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Applicazioni del Formalismo di Superspazio $\mathcal{N} = 2$ in Tre Dimensioni

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Chapter 1.

Introduction

Since the groundbreaking results by Maldacena [1] in the late 90's, high energy theoretical physics has motivated the expansion of 'stringy'-inspired investigation throughout a vast set of all known theoretical physics. The Maldacena conjecture involves a duality between a quantum theory of gravity (String/M-theory) and a quantum field theory in flat space in a strong/weak fashion. As such, studying many different aspects of this duality leads us to novel understanding in quantum gravity aspects such as black holes, holography and M-theory and on the other hand, in quantum field theories, other aspects such as integrability, QCD-like theories, strongly coupled condensed matter physics, among many other examples.

The original and most tested proposal for this conjecture relates Type IIB String Theory in an $AdS_5 \times S^5$ background and four-dimensional $\mathcal{N} = 4$ super Yang-Mills theory. In other words, a UV complete theory of gravity in five dimensions, which in a certain regime can be approximated with super gravity, and the conformal, maximally supersymmetric gauge theory in four dimensions. To arrive to this conjecture, one relates the infrared limit of the worldvolume theory living in a stack of branes ($D3$ in AdS_5/CFT_4 case) with a theory of gravity which has these branes as solutions. After justifying the decoupling limit, the duality between the gauge theory describing the low energy dynamics of the brane and the gravity theory (string, M or pure, as long as it is a full quantum theory of gravity) in the near horizon background is conjectured.

In eleven dimensional supergravity, solitonic solutions with two spatial dimensions (membranes) preserving all or most of the available supersymmetries exist and are called $M2$-branes. The idea of finding a three dimensional conformal gauge field theory (a

Chern-Simons theory) which could be the infrared limit of the theory living in the worldvolume of an $M2$-brane such that it could be conjectured as dual of M-theory in a given background has been for long unfruitfully desired. The works of Bagger, Lambert and Gustavsson [2–4] inspired a lot of research in this direction, since they were able to construct a unique maximally supersymmetric Chern-Simons-matter theory. Unfortunately, this theory has a fixed gauge group which does not admit a planar expansion of the theory and it seemed impossible to extend this theory for at least a one parameter family of groups.

The situation changed in recent times when an $\mathcal{N} = 6$ supersymmetric and conformal Chern-Simons-Matter theory was constructed for a one parameter gauge group $U(N) \times U(N)$ and a two parameter gauge group $U(M) \times U(N)$ called ABJM and ABJ respectively [5, 6]. These theories have a planar perturbative expansion, and their moduli space coincides with that of a stack of M_2-branes proving the singularity of a given orbifold preserving as much supersymmetry as the field theory. From this, a new AdS_4/CFT_3 conjecture was established.

Even if supersymmetry is not a necessary requisite to formulate a gauge/gravity duality, it turns out that most of the finest knowledge of the conjecture aspects in both sides is known in the context of supersymmetric theories. While supersymmetric field theories can be constructed in the usual language of component field theories defined in ordinary space-time, from the mostly theoretical but also practical points of view, the development and application of superspace formulation of supersymmetric theories has led to many general and particular results which would have otherwise been much more difficult if not impossible to achieve. From the proof of conformal invariance of interesting theories such as Beta-deformed Super Yang-Mills theory [7, 8], to the high-loop calculation of wrapping corrections of gauge invariant operators [9], to name a few examples, the amount of results obtained through the different superspace approaches in supersymmetric gauge theories is very vast.

The three dimensional $\mathcal{N} = 6$ Chern-Simons-Matter theory admits a superspace formulation. In particular, a similar formalism to the very well known four dimensional $\mathcal{N} = 1$ superspace formalism exists. This is the three dimensional $\mathcal{N} = 2$ formalism which can be thought as a complexification of the three dimensional $\mathcal{N} = 1$ formalism. In this formalism, only the $\mathcal{N} = 2$ supersymmetry is realized off-shell and more extended supersymmetries may be completely hidden or may be realized as flavor symmetries which do not commute with the supercharges.

There are two main advantages of this formalism over the field theory approach which are immediately observed. Firstly, ultraviolet convergence of the diagrams is improved, and it is possible to derive many non-renormalization theorems for particular or general situations due to this improvement. Secondly, for any given calculation which can be compared to an equivalent field theory calculation, the amount of diagrams which are necessary to be calculated is dramatically reduced.

Compared to higher $\mathcal{N}$ superspace approaches such as $\mathcal{N}=3$ harmonic superspace, the $\mathcal{N}=2$ superspace approach is simpler in the fact that for any diagram, it only involves two steps: algebra and Feynman integrals. The $\mathcal{N}=3$ approach also involves further harmonic integrations making it more difficult and lengthy from this perspective. On the other hand, the $\mathcal{N}=3$ approach has a better UV convergence and the number of diagrams is presumably reduced in any calculation.

One of the advantages of the component approach over the superspace approach is that, due to arguments *a là* Poggio-Quinn [10], since the component approach has only classically marginal couplings, it is known to be infrared safe. In the superspace approach, these arguments are no longer valid and either $d=4$ super Yang-Mills theories formulated in $\mathcal{N}=1$ superspace or $d=3$ Chern-Simons theories formulated in $\mathcal{N}=2$ superspace are plagued by infrared infinities. These infrared divergencies are an artifact of the formalism and are due to the gauge propagator structure. In the most studied case, which is $\mathcal{N}=4$ super Yang-Mills in $\mathcal{N}=1$ superspace, this infrared divergencies can be hidden with a safe gauge choice, but this is not the case for more general Yang-Mills theories [11]. As we will show in this work, the Yang-Mills propagator emerges in quantum loop corrections of Chern-Simons theories and produces the same problem of infrared divergencies as in four dimensions.

This whole thesis work is devoted to the application of $\mathcal{N}=2$ superspace techniques in general and in particular to the recent and highly interesting $\mathcal{N}=6$ supersymmetric Chern-Simons-matter theory. It is based on the author's contributions [12, 13]. The outline of the work is as follows: in chapter 2 we detail the main characteristics of the formalism itself. We motivate Chern-Simons theories and we formulate its supersymmetric version. We review the application of the formalism to the calculation of super-Feynman diagrams, the general arguments of renormalization properties, and as a working example we study the two-loop renormalization of the gauge sector with and without matter. In chapter 3 we review the infrared flow of mass deformed $\mathcal{N} = 4$ Yang-Mills theory to $\mathcal{N} = 3$ Chern-Simons theory so as to derive the construction of ABJ(M) theory. Its full superspace and component formulations are given and compared and the relevance of this theory as a gauge dual of M-theory is discussed. In chapter 4 we study in deepness the problem of infrared divergencies of the formalism by calculating some of its Green functions. We provide a solution to this problem by proposing a non-canonical gauge fixing procedure and we show it at work. In chapter 5 we display the full power of the formalism by calculating the four-loop correction of the anomalous dimension of long operators which are relevant in the AdS/CFT context. Finally in chapter 6 we give our conclusions.

Chapter 2.

$\mathcal{N} = 2$ Superspace

As mentioned in the introduction, our objects of study are three-dimensional supersymmetric Chern-Simons-Matter theories. Even if supersymmetric theories may be defined and quantized with the standard component field formulation, as we will show throughout this work, superspace formalism is very advantageous from many points of view with respect to component field formalism. The price to pay for these advantages is in the understanding of the formalism and its new quantization and super-Feynman rules.

The aim of this chapter is to provide a basic introduction to the formalism, to present the formulation of Chern-Simons-Matter theories in it, to explain the perturbative quantization in superspace and to establish the so-called D-algebra, which is the core of the calculative aspect of this formalism. Since there is a lot of resemblance between $\mathcal{N} = 2$ three-dimensional superspace formalism and the more well-known $\mathcal{N} = 1$ four-dimensional formalism, throughout this chapter, we will emphasize their main differences and similitudes.

As examples of superspace calculations, we provide the two-loop renormalization for pure $\mathcal{N} = 2$ Chern-Simons theory, and for that same theory coupled to chiral matter. Our results agree with the previously known results in the literature obtained by using other formalisms.

2.1. Basics

2.1.1. Supersymmetry and Superspace

At least up to the point where divergent integrals have to be regularized by dimensional reduction, we will work with three-dimensional ($d = 3$) objects and theories. In three-dimensional Minkowski space, the spin-group of its lorentz rotational symmetries ($SO(1,2)$), is $SL(2,\mathbb{R})$, namely the group of real 2×2 matrices with unit determinant. Because of this, as opposed to four-dimensional theories, there will not be a fundamental and anti-fundamental representation of the group. The fundamental representation of $SL(2,\mathbb{R})$ acts on a real two-component spinor Ψ^α, where $\alpha = +,-$. In the spinor language and as opposed to four dimensions, we will not need to differentiate between dotted and undotted spinorial indexes. Spinors will always be taken as Grassmannian variables.

There is only one lorentz-invariant tensor which may be used to define quadratic forms, which is a second-rank antisymmetric tensor proportional to the Levi-Civita tensor. We conveniently choose it to be given by $C^{\alpha\beta} = i\varepsilon^{\alpha\beta}$ in such a way that the quadratic form constructed with a real spinor is hermitian. We always contract indices following $\searrow$. We give all the details in Appendix A.

In spinorial language, a three-dimensional vector such as the gauge vector or the momentum vector is described by symmetric second-rank spinors $A_{\alpha\beta}$, $p_{\alpha\beta}$ or traceless second-rank spinors A_α^β, p_α^β [1] which are related to the usual arbitrary basis description A_μ, p_μ by the Gamma matrices given in Appendix A ($\mu = (0,1,2)$)[2].

Supersymmetry is introduced by grading the Poincaré algebra in a non-trivial way by the use of anticommutators, leading to the so-called super-Poincaré algebra. This grading involves the enlargement of the group by the introduction of $I = 1, \cdots, \mathcal{N}$ spinor supersymmetry generators Q_α^I, called supercharges, which satisfy the fundamental relation

$$\{Q_\alpha^I, Q_\beta^J\} = 2p_{\alpha\beta}\delta^{IJ} = 2\gamma^\mu_{\alpha\beta}p_\mu\delta^{IJ}. \tag{2.1.1}$$

[1] See [14], chapter 2, 'A toy superspace'.

[2] We use letters from the beginning of the Greek alphabet for spinorial indexes and Greek letters from μ onwards for the usual lorentz indexes.

The set of transformations which mix the supercharges while leaving (2.1.1) invariant forms a group, called $\mathcal{R}$-symmetry group which in three dimensions is $SO(\mathcal{N})$ (this is to be contrasted with four-dimensional $\mathcal{R}$-symmetry which is given by the group $SU(\mathcal{N})$).

Global flat superspace is the coset space (Super-Poincaré)/(Lorentz Group). To coordinatize this space, apart from the usual space-time coordinates $x^{\alpha\beta}$, one has to introduce new anticommuting coordinates θ_I^α. These coordinates $(x^{\alpha\beta}, \theta_I^\alpha)$ realize the action of a supersymmetric transformation as a coordinate transformation of the superspace. We may realize the supersymmetry algebra by representing the generators through

$$p_{\alpha\beta} = i\partial_{\alpha\beta}, \quad Q_\alpha^I = i(\partial_\alpha^I - \theta^{I\beta} i\partial_{\beta\alpha}). \tag{2.1.2}$$

These generators act over superfields $\Phi_{\alpha\beta\cdots}(x,\theta)$, which are functions of the superspace coordinates and transform as scalars, spinors and multispinors according to their spinorial structure. From the commuting nature of the momentum generators ($[p_{\alpha\beta}, p_{\gamma\delta}] = 0, [p_{\alpha\beta}, Q_\gamma^I] = 0$), it is possible to see that the spacetime derivative of a superfield $\partial_{\alpha\beta}\Phi$ also carries a representation of supersymmetry; in contrast, a spinor derivative such as $\partial_\alpha^I\Phi$ does not. Therefore, one is led to define covariant derivatives

$$D_\alpha^I = \partial_\alpha^I + i\theta^{I\beta}\partial_{\alpha\beta}, \tag{2.1.3}$$

which commute with the supersymmetry generators

$$[D_\alpha^I, Q_\beta^J] = 0, \quad [D_\alpha^I, p_{\gamma\delta}] = 0, \tag{2.1.4}$$

and satisfy the (anti-)commutation relations

$$\{D_\alpha^I, D_\beta^J\} = 2i\partial_{\alpha\beta}\,\delta^{IJ}; \qquad [D_\alpha^I, \partial_{\beta\gamma}] = 0. \tag{2.1.5}$$

The case $\mathcal{N} = 1$ is called simple supersymmetry while for $\mathcal{N} > 1$ we have extended supersymmetry. Theories with global $\mathcal{N} = 1$, 2, 3, 4, 6 and 8 supersymmetry have been constructed. But independently of the amount of extended supersymmetry a given model has, one may fix $\mathcal{N}$ and define supersymmetric models which by construction will have $\mathcal{N}$ supersymmetries even if the true extended supersymmetry of the theory is larger. In three dimensions, superspace formalisms for $\mathcal{N} = 1$, 2 and 3 exist; see *e.g.* [14–17]. We will focus on the $\mathcal{N} = 2$ formalism.

