm{R}}(X,\nabla h)N=-\overline{\mathrm{R}}(X,T)N+\Theta\overline{\mathrm{R}}(N,X)N$$

Moreover, using the relationship between the curvature tensor of a warped product and the curvature tensor of its base and fiber (1.36) and the fact that $T^* = 0$, we get

$$\overline{\mathrm{R}}(X,T)N = -\frac{\rho''(h)}{\rho(h)}\widehat{\Theta}X.$$

Finally, using again Equation (1.36) we find

$$\begin{aligned}\overline{\mathrm{R}}(N,X)N =& \mathrm{R}_{\mathbb{P}}(N^*,X^*)N^* - (((\log\rho)')^2(h) + (\log\rho)''(h)\Theta^2)X \\ &+ (\log\rho)''(h)\langle X,T\rangle T + (\log\rho)''(h)\Theta\langle X,T\rangle N.\end{aligned}$$

and the conclusion follows observing that

$$\begin{aligned}\widehat{\Theta}\sum_{i=1}^{n}\langle \mathrm{R}_{\mathbb{P}}(N^*,E_i^*)N^*,P_{k-1}E_i\rangle =&\widehat{\Theta}\rho^2(h)\sum_{i=1}^{n}\mu_{i,k-1}\langle \mathrm{R}_{\mathbb{P}}(N^*,E_i^*)N^*,E_i\rangle_{\mathbb{P}} \\ =&\frac{\widehat{\Theta}}{\rho^2(h)}\sum_{i=1}^{n}\mu_{i,k-1}K_{\mathbb{P}}(N^*\wedge E_i^*)\|N^*\wedge E_i\|^2\end{aligned}$$

□

We are now ready to prove the following

Lemma 3.6 (Corollary 8.5 in [**8**])**.** *Let Σ^n be a spacelike hypersurface immersed into a GRW spacetime $-I\times_\rho \mathbb{P}^n$, with angle function Θ and height function h. Let $\widehat{\Theta} = \rho(h)\Theta$. Then, for every $k = 1\dots n$ we have*

$$\begin{aligned}L_{k-1}\widehat{\Theta} =&\binom{n}{k}\rho(h)\langle\nabla h,\nabla H_k\rangle - \widehat{\Theta}(\log\rho)''(h)(\|\nabla h\|^2 c_{k-1}H_{k-1} \\ &- \langle P_{k-1}\nabla h,\nabla h\rangle) + \widehat{\Theta}\binom{n}{k}(nH_1H_k - (n-k)H_{k+1}) \\ &+ \rho'(h)c_{k-1}H_k + \frac{\widehat{\Theta}}{\rho(h)^2}\sum_{i=1}^{n}\mu_{i,k-1}K_{\mathbb{P}}(E_i^*\wedge N^*)\|E_i^*\wedge N^*\|^2.\end{aligned} \tag{3.4}$$

Proof. Since $\mathcal{T} = \rho(t)T$ is a timelike closed conformal vector field, it follows that

$$X\widehat{\Theta} = X\langle N,\rho(h)T\rangle = -\rho(h)\langle AX,T\rangle + \rho'(h)\langle N,X\rangle = \rho(h)\langle AX,\nabla h\rangle,$$

for every $X\in T\Sigma$, so that

$$\nabla\widehat{\Theta} = \rho(h)A\nabla h = A\nabla\sigma(h).$$

Therefore, using the Codazzi Equation and Equation (3.3) we obtain

$$\begin{aligned}\nabla_X\nabla\widehat{\Theta} =&(\nabla_X A)\nabla\sigma(h) + A\nabla_X\nabla\sigma(h) \\ =&\rho(h)(\nabla_{\nabla h}A)(X) + \rho(h)(\overline{\mathrm{R}}(X,\nabla h)N)^T - \rho'(h)AX + \widehat{\Theta}A^2(X).\end{aligned}$$

Then, if we denote by $\{E_1, \cdots, E_n\}$ an arbitrary local orthonormal frame, we get

$$\begin{aligned} L_{k-1}\widehat{\Theta} =& \rho(h)\mathrm{Tr}((\nabla_{\nabla h}A)\circ P_{k-1}) + \rho(h)\sum_{i=1}^{n}\left\langle \overline{\mathrm{R}}(E_i,\nabla h)N, P_{k-1}E_i\right\rangle \\ &- \rho'(h)\mathrm{Tr}(A\circ P_{k-1}) + \widehat{\Theta}\mathrm{Tr}(A^2\circ P_{k-1}) \\ =& -\binom{n}{k}\rho(h)\left\langle \nabla H_k, \nabla h\right\rangle + \rho(h)\sum_{i=1}^{n}\left\langle \overline{\mathrm{R}}(E_i,\nabla h)N, P_{k-1}E_i\right\rangle \\ &- \rho'(h)\mathrm{Tr}(A\circ P_{k-1}) + \widehat{\Theta}\mathrm{Tr}(A^2\circ P_{k-1}), \end{aligned}$$

where we used

$$\mathrm{Tr}(\nabla_X A\circ P_{k-1}) = \mathrm{Tr}(P_{k-1}\circ\nabla_X A) = -\binom{n}{k}\left\langle \nabla H_k, X\right\rangle.$$

The conclusion then follows using Proposition 1.6 and Lemma 3.5. □

Finally, if we assume that the sectional curvature of the fiber $\mathbb{P}^n$ is constant, a straightforward computation gives the following

Corollary 3.7. *Let Σ^n be a spacelike hypersurface immersed into a GRW spacetime $-I\times_\rho\mathbb{P}^n$, with angle function Θ and height function h. Assume that $\mathbb{P}^n$ has constant sectional curvature κ and let $\widehat{\Theta}=\rho(h)\Theta$. Then, for every $k=1,...,n$ we have*

$$\begin{aligned} L_{k-1}\widehat{\Theta} =& \binom{n}{k}\rho(h)\left\langle \nabla h, \nabla H_k\right\rangle + \rho'(h)c_{k-1}H_k \\ &+ \widehat{\Theta}\Big(\frac{\kappa}{\rho^2(h)} - (\log\rho)''(h)\Big)(\|\nabla h\|^2 c_{k-1}H_{k-1} - \left\langle P_{k-1}\nabla h, \nabla h\right\rangle) \\ &+ \widehat{\Theta}\binom{n}{k}(nH_1H_k - (n-k)H_{k+1}). \end{aligned} \tag{3.5}$$

3.2. Uniqueness of spacelike hypersurfaces: compact case

We are now ready to state and prove some uniqueness results for compact spacelike hypersurfaces of constant k-mean curvature, $2\le k\le n$, in GRW spacetimes. The results are very similar to the ones obtained in the Riemannian setting, but with slightly different assumptions. In fact, rather than the convexity of $(\log\rho)(h)$, in the Lorentzian case one requires the convexity of the function $-(\log\rho)(h)$. Moreover, unlike the Riemannian case, in the case of spacelike hypersurfaces in GRW spacetimes no requirements on the sign of the angle function are needed. Indeed, as seen in the preliminary discussion of this chapter, the function Θ never vanishes and its sign only depends on the chosen orientation.

We will start exhibiting some of the results proved in [**8**] for compact spacelike hypersurfaces of constant k-mean curvature, $2\le k\le n$. Before stating these theorems we recall from Proposition 3.2 in [**16**] that if a GRW spacetime admits a compact spacelike hypersurface, then the Riemannian fiber $\mathbb{P}^n$ is necessarily compact. In that case, we say that M^{n+1} is a *spatially closed* GRW spacetime. The first result obtained in [**8**] is the following.

Theorem 3.8 (Corollary 5.2 in [**8**]). *Let* $-I \times_\rho \mathbb{P}^n$ *be a spatially closed GRW spacetime with warping function satisfying* $(\log \rho)'' \leq 0$. *Let* $f : \Sigma^n \to \mathbb{R} \times_\rho \mathbb{P}^n$ *be a compact spacelike hypersurface with* $H_2 > 0$, $\frac{H_2}{H_1} = \text{constant}$. *Then* $f(\Sigma^n)$ *is a slice.*

The proof uses of the constancy of H_2/H_1 and the fact that, by compactness, there exist two points $p_{\max}$ and $p_{\min}$ where the height function h (and hence its primitive $\sigma(h)$) attains its maximum and its minimum, in order to obtain the inequalities

$$\frac{H_2}{H_1} = \frac{H_2}{H_1}(p_{\max}) \leq (\log \rho)'(h(p_{\max})),$$

$$\frac{H_2}{H_1} = \frac{H_2}{H_1}(p_{\min}) \geq (\log \rho)'(h(p_{\min})).$$

Exploiting then the convexity of $-(\log \rho)(h)$ it follows that the function $(\log \rho)'(h)$ is constantly equal to H_2/H_1. Inserting this expression into Equation (3.2) it is easy to see that the function $\sigma(h)$ is subharmonic and then one concludes by applying the classical maximum principle.

The previous theorem can be easily generalized to hypersurfaces of constant k-mean curvature, $3 \leq k \leq n$. In order to do that we need the operators L_j, $1 \leq j \leq k-1$, to be elliptic. As stated in Proposition 1.9, in order to guarantee this we need the existence of an elliptic point $p \in \Sigma$. In the compact case, the following technical lemma gives a geometric condition that implies the existence of such a point.

Lemma 3.9 (Lemma 5.3, [**8**]). *Let* $f : \Sigma^n \to -I \times_\rho \mathbb{P}^n$ *be a compact spacelike hypersurface immersed into a spatially closed GRW spacetime, and assume that* $\rho'(h)$ *does not vanish on* Σ *(equivalently,* $f(\Sigma)$ *is contained in a slab* $[t_1, t_2] \times \mathbb{P}^n$ *on which* ρ' *does not vanish).*

(i) *if* $\rho'(h) > 0$ *on* Σ *(equivalently,* $\rho' > 0$ *on* $[t_1, t_2]$*), then there exists an elliptic point on* Σ *with respect to its future-pointing Gauss map.*
(ii) *if* $\rho'(h) < 0$ *on* Σ *(equivalently,* $\rho' < 0$ *on* $[t_1, t_2]$*), then there exists an elliptic point on* Σ *with respect to its past-pointing Gauss map.*

Since Proposition 1.9 asserts that the existence of an elliptic point, jointly with the assumption $H_k > 0$, imply that any L_j, $0 \leq j \leq k-1$, is elliptic and, in particular, that any H_j is positive. Theorem 3.8 can then be generalized as follows

Theorem 3.10 (Corollary 5.4 in [**8**]). *Let* $-I \times_\rho \mathbb{P}^n$ *be a spatially closed GRW spacetime with warping function satisfying* $(\log \rho)'' \leq 0$.
Let $f : \Sigma^n \to \mathbb{R} \times_\rho \mathbb{P}^n$, $n \geq 3$, *be a compact spacelike hypersurface contained in a slab* $[t_1, t_2] \times \mathbb{P}^n$ *on which* ρ' *does not vanish. If* $H_k > 0$ *on* Σ *and some of the quotients* $\frac{H_j}{H_{j-1}}$ *is constant for some* $3 \leq j \leq k$, *then* $f(\Sigma^n)$ *is necessarily an embedded slice* $\{t_0\} \times \mathbb{P}^n$, *with* $t_0 \in (t_1, t_2)$.

The proof proceed exactly as in the previous theorem with the operator L_1 replaced by L_{j-1}, $2 \leq j \leq k$.

