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Dualities and Representations for Many-Valued Logics in the Hierarchy of Weak Nilpotent Minimum.

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Introduction

In this thesis we study particular subclasses of WNM algebras. The variety of WNM algebras forms the algebraic semantics of the *WNM logic*, a propositional many-valued logic that generalizes some well-known case in the setting of *triangular norms logics*.

A *triangular norm* T is a binary, associative and commutative $[0,1]$-valued operation on the unit square $[0,1]^2$ that is monotone, has 1 as identity, and has 0 as annihilator ($y \leq z$ implies $T(x,y) \leq T(x,z)$, $T(x,1) = x$, and $T(x,0) = 0$).[1] In the theory of fuzzy sets (introduced in [52] by L.A. Zadeh), triangular norms and their duals, triangular conorms, model respectively intersections and unions of fuzzy sets and hence provide natural interpretations for conjunctions and disjunctions of propositions whose truth values range over the unit interval. If a triangular norm T is left-continuous, then the operation $R = \max\{z \mid T(x,z) \leq y\}$, called the *residuum* of T, is the unique binary $[0,1]$-valued operation on the unit square that satisfies the residuation equivalence,

$$T(x,y) \leq z \text{ if and only if } x \leq R(y,z),$$

and hence, arguably acts as the logical implication induced by the interpretation of T as a logical conjunction (for instance, it implies right-distributivity of R over T). The variety of MTL algebras forms the algebraic counterpart of the *MTL logic*, the logic of all left-continuous triangular norms and their residua [23, 34], and the WNM logic lies in the hierarchy of its schematic extensions. Insisting on the continuity of T, the hierarchy of many-valued logics extending Hájek's Basic logic (BL) arises [28].

Among the known schematic extensions of MTL, WNM logic is the biggest one whose corresponding algebraic variety is *locally finite* [46]. That is, the *finitely generated free* WNM algebras are finite. This property allows a combinatorial study of subdirectly irreducible algebras in the variety. Pursuing this analysis, representation of free finitely generated algebras can be obtained, as already shown for several schematic extensions of WNM logic. Namely, Gödel, Nilpotent Minimum and NMG logics in [6] and RDP logic in [49].

[1] A complete treatment of t-norms can be found in the book [37].

Historically, however, the WNM logic has been introduced semantically. In [22], Esteva and Domingo study *weak negation functions* over $[0, 1]$, the *weak nilpotent minimum* left-continuous triangular norm is defined taking a weak negation function n and keeping the value of the minimum t-norm above the graph of n, while forcing the t-norm to be 0 under the graph. This construction is a generalization of Fodor's nilpotent minimum t-norm [24], where he consider an *involutive* negation. In these terms, WNM logic arises as a natural generalization among the family of triangular norm based logics.

In the present work, we extensively study two extension of WNM logic, namely RDP logic and NMG logic, from the point of view of algebraic and categorical logic.

The RDP logic has been introduced semantically, by Jenei. In [32], the author applies a generalization of the ordinal sum theorem of semigroups to the construction of new families of left-continuous triangular norms as ordinal sums of triangular subnorms. As a remarkable example of this machinery, the *Revised Drastic Product* left-continuous triangular norm arises by revising the left-discontinuous drastic product triangular norm [2] in such a way to render it left-continuous, obtaining the revised drastic product as the ordinal sum of the trivial triangular subnorm and the minimum triangular norm. Hence, RDP logic is a natural boundary case of t-norm based logics.

The NMG logic has been introduced in [51], where authors construct the NMG t-norm as an ordinal sum of the nilpotent minimum t-norm and the minimum t-norm. Hence, NMG logic appears as a generalization of two well-studied t-norm based logics, namely NM logic and Gödel logic. For an algebraic and categorical study of these two logics, we refer the interested reader to [14] and [5] for NM logic, and to [26] and [18] for Gödel logic. In [8] and [6] one can find combinatorial representation of free algebras associated to these logics. The latter paper contains also such a study for NMG logic.

Representations of free algebras for many-valued logics have a long history. As an example, in the 1951 [42] McNaughton gives a functional representation of free algebras corresponding to Łukasiewicz logics (an important extension of BL, see [28]). An explicit construction of this representation through normal forms is given by Mundici in [44]. Free BL algebras have been characterized in [2]. A complete survey on this subject is [3].

As the lattice reduct of a (finite) MTL algebra is a (finite) bounded distributive lattice, it is natural to study the dual space of such algebras building upon the Priestley (or Birkhoff, emphasizing finiteness) duality between finite bounded distributive lattices and bounded lattice homomorphisms, and finite posets and monotone maps [19, and references therein].

[2] The drastic product t-norm is the smallest t-norm with respect to the pointwise order.

This line of research has been followed in [5] and [4], where authors give spectral dualities for the variety of NM algebras and locally finite varieties of BL algebras respectively. In this thesis we develop spectral dualities between the varieties of RDP algebras and NMG algebras and suitable defined combinatorial categories. Exploiting these categorical equivalences, we give algorithmic construction of products in the dual categories obtaining concrete description of coproducts for the corresponding finite algebras. Moreover, we give representation theorems for finite algebras and free finitely generated algebras in the considered varieties. This latter characterization is especially useful to provide explicit construction of a number of objects relevant from the point of view of the logical interpretation of the two varieties of algebras: normal forms, strongest deductive interpolants and most general unifiers. Indeed, we show how to build these objects for RDP logic. The results on RDP algebras have previously appeared in [12] and [49].

Collecting this kind of knowledge about extensions of WNM logic will afford a good starting point to accomplish the same agenda for WNM logic itself. Indeed, the study of totally ordered WNM algebras will give the theoretical framework that we will use to develop the categorical equivalences for finite RDP and finite NMG algebras. Moreover, these dualities are a preliminary step in the direction of a duality for finite WNM algebras. As we will see at the end of this thesis, there is a strong belief that a "merging" of the approaches used for finite RDP algebras and for finite NMG algebras, enriched with some additional concept will lead to analogous results for finite WNM algebras.

This work on axiomatic extensions of WNM logic can be seen as a part of the study of algebraizable many-valued logics based on varieties of residuated lattices in the sense of [25]. Furthermore, concrete representations of different logics could be useful tools for other lines of research not explicitly faced in this thesis. In [38], definitions of probability measures are defined for algebraic systems of continuous functions over $[0, 1]$ and recently these definitions have been extended over non-classical events described by Gödel [9] and NM logics [7]. Characterizations of free algebras are useful tools for the prosecution of this field.

The thesis is organized as follows. After a necessary review of basic concepts given in Chapter 1, we start the investigation in Chapter 2 collecting results on finite (totally ordered) WNM algebras. The core of the thesis is represented by Chapter 3 and Chapter 4, where we develop the categorical equivalence for finite RDP algebras and for finite NMG algebras respectively. In Chapter 5 we recall normal forms construction for Gödel logic. This construction will be used in Chapter 6 where we derive normal forms for RDP logic in two ways, one based on totally ordered RDP chains, and one based on the duality developed in Chapter 3.

We conclude the thesis with a discussion on the free singly generated

WNM algebra $\mathbf{F}_1(WNM)$. We will compare this algebra with the free 1-generated RDP algebra $\mathbf{F}_1(RDP)$ and the free 1-generated NMG algebra $\mathbf{F}_1(NMG)$. This comparison is particularly useful to develop a possible approach to the categorical study of finite WNM algebras. Indeed, we will show that at least in the particular case given by $\mathbf{F}_1(WNM)$ its dual can be obtained as an *intelligent* merging of the dual objects of $\mathbf{F}_1(RDP)$ and $\mathbf{F}_1(NMG)$.

The appendixes contain basic notions of universal algebra and category theory.

Background

In the following two chapters we give the necessary background notions to set our work in the frame that fit our purposes. We start the first chapter introducing many-valued t-norm based logics as algebraizable logics in the sense of Blok-Pigozzi [11]. After a brief historical introduction to duality theory, we recall the dual equivalence between a suitably defined combinatorial category and the category of Gödel Algebras. In a sense that will be clear in the following chapters, this duality is the starting point to obtain categorical equivalences for the locally finite varieties corresponding to the t-norm based logics analyzed in this thesis. We will conclude the first chapter with background notions for the poset representations of free finitely generated algebras that will be detailed in Chapter 4.4.

The second chapter will be dedicated to the analysis of the algebras related to Weak Nilpotent Minimum logic. We briefly introduce some known results about the class of WNM algebras and we use them to characterize finite WNM chains, to describe the prime spectrum of the free 1-generated WNM algebra, and to settle some properties of directly indecomposable WNM algebras. The theoretical stuff developed for finite WNM algebras will be refined to RDP algebras in Chapter 3 and to NMG algebras in Chapter 4.

Chapter 1

Basic Notions

1.1 Many-Valued Logics

Let $X = \{x_1, x_2, \dots\}$ be a countable set of *variables*. A *language* over X is a nonempty set L disjoint from X, such that a non-negative integer n is associated to every $c \in L$. Members of L are called *connectives*, and the associated integer is called the *arity* of the connective. The set Fm of formulas over X is inductively defined in the following way,

- every x_i in X is a formula,
- if c is a connective of arity n, and $\varphi_1, \dots, \varphi_n$ are formulas, then $c(\varphi_1, \dots, \varphi_n)$ is a formula.

The concept of language can be used in order to specify the symbols that denote the operations of an algebra. In this case, we will say *type* instead of language and *term* instead of formula (see Sections A.1 and A.5 in Appendix A). By the inductive definition of formula, $\mathbf{Fm} = \langle Fm, c_1, \dots, c_n \rangle$ is an algebra of type $L = \{c_1, \dots, c_n\}$. An *equation* of type L is a pair $\varphi = \psi$ of formulas. We denote Eq the set of equations. We call *substitutions* endomorphisms on $\mathbf{Fm}$. We can apply a substitution σ to a set of formulas Γ, that is $\sigma(\Gamma) = \{\sigma(\varphi) \mid \varphi \in \Gamma\}$. We denote with $\varphi(x_1/\psi_1, \dots, x_n/\psi_n)$ the result of substituting in a uniform manner every occurrence of x_i with an occurrence of ψ_i.

A *consequence relation* over $\mathbf{Fm}$ is a relation $\vdash \subseteq \mathcal{P}(Fm) \times Fm$ [1] that satisfies the following properties,

- $\varphi \vdash \varphi$;
- if $\Gamma \vdash \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash \varphi$;
- if $\Gamma \vdash \varphi$ and $\Delta \vdash \psi$ for every $\psi \in \Gamma$, then $\Delta \vdash \varphi$;

[1] We denote with $\mathcal{P}(S)$ the powerset of a set S.

where Γ and Δ are in $\mathcal{P}(Fm)$

A *propositional logic* is a pair $\mathcal{L} = \langle \mathbf{Fm}, \vdash \rangle$ where $\vdash$ is such that if $\Gamma \vdash \varphi$ and σ is a substitution on $\mathbf{Fm}$, then $\sigma(\Gamma) \vdash \sigma(\varphi)$. A formula φ is a *theorem* of a logic $\mathcal{L}$ when $\emptyset \vdash \varphi$.

An *inference rule* of $\mathbf{Fm}$ is a pair (Γ, φ), where $\Gamma \subseteq Fm$ and $\varphi \in Fm$. We call *axiom* a rule of the form $(\emptyset, \varphi)$. An *Hilbert-style calculus* is a set of inference rules that contains at least an axiom and at least a rule (Γ, φ) where Γ is not the empty set. Given an Hilbert-style calculus H, if $\Delta \cup \psi \subseteq Fm$ then a *proof* of ψ from Δ in H is a finite sequence $\psi_1, \dots, \psi_n$ of formulas, such that $\psi_n = \psi$ and for each ψ_i

- either $\psi_i \in \Delta$;
- or $\psi_i = \sigma(\varphi)$ where σ is a substitution and φ is an axiom;
- or there exist an inference rule (Γ, φ) such that $\psi_i = \sigma(\varphi)$ and $\sigma(\phi) \in \{\psi_1, \dots, \psi_{i-1}\}$ for every ϕ in Γ.

In this case, we say that φ is *derivable* (or *provable*) from Γ in H, denoted as $\Gamma \vdash_H \varphi$. The pair $\langle \mathbf{Fm}, \vdash_H \rangle$ is a logic and has the property that when $\Gamma \vdash_H \varphi$, there exists a finite subset $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash_H \varphi$. Propositional logics having this property are called *finitary*.

Let K be a class of algebras of the same type of $\mathbf{Fm}$, and let $\models$ be the equational consequence relation of K (see Appendix A),

K is an *algebraic semantics* of a logic $\mathcal{L}$ if and only if there is a set of *defining equations* in one variable

$$(\varphi(x) = \psi(x)) := \{\varphi_i(x) = \psi_i(x) \mid i \in I\},$$

such that

$$\Gamma \vdash \alpha \quad \text{if and only if} \quad \{\varphi(\gamma) = \psi(\gamma) \mid \gamma \in \Gamma\} \models \varphi(\gamma) = \psi(\gamma).$$

A logic $\mathcal{L} = \langle \mathbf{Fm}, \vdash \rangle$ is *algebraizable* (in the sense of Blok-Pigozzi [11]) with *algebraic semantics* K, if and only if there exist a set of formulas in two variable $\{\phi_j(x, x')\}_{j \in J}$ and two maps $g(\varphi = \psi) := \{\phi_j(x/\varphi, x'/\psi)\}_{j \in J}$ and $f(\alpha) := \{\varphi_i(\alpha) = \psi_i(\alpha)\}_{i \in I}$, such that for any $\alpha, \varphi, \psi \in Fm$ the following holds,

- $\Gamma \vdash \alpha$ if and only if $f(\Gamma) \models f(\alpha)$;
- $\Sigma \models \varphi = \psi$ if and only if $g(\Sigma) \vdash g(\varphi = \psi)$;
- $\alpha \vdash g(f(\alpha))$ and $g(f(\alpha)) \vdash \alpha$;
- $\varphi = \psi \models f(g(\varphi = \psi))$ and $f(g(\varphi = \psi)) \models \varphi = \psi$;

where $f(\Gamma) = \{f(\gamma) \mid \gamma \in \Gamma\}$ and $g(\Sigma) = \{g(e) \mid e \in \Sigma\}$.

The logics that we consider in this thesis are all built over the finite language $L = \{\wedge, \odot, \to, \perp\}$, where $\perp$ has arity 0 and every other connective has arity 2. In this last case we employ the infix notation, and we call them *binary* connectives.

In the following we denote by $\varphi^{\mathbf{A}}(x_1, \ldots, x_n)$ the result of the application to φ of the homomorphisms $h : \mathbf{Fm} \to \mathbf{A}$ such that $h(x_i) = (a_i)$ for¡ $i \in \{1, \ldots, n\}$, where $\mathbf{A}$ is an algebra, $a_1, \ldots, a_n$ are elements of A, and $\varphi(x_1, \ldots, x_n)$ is a formula containing at most the $x_1, \ldots, x_n$ variables. We write simply $\varphi^{\mathbf{A}}$ when the presence of $x_1, \ldots, x_n$ is clear from the context (or inessential).

1.1.1 Triangular Norms

We start this section recalling definitions and results about weak negation functions detailed in [22].

Definition 1.1.1. *A* weak negation *is a function* $n : [0,1] \to [0,1]$ *such that:*

- $n(1) = 0$*;*
- *if* $x \leq y$ *then* $n(x) \geq n(y)$*;*
- $n(n(x)) \geq x$*;*

for all $x, y \in [0,1]$*. As a consequence we obtain* $n(0) = 1$*.*

A weak negation function is called *involutive* (or *strong negation*) if and only if $x = n(n(x))$ for every x in $[0,1]$. A prototype of every involutive negation is the *standard negation*, defined by $n_s(x) = 1 - x$. Indeed, Trillas [47] has proved that any involutive negation $n_i : [0,1] \to [0,1]$ can be obtained from the standard one in the following way,

$$n_i(x) = (m^{-1} \circ n_s \circ m)(x)$$

where $m : [0,1] \to [0,1]$ is a monotone bijection. Examples of involutive and standard negations are depicted in Figure 1.1.

In [22] authors introduce a notions of symmetry useful to characterize weak negations. A non-increasing function $n : [0,1] \to [0,1]$ is *symmetric* with respect to the identity when it satisfies the following properties:

- if $x \in n([0,1])$ then $n(x) = y$ implies $x = n(y)$;
- if $x \notin n([0,1])$ then:
  - n is constant in the interval $[x, n(n(x))]$ with value $n(x)$,

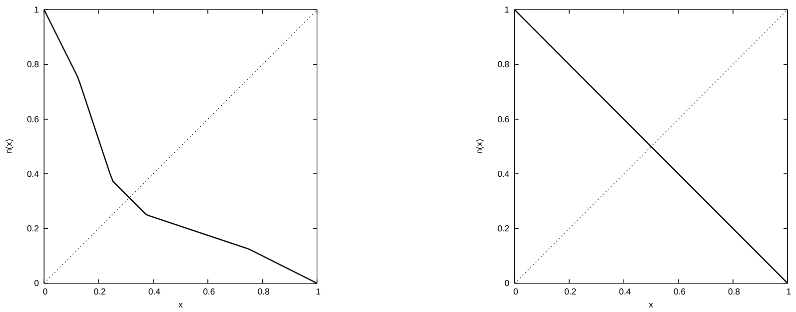


Figure 1.1: An involutive negation and the standard one.

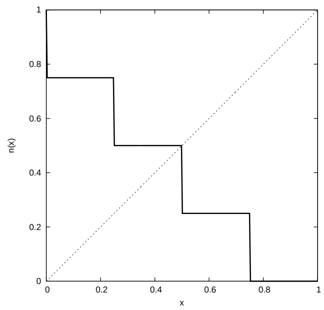


Figure 1.2: A graphic of weak negation with drawn vertical lines corresponding to the jumps.

- for any $y > n(x)$ we have $n(y) < x$, that is $n(x)$ is a discontinuity point on the right with $n(n(x)^-) = n(n(x))$ and $n(n(x)^+) = min\{z \mid n \text{ is constant in } [z, n(n(x))]\} < x$.

Proposition 1.1.1 ([22])**.** *$n : [0,1] \to [0,1]$ is a weak negation function if and only if it is non-increasing and symmetric with respect to the identity.*

This condition forces weak negations to have a behavior that can be graphically explained. Indeed, given the graphic of a weak negation n, if we draw the vertical lines corresponding to the discontinuity jumps of n, then the resulting graphic is symmetrical with respect to $y = x$ (see Figure 1.2).

Definition 1.1.2. *A* triangular norm *(t-norm for short) $*$ is a binary, associative and commutative operation on the unit square $[0,1]^2$ that is monotone, has 1 as identity, and has 0 as annihilator, i.e. for all $x, y, z \in [0,1]$:*

1. $x * y = x * y$,
2. $x * (y * z) = (x * y) * z$,
3. $y \leq z$ *implies* $x * y \leq x * z$,

4. $x * 1 = x$ *and* $x * 0 = 0$.

The three *basic* continuous t-norms are,

- *Łukasiewicz t-norm*: $x *_Ł y = max(x + y - 1, 0)$;
- *Product t-norm*: $x *_P y = x \cdot y$;
- *Gödel t-norm*: $x *_G y = min(x, y)$.

See Figure 1.3 for graphics of Łukasiewicz and Gödel t-norms.

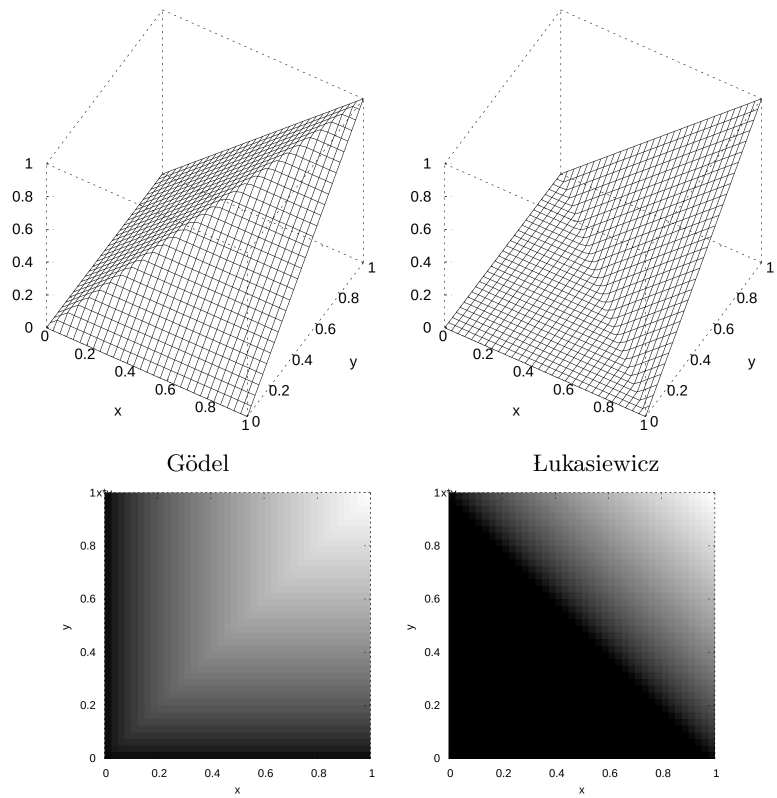


Figure 1.3: Gödel and Łukasiewicz t-norms.

Given increasing sequences $(x_n)_{n \in \mathbb{N}}$,$(y_n)_{n \in \mathbb{N}}$ in $[0, 1]$,, such that:

$$sup\{x_n | n \in \mathbb{N}\} = x \qquad sup\{y_n | n \in \mathbb{N}\} = y$$

a t-norm $*$ is left-continuous if and only if $sup\{x_n * y_n | n \in \mathbb{N}\} = x * y$ [2].

Given a t-norm $*$, left-continuity of $*$ is the necessary and sufficient condition for the existence of the *residuum* of $*$, that is, the operation $\Rightarrow$

[2]If we substitute *sup* with the limit operation $n \to \infty$ we obtain continuity of the t-norm.

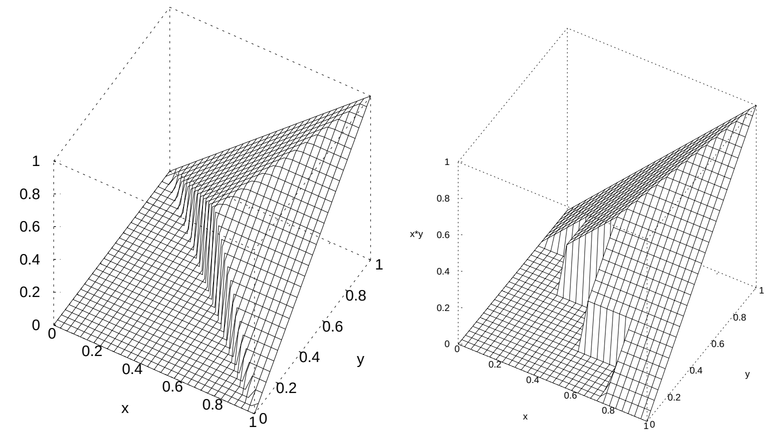


Figure 1.4: On the left the Nilpotent Minimum T-norm. On the right a Weak Nilpotent Minimum t-norm defined by (1.2), where n is the weak negation in Figure 1.2.

that satisfies the *residuation property*,

$$x * z \leq y \text{ if and only if } z \leq x \Rightarrow y. \tag{1.1}$$

In this case, the residuum is defined as $x \Rightarrow y = max\{z \mid x * z \leq y\}$. A couple of operations $(*, \Rightarrow)$ that satisfies the residuation property is called a *residuated pair*.

Continuous t-norms have a nice representation theorem, every continuous t-norm is an ordinal sum whose components are isomorphic to one of the three basic t-norms [40] (see [43] for the same result in a more general setting). A similar characterization for left-continuous t-norms is still lacking.

The first left-continuous t-norms appeared in literature is the *nilpotent minimum* [24],

$$x * y = \begin{cases} min(x, y) & \text{if } x \geq n(y), \\ 0 & \text{otherwise} \end{cases} \tag{1.2}$$

where n is an involutive negation.

Advances in the construction of left-continuous t-norms using involutive negations has been made by Jenei in [33] and in [31], where he generalizes (1.2) using a continuous t-norms instead of the minimum. In [23], authors introduce the *weak nilpotent minimum* t-norm using (1.2), where n is not an involutive negation but it may be any weak negation. See Figure 1.4 as an example. All the logics investigated in this thesis are based on weak nilpotent minimum t-norms.

1.1.2 T-norm based Logics

A logic $\mathcal{L} = \langle L, \vdash \rangle$ is *t-norm based* if and only if there exists a class of algebras K of type L whose support is $[0,1]$ and the interpretation of $\odot$ and $\rightarrow$ is given by a t-norm and its residuum.

In this section we will recall some notions about the weakest t-norm based logic and some of its extensions.

The *Monoidal T-norm based Logic* (MTL) has been introduced in [23] as the logic of all left-continuous t-norms and their residua.

MTL is defined over the language L and has the following Hilbert-style calculus,

(A1) $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$

(A2) $(\varphi \odot \psi) \rightarrow \varphi$

(A3) $(\varphi \odot \psi) \rightarrow (\psi \odot \varphi)$

(A4) $(\varphi \wedge \psi) \rightarrow \varphi$

(A5) $(\varphi \wedge \psi) \rightarrow (\psi \wedge \varphi)$

(A6) $(\varphi \odot (\varphi \rightarrow \psi)) \rightarrow (\varphi \wedge \psi)$

(A7a) $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \odot \psi) \rightarrow \chi)$

(A7b) $((\varphi \odot \psi) \rightarrow \chi)) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$

(A8) $((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi)$

(A9) $\bot \rightarrow \varphi$

where the only inference rules is *modus ponens* $(\{\varphi, \varphi \rightarrow \psi\}, \psi)$. We can write it in the classical fractional form,

$$\frac{\varphi \qquad \varphi \rightarrow \psi}{\psi}$$

We define some derived connectives:

$$\begin{aligned} \varphi \vee \psi &:= ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi); \\ \varphi \leftrightarrow \psi &:= (\varphi \rightarrow \psi) \odot (\psi \rightarrow \varphi); \\ \neg \varphi &:= \varphi \rightarrow \bot; \\ \top &:= \neg \bot. \end{aligned}$$

We will use φ^n to denote the $\odot$-conjunction of φ with itself for n times.

A set Γ of formulas is called a *theory*. A theory Γ is said *consistent* if it is impossible to derive $\bot$ from Γ, in symbols $\Gamma \nvdash_{MTL} \bot$.

The *local deduction theorem* of MTL is the following.

Theorem 1.1.1 ([23]). *Given a theory* Γ *and* φ, ψ *formulas, then there exists a* $n \in \mathbb{N}$ *such that:*

$$\Gamma \cup \{\varphi\} \vdash_{MTL} \psi \textit{ if and only if } \Gamma \vdash_{MTL} \varphi^n \to \psi.$$

We say that a logic $\mathcal{L}$ is a *schematic extension* of MTL if $\mathcal{L}$ is obtained adding some axioms to MTL.

Basic Logic (BL), the logic of all continuous t-norms and their residua [28], is obtained adding *divisibility* axiom to MTL:

$$\varphi \wedge \psi \to \varphi \odot (\varphi \to \psi). \tag{div}$$

Adding the following axiom we have the logic of weak nilpotent minimum:

$$\neg(\varphi \odot \psi) \vee ((\varphi \wedge \psi) \to (\varphi \odot \psi)). \tag{WNM}$$

If we add to WNM the following axiom we obtain NMG logic [51]:

$$(\neg\neg\varphi \to \varphi) \vee ((\varphi \wedge \psi) \to (\varphi \odot \psi)). \tag{NMG}$$

Adding *involutivity* to WNM we obtain the logic of nilpotent minimum (NM):

$$\neg\neg\varphi \to \varphi. \tag{inv}$$

Revised Drastic Product logic [50] is obtained adding to WNM (or to MTL) the axiom:

$$\neg\neg\varphi \vee (\varphi \to \neg\varphi) \tag{RDP}$$

Gödel logic is the schematic extension of WNM (of RDP, NMG, BL and MTL too) obtained adding the *idempotency* axiom:

$$\varphi \to (\varphi \odot \varphi) \tag{id}$$

Adding the *excluded middle* law $\varphi \vee \neg\varphi$ to MTL we obtain the classical propositional logic B.

See Figure 1.5 for a diagram of schematic extensions of MTL.

MTL logic is algebraizable in the sense of Blok-Pigozzi and Lindenbaum-Tarski, and the class of MTL algebras forms the algebraic semantics of MTL logic [23].

An MTL algebra $\mathbf{A} = \langle A, \odot^{\mathbf{A}}, \to^{\mathbf{A}}, \wedge^{\mathbf{A}}, \vee^{\mathbf{A}}, \bot^{\mathbf{A}}, \top^{\mathbf{A}} \rangle$ is a bounded lattice where $(A, \odot^{\mathbf{A}}, \top^{\mathbf{A}})$ is a monoid, $(\wedge^{\mathbf{A}}, \vee^{\mathbf{A}})$ are lattice operations, and for every $x, y \in A$ *prelinearity* holds,

$$(x \to^{\mathbf{A}} y) \vee^{\mathbf{A}} (y \to^{\mathbf{A}} x) = \top^{\mathbf{A}}, \tag{1.3}$$

and $(\odot^{\mathbf{A}}, \to^{\mathbf{A}})$ is a residuated pair, then satisfies the *residuation* condition,

$$x \odot^{\mathbf{A}} y \leq z \text{ if and only if } x \leq y \to^{\mathbf{A}} z. \tag{1.4}$$

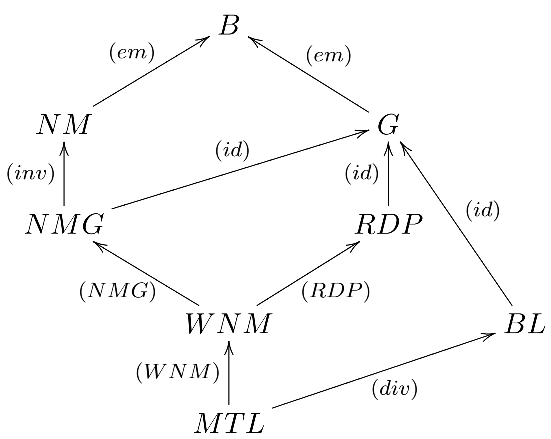


Figure 1.5: Some axiomatic extensions of MTL.

Negation operation is defined as $\neg^{\mathbf{A}} x = x \rightarrow^{\mathbf{A}} \perp^{\mathbf{A}}$. We call *MTL chain* a totally ordered MTL algebra. Examples of MTL chains are given by the *standard* MTL chains,

$$[\mathbf{0}, \mathbf{1}]_* = \langle [0,1], *, \Rightarrow, min, max, 0, 1 \rangle \tag{1.5}$$

where $*$ is a left-continuous t-norm and $\Rightarrow$ is its residuum.

Notation We have used the same symbols for logical connectives and their corresponding algebraic operations, adding to the latter a superscript to denote the associated algebra. When there is no ambiguity we will drop the superscript from the algebraic operations. We will do an exception in standard algebras, as in (1.5), where for t-norms and their associated residua we will adopt symbols different from their corresponding logical connectives.

MTL algebras are definable by a finite set of equations. Hence, by Birkhoff's Theorem A.5.1 they form a variety of algebras (denoted $\mathbb{V}(MTL)$).

Since MTL is an algebraizable logic, for every MTL formula φ we can obtain an algebraic MTL term t_φ replacing $\odot$ by $\odot^{\mathbf{A}}$, $\rightarrow$ by $\rightarrow^{\mathbf{A}}$, $\vee$ by $\vee^{\mathbf{A}}$ and $\wedge$ by $\wedge^{\mathbf{A}}$. Viceversa, given a term t we denote with φ_t the MTL formula obtained with the inverse substitution. Hence, every subvariety of $\mathbb{V}(MTL)$ corresponds to a schematic extension of MTL. Indeed, let $\mathcal{L}$ be an axiomatic extension of MTL obtained adding the set of formulas Γ to the axioms of MTL. Then, its algebraic semantics is given by the variety $\mathbb{V}(\mathcal{L}) \subseteq \mathbb{V}(MTL)$ obtained by the equations $\{t_\varphi = \top \mid \varphi \in \Gamma\}$. Conversely, given a subvariety $\mathbb{V}(\mathcal{L})$ of $\mathbb{V}(MTL)$ defined by terms in Σ, the corresponding logic is obtained from MTL by adding $\{\varphi_t \leftrightarrow \psi_r \mid t, r \in \Sigma\}$ as axioms. Hence, *WNM Algebras* are MTL algebras that satisfy the *weak nilpotent minimum* equation,

$$\neg(x \odot y) \vee ((x \wedge y) \rightarrow (x \odot y)) = \top. \tag{WNM}$$

An *NMG algebra* is a WNM algebra satisfying:

$$(\neg\neg x \to x) \vee (x \wedge y \to x \odot y) = \top. \tag{NMG}$$

If $\neg\neg x = x$ holds in a NMG algebra $\mathbf{A}$, then $\mathbf{A}$ is a *NM algebra*. An *RDP Algebra* is a WNM algebra where the following holds

$$\neg\neg x \vee (x \to \neg x) = \top. \tag{RDP}$$

A *Gödel* algebra is a WNM algebra satisfying *idempotency*, $x \odot x = x$. *Boolean algebras* are MTL algebras that satisfy $x \vee \neg x = \top$.

In this light, the diagram of Figure 1.5 can be seen as a lattice of subvarieties ordered by reverse inclusion.

Let $\mathbf{A}$ be an MTL algebra. An *evaluation* e is a map assigning to each propositional variable v an element of A. Such an evaluation can be uniquely extended to all propositional formulas as follows:

$$\begin{aligned} e(\bot) &= \bot^{\mathbf{A}}; \\ e(\varphi \wedge \psi) &= e(\varphi) \wedge^{\mathbf{A}} e(\psi); \\ e(\varphi \vee \psi) &= e(\varphi) \vee^{\mathbf{A}} e(\psi); \\ e(\varphi \odot \psi) &= e(\varphi) \odot^{\mathbf{A}} e(\psi); \\ e(\varphi \to \psi) &= e(\varphi) \to^{\mathbf{A}} e(\psi). \end{aligned}$$

A formula φ is an $\mathbf{A}$-*tautology* of an MTL algebra $\mathbf{A}$ if and only if $e(\varphi) = \top^{\mathbf{A}}$ for all evaluations e. We say that φ is a *tautology* when φ is a $\mathbf{A}$-*tautology* of all MTL algebras $\mathbf{A}$.

Theorem 1.1.2 ([23])**.** *MTL logic is complete with respect to $\mathbb{V}(MTL)$. That is, for every formula φ of MTL logic, the following statements are equivalent:*

- $\vdash_{MTL} \varphi$;
- *φ is a tautology.*

The variety $\mathbb{V}(MTL)$ is generated by the class of MTL chains.

Theorem 1.1.3 ([23])**.** *Every MTL algebra is a subdirect product of MTL chains.*

As a consequence,

Theorem 1.1.4 ([23])**.** *MTL logic is complete with respect to the class of MTL chains. That is, for every formula φ of MTL logic, the following statements are equivalent:*

- $\vdash_{MTL} \varphi$;

- φ *is a* $\mathbf{A}$*-tautology for each MTL chain* $\mathbf{A}$.

Since we are describing t-norm based logics, another interesting result is given by the completeness with respect to the class of standard MTL algebras. Jenei and Montagna ([34]) have shown that MTL logic is standard complete.

Theorem 1.1.5 ([23])**.** *MTL logic is complete with respect to the class of standard MTL chains. That is, for every formula* φ *of MTL logic, the following statements are equivalent:*

- $\vdash_{MTL} \varphi$;
- φ *is a* $[\mathbf{0},\mathbf{1}]_*$*-tautology for each standard MTL chain* $[\mathbf{0},\mathbf{1}]_*$.

Extensions of MTL whose algebraic semantics is a locally finite variety are amenable to a combinatorial study. Indeed, in these varieties the classes of finitely presented, of finitely generated, and of finite algebras coincide. Hence, by Theorem 1.1.3 and Theorem 1.1.4 we can restrict our attention to finitely generated chains. The basis of this approach will be established in Chapter 2 for WNM logic, and will be further developed in Chapter 3 and Chapter 4 for RDP and NMG logic respectively.

In the following we introduce some key concepts for our categorical investigation of locally finite subvarieties of $\mathbb{V}(MTL)$.

Definition 1.1.3. *Let* $\mathbf{A}$ *be an MTL algebra and* S *be a subset of* A*. Then,* S *is an* upper set *of* $\mathbf{A}$ *when* $\top^{\mathbf{A}} \in S$*, and if* $x \in S$ *and* $x \leq y$ *then* $y \in S$*. Dually,* S *is a* lower set *of* $\mathbf{A}$ *when* $\bot^{\mathbf{A}} \in S$*, and if* $x \in S$ *and* $x \geq y$ *then* $y \in S$*.*

Given an arbitrary subset S *of* A *and* $x \in S$*, we denote* $\uparrow S$ *the smallest upper set containing* S*. Dually, we denote* $\downarrow S$ *the smallest lower set containing* S*. We let* $\uparrow x = \uparrow \{x\}$ *and* $\downarrow x = \downarrow \{x\}$*.*

Definition 1.1.4. *Given a upper set* F *of* $\mathbf{A}$*, if* F *satisfies also the following condition*

$$\text{for all } x, y \in F \text{ we have } x \odot^{\mathbf{A}} y \in F,$$

then F *is called a* filter *of* $\mathbf{A}$*.*

A filter F *is said to be* proper *when* $\bot^{\mathbf{A}} \notin F$*. Given a proper filter* F*, we call* F *a* prime *filter if for all* $x, y \in A$ *either* $x \to^{\mathbf{A}} y \in F$ *or* $y \to^{\mathbf{A}} x \in F$*.*

It can be shown that the family of all filters of an MTL algebra is closed under arbitrary intersection. Hence, given a subset B of A we call *filter generated* by B, the intersection of all filters containing B. When a filter is generated by a single element $x \in A$ we will denote it F_x.

Definition 1.1.5. *We call* prime spectrum *of an MTL algebra* $\mathbf{A}$*, the poset of prime filters of* $\mathbf{A}$ *ordered by reverse inclusion and we denote it with* $\mathsf{Spec}\mathbf{A}$*.*

Let $\mathbf{A}$ be an MTL algebra. For every filter F of $\mathbf{A}$ we can define a congruence as follows,

$$\theta_F := \{(x, y) \in A^2 \mid (x \leftrightarrow^{\mathbf{A}} y) \in F\} \tag{1.6}$$

and for every congruence θ of $\mathbf{A}$ we define a filter as follows,

$$F(\theta) := \{x \in A \mid (x, \top^{\mathbf{A}}) \in \theta\}. \tag{1.7}$$

We denote with $[x]_F$ the equivalence classes of θ_F where $x \in \mathbf{A}$.

The following result will be very useful in the following chapters, where we use the relation between prime filters and chains to characterize prime spectra of the investigated algebras.

Proposition 1.1.2 ([23]). *Let* $\mathbf{A}$ *be an MTL algebra and* F *be a prime filter. Then,* $\mathbf{A}/\theta_F$ *is an MTL chain.*

1.2 Duality Theory

In 1936, M. H. Stone started duality theory by establishing a dual equivalence between the category of Boolean algebras and the category of the so-called Boolean Spaces [3] (see [35]).

The restriction of Stone's duality to finite objects yields the well-known duality between finite Boolean algebras and sets. Indeed, let $\mathbf{B}$ be a finite Boolean algebra and denote with $A(\mathbf{B})$ its set of atoms [4]. Then, the map

$$b \in \mathbf{B} \mapsto \{a \in A(\mathbf{B}) \mid a \leq b\}$$

is an isomorphism of $\mathbf{A}$ onto the powerset of $A(\mathbf{B})$. We can define the inverse of the above map by taking $\bigcup S$ for every S subset of $A(\mathbf{B})$.

The correspondence between finite Boolean algebras and powersets is a very special case of the Birkhoff's representation theorem for finite distributive lattices.

Recall that an element d of a distributive lattice $\mathbf{D}$ is called *join-irreducible* if d is not the bottom of $\mathbf{D}$ and if $d = a \vee b$ then $d = a$ or $d = b$. Denote with $J(\mathbf{D})$ the set of join-irreducible elements of $\mathbf{D}$. Then, $\langle J(\mathbf{D}), \leq\rangle$ is a poset where $\leq$ is the order relation inherited from $\mathbf{D}$. For each poset $\langle \mathbf{P}, \leq\rangle$ we denote $O(\mathbf{P})$ the distributive lattice of all lower sets of $\langle \mathbf{P}, \leq\rangle$, where $\cup$ and $\cap$ plays the role of join and meet respectively. Then, Birkhoff has shown that every finite distributive lattice $\mathbf{D}$ is isomorphic to $\langle O(J(\mathbf{D})), \leq\rangle$, and every poset $\langle \mathbf{P}, \leq\rangle$ is isomorphic to $J(O(\mathbf{P}))$. We call $\langle J(\mathbf{D}), \leq\rangle$ the *dual poset* of $\mathbf{D}$ and $O(\mathbf{P})$ the *dual lattice* of $\langle \mathbf{P}, \leq\rangle$.

[3] A Boolean Space is a compact Hausdorff space having a basis of clopen sets.

[4] An element a of a Boolean algebra is an *atom* if it covers $\perp^{\mathbf{A}}$.