For $\mathcal{N} = 2$, the $\mathcal{R}$ symmetry group is $SO(2)$ which is equal to $U(1)$. Notice that $U(1)$ is also the $\mathcal{R}$-symmetry group of the $\mathcal{N} = 1$ four-dimensional formalism. To make this analogy explicit, we rewrite the algebra and the derivatives in such a way that they satisfy analogue relations to the well known ones from $\mathcal{N} = 1$ superspace in four dimensions. Our superspace has two 2-component anticommuting coordinates θ_1^α and θ_2^α. Then we define new complex anticommuting coordinates

$$\theta^\alpha = \theta_1^\alpha - i\theta_2^\alpha, \quad \bar{\theta}^\alpha = \theta_1^\alpha + i\theta_2^\alpha, \tag{2.1.6}$$

and the spinor derivatives

$$\partial_\alpha = \frac{1}{2}(\partial_\alpha^{(1)} + i\partial_\alpha^{(2)}), \qquad \bar{\partial}_\alpha = \frac{1}{2}(\partial_\alpha^{(1)} - i\partial_\alpha^{(2)}), \tag{2.1.7}$$

such that

$$\partial_\alpha\theta^\beta = \delta_\alpha^\beta, \quad \bar{\partial}_\alpha\bar{\theta}^\beta = \delta_\alpha^\beta, \quad \bar{\partial}_\alpha\theta^\beta = 0, \quad \partial_\alpha\bar{\theta}^\beta = 0. \tag{2.1.8}$$

Supercharges are defined as

$$Q_\alpha = \frac{1}{2}(Q_\alpha^{(1)} + iQ_\alpha^{(2)}), \qquad \bar{Q}_\alpha = \frac{1}{2}(Q_\alpha^{(1)} - iQ_\alpha^{(2)}), \tag{2.1.9}$$

such that their algebra is

$$\{Q_\alpha, \bar{Q}_\beta\} = P_{\alpha\beta}, \quad \{Q_\alpha, Q_\beta\} = 0. \tag{2.1.10}$$

We also define covariant derivatives as

$$D_\alpha = \frac{1}{2}(D_\alpha^{(1)} + iD_\alpha^{(2)}) = \partial_\alpha + \frac{1}{2}\bar{\theta}^\beta i\partial_{\alpha\beta}, \quad \bar{D}_\alpha = \frac{1}{2}(D_\alpha^{(1)} - iD_\alpha^{(2)}) = \bar{\partial}_\alpha + \frac{1}{2}\theta^\beta i\partial_{\alpha\beta}. \tag{2.1.11}$$

We finally find that

$$\{D_\alpha, \bar{D}_\beta\} = i\partial_{\alpha\beta}, \quad \{D_\alpha, D_\beta\} = 0, \quad \{\bar{D}_\alpha, \bar{D}_\beta\} = 0. \tag{2.1.12}$$

Apart from the nature of the vector representation in (2.1.12), and that one does not make distinctions between dotted and un-dotted spinor indexes, (2.1.12) is the same algebra of covariant derivatives of $\mathcal{N}=1$ four-dimensional superspace thus making Feynman supergraph rules very similar to the known rules. On the other hand, one may construct contractions that were not allowed in four dimensions such as $\bar{D}^\alpha D_\alpha$ or $\bar{\theta}^\alpha\theta_\alpha$. The preceding discussion is also valid for theories in wick-rotated Euclidean space (which is more convenient for quantization).

In order to write a superspace action as a functional of superfields we define Berezin integration over 'half' of the superspace $\int d^2\theta = \frac{1}{2}\int d\theta^\alpha d\theta_\alpha$, $\int d^2\bar{\theta} = \frac{1}{2}\int d\bar{\theta}^\alpha d\bar{\theta}_\alpha$, and over the 'full' superspace $\int d^4\theta = \int d^2\theta d^2\bar{\theta}$. Linearity and invariance under translations imposed over the definition of these integrals determines them to be equal to the derivative operation. Thus, up to a total space-time derivative, we have that

$$\int d^2\theta \ldots = D^2 \ldots |_{\theta=\bar{\theta}=0} \quad \text{and} \quad \int d^2\bar{\theta} \ldots = \bar{D}^2 \ldots |_{\theta=\bar{\theta}=0}. \tag{2.1.13}$$

From Berezin integral properties, a suitable definition for a superspace delta-function is straightforward:

$$\delta^{3|4}(z-z') = \delta^3(x-x')\delta^4(\theta-\theta') = \delta^3(x-x')(\theta-\theta')^2(\bar{\theta}-\bar{\theta}')^2, \tag{2.1.14}$$

where $\delta^3(x-x')$ is the usual three-dimensional Dirac delta distribution. The physics derived from an action written as superspace integral of a scalar is guaranteed to be supersymmetric since the supersymmetric variation is a total superspace derivative.

2.1.2. Enter Matter

The simplest superfields we may take to construct $\mathcal{N} = 2$ invariant theories are a real scalar superfield $V(x, \theta, \bar{\theta})$ and a complex chiral scalar superfield $\Phi(x, \theta, \bar{\theta})$. In fact, these two are the only 'ingredients' we will use to construct Chern-Simons-Matter theories (CSM). We leave the discussion of the former ones for the next section where we discuss the Chern-Simons gauge-vector superfield. Chiral superfields, on the other hand, are defined such that they form a scalar irreducible representation of supersymmetry: the scalar multiplet. For this purpose, we constrain the superfield by the condition

$$\bar{D}_\alpha \Phi(x,\theta,\bar{\theta}) = 0. \tag{2.1.15}$$

Since this condition is constructed with the covariant derivative, it is invariant under supersymmetric transformations. To solve this constraint we may introduce the new coordinates

$$x_L^{\alpha\beta} = x^{\alpha\beta} + \tfrac{i}{4}(\theta^\alpha\bar{\theta}^\beta + \theta^\beta\bar{\theta}^\alpha), \quad x_R^{\alpha\beta} = x^{\alpha\beta} - \tfrac{i}{4}(\theta^\alpha\bar{\theta}^\beta + \theta^\beta\bar{\theta}^\alpha), \tag{2.1.16}$$

such that a scalar field with the superspace dependence $\Phi = \Phi(x_L, \theta)$ can be seen to automatically solve the chiral constraint (and analogously an anti-chiral superfield with the dependence $\bar{\Phi} = \bar{\Phi}(x_R, \bar{\theta})$ solves the anti-chiral constraint, which is the hermitian conjugate of (2.1.15)). We Taylor-expand the chiral field over its θ dependence to appreciate the field content of the superfield

$$\Phi(x_L,\theta) = \phi(x_L) + \theta^\alpha\psi_\alpha(x_L) - \theta^2 F(x_L). \tag{2.1.17}$$

We thus have a complex scalar boson ϕ, a two-component complex fermion ψ and a complex scalar F which, as we will see in the following, plays the role of an auxiliary scalar. We shall use these type of multiplets as the matter part of CSM theories. By dimensional analysis, a unique quadratic free action can be constructed for the chiral superfield as an integral over the whole superspace:

$$S_{\text{free}} = \int d^3x d^4\theta\, \bar{\Phi}\Phi. \tag{2.1.18}$$

This action is supersymmetric invariant as can be seen by analyzing its supersymmetric variation. By projecting the $d^4\theta$ integral we obtain

$$S_{\text{free}} = \int d^3x \left(-\partial^\mu\bar{\phi}\partial_\mu\phi + i\bar{\psi}^\alpha\partial_\alpha{}^\beta\psi_\beta + \bar{F}F\right), \tag{2.1.19}$$

from which we see the auxiliary role played by the F scalar field.

In an on-shell analysis from the $\mathcal{N}=1$ perspective, the superspin 0 multiplet is formed by a real scalar and a real two-component fermion. The boson/fermion balance is satisfied on shell where the two-component fermion looses one of its helicities since in three dimensions, in contrast to four dimensions, there is a unique physical helicity. From this perspective, the $\mathcal{N} = 2$ scalar multiplet we have just constructed with the use of an $\mathcal{N} = 2$ chiral-constrained complex scalar superfield is composed of a pair of $\mathcal{N} = 1$ superspin 0 multiplets. Its physical on-shell degrees of freedom are a complex scalar and a complex two-component fermion, both with two physical degrees of freedom since once again, the fermion has only one physical helicity. Notice that the complex nature of the fermion, as opposed to $d = 4$, is not related to it being in the fundamental representation of the lorentz group but it is related to it being part of an $SO(2) = U(1)$ $\mathcal{R}$-symmetry multiplet.

Supersymmetry invariance is also guaranteed if one adds to the action interaction terms expressible as an integral over half of the superspace of a given function $W(\Phi)$ of the chiral field (plus its hermitian conjugate). This function is usually called 'superpotential'. In our case of study we will be interested in superpotentials with dimensionless couplings; straightforward power-counting leads us to a quartic superpotential

$$S_{\text{free}} + S_{\text{pot}} = \int d^3x d^4\theta\ \bar{\Phi}\Phi + \int d^3x d^2\theta\ \frac{\lambda}{4!}\Phi^4 + \text{h.c.}, \tag{2.1.20}$$

where λ is a dimensionless coupling constant. This marginal superpotential should be contrasted with the usual four-dimensional case where it is of the form Φ^3. Projecting the $d^2\theta(d^2\bar{\theta})$ integrals we obtain the component interaction part

$$\frac{\lambda}{3!}\int d^3x(3\psi^2\phi^2 + 3\bar{\psi}^2\bar{\phi}^2 - \phi^3 F - \bar{\phi}^3\bar{F}). \tag{2.1.21}$$

Integrating out the algebraic auxiliary field F by its equation of motion $F = \frac{\lambda}{3!}\bar{\phi}^3$ we obtain a three-dimensional interacting theory of a single chiral $\mathcal{N} = 2$ supersymmetric multiplet

$$S_{\text{free}} + S_{\text{pot}} = \int d^3x \left(-\partial^\mu\bar{\phi}\partial_\mu\phi + i\bar{\psi}^\alpha\partial_\alpha{}^\beta\psi_\beta + \frac{\lambda}{2}(\psi^2\phi^2 + \bar{\psi}^2\bar{\phi}^2) - \left(\frac{\lambda}{3!}\right)^2\phi^3\bar{\phi}^3\right). \tag{2.1.22}$$

Notice the scalar sextic self-interaction potential for the scalar ϕ and the quartic 'Yukawa-like' coupling between the scalar and the fermion. From this we may as well foresee that those are the type of matter self interactions we shall encounter in the matter sector of

CSM theories. Once we introduce the Chern-Simons gauge-vector multiplet we will also see that apart from the interactions we have just deduced, the matter fields will interact minimally with the gauge sector.

2.2. Supersymmetric Chern-Simons-Matter theories

2.2.1. Pure Chern-Simons action and gauge invariance

To motivate Chern-Simons theory as an interesting gauge theory in three dimensions we present its bosonic version and study its properties under gauge invariance. To simplify the notation and calculations of this section, we begin by studying pure bosonic Chern-Simons theory in the language of forms, where gauge invariance is more easily analyzed.

Consider a Lie-group $\mathbf{G}$ with a Lie-algebra $\mathfrak{g}$ and generators $T^I \in \mathfrak{g}$ such that $[T^I, T^J] = if^{IJ}{}_K T^K$. We define a 1-form connection $A = A_\mu dx^\mu$ which is a Lie-algebra-valued field $A = A_I T^I$. Under a Gauge transformation $U \in \mathbf{G}$, the connection transforms inhomogeneously as

$$A' = U^{-1} A U + U^{-1} dU, \tag{2.2.1}$$

where d is the exterior derivative which goes from p-forms to (p+1)-forms. The 2-form field strength

$$F = dA + A \wedge A, \tag{2.2.2}$$

on the contrary, transforms homogeneously $F' = U^{-1} F \, U$ such that in four dimensions, the 4-forms densities

$$\mathrm{tr}(F \wedge F), \quad \mathrm{tr}(*F \wedge F), \tag{2.2.3}$$

are gauge invariant. In this last expression the trace is taken over the Lie-algebra generators in the fundamental representation, and the '*' symbol is the Hodge dual. These two four-dimensional densities are the Yang-Mills density and the so called θ-

term. Unfortunately, in three dimensions a Yang-Mills theory would be possible only if the coupling constant acquires dimensions, leading to a super-renormalizable theory [18]. If one is interested in constructing strictly renormalizable theories with dimensionless couplings, an alternative to Yang-Mills exists in three dimensions. Consider the following action

$$S_{\rm cs}=\frac{ik}{4\pi}\int\limits_{M}\mathrm{tr}\left(A\wedge dA+\tfrac{2}{3}A\wedge A\wedge A\right), \tag{2.2.4}$$

which is written as the integral over a three-dimensional manifold M of the 3-form called Chern-Simons form [19]. The parameter k, called the Chern-Simons level, is dimensionless. Since the theory is not written in terms of homogeneous-transforming fields, gauge invariance is not evident as in the Yang-Mills case. Consider the infinitesimal gauge transformation $U=e^{i\Lambda}\sim 1+i\Lambda$. The transformation of the connection A and its exterior derivative dA under this infinitesimal gauge transformation is

$$\delta A=i[A,\Lambda]+i\,d\Lambda,\quad d\delta A=i(dA\,\Lambda-A\wedge d\Lambda-d\Lambda\wedge A-\Lambda\,dA), \tag{2.2.5}$$

while the variation of the action

$$\begin{aligned}\delta S_{\rm cs}&=\frac{ik}{4\pi}\int\limits_{M}\mathrm{tr}\left(\delta A\wedge dA+A\wedge d\delta A+2\delta A\wedge A\wedge A\right)\\&=-\frac{k}{4\pi}\int\limits_{M}d\left(\mathrm{tr}\left(\Lambda dA\right)\right)=-\frac{k}{4\pi}\int\limits_{\partial M}\mathrm{tr}\left(\Lambda dA\right)=0,\end{aligned} \tag{2.2.6}$$

where we have used Stokes theorem and we have discarded boundary terms[3]. So we see that the Chern-Simons action is invariant under infinitesimal gauge transformations. If instead we consider a 'large' gauge transformation as in (2.2.1), the first part of the Chern-Simons form changes by

$$\begin{aligned}\mathrm{tr}(A'\wedge dA')=&\mathrm{tr}\left(A\wedge dA-2U^{-1}A\wedge A\wedge dU+U^{-1}dA\wedge dU\right.\\&\left.-(U^{-1}dU)^3-3(dU\,U^{-1})^2\wedge A\right),\end{aligned} \tag{2.2.7}$$