Following the ideas of Chapter 2 and introducing a suitable family of elliptic operators, it is not difficult to prove the next

Theorem 3.11. *Let $-I \times_\rho \mathbb{P}^n$ be a spatially closed GRW spacetime with warping function satisfying $(\log\rho)'' \leq 0$. Let $f : \Sigma^n \to \mathbb{R} \times_\rho \mathbb{P}^n$ be a compact spacelike hypersurface of constant positive 2-mean curvature H_2. Then $f(\Sigma^n)$ is a slice.*

Proof. It follows from the basic inequality $H_1^2 \geq H_2 > 0$ that $|H_1| \neq 0$. We may then choose the orientation so that $H_1 > 0$. Assume that $\Theta < 0$ with respect to the chosen orientation and recall that (3.1) gives

$$\Delta h = -(\log\rho)'(h)(n + \|\nabla h\|^2) - n\Theta H_1.$$

As already observed, the compactness of Σ implies the existence of two points $p_{\max}$ and $p_{\min}$ where the height function attains its maximum and minimum values respectively. Then $\nabla h(p_{\max}) = \nabla h(p_{\min}) = 0$ and, since $\Theta < 0$, $\Theta(p_{\max}) = -1$. Moreover

$$0 \geq \Delta h(p_{\max}) = -n(\log\rho)'(\overline{h}) + nH_1(p_{\max}) > -n(\log\rho)'(\overline{h}),$$

which together with the assumption $(\log\rho)'' \leq 0$ implies

$$(\log\rho)'(h) \geq (\log\rho)'(\overline{h}) > 0.$$

Hence the operator

$$\begin{aligned}\widetilde{\mathcal{L}}_1 =& \frac{c_1}{c_0}((\log\rho(h))')\Delta - \Theta L_1 \\ =& \mathrm{Tr}(\widetilde{\mathcal{P}}_1 \circ \mathrm{hess}),\end{aligned}$$

where

$$\widetilde{\mathcal{P}}_1 = (n-1)((\log\rho(h))')I - \Theta P_1.$$

is elliptic. A simple computation using Equation (3.2) gives

$$\widetilde{\mathcal{L}}_1\sigma(h) = -c_1\rho(h)(((\log\rho)'(h))^2 - \Theta^2 H_2).$$

Proceeding as in Theorem 2.13 we obtain the inequalities

$$(\log\rho)'(\underline{h}) \geq H_2^{1/2} \geq (\log\rho)'(\overline{h}).$$

The convexity of $-(\log\rho)(h)$ implies then that $(\log\rho)'(\underline{h}) = (\log\rho)'(\overline{h})$ and hence $H_2^{1/2}$ is constantly equal to $(\log\rho)(h)$. Then

$$\begin{aligned}\widetilde{\mathcal{L}}_1\sigma(h) &= -c_1\rho(h)(((\log\rho)'(h))^2 - \Theta^2 H_2) \\ &= -c_1\rho(h)H_2(1-\Theta^2) \\ &\geq 0\end{aligned}$$

and we reach the desired conclusion using the classical maximum principle. On the other hand, if $\Theta > 0$ with respect to the chosen orientation, then $\Theta(p_{\min}) = 1$ and

$$0 \leq \Delta h(p_{\min}) = -n(\log\rho)'(\underline{h}) - nH_1(p_{\min}) < -n(\log\rho)'(\underline{h}),$$

and we deduce that

$$(\log\rho)'(h) \leq (\log\rho)'(\underline{h}) < 0.$$

Then the proof proceeds as above using the elliptic operator $-\widetilde{\mathcal{L}}_1$ instead of $\widetilde{\mathcal{L}}_1$. □

In the case $3 \leq k \leq n$, we introduce the family of operators

$$\begin{aligned}\widetilde{\mathcal{L}}_{k-1} =& \mathrm{Tr}\Big(\Big[\sum_{i=0}^{k-1}(-1)^i\frac{c_{k-1}}{c_i}((\log\rho)'(h))^{k-1-i}\Theta^i P_i\Big]\circ \mathrm{hess}\Big)\\ =& \sum_{i=0}^{k-1}(-1)^i\frac{c_{k-1}}{c_i}((\log\rho)'(h))^{k-1-i}\Theta^i L_i.\end{aligned}$$

We claim that

$$\widetilde{\mathcal{L}}_{k-1}\sigma(h) = -c_{k-1}\rho(h)(((\log\rho)'(h))^k + (-1)^{k-1}\Theta^k H_k). \tag{3.6}$$

We can prove the claim by induction. It is straightforward to prove that Equation (3.13) holds true for $k = 1$. Hence, assume that it holds for $k-2$ and observe that

$$\begin{aligned}\widetilde{\mathcal{L}}_{k-1}\sigma(h) =& \frac{c_{k-1}}{c_{k-2}}(\log\rho)'(h)\sum_{i=0}^{k-2}(-1)^i\frac{c_{k-2}}{c_i}((\log\rho)'(h))^{k-2-i}\Theta^i L_i\sigma(h)\\ &+ (-1)^{k-1}\Theta^{k-1}L_{k-1}\sigma(h)\\ =& -c_{k-1}\rho(h)(((\log\rho)'(h))^k - (-1)^{k-1}(\log\rho)'(h)\Theta^{k-1}H_{k-1}\\ &+ (-1)^{k-1}(\log\rho)'(h)\Theta^{k-1}H_{k-1} + (-1)^{k-1}\Theta^k H_k)\\ =& -c_{k-1}\rho(h)(((\log\rho)'(h))^k + (-1)^{k-1}\Theta^k H_k).\end{aligned}$$

Then, reasoning as in the previous theorem, we can establish the next

Theorem 3.12 (Theorem 3.1 in [**14**]). *Let $-I \times_\rho \mathbb{P}^n$ be a spatially closed GRW spacetime with warping function satisfying $(\log\rho)'' \leq 0$. Let $f : \Sigma^n \to \mathbb{R} \times_\rho \mathbb{P}^n$, $n \geq 3$, be a compact spacelike hypersurface of constant k-mean curvature H_k, $3 \leq k \leq n$, contained in a slab on which ρ' does not vanish. Then $f(\Sigma^n)$ is a slice.*

The proof proceeds exactly as in Theorem 3.11 once one observes that, since $\rho'(h)$ does not vanish, there exists an elliptic point on Σ and hence, by Proposition 1.9, all the L_j's are elliptic, $1 \leq j \leq k-1$. Thus the operator $\widetilde{\mathcal{L}}_{k-1}$ is elliptic and the conclusion follows as in the previous theorem.

Notice that all the previous theorems have been obtained assuming the concavity condition

$$(\log\rho)'' \leq 0.$$

As observed by Montiel in [**50**], this is a reasonably weak condition on the warping function which is sufficient to obtain the desired uniqueness results. Recall that a spacetime obeys the *timelike convergence condition* (TCC) if its Ricci curvature is non-negative on timelike directions. A direct computation using Equation (1.36) implies that a GRW spacetime $-I \times_\rho \mathbb{P}^n$ obeys the TCC if and only if

$$\mathrm{Ric}_{\mathbb{P}} \geq (n-1)\sup_I((\log\rho)''\rho^2)\,\langle,\rangle_{\mathbb{P}}\,, \tag{3.7}$$

and

$$\rho'' \leq 0, \tag{3.8}$$

where $\mathrm{Ric}_{\mathbb{P}}$ and $\langle , \rangle_{\mathbb{P}}$ are respectively the Ricci and metric tensors of the Riemannian manifold $\mathbb{P}$.

As observed in [**50**], any of the two conditions above implies separately that spacelike slices are the only compact spacelike hypersurfaces of constant mean curvature in $-I \times_\rho \mathbb{P}^n$. In particular, the sole hypothesis (3.8) suffices to guarantee uniqueness, without any other restriction on the curvature of $\mathbb{P}$. Even more, the more general condition $(\log \rho)'' \leq 0$ is sufficient to obtain the uniqueness. Another geometric condition that is relevant from a physical point of view is the so-called *null convergence condition.* Recall that a spacetime obeys the *null convergence condition* if its Ricci curvature is non-negative on null (lightlike) directions (see [**39**] for more details on this physical conditions).
Using again formula (1.36), it is easy to see that a GRW spacetime obeys the null convergence condition if and only if (3.7) holds. Using this condition Montiel has obtained in [**50**] the following uniqueness result for hypersurfaces of constant mean curvature.

Theorem 3.13 (Theorem 6 in [**50**])**.** *Let $-I \times_\rho \mathbb{P}^n$ be a spatially closed GRW spacetime obeying the null convergence condition. Then the only compact spacelike hypersurfaces immersed in $-I \times_\rho \mathbb{P}^n$ with constant mean curvature are the embedded slices $\{t\} \times \mathbb{P}^n$, $t \in I$, unless in the case where $-I \times_\rho \mathbb{P}^n$ is isometric to the de Sitter spacetime in a neighbourhood of Σ, which must be a round umbilical hypersphere. The latter case cannot occur if we assume that the inequality* (3.7) *is strict.*

The main tools that Montiel used to prove this theorem are the Minkowski integral formulas. However this method can be applied neither in the case of constant higher mean curvature, unless one assumes that the fibre $\mathbb{P}^n$ has constant sectional curvature, nor in the complete non-compact case. Nevertheless, as observed by Alías and Colares in [**8**], the same result can be obtained using the formulas for the Laplacian acting on the function σ and on the function $\widehat{\Theta}$ and applying the classical maximum principle. This observations allowed the two authors to extend Montiel's theorem to compact hypersurfaces of constant higher order mean curvature. In order to do that, one need to impose on $-I \times_\rho \mathbb{P}^n$ the stronger condition

$$K_{\mathbb{P}} \geq \sup_I \{\rho^2 (\log \rho)''\}. \tag{3.9}$$

We will refer to (3.9) as the *strong null convergence condition.* Then we can state the following

Theorem 3.14 (Theorem 9.2 in [**8**])**.** *Let $-I \times_\rho \mathbb{P}^n$, $n \geq 3$, be a spatially closed GRW spacetime and assume that the sectional curvature of $\mathbb{P}^n$ satisfy the strong null convergence condition. Let $f : \Sigma^n \to -I \times_\rho \mathbb{P}^n$ be a compact hypersurface of constant k-mean curvature, $2 \leq k \leq n$, contained in a slab on wich ρ' does not vanish. Then $f(\Sigma^n)$ is totally umbilical. Moreover, Σ must be a slice $\{t_0\} \times \mathbb{P}^n$ (necessarily with $\rho'(t_0) \neq 0$), unless in the case* *where $-I \times_\rho \mathbb{P}^n$ has positive constant sectional curvature and Σ is a round umbilical hypersphere. The latter case cannot occur if we assume that the inequality in* (3.9) *is strict.*

PROOF. Let us choose on Σ the future-pointing Gauss map and assume that $\rho'(h) > 0$ on Σ. Then, Lemma 3.9 guarantees the existence of an elliptic point $p_0 \in \Sigma$. In particular, since H_k is constant, it has to be positive. Morever, using Garding inequalities we get

$$H_1 \geq H_2^{1/2} \geq \cdots \geq H_k^{1/k} > 0, \tag{3.10}$$

with equality only at umbilical points. Consider now the function

$$\phi = H_k^{1/k}\sigma(h) + \widehat{\Theta}.$$

Using Equations (3.2) and (3.4), we find

$$\begin{aligned} L_{k-1}\phi =& c_{k-1}\rho'(h)(H_k - H_k^{1/k}H_{k-1}) \\ &+ \binom{n}{k}\widehat{\Theta}(nH_1H_k - (n-k)H_{k+1} - kH_k^{(k+1)/k}) \\ &+ \frac{\widehat{\Theta}}{\rho(h)^2}\sum_{i=1}^n \mu_{i,k-1}K_{\mathbb{P}}(E_i^* \wedge N^*)\|E_i^* \wedge N^*\|^2 \\ &- (\log\rho)''(h)(c_{k-1}H_{k-1}\|\nabla h\|^2 - \langle P_{k-1}\nabla h, \nabla h\rangle). \end{aligned} \tag{3.11}$$

Notice that, by Equation (3.10), the first and the second terms in Equation (3.11) above are non-positive, with equality only at umbilical points. Moreover, since P_{k-1} is elliptic, its eigenvalues are strictly positive and, taking into account the strong null convergence condition

$$\mu_{i,k-1}K_{\mathbb{P}}(N^* \wedge E_i^*)\|N^* \wedge E_i^*\|^2 \geq \mu_{i,k-1}\gamma\|N^* \wedge E_i^*\|^2,$$

where $\gamma = \sup_I\{(\log\rho)''\rho^2\}$. Taking into account the decompositions

$$N = N^* - \Theta T, \qquad E_i = E_i^* - \langle E_i, T\rangle\, T, \qquad T = -\nabla h - \Theta N,$$

a straightforward computation gives

$$\|N^* \wedge E_i^*\|^2 = \|\nabla h\|^2 - \langle E_i, \nabla h\rangle^2\,.$$

Therefore

$$\begin{aligned} \sum_{i=1}^n \mu_{i,k-1}K_{\mathbb{P}}(N^* \wedge E_i^*)\|N^* \wedge E_i^*\|^2 \geq& \gamma(c_{k-1}H_{k-1}\|\nabla h\|^2 - \langle P_{k-1}\nabla h, \nabla h\rangle) \\ \geq& \rho^2(h)(\log\rho)''(h)(c_{k-1}H_{k-1}\|\nabla h\|^2 \\ &- \langle P_{k-1}\nabla h, \nabla h\rangle). \end{aligned}$$

Summarizing, $L_{k-1}\phi \leq 0$. Since L_{k-1} is an elliptic operator and Σ is compact, we conclude by the maximum principle that ϕ has to be constant. Hence $L_{k-1}\phi = 0$ and all the terms in (3.11) vanish identically. In particular, Σ is a totally umbilical hypersurface and, since H_k is a positive constant, all the higher order mean curvatures are constant. In particular, the mean curvature is constant and the conclusion follows applying Montiel's theorem. □