This duality can be extended to the duality between the category of finite bounded distributive lattices and bounded lattices homomorphisms, and the category of finite posets and order-preserving maps. Indeed, given a bounded lattice homomorphism $h : \mathbf{D} \to \mathbf{D}'$ its dual $f_h : J(\mathbf{D}') \to J(\mathbf{D})$ is defined by

$$f_h(x) = min\{y \in J(\mathbf{D}) \mid x \in h(\downarrow y)\},$$

while the dual of an order-preserving map $f : J(\mathbf{D}') \to J(\mathbf{D})$ is given by

$$h_f(a) = f^{-1}(a).$$

Removing finiteness restrictions, it is possible to obtain the Priestley duality between the category of bounded distributive lattices and bounded lattices homomorphisms, and the category of Priestley spaces [5] and continuous order-preserving maps. For more on this subject see [19] and references therein. Following Stone and Priestley dualities, in [21], Esakia establishes a duality for Heyting algebras and their homomorphisms. In the finite case, the dual category consists of finite posets and monotone maps sending lower sets to lower sets (which we call *open* maps here, despite the original terminology); such maps dualize exactly those lattice homomorphisms that preserve the residual of the lattice meet, namely, intuitionistic implication. Diverting the intuitionistic paradigm, the role of many-valued implication over MTL algebras is played by the residual of the monoidal operation $\odot$ discussed above, which is added to the lattice. Therefore, to dualize subvarieties of MTL algebras, plain posets and open maps are not sufficient, even when one restricts attention to finite objects only. Suitable additional structure does become necessary. This line of research has been pursued in [5], where an enriched Priestley duality for the finite objects in a pertinent locally finite subvariety of MTL algebras has been presented. [6] In the same vein, we develop in this thesis Priestley dualities for finite RDP algebras and finite NMG algebras, and prove categorical equivalences between finite classes of algebras and suitably defined combinatorial categories.

[5] A compact totally order-disconnected space is called *Priestley space.*

[6] It is worth mentioning that in recent work, Cabrer and Celani, building on [16, 48], give dualities for several algebraic varieties of bounded distributive lattices with additional (logical) operators, including non locally finite varieties and in particular, MTL algebras [15]. Their very general technique, motivated by the topological characterization of congruences in these varieties, relies upon the systematic translation of the equations defining the target algebraic class into (possibly first-order) relational conditions over the dual Priestley space. We believe that similar dualities can be attained for diverse locally finite subvarieties of MTL algebras, including several subvarieties of WNM algebras. In the spirit of the present work, it would be interesting to understand whether such general methods support explicit descriptions of algebraic coproducts and free algebras on the primal side; this would potentially enlighten widely open problems such as, for instance, a satisfactory representation of free finitely generated MTL algebras.

1.2.1 Finite Gödel Algebras

A *forest* is a finite poset such that for every $x \in F$ the lower set $\downarrow x$ is a chain. A forest with a bottom element is called a *tree* and its bottom element is called *root*. A lower set of a tree is called *subtree*. We denote with $F_\perp$ the tree $\perp \oplus F$ obtained by adding to the forest F a new root $\perp$. A function between forests is *open*, if it carries lower sets to lower sets.

We record the categorical equivalence between the category of finite Gödel algebras and their homomorphisms, FG, and the category of finite forests and open maps, F, presented in [18]. The equivalence is based on the Horn's proof [30] that a finite Gödel algebra is directly indecomposable if and only if its prime spectrum is a tree. Indeed, let F, F' and F'' be prime filters of a finite directly indecomposable Gödel algebra $\mathbf{A}$ such that $F \subseteq F'$ and $F \subseteq F''$. Suppose that F' and F'' are incomparable. Then, there exist $x \in F'$ and $y \in F''$ such that $x \notin F''$ and $y \notin F'$. By definition of prime filters, either $x \to y \in F$ or $y \to x \in F$. Then, by residuation $x \odot (x \to y) \leq y$. Without loss of generality, suppose that $x \to y \in F$. Since $y \notin F'$ it follows $x \odot (x \to y) \notin F'$. By hypothesis $x \in F'$, then $(x \to y) \notin F'$ in contradiction with $F \subseteq F'$.

From the above discussion and the fact that every finite algebra is isomorphic to the direct product of its directly indecomposable factors (Theorem A.4.1), we can state:

Theorem 1.2.1. FG *and* F *are dually equivalent via the contravariant functor* Spec*, defined as follows: for every object* $\mathbf{A}$ *in* FG*,*

$$\mathsf{Spec}(\mathbf{A}) = (\{F \subseteq A \mid F \textit{ prime filter}\}, \supseteq);$$

for every morphism $h\colon \mathbf{A} \to \mathbf{B}$ *in* FG*,* $\mathsf{Spec}(h)$ *is the open map sending each prime filter* F *in* $\mathsf{Spec}(\mathbf{B})$ *to the prime filter in* $\mathsf{Spec}(\mathbf{A})$ *defined as follows:*

$$(\mathsf{Spec}(h))(F) = \{a \in A \mid h(a) \in F\}. \tag{1.8}$$

In Figure 1.6 is depicted a finite Gödel algebra and its prime spectrum.

Adding just a bit of information to each tree, authors in [5] show that the above duality can be extended to the class of NM algebras, the algebraic variety corresponding to NM logic. In the following chapters we extend the above duality in two different ways in order to obtain analogous results for the classes of finite RDP algebras and finite NMG algebras.

In [18] authors give a description of product in F based on partitions associated to trees. Here we recall a more compact definition that is useful when we are only concerned with product of objects and not with the associated projections. Given two forests F and F' their coproduct $F + F'$ is the forest over $F \cup F'$ formed by defining $x \leq y$ if and only if $x, y \in F$ and $x \leq y$ in F or $x, y \in F'$ and $x \leq y$ in F'.

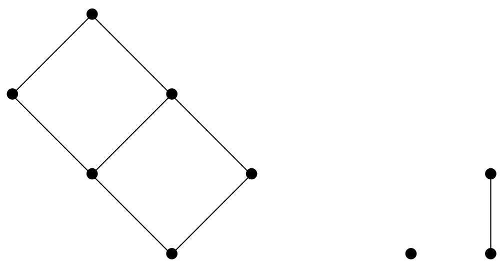


Figure 1.6: The order structure of a finite Gödel algebra (in fact, the free 1-generated Gödel algebra) and its prime spectrum.

Let F,F' and G be forests in F. We define the product of forests by the following rules,

(P1) $F \times F' \cong F'$ when $|F| = 1$;

(P2) $G \times (F \times F') \cong (G \times F) + (G \times F')$;

(P3) $F_\perp \times F'_\perp \cong ((F_\perp \times F') + (F \times F') + (F \times F'_\perp))_\perp$.

See Figure 1.7 as an example of the product of trees.

Lemma 1.2.1 ([3]). *The product $F \times F'$ satisfies the universal property of products in* F.

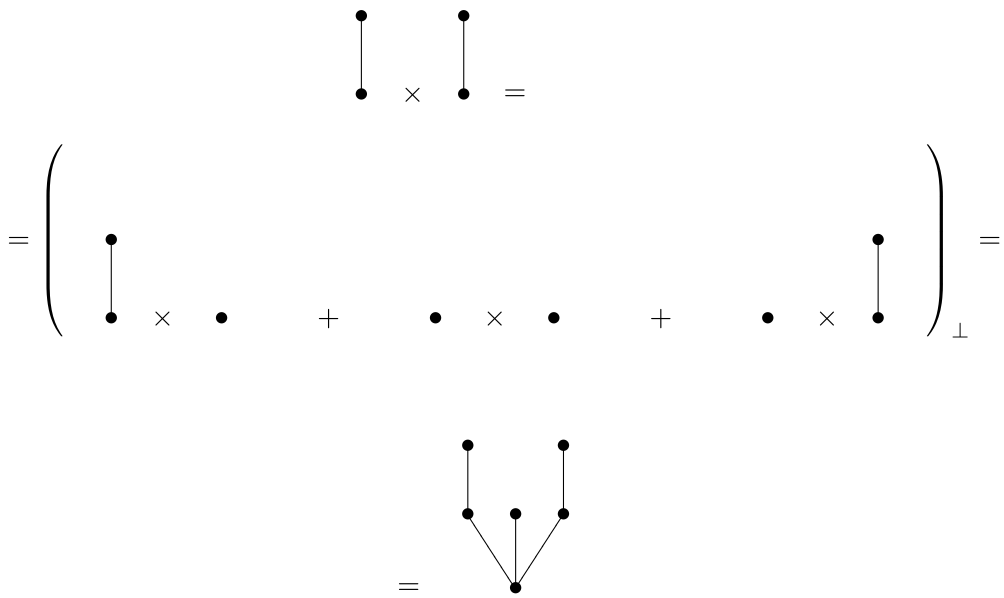


Figure 1.7: The product of two elements chain.

1.3 Notions on Posets and Normal Forms

We conclude the chapter recalling some useful notion on posets and introducing the semantical definitions of minterms and maxterms given in [6]. The following concepts will be useful in Chapter 5 and Chapter 6 when dealing with poset representations and normal forms for Gödel and RDP logics.

Given two disjoint posets $(A, \leq)$ and $(B, \leq)$, their *horizontal sum* $A \sqcup B$ is the poset over $A \cup B$ formed by defining $x \leq y$ if and only if $x, y \in A$ and $x \leq y$ in A or $x, y \in B$ and $x \leq y$ in B. Let $(A, \leq)$ and $(B, \leq)$ be two disjoint poset, their *vertical sum* $A \oplus B$ is the poset over $A \cup B$ obtained by taking the order relation defined in the following way: let x and y be two elements that belong to $A \cup B$, then $x \leq y$ if the pair (x, y) fall in one of the following three mutually disjoint cases; $x \leq y$ if and only if $x, y \in A$ and $x \leq y$ in A, second $x, y \in B$ and $x \leq y$ in B and finally $x \in A$ and $y \in B$. Given a poset $(A, \leq)$ we define its *order dual* as the poset $(A, \leq^\partial)$ defined by: $x \leq^\partial y$ holds in $(A, \leq^\partial)$ if and only if $y \leq x$ holds in $(A, \leq)$. Let x and y be elements of $(A, \leq)$. Then, y *covers* x if $x < y$ and $x \leq z < y$ implies $z = x$.

Definition 1.3.1. *A* maximal antichain *(*chain*, respectively) in a poset is a maximal set of pairwise incomparable (comparable, respectively) points.*

Given a poset $(A, \leq)$, we denote $\mathcal{C}_A$ and $\mathcal{A}_A$ the set of maximal chains in $(A, \leq)$ and the set of maximal antichains in $(A, \leq)$ respectively. An element B of $\mathcal{C}_A$ is called a branch *of $(A, \leq)$. We denote with $[p_B]_{\mathcal{C}_A}$ a maximal antichain over $(A, \leq)$, where $p_B \in B$ for every branch B in $\mathcal{C}_A$.*

The set $\mathcal{A}_A$ can be equipped with an order structure, for any two maximal antichains $[p_B]_{\mathcal{C}_A}$ and $[q_B]_{\mathcal{C}_A}$, we define:

$$[p_B]_{\mathcal{C}_A} \leq [q_B]_{\mathcal{C}_A},$$

if and only if $p_B \leq q_B$, for every $B \in \mathcal{C}_A$.

We denote with $\mathbf{1}$ the poset containing only one element.

The poset obtained by $\mathbf{1} \oplus ... \oplus \mathbf{1}$ applied n times, is isomorphic to a chain of n elements, which we denote by $\mathbf{n}$.

If we replace every element of a poset $(A, \leq)$ with a copy of $\mathbf{1}$ we obtain a poset $(o(A), \leq)$ that is order isomorphic to $(A, \leq)$. We call $(o(A), \leq)$ the *order type* of $(A, \leq)$. The order type provides only the order-theoretic structure of a poset.

Definition 1.3.2. *A* common infix *of two chains C and C' is a chain I such that $C = V \oplus I \oplus W$ and $C' = V' \oplus I \oplus W'$, for some V, V' and W, W'. If the bottom elements of W and W' are distinct, and the top elements of V and V' are distinct, then I is the* longest common infix *of C and C'. If*

$V = V' = \emptyset$ *then* I *is called a* prefix. *If* $W = W' = \emptyset$ *then* I *is called a* postfix.

The notion of prefix will be characterized in two different ways over posets related to Gödel (Chapter 5) and RDP chains (Section 6.1). These characterizations will be used to define suitable terms to obtain normal forms for the corresponding logics. Hence, we recall here the notion of semantical minterms and maxterms.

Definition 1.3.3 ([6]). *Let* A *be a finite poset. Take* $B \in \mathcal{C}_A$ *and* $p_B \in B$. *A* semantical minterm *for* p_B *is the smallest* ϕ_{p_B} *in* $\mathcal{A}_A$ *such that*

$$\phi_{p_B} \cap B = \{p_B\}.$$

A semantical maxterm *for* p_B *is the greatest* Φ_{p_B} *in* $\mathcal{A}_A$ *such that*

$$\Phi_{p_B} \cap B = \{p_B\}.$$

Let s be an element of $\mathcal{A}_A$. Using the above definition we can obtain a *disjunctive normal form* for s,

$$s = \bigvee_{B \in \mathcal{C}_A} \phi_{p_B}, \tag{1.9}$$

and a *conjunctive normal form* for s,

$$s = \bigwedge_{B \in \mathcal{C}_A} \Phi_{p_B}. \tag{1.10}$$

Chapter 2

Weak Nilpotent Minimum Algebras

In this chapter we resume some concept from [46] and we introduce new definitions and results that will be useful when dealing with (finite) WNM chains. Finite WNM chains are the building blocks of any algebra that we will consider in this thesis. Indeed, the varieties of RDP algebras and NMG algebras are subvarieties of $\mathbb{V}(WNM)$. Thus, the characterizations developed in the following sections will be refined in Chapters 3 and 4 to give dualities for finite RDP algebras and finite NMG algebras respectively, and in Chapter 6 to study logical properties of RDP logic.

2.1 WNM Chains

Historically, the study of totally ordered WNM algebras can be traced back to [22], where authors study weak negation functions over the real interval $[0, 1]$. As we have seen in Section 1.1.1, through weak negation functions (Definition 1.1.1) it is possible to define weak nilpotent minimum t-norms and then standard WNM algebras. A standard WNM algebra is a structure

$$[\mathbf{0}, \mathbf{1}]_* = \langle [0, 1], *, \Rightarrow, \wedge, \vee, 0, 1\rangle$$

where, for every $x, y \in [0, 1]$, we let $x \wedge y = \min\{x, y\}$, $x \vee y = \max\{x, y\}$, and for some arbitrary but fixed weak negation function n,

$$x * y = \begin{cases} 0 & \text{if } x \leq \neg y \\ min\{x, y\} & \text{otherwise.} \end{cases} \tag{2.1}$$

$$x \Rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ max\{n(x), y\} & \text{otherwise.} \end{cases} \tag{2.2}$$

Obviously, $*$ is a WNM t-norm (see Definition 1.1.2 and compare with (1.2)) and $\Rightarrow$ is its residuum (1.1). WNM logic is complete respect to standard WNM algebras ([23]).

Among standard WNM chains a particular class it is of our interest, the *generic* standard WNM chains. An algebra is said to be *generic* if it generates its whole variety (see Appendix A.5).

Theorem 2.1.1 ([46]). *Let $[\mathbf{0},\mathbf{1}]_*$ be a standard WNM chain. Then, $[\mathbf{0},\mathbf{1}]_*$ is generic if and only if it satisfies the following condition,*

- $[\mathbf{0},\mathbf{1}]_*$ *has a* negation fixpoint $f = \neg f$ *and* $|\{x \in [\mathbf{0},\mathbf{1}]_* \mid \neg x = f\}| > 1$;
- *either there is an increasing sequence $\langle x_n \in [\mathbf{0},\mathbf{1}]_* \mid n \in \omega\rangle$ of elements such that, $x_n = \neg\neg x_n$ and $x_n < \neg x_n$;*

  or there is an increasing sequence $\langle x_n \in [\mathbf{0},\mathbf{1}]_* \mid n \in \omega\rangle$ of elements such that, $x_n = \neg\neg x_n$ and $x_n > \neg x_n$;

  such that for every $n \geq 0$ there exists $m \geq n$ such that the sets $\{y \in [\mathbf{0},\mathbf{1}]_* \mid \neg y = x_m\}$ and $\{y \in [\mathbf{0},\mathbf{1}]_* \mid \neg y = \neg x_m\}$ have cardinality greater than 1.

Let $\mathbf{C} = \langle C, \odot, \rightarrow, \wedge, \bot, \top\rangle$ be a WNM chain. Operations in C are completely determined by its negation $\neg$ ([46]):

$$x \odot y = \begin{cases} \bot & x \leq \neg y; \\ x & x > \neg y, x \leq y; \\ y & x > \neg y, x > y. \end{cases} \tag{2.3}$$

$$x \rightarrow y = \begin{cases} \top & x \leq y; \\ y & x > y, y > \neg x; \\ \neg x & x > y, y \leq \neg x. \end{cases} \tag{2.4}$$

We denote with $\mathcal{F}_n$ the set

$$\{\bot, x_1, \ldots, x_n, \neg x_1, \ldots, \neg x_n, \neg\neg x_1, \ldots, \neg\neg x_n, \top\}. \tag{2.5}$$

Let $\mathbf{C}$ be a WNM chain generated by $x_1, \ldots, x_n$, and $\varphi(x_1, \ldots, x_n)$ a term. By (2.3) and (2.4), we have $\varphi^{\mathbf{C}} \in \mathcal{F}_n$. Hence, $\mathbf{C}$ is locally finite. Moreover, there is only a finite number of finitely generated WNM chains up to isomorphisms. From this and the fact that by subdirect representation every algebra $\mathbf{A}$ in $\mathbb{V}(WNM)$ is a subdirect product of WNM chains (see Theorem A.4.2), we conclude

Proposition 2.1.1 ([46]). *$\mathbb{V}(WNM)$ is a locally finite variety.*

Therefore, finitely generated WNM algebras and finite WNM algebras coincide. Moreover, every subvariety of $\mathbb{V}(WNM)$ is generated by its finite chains.

Now we introduce a characterization of finite WNM chains that is the starting point for obtaining combinatorial representation of free algebras in subvarieties of $\mathbb{V}(WNM)$ (see [6] for details on this). Moreover, this characterization is useful for verifying terms and when creating visual examples. We will exploit this type of representations for obtaining logical properties of RDP in Chapter 4.4.

Let $\mathbf{C}$ be a finitely n generated WNM chain with generating set $\{x_1, \ldots, x_n\}$. We have seen that $\varphi^{\mathbf{C}} \in \mathcal{F}_n$ for every term $\varphi(x_1, \ldots, x_n)$.

For every element $c_i \in \mathbf{C}$ we call B_i the set of all formulas in $\mathcal{F}_n$ that are interpreted in c_i. That is $B_i = \{\varphi \in \mathcal{F}_n \mid \varphi^{\mathbf{C}}(x_1, ..., x_n) = c_i\}$. Let $\mathcal{B}$ be the chain $B_0 < B_2 < ... < B_k$. Then, $\mathcal{B}$ is isomorphic to $\mathbf{C}$ via the map that sends every c_i to B_i. Note that $\bot \in B_0$ and $\top \in B_k$.

It is easy to see that every $\mathcal{B}$ is a partition of $\mathcal{F}_n$ with a total order inherited by the WNM chain $\mathbf{C}$. Hence, we call $\mathcal{B}$ an *ordered partition* of $\mathcal{F}_n$, and each B_i is called *block* of $\mathcal{B}$. We shall tacitly use this identification throughout the thesis. As a consequence, when we consider the set of all n-generated WNM chains, we identify it with the set of all ordered partitions of $\mathcal{F}_n$ equipped with a structure of a totally ordered WNM algebra. The block B_i of a chain $\mathbf{C}$ that contains the element x_i will be denoted $x_i^{\mathbf{C}}$. As an example of this characterization, see the nine ways of singly generating a WNM chains over 1 generator depicted in Figure 2.1.

Definition 2.1.1. *Given $\mathcal{L}$ an axiomatic extension of WNM, we denote with $\mathcal{C}_n^{\mathcal{L}}$ the set of n-generated $\mathcal{L}$ chains of the form $B_0 < B_2 < ... < B_k$, and we call them* canonical *chains.*

Notation For the sake of a better rendering, when displaying WNM chains in a picture we use the following identification: $\neg x = \dot{x}$ and $\neg\neg x = \ddot{x}$.

Locally finiteness of $\mathbb{V}(WNM)$ is a property inherited by all its subvarieties. This and the fact that (2.3) and (2.4) are completely determined by the negation, means that the 1-generated chains of every subvariety of $\mathbb{V}(WNM)$ will be subsets of $\mathcal{C}_1 = \{\mathbf{C}_i \mid 1 \leq i \leq 9\}$.

Given a filter F (Definition 1.1.4) of a WNM algebra $\mathbf{A}$, we denote $\bar{F}$ the set $\{x \in \mathbf{A} \mid \neg x \in F\}$. The following lemmas characterize prime filters and congruences over WNM chains.

Lemma 2.1.1 ([46]). *Let $\mathbf{A}$ be a WNM algebra and F be a prime filter. Consider the quotient chain $\mathbf{A}/\theta_F$. Then,*

- $[\top]_{\theta_F} = F$;
- $[\bot]_{\theta_F} = \bar{F}$;

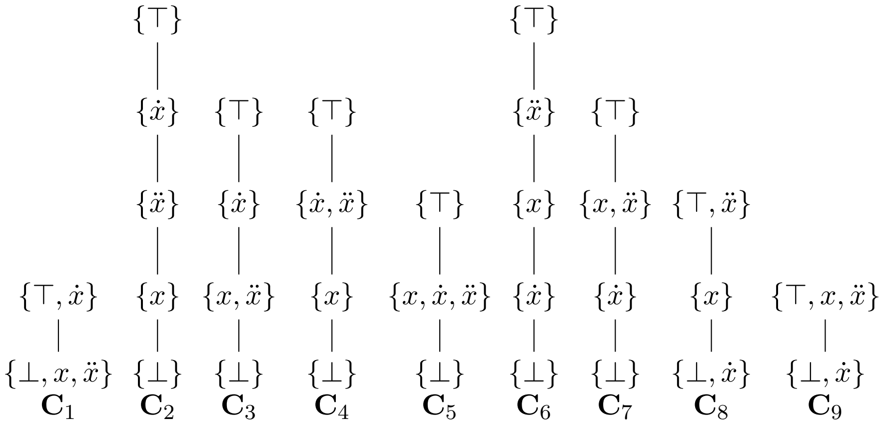


Figure 2.1: The nine ways to 1-generate WNM chains.

- *for every $x, y \in A \setminus (F \cup \bar{F})$ such that $x \neq y$ we have $[x]_{\theta_F} \neq [y]_{\theta_F}$.*

Lemma 2.1.2 ([6])**.** *Let $\mathcal{B} = \{B_0 < B_2 < ... < B_k\}$ be a chain in $\mathbb{V}(\mathcal{L})$, where $\mathcal{L}$ is a schematic extension of WNM. Then, for any congruence θ on $\mathcal{B}$, if $B_i \leq B_j$ and $B_i \theta B_k$ then $B_j \theta B_k$. Moreover, if the equivalence class of B_i under θ is not equal to the singleton $\{B_i\}$, then $B_i \theta B_k$ or $\neg B_i \theta B_k$.*

Let $\mathbf{C}$ be a n-generated canonical WNM chain. Then, $\mathbf{C}$ is a *redundant* chain if there exist a n-generated WNM chain $\mathbf{C}'$ and a congruence θ on $\mathbf{C}'$ such that $\mathbf{C}'/\theta$ is isomorphic to $\mathbf{C}$ via the map that sends every equivalence class $[c']_\theta \in C'/\theta$ in the unique block B of $\mathbf{C}$ such that $[c']_\theta \subseteq B$.

Example 2.1.1. *Consider 1-generated WNM chains in Figure 2.1. It is clear that $\mathbf{C}_8$ is isomorphic to $\mathbf{C}_7/\theta$ where θ is defined by the following equivalence classes: $[\bot]_\theta = \{\{\bot\}, \{\neg x\}\}$, $[x]_\theta = \{\{x\}\}$ and $[\top]_\theta = \{\{\neg\neg x\}, \{\top\}\}$.*

We denote with $\mathcal{K}_n$ the n-generated non-redundant WNM chains (see Figure 2.2). Then, $\mathbf{C}_7$ in Example 2.1.1 belongs to $\mathcal{K}_1$, while $\mathbf{C}_8 \notin \mathcal{K}_1$. Given $\mathcal{L}$, schematic extension of WNM, we denote with $\mathcal{K}_n^{\mathcal{L}}$ the set of n-generated non-redundant $\mathcal{L}$ chains.

We conclude with a key lemma for the combinatorial representations of free algebras in subvarieties of $\mathbb{V}(WNM)$. We present such characterization for free RDP algebras in Section 6.1 and remind the interested reader to [6] for the basis of this technique.

Lemma 2.1.3 ([6])**.** *Let $\mathcal{L}$ be a schematic extension of WNM logic and $\varphi^{\mathbf{F}_n}$,be an element of the free n-generated WNM algebra $\mathbf{F}_n(\mathcal{L})$. Then, the map*

$$\varphi^{\mathbf{F}_n} \mapsto (\varphi^{\mathbf{C}})_{\mathbf{C} \in \mathcal{K}_n} \in \prod_{\mathbf{C} \in \mathcal{K}_n} \mathbf{C}$$

is a monomorphism.

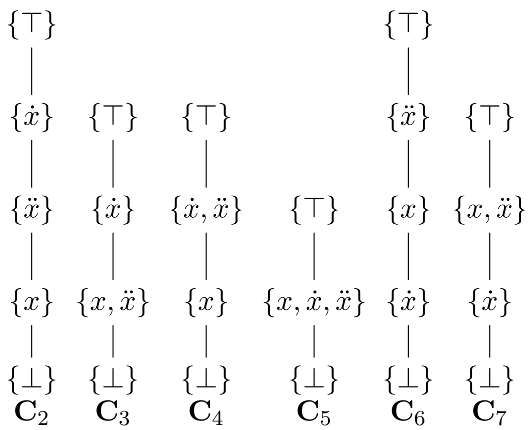


Figure 2.2: The set $\mathcal{K}_1$ composed of all non-redundant chains in $\mathcal{C}_1$.

2.2 Prime Spectrum of the free one generated WNM Algebra

The free 1-generated WNM algebra $\mathbf{F}_1$ is finite. Hence, $\mathbf{F}_1$ is isomorphic to a subdirect product of a finite number of singly generated WNM chains. In the previous section we have seen the nine ways to generate a WNM chain with a single element. That is, the nine chains $\{\mathbf{C}_i\}_{i=1}^{9}$ depicted in Figure 2.1.

There is a subdirect embedding of $\mathbf{F}_1$ into the direct product of the WNM chains $\{\mathbf{C}_i\}_{i=1}^{9}$. Hence, the maps $v_i : \mathbf{F}_1 \to \mathbf{C}_i$ are epimorphisms, and then $\mathbf{C}_i \cong \mathbf{F}_1/\theta_i$, for a congruence θ_i. For every congruence θ_i we can define a filter F_{θ_i} of $\mathbf{F}_1$, via $F = \{y \in \mathbf{F}_1 \mid (\top, y) \in \theta_i\}$ (see (1.7)). Moreover, by Propostion 1.1.2 every prime filter corresponds to a congruence θ such that $\mathbf{F}_1/\theta$ is a chain. Hence, every chain in $\{\mathbf{C}_i\}_{i=1}^{9}$ correspond to a prime filter F_{θ_i} of $\mathbf{F}_1$. We can recover the prime spectrum $\mathsf{Spec}(\mathbf{F}_1)$ simply analyzing the chains $\{\mathbf{C}_i\}_{i=1}^{9}$ and their generating congruences θ_i.

Take a chain $\mathbf{C}_i$ and denote with $B^i \subseteq \mathcal{F}_1$ the block of $\mathbf{C}_i$ such that $\top \in B^i$. The prime filter F_{θ_i} associated to $\mathbf{C}_i$ is equal to the equivalence class of $\top$ in θ_i, which by Lemma 2.1.1 in $\mathbf{C}_i$ is expressed by B^i. Note that, $F_{\theta_i} \subseteq F_{\theta_j}$ if and only if $T_i \supseteq T_j$ and $\mathbf{C}_j = \mathbf{C}_i/\theta$ for a suitable θ. Thus, we have obtained the prime spectrum of $\mathbf{F}_1$. See Figure 2.3.

As an example, consider chains $\mathbf{C}_6$, $\mathbf{C}_7$, $\mathbf{C}_8$ and $\mathbf{C}_9$ in Figure 2.1. The prime filters F_{θ_7} and F_{θ_8} are both in F_{θ_9}, then $F_{\theta_9} \leq F_{\theta_7}$ and $F_{\theta_9} \leq F_{\theta_8}$. Moreover, $\mathbf{C}_6$ is such that $\mathbf{C}_8 = \mathbf{C}_6/\theta$ where $(\neg\neg x, \top) \in \theta$, then $F_{\theta_8} \leq F_{\theta_6}$. Hence, we see $\mathsf{Spec}\mathbf{F}_1$ in Figure 2.3, and the order structure of $\{F_{\theta_j}\}_{j\in\{6,7,8,9\}}$ is the tree on the right, where F_{θ_9} is its root.

In the following chapters we will see that the order structures of the prime spectra of the free 1-generated NMG and RDP algebras are subforests of the

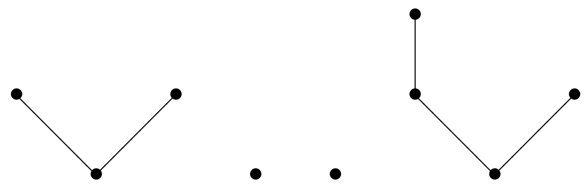


Figure 2.3: The prime spectrum of the free 1-generated WNM algebra.

forest depicted in Figure 2.3.

2.3 Some Properties of WNM Algebras

By the subdirect representation theorem (Theorem A.4.3 in Appendix A.4) and the fact that subdirectly irreducible MTL algebras are chains (see Theorem 1.1.3), every WNM algebra $\mathbf{A}$ is isomorphic to a subdirect product of a family $(\mathbf{C}_i)_{i\in I}$ of WNM chains, for some index set I. When $\mathbf{A}$ is finite and not trivial, then the family $(\mathbf{C}_i)_{i\in I}$ of non trivial chains is essentially unique up to reordering of the finite index set I. Hence, there exist $\pi_i : \mathbf{A} \to \mathbf{C}_i$ such that $\pi_i(x) = x_i$ for every $x \in \mathbf{A}$. We call x_i the ith-projection of x. Then, we can display every element x in $\mathbf{A}$ by means of its projections $(x_i)_{i\in I}$. In other words, every element x_i in every chain $\mathbf{C}_i$ is used as a coordinate in $\mathbf{A}$. Moreover, by taking homomorphic images of $\mathbf{A}$ we can get every $\mathbf{C}_i$.

Definition 2.3.1. *Let $\mathbf{A}$ be a WNM algebra and y be an element of A. If $y > \neg y$, then y is called a* positive *element, otherwise if $y < \neg y$, then y is called a* negative *element. We denote with $P_{\mathbf{A}}$ the set of positive elements of A, that is $P_{\mathbf{A}} = \{x \in A \mid x > \neg x\}$, and with $N_{\mathbf{A}}$ the set of negative elements of A, that is $N_{\mathbf{A}} = \{x \in A \mid x < \neg x\}$. When $y = \neg y$, then y is called* negation fixpoint.

The existence of the fixpoint is equivalent to the existence of the fixpoint in each subdirect factor.

Proposition 2.3.1. *If $\mathbf{A}$ is a WNM algebra, then A has at most one negation fixpoint.*

Proof. Each WNM chain $\mathbf{C}$ has at most one fixpoint, since if x and y are fixpoints of $\mathbf{C}$, say without loss of generality $x \le y$, then $y = \neg y \le \neg x = x$ by antitonicity, and $x = y$. Let $\mathbf{A}$ be an WNM algebra, displayed as the subdirect product of the indexed family $(\mathbf{C}_i)_{i\in I}$ of WNM chains. Now, if x is a fixpoint of A, the ith projection x_i of x is the unique fixpoint of $\mathbf{C}_i$ (for all $i \in I$), and then, x is unique. $\square$

Remember that an algebra is directly indecomposable if it is not representable as the direct product of two nontrivial algebras (see Appendix A.4).

Proposition 2.3.2. *Every element of a finite directly indecomposable WNM algebra* $\mathbf{A}$ *is either positive, negative or a negation fixpoint.*

Proof. By the subdirect representation theorem, we display $\mathbf{A}$ as a family of WNM chains $(\mathbf{C}_i)_{i\in I}$. Suppose that $x \in \mathbf{A}$ is neither positive, negative nor a negation fixpoint and denote with x_i its projection over $\mathbf{C}_i$. Hence without loss of generality, we can assume that $x_j < \neg x_j$ and $x_k > \neg x_k$, for $j \in J$ and $k \in K$, and where J and K are two disjoint subsets of I. A direct computation on $\mathbf{A}$ shows that the element:

$$s = (x \to \neg x) \odot (x \to \neg x),$$

is such that $s_j = \top_j$ and $s_k = \bot_k$ for all $j \in J$ and $k \in K$.

Take two elements a and a' in $\mathbf{A}$, such that $a_k = \bot_k$ and $a'_j = \bot_j$ for all $j \in J$ and $k \in K$. Hence, $e = (s \wedge a) \vee (\neg s \wedge a')$ is an element of $\mathbf{A}$ such that $e_j = a_j$ and $e_k = a'_k$. It follows that $\mathbf{A}$ is isomorphic to the direct product $\mathbf{A}_J \times \mathbf{A}_K$, where $\mathbf{A}_J$ and $\mathbf{A}_K$ are some WNM algebras arising as subdirect product of $(\mathbf{C}_j)_{j\in J}$ and $(\mathbf{C}_k)_{k\in K}$ respectively. In contradiction with the hypothesis of direct indecomposability of $\mathbf{A}$. □

Proposition 2.3.3. *An element* x *of a finite WNM algebra* $\mathbf{A}$ *is positive if and only if* $x \odot x = x$*. Moreover, generators of prime filters of* $\mathbf{A}$ *can only be positive elements.*

Proof. We can display the finite WNM algebra $\mathbf{A}$ as the subdirect product of $(\mathbf{C}_i)_{i\in I}$, where every $\mathbf{C}_i$ is an WNM chain. Since x is positive, then $x_i > \neg x_i$ for each one of its projections x_i over $\mathbf{C}_i$. By (4.5), it follows that $x_i \odot x_i = x_i$. We conclude that $x \odot x = x$.

It is easy to show the second part of the proposition, after noticing that $y \odot y = \bot$ for y not positive. □

Let $\mathbf{A}$ be a finite directly indecomposable WNM algebra. By finiteness, there exists the element $\bigwedge_{x\in P_{\mathbf{A}}} x$ in $\mathbf{A}$. We let,

$$m_{\mathbf{A}} := \bigwedge_{x\in P_{\mathbf{A}}} x \tag{2.6}$$

By negation $\neg m_{\mathbf{A}}$ is the element $\bigvee_{x\in N_{\mathbf{A}}} x$ in $\mathbf{A}$. Hence, $\neg m_{\mathbf{A}} < f_{\mathbf{A}} < m_{\mathbf{A}}$ where $f_{\mathbf{A}}$ is the negation fixpoint of $\mathbf{A}$ if it exists, or $\neg m_{\mathbf{A}} < m_{\mathbf{A}}$ if $\mathbf{A}$ does not have a negation fixpoint. It follows that $m_{\mathbf{A}}$ is join-irreducible and there exists a unique maximal filter in $\mathsf{Spec}\mathbf{A}$.

We have seen in Section 1.2.1 that Horn has showed that prime filters of finite directly indecomposable Gödel Algebras form a tree. Horn's results are valid also for directly indecomposable WNM algebras. Indeed, linearity forces $\mathsf{Spec}\mathbf{A}$ to be a forest (see Section 1.2.1) for any $\mathbf{A} \in \mathbb{V}(WNM)$ (and $\mathbb{V}(MTL)$ too).

This, and the fact that $m_{\mathbf{A}}$ is a join-irreducible element lead to the following statement.

Proposition 2.3.4. *The prime spectrum of a finite directly indecomposable WNM algebra* $\mathbf{A}$ *is a tree and its root is the maximal filter generated by* $m_{\mathbf{A}}$.

Dualities and Representations

In the following two chapters we present spectral dualities for finite RDP algebras and for finite NMG algebras: we define combinatorial categories of finite suitable forests and their morphisms, and we prove that they are dually equivalent to the considered categories of finite algebras. As a benchmark of the manageability and usefulness of the presented dualities, we give algorithmic constructions for finite products in the dual categories and we obtain explicit descriptions of coproducts for the finite algebras. We thus attain an amenable combinatorial representation of free finitely generated algebras for the two considered varieties.

Chapter 3

Finite RDP Algebras

An RDP algebra is a WNM algebra satisfying the *revised drastic product* equation (RDP). Notice that Gödel algebras are idempotent RDP algebras.

In every RDP algebra, the operations $\wedge$ and $\vee$, and the constant $\top$ are definable as term operations over $\odot$, $\to$, $\perp$ [50, Proposition 3.2],

$$
\begin{aligned}
x \vee y :=& ((\varphi \to \psi) \to \varphi) \odot (((\varphi \to \psi) \to \varphi) \to \psi);\\
\text{where} \quad & \varphi := (y \to x) \to x \quad \text{and} \quad \psi := (x \to y) \to y;\\
x \wedge y :=& (x \odot (x \to y)) \vee (y \odot (y \to x));\\
\top :=& \perp \to \perp.
\end{aligned}
$$

A standard RDP algebra is of the form,

$$[\mathbf{0},\mathbf{1}]_* = \langle [0,1], *, \Rightarrow, \wedge, \vee, 0, 1\rangle, \tag{3.1}$$

where, for every $x, y \in [0,1]$, we let $x \wedge y = \min\{x,y\}$, $x \vee y = \max\{x,y\}$, and for some arbitrary but fixed $0 < a < 1$,

$$x * y = \begin{cases} 0 & x, y \leq a, \\ \min\{x,y\} & \text{otherwise}, \end{cases} \tag{3.2}$$

$$x \Rightarrow y = \begin{cases} 1 & x \leq y, \\ a & y < x \leq a, \\ y & \text{otherwise}. \end{cases} \tag{3.3}$$

By direct computation, for every $x \in [0,1]$,

$$\neg x = \begin{cases} 1 & x = 0, \\ a & 0 < x \leq a, \\ 0 & \text{otherwise}. \end{cases} \tag{3.4}$$

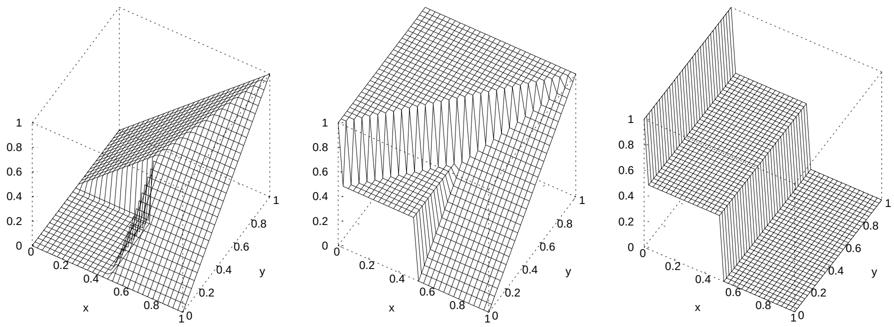
Note that the operation $\Rightarrow$ is the unique binary operation over the real interval $[0,1]$ satisfying the residuation equivalence with respect to $*$.

Recall that RDP logic is a many-valued propositional logic introduced in Chapter 1 as schematic extension of MTL.

Theorem 3.1. *[50, Theorem 3.7 and Theorem 3.8] For every formula φ of RDP logic, the following statements are equivalent:*

- $\vdash_{RDP} \varphi$;
- φ *is a* $[\mathbf{0},\mathbf{1}]_*$*-tautology.*

Hence, the variety of RDP algebras is singly generated by the standard algebra $[\mathbf{0},\mathbf{1}]_*$.



$x * y$. $\qquad$ $x \Rightarrow y$. $\qquad$ $\neg x = x \Rightarrow \bot$.

Figure 3.1: The revised drastic product left-continuous triangular norm and its residual, with $a = 1/2$ in (3.2)-(3.4).

As mentioned in the previous chapter, the variety of WNM algebras is locally finite; it follows that the variety of RDP algebras is locally finite. Therefore, finitely generated RDP algebras and finite RDP algebras coincide. To see this directly, observe that RDP chains are locally finite: Indeed, let $\mathbf{C} = \langle C, \odot, \rightarrow, \wedge, \vee, \bot, \top\rangle$ be a RDP chain generated by $x_1, \ldots, x_n$. Then, since $\mathbf{C}$ is (isomorphic to) a subalgebra of $[\mathbf{0},\mathbf{1}]_*$, for all $x, y \in C$, by equations (3.2), (3.3) and (3.4),

$$x \odot y = \begin{cases} \bot & x, y \leq \neg x, \neg y, \\ \min\{x, y\} & \text{otherwise,} \end{cases} \tag{3.5}$$

$$x \rightarrow y = \begin{cases} \top & x \leq y, \\ \neg x & y < x \leq \neg x, \\ y & \text{otherwise.} \end{cases} \tag{3.6}$$

Let t be a RDP term over variables $x_1, \ldots, x_n$. By induction on t, and direct inspection of equations (3.5) and (3.6),

$$t^C \in \{\bot^C, \top^C, x_i^C, \neg x_i^C \mid 1 \leq i \leq n\};$$

hence, $|C| \leq 2(n+1)$. Note that the above set is a subset of (2.5), the set $\mathcal{F}_n$ defined in Section 2.1.