[3]There are some subtleties in dropping these term which will become relevant in Chapter 3.

while the second by

$$\begin{aligned}\mathrm{tr}(A'\wedge A'\wedge A') =&\mathrm{tr}\left(A\wedge A\wedge A+3\,U^{-1}A\wedge A\wedge dU\right.\\ &\left.+(U^{-1}dU)^3+3(dU\,U^{-1})^2\wedge A\right).\end{aligned} \tag{2.2.8}$$

Putting altogether we find

$$\begin{aligned}S_{\rm cs}[A']-S_{\rm cs}[A] &= -\frac{ik}{24\pi}\int_M \mathrm{tr}(U^{-1}dU)^3-\frac{ik}{4\pi}\int_M d\left(\mathrm{tr}(A\wedge dU\,U^{-1})\right)\\ &= -\frac{ik}{24\pi}\int_M \mathrm{tr}(U^{-1}dU)^3 = ik\,I_{\rm WZ}[U],\end{aligned} \tag{2.2.9}$$

where again, after using Stokes theorem we have discarded a boundary term. This time we see that the difference between the actions after a large gauge transformation is not zero, but is proportional to the integral $I_{\rm WZ}[U]$ which is independent of the gauge field and only depends on the gauge transformation. Classically, this gauge field independence of the difference of both Chern-Simons actions implies the gauge invariance of the theory. In the quantum theory, one needs $\exp(S_{\rm cs})$ to be strictly invariant under these large gauge transformations. It turns out that when the third homotopy group of $\mathbf{G}$ is non trivial then $I_{\rm WZ}[U]=2\pi n$, where n is an integer which measures a topological property of the gauge transformation [20]. In these cases we have that the gaussian measure changes as

$$\exp(S_{\rm cs}[A'])=\exp(S_{\rm cs}[A])\,\exp(i2\pi nk), \tag{2.2.10}$$

forcing us to choose the Chern-Simons level k to be also an integer so that $\exp(i2\pi nk)=1$. In interesting groups such as semi-simple Lie groups, the third homotopy group is always non trivial such that the Chern-Simons level is quantized.

Summarizing, we have shown the gauge invariance under both infinitesimal and large gauge transformations of a three dimensional field theory, called Chern-Simons theory, which is characterized by a gauge group and a dimensionless coupling constant k which is an integer. Since we will work with unitary groups, it is convenient to rescale the gauge field $A\to iA$ such that the components A^I are real. Going back from the language of forms to the component language we have

$$S_{\rm cs}[A] = -\frac{ik}{4\pi}\int d^3x\epsilon^{\mu\nu\rho}{\rm tr}\left(A_\mu\partial_\nu A_\rho + i\frac{2}{3}A_\mu A_\nu A_\rho\right), \qquad (2.2.11)$$

which is invariant under the new gauge transformation

$$A'_\mu = U^\dagger A_\mu U - iU^\dagger\partial_\mu U. \qquad (2.2.12)$$

By varying the action we find the classical equation of motion

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] = 0, \qquad (2.2.13)$$

this is, the field strength must vanish everywhere. Notice that since the action is linear in the derivative of the field, the equation of motion is a first order differential equation, which is an unusual situation for a bosonic field. From (2.2.13) it is said that the field A_μ is 'pure gauge', in the sense that, any solution of (2.2.13), by a suitable gauge transformation, can be made to vanish locally in general and globally when the holonomy of the connection is trivial.

2.2.2. The $\mathcal{N} = 2$ Chern-Simons multiplet

We would like to construct a gauge invariant $\mathcal{N} = 2$ Lagrangian which describes a gauge field and its superpartners such that we obtain Chern-Simons equations of motion for the gauge field. We begin by analyzing the Abelian case. By an on-shell analysis of $\mathcal{N} = 2$ supersymmetry representations one concludes that the simplest multiplet which has a spin-1 field as its highest spin component will be formed by a real scalar, a complex fermion and the gauge field $(\sigma, \chi_\alpha, A_{\alpha\beta})$. As explained in the last section, on shell, the complex fermion χ_α will only have one physical helicity. The gauge field on the other hand, will loose its longitudinal helicity by gauge invariance as usual, but once again of both its transverse helicities, only one of them is physical. Among with the real scalar σ, these gives two bosonic and two fermionic physical degrees of freedom.

It turns out that the simplest description of this superspin $1/2$ multiplet is given in terms of a real scalar unconstrained superfield V which is called prepotential or gauge vector. In terms of component fields, a real scalar unconstrained field possesses sixteen degrees of freedom, but many of these degrees of freedom may be eliminated by a gauge transformation. Defining the Abelian gauge transformation as

$$V' = V + i(\bar{\Lambda} - \Lambda), \tag{2.2.14}$$

where the parameter Λ is a chiral field, it is possible to see that seven (one real scalar, one complex scalar and one fermion) out of the sixteen off-shell degrees of freedom can be gauged away. This is the three-dimensional version of the Wess-Zumino gauge. In this gauge, the expansion of the gauge vector superfield is given by

$$\begin{aligned} V(x,\theta,\bar{\theta}) = {} & \theta^\alpha \bar{\theta}_\alpha \sigma(x) + \theta\gamma^\mu\bar{\theta} A_\mu(x) + \theta^2 \bar{\theta}^\alpha \bar{\chi}_\alpha(x) \\ & + \bar{\theta}^2 \theta^\alpha \chi_\alpha(x) + \theta^2 \bar{\theta}^2 D(x), \end{aligned} \tag{2.2.15}$$

where we have introduced real scalars $\sigma(x)$, $D(x)$, a complex fermion field $\chi(x)$ and the gauge field A_μ ($\theta\gamma^\mu\bar{\theta}$ is short for $\theta_\alpha(\gamma^\mu)^\alpha{}_\beta \bar{\theta}^\beta$). From the on-shell description from above, we expect that at least one of these real scalars should be an auxiliary field in a supersymmetric theory. This is in fact the case in three dimensional Super Yang-Mills theory which may be written in two different but equivalent forms:

$$S_{YM} = -\frac{1}{2g_{YM}^2} \int d^3x d^4\theta \; \Sigma^2 = \frac{1}{g_{YM}^2} \int d^3x d^2\theta \; W^2, \tag{2.2.16}$$

where $\Sigma = \bar{D}^\alpha (e^{-V} D_\alpha e^V)$ is the scalar field strength and $W^2 = \frac{1}{2} W^\alpha W_\alpha$ with $W^\alpha = i\bar{D}^2(e^{-V} D^\alpha e^V)$ the spinorial field strength. The second way of writing (2.2.16) is the usual one from $\mathcal{N} = 1$, $d = 4$ formalism while the first way is particular of the $\mathcal{N} = 2$, $d = 3$ formalism. For the Abelian case, we expand this action to quadratic order

$$S_{YM} = -\frac{1}{2g_{YM}^2} \int d^3x d^4\theta \; \bar{D}^\alpha D_\alpha V \; \bar{D}^\beta D_\beta V = \frac{1}{2g_{YM}^2} \int d^3x d^4\theta \; V \, D^\alpha \bar{D}^2 D_\alpha V, \tag{2.2.17}$$

and projecting the $d^4\theta$ integrals we obtain

$$S_{YM} = \frac{1}{g_{YM}^2} \int d^3x \left(-\frac{1}{2} F_{\mu\nu} F^{\mu\nu} - \partial^\mu \sigma \partial_\mu \sigma + D'^2 + i\, \bar{\chi} \partial\!\!\!/ \chi \right), \tag{2.2.18}$$

i.e. Klein-Gordon, Dirac and Maxwell kinetic terms for the fields $(\sigma, \chi_\alpha, A_{\alpha\beta})$ respectively, while the scalar $D(x)$ turns out to be auxiliary. The $\mathcal{N} = 2$ vector multiplet $(\sigma, \chi_\alpha, A_{\alpha\beta})$ should be contrasted with its $\mathcal{N} = 1$ four-dimensional counterpart formed by the fields $(\chi_\alpha, A_{\alpha\dot\beta})$; the extra scalar in the three-dimensional case compensates the fermion-boson balance due to the different number of on-shell components of the gauge field A. The $\mathcal{N} = 2$ $d = 3$ multiplet can also be thought as the component fields of a $d = 4$ $\mathcal{N} = 1$ vector multiplet dimensionally reduced. The extra σ scalar field would be the A_3 component of the four-dimensional gauge field.

By looking at (2.2.17) it is possible to make an educated guess on the form of a gauge invariant Chern-Simons action for the gauge-vector V in the Abelian case. We consider

$$S_{\rm CS}^{abelian} = \frac{k}{8\pi}\int d^3x d^4\theta\, V \bar{D}^\alpha D_\alpha V, \tag{2.2.19}$$

which is easily seen to be gauge invariant up to boundary terms. The abelian equation of motion is $\bar{D}^\alpha D_\alpha V = 0$. By projecting this equation we obtain $\partial_{[\mu} A_{\nu]} = 0$ and $\chi = \sigma = D = 0$. The vanishing of the abelian field-strength is what we expected for a Chern-Simons theory.

The construction of the superspace action in the non-abelian case is not so straightforward. It is based on the Vainberg construction [21–24] which involves the addition of an extra coordinate "t" such that $t \in [0, 1]$, and the "generalized" gauge vector depends on it $V(x, \theta, \bar\theta) \to \tilde{V}(x, \theta, \bar\theta; t)$ with the boundary conditions

$$\tilde{V}(x, \theta, \bar\theta; 1) = V(x, \theta, \bar\theta) \quad \tilde{V}(x, \theta, \bar\theta; 0) = 0. \tag{2.2.20}$$

The gauge vector and its generalization are now Lie-algebra valued fields $\tilde{V}(t) = \tilde{V}_A(t)\, T^A$. We define the Vainberg direction as

$$\mathcal{V}_t = e^{-\tilde{V}(t)}\partial_t e^{\tilde{V}(t)} = \partial_t \tilde{V}(t) + \frac{1}{2}[\partial_t \tilde{V}(t), \tilde{V}(t)] + \ldots \tag{2.2.21}$$

and the generalized scalar field strength $\tilde\Sigma(t) = \bar{D}^\alpha(e^{-\tilde{V}(t)} D_\alpha e^{\tilde{V}(t)})$. We consider a superspace action for the generalized $\tilde{V}(t)$ field as an integral over the whole superspace and the Vainberg coordinate t

$$S_{\rm CS} = \frac{k}{4\pi}\int d^3x d^4\theta \int_0^1 dt\, {\rm tr}\left(\mathcal{V}_t\, \tilde\Sigma(t)\right). \tag{2.2.22}$$

By expanding the Lagrangian in powers of $\tilde{V}$, it is possible to observe that each term of the expansion is a total derivative in the Vainberg parameter. Thus, even if we are not able to get rid of the t-parametrization in general, we do know that the dynamics do not depend on it and only depend on the boundary values of $\tilde{V}$.

In the abelian limit, the Vainberg parameter becomes $\mathcal{V}_t = \partial_t \tilde{V}(t)$, and the scalar field strength $\tilde{\Sigma}(t) = \bar{D}^\alpha D_\alpha \tilde{V}(t)$, such that we can easily get rid of the t parametrization since

$$\begin{aligned} S_{\text{CS}}^{\text{abelian}} =& \frac{k}{4\pi} \int d^3x d^4\theta \int_0^1 dt \, \text{tr} \left(\partial_t \tilde{V}(t) \, \bar{D}^\alpha D_\alpha \tilde{V}(t) \right) \\ =& \frac{k}{8\pi} \int d^3x d^4\theta \, \text{tr} \, \tilde{V}(t) \, \bar{D}^\alpha D_\alpha \tilde{V}(t) \Big|_0^1 , \end{aligned} \tag{2.2.23}$$

and using the boundary conditions we recover the abelian action (2.2.19). We postulate the transformation of the $\tilde{V}$ superfield under generalized non-abelian gauge transformations as

$$e^{\tilde{V}'(t)} = e^{i\bar{\Lambda}(t)} e^{\tilde{V}(t)} e^{-i\Lambda(t)}, \tag{2.2.24}$$

where as before, $\Lambda(t)$ is a chiral field, but now it is valued in the Lie algebra. Under this transformation, the Vainberg direction changes by

$$\mathcal{V}'_t = e^{i\Lambda} \mathcal{V}_t e^{-i\Lambda} + e^{i\Lambda} e^{-\tilde{V}} e^{-i\bar{\Lambda}} \partial_t e^{i\bar{\Lambda}} e^{\tilde{V}} e^{-i\Lambda} + e^{i\Lambda} \partial_t e^{-i\Lambda}, \tag{2.2.25}$$

while the generalized scalar field strength transforms as a scalar in the adjoint, namely

$$\tilde{\Sigma}' = \bar{D}^\alpha \left(e^{-\tilde{V}'} D_\alpha e^{\tilde{V}'} \right) = e^{i\Lambda} \left[D^{\bar{\alpha}} \left(e^{-\tilde{V}} D_\alpha e^{\tilde{V}} \right) \right] e^{-i\Lambda} = e^{i\Lambda} \tilde{\Sigma} e^{-i\Lambda}, \tag{2.2.26}$$

where the key property $\bar{D}^\alpha D_\alpha = D^\alpha \bar{D}_\alpha$ and the chirality of the gauge transformation parameters $\bar{D}^\alpha \Lambda = D^\alpha \bar{\Lambda} = 0$ were used and will be repeatedly used in the following.