3.3. Uniqueness of spacelike hypersurfaces: complete non-compact case

Our aim now is to give a characterization of spacelike slices in the complete non-compact case by means of the Omori-Yau maximum principle for semi-elliptic operators of the form (1.11), Recall that in order to guarantee the the validity of the generalized Omori-Yau maximum principle it suffices to assume that

$$K_\Sigma(\nabla r, X) \geq -G(r), \tag{3.12}$$

where $r(\cdot)$ is the distance function from a reference point $o \in \Sigma$ and X is any vector field in $T\Sigma$. Moreover, in order to guarantee (3.12), recall the Gauss equation

$$R(X,Y)Z = (\overline{R}(X,Y)Z)^T - \langle AX, Z\rangle AY + \langle AY, Z\rangle AX,$$

for all vector fields X, Y, Z tangent to Σ. Without loss on generality we will assume that X is a unitary vector field orthogonal to ∇r. Then

$$\begin{aligned} K_\Sigma(\nabla r, X) =& \overline{K}(\nabla r, X) - \langle AX, X\rangle \langle A\nabla r, \nabla r\rangle + \langle AX, \nabla r\rangle^2 \\ \geq& \overline{K}(\nabla r, X) - \langle AX, X\rangle \langle A\nabla r, \nabla r\rangle . \end{aligned}$$

Notice that

$$\begin{aligned} \Big| \langle A\nabla r, \nabla r\rangle \Big| \leq& \|A\nabla r\| \|\nabla r\| \\ \leq& \|A\| \|\nabla r\|^2 \\ \leq& nH_1 \|\nabla r\|^2. \end{aligned}$$

Analogously

$$\Big| \langle AX, X\rangle \Big| \leq nH_1 \|X\|^2,$$

hence

$$K_\Sigma(\nabla r, X) \geq \overline{K}(\nabla r, X) - n^2 H_1^2.$$

Thus, if we assume that $\sup_\Sigma |H_1| < +\infty$, the last term in the previous inequality is bounded from below. Moreover, using Equation (1.36) we get

$$\begin{aligned} \overline{K}(\nabla r, X) =& \frac{1}{\rho^2(h)} K_{\mathbb{P}}(\pi_{\mathbb{P}*}\nabla r, \pi_{\mathbb{P}*}X) \|\pi_{\mathbb{P}*}\nabla r \wedge \pi_{\mathbb{P}*}X\|^2 \\ &+ ((\log\rho)')^2(h)(\langle \nabla r, \nabla r\rangle \langle X, X\rangle - \langle \nabla r, X\rangle^2) \\ &+ (\log\rho)''(h) \langle \nabla r, \nabla h\rangle (\langle X, \nabla h\rangle \langle \nabla r, X\rangle - \langle \nabla r, \nabla h\rangle \langle X, X\rangle) \\ &- (\log\rho)''(h)(\langle \nabla r, \nabla r\rangle \langle X, \nabla h\rangle - \langle \nabla r, \nabla h\rangle \langle X, \nabla r\rangle) \langle X, \nabla h\rangle \\ \geq& \frac{1}{\rho^2(h)} K_{\mathbb{P}}(\pi_{\mathbb{P}*}\nabla r, \pi_{\mathbb{P}*}X) \|\pi_{\mathbb{P}*}\nabla r \wedge \pi_{\mathbb{P}*}X\|^2 \\ &+ (\log\rho)''(h)(2\langle \nabla r, \nabla h\rangle \langle X, \nabla h\rangle \langle \nabla r, X\rangle \\ &- \langle \nabla r, \nabla h\rangle^2 - \langle X, \nabla h\rangle^2). \end{aligned}$$

Since by the Cauchy-Schwarz inequality

$$\langle X, \nabla r\rangle \leq |\langle X, \nabla r\rangle| \leq 1$$

we find

$$\begin{aligned}
2\langle\nabla r,\nabla h\rangle\langle X,\nabla h\rangle\langle\nabla r,X\rangle-\langle\nabla r,\nabla h\rangle^2-\langle X,\nabla h\rangle^2 \\
&\leq 2|\langle\nabla r,\nabla h\rangle\langle X,\nabla h\rangle|-\langle X,\nabla h\rangle^2-\langle\nabla r,\nabla h\rangle \\
&=-\left(|\langle X,\nabla h\rangle|-|\langle\nabla r,\nabla h\rangle|\right)^2 \\
&\leq 0.
\end{aligned}$$

Since we will assume that $(\log\rho)''<0$ and that h is a bounded function, if we suppose that the sectional curvature of $\mathbb{P}^n$ is bounded from below, we find that condition (3.12) is met. Summarizing:

Corollary 3.15. *Let $\mathbb{P}^n$ be a Riemannian manifold with sectional curvature bounded from below, and let $f:\Sigma^n\to -I\times_\rho\mathbb{P}^n$ be a complete spacelike hypersurface contained in a slab and satisfying $\sup_\Sigma|H_1|<+\infty$. Then the sectional curvature of Σ is bounded from below and the Omori-Yau maximum principle holds on Σ for any semi-elliptic operator of the form* (1.11).

Our first result is the extension of Theorem 3.8 to the complete case

Theorem 3.16 (Theorem 4.5 in [**13**])**.** *Let $-I\times_\rho\mathbb{P}^n$ be a GRW spacetime whose warping function satisfies $(\log\rho)''\leq 0$, with equality only at isolated points, and suppose that $\mathbb{P}^n$ has sectional curvature bounded from below. Let $f:\Sigma^n\to -I\times_\rho\mathbb{P}^n$ be a complete spacelike hypersurface contained in a slab with $H_k>0$, for some $2\leq k\leq n$, and $\frac{H_{i+1}}{H_i}=$ constant for some $1\leq i\leq k-1$. Assume that $\sup_\Sigma H_1<+\infty$ and, for $k\geq 3$, that there exists an elliptic point in Σ. Then, $f(\Sigma^n)$ is a slice.*

Proof. We consider first the case $k=2$. By the basic inequality $H_1^2\geq H_2>0$, it follows that we the mean curvature H_1 never vanishes and we can orient the hypersurface so that $H_1>0$ on Σ. We define the operator $\widehat{L}_1=\mathrm{Tr}(\widehat{P}_1\circ\mathrm{hess})$ with $\widehat{P}_1=\frac{1}{H_1}P_1$. Note that $\mathrm{Tr}(\widehat{P}_1)=c_1$ and therefore, by Corollary 3.15, we can apply the Omori-Yau maximum principle for the operator $\widehat{L}_1$. We let $\{p_j\}$ and $\{q_j\}$ be two sequences such that

$$\begin{aligned}
&(i)\quad \lim_{j\to+\infty}\sigma(h(p_j))=\sup_\Sigma\sigma(h),\\
&(ii)\quad \|\nabla\sigma(h)(p_j)\|=\rho(h(p_j))\|\nabla h(p_j)\|<\frac{1}{j},\\
&(iii)\quad \widehat{L}_1\sigma(h)(p_j)<\frac{1}{j},
\end{aligned}$$

and

$$\begin{aligned}
&(i)\quad \lim_{j\to+\infty}\sigma(h(q_j))=\inf_\Sigma\sigma(h),\\
&(ii)\quad \|\nabla\sigma(h)(q_j)\|=\rho(h(q_j))\|\nabla h(q_j)\|<\frac{1}{j},\\
&(iii)\quad \widehat{L}_1\sigma(h)(q_j)>-\frac{1}{j},
\end{aligned}$$

Observe that condition (i) implies that $\lim_{j\to+\infty}h(p_j)=h^*=\sup_\Sigma h$ and $\lim_{j\to+\infty}h(q_j)=h_*=\inf_\Sigma h$, because $\sigma(t)$ is strictly increasing. Thus by condition (ii) we also have $\lim_{j\to+\infty}\|\nabla h(p_j)\| = \lim_{j\to+\infty}\|\nabla h(q_j)\| = 0$, and $\lim_{j\to+\infty}\Theta(p_j) = \lim_{j\to+\infty}\Theta(q_j) = \operatorname{sgn}\Theta$. Therefore, using the equation

$$\widehat{L}_1\sigma(h) = -c_1\left(\rho'(h) + \Theta\rho(h)\frac{H_2}{H_1}\right)$$

and taking $j\to+\infty$ we obtain

$$(\log\rho)'(h_*) \le -\operatorname{sgn}\Theta\frac{H_2}{H_1} \le (\log\rho)'(h^*).$$

The assumption $(\log\rho)'' \le 0$ with equality only at isolated points, allows us to conclude that h must be constant.

For the general case $k\ge 3$, first observe that the existence of an elliptic point together with $H_k > 0$ imply that $H_i > 0$ and the operators P_i are positive definite for all $1 \le i \le k-1$. Choose i as in the statement of the Theorem, so that H_{i+1}/H_i is constant and consider the operator $\widehat{L}_i = \operatorname{Tr}(\widehat{P}_i\circ \operatorname{hess})$ with $\widehat{P}_i = \frac{1}{H_i}P_i$. Note that $\operatorname{Tr}(\widehat{P}_i) = c_i$ and therefore, by Corollary 3.15, we can apply the Omori-Yau maximum principle for the operator $\widehat{L}_i$. We conclude then as in the case $k=2$ with the aid of the equation

$$\widehat{L}_i\sigma(h) = -c_i\left(\rho'(h) + \Theta\rho(h)\frac{H_{i+1}}{H_i}\right).$$

□

Under the same hypotheses guaranteeing the validity of the Omori-Yau maximum principle we obtain the following result in case Σ has constant k-mean curvature, $2\le k\le n$, extending Theorems 3.11 and 3.12.

Theorem 3.17 (Theorem 4.6 in [**14**]). *Let $-I\times_\rho\mathbb{P}^n$ be a GRW spacetime whose warping function satisfies $(\log\rho)''\le 0$, with equality only at isolated points, and suppose that $\mathbb{P}^n$ has sectional curvature bounded from below. Let $f:\Sigma^n\to -I\times_\rho\mathbb{P}^n$ be a complete spacelike hypersurface contained in a slab and assume that either*

(i) *H_2 is a positive constant, or*
(ii) *H_k is constant (with $k\ge 3$) and there exists an elliptic point in Σ.*

If $\sup_\Sigma |H_1| < +\infty$, then Σ is a slice.

Proof. Using Proposition 1.8 in case (i) and 1.9 in case (ii), it is easy to see that each P_i, $1\le i\le k-1$, is positive definite. Hence, in particular, P_1 is positive definite and $H_1>0$ for an appropriate choice of the Gauss map. Assume first that $\Theta<0$ with respect to this choice and let us show that $\rho'(h)>0$. To do that, we apply the Omori-Yau maximum principle to the Laplacian to assure the existence of a sequence $\{p_j\}$ with the following properties

$$(i) \quad \lim_{j\to+\infty} h(p_j) = h^*,$$

$$(ii) \quad \|\nabla h(p_j)\| < \frac{1}{j},$$

$$(iii) \quad \Delta h(p_j) < \frac{1}{j}.$$

Therefore, making $j \to +\infty$ in the following inequality

$$\frac{1}{j} > \Delta h(p_j) = -(\log \rho)'(h(p_j))(n + \|\nabla h(p_j)\|^2) - n\Theta(p_j)H_1(p_j)$$

we get

$$(\log \rho)'(h^*) + \liminf_{j\to+\infty} H_1(p_j) \geq 0.$$

Since

$$\liminf_{j\to+\infty} H_1(p_j) \geq \sqrt{H_2} > 0,$$

and $(\log \rho)'(h^*) \leq (\log \rho)'(h)$, it must be $(\log \rho)'(h) > 0$, which means $\rho'(h) > 0$ on Σ. On the other hand, if $\Theta > 0$ with respect to the chosen orientation, then $\rho'(h) < 0$. Indeed, we can apply the Omori-Yau maximum principle to the Laplacian to assure the existence of a sequence $\{q_j\}$ with the following properties

$$(i) \quad \lim_{j\to+\infty} h(q_j) = \inf_\Sigma h = h_*,$$

$$(ii) \quad \|\nabla h(q_j)\| < \frac{1}{j},$$

$$(iii) \quad \Delta h(q_j) > -\frac{1}{j}.$$

Making $j \to +\infty$ in the following inequality

$$-\frac{1}{j} < \Delta h(q_j) = -(\log \rho)'(h(q_j))(n + \|\nabla h(q_j)\|^2) - n\Theta(q_j)H_1(q_j)$$

and reasoning exactly as before we conclude that $\rho'(h) < 0$ on Σ.

