Asymptotic Stability of Solitons.

D. Bambusi[1]

[1]Dipartimento di Matematica "F.Enriques" - Università degli studi di Milano

Ravello, May 23, 2011

Outline

NLS

- Model problem

$$\mathrm{i}\psi_t = -\Delta\psi - \beta'(|\psi|^2)\psi \ , \quad x \in \mathbb{R}^3 \ ,$$

$$\beta'(0) = 0 \ , \quad \left|\beta^{(k)}(u)\right| \le C_k \langle u \rangle^{\tilde{p}-k} \ , \quad \tilde{p} \le 1 + \frac{2}{d-2} \ , \quad d = 3$$

- Symmetries:
  - translations $(\psi, q_i) \mapsto \psi(. - q_i \mathbf{e}^i)$, generated by $-\mathrm{i}\partial_{x_j}$
  - Gauge $(\psi, q_4) \mapsto e^{\mathrm{i}q_4}\psi$, generated by i.
- Conservation laws:

$$\mathcal{P}^j(\psi) := \int \frac{\psi\partial_j\bar{\psi} - \bar{\psi}\partial_j\psi}{2}$$

$$\mathcal{P}^4(\psi) := \int |\psi|^2$$

NLS

- Model problem

$$\mathrm{i}\psi_t = -\Delta\psi - \beta'(|\psi|^2)\psi \ , \quad x \in \mathbb{R}^3 \ ,$$

$$\beta'(0) = 0 \ , \quad \left|\beta^{(k)}(u)\right| \le C_k \langle u\rangle^{\tilde{p}-k} \ , \quad \tilde{p} \le 1 + \frac{2}{d-2} \ , \quad d = 3$$

- Symmetries:
  - translations $(\psi, q_i) \mapsto \psi(. - q_i \mathbf{e}^i)$, generated by $-\mathrm{i}\partial_{x_j}$
  - Gauge $(\psi, q_4) \mapsto e^{\mathrm{i}q_4}\psi$, generated by i.
- Conservation laws:

$$\mathcal{P}^j(\psi) := \int \frac{\psi \partial_j \bar{\psi} - \bar{\psi}\partial_j \psi}{2}$$

$$\mathcal{P}^4(\psi) := \int |\psi|^2$$

NLS

- Model problem

$$\mathrm{i}\psi_t = -\Delta\psi - \beta'(|\psi|^2)\psi \ , \quad x \in \mathbb{R}^3 \ ,$$

$$\beta'(0) = 0 \ , \quad \left|\beta^{(k)}(u)\right| \le C_k \langle u\rangle^{\tilde{p}-k} \ , \quad \tilde{p} \le 1 + \frac{2}{d-2} \ , \quad d = 3$$

- Symmetries:
  - translations $(\psi, q_i) \mapsto \psi(. - q_i \mathbf{e}^i)$, generated by $-\mathrm{i}\partial_{x_j}$
  - Gauge $(\psi, q_4) \mapsto e^{\mathrm{i}q_4}\psi$, generated by i.
- Conservation laws:

$$\mathcal{P}^j(\psi) := \int \frac{\psi\partial_j\bar{\psi} - \bar{\psi}\partial_j\psi}{2}$$

$$\mathcal{P}^4(\psi) := \int |\psi|^2$$

Ground States

- Look for special solutions

$$\psi(x,t) = e^{-\mathrm{i}y_4(t)}\eta_p(x - \mathbf{y}(t)) \ , \quad y_j(t) = \lambda_j t + q_j \ , \quad j = 1, ..., n$$

- η_p is a critical point of

$$H := \int |\nabla\psi|^2 - \beta(|\psi|^2)$$

restricted to

$$\mathcal{S}_p := \{\psi \ : \mathcal{P}^j(\psi) = p^j \ , \ j = 1, ..., 4\}$$

- Ground state. A ground state is the minimum.

Ground States

- Look for special solutions

$$\psi(x,t) = e^{-\mathrm{i}y_4(t)}\eta_p(x - \mathbf{y}(t)) \ , \quad y_j(t) = \lambda_j t + q_j \ , \quad j = 1, \dots, n$$

- η_p is a critical point of

$$H := \int |\nabla\psi|^2 - \beta(|\psi|^2)$$

restricted to

$$\mathcal{S}_p := \left\{\psi \ : \mathcal{P}^j(\psi) = p^j \ , \ j = 1, \dots, 4\right\}$$

- Ground state. A ground state is the minimum.

Ground States

- Look for special solutions

$$\psi(x,t) = e^{-iy_4(t)}\eta_p(x - \mathbf{y}(t)) \ , \quad y_j(t) = \lambda_j t + q_j \ , \quad j = 1, ..., n$$

- η_p is a critical point of

$$H := \int |\nabla\psi|^2 - \beta(|\psi|^2)$$

restricted to

$$\mathcal{S}_p := \left\{\psi \ : \mathcal{P}^j(\psi) = p^j \ , \ j = 1, ..., 4\right\}$$

- Ground state. A ground state is the minimum.

Assumptions

Assumptions

- Linearize: insert

$$\psi(x,t) = e^{-\mathrm{i}y_4(t)}\left[\eta_p(x - \mathbf{y}(t)) + \chi(x - \mathbf{y}(t))\right]$$

  linearize in χ. Denote by

$$\dot{\chi} = L\chi$$

  the linearized equations.
- Linear assumptions
  - $\sigma(L) = \{0, \pm\mathrm{i}\omega_1, ..., \pm\mathrm{i}\omega_K\} \bigcup \pm\mathrm{i}[\Omega, +\infty)$. Minimal multilicity of 0, namely 8.
    $\omega_1 \leq \omega_2, ..., \leq \omega_K < \Omega$
  - For some large enough r, one has $\omega \cdot k \neq \Omega$, $\forall k \in \mathbb{Z}^K$, $|k| \leq r$
- Fermi golden rule (probably generic: work in progress)
- Initial datum $\inf_{p,q} \left\| \psi_0 - e^{-\mathrm{i}q_4}\eta_p(. - q_i\mathbf{e}^i) \right\|_{H^1} \ll 1$

Assumptions

Assumptions

- Linearize: insert

$$\psi(x,t) = e^{-\mathrm{i}y_4(t)}\left[\eta_p(x - \mathbf{y}(t)) + \chi(x - \mathbf{y}(t))\right]$$

  linearize in χ. Denote by

$$\dot{\chi} = L\chi$$

  the linearized equations.