We now establish some useful facts on finite RDP algebras. Let $\mathbf{A}$ be a finite RDP algebra. Since $\mathbb{V}(RDP)$ is a subvariety of $\mathbb{V}(WNM)$, as explained in Section 2.3 by subdirect representation $\mathbf{A}$ is a subdirect product of an indexed family $(\mathbf{C}_i)_{i \in I}$ of RDP chains. For every $y \in A$, we let y_i denote the projection of y over index $i \in I$.

We now record key properties of finite directly indecomposable RDP algebras, with and without a fixpoint: we show that a finite directly indecomposable RDP algebra is either a Gödel algebra, or its nonidempotent elements form a chain below the fixpoint.

Proposition 3.1. *Let $\mathbf{A}$ be a finite directly indecomposable RDP algebra. If x is the fixpoint of $\mathbf{A}$, then $\{y \in A \mid \bot < y \leq x\} = \{y \in A \mid y^2 < y\}$ is a chain. If $\mathbf{A}$ has no fixpoint, then $\{y \in A \mid y^2 < y\}$ is empty.*

Proof. Let $\mathbf{A}$ be the subdirect product of the indexed family $(\mathbf{C}_i)_{i \in I}$ of RDP chains.

For the first part, suppose for a contradiction that the lower set of x is not a chain. Let $y, z \leq x$ be incomparable in the lower set of x. Let J and K be subsets of I such that $y_j \leq z_j$ for all $j \in J$, and $z_k < y_k$ for all $k \in K$. Let $\mathbf{A}'$ and $\mathbf{A}''$ be the nontrivial RDP algebras generated by $\{(a_j)_{j \in J} \mid a \in A\}$ and $\{(a_k)_{k \in K} \mid a \in A\}$ respectively, with coordinatewise defined operations (for nontriviality, notice that there exist $j \in J$ such that $y_j < z_j$ and $k \in K$ such that $z_k < y_k$). We show that $\mathbf{A}$ is the direct product of $\mathbf{A}'$ and $\mathbf{A}''$. A straightforward computation on the subdirect representation of $\mathbf{A}$, using (3.2) and (3.3), shows that the element

$$a = (y \to z) \to \neg(y \to z)$$

of $\mathbf{A}$ is such that $a_j = \bot_j$ for all $j \in J$ and $a_k = \top_k$ for all $k \in K$; thus, $\neg a$ is such that $\neg a_j = \top_j$ for all $j \in J$ and $\neg a_k = \bot_k$ for all $k \in K$. Let $a' \in A'$ and $a'' \in A''$ be any two elements, and let $b' \in A$ and $b'' \in A$ be such that $b'_j = a'_j$ for all $j \in J$ and $b''_k = a''_k$ for all $k \in K$. Notice that b' and b'' exist in A by construction. By direct computation,

$$b = (\neg a \wedge b') \vee (a \wedge b'')$$

is an element of A such that $b_j = b'_j = a'_j$ for all $j \in J$ and $b_k = b''_k = a''_k$ for all $k \in K$. The equality $\{y \in A \mid \bot < y \leq x\} = \{y \in A \mid y^2 < y\}$ is now easy to check on the subdirect representation of $\mathbf{A}$: Every $\bot \neq y \in A$ below x is nonidempotent, and every $y \in A$ strictly above x is idempotent.

For the second part, we show a preliminary fact. Let $\mathbf{C}$ be an RDP chain. We claim that if $\mathbf{C}$ has no fixpoint, then $\mathbf{C}$ is idempotent. Let $w \in C$, so that $w \neq \neg w$. As $\mathbf{C}$ is (isomorphic to) a subalgebra of $[\mathbf{0}, \mathbf{1}]_*$, by (3.2), if

$\neg w < w$, then $w^2 = w$; and if $w < \neg w$, then $w = \bot$ (in fact, $\bot < w < \neg w$ implies $\neg\neg w = \neg w$ by (3.3), contradiction as $\mathbf{C}$ has no fixpoint), so $w^2 = w$.

We now show that if $\mathbf{A}$ is not idempotent, then $\mathbf{A}$ has a fixpoint. Let $J = \{i \in I \mid \mathbf{C}_i \text{ has a fixpoint}\}$ and $K = \{i \in I \mid \mathbf{C}_i \text{ has no fixpoint}\}$. Let $y \in A$ be such that $y^2 < y$, and let $i \in I$ such that $y_i^2 < y_i$. Then $\mathbf{C}_i$ is nonidempotent, and by the preliminary fact, $\mathbf{C}_i$ has a fixpoint; hence $J \neq \emptyset$.

Suppose $J = I$ (or, $K = \emptyset$). We claim that $\mathbf{A}$ has a fixpoint. Indeed, for all $j \in J \neq \emptyset$, let $z_j \in A$ be such that the jth projection $(z_j)_j$ of z_j is the fixpoint of $\mathbf{C}_j$ (such z_j's exist by subdirect representation). Then,

$$f = \bigvee_{j \in J} \neg z_j$$

is the fixpoint of $\mathbf{A}$: For, notice that for all $j \in J$, $(\neg z_j)_j$ is equal to the fixpoint of $\mathbf{C}_j$, and for all $j' \neq j \in J$, $(\neg z_j)_{j'}$ is less than or equal to the fixpoint of $\mathbf{C}_{j'}$, so that, for all $j \in J$, f_j is equal to the fixpoint of $\mathbf{C}_j$.

Otherwise, suppose that $J \subset I$ (or, $K \neq \emptyset$). Let $\mathbf{A}'$ and $\mathbf{A}''$ be the RDP algebras generated by $\{(a_j)_{j \in J} \mid a \in A\}$ and $\{(a_k)_{k \in K} \mid a \in A\}$ respectively, with coordinatewise defined operations. Note that $J \neq \emptyset$ implies that $\mathbf{A}'$ is nontrivial. Also, $|A''| \geq 1$. If $|A''| > 1$, we claim that $\mathbf{A}$ is the direct product of nontrivial RDP algebras $\mathbf{A}'$ and $\mathbf{A}''$. As above, for all $j \in J \neq \emptyset$, let $z_j \in A$ be such that the jth projection $(z_j)_j$ of z_j is the fixpoint of $\mathbf{C}_j$ (such z_j's exist by subdirect representation). Using (3.3) and (3.4), a direct computation on the subdirect representation of $\mathbf{A}$ shows that the element

$$a = \bigvee_{j \in J} (z_j \leftrightarrow \neg z_j)$$

of $\mathbf{A}$ is such that $a_j = \top_j$ for all $j \in J$ and $a_k = \bot_k$ for all $k \in K$; thus, $\neg a$ is such that $\neg a_j = \bot_j$ for all $j \in J$ and $\neg a_k = \top_k$ for all $k \in K$. Let $a' \in A'$ and $a'' \in A''$ be any two elements, and let $b' \in A$ and $b'' \in A$ be such that $b'_j = a'_j$ for all $j \in J$ and $b''_k = a''_k$ for all $k \in K$. Notice that b' and b'' exist in A by construction. By direct computation,

$$b = (a \wedge b') \vee (\neg a \wedge b'')$$

is an element of A such that $b_j = b'_j = a'_j$ for all $j \in J$ and $b_k = b''_k = a''_k$ for all $k \in K$. But this is a contradiction with the fact that $\mathbf{A}$ is directly indecomposable. Then, $|A''| = 1$, and the element f computed above, is again the fixpoint of $\mathbf{A}$: with respect to $k \in K$, simply notice that $f_k = (\neg f)_k$, because $|A''| = 1$ implies $|C_k| = 1$.

This settles the proposition. $\square$

Let $\mathbf{A}$ be a finite directly indecomposable RDP algebra. By Proposition 3.1, we introduce the following terminology. The *type* of $\mathbf{A}$, in symbols type($\mathbf{A}$), is the nonnegative integer uniquely determined by letting,

$$\text{type}(\mathbf{A}) = |\{y \in A \mid y^2 < y\}| = |\{y \in A \mid \perp < y \leq x,\, x \text{ fixpoint of } \mathbf{A}\}|; \tag{3.7}$$

in words, the type of $\mathbf{A}$ is the number of nonidempotent elements in the universe of $\mathbf{A}$, or equivalently, the cardinality of the chain below the fixpoint of $\mathbf{A}$ (excluding the bottom). In particular, the type of $\mathbf{A}$ is equal to 0 if all elements of $\mathbf{A}$ are idempotent, or equivalently, if $\mathbf{A}$ has no fixpoint.

Proposition 3.2. *Let $\mathbf{A}$ and $\mathbf{B}$ be finite directly indecomposable RDP algebras, and let $h\colon \mathbf{A} \to \mathbf{B}$ be a homomorphism. Then,* type($\mathbf{A}$) $\leq$ type($\mathbf{B}$).

Proof. If type($\mathbf{A}$) $= 0$, then the statement holds trivially. Otherwise, suppose type($\mathbf{A}$) > 0. Let y be the fixpoint of A, that is $y = \neg y$. As h is a homomorphism, h is to respect the fixpoint of $\mathbf{A}$, namely, $z = h(y) = h(\neg y) = \neg h(y) = \neg z$. Let z be the fixpoint of $\mathbf{B}$. Also, h is clearly to send each nonidempotent point below the fixpoint of $\mathbf{A}$ to a nonidempotent point below the fixpoint of $\mathbf{B}$. Moreover, h is to respect the chain of nonidempotent elements below the fixpoint of $\mathbf{A}$: For otherwise, suppose for a contradiction that $\perp < x < x' < y$ in $\mathbf{A}$ but $h(x') = w' \leq w = h(x)$ in $\mathbf{B}$. Then, $\top > z = h(y) = h(x' \to x) = h(x') \to h(x) = w' \to w = \top$, contradiction. Then, the cardinality of the chain below the fixpoint of $\mathbf{A}$ is at most equal to the cardinality of the chain below the fixpoint of $\mathbf{B}$, that is, type($\mathbf{A}$) $\leq$ type($\mathbf{B}$). This concludes the proof. $\square$

3.1 Categorical Equivalence

In this section, we prove a Priestley duality between the category of finite RDP algebras and their homomorphisms, FRDP, and the category HF of finite *hall forests*, whose objects are (pairs of) certain finite posets, and whose morphisms are (pairs of) open maps between them. Recall that, if P and Q are posets, an *open* map is a monotone map from P to Q that sends lower sets of P to lower sets of Q. The key lemma (Lemma 3.1.1) establishes a duality between finite directly indecomposable RDP algebras and *hall trees*, yielding the following representation: if $\mathbf{A}$ is a finite directly indecomposable RDP algebra, then the hall tree (T, J), dual to $\mathbf{A}$, is such that the vertical sum $J \oplus T$ of posets J and T (see Section 1.3) is order isomorphic to the prime filters of the lattice reduct of $\mathbf{A}$ ordered by reverse inclusion; and conversely, if (T, J) is a hall tree, then the algebra $\mathbf{A}$, dual to (T, J), is order isomorphic to the lower sets of the poset $J \oplus T$ ordered by inclusion.

The main result of this section exploits the structural resemblance between RDP algebras and Gödel algebras. Let $\mathbf{A}$ be a directly indecomposable RDP algebra. It is possible to describe the prime spectrum of a $\mathbf{A}$ in terms of the prime spectrum of a certain Gödel algebra $\mathbf{A}_G$, specified as follows. First notice that the idempotent elements of $\mathbf{A}$,

$$I(A) = \{x \in A \mid x^2 = x\},$$

form a subuniverse of $\mathbf{A}$ (since the idempotent elements in any RDP chain, $\perp$ or elements x such that $\neg x < x$, are closed under the RDP operations in (3.5) and (3.6), and each RDP algebra is representable as the subdirect product of a family of RDP chains), hence the algebra

$$\mathbf{A}_G = \langle I(A), \wedge, \vee, \odot, \to, \perp, \top \rangle,$$

is a subalgebra of $\mathbf{A}$ and in fact a Gödel algebra. Also, we claim that $\mathbf{A}_G$ is directly indecomposable. Indeed, if $\mathbf{A}$ has no fixpoint, this is trivial because $I(A) = A$ by Proposition 3.1. If x is the fixpoint of $\mathbf{A}$, since $I(A) = \{\perp\} \cup \{y \in A \mid x < y\}$ is a subalgebra of $\mathbf{A}$, it follows straightforwardly that $\{y \in A \mid x < y\}$ is the unique maximal nontrivial filter of $I(A)$, then $\mathbf{A}_G$ is directly indecomposable.

Let $\mathbf{A}$ and $\mathbf{B}$ be directly indecomposable RDP algebras, and let $h \colon \mathbf{A} \to \mathbf{B}$ be a homomorphism. Then, it is straightforward to verify that the restriction of h to $I(A)$, for short h_G, is a homomorphism from $\mathbf{A}_G$ to $\mathbf{B}_G$.

Proposition 3.1.1. *Let $\mathbf{A}$ be a finite directly indecomposable RDP algebra. Then, the prime spectrum of $\mathbf{A}$ is order isomorphic to $\mathsf{Spec}(\mathbf{A}_G)$.*

Proof. The claim is trivial if $\mathbf{A}$ has no fixpoint, because in this case $\mathbf{A} = \mathbf{A}_G$. Let x be the fixpoint of $\mathbf{A}$. It is sufficient to prove that F is a prime filter of $\mathbf{A}$ if and only if F is a prime filter of $\mathbf{A}_G$.

Let F be a prime filter of $\mathbf{A}$, and let $y \in F$. We claim that $y \in I(A)$. Indeed, suppose that y is not in $I(A)$, that is, $\perp < y \leq x$. By Proposition 3.1 the lower set of x in $\mathbf{A}$ is a chain; hence, $y \odot y = \perp$ by (3.5). Thus, $\perp \in F$. But then, $F = A$, and F is not a prime filter, contradiction. Therefore, F is a prime filter of $\mathbf{A}_G$, because the operations of $\mathbf{A}_G$ are the operations of $\mathbf{A}$ restricted to $I(A)$.

Let F be a prime filter of $\mathbf{A}_G$, and let $z \in I(A)$ be the generator of F. Notice that $\perp < z$, as F is prime. Therefore, F is a prime filter of $\mathbf{A}$, because all elements greater than or equal to z in $\mathbf{A}$ are in $I(A)$, and the operations of $\mathbf{A}$, restricted to $I(A)$, behave exactly as the operations of $\mathbf{A}_G$. $\square$

Proposition 3.1.2. *Let $h \colon \mathbf{A} \to \mathbf{B}$ be a homomorphism of finite directly indecomposable RDP algebras $\mathbf{A}$ and $\mathbf{B}$, and let $E(h)$ be the set of homomorphisms h' from $\mathbf{A}$ to $\mathbf{B}$ such that $h_G = h'_G$. If $1 < \mathrm{type}(\mathbf{A}) = n \leq m = \mathrm{type}(\mathbf{B})$, then $|E(h)| = \binom{m}{n}$, otherwise $|E(h)| = 1$.*

Proof. By Proposition 3.1, type($\mathbf{A}$) $\leq$ type($\mathbf{B}$). If type($\mathbf{A}$) $= 0$, then $h = h_G$ and then, $|E(h)| = 1$. If type($\mathbf{A}$) $= 1 <$ type($\mathbf{B}$), then the only extension of h_G to a homomorphism from $\mathbf{A}$ to $\mathbf{B}$ is the unique map that sends the fixpoint of $\mathbf{A}$ to the fixpoint of $\mathbf{B}$. Hence, $|E(h)| = 1$.

If $1 \leq$ type($\mathbf{A}$) $= n \leq m =$ type($\mathbf{B}$), then the extension of h_G to a homomorphism from $\mathbf{A}$ to $\mathbf{B}$ is not unique (unless $n = m$). Each extension sends the fixpoint of $\mathbf{A}$ to the fixpoint of $\mathbf{B}$, each nonidempotent point below the fixpoint of $\mathbf{A}$ to a nonidempotent point below the fixpoint of $\mathbf{B}$, and respects the chain of nonidempotent elements below the fixpoint of $\mathbf{A}$. Since the chain of nonidempotent elements below the fixpoint of $\mathbf{A}$ has n points, and the chain of nonidempotent elements below the fixpoint of $\mathbf{B}$ has $m \geq n$ points, there are exactly $\binom{m}{n}$ mappings that respect the chain of nonidempotent elements below the fixpoint of $\mathbf{A}$. $\square$

In order to achieve a correct definition of the category dual to the category of directly indecomposable finite RDP algebras, it is necessary to consider two facts. First, there exist nonisomorphic directly indecomposable finite RDP algebras $\mathbf{A}$ and $\mathbf{B}$ having order isomorphic prime spectra. For instance, an RDP chain of three elements with fixpoint and an RDP chain of two elements (hence, with no fixpoint) have the same prime spectrum but are not RDP-isomorphic (see Figure 3.2 as another example). Second, by Proposition 3.1.2, there exist distinct homomorphisms h' and h'' of directly indecomposable finite RDP algebras that have the same behavior upon restriction to idempotent elements, and hence induce the same open map between the corresponding prime spectra. For these reasons, objects

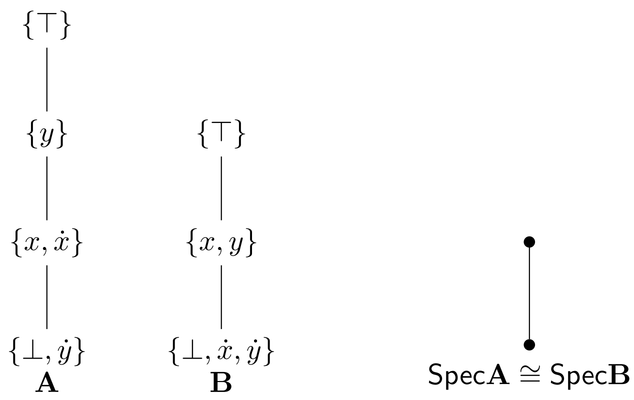


Figure 3.2: Two RDP chains with order isomorphic prime spectra.

in the dual category will be suitable pairs of posets, and morphisms will be suitable pairs of morphisms, acting componentwise, as follows.

Definition 3.1.1 (Hall Forest)**.** *A* (finite) hall tree *is a pair* (T, J) *where* T *is a tree and* J *is a chain. A* (finite) hall forest *is a (finite) multiset* $\{(T_1, J_1), \ldots, (T_n, J_n)\}$ *of (finite) hall trees.* [1]

For every pair (T, J) *and* (T', J') *of hall trees a* morphism (of hall trees) *is a pair* (f, g) *where* $f\colon T \to T'$ *and* $g\colon J \to J'$ *are (partial) open maps, such that* $g(\max(J)) = \max(J')$. [2] *For every pair* F *and* F' *of hall forests, a* morphism (of hall forests) *is a map from the hall trees of* F *to the hall trees of* F'*, acting treewise as a morphism of hall trees.*

For every pair of morphism of hall trees $(f_1, g_1)\colon (T_1, J_1) \to (T_2, J_2)$*, and* $(f_2, g_2)\colon (T_2, J_2) \to (T_3, J_3)$*, the* composition *of* (f_1, g_1) *and* (f_2, g_2) *is the morphism of hall trees*

$$(f, g) = (f_2, g_2) \circ (f_1, g_1)\colon (T_1, J_1) \to (T_3, J_3)$$

such that $f = f_2 \circ f_1$ *and* $g = g_2 \circ g_1$*. The composition of morphisms of hall forests is determined by the treewise composition of the underlying morphism of hall trees.*

Upon noticing that finite posets and open maps form a category, it is easy to check that by Definition 3.1.1 compositions of morphism (of hall forests) are associative and preserve identities. Hence, (finite, hall) forests and their morphisms form a category, HF. We now prove the announced categorical equivalence between FRDP and HF.

First, let HT denote the full subcategory of (finite, hall) trees and their morphisms, and FDRDP denote the category of finite directly indecomposable RDP algebras and their homomorphisms. In light of Proposition 3.1.1, Proposition 3.1.2, and Theorem 1.2.1, we introduce a contravariant functor, Θ, from FDRDP to HT, as follows. Let $\mathbf{A}$ be a finite directly indecomposable RDP algebra. Then,

$$\Theta(\mathbf{A}) = (\mathsf{Spec}(\mathbf{A}_G), \mathbf{A}_P),$$

where

$$\mathbf{A}_P = (\{\{x \in A \mid y \leq x\} \mid \bot < y \leq z, z \text{ fixpoint of } \mathbf{A}\}, \supseteq).$$

In words, $\mathbf{A}_P$ is the structure formed by the filters (with respect to the lattice order of $\mathbf{A}$) generated by the nonidempotent elements of $\mathbf{A}$, ordered by reverse inclusion. By Proposition 3.1, $\mathbf{A}_P$ is a chain, and by (3.7), $|\mathbf{A}_P| = \text{type}(\mathbf{A})$. Let $f\colon \mathbf{A} \to \mathbf{B}$ be a morphism in FDRDP. We let

$$\Theta(f) = (\mathsf{Spec}(f_G), f_P)$$

[1] A *multiset* is a family whose members have multiple instances (a set is a multiset whose members have exactly one instance).

[2] Note that, if $g\colon J \to J'$ is an open map such that $g(\max(J)) = \max(J')$, then $|J'| \leq |J|$.

be the morphism (of hall trees) from $\Theta(\mathbf{B}) = (\mathsf{Spec}(\mathbf{B}_G), \mathbf{B}_P)$ to $\Theta(\mathbf{A}) = (\mathsf{Spec}(\mathbf{A}_G), \mathbf{A}_P)$ such that for every $F \in \mathsf{Spec}(\mathbf{B}_G)$,

$$\mathsf{Spec}(f_G)(F) \in \mathsf{Spec}(\mathbf{A}_G),$$

and, for every $F \in \mathbf{B}_P$,

$$f_P(F) = \{x \in A \mid f(x) \in F\} \in \mathbf{A}_P. \tag{3.8}$$

By Proposition 3.1, the dual of f satisfies the definition of morphism of (finite, hall) trees.

It is routine to verify that Θ is a contravariant functor from FDRDP to HT.

Lemma 3.1.1. *The category* FDRDP *is dually equivalent to the category* HT *via the contravariant functor* Θ*.*

Proof. It is sufficient to show that $\Theta\colon \mathsf{FDRDP} \to \mathsf{LT}$ is full, faithful, and essentially surjective [41, Theorem 4.4.1].

First we prove that Θ is essentially surjective, that is, for every object (T, J) in HT, there exists an object $\mathbf{A}$ in FDRDP such that $\Theta(\mathbf{A})$ is isomorphic to (T, J) in HT. Let (T, J) be in HT. By Theorem 1.2.1, let $\mathbf{B}$ be a finite directly indecomposable Gödel algebra such that $\mathsf{Spec}(\mathbf{B})$ is isomorphic to T in the category of finite forests F. If $|J| = |\emptyset| = 0$, let $\mathbf{A}$ be a finite directly indecomposable RDP algebra such that $\mathbf{A} = \mathbf{A}_G = \mathbf{B}$. Then, (T, J) is isomorphic in HT to $\Theta(\mathbf{A})$. If $|J| > 0$, let $\mathbf{A}$ be the finite directly indecomposable RDP algebra obtained as follows: Replace the minimum element $\bot$ of $\mathbf{B}$ with a chain $\bot < \cdots < x$ of $|J|+1$ elements (whose maximum and minimum are designed respectively as the bottom and the fixpoint of $\mathbf{A}$); define the operations $\odot$ and $\to$ over $\mathbf{A}$ by extending $\odot$ and $\to$ over $\mathbf{B}$ to the new $|J|+1$ elements of $\mathbf{A}$ as follows: if $y, y' \leq x$ in $\mathbf{A}$, then $y \odot y' = \bot$, otherwise $y \odot y' = y \wedge y'$; if $y \leq y'$ in $\mathbf{A}$ then $y \to y' = \top$, otherwise if $y' < y \leq x$ in $\mathbf{A}$ then $y \to y' = x$, otherwise $y \to y' = y'$. By construction, $\mathsf{Spec}(\mathbf{A}_G)$ is order isomorphic to T, and $\mathbf{A}_P$ is order isomorphic to J, so that (T, J) is isomorphic in HT to $\Theta(\mathbf{A})$.

Now we prove that Θ is full, that is, for every morphism (f, g) in HT, there exists a morphism h in FDRDP such that $\Theta(h) = (f, g)$. Let $(f, g)\colon (T, J) \to (T', J')$ be a morphism in HT so that $|J'| \leq |J|$. We construct h, as follows. Since Θ is essentially surjective, there exists objects $\mathbf{A}$ and $\mathbf{B}$ in FDRDP such that $(T, J) = \Theta(\mathbf{B})$ and $(T', J') = \Theta(\mathbf{A})$, that is, $T = \mathsf{Spec}(\mathbf{B}_G)$ and $J = \mathbf{B}_P$, and $T' = \mathsf{Spec}(\mathbf{A}_G)$ and $J' = \mathbf{A}_P$. Note that $\mathrm{type}(\mathbf{A}) \leq \mathrm{type}(\mathbf{B})$. By Theorem 1.2.1, there exists an homomorphism h_G from $\mathbf{A}_G$ to $\mathbf{B}_G$ such that $\mathsf{Spec}(h_G)$ is equal to open map f from T to T'. Now, $h\colon \mathbf{A} \to \mathbf{B}$ is the extension of h_G to nonidempotent elements in $\mathbf{A}$ defined in terms of g, as follows. Let x be a nonidempotent element in

$\mathbf{A}$, and let $F \in \mathbf{A}_P$ be the filter generated by x with respect to the lattice order of $\mathbf{A}$. As $g^{-1}(F) \subseteq \mathbf{B}_P$ is a chain, with respect to the order of $\mathbf{B}_P$, let F' be the maximum in $g^{-1}(F)$, and let y be the generator of F' in $\mathbf{B}$. Then, $h(x) = y$. It is routine to check that, by the definitions, h is a homomorphism from $\mathbf{A}$ to $\mathbf{B}$.

Finally we prove that Θ is faithful, that is, for every pair $f\colon \mathbf{A} \to \mathbf{B}$ and $g\colon \mathbf{A} \to \mathbf{B}$ of morphisms in FDRDP, if $\Theta(f) = \Theta(g)$, then $f = g$. Suppose that f and g are distinct, say $f(y) \neq g(y)$ for some $y \in A$. We distinguish two cases. If $y \in I(A)$, then the open maps that f_G and g_G induce by (1.8) are distinct. But then $\Theta(f) = (\mathsf{Spec}(f_G), \cdot) \neq (\mathsf{Spec}(g_G), \cdot) = \Theta(g)$, because by Theorem 1.2.1, $\mathsf{Spec}(f_G) \neq \mathsf{Spec}(g_G)$. Otherwise, if $y \notin I(A)$, then y lies in the chain below the fixpoint of $\mathbf{A}$ above the bottom (because the homomorphisms f and g are to send the bottom of $\mathbf{A}$ to the bottom of $\mathbf{B}$, and the fixpoint of $\mathbf{A}$ to the fixpoint of $\mathbf{B}$). Also, the length of the chain below the fixpoint of $\mathbf{B}$ is strictly greater than the length of the chain below the fixpoint of $\mathbf{A}$ (because the homomorphisms f and g are to respect the chain below the fixpoint of $\mathbf{A}$, but send the point y to distinct points in the chain below the fixpoint of $\mathbf{B}$). But then, the open maps that f and g induce by (3.8) are distinct. Then, $\Theta(f) = (\cdot, f') \neq (\cdot, g') = \Theta(g)$, because $f' \neq g'$. □

We extend the contravariant functor $\Theta\colon \mathsf{FDRDP} \to \mathsf{HT}$ to the entire category FRDP. For objects, let $\mathbf{A}$ be a finite RDP algebra, and let $(\mathbf{A}_i)_{i\in I}$ be its direct decomposition. Then, $\Theta(\mathbf{A})$ is the hall forest given by the disjoint union (accounting for multiplicity) of the hall trees $\Theta(\mathbf{A}_i)$, for all $i \in I$. For morphisms, let $f\colon \mathbf{A} \to \mathbf{B}$ be a homomorphism of finite RDP algebras. Let $\mathbf{A}$ and $\mathbf{B}$ be directly decomposed by $(\mathbf{A}_i)_{i\in I}$ and $(\mathbf{B}_j)_{j\in J}$ respectively, let $\Theta(\mathbf{B})$ and $\Theta(\mathbf{A})$ be the disjoint union (accounting for multiplicity) of $\Theta(\mathbf{B}_j)$ for $j \in J$ and $\Theta(\mathbf{A}_i)$ for $i \in I$ respectively. Let $j \in J$. If F is a prime lattice filter of $\mathbf{B}_j$, then $G = \{a \in A \mid f(a)_j \in F\}$ is a prime lattice filter of $\mathbf{A}$. By primality, if x is the generator of G, then there exists a unique $i \in I$ such that $\bot_i < x_i$. Moreover, i is independent of the choice of F, that is, if F' is a prime lattice filter of $\mathbf{B}_j$ and x' is the generator of $G' = \{a \in A \mid f(a)_j \in F'\}$, then $\bot_i < x'_i$. Let $f_j\colon \mathbf{A}_i \to \mathbf{B}_j$ be the map defined by $f_j(x) = (f(\bot_1, \ldots, \bot_{i-1}, x, \bot_{i+1}, \ldots, \bot_{|I|}))_j$, for all $x \in \mathbf{A}_i$; it is easy to check that f_j is an RDP-homomorphism, and that $f_j(a_i) = f(a)_j$. The morphism of hall forests $\Theta(f)\colon \Theta(\mathbf{B}) \to \Theta(\mathbf{A})$ is defined treewise by the action of the morphisms of hall trees $\Theta(f_j)$, for all $j \in J$. Compare Example 3.1.3.

Theorem 3.1.1. *The category* FRDP *is dually equivalent to the category* HF *via the contravariant functor* Θ.

Proof. By universal algebraic facts [13, Theorem 7.10], every finite RDP algebra is isomorphic to the direct product of a finite family of directly indecomposable finite RDP algebras, and this direct decomposition is unique (modulo isomorphism). The fact that Θ is full, faithful, and essentially surjective follows by appealing to Lemma 3.1.1. □

Aiming at a combinatorial representation of the free n-generated RDP algebra, we now define explicitly a contravariant functor $\Psi\colon \mathsf{HF} \to \mathsf{FRDP}$, adjoint to $\Theta\colon \mathsf{FRDP} \to \mathsf{HF}$, such that: for every finite hall forest F, $\Psi(F)$ is a finite RDP algebra; and, for every morphism (f, g) from the hall forest F' to the hall forest F'', $\Psi((f, g))$ is a homomorphism from the finite RDP algebra $\Psi(F'')$ to the finite RDP algebra $\Psi(F')$.

We provide a construction in two stages of the finite RDP algebra $\Psi(F)$: first, on the basis of the finite hall forest F, we compute a finite *augmented* forest F'; then, we obtain the finite RDP algebra by equipping the maximal antichains over F' with suitably defined operations.

Step 1: For each hall tree (T, J) in F, the *augmented* forest F' contains an *augmented* tree T'. T' is a copy of T, with the following modifications. If the maximal points of T are $x_1, \dots, x_n$, then T' contains new points $y_1, \dots, y_n$ such that $x_i < y_i$ in T', for all $i \in \{1, \dots, n\}$. Also, if $|J| \geq 1$ and the minimum element of T is y, then the chain J is adjoined below y in T' (that is, y covers the maximal element of J in T'), and in this case, the point y is called the *fixpoint* of T', in symbols, $y = \operatorname{fixpoint} T'$.

Step 2: Let $\mathcal{A}_F$ be the set of maximal antichains in F', and let $\mathcal{C}_F$ be the set of maximal chains in F'. Since each maximal chain $C \in \mathcal{C}_F$ is contained in some augmented tree T' of F', if T' has a fixpoint, then C contains such fixpoint, which we denote by $\operatorname{fixpoint} C$. We interpret the binary operations $\wedge$, $\vee$, $\odot$, and $\to$, and the constants $\perp$ and $\top$ over $\mathcal{A}_F$ as follows ($A, A' \in \mathcal{A}_F$ and $C \in \mathcal{C}_F$):

$$A \wedge_F A' \cap C = \min\{A \cap C, A' \cap C\}, \tag{3.9}$$

$$A \vee_F A' \cap C = \max\{A \cap C, A' \cap C\}, \tag{3.10}$$

$$A \odot_F A' \cap C = \begin{cases} \min C & A \cap C, A' \cap C \leq \operatorname{fixpoint} C, \\ \min\{A \cap C, A' \cap C\} & \text{otherwise}, \end{cases} \tag{3.11}$$

$$A \to_F A' \cap C = \begin{cases} \max C & A \cap C \leq A' \cap C, \\ \operatorname{fixpoint} C & A' \cap C < A \cap C \leq \operatorname{fixpoint} C, \\ A' \cap C & \text{otherwise}, \end{cases} \tag{3.12}$$

$\perp_F \cap C = \min C$, and $\top_F \cap C = \max C$. As maximal antichains in $\mathcal{A}_F$ are uniquely determined by their intersections with maximal chains in $\mathcal{C}_F$, the previous definition is sound. Also, notice the resemblance between (3.11) and (3.12) above and (3.2) and (3.3) respectively.

Example 3.1.1. *If $F = \{(T_1, \emptyset), (T_2, J_2)\}$ is the finite hall forest on the left, then $\mathcal{A}_F$ is the algebra of maximal antichains over the augmented forest $F' =$*

$\{T_1', T_2'\}$ *on the right, where* $\min T_1' = \{\bot, \neg x, \neg y\}$ *and* $\min T_2' = \{\bot, \neg x\}$*; notation is displayed for further reference.*

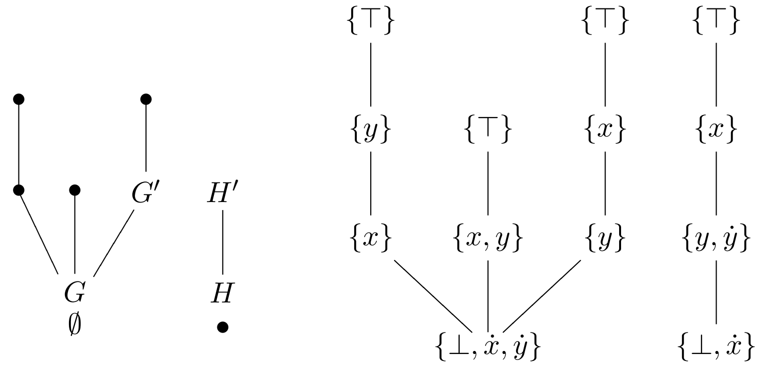


Figure 3.3: Example 3.1.1 and Example 3.1.2.

Let F be a finite hall forest. The key of the construction is to establish a bijection

$$m\colon \mathcal{A}_F \to \hom(F, \Theta(\mathbf{F}_1)), \tag{3.13}$$

from the maximal antichains in $\mathcal{A}_F$, to the morphisms from the hall forest F to the hall forest $\Theta(\mathbf{F}_1)$ corresponding to the prime spectrum of the free 1-generated RDP algebra. For presentation sake, we defer to Proposition 3.3.1 the description of $\mathbf{F}_1$ and the construction of $\Theta(\mathbf{F}_1)$. Here, we assume that $\Theta(\mathbf{F}_1)$ is as in Figure 3.4. The bijection m is defined as follows. Let h be

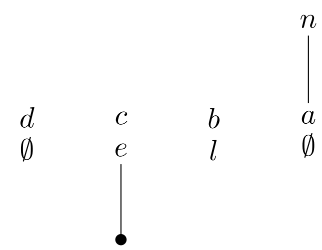


Figure 3.4: $\Theta(\mathbf{F}_1)$ with notation for the discussion of bijection m displayed. For each hall tree (T, J) in $\Theta(\mathbf{F}_1)$, the component J is displayed below T.

a morphism from F to $\Theta(\mathbf{F}_1)$. Let (T, J) be a hall tree in F, and let (f, g) be the morphism implementing the behavior of h on (T, J). Let T' be the augmented tree corresponding to T. Then, the maximal antichain $m^{-1}(h)$, corresponding to the labelled morphism h, restricted to T', satisfies the following conditions. If $f^{-1}(a)$ is empty, then the antichain $m^{-1}(h) \cap T' = \min T'$. Otherwise, if $f^{-1}(b)$ is equal to T, then $m^{-1}(h) \cap T' = \operatorname{fixpoint} T'$.

Otherwise, if $f^{-1}(c)$ is equal to T, then $m^{-1}(h) \cap T'$ is determined by $g^{-1}(e)$, as follows: if the maximum element in $g^{-1}(e)$ is the kth smallest element of J, then $m^{-1}(h) \cap T'$ is the $(k+1)$th smallest element of T'. Otherwise, if $f^{-1}(a)$ is nonempty, $m^{-1}(h) \cap T'$ contains the covers in F' of the maximal points in $f^{-1}(a)$ (these points are in F' by construction). As there are no other cases, the definition of m is complete.

Example 3.1.2. *First compare the hall tree* $(T_1, \emptyset)$ *in Example 3.1.1. By Definition 3.1.1, there are* 19 *morphisms* $h = (f, g)$ *from* $(T_1, \emptyset)$ *to* $\Theta(\mathbf{F}_1)$*, indexed by the* 19 *maximal antichains in* T_1'*. Comparing Figure 3.4, for instance, if* $f(T_1) = d$ *in* $\Theta(\mathbf{F}_1)$*, then* $m^{-1}(h)$ *is the maximal antichain* $\{\bot, \neg x, \neg y\}$ *in* T_1'*; if* $f(T_1) = a$*, then* $m^{-1}(h) = \{\top, \top, \top\}$*; if* $f(\{G, G'\}) = a$ *and* $f(T_1 \setminus \{G, G'\}) = n$*, then* $m^{-1}(h) = \{x, xy, x\}$*.*

Next compare the hall tree (T_2, J_2) *in Example 3.1.1. By Definition 3.1.1, there are* 4 *morphisms* $h = (f, g)$*, from* (T_2, J_2) *to* $\Theta(\mathbf{F}_1)$*, indexed by the* 4 *maximal antichains in* T_2'*, as follows. If* $f(T_2) = d$ *in* $\Theta(\mathbf{F}_1)$*, then* $m^{-1}(h) = \{\bot, \neg x\}$ *in* T_2'*; if* $f(T_2) = b$ *and* $g(J_2) = l$*, then* $m^{-1}(h) = \{y, \neg y\}$*; if* $f(H) = a$ *and* $f(H') = n$*, then* $m^{-1}(h) = \{x\}$*; and, if* $f(T_2) = a$*, then* $m^{-1}(h) = \{\top\}$*.*

Given m, a contravariant functor $\Psi\colon \mathsf{HF} \to \mathsf{FRDP}$ is easily obtained, along the lines of [5], as follows: If F is a finite hall forest, then

$$\Psi(F) = (\mathcal{A}_F, \wedge_F, \vee_F, \odot_F, \to_F, \bot_F, \top_F) \tag{3.14}$$

is a finite RDP algebra. If g is a morphism from the finite hall forest F' to the finite hall forest F'', then $\Psi(g)$ is the homomorphism from $\Psi(F'') = \mathcal{A}_{F''}$ to $\Psi(F') = \mathcal{A}_{F'}$, such that for every $a \in \mathcal{A}_{F''}$,

$$(\Psi(g))(a) = m^{-1}(m(a) \circ g) \in \mathcal{A}_{F'}. \tag{3.15}$$

The verification that $\Psi(g)\colon \mathcal{A}_{F''} \to \mathcal{A}_{F'}$ is an RDP homomorphism is a burdening computation.

Example 3.1.3. *Let* $F' = \{(T_1, J_1), (T_2, J_2)\}$ *and* $F'' = \{(T_3, \emptyset)\}$ *be the hall forests depicted on the left, where* $|T_1| = 1$*,* $|T_2| = 2$*,* $|T_3| = 6$*. Let* $\Psi(F') = \mathcal{A}_{F'}$ *and* $\Psi(F'') = \mathcal{A}_{F''}$ *be the algebras of maximal antichains over the augmented forests* $\{T_1', T_2'\}$ *and* $\{T_3'\}$ *depicted on the right, where* $|T_1'| = 3$*,* $|T_2'| = 4$*,* $|T_3'| = 9$*.*

Let g *be the morphism that sends* T_1 *and* T_2 *to* $\min T_3$*; then,* $\Psi(g)\colon \mathcal{A}_{F''} \to \mathcal{A}_{F'}$ *is defined by (3.15). We compute* $\Psi(g)$ *on two samples.*

Let $a = \{\bot, \neg x, \neg y\} \in \Psi(F'')$*. Along the lines of Example 3.1.2,* $m(a)$ *is a morphism* (f_a, g_a) *from* F'' *to* $\Theta(\mathbf{F}_1)$ *such that* $f_a(T_3) = d$ *(recall Figure 3.4). Then, the composition* $m(a) \circ g$ *is a morphism from* F' *to* $\Theta(\mathbf{F}_1)$ *that sends* T_1 *and* T_2 *to* d*. Then, by the definition of* m*,*

$$(\Psi(g))(a) = m^{-1}(m(a) \circ g) = \{\bot, \bot\}.$$

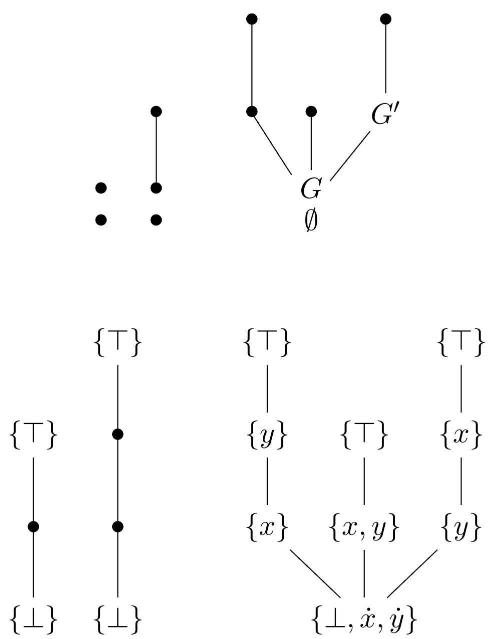


Figure 3.5: Example 3.1.3.