Consider now the operator

$$\begin{aligned}
\widehat{\mathcal{L}}_1 &= -\frac{1}{\Theta}\frac{c_1}{c_0}(\log \rho)'(h)\Delta + L_1 \\
&= \frac{c_1}{c_0}\Big|\frac{(\log \rho)'(h)}{\Theta}\Big|\Delta + L_1 \\
&= \mathrm{Tr}(\widehat{\mathcal{P}}_1 \circ \mathrm{hess}),
\end{aligned}$$

where

$$\widehat{\mathcal{P}}_1 = (n-1)\Big|\frac{(\log \rho)'(h)}{\Theta}\Big| I + P_1$$

and $c_i = (n-i+1)\binom{n}{i-1} = i\binom{n}{i}$. Since $\Theta(\log \rho)'(h) > 0$, $\widehat{\mathcal{P}}_1$ is positive definite. Moreover, since $|1/\Theta| \leq 1$,

$$\mathrm{Tr}\widehat{\mathcal{P}}_1 = c_1\Big(\Big|\frac{(\log \rho)'(h)}{\Theta}\Big| + H_1\Big) \leq c_1(|(\log \rho)'(h)| + \sup_\Sigma H_1) < +\infty,$$

where in the last inequality we used the fact that

$$|(\log\rho)'(h)| \le \begin{cases} (\log\rho)'(h_*) & \text{if } (\log\rho)'(h) > 0, \\ -(\log\rho)'(h^*) & \text{if } (\log\rho)'(h) < 0. \end{cases}$$

Hence $\widehat{\mathcal{L}}_1$ is an elliptic operator and the trace of $\widehat{\mathcal{P}}_1$ is bounded from above. By Corollary 3.15 we can then apply the Omori-Yau maximum principle for $\widehat{\mathcal{L}}_1$. Since $h^* < +\infty$ there exists a sequence $\{p_j\} \subset \Sigma$ such that

$$\begin{aligned} &\lim_{j\to+\infty} (\sigma\circ h)(p_j) = (\sigma\circ h)^* = \sigma(h^*), \\ &\|\nabla(\sigma\circ h)(p_j)\| = \rho(h(p_j))\|\nabla h(p_j)\| < \frac{1}{j}, \\ &\widehat{\mathcal{L}}_1(\sigma\circ h)(p_j) < \frac{1}{j}. \end{aligned}$$

Moreover, by

$$\widehat{\mathcal{L}}_1\sigma(h) = \frac{c_1}{\Theta}\rho(h)((\log\rho)'(h)^2 - \Theta^2 H_2),$$

taking the limit for $j\to+\infty$ we find

$$0 \ge \operatorname{sgn}\Theta((\log\rho)'(h^*)^2 - H_2).$$

On the other hand, since h is bounded from below, we can find a sequence $\{q_j\} \subset \Sigma$ such that

$$\begin{aligned} &\lim_{j\to+\infty} (\sigma\circ h)(q_j) = (\sigma\circ h)_* = \sigma(h_*), \\ &\|\nabla(\sigma\circ h)(q_j)\| = \rho(h(q_j))\|\nabla h(q_j)\| < \frac{1}{j}, \\ &\widehat{\mathcal{L}}_1(\sigma\circ h)(q_j) > -\frac{1}{j} \end{aligned}$$

Hence, proceeding as above we find

$$0 \le \operatorname{sgn}\Theta((\log\rho)'(h_*)^2 - H_2).$$

Thus $(\log\rho)'(h_*) \le (\log\rho)'(h^*)$. Since $(\log\rho)'(h)$ is a decreasing function we conclude that $h^* = h_*$ and hence h must be constant.
For the general case $k \ge 3$, since, as we have observed, each P_i is positive definite, $1 \le i \le k-1$, each H_i is positive and, reasoning as in the case $k = 2$, one can see that Θ and $\rho'(h)$ have opposite sign with respect to a chosen orientation. Furthermore, since, by the Newton inequalities

$$H_i \le H_1^i < +\infty,$$

each H_i is bounded from above. Consider the operator

$$\begin{aligned} \widehat{\mathcal{L}}_{k-1} =& \operatorname{Tr}\Big(\Big[\sum_{i=0}^{k-1} \frac{c_{k-1}}{c_i}\Big|\frac{(\log\rho)'(h)}{\Theta}\Big|^{k-1-i} P_i\Big] \circ \operatorname{hess}\Big) \\ =& \sum_{i=0}^{k-1} \frac{c_{k-1}}{c_i}\Big|\frac{(\log\rho)'(h)}{\Theta}\Big|^{k-1-i} L_i \\ =& \operatorname{Tr}(\widehat{P}_{k-1} \circ \operatorname{hess}). \end{aligned}$$

Since $\widehat{\mathcal{L}}_{k-1}$ is a positive linear combination of the L_i's, it is elliptic. Moreover

$$\begin{aligned}\operatorname{Tr}(\widehat{\mathcal{P}}_{k-1}) =& c_{k-1}\sum_{i=0}^{k-1}\Big|\frac{(\log\rho)'(h)}{\Theta}\Big|^{k-1-i}H_i\\ \leq& c_{k-1}\sum_{i=0}^{k-1}|(\log\rho)'(h)|^{k-1-i}H_1^i\\ \leq& c_{k-1}\sum_{i=0}^{k-1}|(\log\rho)'(h)|^{k-1-i}\sup_\Sigma H_1^i\\ <& +\infty.\end{aligned}$$

It is easy to prove by induction on k that

$$\widehat{\mathcal{L}}_{k-1}\sigma(h) = c_{k-1}\frac{\operatorname{sgn}\Theta}{|\Theta|^{k-1}}\rho(h)(|(\log\rho)'(h)|^k - |\Theta|^k H_k). \tag{3.13}$$

We can then apply the Omori-Yau maximum principle to the operator $\widehat{\mathcal{L}}_{k-1}$. Since $h^* < +\infty$ there exists a sequence $\{p_j\}\subset\Sigma$ such that

$$\begin{aligned}&\lim_{j\to+\infty}(\sigma\circ h)(p_j) = (\sigma\circ h)^* = \sigma(h^*),\\ &\|\nabla(\sigma\circ h)(p_j)\| = \rho(h(p_j))\|\nabla h(p_j)\| < \frac{1}{j},\\ &\widehat{\mathcal{L}}_{k-1}(\sigma\circ h)(p_j) < \frac{1}{j}.\end{aligned}$$

Hence, taking the limit in (3.13) for $j\to+\infty$ and observing that $|\Theta|(p_j)\to 1$ as $j\to+\infty$, we find

$$0 \geq \operatorname{sgn}\Theta(|(\log\rho)'(h^*)|^k - H_k).$$

On the other hand, since h is bounded from below, we can find a sequence $\{q_j\}\subset\Sigma$ such that

$$\begin{aligned}&\lim_{j\to+\infty}(\sigma\circ h)(q_j) = (\sigma\circ h)_* = \sigma(h_*),\\ &\|\nabla(\sigma\circ h)(q_j)\| = \rho(h(q_j))\|\nabla h(q_j)\| < \frac{1}{j},\\ &\widehat{\mathcal{L}}_{k-1}(\sigma\circ h)(q_j) > -\frac{1}{j}\end{aligned}$$

Hence, proceeding as above,

$$0 \leq \operatorname{sgn}\Theta(|(\log\rho)'(h_*)|^k - H_k).$$

Thus $(\log\rho)'(h_*) \leq (\log\rho)'(h^*)$ and we conclude as in case $k=2$. □

In the rest of the section we prove some uniqueness results for complete spacelike hypersurfaces of constant mean and higher order mean curvatures obeying the null convergence condition. For what concerns the mean curvature, it is clear from the proof (see Theorem 9.1 in [**8**]) that the generalization is straightforward if the hypersurface is parabolic, which is always true in the compact case. We can then state the next

Theorem 3.18 (Therorem 5.1 in [**13**])**.** *Let $-I \times_\rho \mathbb{P}^n$ be a GRW spacetime obeying the strict null convergence condition, that is, satisfying*

$$\mathrm{Ric}_{\mathbb{P}} > (n-1)\sup_I((\log\rho)''\rho^2)\,\langle,\rangle_{\mathbb{P}}\,. \tag{3.14}$$

Let $f : \Sigma^n \to -I \times_\rho \mathbb{P}^n$ be a complete spacelike hypersurface of constant mean curvature contained in a slab. Suppose that Σ^n is parabolic and that $\sup_\Sigma |\Theta| < +\infty$*. Then $f(\Sigma^n)$ is a slice.*

Proof. Let us choose on Σ the orientation such that $\Theta < 0$ and consider the function $\phi = H_1\sigma(h) + \widehat{\Theta}$. Since the mean curvature is constant, by Equations (3.2) and (3.4) we have

$$\Delta\phi = \widehat{\Theta}(n(n-1)(H_1^2 - H_2) + \mathrm{Ric}_{\mathbb{P}}(N^*, N^*) - (n-1)(\log\rho)''(h)\|\nabla h\|^2).$$

Reasoning as in the proof of Theorem 9.1 in [**8**], it follows from the hypotheses that $\Delta\phi \le 0$ on Σ. Since $\sup_\Sigma |\Theta| < +\infty$ and Σ is contained in a slab, ϕ is bounded from below. Moreover, Σ being parabolic implies that ϕ must be constant and $\Delta\phi = 0$. In particular

$$\mathrm{Ric}_{\mathbb{P}}(N^*, N^*) - (n-1)(\log\rho)''(h)\|\nabla h\|^2 = 0.$$

Observe that

$$\|\nabla h\|^2 = \|N^*\|^2 = \rho^2(h)\,\langle N^*, N^*\rangle_{\mathbb{P}}$$

and therefore the validity of (3.14) implies $\nabla h = 0$. □

Keeping in mind Remark 2.29, it is straightforward to obtain the following

Corollary 3.19. *Let $-I \times_\rho \mathbb{P}^n$ be a GRW spacetime obeying the strict null convergence condition* (3.14) *and let $f : \Sigma^n \to -I \times_\rho \mathbb{P}^n$ be a complete spacelike hypersurface of constant mean curvature contained in a slab and satisfying*

$$\left(\mathrm{vol}\partial B_t\right)^{-1} \notin L^1(+\infty).$$

Moreover, suppose that $\sup_\Sigma |\Theta| < +\infty$*. Then $f(\Sigma^n)$ is a slice.*

To extend Theorem 3.14 to the complete case we need to restrict ourselves to spacelike hypersurfaces immersed in Robertson-Walker spacetimes (RW). We recall that a RW spacetime is a GRW spacetime whose fibre $\mathbb{P}^n$ is a Riemannian spaceform. These spacetimes are very important from a physical point of view. Indeed they are the only spatially homogeneous spacetime and are taken as realistic models for the universe. Notice that, in case $-I \times_\rho \mathbb{P}^n$ is a RW spacetime and we denote by κ the constant sectional curvature of the fiber $\mathbb{P}^n$, the null convergence condition reads as

$$\kappa \ge \sup_I\{\rho^2(\log\rho)''\}.$$

Consider now the operator

$$\widehat{\mathfrak{L}}_{k-1} f = \mathrm{div}(P_{k-1}\nabla f).$$

It follows by Proposition 1.7 that

$$\begin{aligned}\widehat{\mathfrak{L}}_{k-1}u &= \langle \mathrm{div}P_{k-1}, \nabla u\rangle + L_{k-1}u\\ &= \sum_{j=0}^{k-2}\sum_{i=1}^{n}(-1)^{k-2-j}\left\langle \overline{\mathrm{R}}(E_i, A^{k-2-j}\nabla u)N, P_jE_i\right\rangle + L_{k-1}u\end{aligned}$$

Using the Gauss equation, the expression for the Riemannian tensor for a manifold of constant sectional curvature and Equation (1.36), it follows that, whenever $\mathbb{P}^n$ has constant sectional curvature κ, the operator $\widehat{\mathfrak{L}}_{k-1}$ has the expression

$$\widehat{\mathfrak{L}}_{k-1}u = (n-k+1)\Theta\Big(\frac{\kappa}{\rho^2(h)} - (\log\rho)''(h)\Big)\langle P_{k-2}\nabla h, \nabla u\rangle + L_{k-1}u. \tag{3.15}$$

As in the Riemannian case, we say that the hypersurface $\Sigma^n \hookrightarrow -I\times_\rho \mathbb{P}^n$ is $\widehat{\mathfrak{L}}_{k-1}$-parabolic if the only bounded above C^1 solutions of the differential inequality

$$\widehat{\mathfrak{L}}_{k-1}u \geq 0$$

are constant. Equivalently, we can say that Σ is $\widehat{\mathfrak{L}}_{k-1}$-parabolic if the only bounded below C^1 solutions of

$$\widehat{\mathfrak{L}}_{k-1}v \leq 0$$

are constant.