- Linear assumptions
  - $\sigma(L) = \{0, \pm\mathrm{i}\omega_1, ..., \pm\mathrm{i}\omega_K\} \bigcup \pm\mathrm{i}[\Omega, +\infty)$. Minimal multilicity of 0, namely 8.
    $\omega_1 \leq \omega_2, ..., \leq \omega_K < \Omega$
  - For some large enough r, one has $\omega \cdot k \neq \Omega$, $\forall k \in \mathbb{Z}^K$, $|k| \leq r$
- Fermi golden rule (probably generic: work in progress)
- Initial datum $\inf_{p,q} \left\| \psi_0 - e^{-\mathrm{i}q_4}\eta_p(. - q_i\mathbf{e}^i) \right\|_{H^1} \ll 1$

Assumptions

Assumptions

- Linearize: insert

$$\psi(x,t) = e^{-\mathrm{i}y_4(t)}\left[\eta_p(x-\mathbf{y}(t)) + \chi(x-\mathbf{y}(t))\right]$$

  linearize in χ. Denote by

$$\dot{\chi} = L\chi$$

  the linearized equations.
- Linear assumptions
  - $\sigma(L) = \{0, \pm\mathrm{i}\omega_1, ..., \pm\mathrm{i}\omega_K\} \bigcup \pm\mathrm{i}[\Omega, +\infty)$. Minimal multilicity of 0, namely 8.
    $\omega_1 \leq \omega_2, ..., \leq \omega_K < \Omega$
  - For some large enough r, one has $\omega \cdot k \neq \Omega$, $\forall k \in \mathbb{Z}^K$, $|k| \leq r$
- Fermi golden rule (probably generic: work in progress)
- Initial datum $\inf_{p,q} \left\| \psi_0 - e^{-\mathrm{i}q_4}\eta_p(. - q_i\mathbf{e}^i) \right\|_{H^1} \ll 1$

Assumptions

Assumptions

- Linearize: insert

$$\psi(x,t) = e^{-\mathrm{i}y_4(t)}\left[\eta_p(x-\mathbf{y}(t)) + \chi(x-\mathbf{y}(t))\right]$$

  linearize in χ. Denote by

$$\dot{\chi} = L\chi$$

  the linearized equations.

- Linear assumptions
  - $\sigma(L) = \{0, \pm\mathrm{i}\omega_1, ..., \pm\mathrm{i}\omega_K\} \bigcup \pm\mathrm{i}[\Omega, +\infty)$. Minimal multilicity of 0, namely 8.
    $\omega_1 \leq \omega_2, ..., \leq \omega_K < \Omega$
  - For some large enough r, one has $\omega \cdot k \neq \Omega$, $\forall k \in \mathbb{Z}^K$, $|k| \leq r$
- Fermi golden rule (probably generic: work in progress)
- Initial datum $\inf_{p,q} \left\| \psi_0 - e^{-\mathrm{i}q_4}\eta_p(. - q_i\mathbf{e}^i) \right\|_{H^1} \ll 1$

Assumptions

Assumptions

- Linearize: insert

$$\psi(x,t) = e^{-\mathrm{i}y_4(t)}\left[\eta_p(x-\mathbf{y}(t)) + \chi(x-\mathbf{y}(t))\right]$$

  linearize in χ. Denote by

$$\dot{\chi} = L\chi$$

  the linearized equations.
- Linear assumptions
  - $\sigma(L) = \{0, \pm\mathrm{i}\omega_1, ..., \pm\mathrm{i}\omega_K\} \bigcup \pm\mathrm{i}[\Omega, +\infty)$. Minimal multilicity of 0, namely 8.
    $\omega_1 \leq \omega_2, ..., \leq \omega_K < \Omega$
  - For some large enough r, one has $\omega \cdot k \neq \Omega$, $\forall k \in \mathbb{Z}^K$, $|k| \leq r$
- Fermi golden rule (probably generic: work in progress)
- Initial datum $\inf_{p,q} \left\| \psi_0 - e^{-\mathrm{i}q_4}\eta_p(. - q_i\mathbf{e}^i) \right\|_{H^1} \ll 1$

Theorem

Theorem

There exist functions $p(t)$ $q(t)$ $y(t)$ and a state ψ_∞ such that, one can decompose the solution as

$$\psi(x,t) = e^{-\mathrm{i}q_4(t))}\eta_{p(t)}(x - \mathbf{q}(t)) + e^{-\mathrm{i}y_4(t))}\chi(x - \mathbf{y}(t), t) \ ,$$

one has

$$\lim_{t\to+\infty} \left\| \chi(t) - e^{\mathrm{i}t\Delta}\psi_\infty \right\|_{H^1} = 0$$

The following limits exist

$$\lim_{t\to\infty} \dot{y}(t) \ , \quad \lim_{t\to\infty} \dot{q}(t)$$

Comments

- What's new? Old results when the Floquet spectrum has at most 1 eigenvalue: Weinstein, Soffer-Weinstein, Buslaev-Perelman, Cuccagna, Perelman.
- Ideas from D.B.-Cuccagna (on Klein Gordon), Cuccagna (case with potential), Perelman (no eigenvalues, energy space).
- Key difficulty: the generators of the symmetries are unbounded.
  - Development of reduction theory, Darboux theory and Normal form theory with only continuous transformations.
  - Validity of Strichartz estimates for the relevant operators.

Comments

- What's new? Old results when the Floquet spectrum has at most 1 eigenvalue: Weinstein, Soffer-Weinstein, Buslaev-Perelman, Cuccagna, Perelman.
- Ideas from D.B.-Cuccagna (on Klein Gordon), Cuccagna (case with potential), Perelman (no eigenvalues, energy space).
- Key difficulty: the generators of the symmetries are unbounded.
  - Development of reduction theory, Darboux theory and Normal form theory with only continuous transformations.
  - Validity of Strichartz estimates for the relevant operators.

Comments

- What's new? Old results when the Floquet spectrum has at most 1 eigenvalue: Weinstein, Soffer-Weinstein, Buslaev-Perelman, Cuccagna, Perelman.
- Ideas from D.B.-Cuccagna (on Klein Gordon), Cuccagna (case with potential), Perelman (no eigenvalues, energy space).
- Key difficulty: the generators of the symmetries are unbounded.
  - Development of reduction theory, Darboux theory and Normal form theory with only continuous transformations.
  - Validity of Strichartz estimates for the relevant operators.