Let $a = \{x, xy, x\} \in \Psi(F'')$. *Along the lines of Example 3.1.2,* $m(a)$ *is a morphism* (f_a, g_a) *from* F'' *to* $\Theta(\mathbf{F}_1)$ *such that* $f_a(\{G, G'\}) = a$ *and* $f_a(T_3 \setminus \{G, G'\}) = n$. *Then, the composition* $m(a) \circ g$ *is a morphism from* F' *to* $\Theta(\mathbf{F}_1)$ *that sends* T_1 *and* T_2 *to* a. *By the definition of* m,

$$(\Psi(g))(a) = m^{-1}(m(a) \circ g) = \{\top, \top\}.$$

Let $a = \{x, xy, x\} \in \Psi(F'')$. *In light of the previous computations, we show that* $\Psi(g)$ *preserves the negation of* a,

$$\begin{aligned}
\Psi(g)(\neg_{F''} a) &= \Psi(g)(\neg_{F''}\{x, xy, x\}) \\
&= \Psi(g)(\{\bot, \neg x, \neg y\}) \\
&= \{\bot, \bot\} \\
&= \neg_{F'}\{\top, \top\} \\
&= \neg_{F'}(\Psi(g)(\{x, xy, x\})) \\
&= \neg_{F'}(\Psi(g)(a));
\end{aligned}$$

analogous computations show that in fact, $\Psi(g)$ *is an RDP homomorphism.*

3.2 Coproducts of RDP Algebras

In this section, we describe explicitly the (binary) product operation, $\times$, in the category of finite hall forests. Then, the coproduct of finite RDP algebras $\mathbf{A}$ and $\mathbf{B}$ will be given by

$$\Psi(\Theta(\mathbf{A}) \times \Theta(\mathbf{B})),$$

where Θ and Ψ are the adjoint contravariant functors between finite RDP algebras and finite hall forests given in Section 3.1.

Let F and F' be finite hall forests. We will describe the product $F \times F'$, and the projections π and π' of $F \times F'$ onto F and F' respectively. Each of F and F' is a multiset of finite hall trees, say $F = \{(T_i, J_i) \mid i \in [k]\}$ and $F' = \{(T'_i, J'_i) \mid i \in [k']\}$. In general, the result of the product $F \times F'$, and its projections, are uniquely determined by the result of the individual products $(T_m, J_m) \times (T'_n, J'_n)$ for every pair $(m, n) \in [k] \times [k']$. Hence, it is sufficient to describe the product $(T_m, J_m) \times (T'_n, J'_n)$, and its projections. In the present setting, the result of the product $(T_m, J_m) \times (T'_n, J'_n)$ is uniquely determined by the result of the individual products $T_m \times T'_n$ and $J_m \times J'_n$, and their projections, as follows. The product $T_m \times T'_n$ and its projections is computed in Section 1.2.1, and yields a finite tree S and its projections $\varsigma_{m,n}$ and $\varsigma'_{m,n}$ onto T_m and T'_n respectively. The product $J_m \times J'_n$ and its projections, explained below, yields a finite collection of $N(|J_m|, |J'_n|) \geq 1$ many chains K_o, together with their projections $\rho_{m,n,o}$ and $\rho'_{m,n,o}$ onto J_m and J'_n respectively ($1 \leq o \leq N(|J_m|, |J'_n|)$). Finally, the product $(T_m, J_m) \times (T'_n, J'_n)$ is the finite collection of $N(|J_m|, |J'_n|)$ many hall trees (S, K_o) with projections $(\varsigma_{m,n}, \rho_{m,n,o})$ and $(\varsigma'_{m,n}, \rho'_{m,n,o})$ onto (T_m, J_m) and (T'_n, J'_n) respectively ($1 \leq o \leq N(|J_m|, |J'_n|)$).

Aiming at the proof of the universal property, we give a careful description of the aforementioned chains $K_1, \ldots, K_{N(|J|,|J'|)}$, for a given pair of chains J and J'. If $j \leq 1$ or $j' \leq 1$, then $N(|J|, |J'|) = 1$ and $|K_1| = \max\{j, j'\}$. Otherwise, suppose that $j > 1$ and $j' > 1$. Roughly, given two chains J and J' of cardinality j and j' respectively, the problem is to describe the chains over the points in the union of $J \setminus \max(J)$ and $J' \setminus \max(J')$ that respect the order of J and J'; without loss of generality, $J \cap J' = \emptyset$. Below, we let C_i denote a chain of length i. Clearly, it is possible to obtain chains of minimum length $m = \max\{j, j'\} - 1$ and maximum length $M = j + j' - 2$. Hence, the problem is equivalent to describing the surjective maps f from

$$D = (J \setminus \max(J)) \cup (J' \setminus \max(J'))$$

to chains C_i of length i ranging from m to M that *respect* the order of J and J', that is, if $x < y$ in J or J', then $f(x) < f(y)$ in C_i. We first enumerate these maps, and then, for each such map, we compute the corresponding chain K together with its projections onto J and J'.

The number of maps from $J \setminus \max(J)$ to C_i that respect the order of J is $\binom{i}{j-1}$, and the number of maps from $J' \setminus \max(J')$ to C_i that respect the order of J' is $\binom{i}{j'-1}$, hence the number of maps from D to C_i that respect simultaneously the order of J and J' is

$$OrdPres(i,j,j') = \binom{i}{j-1}\binom{i}{j'-1}.$$

We now establish the number of non-surjective maps from D to C_i that preserve the order of J and J', for short $NotSurj(i,j,j')$, to conclude that

$$N(i,j,j') = OrdPres(i,j,j') - NotSurj(i,j,j').$$

Any non-surjective map from D to C_i neglects k points in C_i, for some k between 1 to $i-m$. Clearly, there are $\binom{i}{k}$ possible choices for these k neglected points, and for each choice, the number of order-preserving non-surjective maps from D to C_i coincide with the number of order-preserving surjective maps from D to C_{i-k}, that is, $N(i-k,j,j')$. Hence, we obtain the recurrence,

$$NotSurj(i,j,j') = \sum_{k=1}^{i-m} \binom{i}{k} N(i-k,j,j'),$$

whose base case is $NotSurj(m,j,j') = 0$, because in this case, the sum is the empty sum. Summarizing, given two chains J and J' of cardinality j and j' respectively, letting $m = \max\{j,j'\}-1$ and maximum length $M = j+j'-2$,

$$N(j,j') = \sum_{i=m}^{M} N(i,j,j').$$

Now, for finite hall forests $F = \{(T_i, J_i) \mid i \in [k]\}$ and $F' = \{(T'_i, J'_i) \mid i \in [k']\}$, let $(m,n) \in [k]\times[k']$, and let J_m and J'_n be the chain components of two hall trees (T_m, J_m) and (T'_n, J'_n). Let f be the oth map in some fixed order over the $N(|J_m|, |J'_n|)$ many surjective order-preserving maps from the union of $J_m \setminus \max(J_m)$ and $J'_n \setminus \max(J'_n)$ to chains of length $\max\{|J_m|, |J'_n|\} - 1 \leq i \leq |J_m| + |J'_n| - 2$. Then, we let the oth chain K_o in the collection of chains returned by $J_m \times J'_n$ be the chain of $i+1$ points, whose projections onto J_m and J'_n are respectively $\rho_{m,n,o}$ and $\rho'_{m,n,o}$, defined as follows. The projection onto the left factor J_m is defined by: $\rho_{m,n,o}(\max(K_o)) = \max(J_m)$; for $x \in K_o$, if $x \in J_m$, then $\rho_{m,n,o}(x)$ is equal to x; otherwise, $\rho_{m,n,o}(x)$ is equal to $\rho_{m,n,o}(y)$ where y is the smallest element of K_o above x such that $y \in J_m$. The projection onto the right factor J'_n is similarly defined by: $\rho_{m,n,o}(\max(K_o)) = \max(J'_n)$; for $x \in K_o$, if $x \in J'_n$, then $\rho_{m,n,o}(x)$ is equal to x; otherwise, $\rho_{m,n,o}(x)$ is equal to $\rho_{m,n,o}(y)$ where y is the smallest element of K_o above x such that $y \in J'_n$.

We now show that the product operation described above has the universal property.

Theorem 3.2.1. *Let $F = \{(T_i, J_i) \mid i \in [k]\}$ and $F' = \{(T'_i, J'_i) \mid i \in [k']\}$ be finite hall forests. Then,*

$$F \times F' = \{(T_m, J_m) \times (T'_n, J'_n) \mid (m, n) \in [k] \times [k']\},$$

with projections π and π' onto F and F' given by,

$$\pi = \{(\varsigma_{m,n}, \rho_{m,n,1}), \ldots, (\varsigma_{m,n}, \rho_{m,n,N(|J_m|,|J'_n|)}) \mid (m, n) \in [k] \times [k']\},$$
$$\pi' = \{(\varsigma'_{m,n}, \rho'_{m,n,1}), \ldots, (\varsigma'_{m,n}, \rho'_{m,n,N(|J_m|,|J'_n|)}) \mid (m, n) \in [k] \times [k']\},$$

is the product of F and F' in the category HF*.*

Proof. The morphisms under consideration split into two components, the first acting on trees as by [18], and the second acting on chains. For the first component we rely upon the universal property of products of finite trees [18]. Hence, we reduce to prove the universal property of products of finite chains. The details follow.

It suffices to prove that if J, J' and J'' are chains, g' and g'' are morphisms from J to J' and J'' respectively, and π' and π'' are the projections of $J' \times J''$ onto J' and J'' respectively, then there exists a unique morphism h from J to $J \times J'$ such that $\pi' \circ h = g'$ and $\pi'' \circ h = g''$.

We establish a bijection between pairs of morphism g' and g'' from J to J' and J'' respectively, and morphisms h from J to $J' \times J''$. The bijection has the property that if h corresponds to g' and g'', then $\pi' \circ h = g'$ and $\pi'' \circ h = g''$. It follows that there exists a unique morphism h that factorizes g' and g'' through π' and π''.

The bijection is given by the following explicit construction of the morphism h, given morphisms g' and g''. The range of h is the chain K_o in $J' \times J''$ defined as follows (h sends J to a single chain in $J' \times J''$, as it is an open map). The chain K_o is the restriction of chain J to the points $x \in J$ such that one of the following four (disjoint and exhaustive) cases occur. Case 1: x is the maximum in $g'^{-1}(y)$ for some $y \in J'$ and x is the maximum in $g''^{-1}(z)$ for some $z \in J''$; in this case, we label x by $\{y, z\}$, and we let $h(x) = \{y, z\}$. Case 2: x is the maximum in $g'^{-1}(y)$ for some $y \in J'$; in this case, we label x by $\{y\}$, and we let $h(x) = \{y\}$. Case 3: x is the maximum in $g''^{-1}(z)$ for some $z \in J''$; in this case, we label x by $\{z\}$, and we let $h(x) = \{z\}$. Case 4: For the remaining $x \in J$, we let $h(x) = h(x')$ where x' is the smallest element above x in J such that $h(x')$ is defined by the above clauses (note that at least, $h(x')$ is defined if $x' = \max(J)$). Clearly, given g' and g'', the map h is uniquely determined. Moreover, by construction, $\pi' \circ h = g'$ and $\pi'' \circ h = g''$.

For injectivity, we prove that if $(f', f'') \neq (g', g'')$ are distinct pairs of morphisms from J to J' and J'' respectively, then the maps obtained from the above construction, say h' and h'', are distinct. If h' and h'' have distinct range, then they are distinct. Otherwise, if they have the same range, we claim that there exists $x \in J$ such that $h'(x) \neq h''(x)$. Suppose for a contradiction that $h' = h''$. Then, $f' = \pi' \circ h' = \pi' \circ h'' = g'$ and $f'' = \pi'' \circ h' = \pi'' \circ h'' = g''$, contradiction. For surjectivity, trivially, if h is a map from J to $J' \times J''$, then there exists a pair of morphisms g' and g'' from J to J' and J'' respectively: simply let, $g' = \pi' \circ h$ and $g'' = \pi'' \circ h$.

The proof is complete. □

It follows that HF has all finite products. In fact, by [41, Proposition 3.5.1], a category has all finite products if it has binary products and a terminal object; but, HF has binary products, and it is easy to check that the finite hall forest $\{(\bullet, \emptyset)\}$ is a terminal object (dually, the Boolean algebra $\{\bot, \top\}$, which obviously is an RDP algebra, homomorphically maps to any RDP algebra). Therefore, for S a finite hall forest in HF, we denote by S^n the product in HF of n copies of S, and by π_i the projection of S^n onto the ith factor S ($n \geq 1$).

In the next section, we will exploit the ability to compute finite coproducts of finitely generated RDP algebras to provide a combinatorial representation of free finitely generated RDP algebras.

3.3 Free Finitely Generated RDP Algebras

In this section, exploiting the categorical machinery developed, we give a combinatorial representation of the free n-generated RDP algebra $\mathbf{F}_n$, for $n \geq 1$.

As a preliminary step, we describe the free 1-generated RDP algebra, $\mathbf{F}_1$ (compare Figure 3.6). Recall from Section 3 that $\mathbf{F}_1$ is finite. Hence, by universal algebraic facts [13, Theorem 9.6], the RDP algebra $\mathbf{F}_1$ is isomorphic to a subdirect product of a finite number of subdirectly irreducible finite RDP algebras. As subdirectly irreducible finite RDP algebras are finite RDP chains, $\mathbf{F}_1$ is isomorphic to a subdirect product of a finite family of singly generated finite RDP chains. Notice that 1-generated RDP chains are a subset of $\mathcal{C}_1$, the set of 1-generated WNM chains characterized in Section 2.1 (compare Figure 2.1). By direct computation over (3.1), there are exactly five ways of singly generating RDP chains: $\{\bot, x, \neg\neg x\} < \{\neg x, \top\}$ is $\mathbf{C}_1$ in $\mathcal{C}_1$, $\{\bot\} < \{x\} < \{\neg x, \neg\neg x\} < \{\top\}$ is $\mathbf{C}_4$ in $\mathcal{C}_1$, $\{\bot\} < \{x, \neg x, \neg\neg x\} < \{\top\}$ is $\mathbf{C}_5$ in $\mathcal{C}_1$, $\{\bot, \neg x\} < \{x\} < \{\top, \neg\neg x\}$ is $\mathbf{C}_8$ in $\mathcal{C}_1$, $\{\bot, \neg x\} < \{x, \neg\neg x, \top\}$ is $\mathbf{C}_9$ in $\mathcal{C}_1$ (where x is the generator). Then, there is a subdirect embedding of $\mathbf{F}_1$ into the direct product of a finite family $\mathbf{A}_1, \ldots, \mathbf{A}_m$ of RDP chains, where each $\mathbf{A}_i$ is either $\mathbf{C}_1$, $\mathbf{C}_4$, $\mathbf{C}_5$, $\mathbf{C}_8$, or $\mathbf{C}_9$. Up to isomorphism, we can remove from the finite family $\mathbf{A}_1, \ldots, \mathbf{A}_m$ all copies of $\mathbf{C}_9$ ($\mathbf{C}_9$ is a proper quotient of $\mathbf{C}_8$, via the map that sends x to $\top$), and multiple copies of $\mathbf{C}_i$ for $i \in \{1, 4, 5, 8\}$. Hence, $\{\mathbf{C}_1, \mathbf{C}_4, \mathbf{C}_5, \mathbf{C}_8\}$ is the set $\mathcal{K}_1^{RDP}$ of non-redundant RDP chains (see Section 2.1).

Summarizing, there is a subdirect embedding of $\mathbf{F}_1$ into the direct product $\mathbf{A} = \mathbf{C}_1 \times \mathbf{C}_4 \times \mathbf{C}_5 \times \mathbf{C}_8$, so that $|\mathbf{F}_1| \leq |\mathbf{A}| = 72$. It is possible to check that $|\mathbf{F}_1| = 72$. The idea is the following: Given a tuple $(a_1, a_2, a_3, a_4) \in \mathbf{A}$, construct an RDP term t over the variable x such that the ith projection of t^A is equal to a_i for $i \in \{1, 4, 5, 8\}$. By direct computation, the RDP terms $x \to \neg x$, $t = \neg((x \leftrightarrow \neg x)^2)$, $t \to \neg\neg x$, and $\neg((\neg x)^2)$ realize respectively $(\top, \top, \top, \bot)$, $(\top, \top, \bot, \top)$, $(\top, \bot, \top, \top)$, and $(\bot, \top, \top, \top)$. Details on the construction of terms using normal forms are given in Section 6.2. As $\mathbf{F}_1$ is the largest singly generated RDP algebra (every singly generated RDP algebra is a quotient of $\mathbf{F}_1$ [13, Corollary 10.11]), we conclude that $\mathbf{F}_1 = \mathbf{A}$.

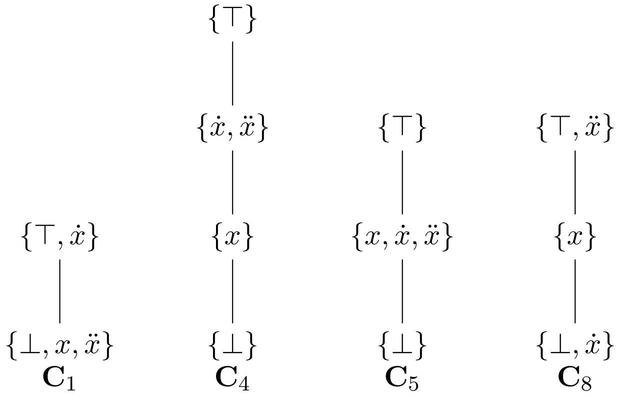


Figure 3.6: The free 1-generated RDP algebra $\mathbf{F}_1$ is the algebra of maximal antichains in the depicted forest, equipped with the operations defined in (3.11)-(3.12).

Proposition 3.3.1. *$\Theta(\mathbf{F}_1) = S_1$ is the finite hall forest displayed in Figure 3.7.*

Proof. We adopt the terminology and notation introduced in the above discussion. Notice that $\mathbf{C}_1$, $\mathbf{C}_4$, $\mathbf{C}_5$, $\mathbf{C}_8$ are finite, directly indecomposable RDP algebras. By definition: $\Theta(\mathbf{C}_1) = (G_1, J_1)$, where G_1 is the prime filter of $\mathbf{F}_1$ generated by $(\neg x, \bot, \bot, \bot)$, and $|J_1| = \text{type}(\mathbf{C}_1) = 0$; $\Theta(\mathbf{C}_4) = (G_2, J_2)$, where G_2 is the prime filter of $\mathbf{F}_1$ generated by $(\bot, \top, \bot, \bot)$, and $|J_2| = \text{type}(\mathbf{C}_4) = 2$; $\Theta(\mathbf{C}_5) = (G_3, J_3)$, where G_3 is the prime filter of $\mathbf{F}_1$ generated by $(\bot, \bot, \top, \bot)$, and $|J_3| = \text{type}(\mathbf{C}_3) = 1$; $\Theta(\mathbf{C}_8) = (G_4 \supseteq G_5, J_4)$, where G_4 and G_5 are the prime filters of $\mathbf{F}_1$ generated respectively by $(\bot, \bot, \bot, x)$ and $(\bot, \bot, \bot, \top)$, and $|J_4| = \text{type}(\mathbf{C}_4) = 0$. As $\Theta(\mathbf{F}_1)$ is the disjoint union of $\Theta(\mathbf{C}_i)$ for $i \in \{1, 4, 5, 8\}$, the statement is proved. $\square$

Lemma 3.3.1. *The prime spectrum $\Theta(\mathbf{F}_n)$ of the free n-generated RDP algebra $\mathbf{F}_n$, over the free generators $x_1, \ldots, x_n$, is the finite hall forest S_1^n.*

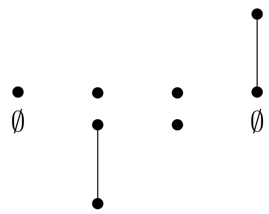


Figure 3.7: The hall forest $S_1 = \Theta(\mathbf{F}_1)$. For each hall tree (T, J) in S_1, the component J is displayed below T.

Proof. As in any variety, the free n-generated RDP algebra, $\mathbf{F}_n$, is the coproduct of n copies of the free 1-generated RDP algebra, $\mathbf{F}_1$. By Proposition 3.3.1, $\Theta(\mathbf{F}_1)$ is the finite hall forest S_1. The statement now follows from the categorical equivalence of HF and FRDP via the contravariant functor Θ (Theorem 3.1.1). $\square$

Theorem 3.3.1. *The free n-generated RDP algebra $\mathbf{F}_n$, over the free generators $x_1, \ldots, x_n$, is isomorphic to $\Psi(S_1^n)$.*

Proof. Note that the functor Ψ is the contravariant adjoint to the functor Θ, and that, by Lemma 3.3.1, the finite hall forest S_1^n is exactly $\Theta(\mathbf{F}_n)$, that is, the prime spectrum of the free n-generated RDP algebra $\mathbf{F}_n$ over the free generators $x_1, \ldots, x_n$. Recall that $\Psi(S_1^n)$ is the algebra of maximal antichains in $\mathcal{A}_{S_1^n}$ specified by (3.14). to identify the maximal antichains in $\mathcal{A}_{S_1^n}$ corresponding to the free generators $x_1, \ldots, x_n$, let π_i be the projection of S_1^n onto the ith factor S_1, and let m be the bijection in (3.13); the maximal antichain corresponding to the free generator x_i of $\mathbf{F}_n$ is $m^{-1}(\pi_i)$, for $1 \leq i \leq n$. $\square$

As an example, see Figure 3.8 for the prime spectrum of the free 2-generated RDP algebra.

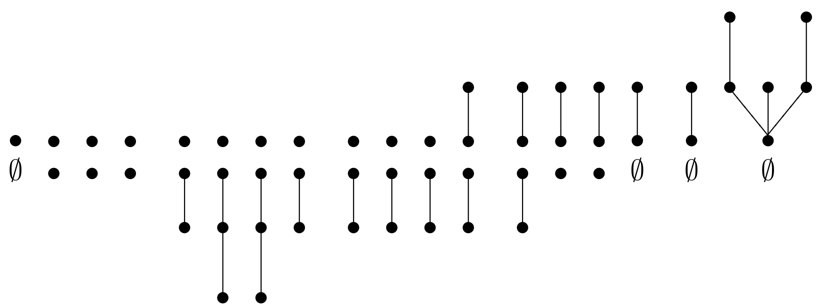


Figure 3.8: The finite hall forest $S_1^2 = S_1 \times S_1$.

Chapter 4

Finite NMG Algebras

As stated in Section 1.1.2, an NMG algebra is a WNM algebra satisfying (NMG). Notice that Gödel algebras are idempotent NMG algebras, and if $\neg\neg x = x$ holds in a NMG algebra $\mathbf{A}$, then $\mathbf{A}$ is a NM algebra. Furthermore, every NMG chain $\mathbf{A}$ can be decomposed as an ordinal sum $\mathbf{B} \oplus \mathbf{C}$ where the first component $\mathbf{B}$ is a NM algebra and the second component $\mathbf{C}$ is a Gödel algebra. For background on ordinal sums we refer to [1], see [45] for ordinal sums in the MTL setting.

A standard NMG algebra $[\mathbf{0}, \mathbf{1}]_*$ is of the form,

$$[\mathbf{0}, \mathbf{1}]_* = \langle [0, 1], *, \Rightarrow, \vee, \wedge, 0, 1 \rangle, \tag{4.1}$$

where, for every $x, y \in [0, 1]$, $x \wedge y = min\{x, y\}$, $x \vee y = max\{x, y\}$, and for arbitrary fixed $0 < a < 1$:

$$x * y = \begin{cases} min(x, y) & \text{if } x + y > a \\ 0 & \text{else.} \end{cases} \tag{4.2}$$

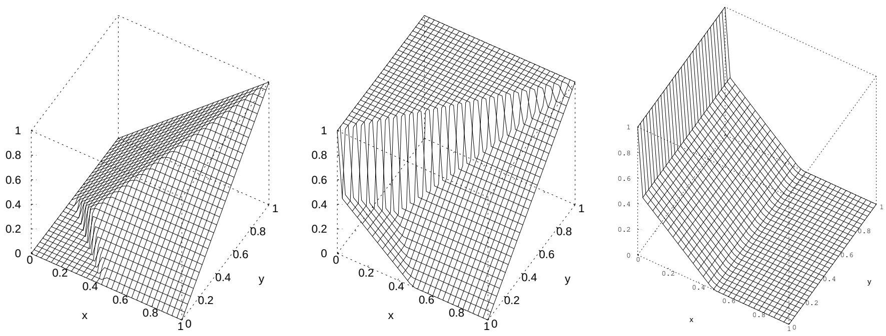
$$x \Rightarrow y = \begin{cases} 1 & \text{if } x \leq y, \\ max(a - x, y) & \text{else.} \end{cases} \tag{4.3}$$

Obviously $*$ is a t-norm and $\Rightarrow$ is its associated residuum (compare with Definition 1.1.2 and (1.1) respectively). We define, for every $x \in [0, 1]$:

$$\neg x := x \to 0 = \begin{cases} 1 & \text{if } x = 0, \\ max(a - x, 0) & \text{else.} \end{cases} \tag{4.4}$$

NMG logic is a many-valued propositional logic introduced in Chapter 1 as schematic extension of MTL.

Theorem 4.1. *[51] For every formula φ of NMG logic, the following statements are equivalent:*



$x * y.$ $\qquad$ $x \Rightarrow y.$ $\qquad$ $\neg x = x \Rightarrow \bot.$

Figure 4.1: A NMG triangular norm and its residuum, with $a = 1/2$ in (4.2) and (4.3).

- $\vdash_{NMG} \varphi$;
- *φ is a $[\mathbf{0}, \mathbf{1}]_*$-tautology.*

Hence, the variety of NMG algebras is singly generated by the standard algebra $[\mathbf{0}, \mathbf{1}]_*$.

The variety of NMG algebras is a subvariety of $\mathbb{V}(WNM)$, then $\mathbb{V}(NMG)$ is locally finite. Hence, we have the same situation of WNM algebras and RDP algebras. That is, every NMG algebra $\mathbf{A}$ is isomorphic to a subdirect product of a family $(\mathbf{C}_i)_{i \in I}$ of NMG chains, for some index set I. When $\mathbf{A}$ is finite and not trivial, then the family $(\mathbf{C}_i)_{i \in I}$ of non trivial chains is essentially unique up to reordering of the finite index set I. Hence, there exist $\pi_i : \mathbf{A} \to \mathbf{C}_i$ such that $\pi_i(a) = a_i$ for every $a \in \mathbf{A}$. We call a_i the ith-projection of a. Then, we can display every element a in $\mathbf{A}$ by means of its projections $(a_i)_{i \in I}$.

Since every finite NMG chain $\mathbf{C} = (C, \odot, \to, \vee, \wedge, \bot, \top)$ is a subalgebra of $[\mathbf{0}, \mathbf{1}]_*$, then for all $x, y \in C$, by (4.2), (4.3) and (4.4),

$$x \odot y = \begin{cases} min(x, y) & x > \neg y; \\ \bot & x \leq \neg y. \end{cases} \tag{4.5}$$

$$x \to y = \begin{cases} \top & x \leq y; \\ max(\neg x, y) & x > y. \end{cases} \tag{4.6}$$

We conclude this section with a discussion on positive and negative elements in NMG algebras. Remember that each prime filter of a WNM algebra (and of an NMG algebra too) is generated by a join-irreducible element.

As defined in Section 2.3, we recall that $P_{\mathbf{A}}$ and $N_{\mathbf{A}}$ denote respectively the set of positive and negative elements of an NMG algebra $\mathbf{A}$. Moreover, an *involutive* element y of $\mathbf{A}$ is such that $y \in \{\neg a \mid a \in \mathbf{A}\}$, otherwise y

is called a *weak* element. Notice that by (NMG) the negation fixpoint and all negative elements are involutive. It follows that weak elements of NMG algebras are all positive. Furthermore, by (NMG) and (4.5) in a finite NMG chain a weak element x is such that $\neg x = \bot$ [1].

Let $\mathbf{C}$ be a NMG chain with a weak element x. Since $\neg\neg x = \top$, it is easy to see that every element $y \geq x$ is weak. Indeed, suppose that $y = \neg\neg y$. By hypothesis $x \leq y = \neg\neg y < \top$, from which $\neg x \geq \neg y \geq \neg x$, that is $\neg x = \neg y$. By properties of the negation connective, we obtain $\neg\neg y = \neg\neg x = \top$, in contradiction with the hypothesis.

Given the finite directly indecomposable NMG algebra $\mathbf{A}$, we denote $G_{\mathbf{A}}$ the set of its weak elements and with $I_{\mathbf{A}}$ the set of its involutive elements.

By the above discussions, we can deduce that negative elements are obtained negating positive involutive elements. Hence,

$$|N_{\mathbf{A}}| = |P_{\mathbf{A}}| - |G_{\mathbf{A}}|. \tag{4.7}$$

Moreover, $N_{\mathbf{A}}$ is the order dual (see Section 1.3) of $(\{P_{\mathbf{A}} \setminus G_{\mathbf{A}}\}, \leq)$,

$$N_{\mathbf{A}} = (\{P_{\mathbf{A}} \setminus G_{\mathbf{A}}\}, \leq^{\partial}), \tag{4.8}$$

where $\leq$ is the order relation inherited from $\mathbf{A}$.

Let $f : \mathbf{A} \to \mathbf{B}$ be a homomorphism between finite directly indecomposable NMG algebras. Since f must preserve operations (negation, in particular) and the order, we can split its behavior in three maps $f = (f^+, f^*, f^-)$, such that $f^+ : P_{\mathbf{A}} \to P_{\mathbf{B}}$, $f^- : N_{\mathbf{A}} \to N_{\mathbf{B}}$ and f^* acting on the negation fixpoint of $\mathbf{A}$ (if it exists) to the negation fixpoint of $\mathbf{B}$ that must exists if it exists in $\mathbf{A}$ (otherwise f^* is the map $\emptyset \to \emptyset$). Given x a positive element of $\mathbf{A}$, by negation it must holds that

$$f^-(\neg x) = \neg f^+(x). \tag{4.9}$$

4.1 Categorical Equivalence

In this section we introduce the spectral duality between the category of finite directly indecomposable NMG algebras and their homomorphisms, DNMG, and the category IGT of finite labelled trees, whose morphisms are open maps with additional constraints. As for RDP algebras, by Birkhoff's representation theorem (see Appendix A) of finite algebras we extend this equivalence to a duality between the category of finite NMG algebras and their homomorphisms, FNMG, and the category of finite labelled forests IGF.

In the previous section we have seen that negative elements of finite RDP algebras have the order structure of a chain, then different finite RDP

[1]Let $\mathbf{C}$ be a NMG chain. The subchain $\{x \mid x \in C \setminus \{\bot, \top\}$ and x is a weak element$\} \cup \{\bot, \top\}$ with the order inherited by $\mathbf{C}$, is a Gödel chain.

algebras may have the same prime spectrum (recall Example 1.2.1). We have seen that negative elements of NMG algebras have a richer order structure than negative elements of finite RDP algebras, but this structure is encoded in the involutive positive elements. Hence, there exist nonisomorphic finite NMG algebras having order isomorphic prime spectra (as showed in the following Example), we can address this problem by labelling the prime filters.

Example 4.1.1. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be the two NMG chains depicted in Figure 4.2. It is easy to see that* $\mathsf{Spec}\mathbf{A}$ *and* $\mathsf{Spec}\mathbf{B}$ *are order isomorphic.*

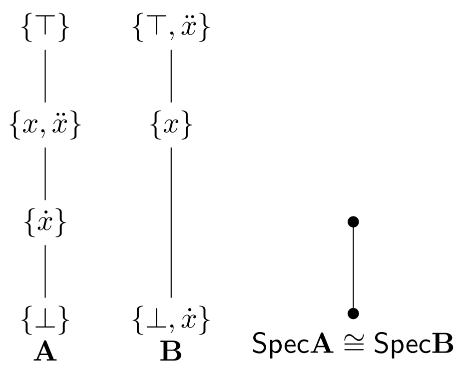


Figure 4.2: Two NMG chains with order isomorphic prime spectrum.

Given a domain L of labels and a poset P, a *labelling function* $\lambda : P \to L$ is such that $\lambda(e) = l$ for $e \in P$ and $l \in L$. We call l the *label* of e.

Let T be a tree, L a set of labels and λ a labelling function having T and L as domain and codomain respectively. Then, we call T a *labelled tree* if $\lambda(t) = l$ for every $t \in T$ and for some $l \in L$.

Definition 4.1.1. *An* IG-tree *is a labelled tree* T *where* $\lambda : T \to \{B, G, I\}$ *is a labelling function such that:*

- *if* r *is the root of* T *then* $\lambda(r) = \{B, I\}$*;*
- *if* $\lambda(t) = B$ *then* t *is the root of* T*;*
- $\{t \mid \lambda(t) = G\}$ *is an upper set of* T.

Let T *and* T' *be two IG-trees whose roots are* r *and* r' *respectively. A* morphism *between* T *and* T' *is an order-preserving open map* $f : T \to T'$ *such that* $t \in T \mapsto f(t)$ *when*

- *if* $\lambda(t) = B$ *then* $\lambda(f(t)) = B$*;*
- *if* $\lambda(t) = I$ *then* $\lambda(f(t)) \in \{B, I\}$*;*

- *if* $\lambda(t) = G$ *then* $\lambda(f(t)) = G$,

  or there exists $t' < t$ *such that* $f(t') = f(t)$ *and* $\lambda(f(t')) \in \{B, I\}$.

An IG-forest *is a disjoint union of IG-trees* $F = \bigsqcup_{i \in I} T_i$, *for* I *a finite index set. A* morphism *between IG-forests* $f : F \to F'$ *is a map from the IG-trees of* $F = \bigsqcup_{i \in I} T_i$ *to the IG-trees of* $F' = \bigsqcup_{j \in J} T'_j$, *such that* $f(T_i) \subseteq T'_j$ *for all* $i \in I$ *and for some* $j \in J$.

It is easy to see that morphisms of IG-trees and IG-forest compose. We denote with IGF the category of IG-forests and their morphisms, and IGT the subcategory of IG-trees and their morphisms.

Let $\mathbf{A}$ be a finite NMG algebra and a be a join-irreducible positive element in $\mathbf{A}$. Then, a generates a prime filter F_a. In the usual way (see (1.6)), starting from F_a we define a congruence $\theta_{F_a} = \{(x, y) \in A^2 \mid (x \leftrightarrow y) \in F_a\}$. We denote with $[x]$ the equivalence classes of θ_{F_a}. Since a is join-irreducible, then it covers (see Section 1.3) a unique element $a' \in \mathbf{A}$. The NMG algebra $\mathbf{A}/\theta_{F_a}$ is a NMG chain of equivalence classes, where $a \in [\top]$ and $[a']$ is the coatom.

Then, for every filter F_a in $\mathsf{Spec}\mathbf{A}$ we define a label:

$$\Lambda(F_a) = \begin{cases} B & \text{if } a = m_{\mathbf{A}} \text{ and } \mathbf{A}/\theta_{F_a} \text{ does not have a negation fixpoint;} \\ I & \text{if } [a'] \text{ is involutive, or} \\ & \text{if } a = m_{\mathbf{A}} \text{ and } \mathbf{A}/\theta_{F_a} \text{ has a negation fixpoint;} \\ G & \text{otherwise.} \end{cases} \tag{4.10}$$

Recall that $m_{\mathbf{A}}$ is the generator of the maximal prime filter in $\mathbf{A}$, see (2.6).

Remark 4.1.1. *Note that, when* $\Lambda(F_a) = B$ *then* $\mathbf{A}/\theta_{F_a}$ *is isomorphic to the 2 element Boolean chain, when* $\Lambda(F_a) = G$ *then* $\mathbf{A}/\theta_{F_a} \cong C \oplus C'$ *where* C *is a NM chain and* C' *a non-trivial Gödel chain, when* $\Lambda(F_a) = I$ *then* $\mathbf{A}/\theta_{F_a}$ *is a NM chain with more than 3 elements and if* $a = m_{\mathbf{A}}$ *is a 3 element NM chain with fixpoint.*

As an example, take the NMG chain $\mathbf{A}$ *in Example 4.2 and let* θ_{F_a} *be a congruence defined by a filter on* $\mathbf{A}$*. If* $(\{\neg x\}, \{\top\}) \in \theta_{F_a}$*, then* $\mathbf{A}/\theta_{F_a}$ *is isomorphic to the 2 element Boolean chain.*

In this way, in case $\mathbf{A}$ is directly indecomposable, we obtain a labelled tree $(\Lambda(\mathsf{Spec}\mathbf{A}), \leq)$ whose root is labelled with I if $\mathbf{A}$ has a fixpoint and with B otherwise, all other nodes are labelled with I and G and $\leq$ is the order inherited by $\mathsf{Spec}\mathbf{A}$. Then, we define a contravariant functor Θ from DNMG to IGT:

$$\Theta(\mathbf{A}) = (\Lambda(\mathsf{Spec}\mathbf{A}), \leq).$$

Example 4.1.2. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be the two NMG chains in Figure 4.2. Now we can distinguish between them looking at their labelled prime spectra depicted in Figure 4.3.*

$$\begin{array}{cc} I & G \\ \Big| & \Big| \\ B & B \\ \Lambda(\mathsf{Spec}\mathbf{A}) & \Lambda(\mathsf{Spec}\mathbf{B}) \end{array}$$

Figure 4.3: The labelled prime spectra of the NMG chains in Figure 4.2.

Notice that, since weak elements are upper sets in NMG chains then the set of elements labelled with G is an upper set in every IG-tree.

Let $h : \mathbf{A} \to \mathbf{B}$ be a homomorphism in DNMG and $F \in \mathsf{Spec}\mathbf{B}$. Then, the dual of h is given by

$$\Theta(h)(F) = \{a \in A \mid h(a) \in F\} \in \Theta(\mathbf{A}). \tag{4.11}$$

Note that, if $a \in \mathbf{A}$ is weak, then either $h(a) = \top^{\mathbf{B}}$ or $h(a)$ is weak. While, if $a \in \mathbf{A}$ is involutive, then $h(a)$ is involutive. Moreover, homomorphisms of NMG algebras with negation fixpoint into NMG algebras without fixpoint do not exist. Hence, duals of homomorphisms in DNMG are well defined morphisms in IGT.

Before showing that the functor Θ establishes a categorical duality between DNMG and IGT, we show how to reconstruct a directly indecomposable NMG algebra from an IG-tree.

Let $\mathbf{A}$ be a finite NMG algebra, and $\mathbf{F}_1$ be the free 1-generated NMG-algebra. Then, the homomorphisms $\mathbf{F}_1 \to \mathbf{A}$ are in bijection with the elements of $\mathbf{A}$, moreover these homomorphisms form an NMG algebra isomorphic to $\mathbf{A}$. This means that, dually we can reconstruct a directly indecomposable NMG algebra using the morphisms from its dual IG-tree T to the prime spectrum of $\mathbf{F}_1$.

Here we assume that the prime spectrum of the free 1-generated NMG algebra is the IG-forest depicted in Figure 4.4, we have *polarized* the labels for reference in the following discussion. Details on the construction of prime spectra for free NMG algebras will be given in Section 4.4.

By Definition 4.1.1, every subtree T' of T can be always mapped to B^- or to B^+. To extend these maps to morphisms from T to F^- and F^+ we have to do some additional consideration. Let t be an element of T and $\prec_T$ be the covering relation in T, we call *G-homogeneous* the set $\uparrow^G t \subseteq (\uparrow t \backslash \{t\})$ where for every $t' \in \uparrow^G t$ we have $\lambda(t') = G$ and $t'' \prec_T t'$ is such that $t'' \in \uparrow^G t$ or $t'' = t$. Note that, since elements labelled with G form an upper set in every IG-tree, if $\uparrow^G t$ exists for $t \in T$, then it is unique.

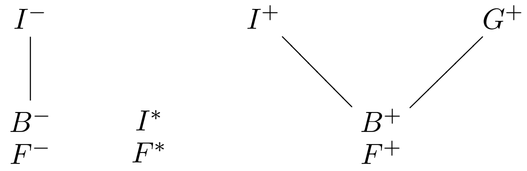


Figure 4.4: The polarized prime spectrum of the free 1-generated NMG algebra.

By Definition 4.1.1, a morphism $f : T \to F^-$ such that $T' \mapsto B^-$ exists if and only if

$$\uparrow^G t \subseteq T' \text{ for all } t \in T'.$$

While a morphism $g : T \to F^+$ such that $T' \mapsto B^+$ exists for every subtree T' of T. Furthermore, for each subtree T' of T there exists a morphism that maps T' to I^* only if the root of T is labelled with I.

Example 4.1.3. *Let T be the IG-tree depicted in Figure 4.5. Then, there exist 6 morphisms $f : T \to F^+$ and each morphism "selects" a subtree S of T such that $S \mapsto B^+$. There exist only 2 morphisms $f : T \to F^-$ and each one "selects" a subtree S of T such that $S \mapsto B^-$.*

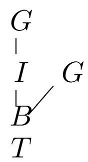


Figure 4.5: The IG-tree T used in Example 4.1.3.

The above discussion justifies the following construction.

Let T be a labelled tree in IGT and call r its root. We denote with $SubT$ the set of all subtrees of T. We set $P_T = SubT \setminus \{\emptyset\}$ and $N_T = \{S \in P_T \mid$ there exists $f : T \to F^-$ such that $S \mapsto B^-\}$. Finally, $A_T = P_T \cup N_T$. If $\lambda(r) = I$ then there exists a morphism such that $T \mapsto I^*$. Since T is already in $P_T \subseteq A_T$, when $\lambda(r) = I$ we add $\{\emptyset\} \in SubT$ to A_T, meaning that $T \not\mapsto B^-$ and $T \not\mapsto B^+$. Note that $\{\emptyset\} \notin P_T \cup N_T$. We denote every element S in P_T as S^+ and every element R in N_T as R^-.