It is not difficult to see that Theorem 2.33 applies in this case and we conclude again that (2.21) is a sufficient condition for $\widehat{\mathfrak{L}}_{k-1}$-parabolicity.

We are now able to establish the following result, which extends Theorem 3.14 to the complete case, at least when $\mathbb{P}^n$ has constant sectional curvature.

Theorem 3.20 (Theorem 5.6 in [**14**])**.** *Let $-I\times_\rho\mathbb{P}^n$ be a RW spacetime and denote by κ the constant sectional curvature of $\mathbb{P}^n$. Let $f:\Sigma^n\to -I\times_\rho\mathbb{P}^n$ be a complete spacelike hypersurface of constant k-mean curvature, $k\geq 2$, contained in a slab $[t_1,t_2]\times\mathbb{P}^n$ on which ρ' does not change sign and*

$$\kappa > \max_{[t_1,t_2]}((\log\rho)''\rho^2). \tag{3.16}$$

Suppose that Σ^n satisfies condition (2.21) *and either*

(i) *$k=2$ and $H_2>0$ or*
(ii) *$k\geq 3$ and there exists an elliptic point $p\in\Sigma^n$.*

If $\sup_\Sigma|H_1|<+\infty$ and $\sup_\Sigma|\Theta|<+\infty$, then $f(\Sigma^n)$ is a slice.

Proof. Assume that $\rho'(h)\geq 0$. Proceeding as in the proof of Theorem 3.17 we realize that $H_k>0$ for the orientation with $\Theta<0$ and the Newton tensors P_j are positive definite for any $1\leq j\leq k-1$. Note that in order to determine the sign of Θ we need to apply the Omori-Yau maximum principle for the Laplacian, as in the proof of Theorem 3.17. This is possible because of the assumption on H_1 (see Corollary 3.15).

Next we consider the function

$$\phi = H_k^{\frac{1}{k}}\sigma(h) + \widehat{\Theta},$$

where $\widehat{\Theta}=\rho(h)\Theta$. Since $\mathbb{P}^n$ has constant sectional curvature κ, it follows by Equation (3.15) that

$$\begin{aligned}\widehat{\mathfrak{L}}_{k-1}\phi =&(n-k+1)\Theta\Big(\frac{\kappa}{\rho^2(h)}-(\log\rho)''(h)\Big)\,\langle P_{k-2}\nabla h,\nabla\phi\rangle+L_{k-1}\phi\\ =&(n-k+1)\widehat{\Theta}\Big(\frac{\kappa}{\rho^2(h)}-(\log\rho)''(h)\Big)\,\langle P_{k-2}\nabla h,\nabla h\rangle\\ &+(n-k+1)\widehat{\Theta}\Big(\frac{\kappa}{\rho^2(h)}-(\log\rho)''(h)\Big)\,\langle P_{k-2}A\nabla h,\nabla h\rangle\\ &+H_k^{\frac{1}{k}}L_{k-1}\sigma(h)+L_{k-1}\widehat{\Theta}.\end{aligned}$$

Using Equation (3.2) and Corollary 2.27 we can write

$$\begin{aligned}\widehat{\mathfrak{L}}_{k-1}\phi =&-c_{k-1}\rho'(h)H_k^{\frac{1}{k}}(H_{k-1}-H_k^{\frac{k-1}{k}})\\ &+(n-k+1)\widehat{\Theta}\Big(\frac{\kappa}{\rho^2(h)}-(\log\rho)''(h)\Big)H_k^{\frac{1}{k}}\,\langle P_{k-2}\nabla h,\nabla h\rangle\\ &+(n-k)\widehat{\Theta}\Big(\frac{\kappa}{\rho^2(h)}-(\log\rho)''(h)\Big)\,\langle P_{k-1}\nabla h,\nabla h\rangle\\ &+\widehat{\Theta}\binom{n}{k}(nH_1H_k-(n-k)H_{k+1}-kH_k^{\frac{k+1}{k}}).\end{aligned}\tag{3.17}$$

Using Garding inequalities it is easy to prove that the first and the last terms are non-negative. By the fact that each P_j is an elliptic operator, $0\le j\le k-1$, and by Equation (3.16) it follows that also all the remaining terms in the previous equation are non-negative and hence $\widehat{\mathfrak{L}}_{k-1}\phi\le 0$. Since $\sup_\Sigma|\Theta|<+\infty$ and the hypersurface is contained in a slab, ϕ is bounded from below. Moreover assumption (2.21) implies that Σ is $\widehat{\mathfrak{L}}_{k-1}$-parabolic. Therefore we conclude that ϕ has to be constant. Thus $\widehat{\mathfrak{L}}_{k-1}\phi=0$ and each term of Equation (3.17) must vanish. In particular, the equality

$$nH_1H_k-(n-k)H_{k+1}-kH_k^{\frac{k+1}{k}}=0$$

implies that Σ is a totally umbilical hypersurface. Moreover, since each P_j, $j=0,...,k-1$ is an elliptic operator and since (3.16) holds, we conclude that $\nabla h=0$ and hence $f(\Sigma)$ is a slice. □

CHAPTER 4

Curvature estimates for spacelike hypersurfaces and a Bernstein-type theorem

The aim of this chapter is to study the geometry, and hence the mean curvatures, of spacelike hypersurfaces bounded by a level set of the Lorentzian distance function from a point.
Estimates for the mean curvature of spacelike hypersurfaces bounded by a level set of the Lorentzian distance function have recently been obtained in [**12**]. There the authors assume the spacetime to have sectional curvature bounded by a constant, obtaining estimates of the mean curvature of the hypersurfaces in terms of that of Lorentzian spheres, that can be viewed, as we will clarify later, as spacelike level sets of the Lorentzian distance function from a point in spaceforms. For what concerns the higher order mean curvatures, a priori estimates in this spirit have been established in [**2**] and [**3**]. In the first paper spacelike hypersurfaces in the Lorentz-Minkowski spacetime are studied assuming they are bounded either by two parallel hyperplanes or by an upper Lorentzian sphere. In the second paper the authors consider spacelike hypersurfaces in the de Sitter space and they are able to obtain lower estimates for the higher order mean curvatures once they are contained in certain unbounded regions of the ambient spacetime. In the first section, assuming that the radial sectional or Ricci curvature of the spacetime is bounded by a radial function we are going to draw conclusions by means of an analysis of the Lorentzian distance function restricted to the hypersurface. In particular, we expect to be able to estimate the k-mean curvature of the spacelike hypersurface in terms of that of the Lorentzian spheres, extending in this way the results mentioned before.
Moreover, both in [**12**] and in [**2**], Bernstein-type theorems are proved. Namely, in [**12**], in the case when the spacetime is a space form and the spacelike hypersurfaces have constant mean curvature, it is proved a characterization of Lorentzian spheres as the only spacelike hypersurfaces bounded by a level set of the Lorentzian distance function and with constant mean curvature. The same characterization is proved in [**2**] for hypersurfaces of constant higher order mean curvature in the Lorentz-Minkowski spacetime. In the last section we will see how, using our estimates, it is possible to extend this characterization results to hypersurfaces of constant k-mean curvature in every spaceform.

4.1. Mean curvature estimates for hypersurfaces bounded by a level set of the Lorentzian distance function from a point

In the following we will consider a spacelike hypersurface $f : \Sigma \to M^{n+1}$. Under suitable bounds on the sectional curvature of M, combining the Hessian and Laplacian comparison theorems established in Chapter 1 and the Omori-Yau maximum principle, we will be able to establish lower and upper bounds for the mean curvature of the hypersurface, generalizing the results in [**12**].

Let us start with some computational results. Since M is time-orientable, there exists a unique future-directed timelike unit normal field N globally defined on Σ. We will refer to that normal field N as the future-pointing Gauss map of the hypersurface. Assume that there exists a point $p \in M$ such that $\mathcal{I}^+(p) \neq \emptyset$ and that $f(\Sigma) \subset \mathcal{I}^+(p)$. Let $r(\cdot) = d_p(\cdot)$ be the Lorentzian distance function from p and let $u = r \circ f : \Sigma \to (0, +\infty)$ be the function r along the hypersurface, which is a smooth function on Σ. Let us calculate the Hessian of u on Σ. Notice that

$$\overline{\nabla} r = \nabla u - \left\langle \overline{\nabla} r, N \right\rangle N.$$

Thus, since $\left\| \overline{\nabla} r \right\|^2 = -1$ and $\left\langle \overline{\nabla} r, N \right\rangle > 0$, we have

$$\left\langle \overline{\nabla} r, N \right\rangle = \sqrt{1 + \|\nabla u\|^2} \geq 1$$

and hence

$$\overline{\nabla} r = \nabla u - \sqrt{1 + \|\nabla u\|^2} N$$

Moreover
(4.1)

$$\overline{\nabla}_X \overline{\nabla} r = \nabla_X \nabla u + \sqrt{1 + \|\nabla u\|^2} AX + \langle AX, \nabla u \rangle N - X(\sqrt{1 + \|\nabla u\|^2}) N$$

for every spacelike $X \in T\Sigma$. Thus

$$\operatorname{Hess} u(X, X) = \overline{\operatorname{Hess}} r(X, X) - \sqrt{1 + \|\nabla u\|^2} \langle AX, X \rangle$$

and, tracing the last expression,

$$\Delta u = \overline{\Delta} r + \overline{\operatorname{Hess}} r(N, N) + n H_1 \sqrt{1 + \|\nabla u\|^2}.$$

Hence, if we assume that $\operatorname{Ric}_M(\overline{\nabla} r, \overline{\nabla} r) \geq -nG(r)$, then, by applying Theorem 1.28 it is not difficult to prove the next

Proposition 4.1. *Let M^{n+1} be an $(n+1)$-dimensional spacetime. Assume that there exists a point $p \in M$ such that $\mathcal{I}^+(p) \neq \emptyset$ and let $r(\cdot) = d_p(\cdot)$ be the Lorentzian distance function from p. Given a smooth even function G on $\mathbb{R}$, let h be a solution of the Cauchy problem*

$$\begin{cases} h'' - Gh = 0 \\ h(0) = 0, \ h'(0) = 1 \end{cases} \tag{4.2}$$

and let $I = [0, r_G) \subset [0, +\infty)$ be the maximal interval where h is positive. Let $f : \Sigma^n \to M^{n+1}$ be a spacelike hypersurface such that $f(\Sigma^n) \subset \mathcal{I}^+(p) \cap$

$B^+(p, r_G)$. *If*

$$\mathrm{Ric}_M(\overline{\nabla} r, \overline{\nabla} r) \geq -nG(r), \tag{4.3}$$

then

$$\Delta u \geq -n\frac{h'}{h}(u) + \overline{\mathrm{Hess}} r(N, N) + nH_1\sqrt{1 + \|\nabla u\|^2}. \tag{4.4}$$

On the other hand, if we take into account the decomposition

$$X = X^* - \langle X, \nabla u\rangle \overline{\nabla} r,$$

where X^* is the component of X orthogonal to $\overline{\nabla} r$, then

$$\langle X^*, X^*\rangle = \langle X, X\rangle + \langle X, \nabla u\rangle^2 .$$

and since

$$\overline{\nabla}_{\overline{\nabla} r}\overline{\nabla} r = 0$$

we find

$$\overline{\mathrm{Hess}} r(X, X) = \overline{\mathrm{Hess}} r(X^*, X^*).$$

Hence, if we assume that $K_M(\Pi) \leq G(r)$, for all timelike planes Π, Theorem 1.27 implies that

$$\begin{aligned}\overline{\mathrm{Hess}} r(X, X) =& \overline{\mathrm{Hess}} r(X^*, X^*) \geq -\frac{h'}{h}(u)\,\langle X^*, X^*\rangle \\ =& -\frac{h'}{h}(u)(\langle X, X\rangle + \langle X, \nabla u\rangle^2),\end{aligned}$$

where h is a solution of the problem (4.2), for all spacelike $X \in T_q\Sigma$, $q \in \mathcal{I}^+(p) \cap B^+(p, r_G)$. Therefore

$$\begin{aligned}\mathrm{Hess}\, u(X, X) \geq& -\frac{h'}{h}(u)(\langle X, X\rangle + \langle X, \nabla u\rangle^2) \\ &- \sqrt{1 + \|\nabla u\|^2}\,\langle AX, X\rangle\end{aligned}$$

and, taking the tracing

$$\Delta u \geq -\frac{h'}{h}(u)(n + \|\nabla u\|^2) + n\sqrt{1 + \|\nabla u\|^2}H_1.$$