Outline

Model probelm

- The equation: $i\psi_t = -\Delta\psi + V\psi + \frac{\delta H_P}{\delta\bar{\psi}(x)}$.
  $\mathcal{H}_0 := -\Delta + V$ with one eigenvalue $\mathcal{H}_0\mathbf{e} = \omega\mathbf{e}$
- Spectral coordinates $\psi = \xi\mathbf{e} + f$:
  $H_0 = \langle\bar{f}; Bf\rangle + \omega|\xi|^2$
- Model nonlinearity

$$H_P := \bar{\xi}^\nu\langle\bar{\Phi}; f\rangle + \xi^\nu\langle\Phi; \bar{f}\rangle$$

- Equations

$$\dot{\xi} = -\mathrm{i}\omega\xi - \mathrm{i}\nu\bar{\xi}^\nu\langle\bar{\Phi}; f\rangle$$
$$\dot{f} = -\mathrm{i}(Bf + \xi^\nu\Phi)$$

- Further decoupling $g = f + \xi^\nu\Psi$: if Ψ is such that

$$(B - \nu\omega)\Psi = \Phi$$

then $\dot{g} = \mathrm{i}Bg + O(|\xi|^\nu|f| + |\xi|^{2\nu-1})$

Model probelm

- The equation: $i\psi_t = -\Delta\psi + V\psi + \frac{\delta H_P}{\delta\bar{\psi}(x)}$.
  $\mathcal{H}_0 := -\Delta + V$ with one eigenvalue $\mathcal{H}_0\mathbf{e} = \omega\mathbf{e}$
- Spectral coordinates $\psi = \xi\mathbf{e} + f$:
  $H_0 = \langle\bar{f}; Bf\rangle + \omega|\xi|^2$
- Model nonlinearity

$$H_P := \bar{\xi}^\nu\langle\bar{\Phi}; f\rangle + \xi^\nu\langle\Phi; \bar{f}\rangle$$

- Equations

$$\dot{\xi} = -\mathrm{i}\omega\xi - \mathrm{i}\nu\bar{\xi}^\nu\langle\bar{\Phi}; f\rangle$$
$$\dot{f} = -\mathrm{i}(Bf + \xi^\nu\Phi)$$

- Further decoupling $g = f + \xi^\nu\Psi$: if Ψ is such that

$$(B - \nu\omega)\Psi = \Phi$$

then $\dot{g} = \mathrm{i}Bg + O(|\xi|^\nu|f| + |\xi|^{2\nu-1})$

Model probelm

- The equation: $i\psi_t = -\Delta\psi + V\psi + \frac{\delta H_P}{\delta\bar{\psi}(x)}$.
  $\mathcal{H}_0 := -\Delta + V$ with one eigenvalue $\mathcal{H}_0\mathbf{e} = \omega\mathbf{e}$
- Spectral coordinates $\psi = \xi\mathbf{e} + f$:
  $H_0 = \langle\bar{f}; Bf\rangle + \omega|\xi|^2$
- Model nonlinearity

$$H_P := \bar{\xi}^\nu\langle\bar{\Phi}; f\rangle + \xi^\nu\langle\Phi; \bar{f}\rangle$$

- Equations

$$\dot{\xi} = -\mathrm{i}\omega\xi - \mathrm{i}\nu\bar{\xi}^\nu\langle\bar{\Phi}; f\rangle$$
$$\dot{f} = -\mathrm{i}(Bf + \xi^\nu\Phi)$$

- Further decoupling $g = f + \xi^\nu\Psi$: if Ψ is such that

$$(B - \nu\omega)\Psi = \Phi$$

  then $\dot{g} = \mathrm{i}Bg + O(|\xi|^\nu|f| + |\xi|^{2\nu-1})$

Dissipation

- Define

$$R_\nu^{\mp} := \lim_{\epsilon \to 0^+} (B - \omega\nu \pm i\epsilon)^{-1} \quad , \quad \Psi = R_\nu^- \Phi$$

- then $\Psi(x) \sim \langle x \rangle^{-1}$. Plug in the equation for ξ:

$$\dot{\xi} = -\mathrm{i}\omega\xi - \mathrm{i}\,|\xi|^{2\nu-1} \langle \bar{\Phi}; R_\nu^+ \Phi \rangle \xi + O(\xi^\nu |\langle \Phi; g \rangle|)$$

- Regularization of the resolvent

$$R_\nu^- \equiv (B - \nu\omega - i0)^{-1} = PV(B - \nu\omega) - i\pi\delta(B - \nu\omega) \ .$$

implies $\langle \bar{\Phi}; R_\nu^- \Phi \rangle = a - \mathrm{i}b$, $b \geq 0$

- $\dot{\xi} = -\mathrm{i}\omega\xi - \mathrm{i}a|\xi|^{2\nu-1}\xi - b|\xi|^{2\nu-1}\xi +$h.o.t.
- $\frac{d}{dt}|\xi|^2 = -2b|\xi|^{2\nu} \quad \Longrightarrow |\xi|^\nu \in L_t^2$
- Use normal form to reduce to the model problem.

Dissipation

- Define

$$R_\nu^{\mp} := \lim_{\epsilon \to 0^+} (B - \omega\nu \pm i\epsilon)^{-1} \ , \quad \Psi = R_\nu^- \Phi$$

- then $\Psi(x) \sim \langle x \rangle^{-1}$. Plug in the equation for ξ:

$$\dot{\xi} = -\mathrm{i}\omega\xi - \mathrm{i}\left|\xi\right|^{2\nu-1} \langle \bar{\Phi}; R_\nu^+ \Phi \rangle \xi + O(\xi^\nu |\langle \Phi; g \rangle|)$$

- Regularization of the resolvent

$$R_\nu^- \equiv (B - \nu\omega - i0)^{-1} = PV(B - \nu\omega) - i\pi\delta(B - \nu\omega) \ .$$

  implies $\langle \bar{\Phi}; R_\nu^- \Phi \rangle = a - \mathrm{i}b, \ b \geq 0$

- $\dot{\xi} = -\mathrm{i}\omega\xi - \mathrm{i}a|\xi|^{2\nu-1}\xi - b|\xi|^{2\nu-1}\xi +$h.o.t.
- $\frac{d}{dt}|\xi|^2 = -2b|\xi|^{2\nu} \quad \Longrightarrow |\xi|^\nu \in L^2_t$
- Use normal form to reduce to the model problem.

Dissipation

- Define

$$R_\nu^{\mp} := \lim_{\epsilon \to 0^+} (B - \omega\nu \pm i\epsilon)^{-1} \ , \quad \Psi = R_\nu^- \Phi$$

- then $\Psi(x) \sim \langle x \rangle^{-1}$. Plug in the equation for ξ:

$$\dot{\xi} = -\mathrm{i}\omega\xi - \mathrm{i}\,|\xi|^{2\nu-1} \langle \bar{\Phi}; R_\nu^+ \Phi \rangle \xi + O(\xi^\nu |\langle \Phi; g \rangle|)$$

- Regularization of the resolvent

$$R_\nu^- \equiv (B - \nu\omega - i0)^{-1} = PV(B - \nu\omega) - i\pi\delta(B - \nu\omega) \ .$$

  implies $\langle \bar{\Phi}; R_\nu^- \Phi \rangle = a - \mathrm{i}b, \ b \geq 0$
- $\dot{\xi} = -\mathrm{i}\omega\xi - \mathrm{i}a|\xi|^{2\nu-1}\xi - b|\xi|^{2\nu-1}\xi +$h.o.t.
- $\dfrac{d}{dt}|\xi|^2 = -2b|\xi|^{2\nu} \quad \Longrightarrow |\xi|^\nu \in L_t^2$
- Use normal form to reduce to the model problem.