In the following we show that A_T can be equipped with the structure of an NMG algebra. We start by defining a partial order $\leq$ over A_T, $R^+ \leq S^+$ if $R \subseteq S$, $R^- \leq S^-$ if $R \supseteq S$, $\{\emptyset\} < S^+$ for every S^+, $\{\emptyset\} > R^-$ for every

R^-, and $R^- < S^+$ for every S^+ and R^-. Next, we define negation operation

$$\begin{aligned}
&\neg_T\{\emptyset\} = \{\emptyset\}\\
&\neg_T S^+ = \left(S \cup \{\uparrow^G s \mid \text{ for all } s \in S^+\}\right)^-\\
&\neg_T S^- = S^+
\end{aligned}$$

Note that, by the above definition $\neg_T T^+ = T^-$ and $\neg_T T^- = T^+$. Hence, we define $\top_T = T^+$ and $\bot_T = T^-$. Finally, we define the monoidal operation and its residuum,

$$S^s \odot_T R^r = \begin{cases} S^s \cap R^r & \text{if } s = r = + \\ \bot_T \setminus \uparrow (\neg_T S \setminus R) & \text{if } s = + \text{ and } r = -, \\ \bot_T \setminus \uparrow (\neg_T R \setminus S) & \text{if } s = - \text{ and } r = +, \\ \bot_T & \text{if } s = -, r = -. \end{cases} \tag{4.12}$$

$$S^s \to_T R^r = \begin{cases} T^+ \setminus \uparrow (S^s \setminus R^r) & \text{if } s = r = +, \\ T^+ \setminus \uparrow (R^r \setminus S^s) & \text{if } s = r = -, \\ T^+ & \text{if } s = - \text{ and } r = +, \\ (\neg_T S \cap R)^- & \text{if } s = + \text{ and } r = -. \end{cases} \tag{4.13}$$

Clearly $(A_T, \wedge_T, \vee_T, \bot_T, \top_T)$ is a bounded lattice where $\wedge_T$ and $\vee_T$ are derived from the partial order $\leq$. By direct inspection on (4.12), $(A_T, \odot, \top_T)$ is a monoid. See Figure 4.6 as an example of the above construction.

To show that the structure $\mathbf{A}_T = (A_T, \odot_T, \to_T, \wedge_T, \vee_T, \bot_T, \top_T)$ is an NMG algebra we have to prove that $(\odot_T, \to_T)$ is a residuated pair and $\mathbf{A}_T$ satisfies (NMG), prelinearity and weak nilpotent minimum equations. By residuation property, $S^s \to_T R^r = max\{Z^z \mid S^s \odot_T Z^z \leq R^r\}$. Let $s = r = +$ and by contradiction suppose there exists $Z^z \geq T^+ \setminus (S^s \setminus R^r)$ such that $S^s \odot_T Z^z \leq R^r$. By definition of $\leq$, we have $Z^z \supseteq T^+ \setminus (S^s \setminus R^r)$. Then, there exists an $y \in Z^z$ such that $y \notin T^+ \setminus \uparrow (S^s \setminus R^r)$. It follows that $y \in S^s \setminus R^r$ or $y \in (\uparrow (S^s \setminus R^r) \setminus (S^s \setminus R^r))$. If $y \in S^s \setminus R^r$ then $y \in S^s$ and y belongs to $S^s \cap Z^z \subseteq R^r$, that is $y \in R^r$. Then, $y \notin (S^s \setminus R^r)$ in contradiction with the hypothesis. If $y \in (\uparrow (S^s \setminus R^r) \setminus (S^s \setminus R^r))$ then there exists an $x \leq y$ such that $x \in (S^s \setminus R^r)$ and $x \notin R^r$. Since Z^z is a lower set of T, then $x \in Z^z$. Hence, $x \in S^s \cap Z^z \subseteq R^r$, contradiction.

An analogous reasoning settles the case $s = r = -$.

Let $s = -$ and $r = +$. Then, $S^- \odot Z^z$ is equal to some $K^- = \bot_T \setminus \uparrow (\neg_T S \setminus Z)$ for every Z^z. Trivially, T^+ is the greatest subtree Z^z in $SubT$ such that $S^- \odot_T Z^z \leq R^+$.

Let $s = +$ and $r = -$. Then, $S^+ \to R^- = (\neg_T S \cap R)^-$. Suppose that there exists a $Z^z \geq (\neg_T S \cap R)^-$ such that $S^+ \odot Z^z \leq R^-$. Then, $z = -$ and $Z^z \subseteq (\neg_T S \cap R)^-$, by definition of $\leq$. Hence, there exists an element $y \notin Z^-$ such that $y \in (\neg_T S \cap R)^-$. It follows that $y \in \neg_T S$, $y \in R^-$ and by definition

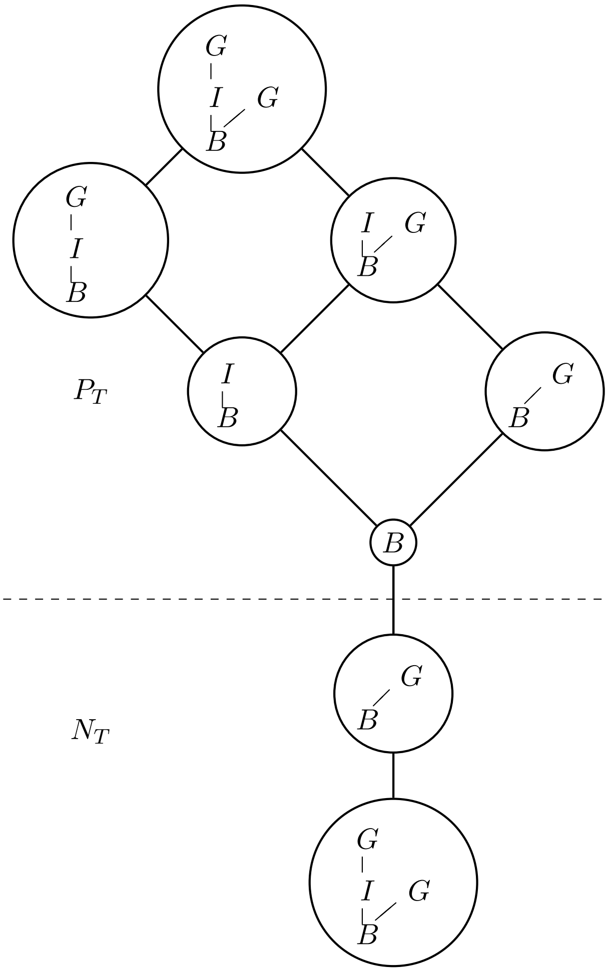


Figure 4.6: The lattice A_T built upon subtrees of the IG-tree T of Figure 4.5.

of $\neg_T$, y belongs also to S. By definition, $S^+ \odot Z^- = \bot_T \setminus \uparrow (\neg_T S \setminus Z)$. Since $y \notin Z$ and $y \in \neg_T S$, then $y \notin S^+ \odot Z^- \subseteq R^-$. In contradiction with $y \in R^-$.

To prove that $\mathbf{A}_T$ satisfies (NMG), we rewrite the equation with subtrees,

$$(\neg_T \neg_T S^s \to S^s) \vee ((S^s \wedge R^r) \to (S^s \odot R^r)) = T^+. \tag{4.14}$$

First, notice that if $s = -$ then $\neg_T \neg_T S^- = S^-$ and by (4.13) we have $(\neg_T \neg_T S^s \to S^s) = T^+$. Moreover, when $r = s = +$ we have $(S^+ \odot R^+) = (S^+ \wedge R^+)$, and then $(S^+ \wedge R^+) \to (S^+ \odot R^+) = T^+$. The last case is $s = +$ and $r = -$. We use (4.12) and (4.13) to rewrite (4.14) in the following form,

$$(T^+ \setminus \uparrow (\neg_T \neg_T S^+ \setminus S^+)) \vee (T^+ \setminus \uparrow ((\bot_T \setminus \uparrow (\neg_T S^+ \setminus R^-)) \setminus R^-)) = T^+. \tag{4.15}$$

Let $A = (T^+ \setminus \uparrow (\neg_T \neg_T S^+ \setminus S^+))$ and $B = (T^+ \setminus \uparrow ((\bot_T \setminus \uparrow (\neg_T S^+ \setminus R^-)) \setminus$

$R^-)$). First notice that when $\neg_T\neg_T S^+ \setminus S^+ = \emptyset$ then $A = T^+$. Thus, we consider the case where $\neg_T\neg_T S^+ \setminus S^+ \neq \emptyset$. It is sufficient to show that if $x \notin A$ then $x \in B$, and viceversa. Let x be an element of T such that $x \notin A$. Then, $x \in \uparrow (\neg_T\neg_T S^+ \setminus S^+)$. Hence, we have either $x \in \neg_T\neg_T S^+$ or $x \in \uparrow (\neg_T\neg_T S^+ \setminus S^+) \setminus \neg_T\neg_T S^+$. By definition of $\neg_T$, we have either $x \in \neg_T S^+$ or $x \in \uparrow (\neg_T S^+ \setminus S^+) \setminus \neg_T S^+$. In both cases, we conclude $x \in B$. Notice that if $x \in R$ then $x \notin \uparrow ((\bot_T \setminus \uparrow (\neg_T S \setminus R)) \setminus R)$ and hence $x \in B$.

Let y be an element of T such that $y \notin B$. Then, $y \in \uparrow ((\bot_T \setminus \uparrow (\neg_T S \setminus R)) \setminus R)$. It follows that $y \notin R$ and either $y \notin \neg_T S^+$ or $y \notin \uparrow (\neg_T S^+ \setminus R^-) \setminus \neg_T S^+$. By definition of $\neg_T$, either $y \notin \neg_T\neg_T S^+$ or $y \notin \uparrow (\neg_T\neg_T S^+ \setminus R) \setminus \neg_T\neg_T S^+$. For the second case note that since $S^+ \subset \neg_T\neg_T S^+$ then $y \notin \uparrow (\neg_T\neg_T S^+ \setminus S^+) \setminus \neg_T\neg_T S^+$. Hence, we conclude $y \in A$.

It is an easy check to show that prelinearity holds in $\mathbf{A}$. Also weak nilpotent minimum equation it is easy settled when $s = r$. Let $s = +$ and $r = -$, we rewrite weak nilpotent minimum equation using (4.12) and (4.13)

$$\neg_T(\bot_T \setminus \uparrow (\neg_T S^+ \setminus R^-)) \vee (T^+ \setminus \uparrow ((\bot_T \setminus \uparrow (\neg_T S^+ \setminus R^-)) \setminus R^-)) = T^+.$$

We adopt the same strategy as above. Let $A = \neg_T(\bot_T \setminus \uparrow (\neg_T S^+ \setminus R^-))$ and $B = \uparrow ((\bot_T \setminus \uparrow (\neg_T S^+ \setminus R^-)) \setminus R^-)$. Let $y \notin T^+ \setminus B$. Then, $y \in B$. It follows that $y \notin R^-$ and then $y \in A$. Let $x \notin A$. Then, $x \in \uparrow (\neg_T S^+ \setminus R^-)$ and $x \notin R^-$. It follows that $x \notin B$ and then $x \in T^+ \setminus B$.

By construction, $\Theta(\mathbf{A}_T)$ is isomorphic to T in IGT. Hence, for every object T in IGT, we have shown that the algebra $\mathbf{A}_T$ is an NMG algebra such that $\Theta(\mathbf{A}_T)$ is isomorphic to T in IGT. It follows that,

Claim 4.1.1. *The functor Θ is essentially surjective.*

Theorem 4.1.1. *The categories DNMG and IGT are dually equivalent.*

Proof. We have already shown that Θ is essentially surjective. By [41, Theorem 4.4.1], it is sufficient to show that $\Theta\colon \mathsf{DNMG} \to \mathsf{IGT}$ is full and faithful.

Claim 4.1.2. *The functor Θ is full.*

Proof. We have to show that for every morphisms $f : T' \to T$ in IGT there exists a morphism h in DNMG such that $\Theta(h) = f$. Since Θ is essentially surjective, there exist $\mathbf{A}, \mathbf{A}' \in \mathsf{DNMG}$ such that $T = \Theta(\mathbf{A})$ and $T' = \Theta(\mathbf{A}')$, and $h : \mathbf{A} \to \mathbf{A}'$. Since $\mathbf{A}$ is finite, every positive element $a \in \mathbf{A}$ is the join of join-irreducible positive elements a_j in $\mathbf{A}$, that is $a = \bigvee_{a_j \leq a} a_j$. Then, every a_j generates a prime filter F_{a_j} in T. We set $X = \{x \mid x \text{ generates } F \in f^{-1}(F_{a_j})\}$. Then, $h(a) = \bigvee X$. In this way we have defined the positive part of h, using (4.9) we can extend h to the negative elements. By the Definition 4.1.1 h is an NMG algebra homomorphism. $\square$

Claim 4.1.3. *The functor Θ is faithful.*

Proof. The functor Θ is faithful when for every pair $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{B}$ of morphisms in DNMG, if $\Theta(f) = \Theta(g)$ then $f = g$. Let f and g be two distinct homomorphisms in DNMG with the same domain and codomain, then there is at least an element $a \in \mathbf{A}$ such that $f(a) \neq g(a)$. As above, we use join-irreducible representation of every element $a \in \mathbf{A}$, that is $a = \bigvee_{a_j \leq a} a_j$. Hence, $f(\bigvee_{a_j \leq a} a_j) \neq g(\bigvee_{a_j \leq a} a_j)$ and $\bigvee_{a_j \leq a} f(a_j) \neq \bigvee_{a_j \leq a} g(a_j)$ since f and g are NMG algebras homomorphisms. Then, there exists at least a join-irreducible element $a_k \in \mathbf{A}$ such that $f(a_k) \neq g(a_k)$. If $f(a_k)$ and $g(a_k)$ are not comparable, then we are done. Otherwise, we assume without loss of generality that $f(a_k) > g(a_k)$. Take the prime filter F in $\mathbf{B}$ generated by $f(a_k)$. Hence, $g(a_k) \notin F$ and $\Theta(f)(F) \neq \Theta(g)(F)$. By (4.11), it follows that the maps $\Theta(f)$ and $\Theta(g)$ are distinct. □

□

Theorem 4.1.2. *The categories* FNMG *and* IGF *are dually equivalent.*

Proof. By universal algebraic facts [13, Theorem 7.10], every finite non-trivial NMG algebra is isomorphic to the direct product of a finite family of non-trivial directly indecomposable finite NMG algebras, and this direct decomposition is unique up to isomorphism. Moreover, coproducts in IGF are disjoint unions. Hence, each IG-forest is a finite coproduct of IG-trees. The fact that Θ is full, faithful, and essentially surjective is a routine verification using Theorem 4.1.1. □

For every T in IGT, we denote with $\mathbf{Sub}T$ the structure $\langle A_T, \odot_T, \to_T, \perp_{\mathbf{A}_T}, \top_{\mathbf{A}_T}\rangle$. By Claim 4.1.1, $\mathbf{Sub}T$ is a finite directly indecomposable NMG algebra. Since every finite NMG algebra is isomorphic to a finite product of finite directly indecomposable NMG algebras, and every IG-forest F is a disjoint union of IG-trees T_i, we can use the structures $\mathbf{Sub}T_i$ to define a contravariant functor from IGF to FNMG. Indeed, let $F = \bigsqcup_{i\in I} T_i$ be an IG-forest. As shown in Claim 4.1.1, every $\mathbf{Sub}T_i$ is a directly indecomposable NMG algebra. Hence, the direct product

$$\mathbf{Sub}F = \prod_{i\in I} \mathbf{Sub}T_i, \tag{4.16}$$

naturally carries an algebraic structure of finite NMG algebra.

To turn $\mathbf{Sub}$ into a functor from IGF to FNMG, we have to define its behavior on morphisms. Let $f : T' \to T$ be an IG-tree morphism. We define $\mathbf{Sub}f : \mathbf{Sub}T \to \mathbf{Sub}T'$ by,

$$S \in \mathbf{Sub}T \mapsto f^{-1}(S) \in \mathbf{Sub}T'.$$

By construction,

$$F \cong \Theta(\mathbf{Sub}F).$$

Example 4.1.4. *Let T and T' be the two IG-trees depicted in Figure 4.7. Take the IG-forest $F = T \sqcup T'$. By* (4.16)*, we can obtain the finite NMG algebra* $\mathbf{Sub}F$*, multiplying the two directly indecomposable NMG algebras* $\mathbf{Sub}T$ *and* $\mathbf{Sub}T'$*. Compare in Figure 4.7 the order structures of* $\mathbf{Sub}T$*,* $\mathbf{Sub}T'$ *and* $\mathbf{Sub}F = \mathbf{Sub}T \times \mathbf{Sub}T'$*.*

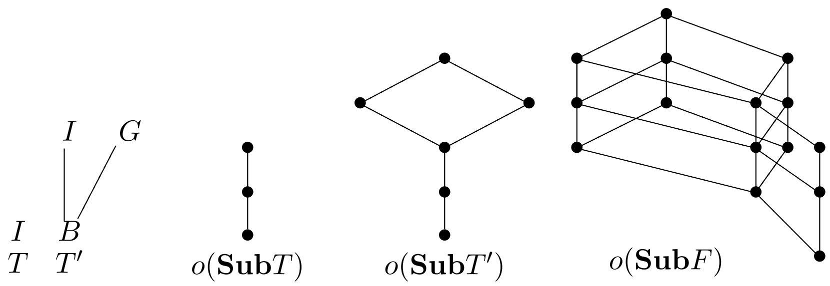


Figure 4.7: Two IG-trees T and T, the order structures $o(\mathbf{Sub}T)$ and $o(\mathbf{Sub}T')$, of their corresponding algebras of subtrees, and the order structure of $\mathbf{Sub}F$ (see Example 4.1.4).

As a consequence we obtain the following representation theorem for finite NMG algebras.

Corollary 4.1.1. *Any finite NMG algebra is isomorphic to* $\mathbf{Sub}F$*, where F is an IG-forest unique up to isomorphisms.*

In [5], authors give a duality between finite NM algebras and a category of forests where to each tree is added a bit of information. They obtain also coproducts of finite NM algebras, amalgamation properties and representation of free finitely generated NM Algebras. Using Corollary 4.1.1, it is easy to obtain a representation theorem for the whole class of finite NM algebras.

Let $\mathbf{A}$ be a finite NMG algebra such that every element is involutive. Obviously, $\mathbf{A}$ is a NM algebra. By (4.10), $\Theta(\mathbf{A})$ is an IG-forest where every node of each tree T in $\Theta(\mathbf{A})$ is labelled with I or B. Hence, we can state:

Corollary 4.1.2. *Any finite NM algebra is isomorphic to* $\mathbf{Sub}F$*, where F is an IG-forest unique up to isomorphisms, where each tree T in F is such that $\lambda(t) = \{B, I\}$ for every t in T.*

The same reasoning as above can be done about Gödel algebras. Let $\mathbf{A}$ be a finite NMG algebra where $x \odot x = x$ for every x in $\mathbf{A}$. That is, $\mathbf{A}$ is a Gödel algebra. By (4.10), $\Theta(\mathbf{A})$ is an IG-forest where every node of each tree T in $\Theta(\mathbf{A})$ is labelled with G or B. Hence, we can state:

Corollary 4.1.3. *Any finite Gödel algebra is isomorphic to* $\mathbf{Sub}F$*, where F is an IG-forest unique up to isomorphisms, where each tree T in F is such that $\lambda(t) = \{B, G\}$ for every t in T.*

4.2 Coproducts of NMG Algebras

In the final part of the previous section we have given a representation for finite NMG algebras, combining products of algebras with our duality. In this section we use the products in the category of IG-forests to recover coproducts of NMG algebras. This construction will be useful for obtaining a representation of finitely generated free NMG algebras.

Given two labels $a, b \in \{B, I, G\}$ we define their product $a \cdot b$ as:

$$a \cdot b = \begin{cases} B & \text{if } a = b = B \\ I & \text{if } a \in \{B, I\}, b = I, \text{ or } b \in \{B, I\}, a = I; \\ G & \text{otherwise.} \end{cases} \tag{4.17}$$

Let $(F, \leq)$ and $(F', \leq')$ be two labelled forests. It is easy to see that their coproduct $F + F'$ is the labelled forest $(F'', \leq')$ where F'' is the disjoint set union of F and F' and $x \leq'' y$ if and only if $x, y \in F$ and $x \leq y$ or $x, y \in F'$ and $x \leq' y$.

Let r be the root of a tree T. If $\lambda(r) = t$ then we write $(T)_t$. When $|T| = 1$ we will write $(\mathbf{1})_t$. We say that $(T)_t$ is determined by a family $\{(T_i)_{t_i}\}_{i=1}^{m}$ if:

$$(T)_t := \left(\sum_{i=1}^{n} (T_i)_{t_i} \right)_t$$

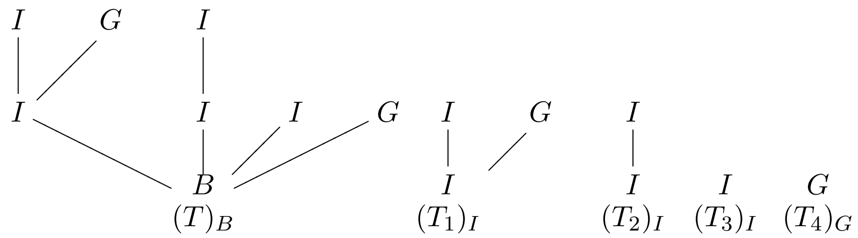


Figure 4.8: An IG-tree $(T)_t$ and the family $\{(T_i)_{t_i}\}_{i=1}^{4}$ of IG-trees that determines $(T)_t$.

Given an IG-tree $(T)_t$ we define

$$(T)_t^G = \begin{cases} (\mathbf{1})_G & \text{if } t = I; \\ (T)_t \setminus \uparrow x \text{ for every } x \text{ such that } \lambda(x) = I & \text{otherwise;} \end{cases} \tag{4.18}$$

Let A and B be two IG-trees, each one composed respectively by trees $\{(A_i)_{a_i}\}_{i=1}^{n}$ and $\{(B_j)_{b_j}\}_{j=1}^{r}$. Their product is inductively defined by the following rules:

(R1) if $a, b \in \{I, G\}$ and $a \neq b$ then $(A)_a \times (B)_b = \emptyset$;

(R2) otherwise

$$
\begin{aligned}
(A)_a \times (B)_b = \Bigg(& \left(\sum_{j, b_j = G} (A)_a^G \times (B_j)_{b_j} \right) + \left(\sum_{j, b_j = I} (A)_a \times (B_j)_{b_j} \right) + \\
& + \left(\sum_{a_i = b_j} (A_i)_{a_i} \times (B_j)_{b_j} \right) + \\
& + \left(\sum_{i, a_i = G} (A_i)_{a_i} \times (B)_b^G \right) + \left(\sum_{i, a_i = I} (A_i)_{a_i} \times (B)_b \right) \Bigg)_{a \cdot b}
\end{aligned}
$$

Moreover, we next define the projections $\pi_A : (A)_a \times (B)_b \to (A)_a$ and $\pi_B : (A)_a \times (B)_b \to (B)_b$ of $(A)_a$ and $(B)_b$ respectively. Let r_0 and s_0 be the roots of $(A)_a$ and $(B)_b$ respectively, and t_0 be the root of $(A)_a \times (B)_b$ labelled with $a \cdot b$. Then, $\pi_A(t_0) = r_0$ and $\pi_B(t_0) = s_0$. Each $x \in ((A)_a \times (B)_b) \setminus \{t_0\}$ has to belongs to one of the trees defined by the five different summands in (R2). Writing $(B_0)_{b_0}$ for $(B)_b$, for every $1 \leq i \leq n$ and for every $0 \leq j \leq r$ we define $\pi_A(x) = \iota_{A_i}(\pi_{A_i}(x))$, where $\pi_{A_i} : (A_i)_{a_i} \times (B_j)_{b_j} \to (A_i)_{a_i}$ is the projection function, and $\iota_{A_i} : (A_i)_{a_i} \to (A)_a$ is the set-theoretic inclusion of $(A_i)_{a_i}$ into $(A)_a$. Note that for x in $(A_i)_{a_i} \times (B)_b^G$, we have $a_i = G$ and by (4.18) and (4.17) the label of the root of each tree $(A_i)_{a_i} \times (B)_b^G$ is equal to G. The projection $\pi_B(x) = \iota_{B_j}(\pi_{B_j}(x))$ is defined in a similar way.

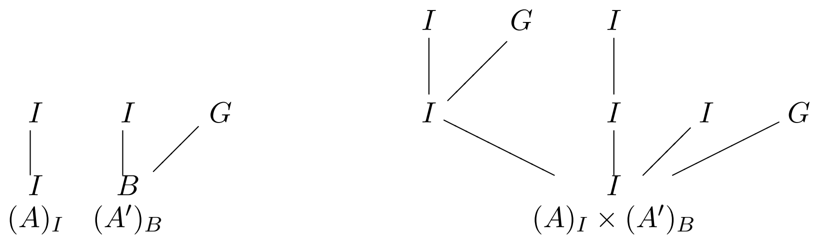


Figure 4.9: Two IG-trees $(A)_I$ and $(A')_B$ and their product.

By (R2) each element x in $(A)_a$ and each element y in $(B)_b$ are involved in the construction of $(A)_a \times (B)_b$. Hence, for each $x \in (A)_a$ and each $y \in (B)_b$ there exists z in $(A)_a \times (B)_b$ such that $\pi_A(z) = x$ and $\pi_B(z) = y$. That is, π_A and π_B are surjective maps.

Remark 4.2.1. *Consider Figure 4.9. Let t be the node of $(A)_I \times (A')_B$ that covers its root and such that $\lambda(t) = G$. Notice that t is obtained by the first summand in (R2). Explicitly, $t = (A)_I \times t'$ where t' is the unique node of A' such that $\lambda(t') = G$. The projection $\pi_{A'}$ and π_A are such that $\pi_{A'}(t) = t'$, and $\pi_A(t) = r$ where r is the root of A. By Definition 4.1.1, π_A is a well defined morphism in* IGT.

In the following theorem we show that the above definition of product of IG-trees satisfies the universal property of products.

Theorem 4.2.1. *Let $(A)_a$, $(B)_b$ and $(T)_t$ be IG-trees, such that there exist $h_A : (T)_t \to (A)_a$ and $h_B : (T)_t \to (B)_b$ Then, there exists a unique IG-tree morphism f that makes the following diagram commutes.*

$$\begin{tikzcd} & (T)_t \arrow[dl, "h_A"'] \arrow[d, dashed, "f"] \arrow[dr, "h_B"] & \\ (A)_a & (A)_a \times (B)_b \arrow[l, "\pi_A"] \arrow[r, "\pi_B"'] & (B)_b \end{tikzcd}$$

Proof. We will construct a map $f : (T)_t \to (A)_a \times (B)_b$ such that $\pi_A \circ f = h_A$ and $\pi_B \circ f = h_B$. First, we partition the tree $(T)_t$ as follows. Let $\bar{A}_0 = h_A^{-1}(a)$ and $\bar{A}_1 = (T)_t \backslash \bar{A}_0$. Analogously, let $\bar{B}_0 = h_B^{-1}(b)$ and $\bar{B}_1 = (T)_t \backslash \bar{B}_0$. Then,

$$\{\bar{A}_0 \cap \bar{B}_0, \bar{A}_0 \cap \bar{B}_1, \bar{A}_1 \cap \bar{B}_0, \bar{A}_1 \cap \bar{B}_1\}$$

is a partition of $(T)_t$. We refine the block $\bar{A}_1 \cap \bar{B}_1$. Let $\bar{A}_2$ be the set of all $x \in \bar{A}_1 \cap \bar{B}_1$ such that there is $y < x$ in $(T)_t$ with $h_A(y) = a$ and $h_B(y) \neq b$. Let $\bar{B}_2$ be the set of all $x \in \bar{A}_1 \cap \bar{B}_1$ such that there is $y < x$ in $(T)_t$ with $h_A(y) \neq a$ and $h_B(y) = b$. Finally, let $V_2 = (\bar{A}_1 \cap \bar{B}_1) \setminus (\bar{A}_2 \cup \bar{B}_2)$. Then,

$$\{\bar{A}_0 \cap \bar{B}_0, \bar{A}_0 \cap \bar{B}_1, \bar{A}_1 \cap \bar{B}_0, \bar{A}_2, \bar{B}_2, V_2\}$$

is a partition of $(T)_t$.

For each $x \in \bar{A}_0 \cap \bar{B}_0$ we let $f(x) = a \cdot b$. Notice that for each $x \in \bar{A}_0 \cap \bar{B}_1$ there is a unique $(B_j)_{b_j}$ such that $h_B(x) \in (B_j)_{b_J}$. Then, we let $f(x)$ be the uniquely determined element z of $(A)_a \times (B_j)_{b_j}$ (or $(A)_a^G \times (B_j)_{b_j}$) such that $\pi_A(z) = h_A(x) = a$ and $\pi_{B_j}(z) = h_B(x)$. Note that the label of b_j has no importance, since the existence of z is given by the fact that h_A, h_B, π_A and π_{B_j} are morphisms in IGT. The case where x is in $\bar{A}_1 \cap \bar{B}_0$ is analogous. For each $x \in \bar{A}_2$ we note that there are uniquely determined trees $(A_i)_{a_i}$ and $(B_j)_{b_j}$ such that $h_A(x) \in (A_i)_{a_i}$ and $h_B = (B_j)_{b_j}$. By construction, there is $y < x$ in $(T)_t$ such that $h_A(y) = a$ and $h_B(y) \neq b$. Morphisms of IG-trees must carry lower sets to lower sets, then $f(x)$ must belong to $(A)_a \times (B_j)_{b_j}$ (or $(A)_a^G \times (B_j)_{b_j}$) and reasoning as above we let $f(x)$ be the uniquely determined element z of $(A)_a \times (B_j)_{b_j}$ (or $(A)_a^G \times (B_j)_{b_j}$) such that $\pi_A(z) = h_A(x)$ and $\pi_{B_j}(z) = h_B(x)$. The case where x is in $\bar{B}_2$ is analogous. The last case occurs when $x \in V_2$. We let $f(x)$ be the uniquely determined element z of $(A_i)_{a_i} \times (B_j)_{b_j}$ such that $\pi_{A_i}(z) = h_A(x)$ and $\pi_{B_j}(z) = h_B(x)$. It is routine to verify that $\pi_A \circ f = h_A$ and $\pi_B \circ f = h_B$.

Finally, we have to show that f is unique. Let $f' : (T)_t \to (A)_a \times (B)_b$ be an IG-tree morphism such that $\pi_A \circ f' = h_A$ and $\pi_B \circ f' = h_B$. For every x in $\bar{A}_0 \cap \bar{B}_0$ it is clear that $f(x) = f'(x)$. If $x \in \bar{A}_0 \cap \bar{B}_1$ then

$h_A(x) = a$ and $h_B(x) \neq b$. Hence $f'(x)$ must belong to $(A)_a \times (B_j)_{b_j}$ for a uniquely determined $(B_j)_{b_j}$. It follows that $f'(x) = f(x)$, otherwise $(A)_a \times (B_j)_{b_j}$ would not be the product of $(A)_a$ and $(B_j)_{b_j}$ in IGT. The cases where x is in $\bar{A}_1 \cap \bar{B}_0$ is the same as above. If $x \in \bar{A}_1 \cap \bar{B}_1$ then there are uniquely determined $(A_i)_{a_i}$ and $(B_j)_{b_j}$, with $i \neq 0 \neq j$ and such that $\pi_{A_i}(f'(x)) = h_A(x)$ and $\pi_{B_j}(f'(x)) = h_B(x)$. If $x \in \bar{A}_2 \subseteq \bar{A}_1 \cap \bar{B}_1$ then there is $y < x$ in $(T)_t$ such that $\pi_A(f'(x)) = a$ and $\pi_B(f'(x)) \neq b$, that is y belongs to $\bar{A}_0 \cap \bar{B}_1$ and hence $f'(y) \in (A)_a \times (B_j)_{b_j}$. Since f' must be an order-preserving and open map, x belongs to the isomorphic copy of $(A_i)_{a_i}$ included in $(A)_a$. It follows that $f'(x) = f(x)$, otherwise $(A)_a \times (B_j)_{b_j}$ would not be the product of $(A)_a$ and $(B_j)_{b_j}$ in IGT. Similar arguments holds when $x \in \bar{B}_2$ and $x \in V_2$. We conclude that $f' = f$. $\square$

This proof is an adaptation to NMG algebras of the argument given in [3] for coproducts of Gödel algebras (see also [18] for more on this subject). Indeed, consider the rule (R2) of the above defined product and take $(A)_a$ and $(B)_b$ as IG-trees where every node is labelled only with G or B. It is easy to see that (R2) reducts to the rule (P3) in Section 1.2.1. Then, we obtain exactly the product of trees as stated in Lemma 1.2.1.

An analogous reasoning can be done if we restrict our attention to IG-trees whose nodes are labelled only with I or B. In this case we are dealing with IG-trees whose primal objects are directly indecomposable NMG algebras satisfying involutivity, that is directly indecomposable NM algebras. Hence, we obtain a definition of products that can be safely used to recover coproducts of finite NM algebras (see [5]).

4.3 Amalgamation Property

Let $\mathbf{A}$ and $\mathbf{B}$ be two algebras in FNMG. We say that FNMG has the *amalgamation property* if and only if for any monomorphisms $i_1 : \mathbf{C} \to \mathbf{A}$ and $i_2 : \mathbf{C} \to \mathbf{B}$ there exists a finite NMG algebra $\mathbf{S}$ with monomorphisms $f_1 : \mathbf{A} \to \mathbf{S}$ and $f_2 : \mathbf{B} \to \mathbf{S}$ such that the following diagram commutes

$$\begin{tikzcd} \mathbf{C} \arrow[r, "i_1"] \arrow[d, "i_2"] & \mathbf{A} \arrow[d, "f_1"] \\ \mathbf{B} \arrow[r, "f_2"] & \mathbf{S} \end{tikzcd}$$

Since NMG algebras form a variety, FNMG has finite colimits, in particular fibred coproducts. We recall that a fibred coproduct is a pushout square,

$$\begin{tikzcd} \mathbf{C} \arrow[r, "m_1"] \arrow[d, "m_2"] & \mathbf{A} \arrow[d, "f"] \\ \mathbf{B} \arrow[r, "g"] & \mathbf{A} +_{\mathbf{C}} \mathbf{B} \end{tikzcd}$$

When all the maps in the pushout square are monic, that is monomorphisms of finite NMG algebras, we obtain a *free product of* $\mathbf{A}$ *and* $\mathbf{B}$ *with amalgamated subobject* $\mathbf{C}$. Dually, we have to consider fibred products of IG-forests.

Now we restrict our attention to the category of IG-trees, appealing at the proof of Theorem 4.1.2 extension of the following results to IGF is straightforward. Then, a fibred product in IGT is just a pullback square,

$$\begin{tikzcd}
(A)_a \times_C (B)_b \arrow[r, "\pi_A^C"] \arrow[d, "\pi_B^C"] & (A)_a \arrow[d, "s_A"] \\
(B)_b \arrow[r, "s_B"] & (C)_c
\end{tikzcd}$$

We have to show that if s_A and s_B are epimorphisms then π_A^C and π_B^C are epimorphisms too.

It is well known that every fibred product has an associated equaliser. Hence, consider the following diagram,

$$\begin{tikzcd}
(A)_a \times_C (B)_b \arrow[rr, "\pi_A^C"] \arrow[dd, "\pi_B^C"'] \arrow[dr, "eq"'] & & (A)_a \arrow[dd, "s_A"] \\
& (A)_a \times (B)_b \arrow[ur, "\pi_A"] \arrow[dl, "\pi_B"] & \\
(B)_b \arrow[rr, "s_B"] & & (C)_c
\end{tikzcd}$$

where π_A and π_B are the projections of the product $(A)_a \times (B)_b$, and $eq : (A)_a \times_C (B) \to (A)_a \times (B)$ is the equaliser of $s_A \circ \pi_A$ and $s_B \circ \pi_B$.

Let $c' \in (C)_c$, since s_A and s_B are epimorphisms there exists $a' \in (A)_a$ and $b' \in (B)_b$ such that $s_A(a') = c'$ and $s_B(b') = c'$. Since π_A and π_B are projection maps, there exists an element z in $(A)_a \times (B)_b$ such that $\pi_A(z) = a'$ and $\pi_B(z) = b'$. Moreover, eq is an equalizer, then it is a monomorphism, that is an injective map. It follows that there exists $y = eq^{-1}(z)$ in $(A)_a \times_C (B)_b$ such that $\pi_A^C(y) = (\pi_A \circ eq)(y) = a'$ and $\pi_B^C(y) = (\pi_B \circ eq)(y) = b'$.

By the above discussion, we conclude that

Corollary 4.3.1. DNMG *has free products with amalgamation.*

4.4 Free Finitely Generated NMG Algebras

In this section we show how to recover the spectra of free finitely generated NMG algebras. For a combinatorial description of free n-generated NMG algebras we refer the reader to [6].

The free 1-generated NMG algebra $\mathbf{F}_1$ is finite. Hence, $\mathbf{F}_1$ is isomorphic to a subdirect product of a finite number of singly generated NMG chains. By direct computation over (4.1), the six ways to generate a NMG chain with a single element are depicted in Figure 4.10 (compare with the nine singly generated WNM chains depicted in Figure 2.1).

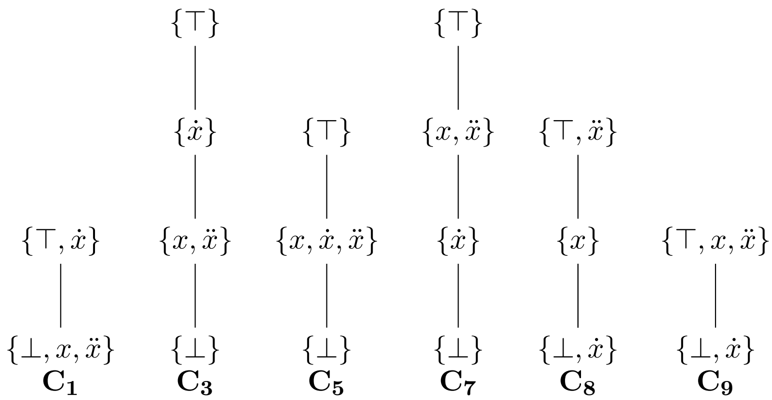


Figure 4.10: The six 1-generated NMG chains, where x is the generator.

Notice that every singly generated NMG chain belongs to $\mathcal{C}_1$, the set of 1-generated WNM chains characterized in Section 2.1.

Authors in [6] have shown that $\mathbf{F}_1$ has 72 elements. Instead of studying $\mathbf{F}_1$ and applying the functor Θ to this huge structure, we build $\Theta(\mathbf{F}_1)$ working in the opposite direction. As explained in Section 2.2 for WNM chains, every chain $\mathbf{C}_i$ with $i \in \{1, 3, 5, 7, 8, 9\}$ corresponds to a prime filter F_{θ_i} of $\mathbf{F}_1$. Hence, we can recover the prime spectrum $\Theta(\mathbf{F}_1)$ simply analyzing the chains $\mathbf{C}_i$ in Figure 4.10 and their generating congruences θ_i, with $i \in \{1, 3, 5, 7, 8, 9\}$. Moreover, following the constructions in Section 4.1, the label of F_{θ_i} is given by the element in $\mathbf{C}_i$ covered by the equivalence class of $\top$. As an example, $\Lambda(F_{\theta_2}) = I$ since $\{\neg x\}$ is involutive. See Figure 4.11 for the prime spectrum of $\mathbf{F}_1$. Comparing the order structure of $\Theta(\mathbf{F}_1)$ with the prime spectrum of the free 1-generated WNM algebra depicted in Figure 2.3, it is clear that the former is a subforest of the latter.

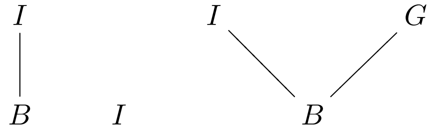


Figure 4.11: The prime spectrum of the free 1-generated NMG algebra.

The free n-generated NMG algebra $\mathbf{F}_n$ is the coproduct of n copies of the free 1-generated NMG algebra. Dually, we can describe the prime spectrum of $\mathbf{F}_n$ with

$$\Theta(\mathbf{F}_n) = \prod^{n} \Theta(\mathbf{F}_1).$$

As an example see Figure 4.12

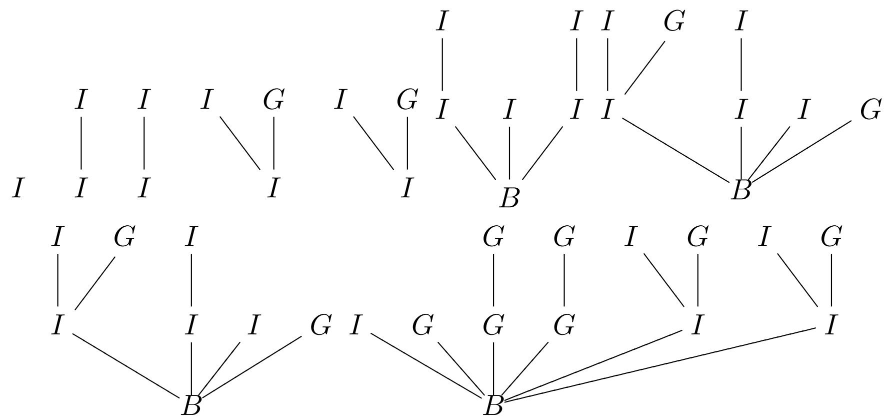


Figure 4.12: $\Theta(\mathbf{F}_2) = \Theta(\mathbf{F}_1) \times \Theta(\mathbf{F}_1)$.