Inserting (2.2.25-2.2.26) into the action we obtain

$$\Delta S_{\rm CS} = S_{\rm CS}[\tilde{V}'] - S_{\rm CS}[\tilde{V}] = \frac{k}{4\pi}\int d^3x d^4\theta \int_0^1 dt\ \mathrm{tr}\left(e^{-\tilde{V}}e^{-i\bar{\Lambda}}\partial_t e^{i\bar{\Lambda}}e^{\tilde{V}}\bar{D}^\alpha(e^{-\tilde{V}}D_\alpha e^{\tilde{V}}) + \partial_t e^{-i\Lambda}e^{i\Lambda}\bar{D}^\alpha(e^{-\tilde{V}}D_\alpha e^{\tilde{V}})\right). \tag{2.2.27}$$

The second term in the *r.h.s.* of (2.2.27) is easily seen to be a total anti-chiral derivative from the chiral nature of the gauge parameter Λ. To arrange the first term in the *r.h.s.* of (2.2.27) we use the identity

$$e^{\tilde{V}}\bar{D}^\alpha(e^{-\tilde{V}}D_\alpha e^{\tilde{V}}) = \bar{D}^\alpha D_\alpha e^{\tilde{V}} - \bar{D}^\alpha e^{\tilde{V}}e^{-\tilde{V}}D_\alpha e^{\tilde{V}} = \bar{D}^\alpha D_\alpha e^{\tilde{V}} + \bar{D}^\alpha e^{\tilde{V}}D_\alpha e^{-\tilde{V}}e^{\tilde{V}}, \tag{2.2.28}$$

to obtain

$$\begin{aligned}\Delta S_{\rm CS} =& \frac{k}{4\pi}\int d^3x d^4\theta \int_0^1 dt\ \mathrm{tr}\left(e^{-i\bar{\Lambda}}\partial_t e^{i\bar{\Lambda}}\bar{D}^\alpha D_\alpha e^{\tilde{V}}e^{-\tilde{V}} + e^{-i\bar{\Lambda}}\partial_t e^{i\bar{\Lambda}}\bar{D}^\alpha e^{\tilde{V}}D_\alpha e^{-\tilde{V}}\right)\\ =& \frac{k}{4\pi}\int d^3x d^4\theta \int_0^1 dt\ D^\alpha\, \mathrm{tr}\left(e^{-i\bar{\Lambda}}\partial_t e^{i\bar{\Lambda}}\bar{D}_\alpha e^{\tilde{V}}e^{-\tilde{V}}\right) = 0,\end{aligned} \tag{2.2.29}$$

thus proving the gauge invariance of the non-abelian action.

Up to now, we have proposed a non-abelian gauge invariant superspace action for the $\mathcal{N} = 2$ vector multiplet described in terms of the superfield V, such that this action leads to the abelian Chern-Simons theory in the abelian limit. Therefore, to complete the construction, it remains to be shown that the general (non-abelian) Chern-Simons dynamics emerges from this action. To do this we would like to expand the action in terms of its component fields.

Now that we have shown gauge invariance, we may simplify the action a bit. Even if we cannot get rid of the t-parametrization of the Chern-Simons action in the superfield language, we may use the simplest choice of the generalized field which fulfills the boundary conditions, *i.e.* $\tilde{V}(t) = tV$. The Vainberg direction with this choice is $\mathcal{V}_t = V$, and the action simplifies to

$$S_{\rm CS} = \frac{k}{4\pi}\int d^3x d^4\theta \int_0^1 dt\, \mathrm{tr}\left(V\, \bar{D}^\alpha (e^{-tV} D_\alpha e^{tV})\right), \tag{2.2.30}$$

while the gauge transformation becomes

$$e^{tV'} = e^{i\bar{\Lambda}} e^{tV} e^{-i\Lambda}. \tag{2.2.31}$$

Before projecting the fields in the Wess-Zumino gauge, we notice that in this gauge all the terms in the Lagrangian with powers of V greater than three will drop. Therefore, we just need

$$S_{\rm CS} = \frac{k}{4\pi}\int d^3x d^4\theta\, \mathrm{tr}\left(\frac{1}{2} V \bar{D}^\alpha D_\alpha V + \frac{1}{3!}\bar{D}^\alpha V[V, D_\alpha V] + \mathcal{O}(V^4)\right). \tag{2.2.32}$$

After projecting the $d^4\theta$ integral we obtain

$$S_{\rm CS} = \frac{k}{4\pi}\int d^3x\, \mathrm{tr}\left(-i\epsilon^{\mu\nu\rho}(A_\mu \partial_\nu A_\rho + i\tfrac{2}{3}A_\mu A_\nu A_\rho) - 2D\sigma - \bar{\chi}^\beta \chi_\beta\right). \tag{2.2.33}$$

We find as expected, the kinetic and self interaction term of the Chern-Simons action for the gauge field A. Notice that the fields $D(x)$, $\sigma(x)$ and $\chi(x)$ are algebraic fields which may be integrated out. This is, as opposed to Yang-Mills theory, where the fields (σ, χ, A) are dynamical, in the Chern-Simons case, the superpartners of the gauge field are auxiliary. This fact could seem somehow surprising, but the fact is that, as we explained when we analyzed bosonic pure CS theory, the gauge field itself is in some sense 'topological' or pure gauge, and therefore non-physical. We interpret that supersymmetry reflects this fact in the $\mathcal{N} = 2$ dynamics by making the gauge field super-partners non dynamical. We conclude that $\mathcal{N} = 2$ pure Chern-Simons theory is classically equivalent to bosonic pure Chern-Simons theory. As noted in [22], after quantization, the coupling between algebraic fields and ghosts will affect quantum corrections of the theory. Also, as we will see later, when we add matter and we go from $\mathcal{N} = 2$ supersymmetric formulation to components formulation, the precise coupling between the algebraic fields and the matter fields will determine the precise form of the matter interaction terms.

2.2.3. Pure $\mathcal{N} = 2$ CS perturbative quantization and renormalization

Quantization

Having found a suitable superspace formulation for a supersymmetric Chern-Simons theory we will describe its perturbative quantization. In Euclidean space, we quantize the theory with a path integral measure of the form $\int D\phi\ e^{S[\phi]}$. From the infinite series of terms in the pure Chern-Simons action (2.2.30), those that we will use in what follows are

$$S_{\rm CS} = \frac{k}{4\pi}\int d^3x d^4\theta \left(\frac{1}{2}V^I \bar{D}^\alpha D_\alpha V^I + \frac{i}{3!} f^{IJK}\bar{D}^\alpha V^I V^J D_\alpha V^K \right. \\ \left. -\frac{1}{4!} f^{IJM} f^{KLM} \bar{D}^\alpha V^I V^J D_\alpha V^K V^L + \mathcal{O}(V^5)\right), \qquad (2.2.34)$$

where uppercase Latin indexes from I onwards count the generators of the group and we have explicitly taken the trace by normalizing the generators through ${\rm tr}(T^I T^J) = \delta^{IJ}$ [4]. Since it will be the relevant case in the subsequent chapters, we consider the gauge group to be $U(N)$. To quantize the theory we re-scale the vector gauge field $V \to \sqrt{\frac{4\pi}{k}}V \equiv gV$ so that the kinetic term has the standard normalization (this scaling can be undone at the end of any calculation by re-scaling back); we will take g small -or equivalently k large- and we make perturbation theory in powers of g. We choose the gauge fixing functions $F = \bar{D}^2 V$, $\bar{F} = D^2 V$ which satisfy the same constraints as the chiral gauge transformation parameters. The standard procedure in $d = 3$ is to introduce in the functional integral the factor:

$$\int \mathcal{D}f\, \mathcal{D}\bar{f}\, \Delta(V)\Delta^{-1}(V)\, \exp\left(\frac{1}{2\alpha}\int d^3x\, d^2\theta\, {\rm tr}\,(f\, f)\right) \exp\left(\frac{1}{2\bar{\alpha}}\int d^3x\, d^2\bar{\theta}\, {\rm tr}\left(\bar{f}\, \bar{f}\right)\right), \qquad (2.2.35)$$

where

$$\Delta(V) = \int d\Lambda\, d\bar{\Lambda}\, \delta(F(V, \Lambda\, \bar{\Lambda}) - f)\, \delta(\bar{F}(V, \Lambda\, \bar{\Lambda}) - \bar{f}), \qquad (2.2.36)$$

[4] We refer the reader to appendix B for useful properties of the structure constants.

with Λ the chiral superfield of the gauge transformation we introduced in the last section and α a dimensionless parameter. Notice that, following [25, 26], we are introducing a gauge averaging given by gaussian weights with chiral integrals of the form $\sim e^{\int ff} e^{\int \bar{f}\bar{f}}$. This is to be contrasted with the standard $d = 4$, $\mathcal{N} = 1$ superspace procedure where one averages with a non-chiral (whole superspace) gaussian weight of the form $e^{\int \bar{f} f}$. The average produces the canonical gauge fixed action [27, 28]

$$\mathcal{S}_{gf}^{(\alpha)} = \frac{1}{2} \int d^3x d^4\theta \ \mathrm{tr} \left[V \left(\bar{D}^\gamma D_\gamma + \frac{1}{\bar{\alpha}} D^2 + \frac{1}{\alpha} \bar{D}^2 \right) V + \mathcal{O}(g) \right] \tag{2.2.37}$$

such that, after inverting the quadratic kinetic operator we obtain the gauge field propagators in momentum space

$$\left\langle V^I(p) V^J(-p) \right\rangle = \frac{1}{p^2} \left(\bar{D}^\alpha D_\alpha + \alpha D^2 + \bar{\alpha} \bar{D}^2 \right) \delta^4_{(\theta,\theta')} \delta^{IJ}, \tag{2.2.38}$$

with $\delta^4_{(\theta,\theta')} = \delta^4(\theta - \theta')$. From now on we shall call this gauge fixing procedure as the "α-gauge". This fixing simplifies greatly with the choice $\alpha \to 0$ (Landau Gauge), which is the one we make in this section. Notice that even in the simplest gauge choice $\alpha \to 0$, the gauge vector propagator contains a quadratic D-operator acting on the line. From the calculative point of view, this presents a major difference between CS theories in $\mathcal{N} = 2$ superspace and four-dimensional Yang-Mills theories in $\mathcal{N} = 1$ superspace, since in the latter case, the Fermi-Feynman gauge produces a simple propagator $1/p^2$. In chapter 4 we will show an alternative gauge fixing which solves the problem of infrared divergencies, but for the moment we keep this gauge fixing procedure.

To complete the gauge fixing procedure, we should rewrite the $\Delta^{-1}(V)$ factor in the path integral by introducing Fadeev-Popov ghosts. Since the gauge transformation parameters Λ are chiral, we introduce Grassmanian $(b - c)$ chiral ghosts in the same representation of the gauge group as the gauge vector. From the infinitesimal (non-linear) variation of the V field

$$\delta V = -i L_{\frac{gV}{2}} [\bar{\Lambda} + \Lambda + \coth(L_{\frac{gV}{2}})(\Lambda - \bar{\Lambda})], \tag{2.2.39}$$

we find the ghost action [25, 27, 28]

$$S_{\rm fp} = \int d^3x d^4\theta\, (b+\bar{b}) L_{\frac{gV}{2}} [(\bar{c}+c) + \coth(L_{\frac{gV}{2}})(c-\bar{c})], \tag{2.2.40}$$

with $L_{\frac{gV}{2}} = \frac{g}{2}[V,\ \ ]$. Expanding in powers of g up to the terms we will use in this chapter we get

$$\begin{aligned}\mathcal{S}_{fp} = \int d^3x d^4\theta\ \Big(&\bar{b}^I\, c^I + \bar{c}^I\, b^I + \frac{ig}{2} f^{IJK}(b^I+\bar{b}^I)V^J(c^K+\bar{c}^K) \\ &+\frac{g^2}{12} f^{IJM} f^{KLM}(b^I+\bar{b}^I)V^J(c^K-\bar{c}^K)V^L + \mathcal{O}(g^4)\Big)\,.\end{aligned} \tag{2.2.41}$$

The entire action $S_{\rm gf} + S_{fp}$ is invariant under the BRST transformations

$$\begin{aligned}&\delta_B b = -\frac{1}{\alpha}\zeta\bar{D}^2 V, \quad \delta_B\bar{b} = -\frac{1}{\bar{\alpha}}\zeta D^2 V, \\ &\delta_B c = -g\zeta c^2, \quad \delta_B\bar{c} = -g\zeta\bar{c}^2, \\ &\delta_B V = \zeta L_{\frac{gV}{2}}[(\bar{c}+c) + \coth(\tfrac{gV}{2})(c-\bar{c})],\end{aligned} \tag{2.2.42}$$

where ζ is the anti-commuting parameter of the transformation. We then have the ghost field c, which deals with the gauge transformation, and the anti-ghost field b, which deals with the gauge-fixing[5]. The usual path integral approach for chiral-constrained scalar fields [14] applies to the ghost fields (not forgetting their anti-commutative nature). We obtain the propagators

$$\left\langle \bar{b}^I(p)c^J(-p)\right\rangle = \left\langle \bar{c}^I(p)b^J(-p)\right\rangle = \frac{1}{p^2}\delta^4_{(\theta,\theta')}\delta^{IJ}. \tag{2.2.43}$$