Outline

Marsden-Weinstein reduction - generel facts

Connection between the two problems: Marsden-Weinstein reduction.

- Symplectic manifold: $(\mathcal{M}, \omega)$, J Poisson tensor
- Symmetry group: $(q, u) \mapsto e^{qJA}u$, generated by

$$\mathcal{P}(u) := \frac{1}{2}\langle u; Au\rangle \ ,$$

- Invariant Hamiltonian: H, s.t. $H(u) = H(e^{qJA}u)$,
- Reduced system: $\mathcal{S}_p := \{u \in \mathcal{M} \ : \ \mathcal{P}(u) = p\}$ and

$$\mathcal{M}_p := \mathcal{S}_p / \simeq \ , \quad \left(u \sim u' \iff u' = e^{qJA}u\right)$$

- Explicit construction: see the blackboard!
- $\Omega := i^*\omega$, and $H_r := i^*H = H \circ i$.
- Ground state, η_p: minimum of H_r!

Marsden-Weinstein reduction - generel facts

Connection between the two problems: Marsden-Weinstein reduction.

- Symplectic manifold: $(\mathcal{M}, \omega)$, J Poisson tensor
- Symmetry group: $(q, u) \mapsto e^{qJA}u$, generated by

$$\mathcal{P}(u) := \frac{1}{2}\langle u; Au\rangle \ ,$$

- Invariant Hamiltonian: H, s.t. $H(u) = H(e^{qJA}u)$,
- Reduced system: $\mathcal{S}_p := \{u \in \mathcal{M} \ : \ \mathcal{P}(u) = p\}$ and

$$\mathcal{M}_p := \mathcal{S}_p / \simeq \ , \quad \left(u \sim u' \iff u' = e^{qJA}u\right)$$

- Explicit construction: see the blackboard!
- $\Omega := i^*\omega$, and $H_r := i^*H = H \circ i$.
- Ground state, η_p: minimum of H_r!

Marsden-Weinstein reduction - generel facts

Connection between the two problems: Marsden-Weinstein reduction.

- Symplectic manifold: $(\mathcal{M}, \omega)$, J Poisson tensor
- Symmetry group: $(q, u) \mapsto e^{qJA}u$, generated by

$$\mathcal{P}(u) := \frac{1}{2}\langle u; Au\rangle \ ,$$

- Invariant Hamiltonian: H, s.t. $H(u) = H(e^{qJA}u)$,
- Reduced system: $\mathcal{S}_p := \{u \in \mathcal{M} \ : \ \mathcal{P}(u) = p\}$ and

$$\mathcal{M}_p := \mathcal{S}_p / \simeq \ , \quad \left(u \sim u' \iff u' = e^{qJA}u\right)$$

- Explicit construction: see the blackboard!
- $\Omega := i^*\omega$, and $H_r := i^*H = H \circ i$.
- Ground state, η_p: minimum of H_r!

Difficulty

Only continuous group actions:
$u(.) \mapsto u(. - q)$, generated by $\partial_t u = -\partial_x u$.

(1) Does reduction theory holds, and in particular
- Is the reduced manifold a manifold?
- Can one define the reduced system?

(2) Canonical coordinates are needed: is it possible to prove Darboux theorem?

(3) develop transformation theory with unbounded generators

(4) Dispersive estimates: do Strichartz estimates persist under unbounded perturbation?

Outline

Framework

- Phase space:
  - H^k, $k \in \mathbb{Z}$ Scale of Hilbert spaces, $\langle .; . \rangle$ scalar prod of H^0
  - $J : H^k \mapsto H^{k-d}$ Poisson tensor
  - $E := J^{-1}$, $\omega(U_1, U_2) := \langle EU_1; U_2 \rangle$
  - $X_H := J\nabla H$ Hamiltonian vector field
- The system: $H := \mathcal{P}^0(\phi) + H_P(u)$; $\mathcal{P}^0(\phi) := \frac{1}{2}\langle A^0 u; u \rangle$,
- Symmetries: $\mathcal{P}^j(u) := \frac{1}{2}\langle A^j u; u \rangle$, $j = 1, ..., n$;
- Linear assumptions $A^\mu : H^k \to H^{k-d_\mu}$, $d_0 \geq d_j$, $j = 1, ..., n$ JA^μ generate a flow: $e^{tJA^\mu} : H^\infty \mapsto H^\infty$, $\mu = 0, ..., n$.
- Nonlinear assumption $\exists k_0$: $H_P \in C^\infty(H^{k_0}, \mathbb{R})$

Framework

- Phase space:
  - H^k, $k \in \mathbb{Z}$ Scale of Hilbert spaces, $\langle .; . \rangle$ scalar prod of H^0
  - $J : H^k \mapsto H^{k-d}$ Poisson tensor
  - $E := J^{-1}$, $\omega(U_1, U_2) := \langle EU_1; U_2 \rangle$
  - $X_H := J\nabla H$ Hamiltonian vector field
- The system: $H := \mathcal{P}^0(\phi) + H_P(u)$; $\mathcal{P}^0(\phi) := \frac{1}{2}\left\langle A^0 u; u \right\rangle$,
- Symmetries: $\mathcal{P}^j(u) := \frac{1}{2}\left\langle A^j u; u \right\rangle$, $j = 1, ..., n$;
- Linear assumptions $A^\mu : H^k \to H^{k-d_\mu}$, $d_0 \geq d_j$, $j = 1, ..., n$ JA^μ generate a flow: $e^{tJA^\mu} : H^\infty \mapsto H^\infty$, $\mu = 0, ..., n$.
- Nonlinear assumption $\exists k_0$: $H_P \in C^\infty(H^{k_0}, \mathbb{R})$