Summarizing, we have proved the next

Proposition 4.2. *Let M^{n+1} be an $(n+1)$-dimensional spacetime. Assume that there exists a point $p \in M$ such that $\mathcal{I}^+(p) \neq \emptyset$ and let $r(\cdot) = d_p(\cdot)$ be the Lorentzian distance function from p. Given a smooth even function G on $\mathbb{R}$, let h be a solution of the Cauchy problem* (4.2) *and let $I = [0, r_G) \subset [0, +\infty)$ be the maximal interval where h is positive. Let $f : \Sigma^n \to M^{n+1}$ be a spacelike hypersurface such that $f(\Sigma^n) \subset \mathcal{I}^+(p) \cap B^+(p, r_G)$. If*

$$K_M(\Pi) \leq G(r) \tag{4.5}$$

for all timelike planes Π, then

$$\Delta u \geq -\frac{h'}{h}(u)(n + \|\nabla u\|^2) + n\sqrt{1 + \|\nabla u\|^2}H_1. \tag{4.6}$$

On the other hand, if we assume that $K_M(\Pi) \geq G(r)$ for all timelike planes Π in M, the same computations yield the following

Proposition 4.3. *Let M^{n+1} be an $(n+1)$-dimensional spacetime. Assume that there exists a point $p \in M$ such that $\mathcal{I}^+(p) \neq \emptyset$ and let $r(\cdot) = d_p(\cdot)$ be the Lorentzian distance function from p. Given a smooth even function G on $\mathbb{R}$, let h be a solution of the Cauchy problem* (4.2) *and let $I = [0, r_G) \subset [0, +\infty)$ be the maximal interval where h is positive. Let $f : \Sigma^n \to M^{n+1}$ be a spacelike hypersurface such that $f(\Sigma^n) \subset \mathcal{I}^+(p) \cap B^+(p, r_G)$. If*

$$K_M(\Pi) \geq G(r) \tag{4.7}$$

for all timelike planes Π, then

$$\Delta u \leq -\frac{h'}{h}(u)(n + \|\nabla u\|^2) + n\sqrt{1 + \|\nabla u\|^2} H_1. \tag{4.8}$$

Using the previous results, we can prove the following mean curvature estimates.

Theorem 4.4 (Theorem 12 in [**41**])**.** *Let M^{n+1} be an $(n+1)$-dimensional spacetime. Assume that there exists a point $p \in M$ such that $\mathcal{I}^+(p) \neq \emptyset$ and let $r(\cdot) = d_p(\cdot)$ be the Lorentzian distance function from p. Given a smooth even function G on $\mathbb{R}$, let h be a solution of the Cauchy problem* (4.2) *and and let $I = [0, r_G) \subset [0, +\infty)$ be the maximal interval where h is positive. Let $f : \Sigma^n \to M^{n+1}$ be a spacelike hypersurface such that $f(\Sigma^n) \subset \mathcal{I}^+(p) \cap B^+(p, \delta)$, for some $\delta \leq r_G$. If*

$$\mathrm{Ric}_M(\overline{\nabla} r, \overline{\nabla} r) \geq -nG(r), \tag{4.9}$$

and the Omori-Yau maximum principle holds on Σ then

$$\inf_\Sigma H_1 \leq \Big(\frac{h'}{h}\big(\sup_\Sigma u\big)\Big),$$

where u denotes the Lorentzian distance d_p along the hypersurface.

Proof. Since $u^* = \sup_\Sigma u < \delta$ and the Omori-Yau maximum principle for the Laplacian holds, we can find a sequence $\{p_i\}_{i \in \mathbb{N}} \subset \Sigma$ such that

$$(i)\ u(p_i) > u^* - \frac{1}{i},\ (ii)\ \|\nabla u(p_i)\| < \frac{1}{i},\ (iii)\ \Delta u(p_i) < \frac{1}{i}.$$

Using Equation (4.6)

$$\begin{aligned} \Delta u(p_i) \geq & -n\frac{h'}{h}(u(p_i)) + \overline{\mathrm{Hess}} r(N(p_i), N(p_i)) + n\sqrt{1 + \|\nabla u(p_i)\|^2} H_1(p_i) \\ \geq & \frac{h'}{h}(u(p_i)) + \overline{\mathrm{Hess}} r(N(p_i), N(p_i)) + n\sqrt{1 + \|\nabla u(p_i)\|^2} \inf_\Sigma H_1. \end{aligned}$$

By the decomposition

$$N(p_i) = N^*(p_i) - \big\langle N(p_i), \overline{\nabla} r(p_i)\big\rangle \overline{\nabla} r(p_i)$$

and by the fact that

$$\big\|\overline{\nabla} r(p_i)\big\|^2 = \|N(p_i)\|^2 = -1, \quad \overline{\nabla} r(p_i) = \nabla u(p_i) - \big\langle \overline{\nabla} r(p_i), N(p_i)\big\rangle N(p_i),$$

it follows that

$$\|N^*(p_i)\|^2 = \|\nabla u(p_i)\|^2$$

and hence $\lim_{i\to+\infty}\|N^*(p_i)\| = 0$, that is $\lim_{i\to+\infty} N^*(p_i) = 0$.Taking into account that

$$\overline{\text{Hess}}r(N(p_i), N(p_i)) = \overline{\text{Hess}}r(N^*(p_i), N^*(p_i))$$

we observe that $\lim_{i\to+\infty}\overline{\text{Hess}}r(N(p_i), N(p_i)) = 0$. The conclusion then follows by taking the limit for $i \to +\infty$ in the inequality above. □

On the other hand, if we assume that the sectional curvature of timelike planes is bounded from below we obtain

Theorem 4.5 (Theorem 13 in [**41**]). *Let M^{n+1} be an $(n+1)$- dimensional spacetime. Assume that there exists a point $p \in M$ such that $\mathcal{I}^+(p) \neq \emptyset$ and let $r(\cdot) = d_p(\cdot)$ be the Lorentzian distance function from p. Given a smooth even function G on $\mathbb{R}$, let h be a solution of the Cauchy problem* (4.2) *and let $I = [0, r_G) \subset [0, +\infty)$ be the maximal interval where h is positive. Let $f : \Sigma^n \to M^{n+1}$ be a spacelike hypersurface such that $f(\Sigma^n) \subset \mathcal{I}^+(p) \cap B^+(p, r_G)$. If*

$$K_M(\Pi) \geq G(r) \tag{4.10}$$

for all timelike planes Π and if the Omori-Yau maximum principle holds on Σ, then

$$\sup_\Sigma H_1 \geq \Big(\frac{h'}{h}\big(\inf_\Sigma u\big)\Big),$$

where u denotes the Lorentzian distance d_p along the hypersurface Σ.

Proof. Since $u_* = \inf_\Sigma u > 0$ and the Omori-Yau maximum principle for the Laplacian holds, we can find a sequence $\{q_i\}_{i\in\mathbb{N}} \subset \Sigma$ such that

$$(i)\ u(q_i) < u_* + \frac{1}{i},\ (ii)\ \|\nabla u(q_i)\| < \frac{1}{i},\ (iii)\ \Delta u(q_i) > -\frac{1}{i}.$$

Using Equation (4.8) and taking the limit for $i \to +\infty$ we then find

$$0 \leq -n\frac{h'}{h}(\inf_\Sigma u) + n\sup_\Sigma H_1.$$

and the conclusion follows. □

Remark 4.6. It is worth pointing out that, for a fixed r, the function $h'/h(r)$ is the mean curvature of the Lorentzian sphere in the Lorentzian model M_G (see Section 1.4), that is the level set

$$\Sigma_G(r) = \{q \in \mathcal{I}^+(p) | d_p(q) = r\}.$$

This can be proved by representing the model space as the warped product $-(0, r_G) \times_h \mathbb{H}^n$, where r_G is the first zero of the function h. Then the conclusion follows observing that the level set $\Sigma_G(r)$ is nothing but a slice $\{r\} \times \mathbb{H}^n$ (see the introduction of Chapter 3 for more details).

In the case of sectional or Ricci curvature bounded by a constant we recover the estimates in Theorems 4.1 and 4.2 in [**12**]. Namely, as an application of Theorems 4.4 and 4.5 we obtain the following

Corollary 4.7 (Theorem 4.1 in [**12**])**.** *Let M^{n+1} be an $(n+1)$- dimensional spacetime, such that $\mathrm{Ric}_M(\overline{\nabla}r, \overline{\nabla}r) \geq -nc$, $c \in \mathbb{R}$, for all timelike planes Π. Assume that there exists a point $p \in M$ such that $\mathcal{I}^+(p) \neq \emptyset$ and let $f : \Sigma^n \to M^{n+1}$ be a spacelike hypersurface such that $f(\Sigma^n) \subset \mathcal{I}^+(p) \cap B^+(p, \delta)$ for some $\delta > 0$ (with $\delta \leq \pi/\sqrt{-c}$ if $c < 0$). If the Omori-Yau maximum principle for the Laplacian holds on Σ, then*

$$\inf_\Sigma H_1 \leq f_c\big(\sup_\Sigma u\big),$$

where u denotes the Lorentzian distance d_p along the hypersurface.

Corollary 4.8 (Theorem 4.2 in [**12**])**.** *Let M^{n+1} be an $(n+1)$- dimensional spacetime, such that $K_M(\Pi) \geq c$, $c \in \mathbb{R}$, for all timelike planes Π. Assume that there exists a point $p \in M$ such that $\mathcal{I}^+(p) \neq \emptyset$ and let $f : \Sigma^n \to M^{n+1}$ be a spacelike hypersurface such that $f(\Sigma^n) \subset \mathcal{I}^+(p)$. Moreover, suppose that $\inf_\Sigma u < \pi/\sqrt{-c}$ if $c < 0$. If the Omori-Yau maximum principle for the Laplacian holds on Σ, then*

$$\sup_\Sigma H_1 \geq f_c\big(\inf_\Sigma u\big),$$

where u denotes the Lorentzian distance d_p along the hypersurface.

4.2. Higher order mean curvature estimates for hypersurfaces bounded by a level set of the Lorentzian distance function from a point

We are now going to extend the previous estimates to the higher order constant mean curvature, generalizing the results in [**2**], [**3**]. We observe that in the latter works, the estimates are proved by means of the Omori-Yau maximum principle. In particular, there the authors found estimates for the principal curvatures of the immersion and then combine them to estimate the higher order mean curvatures. However, this does not allow, for instance, to obtain upper bounds in the case of spacelike hypersurfaces in the de Sitter spacetime. Here we will show how, introducing a family of elliptic operators and using the general version of the Omori-Yau maximum principle introduced in Section 1.3, it is possible to recover these results and to generalize them.
Toward this aim, first observe that, using Equation (4.1), it is easy to see that

$$\operatorname{Hess} u(X, P_{k-1}X) = \overline{\operatorname{Hess}} r(X, P_{k-1}X) - \sqrt{1 + \|\nabla u\|^2}\,\langle P_{k-1}AX, X\rangle$$

On the other hand, we have the following decompositions

$$\begin{aligned} X &= X^* - \langle X, \nabla u\rangle\,\overline{\nabla}r \\ P_{k-1}X &= (P_{k-1}X)^* - \langle X, P_{k-1}\nabla u\rangle\,\overline{\nabla}r, \end{aligned}$$

and

$$\langle X^*, (P_{k-1}X)^*\rangle = \langle X, P_{k-1}X\rangle + \langle X, P_{k-1}\nabla u\rangle\,\langle X, \nabla u\rangle\,.$$

Taking into account that

$$\overline{\nabla}_{\overline{\nabla}r}\overline{\nabla}r = 0$$

we find

$$\overline{\mathrm{Hess}}r(X, P_{k-1}X) = \overline{\mathrm{Hess}}r(X^*, (P_{k-1}X)^*).$$

Hence, if we assume that $K_M(\Pi) \leq G(r)$ for all timelike planes Π, then

$$\begin{aligned}\overline{\mathrm{Hess}}r(X, P_{k-1}X) =&\overline{\mathrm{Hess}}r(X^*, (P_{k-1}X)^*) \geq -\frac{h'}{h}(u)\left\langle X^*, (P_{k-1}X)^*\right\rangle \\ =&-\frac{h'}{h}(u)(\langle X, P_{k-1}X\rangle + \langle X, \nabla u\rangle \langle X, P_{k-1}\nabla u\rangle),\end{aligned}$$

where h is a solution of the problem (4.2). Therefore

$$\begin{aligned}\mathrm{Hess}\, u(X, P_{k-1}X) \geq& -\frac{h'}{h}(u)(\langle X, P_{k-1}X\rangle + \langle X, \nabla u\rangle \langle X, P_{k-1}\nabla u\rangle) \\ &- \sqrt{1 + \|\nabla u\|^2}\left\langle P_{k-1}AX, X\right\rangle.\end{aligned}$$

and, taking the trace,

$$L_{k-1}u \geq -\frac{h'}{h}(u)(c_{k-1}H_{k-1} + \langle \nabla u, P_{k-1}\nabla u\rangle) + \sqrt{1 + \|\nabla u\|^2}c_{k-1}H_k.$$