Normal Forms and Logical Properties

In the previous chapter we have defined combinatorial categories useful to obtain explicit description of coproduct and representations of algebras in the primal side. Moreover, we have settled amalgamation property for NMG algebras. Amalgamation is related to deductive interpolation property of the corresponding logic. In Chapter 6 we use concepts given in Section 1.3 to solve constructively some logical properties related to RDP logic (including interpolation), characterizing the corresponding free algebras as algebras of antichains over suitable defined posets. We will see how to derive this representation using two different strategies, one from the duality presented in Section 3, and another studying finite RDP chains. As a preliminary of the latter technique we introduce in Chapter 5 an analogous representation for free Gödel algebras.

Chapter 5

Gödel Logic Normal Forms

In this section we recall the combinatorial characterization of free n-generated Gödel algebras found in [6].

As mentioned in Section 1.1.2, Gödel algebras are WNM algebras where $x \odot x = x$ holds. By Proposition 2.3.3, every element $x \neq \bot$ of a Gödel algebra $\mathbf{A}$ is positive, that is $x > \neg x$. Then, in each Gödel chain represented as an ordered partitions of $\mathcal{F}_n$ (2.5), the block to which $\neg x_i$ belongs is determined by the block that contains x_i, negation of elements do not brings any information. Hence we can eliminate the elements $\neg x_i$ in every ordered partition that represents a Gödel chain. Throughout this section, by ordered partition we mean an ordered partition of the set $\{\bot, x_1, ..., x_n, \top\}$. Recall that $\mathcal{K}_n^G$ is the set of n-generated non-redundant Gödel chains (see Section 2.1).

Lemma 5.1 ([6]). *A Gödel chain* $\mathbf{C}$ *represented as an ordered partition* $\{B_0 < ... < B_k\}$*, belongs to* $\mathcal{K}_n^G$ *if and only if* $B_k = \{\top\}$.

Remark 5.1. *There are exactly three ways to* 1*-generate a Gödel chain, that is* $\mathbf{C}_1$*,* $\mathbf{C}_8$ *and* $\mathbf{C}_9$ *in Figure 2.1. Non-redundant singly generated Gödel chains are* $\mathcal{K}_1^G = \{\mathbf{C}_1, \mathbf{C}_8\}$.

Lemma 5.2 ([6]). *For any formula* $\varphi(x_1, ..., x_n)$*,*$(\varphi^C)_{C \in \mathcal{K}_n^G}$ *is an element of* $\prod_{C \in \mathcal{K}_n^G} C$ *that satisfies the* prefix property*:*

> *let* C, C' *be chains in* $\mathcal{K}_n^G$ *with a common prefix* $B_0 < ... < B_h$*, and* $\varphi^C = B_i$ *for some* $0 \leq i \leq h$*, then* $\varphi^{C'} = B_i$.

We recall that $\sqcup$ and $\oplus$ denote respectively horizontal and vertical sums of posets as detailed in Section 1.3.

Given a set of Gödel chains $\mathcal{C}$, we construct a poset $\mathcal{M}(\mathcal{C})$. Let $\mathbf{C}_1 = P \oplus C_1'$ and $\mathbf{C}_2 = P \oplus C_2'$ be two chains in $\mathcal{C}$. The subchain P is their longest common prefix (see Definiton 1.3.2). Then, the poset $P \oplus (C_1' \sqcup C_2')$

belongs to $\mathcal{M}(\mathcal{C})$. For each chain $\mathbf{C}$ in $\mathcal{C}$ there is a unique branch [1] of $\mathcal{M}(\mathcal{C})$ that is a unique copy of $\mathbf{C}$ and every branch of $\mathcal{M}(\mathcal{C})$ is a copy of a unique chain in $\mathcal{C}$. We call G_n the poset $\mathcal{M}(\mathcal{K}_n^G)$, and by Definition 1.3.1 $\mathcal{A}_{G_n}$ and $\mathcal{C}_{G_n}$ denote respectively the poset of maximal antichains and the poset of maximal chains over G_n. Note that $\mathbf{S}(G_n) = \langle \mathcal{A}_{G_n}, \wedge, \rightarrow, \bot \rangle$ is a Gödel algebra, where $\bot$ is the least maximal antichain in $\mathcal{A}_{G_n}$ and the operations are defined branch-wise. That is, $[p_B]_{\mathcal{C}_{G_n}} \wedge [q_B]_{\mathcal{C}_{G_n}} = [p_B \wedge q_B]_{\mathcal{C}_{G_n}}$, $[p_B]_{\mathcal{C}_{G_n}} \rightarrow [q_B]_{\mathcal{C}_{G_n}} = [p_B \rightarrow q_B]_{\mathcal{C}_{G_n}}$.

Normal Forms

In order to describe Gödel maxterms consider maximal chains of the form $\mathbf{B} = \{B_0 < ... < B_w < B_{w+1} = \{\top\}\}$ and define formulas for $i \in \{1, ..., w\}$:

$$\rho_{B_i} := \bigvee_{y \in B_i, y \neq z_i} (z_i \leftrightarrow y) \rightarrow z_i$$

The behavior of ρ_{B_i} is such that $\rho_{B_i} = z_i$ over the Gödel chain $\mathbf{B}$, since every y belongs to B_i. For every other Gödel chain $\mathbf{B}' \neq \mathbf{B}$, $\rho_{B_i}^{\mathbf{B}'} = \top^{\mathbf{B}'}$.

Let $\mathbf{B}$ be the same Gödel chain as above, we define terms for each $i \in \{0, ..., w-1\}$ and each $j \in \{0, ..., w\}$, $k \in \{j+1, ..., w+1\}$:

$$\rho_{B_j}^k := \bigvee_{y \in B_k} (y \rightarrow z_j) \quad \rho'_{B_i} := z_{i+1} \rightarrow z_i.$$

The behavior of the above defined terms are such that $\rho'_{B_i} = z_i$ and $\rho_{B_i}^k = z_j$ when evaluated over $\mathbf{B}$. Otherwise $\rho'_{B_i} = \top$ if $z_{i+1} \leq z_i$ and $\rho_{B_i}^k = \top$ if $z_j \geq y$ for some y. This could be the case for some Gödel chain $\mathbf{B}' \neq \mathbf{B}$.

Note that $\rho_{B_j}^{w+1} \equiv z_j$,for each element $p \in\in B$, with $p \in B_j$ for some $j \in \{0, ..., w\}$ we set:

$$\Phi_{p_B} := p_B \vee \bigvee_{i=0}^{j} \rho_{B_i} \vee \bigvee_{i=0}^{j-1} \rho'_{B_i} \vee \bigvee_{i=j+1}^{w} \rho_{B_j}^i.$$

Let A be a maximal antichain in $\mathbf{S}(G_n)$ and let $\mathbf{B}_1, \ldots, \mathbf{B}_k$ be the maximal chains in G_n such that $A \cap \mathbf{B}_i = p_{B_i}$ for $i \in [k]$. Then,

$$t_A = \Phi_{p_{B_1}} \wedge \cdots \wedge \Phi_{p_{B_k}}$$

is such that $t_A^{\mathcal{S}(G_n)} = A$. That is, t_A is a conjunctive normal form for A (see (1.10) in Section 1.3). As a consequence,

Theorem 5.1. *[6] The free algebra $\mathbf{F}_n(G)$ is isomorphic to the algebra of antichains $\mathbf{S}(G_n)$.*

Analogous kinds of Gödel logic normal forms can be found in [10], [29] and [17].

[1] Remember that a *branch* is a maximal chain in a poset, see Section 1.3.

Chapter 6

RDP Logic Normal Forms

In this section, we give a combinatorial representation of free finitely generated RDP algebras to obtain a number of results on the logical counterpart of RDP algebras. This representation can be obtained in two way, starting from the analysis of finite RDP chains and using normal forms to characterize the free n-generated algebras, or applying the theory of finitely generated RDP algebras developed in the previous chapter and defining subsequently the normal forms. The former will be detailed in Section 6.1 where we define conjunctive normal forms using maxterms, and the latter in Section 6.2 where disjunctive normal forms are defined starting from minterms.

6.1 A Bottom-Up Approach

Appealing at Section 2.1, we represent RDP chains generated by $\{x_1, ..., x_n\}$ as ordered partitions of the set $\mathcal{F}_n$ (2.5). Before introducing a characterization of RDP chains, we give an example of how negations and double negations behave in RDP chains.

Example 6.1.1. *Given a RDP chain* $\mathbf{C} = \{B_0 < B_1 < B_2 < B_3 < B_4\}$ *generated by* $\{x_1, x_2, x_3\}$*. Then,* $\perp \in B_0$ *and* $\top \in B_4$*. Let* $x_1 \in B_1$*,* $x_2 \in B_2$*,* $x_3 \in B_3$ *and* $x_2 = \neg x_2$*. Then, by equation (3.4),* $x_1 < \neg x_1$ *and* $\neg x_3 = \top$*. Hence,* $B_1 = \{\perp, \neg x_3\}$*,* $B_1 = \{x_1\}$*,* $B_2 = \{x_2, \neg x_2, \neg\neg x_2, \neg x_1, \neg\neg x_1\}$*,* $B_3 = \{x_3\}$ *and* $B_4 = \{\top, \neg\neg x_3\}$*.*

Every ordered partition of $\mathcal{F}_n$ will be of the form $\{B_0 < ... < B_M < ... < B_k\}$ where:

$$\begin{gathered} x_i \in B_0 \text{ if and only if } \neg x_i \in B_k \text{ if and only if } \neg\neg x_i \in B_0; \\ \text{if } x_i > \neg x_i \text{ then } \neg x_i \in B_0, \neg\neg x_i \in B_k \text{ and } x_i \in B_j \text{ for some } j > M; \\ \text{if } \perp < x_i \leq \neg x_i \text{ then } \neg x_i, \neg\neg x_i \in B_M. \end{gathered} \tag{6.1}$$

From these three conditions, it follows that the set B_M is given by $\{x_i \mid x_i = \neg x_i\} \cup \{\neg x_i, \neg\neg x_i \mid \bot < x_i \leq \neg x_i\}$. If $B_M = \emptyset$ then every x_i belongs to some B_j such that $M < j$ or $j = 0$. Moreover, for any generator x_i, the block which $\neg\neg x_i$ belongs to is uniquely determined by the blocks that contain x_i and $\neg x_i$. Therefore we can remove double negations from $\mathcal{F}_n$. Throughout the rest of this section, by ordered partition we mean an ordered partition of the set $\{\bot, x_1, ...x_n, \neg x_1, ..., \neg x_n, \top\}$. We let $\mathcal{C}_n^{RDP}$ the set of all RDP chains generated by $\{x_1, ..., x_n\}$. Moreover, recall that $\mathcal{K}_n^{RDP}$ denote the set of non-redundant n-generated RDP chains (see Section 2.1).

Lemma 6.1.1. *Let $\mathbf{C} = \{B_0 < ... < B_M < ... < B_k\}$ be in $\mathcal{C}_n^{RDP}$. Then, $\mathbf{C} \in \mathcal{K}_n^{RDP}$ if and only if $\{x_1, ..., x_n\} \cap B_k = \emptyset$.*

Proof. Given $\mathbf{C} = \{B_0 < ... < B_M < ... < B_k\}$, suppose that $x_i \in B_k$ then by Lemma 2.1.2, there exists a chain $\mathbf{C}'$ of the form $\{B_0 < ... < \{x_i\} \cup S < B_k\}$ for some S, and a congruence θ on $\mathbf{C}'$ such that $\mathbf{C} = \mathbf{C}'/\theta$ and $x_i \theta \top$. Hence $C \notin \mathcal{K}_n^{RDP}$.

Let $\mathbf{C} = \{B_0 < ... < B_M < ... < B_k\}$ be a RDP chain in $\mathcal{C}_n^{RDP}$ with $\{x_1, ..., x_n\} \cap B_k = \emptyset$. If $\mathbf{C}$ is a quotient of some RDP chain $\mathbf{C}'$ by a congruence θ, then by Lemma 2.1.2 $\mathbf{C} = \mathbf{C}'$. Hence $\mathbf{C} \in \mathcal{K}_n^{RDP}$.

$\square$

Remark 6.1.1. *There are exactly five ways to 1-generate an RDP chain, thas is $\mathbf{C}_1$, $\mathbf{C}_4$, $\mathbf{C}_5$, $\mathbf{C}_8$ and $\mathbf{C}_9$ in Figure 2.1. Non-redundant singly generated RDP chains are $\mathcal{K}_1^{RDP} = \{\mathbf{C}_1, \mathbf{C}_4, \mathbf{C}_5, \mathbf{C}_8\}$, compare Figure 2.2 with Figure 6.1. Moreover, these chains have been used in Section 3.3 to characterize the prime spectrum of the free 1-generated RDP algebra. Indeed, as the following pages show $\mathcal{K}_1^{RDP}$ is a fundamental construction to characterize the free 1-generated RDP algebras as an algebra of antichains (Theorem 6.1.2).*

Every chain $\mathbf{C}$ in $\mathcal{C}_n^{RDP}$ has the form $\mathbf{C} = B_0 \oplus C_\downarrow \oplus B_M \oplus C_\uparrow$, where B_M may be empty. Further, consider the subchain $B_0 \oplus C_\uparrow$. Then, $B_0 \oplus C_\uparrow$ is a Gödel subalgebra of $\mathbf{C}$ generated by $\{x_1, ..., x_n\} \cap \bigcup B_l$, with $l = 0$ or $l > M$.

Lemma 6.1.2. *For any formula $\varphi(x_1, ..., x_n)$, $(\varphi^{\mathbf{C}})_{\mathbf{C} \in \mathcal{K}_n}$ is an element of $\prod_{\mathbf{C} \in \mathcal{K}_n} \mathbf{C}$ satisfying the following form of the* prefix property*:*

if $\mathbf{U}$ and $\mathbf{V}$ are chains in $\mathcal{K}_n^{RDP}$ with a common prefix $B_0 < ... < B_h$ for some $h \geq M$ (hence $U_\downarrow = V_\downarrow$), and $\varphi^{\mathbf{U}} = B_i$ for some $i \leq h$, then $\varphi^{\mathbf{V}} = B_i$.

Proof. With $\varphi^{\mathbf{C}}$ we mean the block in $\mathbf{C}$ containing the value of φ. The proof is by induction on the structure of φ. The base case is $\varphi = x_i$. Let $\varphi^{\mathbf{U}} = B_l$, hence B_l is the unique block in $\mathbf{U}$ that contains x_i. If $l \leq h$, then B_l is the unique block of $\mathbf{V}$ that contains x_i, hence $\varphi^{\mathbf{V}} = B_l$.

We suppose that the prefix property holds for ψ_1, ψ_2.

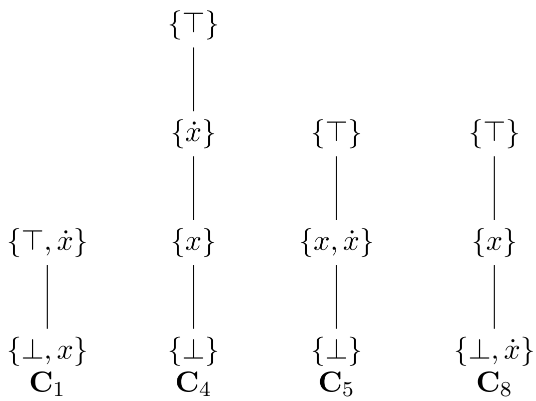


Figure 6.1: The four chains in $\mathcal{K}_1^{RDP}$. Note that B_M is empty in the first chain and in the last chain, B_M is equal to $\{\neg x\}$ and $\{x, \neg x\}$ in the second and in the third chain respectively.

$\underline{\varphi = \psi_1 \wedge \psi_2}$. Let $\psi_1^{\mathbf{U}} = B_i$ and $\psi_2^{\mathbf{U}} = B_j$. Without loss of generality we suppose that $B_i \leq B_j$, then $\varphi^{\mathbf{U}} = B_i$. If $\psi_1^{\mathbf{U}} \leq \psi_2^{\mathbf{U}} \leq B_h$, then by induction $\psi_1^{\mathbf{V}} \leq \psi_2^{\mathbf{V}} \leq B_h$. We conclude that $\varphi^{\mathbf{V}} = B_i$. If $\psi_1^{\mathbf{U}} \leq B_h < \psi_2^{\mathbf{U}}$, then by induction $B_i = \psi_1^{\mathbf{V}} < \psi_2^{\mathbf{V}} = B_j$, for some $j > h$. We deduce that $\varphi^{\mathbf{V}} = B_i$.

$\underline{\varphi = \psi_1 \odot \psi_2}$, then we have:

- if $\psi_1^{\mathbf{U}}, \psi_2^{\mathbf{U}} \leq B_M$ then $\psi_1^{\mathbf{U}} = B_i, \psi_2^{\mathbf{U}} = B_j$ with $i, j \leq M$, so $\varphi^{\mathbf{U}} = B_0$. Then, by induction $\psi_1^{\mathbf{V}}, \psi_2^{\mathbf{V}} \leq B_M$, so $\varphi^{\mathbf{V}} = B_0$.
- if $\psi_1^{\mathbf{U}} \leq B_h$ and $\psi_2^{\mathbf{U}} > B_h$, the formula φ reduces to $min(\psi_1, \psi_2)$ and $\varphi^{\mathbf{U}} = \psi_1^{\mathbf{U}} = B_i$. By induction $B_i = \psi_1^{\mathbf{V}} \leq B_h$ and $\psi_2^{\mathbf{V}} > B_h$, then $\varphi^{\mathbf{V}} = B_i$. The case $\psi_2^{\mathbf{U}} \leq B_h$ and $\psi_1^{\mathbf{U}} > B_h$ is analogous.

$\underline{\varphi = \psi_1 \to \psi_2}$, then:

- if $\psi_2^{\mathbf{U}} < \psi_1^{\mathbf{U}} \leq B_M$, then $\varphi^{\mathbf{U}} = B_M \leq B_h$. By induction also $\psi_2^{\mathbf{V}} < \psi_1^{\mathbf{V}} \leq B_M$ so $\varphi^{\mathbf{V}} = B_M \leq B_h$.
- If $\psi_1^{\mathbf{U}} > B_h$ and $\psi_2^{\mathbf{U}} = B_j \leq B_h$ then $\varphi^{\mathbf{U}} = \psi_2^{\mathbf{U}} = B_j$. By induction $\psi_1^{\mathbf{V}} > B_h$ and $\psi_2^{\mathbf{V}} = B_j$ so $\varphi^{\mathbf{V}} = B_j$.

$\square$

We call RDP_n the poset obtained by $\mathcal{M}(\mathcal{K}_n^{RDP})$, using the prefix property stated in the previous Lemma (the operator $\mathcal{M}$ is defined in Section 5). See Figure 6.2 as an example.

By Definition 1.3.1, $\mathcal{A}_{RDP_n}$ is the poset of maximal antichains over RDP_n and $\mathcal{C}_{RDP_n}$ is the poset of maximal chains (or branches) over RDP_n. The

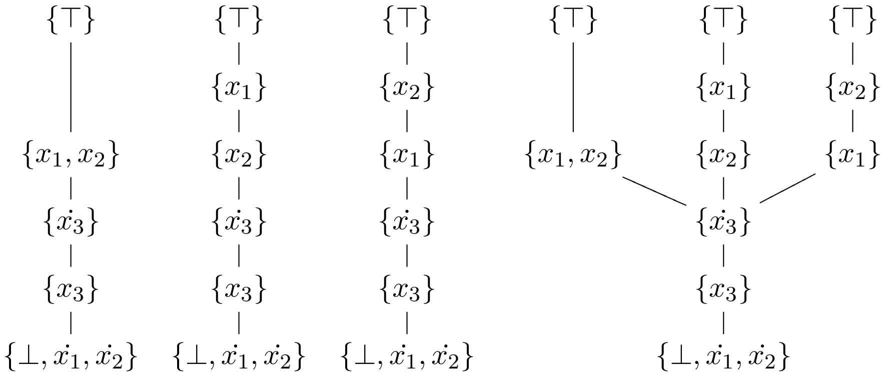


Figure 6.2: Three RDP chains in $\mathcal{K}_3^{RDP}$ and the poset resulting by merging their common prefix.

algebraic structure of each chain in $\mathcal{K}_n^{RDP}$ is preserved in the corresponding branch in RDP_n. The algebra $\mathbf{S}(RDP_n) = \langle \mathcal{A}_{RDP_n}, \odot, \wedge, \to, \bot \rangle$, is an RDP algebra where $\bot$ is the least section in $\mathcal{A}_{RDP_n}$ and the operations $\odot, \wedge, \to$ are defined componentwise. That is, $[p_B]_{\mathcal{C}_{RDP_n}} \odot [q_B]_{\mathcal{C}_{RDP_n}} = [p_B \odot q_B]_{\mathcal{C}_{RDP_n}}$, $[p_B]_{\mathcal{C}_{RDP_n}} \wedge [q_B]_{\mathcal{C}_{RDP_n}} = [p_B \wedge q_B]_{\mathcal{C}_{RDP_n}}$, $[p_B]_{\mathcal{C}_{RDP_n}} \to [q_B]_{\mathcal{C}_{RDP_n}} = [p_B \to q_B]_{\mathcal{C}_{RDP_n}}$. By construction:

Lemma 6.1.3. *The map*

$$(\varphi^{\mathbf{C}})_{\mathbf{C} \in \mathcal{K}_n^{RDP}} \to [\varphi^B]_{\mathcal{C}_{RDP_n}}$$

is a monomorphism from $\prod_{C \in \mathcal{K}_n^{RDP}}$ to $\mathcal{A}_{RDP_n}$. Hence, by Lemma 2.1.3 the free algebra $\mathbf{F}_n(RDP)$ can be embedded in the algebra $\mathbf{S}(RDP_n)$ of maximal antichains over RDP_n.

Normal Forms

In this section we will show that the free n-generated RDP algebra and the algebra of maximal antichains $\mathbf{S}(RDP_n)$ are isomorphic. We have already shown that there is an embedding (Lemma 6.1.3) between this two algebras . In order to obtain an isomorphism, we need to show that for every maximal antichain $[p_B]_{\mathcal{C}_{RDP_n}}$ in $\mathbf{S}(RDP_n)$ it is possible to construct a RDP logic formula that computes $[p_B]_{\mathcal{C}(RDP_n)}$ when evaluated on $\mathbf{S}(RDP_n)$. We will build these logical formulas through normal forms, hence we need to introduce maxterms for RDP logic.

Let $\mathbf{B} = \{B_0 < ... < B_M < ...B_k\}$ be a RDP chain in $\mathcal{C}_{RDP_n}$. For every block B_i with $i > 0$, we fix an element z_i. Moreover, we set $z_0 = \bot$. We define the following formulas:

$$\delta_{B_i} := (z_{i+1} \to z_i) \to (\neg(z_{i+1} \to z_i))$$

$$\delta'_{B_i} := z_i \to \neg z_i$$

$$\delta''_{B_i} := \bigvee_{y \in B_i, y \neq z_i} (z_i \leftrightarrow y) \to \neg(z_i \leftrightarrow y)$$

The behavior of δ_{B_i}, δ'_{B_i} and δ''_{B_i} over B are such that $\delta_{B_i} = \delta'_{B_i} = \delta''_{B_i} = \top$ if and only if $i \leq M$ otherwise $\delta_{B_i} = \delta'_{B_i} = \delta''_{B_i} = \bot$.

We have seen that each RDP chain $\mathbf{C}$ could be represented as a vertical sum $B_0 \oplus C_\downarrow \oplus B_M \oplus C_\uparrow$, where $B_0 \oplus C_\uparrow$ is a Gödel chain. Thanks to this, we can safely apply the terms ρ, ρ^k and ρ' already defined in Chapter 5 on elements of $\mathbf{C}$ that belongs to blocks B_i where $i > M$.

For each $p \in\in B$, with $p \in B_l$, if $l > M$ then we set $j = l$ and $h = w$, otherwise we set $j = 0$ and $h = 0$. We define:

$$\Phi_{p_B} := p_B \vee \bigvee_{i=M+1}^{j} \rho_{B_i} \vee \bigvee_{i=M+1}^{j-1} \rho'_i \vee \bigvee_{i=j+1}^{h} \rho^i_{B_j} \vee \bigvee_{i=M+1}^{w} \delta'_{B_i} \vee \bigvee_{i=0}^{M} \neg\delta'_{B_i} \vee \bigvee_{i=0}^{M-1} \neg\delta_{B_i} \vee \bigvee_{i=0}^{M} \delta''_{B_i}$$

Theorem 6.1.1. *For any $\mathbf{B} = \{B_0 < ... < B_w < \top\} \in \mathcal{C}_{RDP_n}$, Φ_{p_B} is a syntactical maxterm for RDP_n.*

Proof. By direct inspection $\Phi^{\mathbf{B}}_{p_B} = p_B$. Moreover, in order to show that Φ_{p_B} is a syntactical maxterm, we have to prove that for any branch $\mathbf{B}' \neq \mathbf{B}$ in $\mathcal{C}_{RDP_n}$, we have $\Phi^{\mathbf{B}'}_{p_B} = \top$. Let $\mathbf{B}' = \{B_0 < ... < B_{k-1} < V_k < ... < V_v\}$ with $V_k \neq B_k$, where $\top \in V_v$. Hence, $\{B_0 < ... < B_{k-1}\}$ is the longest common prefix of $\mathbf{B}$ and $\mathbf{B}'$. We have to distinguish two cases:

- $\mathbf{B}$ and $\mathbf{B}'$ share a common prefix of length $k - 1 \geq M$.

  $\mathbf{B}'$ differs from $\mathbf{B}$ on elements belonging to blocks B_l where $l > k-1 \geq M$. Since $B_0 \oplus C_\uparrow$ and $W'_0 \oplus C'_\uparrow$ (the Gödel subchain of C') are Gödel chains, there is an index $M < t \leq w$ such that $\pi^{\mathbf{B}'}_t = V_v$ or $\rho^{\mathbf{B}'}_t = V_v$ (see Section 5). Hence, $\Phi^{\mathbf{B}'}_{p_B} = V_v = \top^{\mathbf{B}'}$.

- $\mathbf{B}$ and $\mathbf{B}'$ share a common prefix of length $0 \leq k < M$.

  We now assume that $z^{\mathbf{B}'}_k < z^{\mathbf{B}'}_{k+1} < ... < z^{\mathbf{B}'}_w$. Otherwise, there will be an index t, $k \leq t \leq M$ such that $\neg\delta^{\mathbf{B}'}_t = V_v$, or an index $M < t \leq w$ such that $\rho^{\mathbf{B}'}_t = V_v$. We know $\mathbf{B} \neq \mathbf{B}'$, so:

  - either there exists an index $k \leq t \leq w$ such that $z^{\mathbf{B}}_t \leq B_M$ and $z^{\mathbf{B}'}_t > B_M$, hence $\neg s^{\mathbf{B}'}_t = V_v$. Viceversa, $z^B_t > B_M$ and $z^{\mathbf{B}'}_t \leq B_M$, hence $s^{\mathbf{B}'}_t = V_v$;
  - or there exists an index $k \leq t \leq w$ such that $x \in B^{\mathbf{B}}_t$ but $x \notin B^{\mathbf{B}'}_t$, so $C^{\mathbf{B}'}_t = V_v$.

We have shown that the maxterm Φ_{p_B}, when evaluated on branches $\mathbf{B}' \neq \mathbf{B}$, is always equal to the top element. This settles the claim. $\square$

Hence, we can define normal forms for RDP logic. Let A be a maximal antichain in $\mathbf{S}(RDP_n)$ and let $\mathbf{B}_1, \ldots, \mathbf{B}_k$ be the maximal chains in RDP_n such that $A \cap \mathbf{B}_i = p_{B_i}$ for $i \in [k]$. Then,

$$t_A = \Phi_{p_{B_1}} \wedge \cdots \wedge \Phi_{p_{B_k}}$$

is such that $t_A^{\mathcal{F}_n(G)} = A$. That is, t_A is a conjunctive normal form for A (see (1.10) in Section 1.3).

Theorem 6.1.2. *The free algebra $\mathbf{F}_n(RDP)$ is isomorphic to the algebra of maximal antichains $\mathbf{S}(RDP_n)$.*

Proof. By Lemma 2.1.3 and Lemma 6.1.3, $\mathbf{F}_n(RDP)$ can be embedded in $\mathbf{S}(RDP_n)$. We have to show that for any section $[p_B]_{\mathcal{C}_{RDP_n}}$ there is a term φ in n variables such that:

$$[\varphi^B]_{\mathcal{C}_{RDP_n}} = [p_B]_{\mathcal{C}_{RDP_n}}.$$

By Theorem 6.1.1 $\Phi_{p_B}^{\mathbf{S}(RDP_n)} = [\Phi_{p_B}^{\mathbf{B}'}]_{\mathbf{B}' \in \mathcal{C}_{RDP_n}}$ is the semantical maxterm taking value p over B. Then, for any maximal antichain $[p_B]_{\mathcal{C}_{RDP_n}}$ over RDP_n,

$$\left(\bigwedge_{\mathbf{B} \in \mathcal{C}_{RDP_n}} \Phi_{p_B} \right)^{\mathbf{S}(RDP_n)} = \left[\bigwedge_{\mathbf{B} \in \mathcal{C}_{RDP_n}} \Phi_{p_B}^{\mathbf{B}'} \right]_{\mathbf{B}' \in \mathcal{C}_{RDP_n}} = [p_B]_{\mathcal{C}_{RDP_n}}$$

$\square$

6.2 A Top-Down Approach

In Section 3.3 Theorem 3.3.1, we characterize the free n-generated RDP algebra $\mathbf{F}_n$ as the algebra $\Psi(S_1^n)$, that is, the algebra of maximal antichains in $\mathcal{A}_{S_1^n}$ over the augmented forest of S_1^n specified by (3.14). To sample the general case, we now describe in a sequence of examples the product of two copies of the finite hall forest S_1 depicted in Figure 6.3, namely, the product $F \times F'$ where

$$\begin{aligned}
F &= \{(T_1, J_1), (T_2, J_2), (T_3, J_3), (T_4, J_4)\} \\
&= \{(\{\bot\}, \emptyset), (\{\bot\}, \{x < \dot{x}\}), (\{\bot\}, \{x = \dot{x}\}), (\{\bot < x\}, \emptyset)\}; \\
F' &= \{(T_1', J_1'), (T_2', J_2'), (T_3', J_3'), (T_4', J_4')\} \\
&= \{(\{\bot\}, \emptyset), (\{\bot\}, \{y < \dot{y}\}), (\{\bot\}, \{y = \dot{y}\}), (\{\bot < y\}, \emptyset)\}.
\end{aligned}$$

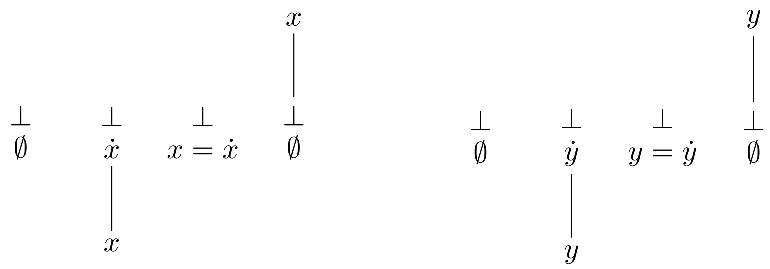


Figure 6.3: Two copies of S_1 suitably labelled in view of the description of $S_1 \times S_1$. For each hall tree (T, J) in S_1, the component J is displayed below T.

The adopted labelling of factors is useful to describe the product operation and the projection maps.

The general behavior of products of trees is described in [18]. In the sample case under consideration, we have the following.

Example 6.2.1. *We study the action of product $F \times F'$ over the tree components of pairs of hall trees in F and F'. Precisely, for each $(m, n) \in [4] \times [4]$, we compute the product $T_m \times T'_n$, together with its projections onto the left and right factor. The result is the following.*

For $j = 1, 2, 3$ and $i = 1, 2, 3$, $T_j \times T'_i$ yields the tree $S_{j,i} = \{\perp\}$, whose projection $\varsigma_{j,i}$ onto T_j is $\perp \mapsto \perp$, and whose projection $\varsigma'_{j,i}$ onto T'_i is $\perp \mapsto \perp$.

For $j = 1, 2, 3$, $T_j \times T'_4$ yields the tree $S_{j,4} = \{\perp < y\}$, whose projections $\varsigma_{j,4}$ and $\varsigma'_{j,4}$ are respectively, $\perp \mapsto \perp, y \mapsto \perp$, and $\perp \mapsto \perp, y \mapsto y$.

For $i = 1, 2, 3$, $T_4 \times T'_i$ yields the tree $S_{4,i} = \{\perp < x\}$, whose projections $\varsigma_{4,i}$ and $\varsigma'_{4,i}$ are respectively, $\perp \mapsto \perp, x \mapsto \perp$, and $\perp \mapsto \perp, x \mapsto x$.

$T_4 \times T'_4$ yields the tree $S_{4,4}$ given by the chains $\perp < \{x = y\}$, $\perp < x < \{x < y\}$, $\perp < y < \{y < x\}$, whose projections $\varsigma_{4,4}$ and $\varsigma'_{4,4}$ are respectively, $\perp \mapsto \perp, \{x = y\} \mapsto x, x \mapsto x, \{x < y\} \mapsto \perp, y \mapsto \perp, \{y < x\} \mapsto x$, and $\perp \mapsto \perp, \{x = y\} \mapsto y, x \mapsto \perp, \{x < y\} \mapsto y, y \mapsto y, \{y < x\} \mapsto \perp$.

The action of the product $F \times F'$ over the chain components of pairs of hall trees in F and F' is the following.

Example 6.2.2. *We study the action of product $F \times F'$ over the chain components of pairs of hall trees in F and F'. Precisely, for each $(m, n) \in [4] \times [4]$, we compute the product $J_m \times J'_n$, together with its projections onto the left and right factor. The result is the following.*

$J_1 \times J'_1$ yields the chain $K_{1,1} = \emptyset$, whose projection $\rho_{1,1}$ onto J_1 is the empty function, and whose projection $\rho'_{1,1}$ onto J'_1 is the empty function.

$J_1 \times J'_2$ yields $K_{1,2} = \{y < \dot{y}\}$, whose projections $\rho_{1,2}$ and $\rho'_{1,2}$ are respectively, the empty function, and $y \mapsto y, \dot{y} \mapsto \dot{y}$.

$J_1 \times J'_3$ yields $K_{1,3} = \{\{y = \dot{y}\}\}$, whose projections $\rho_{1,3}$ and $\rho'_{1,3}$ are respectively, the empty function, and $\{y = \dot{y}\} \mapsto \{y = \dot{y}\}$.

$J_1 \times J'_4$ yields $K_{1,4} = \emptyset$, whose projections $\rho_{1,4}$ and $\rho'_{1,4}$ are respectively, the empty function, and the empty function.

$J_2 \times J'_1$ yields $K_{2,1} = \{x < \dot{x}\}$, whose projections $\rho_{2,1}$ and $\rho'_{2,1}$ are respectively, $x \mapsto x, \dot{x} \mapsto \dot{x}$, and the empty function.

$J_2 \times J'_2$ yields the following three chains: $K_{2,2,1} = \{x = y < \dot{x} = \dot{y}\}$, whose projections $\rho_{2,2,1}$ and $\rho'_{2,2,1}$ are respectively, $x = y \mapsto x, \dot{x} = \dot{y} \mapsto \dot{x}$, and $x = y \mapsto y, \dot{x} = \dot{y} \mapsto \dot{y}$; $K_{2,2,2} = \{x < y < \dot{x} = \dot{y}\}$, whose projections $\rho_{2,2,2}$ and $\rho'_{2,2,2}$ are respectively, $x \mapsto x, y \mapsto \dot{x}, \dot{x} = \dot{y} \mapsto \dot{x}$, and $x \mapsto y, y \mapsto y, \dot{x} = \dot{y} \mapsto \dot{y}$; and $K_{2,2,3} = \{y < x < \dot{x} = \dot{y}\}$, whose projections $\rho_{2,2,3}$ and $\rho'_{2,2,3}$ are respectively, $y \mapsto x, x \mapsto x, \dot{x} = \dot{y} \mapsto \dot{x}$, and $y \mapsto y, x \mapsto \dot{y}, \dot{x} = \dot{y} \mapsto \dot{y}$.

$J_2 \times J'_3$ yields $K_{2,3} = \{x < \dot{x} = y = \dot{y}\}$, whose projections $\rho_{2,3}$ and $\rho'_{2,3}$ are respectively, $x \mapsto x, \dot{x} = y = \dot{y} \mapsto \dot{x}$, and $x \mapsto y = \dot{y}, \dot{x} = y = \dot{y} \mapsto y = \dot{y}$.

$J_2 \times J'_4$ yields $K_{2,4} = \{x < \dot{x}\}$, whose projections $\rho_{2,4}$ and $\rho'_{2,4}$ are respectively, $x \mapsto x, \dot{x} \mapsto \dot{x}$, and the empty function.

$J_3 \times J'_1$ yields $K_{3,1} = \{x = \dot{x}\}$, whose projections $\rho_{3,1}$ and $\rho'_{3,1}$ are respectively, $x = \dot{x} \mapsto x = \dot{x}$, and the empty function.

$J_3 \times J'_2$ yields $K_{3,2} = \{y < x = \dot{x} = \dot{y}\}$, whose projections $\rho_{3,2}$ and $\rho'_{3,2}$ are respectively, $y \mapsto x = \dot{x}, x = \dot{x} = \dot{y} \mapsto x = \dot{x}$, and $y \mapsto y, x = \dot{x} = \dot{y} \mapsto \dot{y}$.

$J_3 \times J'_3$ yields $K_{3,3} = \{x = \dot{x} = y = \dot{y}\}$, whose projections $\rho_{3,3}$ and $\rho'_{3,3}$ are respectively, $x = \dot{x} = y = \dot{y} \mapsto x = \dot{x}$, and $x = \dot{x} = y = \dot{y} \mapsto y = \dot{y}$.

$J_3 \times J'_4$ yields $K_{3,4} = \{x = \dot{x}\}$, whose projections $\rho_{3,4}$ and $\rho'_{3,4}$ are respectively, $x = \dot{x} \mapsto x = \dot{x}$, and the empty function.

$J_4 \times J'_1$ yields $K_{4,1} = \emptyset$, whose projections $\rho_{4,1}$ and $\rho'_{4,1}$ are respectively, the empty function, and the empty function.

$J_4 \times J'_2$ yields $K_{4,2} = \{y < \dot{y}\}$, whose projections $\rho_{4,2}$ and $\rho'_{4,2}$ are respectively, the empty function, and $y \mapsto y, \dot{y} \mapsto \dot{y}$.

$J_4 \times J'_3$ yields $K_{4,3} = \{y = \dot{y}\}$, whose projections $\rho_{4,3}$ and $\rho'_{4,3}$ are respectively, the empty function, and $y = \dot{y} \mapsto y = \dot{y}$.

$J_4 \times J'_4$ yields $K_{4,4} = \emptyset$, whose projections $\rho_{4,4}$ and $\rho'_{4,4}$ are respectively, the empty function, and the empty function.

Figure 6.4 displays $F \times F'$. The projections π and π' of $F \times F'$, onto F and F' respectively, are uniquely determined by their restrictions to each pair of hall trees, as specified in the following example.

Example 6.2.3. *For each $(m, n) \in [4] \times [4]$, we compute the product $(T_m, J_m) \times (T'_n, J'_n)$, together with its projections onto the left and right factor. The result is the following.*

If $m = n = 2$, $(T_2, J_2) \times (T'_2, J'_2)$ yields three hall trees, namely, for $j = 1, 2, 3$, $(S_{2,2}, K_{2,2,j})$, whose projections are $\pi_{2,2,j} = (\varsigma_{2,2}, \rho_{2,2,j})$ and $\pi'_{2,2,j} = (\varsigma'_{2,2}, \rho'_{2,2,j})$. Otherwise, $(T_m, J_m) \times (T'_n, J'_n)$ yields the hall tree $(S_{m,n}, K_{m,n})$ whose projections are $\pi_{m,n} = (\varsigma_{m,n}, \rho_{m,n})$ and $\pi'_{m,n} = (\varsigma'_{m,n}, \rho'_{m,n})$.

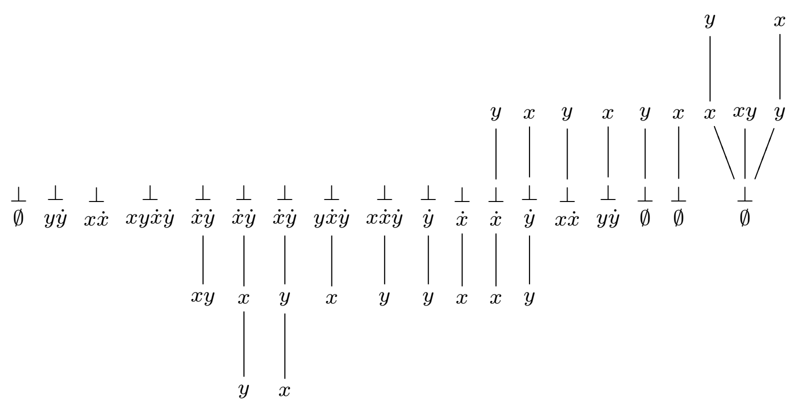


Figure 6.4: The finite hall forest $S_1^2 = S_1 \times S_1$. The labelling allows for recovering the projection maps of the first and second factor, displayed in Figure 6.4. For each hall tree (T, J) in S_1, the component J is displayed below T.