Framework

- Phase space:
  - H^k, $k \in \mathbb{Z}$ Scale of Hilbert spaces, $\langle .; . \rangle$ scalar prod of H^0
  - $J : H^k \mapsto H^{k-d}$ Poisson tensor
  - $E := J^{-1}$, $\omega(U_1, U_2) := \langle EU_1; U_2 \rangle$
  - $X_H := J\nabla H$ Hamiltonian vector field
- The system: $H := \mathcal{P}^0(\phi) + H_P(u)$;
  $\mathcal{P}^0(\phi) := \frac{1}{2}\left\langle A^0 u; u \right\rangle$,
- Symmetries: $\mathcal{P}^j(u) := \frac{1}{2}\left\langle A^j u; u \right\rangle$, $j = 1, ..., n$;
- Linear assumptions $A^\mu : H^k \to H^{k-d_\mu}$, $d_0 \geq d_j$, $j = 1, ..., n$
  JA^μ generate a flow: $e^{tJA^\mu} : H^\infty \mapsto H^\infty$, $\mu = 0, ..., n$.
- Nonlinear assumption $\exists k_0$: $H_P \in C^\infty(H^{k_0}, \mathbb{R})$

Framework

- Phase space:
  - H^k, $k \in \mathbb{Z}$ Scale of Hilbert spaces, $\langle .; . \rangle$ scalar prod of H^0
  - $J : H^k \mapsto H^{k-d}$ Poisson tensor
  - $E := J^{-1}$, $\omega(U_1, U_2) := \langle EU_1; U_2 \rangle$
  - $X_H := J\nabla H$ Hamiltonian vector field
- The system: $H := \mathcal{P}^0(\phi) + H_P(u)$;
  $\mathcal{P}^0(\phi) := \frac{1}{2}\left\langle A^0 u; u \right\rangle$,
- Symmetries: $\mathcal{P}^j(u) := \frac{1}{2}\left\langle A^j u; u \right\rangle$, $j = 1, ..., n$;
- Linear assumptions $A^\mu : H^k \to H^{k-d_\mu}$, $d_0 \geq d_j$, $j = 1, ..., n$
  JA^μ generate a flow: $e^{tJA^\mu} : H^\infty \mapsto H^\infty$, $\mu = 0, ..., n$.
- Nonlinear assumption $\exists k_0$: $H_P \in C^\infty(H^{k_0}, \mathbb{R})$

Framework

- Phase space:
  - H^k, $k \in \mathbb{Z}$ Scale of Hilbert spaces, $\langle .; . \rangle$ scalar prod of H^0
  - $J : H^k \mapsto H^{k-d}$ Poisson tensor
  - $E := J^{-1}$, $\omega(U_1, U_2) := \langle EU_1; U_2 \rangle$
  - $X_H := J\nabla H$ Hamiltonian vector field
- The system: $H := \mathcal{P}^0(\phi) + H_P(u)$;
  $\mathcal{P}^0(\phi) := \frac{1}{2}\left\langle A^0 u; u \right\rangle$,
- Symmetries: $\mathcal{P}^j(u) := \frac{1}{2}\left\langle A^j u; u \right\rangle$, $j = 1, ..., n$;
- Linear assumptions $A^\mu : H^k \to H^{k-d_\mu}$, $d_0 \geq d_j$, $j = 1, ..., n$
  JA^μ generate a flow: $e^{tJA^\mu} : H^\infty \mapsto H^\infty$, $\mu = 0, ..., n$.
- Nonlinear assumption $\exists k_0$: $H_P \in C^\infty(H^{k_0}, \mathbb{R})$

Ground state and decomposition of the space

- Ground states:

$$A^0\eta_p + \nabla H_P(\eta_p) - \sum_j \lambda_j(p) A^j \eta_p = 0$$

- Assumptions:
  (1) $\mathbb{R}^n \supset I \ni p \mapsto \eta_p \in H^\infty$ is smooth.
  Normalization condition $\mathcal{P}^j(\eta_p) = p^j$.
  (2) $\bigcup_{p\in I} \{\eta_p\}$ is isotropic
- Soliton manifold: $\mathcal{T} := \bigcup_{q,p} e^{\sum q_j JA^j} \eta_p$,
- Natural decomposition: $H^0 \equiv T_{\eta_p} H^0 \simeq T_{\eta_p}\mathcal{T} \oplus T^\omega_{\eta_p}\mathcal{T}$
  with $T^\omega_{\eta_p}\mathcal{T} := \left\{ U \ : \ \omega(U; X) = 0 \ , \ \forall X \in T_{\eta_p}\mathcal{T} \right\}$
  Projection: $H \ni U \mapsto \Pi_p U \in T^\omega_{\eta_p}\mathcal{T}$ with

$$\Pi_p U := U - \langle A_j \eta_p; U \rangle \frac{\partial \eta_p}{\partial p_j} + \langle E \frac{\partial \eta_p}{\partial p_j}; U \rangle JA_j \eta_p \ . \qquad (1)$$

Ground state and decomposition of the space

- Ground states:

$$A^0\eta_p + \nabla H_P(\eta_p) - \sum_j \lambda_j(p) A^j \eta_p = 0$$

- Assumptions:
  (1) $\mathbb{R}^n \supset I \ni p \mapsto \eta_p \in H^\infty$ is smooth.
  Normalization condition $\mathcal{P}^j(\eta_p) = p^j$.
  (2) $\bigcup_{p\in I} \{\eta_p\}$ is isotropic

- Soliton manifold: $\mathcal{T} := \bigcup_{q,p} e^{\sum q_j JA^j}\eta_p$,

- Natural decomposition: $H^0 \equiv T_{\eta_p}H^0 \simeq T_{\eta_p}\mathcal{T} \oplus T^\omega_{\eta_p}\mathcal{T}$
  with $T^\omega_{\eta_p}\mathcal{T} := \left\{ U \ : \ \omega(U;X) = 0 \ , \ \forall X \in T_{\eta_p}\mathcal{T} \right\}$
  Projection: $H \ni U \mapsto \Pi_p U \in T^\omega_{\eta_p}\mathcal{T}$ with

$$\Pi_p U := U - \langle A_j\eta_p; U\rangle \frac{\partial \eta_p}{\partial p_j} + \langle E\frac{\partial \eta_p}{\partial p_j}; U\rangle JA_j\eta_p \ . \qquad (1)$$

Ground state and decomposition of the space

- Ground states:

$$A^0\eta_p + \nabla H_P(\eta_p) - \sum_j \lambda_j(p) A^j \eta_p = 0$$

- Assumptions:
  (1) $\mathbb{R}^n \supset I \ni p \mapsto \eta_p \in H^\infty$ is smooth.
  Normalization condition $\mathcal{P}^j(\eta_p) = p^j$.
  (2) $\bigcup_{p\in I} \{\eta_p\}$ is isotropic
- Soliton manifold: $\mathcal{T} := \bigcup_{q,p} e^{\sum q_j JA^j} \eta_p ,$
- Natural decomposition: $H^0 \equiv T_{\eta_p} H^0 \simeq T_{\eta_p}\mathcal{T} \oplus T^\omega_{\eta_p}\mathcal{T}$
  with $T^\omega_{\eta_p}\mathcal{T} := \{ U \ : \ \omega(U; X) = 0 \ , \ \forall X \in T_{\eta_p}\mathcal{T} \}$
  Projection: $H \ni U \mapsto \Pi_p U \in T^\omega_{\eta_p}\mathcal{T}$ with

$$\Pi_p U := U - \langle A_j \eta_p; U\rangle \frac{\partial \eta_p}{\partial p_j} + \langle E \frac{\partial \eta_p}{\partial p_j}; U\rangle JA_j \eta_p \ . \qquad (1)$$