Summarizing, we have proved the following

Proposition 4.9. *Let M^{n+1} be an $(n+1)$-dimensional spacetime. Assume that there exists a point $p \in M$ such that $\mathcal{I}^+(p) \neq \emptyset$ and let $r(\cdot) = d_p(\cdot)$ be the Lorentzian distance function from p. Given a smooth even function G on $\mathbb{R}$, let h be a solution of the Cauchy problem* (4.2) *and let $I = [0, r_G) \subset [0, +\infty)$ be the maximal interval where h is positive. Let $f : \Sigma^n \to M^{n+1}$ be a spacelike hypersurface such that $f(\Sigma^n) \subset \mathcal{I}^+(p) \cap B^+(p, r_G)$. If*

$$K_M(\Pi) \leq G(r) \tag{4.11}$$

for all timelike planes Π, then

$$L_{k-1}u \geq -\frac{h'}{h}(u)(c_{k-1}H_{k-1} + \langle \nabla u, P_{k-1}\nabla u\rangle) + \sqrt{1 + \|\nabla u\|^2}c_{k-1}H_k. \tag{4.12}$$

On the other hand, if we assume that $K_M(\Pi) \geq G(r)$ for all timelike planes in M, the same computations yield the following

Proposition 4.10. *Let M^{n+1} be an $(n+1)$-dimensional spacetime. Assume that there exists a point $p \in M$ such that $\mathcal{I}^+(p) \neq \emptyset$ and let $r(\cdot) = d_p(\cdot)$ be the Lorentzian distance function from p. Given a smooth even function G on $\mathbb{R}$, let h be a solution of the Cauchy problem* (4.2) *and let $I = [0, r_G) \subset [0, +\infty)$ be the maximal interval where h is positive. Let $f : \Sigma^n \to M^{n+1}$ be a spacelike hypersurface such that $f(\Sigma^n) \subset \mathcal{I}^+(p) \cap B^+(p, r_G)$. If*

$$K_M(\Pi) \geq G(r) \tag{4.13}$$

for all timelike planes Π, then

$$L_{k-1}u \leq -\frac{h'}{h}(u)(c_{k-1}H_{k-1} + \langle \nabla u, P_{k-1}\nabla u\rangle) + \sqrt{1 + \|\nabla u\|^2}c_{k-1}H_k. \tag{4.14}$$

We are now ready to extend the estimates of the previous section to the higher order mean curvatures. To do that we will use the Omori-Yau maximum principle for semi-elliptic operators of the form (1.11). For simplicity, we will refer to that as the generalized Omori-Yau maximum principle.

Theorem 4.11 (Theorem 14 in [**41**]). *Let M^{n+1} be an $(n+1)$-dimensional spacetime. Assume that there exists a point $p \in M$ such that $\mathcal{I}^+(p) \neq \emptyset$ and let $r(\cdot) = d_p(\cdot)$ be the Lorentzian distance function from p. Given a smooth even function G on $\mathbb{R}$, let h be a solution of the Cauchy problem (4.2) and let $I = [0, r_G) \subset [0, +\infty)$ be the maximal interval where h is positive. Let $f : \Sigma^n \to M^{n+1}$ be a spacelike hypersurface such that $f(\Sigma^n) \subset \mathcal{I}^+(p) \cap B^+(p, \delta,$ for some $\delta \leq r_G$. Assume that $H_2 > 0$ and that $\sup_\Sigma H_1 < +\infty$. If*

$$K_M(\Pi) \leq G(r) \tag{4.15}$$

for all timelike planes Π and if the generalized Omori-Yau maximum principle holds on Σ, then

$$\inf_\Sigma H_2^{\frac{1}{2}} \leq \Big|\frac{h'}{h}\big(\sup_\Sigma u\big)\Big|,$$

where u denotes the Lorentzian distance d_p along the hypersurface.

Proof. Consider the operator

$$\begin{aligned}\mathcal{L} =& L_1 + (n-1)\frac{1}{\sqrt{1+\|\nabla u\|^2}}\Big(\Big|\frac{h'}{h}(u)\Big|\Big)\Delta \\ =& \operatorname{Tr}(\mathcal{P} \circ \operatorname{hess}),\end{aligned}$$

where

$$\mathcal{P} = P_1 + (n-1)\frac{1}{\sqrt{1+\|\nabla u\|^2}}\Big(\Big|\frac{h'}{h}(u)\Big|\Big)I.$$

Notice that, since $H_2 > 0$, the operator L_1 is elliptic and so is $\mathcal{L}$. Since $0 < u < \sup_\Sigma u < \delta$, $h'/h(u)$ is bounded. Furthermore, $\sup_\Sigma H_1 < +\infty$ and $1/\sqrt{1+\|\nabla u\|^2} \leq 1$, hence we can apply the Omori-Yau maximum principle for the operator $\mathcal{L}$. We can then find a sequence $\{p_i\}_{i\in\mathbb{N}} \subset \Sigma$ such that

$$(i)\ u(p_i) > u^* - \frac{1}{i},\ (ii)\ \|\nabla u(p_i)\| < \frac{1}{i},\ (iii)\ \mathcal{L}u(p_i) < \frac{1}{i}.$$

A simple computation shows that

$$\begin{aligned}\mathcal{L}u \geq& -(n-1)\frac{1}{\sqrt{1+\|\nabla u\|^2}}\Big(\frac{h'}{h}(u)\Big)^2(n+\|\nabla u\|^2)- \\ &-\Big(\Big|\frac{h'}{h}(u)\Big|\Big)\langle P_1\nabla u, \nabla u\rangle + n(n-1)\sqrt{1+\|\nabla u\|^2}H_2.\end{aligned}$$

Hence

$$
\begin{aligned}
\frac{1}{i} > \mathcal{L}u(p_i) \geq & -(n-1)\frac{1}{\sqrt{1+\|\nabla u(p_i)\|^2}}\Big(\frac{h'}{h}(u(p_i))\Big)^2(n+\|\nabla u(p_i)\|^2)-\\
& -\Big(\Big|\frac{h'}{h}(u(p_i))\Big|\Big)\langle P_1\nabla u(p_i),\nabla u(p_i)\rangle\\
& +n(n-1)\sqrt{1+\|\nabla u(p_i)\|^2}H_2(p_i)\\
\geq & -(n-1)\frac{1}{\sqrt{1+\|\nabla u(p_i)\|^2}}\Big(\frac{h'}{h}(u(p_i))\Big)^2(n+\|\nabla u(p_i)\|^2)-\\
& -\Big(\Big|\frac{h'}{h}(u(p_i))\Big|\Big)\langle P_1\nabla u(p_i),\nabla u(p_i)\rangle\\
& +n(n-1)\sqrt{1+\|\nabla u(p_i)\|^2}\inf_{\Sigma} H_2
\end{aligned}
$$

Taking the limit for $i \to +\infty$ $\lim_{i\to\infty} H_2(p_i) \geq \inf_\Sigma H_2$, we find

$$0 \geq -n(n-1)\Big(\frac{h'}{h}(\sup_\Sigma u)\Big)^2 + n(n-1)\inf_\Sigma H_2.$$

and the conclusion follows. □

When $3 \leq k \leq n$ we obtain the next

Theorem 4.12 (Theorem 15 in [**41**]). *Let M^{n+1} be an $(n+1)$- dimensional spacetime, $n \geq 3$. Assume that there exists a point $p \in M$ such that $\mathcal{I}^+(p) \neq \emptyset$ and let $r(\cdot) = d_p(\cdot)$ be the Lorentzian distance function from p. Given a smooth even function G on $\mathbb{R}$, let h be a solution of the Cauchy problem (4.2) and let $I = [0, r_G) \subset [0, +\infty)$ be the maximal interval where h is positive. Let $f : \Sigma^n \to M^{n+1}$ be a spacelike hypersurface such that $f(\Sigma^n) \subset \mathcal{I}^+(p) \cap B^+(p, \delta)$, for some $\delta \leq r_G$. Assume that there exists an elliptic point $p_0 \in \Sigma$, that $H_k > 0$, $3 \leq k \leq n$, and that $\sup_\Sigma H_1 < +\infty$. If*

$$K_M(\Pi) \leq G(r) \tag{4.16}$$

for all timelike planes Π and if the generalized Omori-Yau maximum principle holds on Σ, then

$$\inf_\Sigma H_k^{\frac{1}{k}} \leq \Big|\frac{h'}{h}\big(\sup_\Sigma u\big)\Big|,$$

where u denotes the Lorentzian distance d_p along the hypersurface.

Proof. Consider the operator

$$\mathcal{L} = \sum_{j=0}^{k-1}(1+\|\nabla u\|^2)^{-\frac{k-1-j}{2}}\Big(\Big|\frac{h'}{h}(u)\Big|\Big)^{k-1-j}\frac{c_{k-1}}{c_j}L_j$$

Notice that, since there exists an elliptic point $p_0 \in \Sigma$ and $H_k > 0$, $3 \leq k \leq n$, the operators L_j are elliptic for all $1 \leq j \leq k-1$. Since $0 < u < \sup_\Sigma u < \delta$, $1/\sqrt{1+\|\nabla u\|^2} \leq 1$ and $\sup_\Sigma H_1 < +\infty$, we can apply the

Omori-Yau maximum principle for the operator $\mathcal{L}$. Hence, we can find a sequence $\{p_i\}_{i\in\mathbb{N}} \subset \Sigma$ such that

$$(i)\ u(p_i) > u^* - \frac{1}{i},\ (ii)\ \|\nabla u(p_i)\| < \frac{1}{i},\ (iii)\ \mathcal{L}u(p_i) < \frac{1}{i}.$$

A straightforward computation using Proposition 4.9 shows that

$$\begin{aligned}\mathcal{L}u \geq & -\sum_{j=1}^{k-1}(1+\|\nabla u\|^2)^{-\frac{k-1-j}{2}}\left(\Big|\frac{h'}{h}(u)\Big|\right)^{k-j}\frac{c_{k-1}}{c_j}\langle P_j\nabla u, \nabla u\rangle \\ & - c_{k-1}\frac{1}{(1+\|\nabla u\|^2)^{k/2}}\left(\Big|\frac{h'}{h}(u)\Big|\right)^k + \sqrt{1+\|\nabla u\|^2}c_{k-1}H_k.\end{aligned}$$

Hence

$$\begin{aligned}\frac{1}{i} > \mathcal{L}u(p_i) \geq & - c_{k-1}\frac{1}{\sqrt{1+\|\nabla u(p_i)\|^2}^k}\left(\Big|\frac{h'}{h}(u(p_i))\Big|\right)^k \\ & - \sum_{j=1}^{k-1}(1+\|\nabla u(p_i)\|^2)^{-\frac{k-1-j}{2}}\left(\Big|\frac{h'}{h}(u(p_i))\Big|\right)^{k-j}\frac{c_{k-1}}{c_j}\langle P_j\nabla u, \nabla u\rangle(p_i) \\ & + \sqrt{1+\|\nabla u(p_i)\|^2}c_{k-1}\inf_\Sigma H_k.\end{aligned}$$

Taking the limit for $i \to +\infty$ we find

$$0 \geq -c_{k-1}\left(\Big|\frac{h'}{h}(\sup_\Sigma u)\Big|\right)^k + c_{k-1}\inf_\Sigma H_k.$$

□

On the other hand, if we assume that the sectional curvature of timelike planes is bounded from below we find the following estimates

Theorem 4.13 (Theorem 16 in [**41**]). *Let M^{n+1} be an $(n+1)$-dimensional spacetime. Assume that there exists a point $p \in M$ such that $\mathcal{I}^+(p) \neq \emptyset$ and let $r(\cdot) = d_p(\cdot)$ be the Lorentzian distance function from p. Given a smooth even function G on $\mathbb{R}$, let h be a solution of the Cauchy problem (4.2) and let $I = [0, r_G) \subset [0, +\infty)$ be the maximal interval where h is positive. Let $f : \Sigma^n \to M^{n+1}$ be a spacelike hypersurface such that $f(\Sigma^n) \subset \mathcal{I}^+(p) \cap B^+(p, r_G)$. Assume that $H_2 > 0$ and that $\sup_\Sigma H_1 < +\infty$. If*

$$K_M(\Pi) \geq G(r) \tag{4.17}$$

for all timelike planes Π and if the generalized Omori-Yau maximum principle holds on Σ, then

$$\sup_\Sigma H_2^{\frac{1}{2}} \geq \left(\frac{h'}{h}\big(\inf_\Sigma u\big)\right),$$

where u denotes the Lorentzian distance d_p along the hypersurface.