In the rest of this section, it is convenient to adopt a labelled display of the augmented forest of S_1^n, where each point is labelled with subsets of $\{\bot, \top, x_1, \neg x_1, \dots, x_n, \neg x_n\}$, satisfying the following conditions:

(*i*) x_i belongs to the label of each point in the maximal antichain corresponding to the free generator x_i of $\mathbf{F}_n$ (compare Theorem 3.3.1).

(*ii*) The label of each root contains $\bot$, and the label of each leaf contains $\top$.

(*iii*) $\neg x_i$ belongs to the label of each point in the negation in $\mathcal{A}_{S_1^n}$ of the antichain corresponding to the free generator x_i.

Thanks to the above defined labelling procedure, we can obtain a combinatorial representation analogous to the one presented in Section 6.1. We conclude by displaying in Figure 6.5 the labelled version of $\Psi(S_1^2)$, paralleling Figure 3.6 in the 2-generated case. This labelling method will allow for a streamlined investigation of several logical problems related to the free finitely generated RDP algebra.

The combinatorial representation of $\mathbf{F}_n$ achieved is amenable for investigation under several respects, substantially sampled by the logical applications in the next maximal antichains. In addition, we mention that the given representation yields a recurrence relation for the computation the cardinality of $\mathbf{F}_n$. We omit the details [49], and limit to report that, for instance, $|\mathbf{F}_1| = 72$, $|\mathbf{F}_2| = 94556160000$, $|\mathbf{F}_3| \sim 4.06 \cdot 10^{71}$, and $|\mathbf{F}_4| \sim 1.478733152865106 \cdot 10^{543}$. The first two statements are easy to check by directly count the maximal antichains in the forests displayed in Figure 3.6 and Figure 6.5.

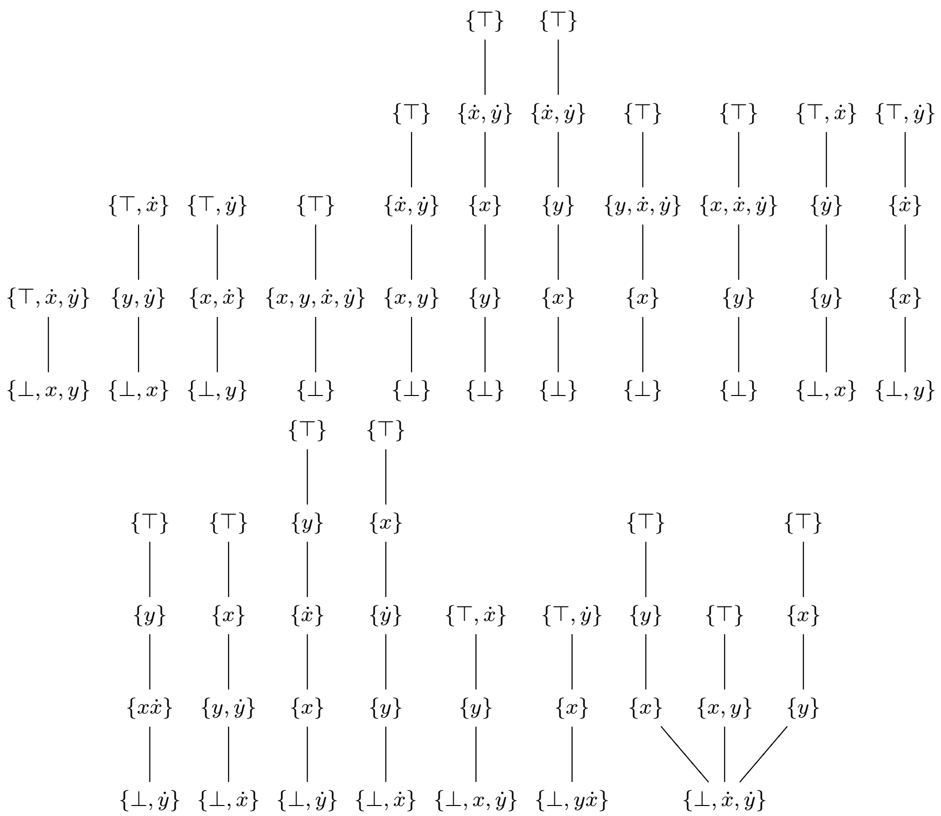


Figure 6.5: Display of $\Psi(S_1^2)$, by Theorem 3.3.1 isomorphic to $\mathbf{F}_2$, where the maximal antichains corresponding to the free generators x and y of $\mathbf{F}_2$ are those containing points whose label include x and y respectively.

As a first example of the strength of the above representation, we settle the tautology problem for RDP logic. Let $\mathbf{C} \in \mathcal{C}_{S_1^n}$ be a maximal chain in the labelled augmented forest of S_1^n. Note that $\mathbf{C}$ is a homomorphic image of $\mathbf{F}_n$; indeed, the map $h\colon \mathcal{A}_{S_1^n} \to \mathbf{C}$ such that for every $A \in \mathcal{A}_{S_1^n}$ and $c \in \mathbf{C}$, $h(A) = c$ if and only if $A \cap C = c$ is a surjective RDP homomorphism. Hence, $\mathbf{C}$ is an RDP chain. In the adopted display, $\mathbf{C}$ is an ordered partition $\{B_1 < \cdots < B_k\}$ of $\{\bot, \top, x_1, \neg x_1, \ldots, x_n, \neg x_n\}$, such that: $\bot \in B_1$ (the bottom of $\mathbf{C}$), $\top \in B_k$ (the top of $\mathbf{C}$), there exists at most one index $1 < f < k$ such that some $\neg x_i$'s belong to B_f (the fixpoint of $\mathbf{C}$), and each B_i that is neither the bottom, nor the fixpoint, nor the top of $\mathbf{C}$ contains at least one of $x_1, \ldots, x_n$. Note that any point $c \in \mathbf{C}$ can be regarded as a block amongst $B_1, \ldots, B_k$.

Now, let $t(x_1, \ldots, x_n)$ be a RDP term over variables $x_1, \ldots, x_n$. Then, the maximal antichain $t^{\mathbf{F}_n}$ that corresponds to t in the labelled display of $\mathbf{F}_n$ is inductively defined as follows. For every $\mathbf{C} = \{B_1 < \cdots < B_k\} \in \mathcal{C}_{S_1^n}$:

If $t = x_j$, then $x_j \in t^{\mathbf{F}_n} \cap C$; if $t = \bot$, then $\bot \in t^{\mathbf{F}_n} \cap C$; for $\circ \in \{\odot, \to\}$, if $t = t' \circ t''$, $t'^{\mathbf{F}_n} \cap C = B'$, and $t''^{\mathbf{F}_n} \cap C = B''$, then $t'^{\mathbf{F}_n} \cap C = B' \circ B''$, where the operation $\circ$ on $\{B_1, \dots, B_k\}$ is defined by making the block that contains x (respectively, $\neg x$, y, $\neg y$, $\bot$, $\top$) acting as x (respectively, $\neg x$, y, $\neg y$, $\bot$, $\top$) in (3.5) and (3.6). Compare Figure 6.6.

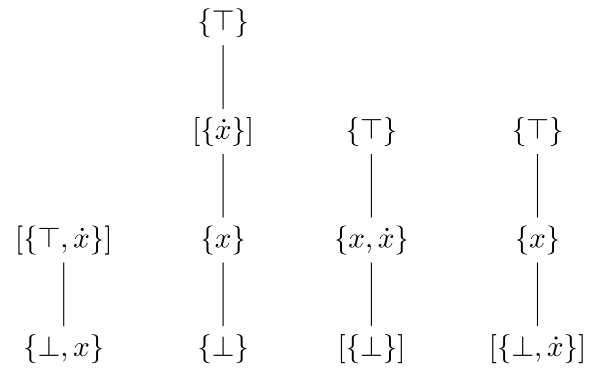


Figure 6.6: Displaying terms in $\mathbf{F}_1$ as maximal antichains in the labelled augmented forest of S_1: $(\neg(\neg x \to x))^{\mathbf{F}_1}$ is the braced maximal antichain in the diagram.

For the sake of notation, in the sequel we let

$$t(C) = t^{\mathbf{F}_n} \cap C.$$

A routine induction on t shows that t is a tautology of RDP logic if and only if $t(C) = \max C$ for every maximal chain $\mathbf{C} \in \mathcal{C}_{S_1^n}$, and by the standard completeness theorem [50], it follows that t is a theorem of RDP logic, in symbols $\vdash_{RDP} t$.

The computational complexity of deciding the tautology problem of RDP logic is as expected.

Proposition 6.2.1. *The RDP tautology problem is coNP-complete (under logspace many-one reductions).*

Proof. Let t be an RDP term on the variables $x_1, \dots, x_n$. For the upper bound, the algorithm receives in input a maximal chain in $\mathcal{C}_{S_1^n}$ and returns in output "Yes" if $t(C) = \max C$, and "No" otherwise. For the lower bound, we interpret the Boolean tautology problem. The reduction, given a Boolean term $t(x_1, \dots, x_n)$, say on conjunction $\odot$, implication $\to$, and zero $\bot$, outputs the RDP term $s = t(r_1, \dots, r_n)$, obtained by replacing uniformly variable x_i with term $r_i = (\neg\neg x_i) \odot (\neg\neg x_i)$ in t, for all $i \in [n]$ [1]. The substitution is feasible in logspace, and it is easy to check that t is a Boolean tautology (that is, $t = \top$ in $\mathbf{2}$) if and only if s is an RDP tautology (that is, $s = \top$ in the generic RDP algebra $[0, 1]$ given by (3.1)).

[1] As a notation, for $n \geq 1$, we let $[n] = \{1, \dots, n\}$.

Indeed, assume that t is a Boolean tautology. Let $\mathbf{a} \in [0,1]^n$. Noticing that $(r_1^{[0,1]}(\mathbf{a}), \ldots, r_n^{[0,1]}(\mathbf{a})) = \mathbf{b} \in \{0,1\}^n$, and that for any term q, the operations $q^{\mathbf{2}}$ and $q^{[0,1]}$ coincide upon restriction to $\{0,1\}$, we have,

$$s^{[0,1]}(\mathbf{a}) = t^{[0,1]}(r_1^{[0,1]}(\mathbf{a}), \ldots, r_n^{[0,1]}(\mathbf{a})) = t^{[0,1]}(\mathbf{b}) = t^{\mathbf{2}}(\mathbf{b}) = \top^{\mathbf{2}} = \top^{[0,1]},$$

so s is an RDP tautology. Conversely, if t is not a Boolean tautology, say $t^{\mathbf{2}}(\mathbf{b}) = \perp^{\mathbf{2}}$ for $\mathbf{b} = (b_1, \ldots, b_n) \in \{0,1\}^n$, since $r_i^{[0,1]}(\mathbf{b}) = b_i$ for all $i \in [n]$, we similarly have,

$$s^{[0,1]}(\mathbf{b}) = t^{[0,1]}(r_1^{[0,1]}(\mathbf{b}), \ldots, r_n^{[0,1]}(\mathbf{b})) = t^{[0,1]}(\mathbf{b}) = t^{\mathbf{2}}(\mathbf{b}) = \perp^{\mathbf{2}} = \perp^{[0,1]},$$

so s is not an RDP tautology. $\square$

Let r and s be MTL terms over the variables $x_1, \ldots, x_n$. Since the equation $x^3 = x^2$ holds in every WNM algebra, the local deduction theorem (see Theorem 1.1.1) holds in RDP logic with $n = 2$, namely,

$$r \vdash_{RDP} s \text{ if and only if } \vdash_{RDP} r^2 \to s. \tag{6.2}$$

In this light, we say that RDP logic proves s from r ($r \vdash_{RDP} s$), if $r^2 \to s$ is a theorem of RDP logic.

Normal Forms

In this section, we compute disjunctive *normal forms* for the elements of the free n-generated RDP algebra $\mathbf{F}_n$. The construction naturally generalizes disjunctive normal forms for the elements of the free n-generated Boolean algebra, exploiting the representation of $\mathbf{F}_n$ as the algebra of maximal antichains in the augmented forest of S_1^n specified by (3.14).

Let $\mathbf{C}$ be a maximal chain in the augmented forest of S_1^n, let c be a point in $\mathbf{C}$, and let A' be the smallest maximal antichain in $\mathcal{A}_{S_1^n}$ satisfying $A' \cap C = c$. An n-ary *RDP minterm* is an RDP term t_c over the variables $x_1, \ldots, x_n$ such that $t_c^{\mathbf{F}_n} = A'$. Now, let A be any maximal antichain in $\mathcal{A}_{S_1^n}$, let $\mathbf{C}_1, \ldots, \mathbf{C}_k$ be the maximal chains in $\mathcal{C}_{S_1^n}$, and let $A \cap C_i = c_i$ for $i \in [k]$. Then, the RDP term

$$t_A = t_{c_1} \vee \cdots \vee t_{c_k} \tag{6.3}$$

provides the desired disjunctive normal form for A, indeed, $t_A^{\mathbf{F}_n} = A$.

In light of the previous remark, it is sufficient to provide an explicit construction of the RDP minterm t_c for every maximal chain $\mathbf{C} \in \mathcal{C}_{S_1^n}$ and every $c \in \mathbf{C}$.

Fix an RDP chain $\mathbf{C} = \{B_1 < \cdots < B_f < \cdots < B_k\}$ in $\mathcal{C}_{S_1^n}$, and let B_f be the fixpoint of $\mathbf{C}$, where $f > 1$; if $\mathbf{C}$ has no fixpoint, we stipulate that $f = 0$. For $i = 1, \ldots, f$, fix a point $z_i \in B_i$, and define the following RDP terms:

(N1) $\xi_{B_i} = \bigwedge_{x \in B_i} \neg((z_i \leftrightarrow x) \rightarrow \neg(z_i \leftrightarrow x))$;

(N2) $\xi'_{B_i} = (z_{i+1} \rightarrow z_i) \rightarrow \neg(z_{i+1} \rightarrow z_i)$;

(N3) $\xi''_{B_i} = z_i \rightarrow \neg z_i$.

For $i = f+1, \ldots, k$, fix a point $z_i \in B_i$, and define the following RDP terms:

(I1) $\zeta_{B_i} = \bigwedge_{x \in B_i} (z_i \leftrightarrow x)$;

(I2) $\zeta'_{B_i} = (z_{i+1} \rightarrow z_i) \rightarrow z_{i+1}$ for $i < k$;

(I3) $\zeta''_{B_i} = \neg(z_i \rightarrow \neg z_i)$ for $i > 1$.

Example 6.2.4 ($n = 3$). *We construct the terms in (N1)-(N3) and (I1)-(I3) picking two samples* $\mathbf{C}$ *in* $\mathcal{C}_{S_1^3}$. *The first sample is an RDP chain* $\mathbf{C}$ *with fixpoint,* $\mathbf{C} = \{\{\bot, \dot{x_2}, \dot{x_3}\} < \{x_1\} < \{\dot{x_1}\} < \{x_2\} < \{x_3\} < \{\top\}\}$. *Fix* $z_1 = \bot$, $z_2 = x_1$, $z_3 = \dot{x_1}$, $z_4 = x_2$, $z_5 = x_3$ *and* $z_6 = \top$. *Then:*

(N1) $\xi_{\bot \dot{x_2} \dot{x_3}} = \neg((\bot \leftrightarrow \neg x_2) \rightarrow \neg(\bot \leftrightarrow \neg x_2)) \wedge \neg((\bot \leftrightarrow \neg x_3) \rightarrow \neg(\bot \leftrightarrow \neg x_3))$;

$\xi_{x_1} = \neg((x_1 \leftrightarrow x_1) \rightarrow \neg(x_1 \leftrightarrow x_1))$;

$\xi_{\dot{x_1}} = \neg((\neg x_1 \leftrightarrow \neg x_1) \rightarrow \neg(\neg x_1 \leftrightarrow \neg x_1))$;

(N2) $\xi'_{\bot \dot{x_2} \dot{x_3}} = (x_1 \rightarrow \bot) \rightarrow \neg(x_1 \rightarrow \bot)$;

$\xi'_{x_1} = (\neg x_1 \rightarrow x_1) \rightarrow \neg(\neg x_1 \rightarrow x_1)$;

$\xi'_{\dot{x_1}} = (x_2 \rightarrow \neg x_1) \rightarrow \neg(x_2 \rightarrow \neg x_1)$;

(N3) $\xi''_{\bot \dot{x_2} \dot{x_3}} = \bot \rightarrow \top$;

$\xi''_{x_1} = x_1 \rightarrow \neg x_1$;

$\xi''_{\dot{x_1}} = \neg x_1 \rightarrow \neg\neg x_1$.

(I1) $\zeta_{x_2} = (x_2 \leftrightarrow x_2)$;

$\zeta_{x_3} = (x_3 \leftrightarrow x_3)$;

$\zeta_{\top} = (\top \leftrightarrow \top)$;

(I2) $\zeta'_{x_2} = (x_3 \rightarrow x_2) \rightarrow x_3$;

$\zeta'_{x_3} = (\top \rightarrow x_3) \rightarrow \top$;

(I3) $\zeta''_{x_2} = \neg(x_2 \rightarrow \neg x_2)$;

$\zeta''_{x_3} = \neg(x_3 \rightarrow \neg x_3)$.

$\zeta''_{\top} = \neg(\top \to \bot).$

The second sample is an RDP chain $\mathbf{D}$ *with no fixpoint,* $\mathbf{D} = \{\{\bot, \dot{x_1}, \dot{x_2}, \dot{x_3}\} < \{x_1\} < \{x_2\} < \{x_3\} < \{\top\}$. *Note that in this case, the terms (N1)-(N3) do not exist. Fix* $z_1 = \bot$, $z_2 = x_1$, $z_3 = x_2$, $z_4 = x_3$ *and* $z_5 = \top$. *Then:*

(I1) $\zeta_{\bot\dot{x_1}\dot{x_2}\dot{x_3}} = (\bot \leftrightarrow \neg x_1) \wedge (\bot \leftrightarrow \neg x_2) \wedge (\bot \leftrightarrow \neg x_3);$

$\zeta_{x_1} = (x_1 \leftrightarrow x_1);$

$\zeta_{x_2} = (x_2 \leftrightarrow x_2);$

$\zeta_{x_3} = (x_3 \leftrightarrow x_3);$

$\zeta_{\top} = (\top \leftrightarrow \top);$

(I2) $\zeta'_{\bot\dot{x_1}\dot{x_2}\dot{x_3}} = (x_1 \to \bot) \to x_1;$

$\zeta'_{x_1} = (x_2 \to x_1) \to x_2;$

$\zeta'_{x_2} = (x_3 \to x_2) \to x_3;$

$\zeta'_{x_3} = (\top \to x_3) \to \top;$

(I3) $\zeta''_{x_1} = \neg(x_1 \to \neg x_1);$

$\zeta''_{x_2} = \neg(x_2 \to \neg x_2);$

$\zeta''_{x_3} = \neg(x_3 \to \neg x_3);$

$\zeta''_{\top} = \neg(\top \to \bot).$

The following facts hold by direct computation of the value of the involved RDP terms over the involved RDP chains. First, we study how the terms in (N1)-(N3) and (I1)-(I3) behave on $\mathbf{C}$.

Fact 6.2.1. *The terms in (N1)-(N3) and (I1)-(I3) evaluate to* $\max C$ *over* $\mathbf{C}$.

Example 6.2.5 ($n = 3$)**.** *Let* $\mathbf{C}$ *be the RDP chain in Example 6.2.4. For instance, we evaluate the term* $\xi_{\{\bot,\dot{x_2},\dot{x_3}\}}$ *over* $\mathbf{C}$*:*

$$\begin{aligned}
\xi_{\{\bot,\dot{x_2},\dot{x_3}\}}(C) &= \neg((\bot(C) \leftrightarrow \neg x_2(C)) \to \neg(\bot(C) \leftrightarrow \neg x_2(C)))\wedge \\
&\quad \neg((\bot(C) \leftrightarrow \neg x_3(C)) \to \neg(\bot(C) \leftrightarrow \neg x_3(C))) \\
&= \neg((\top(C) \to \neg\top(C))) \wedge \neg((\top(C) \to \neg\top(C))) \\
&= \neg\bot(C) \wedge \neg\bot(C) \\
&= \neg\bot(C) \\
&= \top(C) = \max C.
\end{aligned}$$

Also,

$$\begin{aligned}\zeta_{x_2}(C) &= (x_2 \leftrightarrow x_2)\\ &= (x_2 \to x_2) \wedge (x_2 \to x_2)\\ &= \top(C) = \max C.\end{aligned}$$

Next, we study how RDP terms in (N1)-(N3) and (I1)-(I3) behave on an RDP chain $\mathbf{C}' \in \mathcal{C}_{S_1^n}$ different from $\mathbf{C}$, entering an exhaustive case distinction.

The first case we consider is the following: Either $\mathbf{C}$ has a fixpoint B_f, $\mathbf{C}'$ has a fixpoint $B_{f'}$, and the first f' blocks of C' are equal to the first f blocks of $\mathbf{C}$; or, $\mathbf{C}$ and $\mathbf{C}'$ have no fixpoint. In this case, by [6, Theorem 5.5], we have

Fact 6.2.2. *The terms in (N1)-(N3) and (I3) evaluate to* $\max C'$ *over* $\mathbf{C}'$*; the terms in (I1) and (I2) evaluate to the smallest* $c' \in \mathbf{C}'$ *such that* $c' \parallel \max C$ *in the augmented forest of* S_1^n*.*

Example 6.2.6 ($n = 3$)**.** *Let* $\mathbf{C}$ *be the RDP chain in Example 6.2.4, and let* $\mathbf{C}' \in \mathcal{C}_{S_1^3}$ *be the RDP chain* $\{\{\bot, \dot{x_2}, \dot{x_3}\} < \{x_1\} < \{\dot{x_1}\} < \{x_3\} < \{x_2\} < \{\top\}\}$*, so that* C *and* C' *share the lower set of the fixpoint. Then,* $\xi_{\{\bot,\dot{x_2},\dot{x_3}\}}$ *evaluates to* $\max C'$ *over* $\mathbf{C}'$*,*

$$\begin{aligned}\xi_{\{\bot,\dot{x_2},\dot{x_3}\}}(C') = &\neg((\bot(C') \leftrightarrow \neg x_2(C')) \to \neg(\bot(C') \leftrightarrow \neg x_2(C')))\wedge\\ &\neg((\bot(C') \leftrightarrow \neg x_3(C')) \to \neg(\bot(C') \leftrightarrow \neg x_3(C')))\\ &= \neg((\top(C') \to \neg\top(C'))) \wedge \neg((\top(C') \to \neg\top(C')))\\ &= \neg\bot(C') \wedge \neg\bot(C')\\ &= \neg\bot(C')\\ &= \top(C') = \max C';\end{aligned}$$

and, ζ'_{x_2} *evaluates to the smallest* $c' \in \mathbf{C}'$ *such that* $c' \parallel \max C$*, namely,*

$$\begin{aligned}\zeta'_{x_2}(C') &= (x_3(C') \to x_2(C')) \to x_3(C')\\ &= \top(C') \to x_3(C')\\ &= x_3(C').\end{aligned}$$

The second case we consider is the following: Either $\mathbf{C}$ has a fixpoint B_f, $\mathbf{C}'$ has a fixpoint $B_{f'}$, and the first f' blocks of $\mathbf{C}'$ are not equal to the first f blocks of $\mathbf{C}$; or, $\mathbf{C}$ has a fixpoint B_f, and $\mathbf{C}'$ has no fixpoint.

Fact 6.2.3. *At least one term in (N1)-(N3) or in (I3) evaluates to* $\min C'$ *over* $\mathbf{C}'$*.*

Example 6.2.7 ($n = 3$)**.** *Let* $\mathbf{C}$ *be the RDP chain in Example 6.2.4, and let* $\mathbf{C}' \in \mathcal{C}_{S_1^3}$ *be the RDP chain* $\{\{\bot, \dot{x_3}\} < \{x_1\} < \{x_2, \dot{x_2}, \dot{x_1}\} < \{x_3\} < \{\top\}\}$*.*

Then, C and C' have fixpoint, but the lower sets of the fixpoints is not equal. Indeed, $\xi_{\{\perp,\dot{x_2},\dot{x_3}\}}$ evaluates to $\min C'$ *over* $\mathbf{C}'$*,*

$$\begin{aligned}\xi_{\{\perp,\dot{x_2},\dot{x_3}\}}(C') &= \neg((\perp(C') \leftrightarrow \neg x_2(C')) \to \neg(\perp(C') \leftrightarrow \neg x_2(C')))\wedge\\ &\neg((\perp(C') \leftrightarrow \neg x_3(C')) \to \neg(\perp(C') \leftrightarrow \neg x_3(C')))\\ &= \neg((\perp(C') \to \neg\perp(C'))) \wedge \neg((\top(C') \to \neg\top(C')))\\ &= \neg\top(C') \wedge \neg\perp(C')\\ &= \perp(C') \wedge \top(C')\\ &= \perp(C') = \min C'.\end{aligned}$$

The last case is where $\mathbf{C}$ has no fixpoint and $\mathbf{C}'$ has a fixpoint.

Fact 6.2.4. *At least one term in (I1)-(I3) evaluates to* $\min C'$ *over* $\mathbf{C}'$*.*

Example 6.2.8 ($n = 3$)**.** *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be the RDP chains in Example 6.2.4, so that* $\mathbf{C}$ *has a fixpoint and* $\mathbf{D}$ *has no fixpoint. Indeed, ζ''_{x_1}, defined in the second part of Example 6.2.4, evaluates to* $\min C$ *over* $\mathbf{C}$*,*

$$\begin{aligned}\zeta''_{x_1}(C) &= \neg(x_1(C) \to \neg x_1(C))\\ &= \neg\top(C)\\ &= \perp(C) = \min C.\end{aligned}$$

In light of the previous facts, we complete the construction of the RDP minterm t_c, and prove its correctness.

If $c = B_1$, then $t_c = \perp$; otherwise, if $c = B$ and x_j belongs to B, we let

$$t_C = \bigwedge_{i=1}^{f} \xi_{B_i} \wedge \bigwedge_{i=1}^{f-1} \xi'_{B_i} \wedge \bigwedge_{i=1}^{f} \xi''_{B_i} \wedge \bigwedge_{i=f+1}^{k} \zeta_{B_i} \wedge \bigwedge_{i=f+1}^{k-1} \zeta'_{B_i} \wedge \bigwedge_{i=f+1}^{k} \zeta''_{B_i}, \tag{6.4}$$

and

$$t_c = x_j \wedge t_C. \tag{6.5}$$

Proposition 6.2.2. *Let $C \in \mathcal{C}_{S_1^n}$, let $c \in C$, and let $A \in \mathcal{A}_{S_1^n}$ be the smallest maximal antichain such that $A \cap C = c$. Then,*

$$t_c^{\mathbf{F}_n} = A.$$

Proof. By Fact 6.2.1, $t_C(C) = \max C$ hence,

$$t_c^{\mathbf{F}_n} \cap C = t_c(C) = (x_j \wedge t_C)(C) = x_j(C) \wedge t_C(C) = B \wedge B_k = c \wedge \max C = c.$$

Also, let $C' \in \mathcal{C}_{S_1^n}$ be different from C. Then, by either Fact 6.2.3, or Fact 6.2.4, or Fact 6.2.2, $t_C(C')$ evaluates to either $\min C'$ or to the smallest $c' \in C'$ such that $c' \parallel \max C$, and hence $c' \parallel c$, in the augmented forest of S_1^n. In both cases, $t_C(C') \leq x_j(C')$, so that $t_c(C') = t_C(C')$. Summarizing, for each $C' \in \mathcal{C}_{S_1^n}$ different from C, $t_c^{\mathbf{F}_n} \cap C'$ is equal to the smallest $c' \in C'$ such that $c' \parallel c$ in the augmented forest of S_1^n.

□

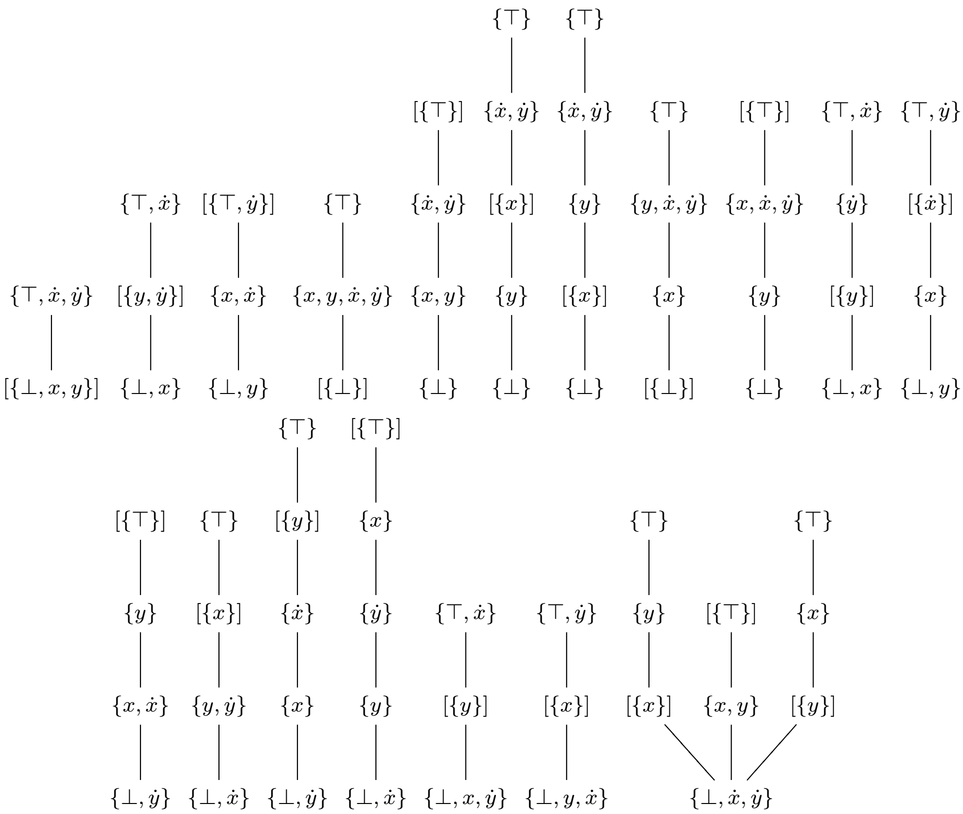


Figure 6.7: Sampling Proposition 6.2.2. The RDP term $t(x,y) = t_{\{\bot,x,y\}} \vee t_{\{y,\dot{y}\}} \vee t_{\{\top,\dot{y}\}} \vee t_{\{\bot\}} \vee t_{\{\top\}} \vee t_{\{x\}} \vee t_{\{x\}} \vee t_{\{\bot\}} \vee t_{\{\top\}} \vee t_{\{y\}} \vee t_{\{\dot{x}\}} \vee t_{\{\top\}} \vee t_{\{x\}} \vee t_{\{y\}} \vee t_{\{\top\}} \vee t_{\{y\}} \vee t_{\{x\}} \vee t_{\{x\}} \vee t_{\{\top\}} \vee t_{\{y\}}$, is such that $t^{\mathbf{F}_2}$ is the maximal antichain highlighted (braced) in the labelled augmented forest S_1^2 in the figure.

6.3 Interpolation Properties

In this section, we prove that RDP logic has the deductive interpolation property, and provide an explicit construction of strongest deductive interpolants.

Let X, Y, and Z be pairwise disjoint sets of variables. Let r and s be RDP terms over $X \cup Z$ and $Y \cup Z$ respectively. The pair $r = x \wedge \neg x$ and $s = y \vee \neg y$ witnesses the failure of *Craig interpolation* in RDP logic, as direct inspection of $\mathbf{F}_2$ in Figure 6.5 shows: indeed, $\vdash_{RDP} r \to s$, but there not exists a ground term t such that $\vdash_{RDP} r \to t$ and $\vdash_{RDP} t \to s$. However, building upon the representation of free finitely generated RDP algebras given in Section 3.3, and the construction of normal forms given in Section 6.2, we now provide a constructive proof that RDP logic enjoys a weaker interpolation property, the *deductive interpolation* property:

If $r \vdash_{RDP} s$, then there exists an RDP term t over the variables Z such that $r \vdash_{RDP} t$ and $t \vdash_{RDP} s$. We describe an explicit construction of the *strongest* deductive interpolant t to r and s in RDP logic, namely, a deductive interpolant t to r and s such that for every deductive interpolant t' to r and s, $t \vdash_{RDP} t'$.

For W a set of variables, we display the free $|W|$-generated RDP algebra $\mathbf{F}_W$ as the RDP algebra of labelled maximal antichains over the augmented forest of S_1^W discussed in the introduction of Section 6.2. If t is an RDP term on W, we let $A_t \in \mathcal{A}_{S_1^W}$ denote the maximal labelled antichain in $\mathbf{F}_W$ corresponding to t, that is, $t^{\mathbf{F}_W} = A_t$. Let $V \subseteq W$. If $B \subseteq \{\bot, \top, x, \neg x \mid x \in W\}$, we let $B|_V = B \setminus \{x, \neg x \mid x \in W \setminus V\}$ denote the V*-structure* of B. Let $D = D_1 < \cdots < D_m \in \mathcal{C}_{S_1^V}$. Then, $C = C_1 < \cdots < C_n \in \mathcal{C}_{S_1^W}$ is said to be V*-equivalent* to D if $C_1|_V < \cdots < C_n|_V$, after eliminating empty blocks, is equal to $D_1 < \cdots < D_m$. Let $A' \in \mathcal{A}_{S_1^V}$. Then, $A \in \mathcal{A}_{S_1^W}$ is said the *cylindrification* of A' over $W \setminus V$ if for all $D \in \mathcal{C}_{S_1^V}$, for all $C \in \mathcal{C}_{S_1^W}$ V-equivalent to D, it holds that $(A \cap C)|_V = A' \cap D$; note that $A' \in \mathcal{A}_{S_1^V}$ guarantees that the right hand side of the equality is nonempty.

Assume $r \vdash_{RDP} s$, or equivalently, $\vdash_{RDP} r^2 \to s$, where r and s are specified as above. Let $W = X \cup Y \cup Z$. Then,

$$A_{r^2} \leq A_s$$

holds in $\mathbf{F}_W$. Let A_t be the *smallest* maximal antichain in $\mathcal{A}_{S_1^Z}$ such that

$$A_{r^2} \leq A_t$$

holds in $\mathbf{F}_W$; here, with slight abuse of notation, $A_t \in \mathcal{A}_{S_1^W}$ denotes the cylindrification of $A_t \in \mathcal{A}_{S_1^Z}$ over $X \cup Y$. We now show that A_t corresponds to the desired interpolant.

Claim 6.3.1. *$A_{t^2} \leq A_s$ in $\mathbf{F}_W$.*

Proof. Suppose for a contradiction that $A_{t^2} \leq A_s$ does not hold in $\mathbf{F}_W$. Then, there exists $C \in \mathcal{C}_{S_1^W}$ such that $A_{t^2} \cap C > A_s \cap C$ over C. By the choice of A_t, $A_t \cap C$ is the smallest point $d \in C$ such that $A_{r^2} \cap C \leq d$ and $d|_Z \neq \emptyset$; in words, d is the smallest point in C lying above $A_{r^2} \cap C$ and having nonempty Z-structure (otherwise, if $d' \in C$ is a point such that $A_{r^2} \cap C \leq d' < d$ and $d'|_Z \neq \emptyset$, the maximal antichain $A_{t'}$ such that $A_{t'} \cap D = d'$ for all maximal chains $D \in \mathcal{C}_{S_1^W}$ that are $X \cup Z$-equivalent to C, and equal to A_t otherwise, would satisfy $A_{r^2} \leq A_{t'} < A_t$, contradicting the minimality of A_t).

Observe that $\min C < A_{r^2} \cap C = A_r \cap C$: Indeed, if $\min C = A_{r^2} \cap C$, then $A_t \cap C = \min C$ (as $\min C$ has nonempty Z-structure, since $\bot \in \min C$); but $A_t \cap C = \min C$ implies $A_{t^2} \cap C = \min C$, contradiction with $A_{t^2} \cap C > A_s \cap C$.

Moreover, $A_{r^2} \cap C < A_r \cap C$ implies $\min C = A_{r^2} \cap C$, again impossible along the above lines.

By the previous observation $A_{r^2} \cap C$ is idempotent, and since $A_{r^2} \cap C \leq A_t \cap C$ by the choice of A_t, we have $A_{t^2} \cap C = A_t \cap C$. The choice of $A_t \cap C$ is such that the right-open interval $\mathcal{I} = [A_{r^2} \cap C, A_{t^2} \cap C)$ in C has no Z-structure, that is, each point in the interval has empty Z-structure. Note that $A_{r^2} \cap C \leq A_s \cap C < A_{t^2} \cap C$ implies that $A_s \cap C$ lies in $\mathcal{I}$; also, by the observation in the previous paragraph, the interval $\mathcal{I}$ lies above the fixpoint of C if such fixpoint exists, or above $\min C$ if such fixpoint does not exists. Say that $\mathcal{I}$ has the form

$$A_{r^2} \cap C = B_1 < \cdots < B_n < A_{t^2} \cap C,$$

with $B_i = X_i \cup Y_i$, where X_i and Y_i denote the X-structure and the Y-structure of B_i respectively, for $i \in [n]$; note that $\bot \notin B_1$ and $\top \notin B_n$, as $\mathcal{I}$ lies above the bottom of C and below $A_{t^2} \cap C \leq \max C$, thus the X-structure and Y-structure of each B_i are disjoint. We know that $A_{r^2} \cap C = B_1$; suppose that $A_s \cap C = B_i$ for some $1 \leq i \leq n$. Let C' be the maximal chain in $\mathcal{C}_{S_1^W}$, obtained by replacing in C the interval $B_1 < \cdots < B_n$ with the interval (for instance)

$$Y_1 < \cdots < Y_i < \cdots < Y_n < X_1 < \cdots < X_n,$$

disregarding empty X_k's and Y_k's; by the above, Y_i and X_1 are nonempty. By construction, C' is $X \cup Z$-equivalent and $Y \cup Z$-equivalent to C. But then, $A_s \cap C' = Y_i < X_1 = A_{r^2} \cap C'$, contradiction with the fact that $A_{r^2} \leq A_s$ holds in $\mathbf{F}_W$, and hence in particular over C'. □

Therefore, $A_{r^2} \leq A_t$ by the choice of A_t, and $A_{t^2} \leq A_s$ by the claim. We use the normal forms construction in Section 6.2 to compute an RDP term over variables in Z that corresponds to A_t; with slight abuse of notation, let t denote such term, that is, $t^{\mathbf{F}_Z} = A_t$. We immediately have $\vdash_{RDP} r^2 \to t$ and $\vdash_{RDP} t^2 \to s$, and by (6.2), $r \vdash_{RDP} t$ and $t \vdash_{RDP} s$. So, t is a deductive interpolant to r and s in RDP logic, in fact the strongest such, by the choice of A_t. Summarizing,

Theorem 6.3.1. *RDP logic has the deductive interpolation property.* [2]

6.4 Unification Type

In this section, we prove that the variety of RDP algebras has unitary unification type. If a given RDP unification instance is solvable, we provide an explicit exponential-time construction of the most general RDP unifier (which is likely to be optimal, since the problem in NP-hard).

[2]Equivalently, RDP algebras enjoy the *injective generalized amalgamation* property [36].

Let T_n denote the RDP algebra of terms over the variables $x_1, \ldots, x_n$. An instance to the RDP unification problem is a term $t \in T_n$, and the question is whether there exists a *unifier* for t, that is, an endomorphism h of T_n such that

$$\vdash_{RDP} h(t).$$

A unifier h for $t \in T_n$ such that $h(x_i) \in \{\bot, \top\}$ for $i \in [n]$ is said *ground.*

Proposition 6.4.1. *Let $t \in T_n$. Then, t is unifiable if and only if t has a ground unifier.*

Proof. Let h be a unifier for t, and let C in $\mathcal{C}_{S_1^n}$ be the labelled maximal chain of the form $\{\bot, x_1, \ldots, x_n\} < \{\top, \neg x_1, \ldots, \neg x_n\}$. Let h' be the endomorphism of T_n such that, for $i \in [n]$,

$$h'(x_i) = \begin{cases} \bot & \text{if } \bot \in (h(x_i))(C), \\ \top & \text{if } \top \in (h(x_i))(C). \end{cases} \tag{6.6}$$

It is easy to check that h' is a ground unifier for t. The converse is trivial. $\square$

Let h and h' be unifiers for t. Then, h' is *less general* than h, in symbols $h' \leq h$, if there exists an endomorphism h'' of T_n such that

$$\vdash_{RDP} h'(x_i) \leftrightarrow h''(h(x_i))$$

for $i \in [n]$. A unifier h for t such that every unifier for t is less general than h is said a *most general* unifier for t.

In the rest of this section, we prove that the type of RDP unification is *unitary*, that is, every unifiable RDP term has a most general unifier. The proof provides an explicit construction of most general unifiers.

An RDP term $t \in T_n$ is said to be *projective* if there exists a unifier h for t such that, for $i \in [n]$,

$$t \vdash_{RDP} x_i \leftrightarrow h(x_i). \tag{6.7}$$

Proposition 6.4.2. *Let $t \in T_n$. If t is projective, then t has a most general unifier.*

Proof. Suppose that t is projective with h witnessing (6.7), and let h' be a unifier for t. It is easy to check that $h' \leq h$. Indeed, by instantiating (6.7) through h', $h'(t) \vdash_{RDP} h'(x_i \leftrightarrow h(x_i))$; as h' commutes over the RDP signature, $h'(t) \vdash_{RDP} h'(x_i) \leftrightarrow h'(h(x_i))$; as $\vdash_{RDP} h'(t)$, we conclude that $\vdash_{RDP} h'(x_i) \leftrightarrow h'(h(x_i))$. Therefore, h is a most general unifier for t. $\square$

The following characterization of projectivity, which parallels the Boolean case, is key to prove that RDP unification is unitary.