Ground state and decomposition of the space

- Ground states:

$$A^0\eta_p + \nabla H_P(\eta_p) - \sum_j \lambda_j(p) A^j \eta_p = 0$$

- Assumptions:
  (1) $\mathbb{R}^n \supset I \ni p \mapsto \eta_p \in H^\infty$ is smooth.
  Normalization condition $\mathcal{P}^j(\eta_p) = p^j$.
  (2) $\bigcup_{p\in I} \{\eta_p\}$ is isotropic
- Soliton manifold: $\mathcal{T} := \bigcup_{q,p} e^{\sum q_j JA^j} \eta_p$,
- Natural decomposition: $H^0 \equiv T_{\eta_p} H^0 \simeq T_{\eta_p}\mathcal{T} \oplus T^\omega_{\eta_p}\mathcal{T}$
  with $T^\omega_{\eta_p}\mathcal{T} := \left\{ U \ : \ \omega(U;X) = 0 \ , \ \forall X \in T_{\eta_p}\mathcal{T} \right\}$
  Projection: $H \ni U \mapsto \Pi_p U \in T^\omega_{\eta_p}\mathcal{T}$ with

$$\Pi_p U := U - \langle A_j \eta_p; U\rangle \frac{\partial \eta_p}{\partial p_j} + \langle E \frac{\partial \eta_p}{\partial p_j}; U\rangle JA_j \eta_p \ . \qquad (1)$$

Ground state and decomposition of the space

- Ground states:

$$A^0\eta_p + \nabla H_P(\eta_p) - \sum_j \lambda_j(p) A^j \eta_p = 0$$

- Assumptions:
  (1) $\mathbb{R}^n \supset I \ni p \mapsto \eta_p \in H^\infty$ is smooth.
  Normalization condition $\mathcal{P}^j(\eta_p) = p^j$.
  (2) $\bigcup_{p\in I} \{\eta_p\}$ is isotropic
- Soliton manifold: $\mathcal{T} := \bigcup_{q,p} e^{\sum q_j JA^j} \eta_p$,
- Natural decomposition: $H^0 \equiv T_{\eta_p} H^0 \simeq T_{\eta_p}\mathcal{T} \oplus T^\omega_{\eta_p}\mathcal{T}$
  with $T^\omega_{\eta_p}\mathcal{T} := \left\{U \ : \ \omega(U;X) = 0 \ , \ \forall X \in T_{\eta_p}\mathcal{T}\right\}$
  Projection: $H \ni U \mapsto \Pi_p U \in T^\omega_{\eta_p}\mathcal{T}$ with

$$\Pi_p U := U - \langle A_j \eta_p; U\rangle \frac{\partial \eta_p}{\partial p_j} + \langle E \frac{\partial \eta_p}{\partial p_j}; U\rangle JA_j \eta_p \ . \qquad (1)$$

Explicit construction of the reduced system

Fix $p_0 \in I$.

- Level surface of $\mathcal{P}$: $\mathcal{S}_{p_0} := \left\{ u \ : \mathcal{P}^j(u) = p_0^j \ , \ \forall j \right\} \ni \eta_{p_0}$
- Local model: $\mathcal{V}^k := \Pi_{p_0} H^k \simeq H^k \cap T^{\omega}_{\eta_{p_0}} \mathcal{T}$.
- Construct $\mathcal{V}^k \ni \phi \mapsto p(\phi)$ s.t.

$$u := \eta_{p(\phi)} + \Pi_{p(\phi)} \phi \in \mathcal{S}_{p_0} \ .$$

- Reduced space: The map $\mathcal{V}^k \ni \phi \mapsto i(\phi) := \eta_{p(\phi)} + \Pi_{p(\phi)} \phi$ is a coordinate system for the reduced space.
- Reduced Hamiltonian

$$H_r := i^* H \ , \quad \Omega := i^* \omega \ .$$

- Problem Ω does not vary smoothly.

Explicit construction of the reduced system

Fix $p_0 \in I$.

- Level surface of $\mathcal{P}$: $\mathcal{S}_{p_0} := \left\{ u \ : \mathcal{P}^j(u) = p_0^j \ , \ \forall j \right\} \ni \eta_{p_0}$
- Local model: $\mathcal{V}^k := \Pi_{p_0} H^k \simeq H^k \cap T^{\omega}_{\eta_{p_0}} \mathcal{T}$.
- Construct $\mathcal{V}^k \ni \phi \mapsto p(\phi)$ s.t.

$$u := \eta_{p(\phi)} + \Pi_{p(\phi)} \phi \in \mathcal{S}_{p_0} \ .$$

- Reduced space: The map $\mathcal{V}^k \ni \phi \mapsto i(\phi) := \eta_{p(\phi)} + \Pi_{p(\phi)} \phi$ is a coordinate system for the reduced space.
- Reduced Hamiltonian

$$H_r := i^* H \ , \quad \Omega := i^* \omega \ .$$

- Problem Ω does not vary smoothly.

Explicit construction of the reduced system

Fix $p_0 \in I$.

- Level surface of $\mathcal{P}$: $\mathcal{S}_{p_0} := \left\{ u \ : \mathcal{P}^j(u) = p_0^j \ , \ \forall j \right\} \ni \eta_{p_0}$
- Local model: $\mathcal{V}^k := \Pi_{p_0} H^k \simeq H^k \cap T^\omega_{\eta_{p_0}} \mathcal{T}$.
- Construct $\mathcal{V}^k \ni \phi \mapsto p(\phi)$ s.t.

$$u := \eta_{p(\phi)} + \Pi_{p(\phi)} \phi \in \mathcal{S}_{p_0} \ .$$

- Reduced space: The map $\mathcal{V}^k \ni \phi \mapsto i(\phi) := \eta_{p(\phi)} + \Pi_{p(\phi)} \phi$ is a coordinate system for the reduced space.
- Reduced Hamiltonian

$$H_r := i^* H \ , \quad \Omega := i^* \omega \ .$$

- Problem Ω does not vary smoothly.

Explicit construction of the reduced system

Fix $p_0 \in I$.