Super-Feynman rules now follow in a similar way as in $\mathcal{N}=1$ four-dimensional formalism. We explain them emphasizing some of the similarities and differences between both formalisms. We call '$\mathcal{D}$-Algebra' to the graphical technique of integration by parts that permits us to go from a supergraph with momenta and $d^4\theta$ integrations to a simple covariant Feynman integral. Interaction vertices are derived from the interaction terms in the Lagrangian by making functional derivatives. After constructing a diagram rele-

[5]In this context, the system of ghosts/anti-ghosts we called (b,c) are sometimes called (c',c) in the literature.

vant for a certain calculation a set of rules should be followed in order to determine the operators acting on the diagram:

- For every (anti)-chiral line arriving to a vertex, a (D^2) $\bar{D}^2$ operator acts on the propagator of the line.
- An exception should be made on purely (anti)-chiral vertices such as the marginal superpotential type presented in (2.1.20), where one out of the four (D^2) $\bar{D}^2$ operators should be omitted. The freedom of choosing which one is omitted is usually used to simplify the procedure.
- Particularly for CS theories in the α-gauge, a $\bar{D}^\alpha D_\alpha + \bar{\alpha}\bar{D}^2 + \alpha D^2$ operator acts on the propagator lines of gauge vectors. For obvious reasons the Landau gauge is the simplest
- Gauge-vector self interaction terms contain D derivatives acting on the different lines arriving to the vertices whose detailed structure should be functionally derived from the Lagrangian.
- For every vertex on the graph there is a $\int d^4\theta$ integral and for every loop there is, as usual, a $\int d^3p$ integral.
- One may transfer D-operators along a line producing a minus sign for each D operator transferred. Notice that this does not produce any ambiguity in the construction of the diagram since the propagators always come with an even number of derivatives such that their derivatives may be thought to be acting in any of the two vertices which the line connects.

After constructing the diagram with all its D-operators, the process of integration by parts begins. The idea is to choose a given line and integrate by parts the operators acting on it in order to perform one of the $\int d^4\theta$ integrals of one of its vertices without producing a vanishing result. After integration, lines are contracted to points and if the process is continued one ends up reducing the whole diagram to a point with a single integral on a θ variable in which the external fields are valued. Some very useful rules of the process of integration by parts are the following:

- The product of a $\delta^4(\theta - \theta') = (\theta - \theta')^2(\bar{\theta} - \bar{\theta}')^2$ from a given line, multiplying an equal delta of another line is zero. This fact is used repeatedly when two lines connect the same two vertices such that after freeing one of the lines of operators, the other line should have two D's and two $\bar{D}$'s either in the form $D^2\bar{D}^2$ or $\bar{D}^2 D^2$. If it has less, it is zero. If it has more, the properties (A.0.7) are very useful to reduce the number of operators converting them in momenta. This rule is essentially the same as in $\mathcal{N} = 1$ four-dimensional formalism.

- When reducing D-operators one produces momenta bi-spinors and δ^α_β (or $C^{\alpha\beta}$) tensors. A typical situation comes about when one of the possible paths of integration by parts produces products of the form $p_{\alpha\beta}C^{\alpha\beta}$, which, from the symmetric nature of the vector representation is zero. This situation, which reduces the number of Feynman integrals produced by a given diagram, is an intrinsic three-dimensional phenomena of the technique.

- Another fact which is pretty useful, and which is very particular of three dimensions, is the fact that gauge vector propagator contraction $\bar{D}^\alpha D_\alpha$ satisfies $\bar{D}^\alpha D_\alpha = D^\alpha \bar{D}_\alpha$. This permits to have the freedom of combining this operator with the contiguous operators acting on it and usually leading to reductions.

- Signs are one of the most tricky parts of the $\mathcal{D}$-algebra. One should keep track of all the signs coming from transfers and integration by parts and at the end count the number of transpositions of spinorial indexes that the D-algebra produced. An (odd) even number of transpositions produces a (minus) plus sign.

Apart from this rules, the usual field theory rules apply for the diagrams: for every diagram one should calculate the corresponding color and flavor factors coming from the specific nature of the interaction and the symmetry factor that compensates for the overcounting of equivalent diagrams[6].

Since ultraviolet (and eventually infrared) divergences will appear in the final Feynman integrals after the D-Algebra is complete we use the 'dimensional reduction' prescription for regularization. This is, we perform all the manipulations in $d = 3$ dimensions and only when we arrive to the final Feynman integral we regularize divergences by turning $d = 3 - 2\epsilon$. It turns out that the simplest D-algebras are those where only chiral lines are involved in a given diagram. Once we introduce gauge-vectors interacting with chiral matter (or ghosts) or self-interacting with themselves, the D-algebra becomes

[6]There are many different prescriptions to calculate symmetry factors in field theories and many of them lead to ambiguities. We refer the reader to [29] which we believe is the best.

more complicated. Even if the calculation of relevant quantities through superspace formalism has this extra difficulty of having to perform the D-algebra for every diagram one writes down, we emphasize that every calculation performed with this formalism involves a highly reduced number of diagrams when compared to an equivalent component formalism.

As a non-trivial example of a superspace calculation, in the remaining of the section we study renormalization properties of pure $\mathcal{N} = 2$ Chern-Simons theory.

2-loop renormalization

To face the task of studying renormalization properties of a pure $\mathcal{N} = 2$ Chern-Simons theory we may advance some conclusions by making an exhaustive power-counting of the Feynman integrals that will appear for any diagram of the theory. Consider any general diagram Γ with E external lines, V vertices, I internal lines and L loops. Topological considerations permit to affirm that

$$L = I - V + 1 \qquad 2I = \sum_{n=2} nV_n - E, \tag{2.2.44}$$

where V_n is the number of n-vertices. We define V_{vc} as the number of chiral/gauge-vector interaction vertices. Notice that in the case of ghosts, there are always two (anti)-chiral lines leaving that type of vertex; as we will see in the next section, when we couple matter with the gauge vector this will be also the case for matter/gauge-vector interactions. We also define V_c as the number of purely (anti)-chiral 4-vertices coming from the superpotential (this are absent in pure CS theory but we make the counting more general).

If we consider any given diagram in a CSM theory with I_c (E_c) internal (external) chiral lines and we delete from the diagram all the gauge-vector lines, we are left with a diagram with only chiral lines with chiral 4-vertices and 2-vertices where chiral lines used to interact with gauge vectors before we deleted them. From this subtracted diagram we conclude using the second equation of (2.2.44) that

$$2I_c = 2V_{vc} + 4V_c - E_c. \tag{2.2.45}$$

The deletion of gauge-vector lines was just an artifact to count the chiral 'skeleton' of the diagram, but (2.2.45) is valid in general even in the presence of gauge-vectors.

We now use the rules we constructed in the last section to count the superficial degree of divergence of a super-graph. Each of the I_c chiral propagators scale as $\sim 1/p^2$ while each of the I_v gauge-vector propagators as $\sim 1/p$. From the rule of putting a $(\bar{D}^2)$ D^2 operator on each (anti)-chiral line leaving a vertex, we have that chiral/gauge-vector vertices scale as $D^2\bar{D}^2 \sim p^2$. Since we have to omit one of those operators from purely chiral interactions, we have that each of the V_c superpotential vertices scale as $\sim p^3$. On the other hand, we think of external (anti)-chiral lines as already having one of the (D^2) $\bar{D}^2$ and since those operators do not scale with the loop momenta, we should compensate the overcounting of $D's$ by subtracting E_C. Each of the V_v gluon self-interactions, independently of their number, contain a $\bar{D}^\alpha D_\alpha$ operator, so they scale as $\sim p$. For each loop we have a $\int d^3p \sim p^3$ integral. We also have one delta $\delta^4(\theta-\theta')$ for each of the I internal lines which scale as $\sim 1/p^2$ and one integral $\int d^4\theta$ for each of the total V vertices which scale as $\sim p^2$. The $\delta^4(\theta)$'s and θ-integrals will end up canceling among themselves and producing momenta after the D-algebra is performed such that we reduce the diagram to a single θ-integral on a point in θ-space where the external fields are valued; namely, from the V θ-integrals we have, only $V-1$ contribute to the power counting. With all these elements taken into account, we define the superficial degree of UV divergence as

$$\omega(\Gamma) = V_v + 2V_{vc} + 3V_c - I_v - 2I_c + 3L + 2(V-1) - 2I - E_c. \tag{2.2.46}$$

From the fist equation of (2.2.44) we have

$$L = I - V + 1 = I_v + I_c - V_v - V_{vc} - V_c + 1. \tag{2.2.47}$$

Inserting (2.2.45) and (2.2.47) in (2.2.46) we obtain [30]

$$\omega(\Gamma) = 1 - \frac{E_c}{2}. \tag{2.2.48}$$

Quite surprisingly, the superficial degree of divergence depends exclusively on the number of external chiral lines, either ghost or matter lines[7].

[7] Of course, if one already knew the $d=4$ result for super Yang-Mills theories [31] which is $\omega(\Gamma) = 2 - E_c$, this result turns out to be not so surprising from the similarities of both formalisms.

We may draw many conclusions from (2.2.48). For example, diagrams with more than two external chiral lines, such as quantum corrections to matter superpotentials, do not have ultraviolet divergencies. This makes the renormalization analysis of the matter sector much simpler since in order to derive the renormalization flow of the superpotential coupling one may only study self-energy diagrams of matter fields. In fact, self-energy diagrams of matter fields, ghost fields and chiral/gauge-vector vertex corrections, according to (2.2.48), are at most logarithmically divergent. On the other hand, diagrams that contain only external gauge-vector lines, such as gauge-vector self energy diagrams, will be at most linearly divergent.

When we say 'at most' is because the naive power-counting we just did can be further refined. In fact, in the process of performing the D-algebra, it may happen many times that we are forced, so as not to obtain a null result, to extract D operators outside the graph such that they act on the external fields. In such cases, the extracted D's no longer contribute to the scaling of the Feynman integrals and therefore the degree of UV divergence of the graph is reduced by

$$\omega_{ref.}(\Gamma) = 1 - \frac{E_c + D_{out}}{2}, \tag{2.2.49}$$

where D_{out} is the number of D operators that have been extracted after performing the D-algebra. Notice that this argument, even if it may meliorate the UV behavior of the theory in the cases where D-operators are extracted, it worsens the IR behavior such that if one does not know a priory how many D's will be extracted from the diagram, general power counting arguments for IR divergencies do not apply. We will study this situation in chapter 4.

This refinement of the power counting is essential to study the renormalization of the gauge-vector sector of CSM theories. Consider gauge-vector self-energy diagrams. BRST symmetry of the α-gauge fixed action provides a Slavnov-Taylor identity which determines that quantum corrections of the gauge-vector self-energy should be orthogonal to the superspin 0 projector $\mathcal{P}_0$ introduced in appendix A [8]. This leaves us with two possibilities:

$$V(-p)\,\bar{D}^\alpha D_\alpha V(p) \qquad \text{or} \qquad V(-p)\,\mathcal{P}_{1/2} V(p). \tag{2.2.50}$$

[8] In other words, this is the decoupling of the longitudinal modes of the gauge-vector.

In both cases some D operators should have been extracted in the D-algebra process in order to arrive to those structures. In the first case we have two operators, so the diagram producing that structure will be logarithmically divergent. In the second case, the operator $\mathcal{P}_{1/2}$ contains four operators such that any diagram producing that structure will be UV finite.

Moreover, parity arguments lead us to conclude that the first structure, which coincides with the structure of the propagator in the Landau gauge, appears only with odd powers of k [9] corresponding to even-loop self-energy diagrams. On the other hand, the second structure in (2.2.50) may appear only with even powers of k corresponding to odd-loop self energy diagrams.

We thus see that the first non trivial diagrams which contribute to the gauge-vector anomalous dimension appear at two loops. One loop self energy corrections will be finite and we study them in detail in chapter 4. We define Z_V and Z_{bc} the wave-function renormalization constants of the gauge-field and the ghosts respectively. If we want to study the renormalization of the gauge vector, ghost and their interactions, we find an infinite series of terms in the Lagrangian which should be renormalized. Notice that the gauge vector interacts with the ghosts by a three-vertex and from the expansion of the function $x \coth(x)$, the rest of the interactions are even in the number of V fields. Therefore, we consider Z_g as the renormalization constant of this three-vertex and $Z_{g^{2n}}$ with $n = 1, 2, \ldots$ for the even vertexes. From gauge invariance we have that

$$Z_{g^{2n}} = Z_{bc} \left(\frac{Z_g}{Z_{bc}} \right)^{2n}, \tag{2.2.51}$$

from which we see that we only need to calculate Z_V, Z_{bc} and Z_g.

After discarding many potentially contributing diagrams by inspection, we have found [10] all the two-loop diagrams contributing to self-energy corrections of the gauge vector and we depict them in figure 2.1. As discussed before, the general form of these

[9] Odd powers after re-scaling back the Chern-Simons level

[10] As far as we know, this calculation has never been carried out in this formalism. Even if by itself it might seem not extremely relevant since we arrive to essentially the same results as in component field theory [32] [28], we believe it is illustrative of the whole formalism and among with the calculation of the next section, it poses some relevant questions on the renormalization of the theory. Moreover, we plan to use it as part of an ongoing work [33].