Theorem 4.14 (Theorem 17 in [**41**]). *Let M^{n+1} be an $(n+1)$- dimensional spacetime, $n \geq 3$. Assume that there exists a point $p \in M$ such that $\mathcal{I}^+(p) \neq \emptyset$ and let $r(\cdot) = d_p(\cdot)$ be the Lorentzian distance function from p. Given a smooth even function G on $\mathbb{R}$, let h be a solution of the Cauchy problem*

(4.2) and let $I = [0, r_G) \subset [0, +\infty)$ be the maximal interval where h is positive. Let $f : \Sigma^n \to M^{n+1}$ be a spacelike hypersurface such that $f(\Sigma^n) \subset \mathcal{I}^+(p) \cap B^+(p, r_G)$. Assume that there exists an elliptic point $p_0 \in \Sigma$, that $H_k > 0$, $3 \le k \le n$, and that $\sup_\Sigma H_1 < +\infty$. If

$$K_M(\Pi) \ge G(r) \tag{4.18}$$

for all timelike planes Π and if the generalized Omori-Yau maximum principle holds on Σ, then

$$\sup_\Sigma H_k^{\frac{1}{k}} \ge \Big(\frac{h'}{h}\big(\inf_\Sigma u\big)\Big),$$

where u denotes the Lorentzian distance d_p along the hypersurface.

We will only prove Theorem 4.14. The proof of Theorem 4.13 proceed exactly in the same way.

Proof of Theorem 4.14. If $h'/h(\inf_\Sigma u) \le 0$, the result is trivial since

$$\frac{h'}{h}\Big(\inf_\Sigma u\Big) \le 0 < \sup_\Sigma H_k^{\frac{1}{k}}.$$

Conversely, assume $h'/h(\inf_\Sigma u) > 0$. Since $u \ge u_* := \inf_\Sigma u \ge 0$, we want to apply the Omori-Yau maximum principle for a suitable elliptic operator with trace bounded above. Notice that it must be $\inf_\Sigma u > 0$. Indeed, if $\inf_\Sigma u = 0$, since $\lim_{s\to 0} h'/h(s) = +\infty$, it follows by the estimate in Theorem 4.5 that $\sup_\Sigma H_1 = +\infty$, which contradicts our assumptions. The operator that we consider is the following

$$\begin{aligned}\widehat{\mathcal{L}} =& \sum_{j=0}^{k-1}(1+\|\nabla u\|^2)^{-\frac{k-j-1}{2}}\Big(\frac{h'}{h}(\inf_\Sigma u)\Big)^{k-j-1}\frac{c_{k-1}}{c_j}L_j\\ =&\mathrm{Tr}(\widehat{\mathcal{P}}\circ \mathrm{hess}),\end{aligned}$$

where

$$\widehat{\mathcal{P}} = \sum_{j=0}^{k-1}(1+\|\nabla u\|^2)^{-\frac{k-j-1}{2}}\Big(\frac{h'}{h}(\inf_\Sigma u)\Big)^{k-j-1}\frac{c_{k-1}}{c_j}P_j.$$

Notice that, since there exists an elliptic point $p_0 \in \Sigma$ and $H_k > 0$, $3 \le k \le n$, the operators L_j are elliptic for all $1 \le j \le k-1$ and so $\widehat{\mathcal{L}}$ is elliptic as well. Furthermore, we observe that

$$\mathrm{Tr}\widehat{\mathcal{P}} = \sum_{j=0}^{k-1}(1+\|\nabla u\|^2)^{-\frac{k-j-1}{2}}\Big(\frac{h'}{h}(\inf_\Sigma u)\Big)^{k-j-1}\frac{c_{k-1}}{c_j}H_j.$$

Since $1/\sqrt{1+\|\nabla u\|^2} \le 1$, $h'/h(\inf_\Sigma u) < +\infty$ and, by the Newton inequalities

$$H_j \le H_1^j < +\infty$$

we conclude that $\widehat{\mathcal{P}}$ has trace bounded above and we can apply the Omori-Yau maximum principle for the operator $\widehat{\mathcal{L}}$. Hence, we can find a sequence

$\{q_i\}_{i\in\mathbb{N}} \subset \Sigma$ such that

$$(i)\ u(q_i) < u_* + \frac{1}{i},\ (ii)\ \|\nabla u(q_i)\| < \frac{1}{i},\ (iii)\ \mathcal{L}u(q_i) > -\frac{1}{i}. \tag{4.19}$$

A straightforward computation using Proposition 4.10 shows that

$$\begin{aligned}\widehat{\mathcal{L}}u \leq &- \frac{h'}{h}(u)\sum_{j=0}^{k-1}(1+\|\nabla u\|^2)^{-\frac{k-1-j}{2}}\Big(\frac{h'}{h}(\inf_\Sigma u)\Big)^{k-j-1}\frac{c_{k-1}}{c_j}\langle P_j\nabla u, \nabla u\rangle \\ &- c_{k-1}\frac{h'}{h}(u)\frac{1}{(1+\|\nabla u\|^2)^{(k-1)/2}}\Big(\frac{h'}{h}(\inf_\Sigma u)\Big)^{k-1} + \sqrt{1+\|\nabla u\|^2}c_{k-1}H_k \\ &+ c_{k-1}\sum_{j=1}^{k-1}(1+\|\nabla u\|^2)^{-\frac{k-1-j}{2}}\Big(\frac{h'}{h}(\inf_\Sigma u)\Big)^{k-j-1}\Big(\frac{h'}{h}(\inf_\Sigma u) - \frac{h'}{h}(u)\Big)H_j.\end{aligned}$$

Evaluating the previous expression at q_i, using condition (iii) in (4.19) and taking the limit for $i \to +\infty$, we find

$$0 \leq -c_{k-1}\Big(\frac{h'}{h}(\inf_\Sigma u)\Big)^k + c_{k-1}\sup_\Sigma H_k$$

and this concludes the proof. □

In case $G(r) \equiv c$ the previous theorems read as

Corollary 4.15. *Let M^{n+1} be an $(n+1)$- dimensional spacetime, $n \geq 3$, such that $K_M(\Pi) \leq c$, $c \in \mathbb{R}$, for all timelike planes Π. Assume that there exists a point $p \in M$ such that $\mathcal{I}^+(p) \neq \emptyset$ and let $f : \Sigma^n \to M^{n+1}$ be a spacelike hypersurface such that $f(\Sigma^n) \subset \mathcal{I}^+(p) \cap B^+(p,\delta)$ for some $\delta > 0$ (with $\delta \leq \pi/\sqrt{-c}$ if $c < 0$). Assume that either*

- (i) *$k = 2$ and H_2 is a positive function*
  or
- (ii) *H_k is a positive function, $3 \leq k \leq n$, and there exists an elliptic point $p_0 \in \Sigma$.*

Moreover, suppose that $\sup_\Sigma H_1 < +\infty$. If the generalized Omori-Yau maximum principle holds on Σ, then

$$\inf_\Sigma H_k^{\frac{1}{k}} \leq f_c\big(\sup_\Sigma u\big),$$

where u denotes the Lorentzian distance d_p along the hypersurface.

Corollary 4.16. *Let M^{n+1} be an $(n+1)$- dimensional spacetime, $n \geq 3$, such that $K_M(\Pi) \geq c$, $c \in \mathbb{R}$, for all timelike planes Π. Assume that there exists a point $p \in M$ such that $\mathcal{I}^+(p) \neq \emptyset$ and let $f : \Sigma^n \to M^{n+1}$ be a spacelike hypersurface such that $f(\Sigma^n) \subset \mathcal{I}^+(p)$. Assume that either*

- (i) *$k = 2$ and H_2 is a positive function*
  or
- (ii) *H_k is a positive function, $3 \leq k \leq n$, and there exists an elliptic point $p_0 \in \Sigma$.*

Moreover, suppose that $\sup_\Sigma H_1 < +\infty$ *and that* $\inf_\Sigma u < \pi/\sqrt{-c}$ *if* $c < 0$. *If the generalized Omori-Yau maximum principle holds on* Σ, *then*

$$\sup_\Sigma H_k^{\frac{1}{k}} \geq f_c\big(\inf_\Sigma u\big),$$

where u *denotes the Lorentzian distance* d_p *along the hypersurface.*

4.3. Bernstein-type theorems

We conclude the chapter proving some Bernstein-type theorems for spacelike hypersurfaces of constant k-mean curvature, $1 \leq k \leq n$, which are bounded by a level set of the Lorentzian distance function. Toward this aim we will focus now on the case where M^{n+1} is a Lorentzian space form of constant sectional curvature c. Applying the curvature estimates found in the previous section we are able to obtain the main results of this section, that extends Corollary 4.6 in [**12**] to spacelike hypersurfaces of constant higher order mean curvature.

The restriction to spacetimes of constant sectional curvature as ambient spaces is motivated by the next observation.

Using Gauss equation it is straightforward to see that the sectional curvature of Σ satisfies

$$K(X,Y) \geq K_M(X,Y) - n^2 H_1^2,$$

where the last term follows by applying the Cauchy-Schwartz inequality. In particular, if M^{n+1} is a Lorentzian space form of constant sectional curvature c, then

$$K(X,Y) \geq c - n^2 H_1^2. \tag{4.20}$$

Hence, if the mean curvature is constant, then the Omori-Yau maximum principle for the Laplacian holds on Σ. Therefore, using the estimates on the mean curvature obtained in the previous section it is easy to prove the following

Theorem 4.17 (Corollary 4.6 in [**12**]). *Let* M_c^{n+1} *be a Lorentzian spaceform of constant sectional curvature* c *and let* $p \in M_c^{n+1}$. *Let* Σ *be a complete spacelike hypersurface with constant mean curvature which is contained in* $\mathcal{I}^+(p)$. *If* Σ *is bounded from above by a level set of the Lorentzian distance function* d_p *(with* $d_p < \pi/\sqrt{-c}$ *if* $c < 0$*), then* Σ *is necessarily a level set of* d_p.

Proof. By hypotheses Σ is contained in $\mathcal{I}^+(p) \cap B^+(p,\delta)$, with $\delta \leq \pi/\sqrt{-c}$ when $c < 0$. Moreover, since the mean curvature is constant, it follows by Equation (4.20) that Σ has sectional curvature bounded from below and thus the Omori-Yau maximum principle holds on Σ for the Laplacian. Corollaries 4.7 and 4.8 imply then

$$f_c(\sup_\Sigma u) \geq H_1 \geq f_c(\inf_\Sigma u).$$

Hence, since f_c is a decreasing function, $\sup_\Sigma u = \inf_\Sigma u = f_c^{-1}(H_1)$ and Σ is necessarily the level set $d_p = f_c^{-1}(H_1)$. $\square$

More generally, if $\sup_\Sigma H_1 < +\infty$ the Omori-Yau maximum principle holds on Σ for semi-elliptic operators of the form (1.11), where P is any symmetric operator with trace bounded above. Then, as a direct application of Corollaries 4.15 and 4.16 we get

Theorem 4.18 (Theorem 20 in [**41**])**.** *Let M_c^{n+1} be a Lorentzian spaceform of constant sectional curvature c, $n \geq 3$, and let $p \in M_c^{n+1}$. Let Σ be a complete spacelike hypersurface which is contained in $\mathcal{I}^+(p)$ such that either*

(*i*) *$k = 2$ and H_2 is a positive constant*
or
(*ii*) *H_k is constant, $3 \leq k \leq n$, and there exists an elliptic point $p_0 \in \Sigma$.*

Moreover, assume that $\sup_\Sigma H_1 < +\infty$. If Σ is bounded from above by a level set of the Lorentzian distance function d_p (with $d_p < \pi/\sqrt{-c}$ if $c < 0$), then Σ is necessarily a level set of d_p.

Proof. The proof proceeds exactly as in case $k = 1$. Indeed, our hypotheses imply that Σ is contained in $\mathcal{I}^+(p) \cap B^+(p, \delta)$, with $\delta \leq \pi/\sqrt{-c}$ when $c < 0$ and that Σ has sectional curvature bounded from below. Hence the generalized Omori-Yau maximum principle holds on Σ and we can apply Corollaries 4.15 and 4.16 to obtain

$$f_c(\sup_\Sigma u) \geq H_k^{\frac{1}{k}} \geq f_c(\inf_\Sigma u).$$

As above, since f_c is a decreasing function, $\sup_\Sigma u = \inf_\Sigma u = f_c^{-1}(H_k^{\frac{1}{k}})$ and Σ is necessarily the level set $d_p = f_c^{-1}(H_k^{\frac{1}{k}})$. $\square$

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