Lemma 6.4.1. *Let $t \in T_n$. Then, t is unifiable if and only if t is projective.*

Proof. Suppose that t is unifiable (the other direction is trivial). By Proposition 6.4.1, t has a ground unifier g. We prove that the endomorphism h_t of T_n such that, for $i \in [n]$,

$$h_t(x_i) = (t^2 \to x_i) \odot (\neg t^2 \to g(x_i)) \tag{6.8}$$

is a witnesses of the projectivity of t, and in fact, by Proposition 6.4.2, a most general unifier for t. [3]

Claim 6.4.1. *$\vdash_{RDP} h_t(t)$, that is, $(h_t(t))(C) = \max C$ for every $C \in \mathcal{C}_{S_1^n}$; and $\vdash_{RDP} t^2 \to (x_i \leftrightarrow h_t(x_i))$, that is $t^2(C) \le (x_i \leftrightarrow h_t(x_i))(C)$ for every $C \in \mathcal{C}_{S_1^n}$.*

Proof. Let $C \in \mathcal{C}_{S_1^n}$. We enter a case distinction.

Case 1. Assume $\bot(C) = t(C)$ or $\bot(C) = t^2(C)$. In this case, for $i \in [n]$,

$$\begin{aligned}(h_t(x_i))(C) &= ((t^2 \to x_i) \odot (\neg t^2 \to g(x_i)))(C)\\ &= (\bot(C) \to x_i(C)) \odot (\top(C) \to g(x_i)(C))\\ &= \top(C) \odot g(x_i)(C)\\ &= g(x_i)(C).\end{aligned}$$

Then, $(h_t(t))(C) = t(h_t(x_1), \ldots, h_t(x_n))(C) = t(g(x_1), \ldots, g(x_n))(C) = (g(t))(C) = \max C$, as g is a unifier for t. Clearly, $\bot(C) = t^2(C) \le (x_i \leftrightarrow h_t(x_i))(C)$ for $i \in [n]$.

Case 2. Assume $t(C) = \top(C)$. In this case, for $i \in [n]$,

$$\begin{aligned}(h_t(x_i))(C) &= ((t^2 \to x_i) \odot (\neg t^2 \to g(x_i)))(C)\\ &= (\top(C) \to x_i(C)) \odot (\bot(C) \to g(x_i)(C))\\ &= x_i(C) \odot \top(C)\\ &= x_i(C).\end{aligned}$$

Then, $(h_t(t))(C) = t(h_t(x_1), \ldots, h_t(x_n))(C) = t(x_1, \ldots, x_n)(C) = t(C) = \top(C) = \max C$. Also, $t^2(C) = \top(C) = (x_i \leftrightarrow h_t(x_i))(C)$ for $i \in [n]$.

Case 3. Assume $\bot(C) < t^2(C) = t(C) < \top(C)$. We prove that, for $i \in [n]$,

$$(h_t(x_i))(C) = \begin{cases} x_i(C) & \text{if } x_i(C) < t(C),\\ \top(C) & \text{if } t(C) \le x_i(C).\end{cases} \tag{6.9}$$

[3] This application of (6.8) generalizes previous work of Dzik [20].

Suppose that $\perp(C) \leq x_i(C) < t(C)$. Then,

$$\begin{aligned}
(h_t(x_i))(C) &= ((t^2 \to x_i) \odot (\neg t^2 \to g(x_i)))(C) \\
&= (t(C) \to x_i(C)) \odot (\neg t(C) \to g(x_i)(C)) \\
&= (t(C) \to x_i(C)) \odot (\perp(C) \to g(x_i)(C)) \\
&= x_i(C) \odot \top(C) \\
&= x_i(C).
\end{aligned}$$

Now suppose that $\perp(C) < t(C) \leq x_i(C)$. Then,

$$\begin{aligned}
(h_t(x_i))(C) &= ((t^2 \to x_i) \odot (\neg t^2 \to g(x_i)))(C) \\
&= (t(C) \to x_i(C)) \odot (\neg t(C) \to g(x_i)(C)) \\
&= \top(C) \odot (\perp(C) \to g(x_i)(C)) \\
&= \top(C) \odot \top(C) \\
&= \top(C).
\end{aligned}$$

For the first part, we prove that $(h_t(t))(C) = \max C$. Suppose for a contradiction that $(h_t(t))(C) < \top(C)$. Now, $\perp(C) < t(C) < \top(C)$ implies $t(C) = x_i(C)$ or $t(C) = (\neg x_i)(C)$ for some $i \in [n]$. However, the first case does not occur (if $t(C) = x_i(C)$ for some $i \in [n]$, then $(h_t(t))(C) = (h_t(x_i))(C) = \top(C)$ by the above), therefore $t(C) = (\neg x_i)(C)$ for some $i \in [n]$. But $(\neg x_i)(C) < \top(C)$ implies $\perp(C) = ((\neg x_i)^2)(C)$, contradiction with $\perp(C) < t^2(C)$.

For the second part, we prove that $t^2(C) \leq (x_i \leftrightarrow h_t(x_i))(C)$. By (6.9), we distinguish two cases. Let $i \in [n]$. If $x_i(C) < t(C)$, then $(h_t(x_i))(C) = x_i(C)$ so that $t^2(C) \leq \top(C) = (x_i \leftrightarrow h_t(x_i))(C)$. If $t(C) \leq x_i(C)$, then $(h_t(x_i))(C) = \top(C)$ so that $x_i(C) \leq (x_i \leftrightarrow h_t(x_i))(C)$, and we are done noticing that $t^2(C) = t(C) \leq x_i(C)$.

The claim is settled. □

The lemma is settled. □

Theorem 6.4.1. *RDP unification is unitary.*

Proof. Every RDP term $t \in T_n$ has at most one most general unifier, indeed if t is unifiable, then t has a ground unifier by Proposition 6.4.1, then t is projective by Lemma 6.4.1, and hence, t has a most general unifier by Proposition 6.4.2. □

Note that the complexity of computing the most general unifier h for t via (6.8) is dominated by the complexity of computing the ground unifier g for t. It is easy to check that t has a ground unifier (as an RDP term) if and only if t is satisfiable (as a Boolean term), hence, by Proposition 6.4.1, deciding the RDP unification problem is NP-hard, and in fact, NP-complete: given a ground unifier h for t, it is sufficient to check if the equation $h(t) = \top$ holds.

Concluding Remarks

In this thesis we have investigated two subvarieties of WNM algebras, namely the variety of RDP algebras and the variety of NMG algebras. Starting from the study of subdirectly irreducible members of these varieties we have shown how to build spectral dualities between their corresponding categories and combinatorial categories of enriched forests. From these dualities we have derived useful representation theorems and we have obtained explicit descriptions of coproducts for the finite algebras in our investigated classes. In this light, we have given a combinatorial characterization of the free finitely generated algebras for both varieties. In the case of free RDP algebras, we have exploited this representation to construct normal forms, strongest deductive interpolants and most general unifiers.

Our future aim is to generalize all the techniques presented here to the whole variety of WNM algebras. As shown a combinatorial representation of free WNM algebras is desirable to settle logical properties. Since every WNM algebra is isomorphic to a subdirect product of WNM chains, to sample the general case it make sense to start this investigation studying singly generated WNM chains. Hence, for readability purposes we report in the following figure the canonical WNM chains presented in Section 2.1.

$$
\begin{array}{ccccccccc}
 & \{\top\} & & & & \{\top\} & & & \\
 & | & & & & | & & & \\
 & \{\dot{x}\} & \{\top\} & \{\top\} & & \{\ddot{x}\} & \{\top\} & & \\
 & | & | & | & & | & | & & \\
 & \{\ddot{x}\} & \{\dot{x}\} & \{\dot{x},\ddot{x}\} & \{\top\} & \{x\} & \{x,\ddot{x}\} & \{\top,\ddot{x}\} & \\
 & | & | & | & | & | & | & | & \\
\{\top,\dot{x}\} & \{x\} & \{x,\ddot{x}\} & \{x\} & \{x,\dot{x},\ddot{x}\} & \{\dot{x}\} & \{\dot{x}\} & \{x\} & \{\top,x,\ddot{x}\} \\
| & | & | & | & | & | & | & | & | \\
\{\bot,x,\ddot{x}\} & \{\bot\} & \{\bot\} & \{\bot\} & \{\bot\} & \{\bot\} & \{\bot\} & \{\bot,\dot{x}\} & \{\bot,\dot{x}\} \\
\mathbf{C}_1 & \mathbf{C}_2 & \mathbf{C}_3 & \mathbf{C}_4 & \mathbf{C}_5 & \mathbf{C}_6 & \mathbf{C}_7 & \mathbf{C}_8 & \mathbf{C}_9
\end{array}
$$

Figure 7.1: The nine ways to 1-generate WNM chains.

We recall that to each WNM chain $\{\mathbf{C}_i\}_{i=1}^{9}$ is associated a congruence θ_i and hence a prime filter F_{θ_i}. As detailed in Section 2.2, thanks to these filters we can recover the prime spectrum of the free 1-generated WNM algebra $\mathsf{Spec}(\mathbf{F}_1(WNM))$. In Figure 7.2 we see the prime spectrum of $\mathbf{F}_1(WNM)$ where each node is labelled with the corresponding prime filter (compare with Figure 2.3).

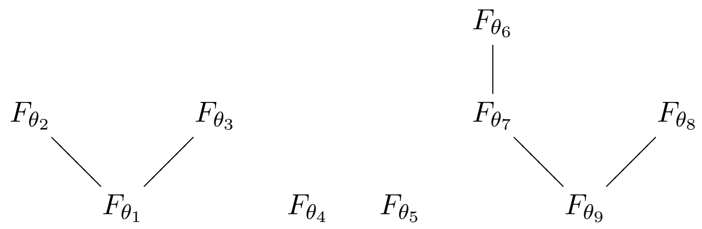


Figure 7.2: The prime spectrum of $\mathbf{F}_1(WNM)$. Each node is labelled with the corresponding prime filter F_{θ_i} for $i \in \{1, \dots, 9\}$.

As shown for the investigated subvarieties, prime spectra of finite WNM algebras are not sufficient to fully describe the primal corresponding algebra. We have faced this problem in two different ways for finite RDP and for finite NMG algebras. In the former case we have enriched the prime spectrum of a directly indecomposable RDP algebra with a chain associated to the negative elements of the algebras (see Section 3.1). In the latter case we have enriched the prime spectrum of a directly indecomposable NMG algebra with labels to distinguish between involutive and weak elements (see Section 4.1). Comparing the prime spectra of the free 1-generated RDP algebra $\mathbf{F}_1(RDP)$ and the free 1-generated NMG algebra $\mathbf{F}_1(NMG)$ obtained in Section 3.3 and Section 4.4 respectively, with the prime spectrum of $\mathbf{F}_1(WNM)$ depicted in the above figure, we can understand how to merge these two different approaches. As shown in Section 3.3 and Section 4.4, $\mathbf{F}_1(RDP)$ is isomorphic to a subdirect product of $\{\mathbf{C}_i \mid i \in \{1, 4, 5, 8, 9\}\}$, and $\mathbf{F}_1(NMG)$ is isomorphic to a subdirect product of $\{\mathbf{C}_j \mid j \in \{1, 3, 5, 7, 8, 9\}\}$. Hence, paralleling Figure 7.2 we report in Figure 7.3 the prime spectra of $\mathbf{F}_1(RDP)$ and $\mathbf{F}_1(NMG)$ labelled with the prime filters associated to each $\mathbf{C}_i$ and $\mathbf{C}_j$ respectively.

Remark 7.1. *Consider the two prime spectra in Figure 7.3. Notice that every component of the prime spectrum of* $\mathbf{F}_1(RDP)$ *is enriched with a chain. Indeed, we have depicted* $\Theta(\mathbf{F}_1(RDP))$*, where* Θ *is the contravariant functor defined in Section 3.1. We can recover the enriched prime spectrum of* $\mathbf{F}_1(NMG)$ *in the sense of Section 4.1, simply labelling the prime filters* F_{θ_i} *of* $\mathsf{Spec}(\mathbf{F}_1(NMG))$ *in Figure 7.3 with the labelling function* (4.10) *(see also Section 4.4), that is:* $\Lambda(F_{\theta_3}) = I$, $\Lambda(F_{\theta_1}) = B$, $\Lambda(F_{\theta_5}) = I$, $\Lambda(F_{\theta_8}) = I$, $\Lambda(F_{\theta_7}) = G$ *and* $\Lambda(F_{\theta_9}) = B$ *(compare with Figure 4.11).*

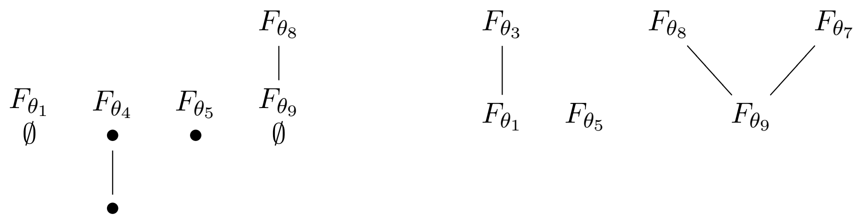


Figure 7.3: The prime spectra $\Theta(\mathbf{F}_1(RDP))$ and $\mathsf{Spec}(\mathbf{F}_1(NMG))$ labelled with the prime filters F_{θ_i} and F_{θ_j}, for $i \in \{1,4,5,8,9\}$ and $j \in \{1,3,5,7,8,9\}$ respectively.

Simply looking at Figure 7.3 we realize that the prime spectra of $\mathbf{F}_1(RDP)$ and $\mathbf{F}_1(NMG)$ are subforests of $\mathsf{Spec}(\mathbf{F}_1(WNM))$. Hence, we can safely enrich $\mathsf{Spec}(\mathbf{F}_1(WNM))$ with the chains associated to $F_{\theta_i} \in \Theta(\mathbf{F}_1(RDP))$ for $i \in \{1,4,5,9\}$, and we can safely label every F_{θ_j} in $\mathsf{Spec}(\mathbf{F}_1(NMG))$ for $j \in \{1,3,4,5,7,8,9\}$ with the labelling function Λ (4.10). The result of this two-step procedure will be an enriched labelled prime spectrum of $\mathbf{F}_1(WNM)$, see the following figure.

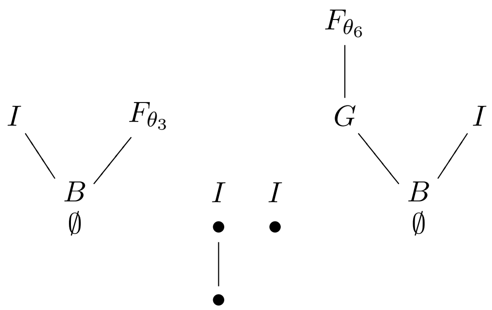


Figure 7.4: An enriched prime spectrum of $\mathbf{F}_1(WNM)$.

The enriched prime spectrum depicted in Figure 7.4 contains redundant information about negative elements of the WNM chains $\mathbf{C}_4$ and $\mathbf{C}_5$ corresponding to prime filters F_{θ_4} and F_{θ_5}. Indeed, the label I associated to F_{θ_5} means that the WNM chain $\mathbf{C}_5$ has a negation fixpoint. Hence, the 1-element chain added to F_{θ_5} does not bring any new information. For the same reasons, we need only a 1-element chain added to F_{θ_4}. Moreover, the labelling function Λ defined for filters of finite NMG algebras it is not defined on filters such as F_{θ_2} and F_{θ_6}. Hence, we have to find a generalization of Λ in order to be able to cope with these cases. Appealing at the definition (4.10) of Λ, we propose the following.

Let $\mathbf{A}$ be a finite WNM algebra and a be a join-irreducible positive element in $\mathbf{A}$. Then, a generates a prime filter F_a and an associated congruence θ_{F_a} (see (1.6)). Recall that $[x]$ is an equivalence class of θ_{F_a} for $x \in \mathbf{A}$. Since

a is join-irreducible, then it covers a unique element $a' \in \mathbf{A}$. The WNM algebra $\mathbf{A}/\theta_{F_a}$ is a WNM chain of equivalence classes, where $a \in [\top]$ and $[a']$ is the coatom. Denote with $\prec$ the covering relation in $\mathbf{A}/\theta_{F_a}$.

Then, for every filter F_a in $\mathsf{Spec}\mathbf{A}$ we define a label:

$$\Lambda'(F_a) = \begin{cases} B & \text{if } a = m_{\mathbf{A}} \text{ and } \mathbf{A}/\theta_{F_a} \text{ does not have a negation fixpoint;} \\ I & \text{if } [a'] \text{ is involutive, or} \\ & \text{if } a = m_{\mathbf{A}} \text{ and } \mathbf{A}/\theta_{F_a} \text{ has a negation fixpoint;} \\ W & \text{if } [b] \prec [a'] \text{ is weak (then } \neg\neg[b] = [a']), \\ U & \text{if } [a'] = \neg[b] \text{ where } [b] \text{ is weak,} \\ G & \text{otherwise.} \end{cases}$$

Denote with $\mathsf{Spec}'(\mathbf{F}_1(WNM))$ the prime spectrum of $\mathbf{F}_1(WNM)$ enriched with the chains associated to the negative elements of $\mathbf{C}_1$, $\mathbf{C}_4$, $\mathbf{C}_5$ and $\mathbf{C}_9$ in the sense of the contravariant functor Θ (see Section 3.1). Applying the labelling function Λ' to $\mathsf{Spec}'(\mathbf{F}_1(WNM))$ we obtain the enriched labelled prime spectrum $\Lambda'(\mathsf{Spec}'(\mathbf{F}_1(WNM)))$ depicted in the following figure.

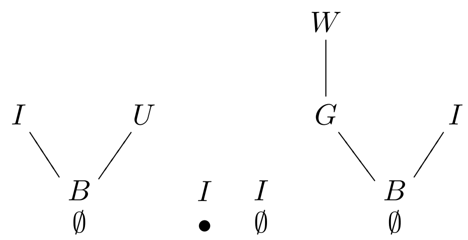


Figure 7.5: The enriched prime spectrum $\Lambda'(\mathbf{Spec}'(\mathbf{F}_1(WNM)))$.

Analyzing the simple case given by the free 1-generated WNM algebra, it is clear that duality and representations for finite WNM algebras are deeply based on the investigations done in this thesis on RDP and NMG algebras. Although the techniques developed in Chapter 3 and Chapter 4 can be safely applied with minor changes to $\mathbf{F}_1(WNM)$ as the above discussion show, for a generalization to the n-generated case and to the whole class of finite WNM algebras additional research has to be done.

We conclude mentioning that a duality will not bring only representation theorems, but it is very useful for the characterization of projective objects. As detailed in [27], projective objects are related to the unification type of the considered logic. As a work in progress in this field, we can mention the identification of the proper subvariety of WNM where the unification is projective. We conjecture that the unification type of the whole variety of WNM algebras is unitary albeit not projective.

Appendix A

Universal Algebra

A.1 Algebraic Structures

Given a set A, we call *n-ary operation* over A a function $f : A^n \rightarrow A$, where A^n is a set of n elements. We call n the *arity* of f. If f has arity 0 then is called *nullary* or *constant.* A nullary operation is a function $c : A^0 \rightarrow A$, where A^0 is the empty set. Hence, c choose a element of A.

The *type* of an algebra is a set of function symbols $\mathcal{F}$, where every symbols $f \in \mathcal{F}$ has an associated integer n that gives the arity of f.

An *algebra* $\mathbf{A}$ is a couple $\langle A, F \rangle$, where F is a set of finitary operations over A such that for every e n-ary $f \in \mathcal{F}$ there is a corresponding n-ary operation $f^{\mathbf{A}}$ over A. The set A is called the *support* (or *universe*) of $\mathbf{A}$. If F is finite we write $\mathbf{A} = \langle A, f_1^{\mathbf{A}}, \ldots, f_k^{\mathbf{A}} \rangle$, where $\{f_1^{\mathbf{A}}, ..., f_k^{\mathbf{A}}\} = \mathcal{F}$. We will drop the superscript when $\mathbf{A}$ is clear from the context.

Let $\mathbf{A} = (A, F)$ and $\mathbf{B} = (B, F)$ be two algebras of the same type $\mathcal{F}$. Then, a function $h : A \rightarrow B$ is an *homomorphism* from $\mathbf{A}$ to $\mathbf{B}$ if for every n-ary function $f \in \mathcal{F}$:

$$h(f^{\mathbf{A}}(a_1, ..., a_n)) = f^{\mathbf{B}}(h(a_1), ..., h(a_n))$$

where $a_1, ..., a_n \in A$.

An homomorphism $h : X \rightarrow Y$ is called:

monomorphism	when h is injective,
epimorphism	when h is surjective,
isomorphism	when h is bijective.

Given two algebras $\mathbf{A}$ and $\mathbf{B}$ we say that $\mathbf{A}$ is *isomorphic* to $\mathbf{B}$, when there exists an isomorphism $h : A \rightarrow B$, in symbols $\mathbf{A} \cong \mathbf{B}$, For short, we denote with $h : \mathbf{A} \rightarrow \mathbf{B}$ an *isomorphism.*

A.2 Subalgebras

Let **A** and **B** be two algebras of the same type $\mathcal{F}$. Then **B** is a *subalgebra* of **A** if $B \subseteq A$ and every operation of **B** is the restriction of the corresponding operation over **A**. That is, for every function symbol $f \in \mathcal{F}$, $f^{\mathbf{B}}$ is the restriction to B of $f^{\mathbf{A}}$. We write $\mathbf{B} \leq \mathbf{A}$ to denote that **B** is a subalgebra of **A**.

We call B *subuniverse* of **A** if $B \subseteq A$ and B is closed with respect to the operation of **A**. That is, if f is an n-ary operation of **A** then $f(a_1, ..., a_n) \in B$ con $a_1, ..., a_n \in B$. Clearly, if **B** is a subalgebra of **A**, then $B$is a subuniverse of **A**.

Given an algebra **A**, we denote with $\mathrm{Sg}(X)$ the subuniverse generated by the set X, that is the set:

$$\mathrm{Sg}(X) = \bigcap\{B \mid X \subseteq B, \text{ with } B \text{ subuniverse of } \mathbf{A}\}$$

for every $X \subseteq A$. Let $X \subseteq A$. we say that X *generates* **A** when $\mathrm{Sg}(X) = A$. For any $X \subseteq A$, we can generate **A** in the following way. Define $E^n(X) = X \cup \{f(a_1, ..., a_n)$, for every n-ary operation $f \in F$ over A and $a_1, ..., a_n \in X\}$, and $E^n(X)$ for $n \geq 0$ by,

$$E^0(X) = X$$
$$E^{n+1}(X) = E(E^n(X))$$

Starting from X, we obtain:

$$\mathrm{Sg}(X) = X \cup E^1(X) \cup E^2(x) \cup ...$$

Let **A** and **B** be two algebras of the same type $\mathcal{F}$. a function $h : A \to B$ is an *embedding* of **A** into **B** if it is a monomorphism. We write $h : \mathbf{A} \to \mathbf{B}$ for h embedding of **A** into **B**. By the definition of homomorphism, $h(A)$ is the subuniverse of **A**. Then, if $h : \mathbf{A} \to \mathbf{B}$ is an embedding, we denote with $h(\mathbf{A})$ the subalgebra of **B** with support $h(A)$.

A.3 Quotient Algebras

Let **A** be an algebra of type $\mathcal{F}$, θ be an equivalence relation over A. We call θ a *congruence* over **A** when, for each n-ary function symbols $f \in \mathcal{F}$ and for each $a_i, b_i \in A$, if $a_i \theta b_i$ holds for $1 \leq i \leq n$ then

$$f^{\mathbf{A}}(a_1, ..., a_n) \theta f^{\mathbf{A}}(b_1, ..., b_n).$$

We denote with $\mathrm{Con}\mathbf{A}$ the set of all congruence on **A**.

Given a congruence θ on **A**, we denote with $\mathbf{A}/\theta$ the algebra whose universe is A/θ and for each n-ary function $f \in \mathcal{F}$ the following holds:

$$f^{\mathbf{A}}(a_1/\theta, ..., a_n/\theta) = f^{\mathbf{A}}(a_1, ..., a_n)/\theta$$

where $a_1, ..., a_n \in A$. We call $\mathbf{A}/\theta$ the *quotient algebra* of $\mathbf{A}$ by θ,

Given an homomorphism $h : \mathbf{A} \to \mathbf{B}$, we define the *kernel* of h:

$$\ker(h) = \{(a, b) \in A^2 : h(a) = h(b)\}$$

Theorem A.3.1. *Let $h : \mathbf{A} \to \mathbf{B}$ be an homomorphism. Then ker(h) is a congruence on $\mathbf{A}$.*

Theorem A.3.2. *Let θ be a congruence on the algebra $\boldsymbol{A}$. The map $\nu_\theta : A \to A/\theta$, defined by $\nu_\theta(a) = a/\theta$ is an epimorphism.*

We call ν the natural homomorphism.

Theorem A.3.3 (First Isomorphism Theorem)**.** *Let $h : \mathbf{A} \to \mathbf{B}$ be an epimorphism. Then, there exists an isomorphism $\beta : \mathbf{A}/ker(h) \to \mathbf{B}$ defined by $h = \beta \circ \nu$, where ν is the natural homomorphism from $\boldsymbol{A}$ to $\boldsymbol{A}/ker(h)$.*

A.4 Varieties of Algebras

Given an epimorphism $h : A \to B$ between two algebras $\mathbf{A}$ and $\mathbf{B}$ of the same type $\mathcal{F}$, we call $\mathbf{B}$ an *homomorphic image* of $\mathbf{A}$.

Let $\alpha : \mathbf{A} \to \mathbf{B}$ be an embedding. Then, $\alpha(A) \in \mathbf{B}$ is an homomorphic image of $\mathbf{A}$.

Hence, every quotient algebra $\mathbf{A}/\theta$ is an homomorphic image of $\mathbf{A}$, viceversa every homomorphic image of $\mathbf{A}$ is isomorphic to $\mathbf{A}/\theta$, for some congruence θ.

Let $(\mathbf{A}_i)_{i\in I}$ be a family of algebras of type $\mathcal{F}$. The *direct product* $\mathbf{A} = \prod_{i\in I} \mathbf{A}_i$ is an algebra with support $\prod_{i\in I} A_i$, such that for every $f \in \mathcal{F}_n$

$$f^{\mathbf{A}}(a_1, ..., a_n)(i) = f^{\mathbf{A}_i}(a_1(i), ..., a_n(i))$$

with $a_1, ..., a_n \in \prod_{i\in I} A_i$ and $i \in I$.

An algebra $\mathbf{A}$ is *directly indecomposable*, if it is not isomorphic to a direct product of two nontrivial algebras.

Theorem A.4.1. *Every finite algebra is isomorphic to a direct product of directly indecomposable algebras.*

An algebra $\mathbf{A}$ is a *subdirect product* of a family $(\mathbf{A}_i)_{i\in I}$ of algebras of the same type, when:

- $\mathbf{A}$ is subalgebra of the direct product $\prod_{i\in I} \mathbf{A}_i$;
- there exists the epimorphism $\pi_i : \mathbf{A} \to \mathbf{A}_i$, for every $i \in I$.

We call *subdirectly irreducible* an algebra $\mathbf{A}$, if for every embedding $h : \mathbf{A} \to \prod_{i\in I} \mathbf{A}_i$, such that $h(\mathbf{A})$ is subdirect product of $\prod_{i\in I} \mathbf{A}_i$, there exists $i \in I$ such that:

$$\pi_i \circ h : \mathbf{A} \to \mathbf{A}_i$$

is an isomorphism.

Theorem A.4.2 (Birkhoff)**.** *Every algebra is isomorphic to a subdirect product of subdirectly irreducible algebras.*

Let K be a family of algebras of the same type. We define the following classes:

$\mathbf{A} \in S(K)$	if and only if	$\mathbf{A}$ is a subalgebra of some member of K
$\mathbf{A} \in H(K)$	if and only if	$\mathbf{A}$ is homomorphic image of some member of K
$\mathbf{A} \in P(K)$	if and only if	$\mathbf{A}$ is direct product of some member of K

A family of algebras of the same type $\mathcal{F}$ is called *variety* if it is closed with respect to subalgebras, homomorphic images and direct products.

Given a class K of algebras of the same type, we denote $\mathbb{V}(K)$ the smallest variety containing K. We call $\mathbb{V}(K)$ the *variety generated by* K.

Theorem A.4.3 (Birkhoff)**.** *If K is a variety, then every member of K is isomorphic to a subdirect product of subdirectly irreducible members of K.*

Thus, a variety is determined by its subdirectly irreducible members.

A.5 Free and Generic Algebras

Given a set of variables V, we define the set of *terms* T as:

- every $x \in V$ is a term,
- if $f_n \in \mathcal{F}$ and $t_1, ..., t_n$ are terms, then $f_n(t_1, ..., t_n)$ is a term.

We write $t(x_1, ..., x_n)$ when the variables occurring in t are among $x_1, ..., x_n$.

Given $\mathcal{F}$ and V, the *terms algebra* $\mathbf{T}(V)$ of type $\mathcal{F}$ over V, is the algebra whose support is T and whose operations satisfy:

$$f^{\mathrm{T}(V)}(t_1, ..., t_n) = f(t_1, ..., t_n) \in T$$

where $f \in \mathcal{F}^n$, $t_i \in T$ for $1 \le i \le n$.

Given $p, q \in T$, we call $p = q$ an *identity* . An algebra $\mathbf{A}$ *satisfies* an identity

$$p(x_1, ..., x_n) = q(x_1, .., x_n)$$

when, for every choice of $a_1, ..., a_n \in A$ we have

$$p^{\mathbf{A}}(a_1, ..., a_n) = q^{\mathbf{A}}(a_1, .., a_n),$$

in symbols $\mathbf{A} \models p = q$. Given a class of algebra K we wrote $K \models p = q$ if $\mathbf{A} \models p = q$ for each $\mathbf{A}$ in K. Given a set Σ of identity of type $\mathcal{F}$, we define $M(\Sigma)$ as the class of algebras that satisfy the identities in Σ. A class of algebras K is an *equational class* if there exists a set Σ such that $K = M(\Sigma)$. We say that K is *axiomatized* by Σ.

Theorem A.5.1 (Birkhoff)**.** *A class of algebras K is an* equational class *if and only if K is a variety.*

Let $\mathbf{A} \in \mathbb{V}(K)$ be an algebra generated by $X = \{x_1, ..., x_n\}$. We call $\mathbf{A}$ a *free algebra* over X in $\mathbb{V}(K)$, if for every $\mathbf{B} \in \mathbb{V}(K)$ and for every $h : X \to B$ there exists a unique homomorphism $f : \mathbf{A} \to \mathbf{B}$ which extends h, that is $h(x_j) = f(x_j)$. The set X is called set of *free generators*. Since the algebra does not depend from X, but only from the cardinality of X, we denote with $\mathbf{F}_n(K)$ the free n-generated algebra in $\mathbb{V}(K)$. We use $\mathbf{F}_n$ when $\mathbb{V}(K)$ is clear from the context.

Given a set of identities Σ, we denote with Θ_Σ the relation in $\mathbf{T}(V)$ such that

$$t_1 \Theta_\Sigma t_2 \text{ if and only if } t_1 = t_2 \text{ holds in } \Sigma.$$

Let K be a variety such that $K = M(\Sigma)$ for some Σ. Then, given X the set of n generators, the algebra $\mathbf{T}(X)/\Theta_\Sigma$ is the free n-generated algebra in the variety K.

An algebra $\mathbf{A}$ is said *generic* for a variety, if it generates the whole variety.

If $\mathbf{A}$ is generic for the variety $\mathbb{V}(K)$, then the free algebra over n generators $\mathbf{F}_n(K)$ is the subalgebra of $\mathbf{A}^{A^n}$ generated by the projections $(x_1, ..., x_n) \mapsto x_i$, where A^n is the set of n-ary function over A.

A class of algebras K is *locally finite* if and only if, for every $\mathbf{A} \in K$ and for every finite set $B \subseteq A$, the subalgebra generated by B is finite. This property is inherited by all subclasses of K.

Appendix B

Category Theory

B.1 Categories and Functors

A *category* C is composed by

- a class of *objects*;
- for any pair of objects A and B in C, a set $\mathsf{C}[A, B]$ of *morphisms* (or *arrows*) from A to B;
- for any triple of objects A, B and C in C, a *composition law* is defined by,

  $$\mathsf{C}[A, B] \times \mathsf{C}[B, C] \longrightarrow \mathsf{C}[A, C]$$

  we will write $g \circ f$ for the composite of the pair of morphisms (f, g);
- for every object A in C, a morphisms $id_A \in \mathsf{C}[A, A]$ is called *identity* on A.

Morphisms respect the following axioms,

$$(f \circ g) \circ h = f \circ (g \circ h),$$
$$id_B \circ h = h \quad id_B \circ g = g,$$

for $h \in \mathsf{C}[A, B]$, $g \in \mathsf{C}[B, C]$ and $f \in \mathsf{C}[C, D]$.

Given a category A, a *subcategory* B of A is composed by:

- a subclass of the objects of A;
- for every pair $A, A' \in \mathsf{A}$, a subset $\mathsf{B}[A, A'] \subseteq \mathsf{A}[A, A']$ such that
  - if $f \in \mathsf{A}[A, A']$ and $g \in \mathsf{A}[A', A'']$ then $g \circ f \in \mathsf{B}[A, A'']$;
  - $1_A \in \mathsf{B}$, for every A in B.

Given a category A, we define its *dual category* A^{op} as the category that has the same objects of A, and for every morphisms $f : A \to B$ in A, there exists a morphism $f^{op} : B \to A$ in A^{op}. The composition law is given by,

$$f^{op} \circ g^{op} = (g \circ f)^{op}$$

A morphism $f : A \to B$ in a category C is called *monomorphism* when f is *left cancellable*, that is for every $C \in \mathsf{C}$ and $h, g \in \mathsf{C}[C, A]$,

$$f \circ g = f \circ h \text{ implies } g = h.$$

The composite of two monomorphisms is a monomorphism. If the composite $g \circ f$ of two morphisms is a monomorphism, then f is a monomorphism.

A morphism $f : A \to B$ in a category C is called *epimorphism* when f is *right cancellable*, that is for every $C \in \mathsf{C}$ and $h, g \in \mathsf{C}[B, C]$,

$$g \circ f = h \circ f \text{ implies } g = h.$$

The composite of two epimorphisms is an epimorphism. If the composite $f \circ g$ of two morphisms is an epimorphism, then f is a epimorphism.

A morphism $f : A \to B$ in a category C is called *isomorphism* when there exists a morphism $g : B \to A$ in C such that,

$$f \circ g = id_B, \quad g \circ f = id_A.$$

The composite of two isomorphisms is an isomorphism. An isomorphism is both a monomorphism and an epimorphism.

A *functor* $\Phi : \mathsf{A} \to \mathsf{B}$ between categories A and B consists of the following,

- a map between the classes of objects of A and B, we write $\Phi(A)$ for the image of $A \in \mathsf{A}$;
- for every pair of objects $A, A' \in \mathsf{A}$, a map

  $$\mathsf{A}[A, A] \longrightarrow \mathsf{B}[\Phi(A), \Phi(A')], \tag{B.1}$$

  we write $\Phi(f)$ for the image of $f \in \mathsf{A}[A, A']$;

A functor Φ respect the following axioms,

$$\Phi(g \circ f) = \Phi(g) \circ \Phi(f), \tag{B.2}$$
$$\Phi(id_A) = id_{\Phi(A)}. \tag{B.3}$$

where $f \in \mathsf{A}[A, A']$ and $g \in \mathsf{A}[A', A'']$. A functor of this type is called *covariant*.

We can obtain another functor Φ between A and B, substituing B.1 with

$$\mathsf{A}[A, A] \longrightarrow \mathsf{B}[\Phi(A'), \Phi(A)],$$

and B.2 with

$$\Phi(g \circ f) = \Phi(f) \circ \Phi(g).$$

This type of functors are called *contravariant*.

Note that a contravariant functor $\Phi : \mathsf{A} \to \mathsf{B}$ is a covariant functor from A^{op} to B.

B.2 Equivalence and Duality

Let $\Phi : \mathsf{A} \to \mathsf{B}$ be a functor between the categories A and B, for every pair of objects $A, A' \in \mathsf{A}$ consider the maps

$$\begin{aligned} \mathsf{A}[A, A] &\longrightarrow \mathsf{B}[\Phi(A), \Phi(A')], \\ f &\mapsto \Phi(f). \end{aligned}$$

If the two above maps are injective for all A, A', then Φ is called *faithful.* If the two above maps are surjective for all A, A', then Φ is called *full.* If for every object $B \in \mathsf{B}$ there exists an object in A whose image is isomorphic to B, then Φ is called *essentially surjective.*

Note that, if B is a subcategory of A, then there exists always a faithful functor from B to A.

Two categories A and B are called *equivalent* provided that there exists faithful, full and essentially surjective functor $\Phi : \mathsf{A} \to \mathsf{B}$. If Φ is contravariant then A and B are called *dually equivalent.*

B.3 Limits and Colimits

Given two objects A and B in a category C, their *product* is an object $A \times B$ in C and two maps $\pi_A : A \times B \to A$ and $\pi_B : A \times B \to B$ such that the following diagram commutes,

$$\begin{tikzcd} & C \arrow[dl, "h_A"'] \arrow[d, dashed, "f"] \arrow[dr, "h_B"] & \\ A & A \times B \arrow[l, "\pi_A"] \arrow[r, "\pi_B"'] & B \end{tikzcd}$$

for every other object $C \in \mathsf{C}$ and morphisms h_A, h_B.

In a category with products the following isomorphisms hold,

$$\begin{aligned} A \times B &\cong B \times A, \\ A \times (B \times C) &\cong (A \times B) \times C. \end{aligned}$$

Given two objects A and B in a category C, their *coproduct* is an object $A+B$ in C and two maps $i_A : A \to A+B$ and $i_B :\to A \times B$ such that the following diagram commutes,

$$\begin{tikzcd}
A \arrow[r, "i_A"'] \arrow[dr, "h_A"'] & A \times B \arrow[d, dashed, "f"'] & B \arrow[l, "i_B"] \arrow[dl, "h_B"] \\
& C &
\end{tikzcd}$$

for every other object $C \in \mathsf{C}$ and morphisms h_A, h_B.

An object $\mathbf{1}$ in C is called *terminal* if for every object A in C, there exists a unique morphism $f : A \to \mathbf{1}$. An object $\mathbf{0}$ in C is called *initial* if for every object A in C, there exists a unique morphism $f : \mathbf{0} \to A$.

Given two morphisms $f, g : A \to B$ in a category C and an object $E \in \mathsf{C}$, a morphism $e : E \to A$ is an *equalizer* of f and g when $f \circ e = g \circ e$, and for every other object $K \in \mathsf{C}$ and morphism $k : K \to A$ such that $f \circ k = g \circ k$, there exists a unique morphism $m : K \to E$ such that $k = e \circ m$.

If e is an equalizer of two morphisms, then e is a monomorphism.

Given two morphisms $f, g : A \to B$ in a category C and an object $E \in \mathsf{C}$, a morphism $q : B \to E$ is an *coequalizer* of f and g when $q \circ f = q \circ g$, and for every other object $K \in \mathsf{C}$ and morphism $k : B \to K$ such that $k \circ f = k \circ g$, there exists a unique morphism $m : E \to K$ such that $k = m \circ q$.

If q is an equalizer of two morphisms, then q is an epimorphism.

Given two morphisms $f : A \to C$ and $g : B \to C$ in a category C, a *pullback* of f and g is composed by,

- an object $P \in \mathsf{C}$;
- two morphisms $f' : P \to B$ and $g' : P \to A$ such that $f \circ g' = g \circ f'$;

and for every other object $K \in \mathsf{C}$ and morphisms $f'' : K \to B$ and $g'' : K \to A$ such that $f \circ g' = g \circ f'$, there exists a unique morphism $k : K \to P$ such that $f'' = f' \circ k$ and $g'' = g' \circ k$. That is, the following diagram commutes,

$$\begin{tikzcd}
K \arrow[dr, dashed, "k"] \arrow[drr, "f''"] \arrow[ddr, "g''"'] & & \\
& P \arrow[r, "f'"'] \arrow[d, "g'"] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

If g is a monomorphism, then g' is a monomorphism. We denote with $A \times_C B$ the object P and we call it the *fibered product* of A and B over C.

Given two morphisms $f : C \to A$ and $g : C \to B$ in a category C, a *pushout* of f and g is composed by,

- an object $P \in \mathsf{C}$;
- two morphisms $f' : B \to P$ and $g' : A \to P$ such that $g' \circ f = f' \circ g$;

and for every other object $K \in \mathsf{C}$ and morphisms $f'' : B \to K$ and $g'' : A \to K$ such that $f'' \circ g = g'' \circ f$, there exists a unique morphism $k : P \to K$ such that $f'' = k \circ f'$ and $g'' = k \circ g'$. That is, the following diagram commutes,

$$\begin{tikzcd}
C \arrow[r, "f"'] \arrow[d, "g"] & A \arrow[d, "g'"'] \arrow[ddr, "g''"] & \\
B \arrow[r, "f'"] \arrow[drr, "f''"'] & P \arrow[dr, dashed, "k"] & \\
 & & K
\end{tikzcd}$$

If f is an epimorphism, then f' is an epimorphism. We denote with $A +_C B$ the object P and we call it the *fibered coproduct* of A and B over C.

The above defined constructions are all special cases of the general notions of *limits* and *colimits*. For the purposes of this thesis it is sufficient this level of generality, we refer the interested reader to standard books as [41] or [39].

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