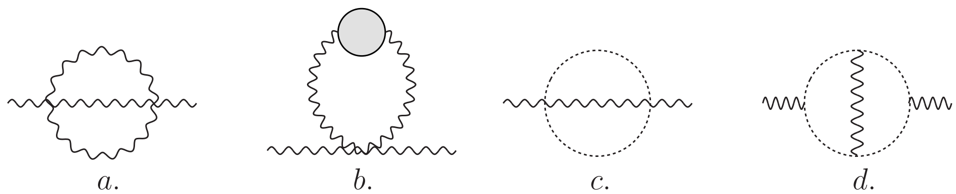


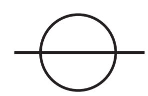
Figure 2.1.: Two-loop UV divergent gauge vector self-energy diagrams. Wavy lines correspond to gauge vectors while dashed lines represent ghosts. The blob in $b.$ is the sum of 1-loop gauge vector and ghost contributions.

diagrams will be

$$\Delta^{(2)}_{V,i} = \frac{1}{2} \int \frac{d^3p\, d^4\theta}{(2\pi)^3} v_i^{AB}(p,\epsilon) V^A \bar{D}^\alpha D_\alpha V^B, \quad \text{with}\, i = a, b, c, d. \tag{2.2.52}$$

The simplest diagram is $c.$ since its D-algebra is trivial. It is composed of four possible ghost circulations and results in

$$v_c^{AB} = -\frac{1}{3}\left(\frac{4\pi N}{k}\right)^2 (\delta^{AB} - \delta^{A\,0}\delta^{B\,0})$$

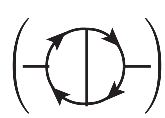


$$= -\frac{1}{12}\lambda^2(\delta^{AB} - \delta^{A\,0}\delta^{B\,0})\frac{1}{\epsilon} + \mathcal{O}(1), \tag{2.2.53}$$

where we have expanded in powers of $\epsilon = 3/2 - d/2$ and retained only the poles. We also defined the 't Hooft coupling as $\lambda = \frac{N}{k} = \frac{g^2 N}{4\pi}$.

Diagram $d.$ on the other hand, has a non so trivial D-algebra that produces only one Feynman integral. Of the four possible ghost circulations, only two of them produce non vanishing results

$$v_d^{AB} = \frac{1}{8}(4\pi\lambda)^2(\delta^{AB} - \delta^{A\,0}\delta^{B\,0}) \;\; \mathrm{tr}$$



$$= \frac{1}{16}\lambda^2(\delta^{AB} - \delta^{A\,0}\delta^{B\,0})\frac{1}{\epsilon} + \mathcal{O}(1), \tag{2.2.54}$$

where the trace is taken cyclically over the momenta in the internal lines (see appendix A for trace properties of the γ matrices and appendix D for properties of Feynman integrals in three dimensions).

From the appearance of diagram $b.$, one expects to obtain a tadpole-like diagram since evidently no external momenta may circulate through the internal lines. In fact, one obtains a tadpole-like diagram which is simultaneously ultraviolet and infrared divergent $\sim \int d^3k/|k|^3$. Since we are only interested in the UV divergence, we reshuffle the external momenta so as to extract only the UV divergence. We delay the analysis of infrared issues until chapter 4. We obtain

$$v_b^{AB} = \frac{1}{24}\lambda^2(\delta^{AB} - \delta^{A\,0}\delta^{B\,0})\frac{1}{\epsilon} + \mathcal{O}(1). \tag{2.2.55}$$

Finally, diagram $a.$ is the most difficult one. When deriving the Feynman rule for the self-interacting gauge vector four-vertex we obtain a total of $4! = 24$ terms with different distributions of the D and $\bar{D}$ operators on the lines. Since we have two four-vertices we have a total of $4!4! = 576$ combinations. By making the product of possibilities with an algebraic computer tool, after cancelations, a total of 144 terms survive. Of these 144, 60 may be discarded and the remaining 84 can be related to each other in three groups by using the symmetries of the different diagrams. In the end, after this process, only three simple D-algebras have to be performed. The result is

$$v_a^{AB} = -\frac{1}{48}\lambda^2(\delta^{AB} - \delta^{A\,0}\delta^{B\,0})\frac{1}{\epsilon} + \mathcal{O}(1). \tag{2.2.56}$$

The sum of all diagrams turns out to be zero. Thus $Z_V = 1$.

The contributions to self-energy two-loop corrections of the ghost fields will have the form

$$\Delta^{(2)}_{bc,i} = \int \frac{d^3p\, d^4\theta}{(2\pi)^3} h_i^{AB}(p,\epsilon)(\bar{b}^A c^B + \bar{c}^A b^B), \quad \text{with}\, i = a, b. \tag{2.2.57}$$

We depict the contributing diagrams in figure 2.2. Diagram $b.$ contributes with

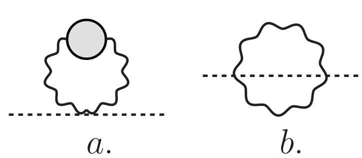


Figure 2.2.: Two-loop UV divergent ghosts self-energy diagrams. The blob in $a.$ is again the sum of a gauge vector simple 1-loop and four ghost 1-loop contributions.

$$h_b^{AB} = -\frac{1}{24}\lambda^2(\delta^{AB} - \delta^{A\,0}\delta^{B\,0})\frac{1}{\epsilon} + \mathcal{O}(1). \tag{2.2.58}$$

Diagram $a.$, once again, is UV and IR divergent and we extract the UV divergence. We obtain

$$h_a^{AB} = \frac{1}{24}\lambda^2(\delta^{AB} - \delta^{A\,0}\delta^{B\,0})\frac{1}{\epsilon} + \mathcal{O}(1). \tag{2.2.59}$$

The sum of both diagrams is zero and therefore $Z_{bc} = 1$. We have then obtained that both anomalous dimensions γ_V and γ_{bc} are zero at two loops. We have also checked that the sum of two loop diagrams correcting the ghost/gauge-vector three-vertex is also zero and therefore $Z_g = 1$. In this way, the whole theory at two loops is finite and conformal since $\beta(g) = 0$. This is the same two-loop result that was obtained for pure Chern-Simons theory in component formalism [32]. As we will see in the next section, once we add matter, this result is modified non trivially.

2.2.4. $\mathcal{N}=2$ Chern-Simons-Matter theories

Matter couplings

There are three irreducible ways in which we may couple chiral multiplets in a gauge-invariant way with the Chern-Simons multiplets. We consider the general case where there could be more than one $\mathcal{N}=2$ Chern-Simons multiplet with its corresponding $\mathcal{N}=2$ Chern-Simons action (for example, two Chern-Simons V and $\hat{V}$). Under the gauge transformations

$$e^{V'} = e^{i\bar{\Lambda}_1}e^{V}e^{-i\Lambda_1}, \quad e^{\hat{V}'}e^{i\bar{\Lambda}_2}e^{\hat{V}}e^{-i\Lambda_2}, \tag{2.2.60}$$

the possible gauge transformations of chiral multiplets are

$$\begin{aligned} &\text{Adjoint:} \quad \Phi' = e^{i\Lambda_1}\Phi e^{-i\Lambda_1} \quad \text{and} \quad \bar{\Phi}' = e^{i\bar{\Lambda}_1}\bar{\Phi}e^{-i\bar{\Lambda}_1} \\ &\text{Fundamental:} \quad \Phi' = e^{i\Lambda_1}\Phi \quad \text{and} \quad \bar{\Phi}' = \bar{\Phi}e^{-i\bar{\Lambda}_1} \\ &\text{Bifundamental:} \quad \Phi' = e^{i\Lambda_1}\Phi e^{-i\Lambda_2} \quad \text{and} \quad \bar{\Phi}' = e^{i\bar{\Lambda}_2}\bar{\Phi}e^{-i\bar{\Lambda}_1} \end{aligned} \tag{2.2.61}$$

Notice that the (anti)-chiral nature of the ($\bar{\Lambda}$) Λ parameters guarantees that the gauge transformation does not change the (anti)-chiral nature of the ($\bar{\Phi}$) Φ field. The invariant Lagrangians in each case are

$$\begin{aligned} &\text{Adjoint:} \quad \int d^3x d^4\theta \ \mathrm{tr}\left(e^{-V}\bar{\Phi}e^{V}\Phi\right) \\ &\text{Fundamental:} \quad \int d^3x d^4\theta \ \bar{\Phi}e^{V}\Phi \\ &\text{Bifundamental:} \quad \int d^3x d^4\theta \ \mathrm{tr}\left(e^{-\hat{V}}\bar{\Phi}e^{V}\Phi\right) \end{aligned} \tag{2.2.62}$$

By expanding each of these actions in powers of the gauge fields, the first term of the expansion corresponds to the kinetic term of the chiral field we had already presented in (2.1.18). The rest of the terms determine the interaction between scalar multiplets and gauge-vectors.

Adjoint-coupled chiral fields are key in the formulation of $d = 4$ maximally supersymmetric Yang-Mills theory formulated in $\mathcal{N} = 1$ formalism. Bifundamental matter will be essential in the formulation of $\mathcal{N} = 6$ Chern-Simons matter theory in $\mathcal{N} = 2$ language as we will present in the next chapter. In this section, we consider fundamental matter. We shall show how the component action of this Chern-Simons-matter theory is in order to illustrate the form of the interactions, and we study the influence of fundamental matter in the renormalization of the Chern-Simons sector.

Consider a $U(N)_k$ $\mathcal{N} = 2$ Chern-Simons Lagrangian coupled to one $\mathcal{N} = 2$ chiral scalar multiplet in the fundamental representation without superpotential (to have a superpotential one would need at least two chiral multiplets in the fundamental/anti-fundamental). Its action is

$$S^{(1)}_{\mathrm{csm}} = \frac{k}{4\pi}\int d^3x d^4\theta \int_0^1 dt \, \mathrm{tr}\left(V\,\bar{D}^{\alpha}(e^{-tV}D_{\alpha}e^{tV})\right) + \int d^3x d^4\theta \ \bar{\Phi}e^{V}\Phi. \tag{2.2.63}$$

Scaling $V \to gV$ and projecting the $d^4\theta$ integrals in the Wess-Zumino gauge (*c.f.*(2.2.15)) we obtain

$$\begin{aligned} S^{(1)}_{\text{csm}} = \int d^3x \, \Big[&\text{tr}\left(-i\epsilon^{\mu\nu\rho}(A_\mu\partial_\nu A_\rho + ig\tfrac{2}{3}A_\mu A_\nu A_\rho) - 2D\sigma - \bar{\chi}^\beta\chi_\beta\right) \\ &-(\mathcal{D}^\mu\phi)^\dagger(\mathcal{D}_\mu\phi) + i\bar{\psi}\not{\mathcal{D}}\psi + \bar{F}F + g\,\bar{\phi}D'\phi - g^2\,\bar{\phi}\sigma^2\phi \\ &-g\,\bar{\psi}^\alpha\bar{\chi}_\alpha\phi - g\,\bar{\phi}\chi^\alpha\psi_\alpha + g\,\bar{\psi}^\alpha\sigma\psi_\alpha\Big], \end{aligned} \tag{2.2.64}$$

where we have defined the covariant derivative $\mathcal{D}_\mu$ acting on matter fields as

$$\mathcal{D}_\mu\phi = \partial_\mu\phi + igA_\mu\phi, \quad \text{and} \quad (\mathcal{D}_\mu\phi)^\dagger = \partial_\mu\bar{\phi} - ig\bar{\phi}A_\mu. \tag{2.2.65}$$

Recall that the fields ϕ, ψ^α and F belong to the chiral multiplet. As such they are all in the fundamental representation such that we may label them as $U(N)$ column vectors in the fundamental, namely ϕ_a, $(\psi^\alpha)_a$, F_a, and $U(N)$ row vectors in the anti-fundamental, that is $\bar{\phi}^a$, $(\bar{\psi}^\alpha)^a$ and $\bar{F}^a$, where lowercase Latin letters run from 1 to N. On the other hand, the fields σ, χ^α, $A_{\alpha\beta}$ and D' belong to the gauge-vector multiplet and so they are in the adjoint representation. We may write them in matrix notation as ${\sigma_a}^b$, ${(\chi^\alpha)_a}^b$, ${(A_{\alpha\beta})_a}^b$ and ${D'_a}^b$. With this notation, from (2.2.64) we find the equations of motion for the auxiliary fields σ, D', χ and F

$$\begin{aligned} \frac{\delta S^{(1)}_{\text{csm}}}{\delta D'} &= 0 \;\to\; {\sigma_a}^b = \frac{g}{2}\bar{\phi}^b\phi_a, \qquad \frac{\delta S^{(1)}_{\text{csm}}}{\delta F} = 0 \;\to \bar{F} = 0, \\ \frac{\delta S^{(1)}_{\text{csm}}}{\delta \sigma} &= 0 \;\to\; {D'_a}^b = \frac{1}{2}(-g^2\bar{\phi}^b{\sigma_a}^c\phi_c - g^2\bar{\phi}^c{\sigma_c}^b\phi_a + g(\bar{\psi}^\alpha)^b(\psi_\alpha)_a) \\ &\qquad\qquad\quad = \frac{g}{2}(\bar{\psi}^\alpha)^b(\psi_\alpha)_a - \frac{g^3}{2}\bar{\phi}^b\phi_a\bar{\phi}^c\phi_c, \\ \frac{\delta S^{(1)}_{\text{csm}}}{\delta \chi^\alpha} &= 0 \;\to\; {(\bar{\chi}^\alpha)_a}^b = -g\bar{\phi}^b(\psi^\alpha)_a. \end{aligned} \tag{2.2.66}$$