- Level surface of $\mathcal{P}$: $\mathcal{S}_{p_0} := \left\{ u \ : \mathcal{P}^j(u) = p_0^j \ , \ \forall j \right\} \ni \eta_{p_0}$
- Local model: $\mathcal{V}^k := \Pi_{p_0} H^k \simeq H^k \cap T^{\omega}_{\eta_{p_0}} \mathcal{T}$.
- Construct $\mathcal{V}^k \ni \phi \mapsto p(\phi)$ s.t.

$$u := \eta_{p(\phi)} + \Pi_{p(\phi)} \phi \in \mathcal{S}_{p_0} \ .$$

- Reduced space: The map $\mathcal{V}^k \ni \phi \mapsto i(\phi) := \eta_{p(\phi)} + \Pi_{p(\phi)} \phi$ is a coordinate system for the reduced space.
- Reduced Hamiltonian

$$H_r := i^* H \ , \quad \Omega := i^* \omega \ .$$

- Problem Ω does not vary smoothly.

Explicit construction of the reduced system

Fix $p_0 \in I$.

- Level surface of $\mathcal{P}$: $\mathcal{S}_{p_0} := \left\{ u \ : \mathcal{P}^j(u) = p_0^j \ , \ \forall j \right\} \ni \eta_{p_0}$
- Local model: $\mathcal{V}^k := \Pi_{p_0} H^k \simeq H^k \cap T^{\omega}_{\eta_{p_0}} \mathcal{T}$.
- Construct $\mathcal{V}^k \ni \phi \mapsto p(\phi)$ s.t.

$$u := \eta_{p(\phi)} + \Pi_{p(\phi)} \phi \in \mathcal{S}_{p_0} \ .$$

- Reduced space: The map $\mathcal{V}^k \ni \phi \mapsto i(\phi) := \eta_{p(\phi)} + \Pi_{p(\phi)} \phi$ is a coordinate system for the reduced space.
- Reduced Hamiltonian

$$H_r := i^* H \ , \quad \Omega := i^* \omega \ .$$

- Problem Ω does not vary smoothly.

Explicit construction of the reduced system

Fix $p_0 \in I$.

- Level surface of $\mathcal{P}$: $\mathcal{S}_{p_0} := \left\{ u \ : \mathcal{P}^j(u) = p_0^j \ , \ \forall j \right\} \ni \eta_{p_0}$
- Local model: $\mathcal{V}^k := \Pi_{p_0} H^k \simeq H^k \cap T^{\omega}_{\eta_{p_0}} \mathcal{T}$.
- Construct $\mathcal{V}^k \ni \phi \mapsto p(\phi)$ s.t.

$$u := \eta_{p(\phi)} + \Pi_{p(\phi)} \phi \in \mathcal{S}_{p_0} \ .$$

- Reduced space: The map $\mathcal{V}^k \ni \phi \mapsto i(\phi) := \eta_{p(\phi)} + \Pi_{p(\phi)} \phi$ is a coordinate system for the reduced space.
- Reduced Hamiltonian

$$H_r := i^* H \ , \quad \Omega := i^* \omega \ .$$

- Problem Ω does not vary smoothly.

Outline

The Darboux theorem

$$\Omega_\phi(\Phi;\Phi) = \langle \mathcal{E}(\phi)\Phi;\Phi\rangle \ , \qquad \mathcal{E}(\phi) = E + O(\phi^2) \ .$$

Darboux theorem

There exists a map of the form

$$\mathcal{F}(\phi) = e^{\sum_j q_j(N,\phi)JA^j}(\phi + S(N,\phi)) \ , \quad N^j := \mathcal{P}^j(\phi') \ , \tag{2}$$

with the following properties

1. $q_i : \mathbb{R}^n \times \mathcal{V}^{-\infty} \to \mathbb{R}$
2. $S : \mathbb{R}^n \times \mathcal{V}^{-\infty} \to \mathcal{V}^{\infty}$ is smoothing.
3. $\mathcal{F}^*\Omega = \Omega_0$.

The function $e^{q_j JA^j}\phi$ is only continuous in q. **Not differentiable**!

The Darboux theorem

$$\Omega_\phi(\Phi;\Phi) = \langle \mathcal{E}(\phi)\Phi;\Phi\rangle \ , \qquad \mathcal{E}(\phi) = E + O(\phi^2) \ .$$

Darboux theorem

There exists a map of the form

$$\mathcal{F}(\phi) = e^{\sum_j q_j(N,\phi) JA^j}(\phi + S(N,\phi)) \ , \quad N^j := \mathcal{P}^j(\phi') \ , \tag{2}$$

with the following properties

1. $q_i : \mathbb{R}^n \times \mathcal{V}^{-\infty} \to \mathbb{R}$
2. $S : \mathbb{R}^n \times \mathcal{V}^{-\infty} \to \mathcal{V}^{\infty}$ is smoothing.
3. $\mathcal{F}^*\Omega = \Omega_0$.

The function $e^{q_j JA^j}\phi$ is only continuous in q. **Not differentiable**!

The Darboux theorem

$$\Omega_\phi(\Phi;\Phi) = \langle \mathcal{E}(\phi)\Phi;\Phi\rangle \ , \qquad \mathcal{E}(\phi) = E + O(\phi^2) \ .$$

Darboux theorem

There exists a map of the form

$$\mathcal{F}(\phi) = e^{\sum_j q_j(N,\phi)JA^j}(\phi + S(N,\phi)) \ , \quad N^j := \mathcal{P}^j(\phi') \ , \qquad (2)$$

with the following properties

1. $q_i : \mathbb{R}^n \times \mathcal{V}^{-\infty} \to \mathbb{R}$
2. $S : \mathbb{R}^n \times \mathcal{V}^{-\infty} \to \mathcal{V}^{\infty}$ is smoothing.
3. $\mathcal{F}^*\Omega = \Omega_0$.

The function $e^{q_j JA^j}\phi$ is only continuous in q. **Not differentiable**!