Substituting these equations in (2.2.64) we obtain

$$\begin{aligned} S^{(1)}_{\text{csm}} = \int d^3x \, \Big[&-i\epsilon^{\mu\nu\rho}\text{tr}(A_\mu\partial_\nu A_\rho + ig\frac{2}{3}A_\mu A_\nu A_\rho) - (\mathcal{D}^\mu\phi)^\dagger(\mathcal{D}_\mu\phi) + i\bar{\psi}\not{\mathcal{D}}\psi \\ &+g^2(\bar{\psi}^\alpha\psi_\alpha)(\bar{\phi}\phi) + \frac{g^2}{2}(\bar{\psi}^\alpha\phi)(\bar{\phi}\psi_\alpha) - \frac{g^4}{4}(\bar{\phi}\phi)^3\Big]. \end{aligned} \tag{2.2.67}$$

Notice that the matter self interaction terms are of the same type we found from a chiral superpotential; namely, sextic scalar self-interactions and 'Yukawa-like' interactions between bosonic and fermionic matter. This is quite interesting since our starting point was an $\mathcal{N} = 2$ action without superpotential. In general, the self-interaction terms coming from the superpotential are called F-terms, while those coming from the coupling between the gauge-vector multiplet and the matter fields are called D-terms. The D-terms however, have coupling constants completely determined by $\mathcal{N} = 2$ supersymmetry and gauge invariance. Apart from both those types of matter self-interaction, the minimal coupling between bosonic and fermionic matter with the gauge vector, which is realized through the covariant derivatives $\mathcal{D}$, naturally emerges from the gauge invariant $\mathcal{N} = 2$ matter lagrangian we started with. We emphasize the fact that, even if pure $\mathcal{N} = 2$ CS theory seemed trivially identical to pure bosonic CS theory since the superpartners of the gauge field A were auxiliary fields, the lifting of the bosonic theory to an $\mathcal{N} = 2$ theory, when coupled to matter, has produced non trivial matter self-interactions.

Gauge sector renormalization in presence of matter

We would like to make the same calculation we did in section (2.2.3) but this time, we couple the CS multiplet to N_f and $\tilde{N}_f$ chiral multiplets in the fundamental and anti-fundamental respectively. This is

$$S_{mat}^{N_f} = \int d^3x d^4\theta \; \left(\bar{A}_k e^{gV} A_k + B_{\dot{k}} e^{-gV} \bar{B}_{\dot{k}}\right), \tag{2.2.68}$$

with gauge transformations given by ($k = 1 \ldots N_f$ and $\dot{k} = 1 \ldots \tilde{N}_f$)

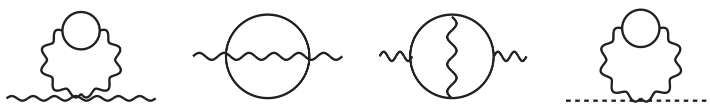
$$A'_k = e^{i\Lambda} A_k, \qquad B'_{\dot{k}} = B_k e^{-i\Lambda}, \qquad \bar{A}'_k = \bar{A}_k e^{-i\bar{\Lambda}}, \qquad \bar{B}'_{\dot{k}} = e^{i\bar{\Lambda}} \bar{B}_{\dot{k}} \tag{2.2.69}$$

Notice that since now we have more than one fundamental chiral multiplet coupled to the CS gauge vector, we could have a gauge invariant superpotential. For example

$$\lambda_1 \int d^3x d^2\theta \; B_{\dot{k}} \, A_k \, B_{\dot{k}} \, A_k + \text{h.c.}. \tag{2.2.70}$$

However, as one can readily see, the two loop calculation of the CS renormalization properties do not depend on the form of the superpotential and only depend on their coupling to matter fields; at this order we do not need to assume anything about the superpotential.

To calculate the anomalous dimensions of the ghost field and the vector field in the presence of fundamental matter, we just need to calculate those diagrams which effectively contain matter interactions. This is because as we saw in section (2.2.3), the sum of purely vector/ghost-corrected diagrams is zero independently for γ_v and for γ_{bc}. We display both gauge vector and ghost two-loop self-energy diagrams with matter interactions in figure 2.3.



a. *b.* *c.* *d.*

Figure 2.3.: Two-loop matter contribution to UV divergent gauge vector (a, b, c) and ghost (d) self-energy diagrams. Wavy: gauge vector, straight: matter, dashed: ghost.

Once again, diagram a is a UV/IR tadpole from which we extract the UV divergence

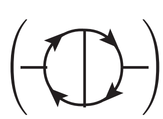
$$v^{AB}_{\text{mat},a} = -\frac{1}{24}\frac{(N_f + \tilde{N}_f)N}{k^2}(\delta^{AB} - \delta^{A\,0}\delta^{B\,0})\frac{1}{\epsilon} + \mathcal{O}(1). \tag{2.2.71}$$

In a similar way, diagram b contributes with

$$v^{AB}_{\text{mat},b} = \frac{1}{8}\frac{(N_f + \tilde{N}_f)N}{k^2}(\delta^{AB} + \delta^{A\,0}\delta^{B\,0})\frac{1}{\epsilon} + \mathcal{O}(1), \tag{2.2.72}$$

but notice how the color parenthesis has a relative sign different from diagram a. Finally, diagram c contributes only to the abelian part of the gauge-vector self energy (this is the direction '0' in the Lie Algebra, *c.f* Appendix B)

$$v^{AB}_{\text{mat},c} = -\frac{1}{2}\left(\frac{4\pi}{k}\right)^2 N(N_f + \tilde{N}_f)\delta^{A\,0}\delta^{B\,0}\ \ \mathrm{tr}$$



$$= -\frac{1}{4}\frac{(N_f + \tilde{N}_f)N}{k^2}\delta^{A\,0}\delta^{B\,0}\frac{1}{\epsilon} + \mathcal{O}(1). \tag{2.2.73}$$

the sum of the three diagrams gives

$$v^{AB}_{\text{mat},a}+v^{AB}_{\text{mat},b}+v^{AB}_{\text{mat},c}=\frac{1}{12}\frac{(N_f+\tilde{N}_f)N}{k^2}(\delta^{AB}-\delta^{A\,0}\delta^{B\,0})\frac{1}{\epsilon}+\mathcal{O}(1). \tag{2.2.74}$$

This time, due to the presence of matter, the result is non zero. An important remark follows (2.2.74): Notice that when $A=B=0$, the sum is zero while for $A=i$, $B=j$ with i,j labeling the $SU(N)$ part of $U(N)$, the sum is diagonal (proportional to δ^{ij}). This in fact is what one expects from gauge invariance since this reveals that the abelian $U(1)$ normal subgroup of $U(N)$ is not involved neither in the gauge-vector self interactions nor in the ghost/gauge-vector interactions; its anomalous dimension should be vanishing [27]. On the other hand, the anomalous dimension of the non-abelian part of the gauge vector will not be zero in the current case. We define the modified renormalization constant for the non-abelian part of the kinetic term of the gauge field as $\tilde{Z}_v$. From (2.2.74), in the minimal substraction scheme we have obtained

$$\tilde{Z}_v=1-\frac{1}{12}\frac{(N_f+\tilde{N}_f)N}{k^2}\frac{1}{\epsilon}. \tag{2.2.75}$$

The only contributing diagram to the two-loop ghost self energy is diagram d in figure 2.3. It gives

$$h^{AB}_{\text{mat},d}=-\frac{1}{24}\frac{(N_f+\tilde{N}_f)N}{k^2}(\delta^{AB}-\delta^{A\,0}\delta^{B\,0})\frac{1}{\epsilon}+\mathcal{O}(1), \tag{2.2.76}$$

which, as in the gauge-vector case, is zero for the abelian direction '0' and non vanishing for the non abelian parts. We obtain the modified wave-function renormalization constant $\tilde{Z}_{bc}$ for the ghost fields

$$\tilde{Z}_{bc}=1+\frac{1}{24}\frac{(N_f+\tilde{N}_f)N}{k^2}\frac{1}{\epsilon}. \tag{2.2.77}$$

In order to study the renormalization flow of the g coupling, we still need the renormalization constant for the ghost/gauge-vector three vertex. It is easy to show that the sum of two loop diagrams correcting that vertex is zero [11], and therefore $\tilde{Z}_g=1$. With

[11] It seems curious that even if Z_v and Z_{bc} are modified in the presence of matter Z_g is still equal to 1. In [34] a theorem proving the finiteness of the ghost/gauge-field three vertex in presence of both bosonic and fermionic matter was given. Even if it seems to work in our formalism, it is not clear to us if the theorem is still valid in this context.

these results and using identity (2.2.51) we may derive the renormalization constant for the b-c-V^{2n} vertexes

$$\tilde{Z}_{g^{2n}} = 1 + (1-2n)\frac{1}{24}\frac{(N_f+\tilde{N}_f)N}{k^2}\frac{1}{\epsilon}. \tag{2.2.78}$$

The renormalization flow equation for the coupling g is

$$\tfrac{dg}{d(\log\mu)} = g\left(\gamma_v + 2\gamma_{bc} - \tfrac{d\log\tilde{Z}_g}{d\log\mu}\right), \tag{2.2.79}$$

but since $\tilde{Z}_g$ is trivial it simplifies to

$$\beta(g) = g\left(\gamma_v + 2\gamma_{bc}\right). \tag{2.2.80}$$

We may obtain the anomalous dimensions through the usual formulas valid to this order: $\gamma_i = -\lim_{\epsilon\to 0}\frac{\epsilon g}{2}(Z_i(g))'$. We obtain

$$\gamma_v = \frac{1}{6}(N_f+\tilde{N}_f)\frac{N}{k^2} \qquad , \gamma_{bc} = -\frac{1}{12}(N_f+\tilde{N}_f)\frac{N}{k^2}, \tag{2.2.81}$$

such that

$$\gamma_v + 2\gamma_{bc} = 0 \quad \Rightarrow \quad \beta(g) = 0. \tag{2.2.82}$$

Thus, even if the gauge-vector superfield and the ghosts have acquired an anomalous dimension, the coupling constant still does not flow in the presence of matter. This is consistent with the fact that k must be an integer so that large gauge transformations leave the theory invariant. A renormalization group flow for that constant would have driven it to non integer values. On the other hand, the perturbative analysis is completely independent of the integer-k condition, so it is non trivial that still, perturbation theory respects that condition.

We found several times in the literature the misbelief that the $\mathcal{N}=2$ Chern-Simons sector is finite. As was shown in [32] for bosonic Chern-Simons theory coupled to matter, in [28] for $\mathcal{N}=1$ and $\mathcal{N}=2$ supersymmetric CSM theories using component and $\mathcal{N}=1$ formalisms, and here for $\mathcal{N}=2$ Chern-Simons-matter theory in $\mathcal{N}=2$ superspace, while the introduction of matter does not change the beta function which is still vanishing making the theory a conformal candidate, it does introduce anomalous dimensions to the fields and the necessity of an infinite renormalization of the gauge-vector self-interaction and gauge-vector/ghosts interaction vertexes. This is not inconsistent with the k-integer condition since all those infinite renormalizations of wave-functions and vertexes smoothly add up so that the Chern-Simons level does not flow.

In the next section, when we introduce $\mathcal{N} = 6$ CSM theory, we will couple the Chern-Simons multiplets with matter in the bifundamental. We have explicitly verified that the influence of matter in the bifundamental in the renormalization of the gauge sector is qualitatively the same as matter in the fundamental: the gauge-vector and the ghost wave-functions receive an infinite renormalization but still the Chern-Simons level does not flow making the theory a conformal candidate.

It could be interesting to repeat the preceding calculation in a general gauge with $\alpha \neq 0$ and see if there is a gauge choice in which the gauge-vector and ghost field have a vanishing anomalous dimension. Since the $\beta(g)$ function, which is gauge invariant in the minimal substraction scheme, is the sum of both anomalous dimensions ($\sim \gamma_v + 2\gamma_{bc}$), each anomalous dimension need not be gauge invariant and it could be the case that there is gauge choice which makes them both vanish. We suspect, though we do not have compulsive evidence, that this is not the case; there is not a gauge choice in which the anomalous dimension of both fields vanish, making the renormalization non-trivial and the theory not finite in any gauge.

In chapter 4, we will introduce a different gauge fixing procedure from the one used in this section with the objective of showing the possibility of eliminating infrared divergencies from the perturbative calculations. As a non-trivial check of the consistency of the gauge fixing procedure, at the end of that chapter, we will perform the exact same calculation of this section in the new gauge and show that the same conclusions of this section hold: the gauge-vector and the ghosts have a non-vanishing anomalous dimension while still the Chern-Simons level does not flow.

Chapter 3.

$\mathcal{N} = 6$ Chern-Simons-Matter theory

After reviewing characteristics and general properties of the superspace formulation of $\mathcal{N} = 2$ theories, we devote this chapter to the review of higher supersymmetric Chern-Simons-Matter theories. We begin by revisiting the infrared flow of mass-deformed $\mathcal{N} = 4$ Yang-Mills matter theories into $\mathcal{N} = 3$ CSM theories. Then, we use that construction in order to derive $\mathcal{N} = 6$ supersymmetric CSM theory. The latter, also called ABJ(M) theory is the main focus of the remaining chapters of this work. We end the chapter by describing the gravity dual proposed for this theory.