Almost smooth maps

- Smoothing maps: $S : \mathbb{R}^n \times \mathcal{V}^{-\infty} \mapsto S(N, \phi) \in \mathcal{V}^{\infty}$ or $\mathbb{R}$.
- Almost smooth maps: A map of the form $\mathcal{F}(\phi) = e^{\sum_j q_j(N,\phi)JA^j}(\phi + S(N, \phi))$ with smoothing q_j and S is said to be **almost smooth**.
- Recovering smoothness:

Lemma

If H is symmetric, namely $H(e^{q_j JA^j} u) = H(u)$, then there exists a smoothing $\tilde{S}$:

$$H(\mathcal{F}(\phi)) = H(\phi + \tilde{S}(N, \phi))$$

Almost smooth maps

- Smoothing maps: $S : \mathbb{R}^n \times \mathcal{V}^{-\infty} \mapsto S(N, \phi) \in \mathcal{V}^{\infty}$ or $\mathbb{R}$.
- Almost smooth maps: A map of the form $\mathcal{F}(\phi) = e^{\sum_j q_j(N,\phi) JA^j}(\phi + S(N, \phi))$ with smoothing q_j and S is said to be **almost smooth**.
- Recovering smoothness:

Lemma

If H is symmetric, namely $H(e^{q_j JA^j} u) = H(u)$, then there exists a smoothing $\tilde{S}$:

$$H(\mathcal{F}(\phi)) = H(\phi + \tilde{S}(N, \phi))$$

Almost smooth maps

- Smoothing maps: $S : \mathbb{R}^n \times \mathcal{V}^{-\infty} \mapsto S(N, \phi) \in \mathcal{V}^{\infty}$ or $\mathbb{R}$.
- Almost smooth maps: A map of the form $\mathcal{F}(\phi) = e^{\sum_j q_j(N,\phi) JA^j}(\phi + S(N, \phi))$ with smoothing q_j and S is said to be **almost smooth**.
- Recovering smoothness:

Lemma

If H is symmetric, namely $H(e^{q_j JA^j} u) = H(u)$, then there exists a smoothing $\tilde{S}$:

$$H(\mathcal{F}(\phi)) = H(\phi + \tilde{S}(N, \phi))$$

Almost smooth maps

- Hamiltonian Flows: Let χ be a smoothing Hamiltonian with a zero of second order at $\phi = 0$, then

$$X_\chi = S(N, \phi) + w_j(N, \phi) J A^j \phi$$

with smoothing w_j and S.

Lemma

The corresponding time 1 flow is well defined in a neighbourhood of the origin and is an almost smooth map.

Almost smooth maps

- Hamiltonian Flows: Let χ be a smoothing Hamiltonian with a zero of second order at $\phi = 0$, then

$$X_\chi = S(N,\phi) + w_j(N,\phi) J A^j \phi$$

with smoothing w_j and S.

Lemma

The corresponding time 1 flow is well defined in a neighbourhood of the origin and is an almost smooth map.

Linearization of the Hamiltonian

- The Hamiltonian $H = H_{L0} + H_{L1} + H_N$,

$$H_{L0}(\phi) = \mathcal{P}^0(\phi) + \frac{1}{2} d^2 H_P(\eta_{p_0})(\phi, \phi) - \lambda_j(p_0)\mathcal{P}^j(\phi)$$

$$H_N(\phi) = H_P^3(\eta_{p_0 - N}; \phi + S(N, \phi))$$

- Assumption There exist coordinates such that

$$H_{L0} = \sum_{l=1}^{K} \omega_j |\xi_j|^2 + \langle \bar{f}; Bf \rangle$$

(In NLS it follows from the assumptions on the spectrum of L)

Linearization of the Hamiltonian

- The Hamiltonian $H = H_{L0} + H_{L1} + H_N$,

$$H_{L0}(\phi) = \mathcal{P}^0(\phi) + \frac{1}{2} d^2 H_P(\eta_{p_0})(\phi, \phi) - \lambda_j(p_0)\mathcal{P}^j(\phi)$$

$$H_N(\phi) = H_P^3(\eta_{p_0 - N}; \phi + S(N, \phi))$$

- Assumption There exist coordinates such that

$$H_{L0} = \sum_{l=1}^{K} \omega_j |\xi_j|^2 + \langle \bar{f}; Bf \rangle$$

(In NLS it follows from the assumptions on the spectrum of L)

Linearization of the Hamiltonian

- The Hamiltonian $H = H_{L0} + H_{L1} + H_N$,

$$H_{L0}(\phi) = \mathcal{P}^0(\phi) + \frac{1}{2} d^2 H_P(\eta_{p_0})(\phi, \phi) - \lambda_j(p_0)\mathcal{P}^j(\phi)$$

$$H_N(\phi) = H_P^3(\eta_{p_0 - N}; \phi + S(N, \phi))$$

- Assumption There exist coordinates such that

$$H_{L0} = \sum_{l=1}^{K} \omega_j |\xi_j|^2 + \langle \bar{f}; Bf \rangle$$

(In NLS it follows from the assumptions on the spectrum of L)

Outline

Normal form.

- Definition: Normal form A function $Z(N,\phi)$ is in normal form at order r, if the following holds

$$\{\omega\cdot(\mu-\nu)\neq 0\ \&\ |\mu|+|\nu|\leq r\} \Longrightarrow \frac{\partial^{|\mu|+|\nu|}Z}{\partial\xi^{\mu}\partial^{\nu}\bar{\xi}}(M,0)=0$$

$$\{\omega\cdot(\mu-\nu)<\Omega\ \&\ |\mu|+|\nu|\leq r-1\} \Longrightarrow d_f\frac{\partial^{|\mu|+|\nu|}Z}{\partial\xi^{\mu}\partial^{\nu}\bar{\xi}}(M,0)=0$$

$$\{-\omega\cdot(\mu-\nu)>\Omega\ \&\ |\mu|+|\nu|\leq r-1\} \Longrightarrow d_{\bar{f}}\frac{\partial^{|\mu|+|\nu|}Z}{\partial\xi^{\mu}\partial^{\nu}\bar{\xi}}(M,0)=0$$

Theorem

For any $r\geq 2$ there exists an almost smooth canonical transformation T_r such that $H\circ T_N$ is in normal form at order r.

Normal form.

- Definition: Normal form A function $Z(N,\phi)$ is in normal form at order r, if the following holds

$$\{\omega\cdot(\mu-\nu)\neq 0 \ \&\ |\mu|+|\nu|\leq r\} \Longrightarrow \frac{\partial^{|\mu|+|\nu|}Z}{\partial\xi^{\mu}\partial^{\nu}\bar{\xi}}(M,0)=0$$

$$\{\omega\cdot(\mu-\nu)<\Omega \ \&\ |\mu|+|\nu|\leq r-1\} \Longrightarrow d_f\frac{\partial^{|\mu|+|\nu|}Z}{\partial\xi^{\mu}\partial^{\nu}\bar{\xi}}(M,0)=0$$

$$\{-\omega\cdot(\mu-\nu)>\Omega \ \&\ |\mu|+|\nu|\leq r-1\} \Longrightarrow d_{\bar{f}}\frac{\partial^{|\mu|+|\nu|}Z}{\partial\xi^{\mu}\partial^{\nu}\bar{\xi}}(M,0)=0$$

Theorem

For any $r\geq 2$ there exists an almost smooth canonical transformation T_r such that $H\circ T_N$ is in normal form at order r.

THE END

THANK YOU