3.1. $\mathcal{N} = 3$ theories

To study $\mathcal{N} = 3$ CS theories we begin by analyzing deformations of $\mathcal{N} = 4$ theories. As mentioned in the previous chapter, an easy way to determine the physical fields of higher supersymmetry multiplets is to consider $\mathcal{N}/2$ multiplets in $d = 4$ and dimensionally reducing them to $d = 3$.

Therefore $\mathcal{N} = 4$ three-dimensional multiplets can be thought as $\mathcal{N} = 2$ four dimensional ones dimensionally reduced. The two lower superspin multiplets in $\mathcal{N} = 2$, $d = 4$ are the vector multiplet (gauge) and the scalar hypermultiplet (matter). In the language of $\mathcal{N} = 1$, $d = 4$, the $\mathcal{N} = 2$ vector multiplet is formed by an $\mathcal{N} = 1$ vector multiplet (V) and a chiral scalar multiplet (Φ), both in the adjoint representation of the gauge group. As we saw in the last Chapter, when dimensionally reducing the $\mathcal{N} = 1$ vector multiplet V we obtain the $\mathcal{N} = 2$, $d = 3$ vector multiplet with a neutral scalar σ, the complex two-component gaugino χ and the gauge vector A_μ. The $\mathcal{N}=1$, $d=4$ chiral scalar on the other hand, reduces to an $\mathcal{N}=2$, $d=3$ chiral scalar with a complex scalar (ϕ) and a complex fermion (ψ) as component fields all in the adjoint.

So in all, our $\mathcal{N}=2$, $d=4$ gauge-vector multiplet reduces to an $\mathcal{N}=4$, $d=3$ vector multiplet which in the $\mathcal{N}=2$, $d=3$ language can be expressed in terms of gauge-vector V and a chiral scalar Φ. The $\mathcal{R}$ symmetry group of $\mathcal{N}=4$ supersymmetry in three dimensions is $SO(4)\simeq SU(2)_R\times SU(2)_N$. The component fields are the three real scalars (the real σ and the complex ϕ) that form a triplet under $SU(2)_R$, the four Majorana fermions (the complex gaugino χ and the complex ψ) which is a four component simultaneous doublet of each of the $SU(2)$'s and finally the gauge vector A_μ.

The $\mathcal{N}=2$, $d=4$ scalar matter hypermultiplet on the other hand, from the $\mathcal{N}=1$, $d=4$ perspective is formed by two chiral multiplets (we shall call them W and Z), which if coupled to the gauge fields, should come in conjugate representations of the gauge group (opposite charges). When dimensionally reducing these multiplets we arrive to the $\mathcal{N}=4$, $d=3$ scalar hypermultiplet which will be formed by two $\mathcal{N}=2$ chiral multiplets in conjugate representations. The component fields are two complex scalars which form a doublet of $SU(2)_N$ and two complex spinors which form a doublet of $SU(2)_R$

With the $\mathcal{N}=4$ vector multiplets and hypermultiplets one can construct an $\mathcal{N}=4$ supersymmetric Yang-Mills matter theory. In $\mathcal{N}=2$ language it has the standard Super Yang-Mills term for the V field and standard kinetic term for the chiral Φ in the adjoint (*c.f.* 2.2.16)

$$S_{SYM}^{\mathcal{N}=4}=\frac{1}{g_{YM}^2}\int d^3xd^4\theta\,\mathrm{tr}\left(-\frac{1}{2}\Sigma^2+\tilde{\Phi}\Phi\right). \tag{3.1.1}$$

Moreover we have an even number ($2N_f$) of matter fields W_A and Z^A (each pair forms an $\mathcal{N}=4$ hyper) in a given representation of the gauge group and the conjugate one respectively, which interact with the $\mathcal{N}=4$ vector multiplet. For simplicity, we start by considering matter in the fundamental (Z^A) and anti-fundamental (W_A). The interactions are the minimal ones with the V field and an interaction superpotential between the matter fields and the chiral Φ $(A = 1 \dots N_f)$.

$$S_{mat}^{\mathcal{N}=4} = \int d^3x d^4\theta \left(W_A e^{-V} \bar{W}^A + \bar{Z}_A e^V Z^A\right) + \int d^3x d^2\theta \, W_A \Phi Z^A + \text{h.c.} \qquad (3.1.2)$$

The full theory $S_{SYM}^{\mathcal{N}=4} + S_{mat}^{\mathcal{N}=4}$ is $\mathcal{N} = 4$ supersymmetric [35]. Notice that the superpotential coupling can be read from the action by putting the $\mathcal{N} = 4$ vector multiplet in the canonical kinetic normalization by scaling $(V, \Phi) \to g_{YM}(V, \Phi)$. In this way, the superpotential coupling is g_{YM} and is fixed by $\mathcal{N} = 4$ supersymmetry.

We now deform this theory by giving masses to the $\mathcal{N} = 4$ vector multiplet. The addition of masses breaks the $SU(2)_R \times SU(2)_N$ $\mathcal{R}$-symmetry down to $SU(2)_D$ and $\mathcal{N} = 4$ supersymmetry down to $\mathcal{N} = 3$ [35]. The physical content is the same but now the fields are organized in a different way under the new $\mathcal{R}$-symmetry group. The hypermultiplets will have $SU(2)_D$ doublets of complex scalars and $SU(2)_D$ doublets of spinors. The vector multiplet on the other hand maintains a triplet of scalars (the real σ and the complex ϕ) under $SU(2)_D$ while the four spinors become a triplet and a singlet of $SU(2)_D$.

From the Yang-Mills point of view, the addition of a mass in a gauge invariant way can be achieved by introducing a Chern-Simons term in the action while for the Φ field we add a $\frac{1}{2} im \, \text{tr}\Phi^2$ term to the superpotential [1]. To maintain $\mathcal{N} = 3$ supersymmetry, both masses must be the same. The Chern-Simons term with level k provides a mass $g_{YM}^2 \frac{k}{4\pi}$ to the gauge-vector so we add to our $\mathcal{N} = 4$ action the terms [2]

$$\frac{k}{4\pi} \int d^3x d^4\theta \int_0^1 dt \, \text{tr} V \tilde{\Sigma}(t) + i \frac{k}{8\pi} \int d^3x d^2\theta \, \text{tr} \, \Phi^2 + \text{h.c.} \qquad (3.1.3)$$

At low energies compared to this mass scale, the vector multiplet fields may be integrated out. A very well known effect of the Chern-Simons level shifting is produced by the integration of the four massive fermions, but this effect is canceled by the contribution

[1] The phase in the mass term is completely irrelevant once auxiliary fields are integrated out. We choose it to be $e^{i\pi/2} = i$ for later convenience.

[2] Notice that the full scale $g_{YM}^2 \frac{k}{4\pi}$ is not explicit in the action since the fields (V, Φ) do not have the standard kinetic term normalization.

of the gauge field [36]. The effective action in this limit becomes

$$
\begin{aligned}
S_{CS}^{\mathcal{N}=3} =& \frac{k}{4\pi}\int d^3x d^4\theta \int_0^1 dt\, \mathrm{tr} V\tilde{\Sigma}(t) + \int d^3x d^4\theta \left(W_A e^{-V}\bar{W}^A + \bar{Z}_A e^V Z^A\right) \\
&+ \int d^3x d^2\theta \left(W_A \Phi Z^A + i\frac{k}{8\pi}\mathrm{tr}\,\Phi^2\right) + \text{h.c.}
\end{aligned}
\tag{3.1.4}
$$

Since the Φ superfield has no kinetic term, we take its equation of motion

$$
\Phi^b{}_a = i\frac{4\pi}{k}(W_A)^b(Z^A)_a, \tag{3.1.5}
$$

and we substitute it back on the action to obtain the superpotential

$$
S_{CS,sup}^{\mathcal{N}=3} = i\frac{2\pi}{k}\int d^3x d^2\theta\, (W_A Z^B)(W_B Z^A) + \text{h.c.} \tag{3.1.6}
$$

So we see that from the $\mathcal{N}=2$ point of view, a Chern-Simons matter theory with $\mathcal{N}=3$ supersymmetry can be written as a theory of an even number of matter multiplets minimally coupled to the vector multiplet V in conjugate representations, and with a superpotential whose coefficient is completely determined by $\mathcal{N}=3$ supersymmetry. Since the $SU(2)$ $\mathcal{R}$-charge cannot be renormalized and the superpotential coefficient is basically the inverse of the Chern-Simons level we expect the $\mathcal{N}=3$ CSM theory to be exactly conformal [37].

3.2. ABJ(M) theory

The construction of $\mathcal{N}=6$ Chern-Simons matter theory can be seen as a particular example of the $\mathcal{N}=3$ construction of the last section. Surprisingly, a particular choice of that construction produces an enhancement of the $\mathcal{R}$-symmetry such that a highly supersymmetric theory emerges [5].

We begin by considering a pair of $\mathcal{N}=4$ hypermultiplets composed of two pairs of $\mathcal{N}=2$ scalar multiplets in the bifundamental and in the anti-bifundamental respectively;

under gauge transformations they transform as

$$Z'^A = e^{i\Lambda_1} Z_A e^{-i\Lambda_2}, \qquad W'_A = e^{i\Lambda_2} W_A e^{-i\Lambda_1}, \quad A = 1, 2. \tag{3.2.1}$$

We also have two $\mathcal{N}=4$ vector multiplets (V, Φ_1) and $(\hat{V}, \Phi_2)$ transforming in the adjoint of the two gauge groups $U(M)$ and $U(N)$ with gauge transformations parameterized by Λ_1 and Λ_2. They interact with the hypermultiplets as before such as to have an $\mathcal{N}=4$ theory. Once again, we deform the theory by adding masses to the $\mathcal{N}=4$ vector multiplets. Since we have two vector multiplets we have the freedom of choosing two different masses $g^2_{YM}\frac{k_1}{4\pi}$ and $g^2_{YM}\frac{k_2}{4\pi}$ for each multiplet. We choose the Chern-Simons levels to be equal in modulus but of opposite sign, that is $k_1 = -k_2 = k$.

After flowing to energy scales small compared to the mass scale, we find as before a Chern-Simons theory coupled to the hypermultiplets with a superpotential given by

$$\int d^3x d^2\theta \,\mathrm{tr}\left(W_A \Phi_1 Z^A + Z^A \Phi_2 W_A + i\frac{k}{8\pi}({\Phi_1}^2 - {\Phi_2}^2)\right) + \text{h.c.}, \tag{3.2.2}$$

with both Φ fields that became auxiliary. Their equations of motion are

$$\Phi_1 = i\frac{4\pi}{k} Z^A W_A, \qquad \Phi_2 = -i\frac{4\pi}{k} W_A Z^A. \tag{3.2.3}$$

Inserting them back in the action we obtain the superpotential

$$\begin{aligned} S^{\mathcal{N}=6}_{sup} =& i\frac{2\pi}{k}\int d^3x d^2\theta \,\left(\mathrm{tr}(Z^A W_A Z^B W_B) - \mathrm{tr}(W_A Z^A W_B Z^B)\right) + \text{h.c.} \\ =& \, i\frac{4\pi}{k}\int d^3x d^2\theta \,\left(\mathrm{tr}(Z^1 W_1 Z^2 W_2) - \mathrm{tr}(Z^1 W_2 Z^2 W_1)\right) + \text{h.c.} \end{aligned} \tag{3.2.4}$$

Notice that in the Abelian case ($N = M = 1$) this superpotential vanishes. From the construction itself, we know this theory has at least $\mathcal{N}=3$ supersymmetry where the hypermultiplets organize themselves in pairs $(Z^1, \bar{W}^1)$ and $(Z^2, \bar{W}^2)$ as doublets of the $SU(2)_D$ $\mathcal{R}$-symmetry. The key observation is that the superpotential can be written in the following way

$$S^{\mathcal{N}=6}_{sup} = i\frac{2\pi}{k}\int d^3x d^2\theta \,\epsilon_{AB}\epsilon^{CD}\mathrm{tr}\left(Z^A W_C Z^B W_D\right) + \text{h.c.}, \tag{3.2.5}$$

where ϵ^{AB} is the Levi-Civita tensor. In this way we make evident the existence of a global $SU(2)\times SU(2)$ flavor symmetry which acts separately on the W fields and the Z fields. The kinetic terms of the matter fields clearly have this symmetry if we accommodate conjugate fields in the conjugate representation of the $SU(2)\times SU(2)$. But since the $SU(2)_D$ $\mathcal{R}$-symmetry mixes the W and Z fields into one another, it is obvious that the new symmetry does not commute with the original $\mathcal{R}$-symmetry. Combining both symmetries we have an enhanced $SU(4)$ symmetry such that the 4-tuple $(Z^1, Z^2, \bar{W}^1, \bar{W}^2)$ transforms in the $\mathbf{4}$ of the group. Since the supercharges cannot be a singlet under this enhanced symmetry, it seems that we have at least $\mathcal{N}=6$ supersymmetry with an $\mathcal{R}$-symmetry group $SO(6)\sim SU(4)_R$. In the $\mathcal{N}=2$ formulation, only the $U(1)_R\times SU(2)\times SU(2)$ symmetry is evident (the first $U(1)_R$ is the $\mathcal{R}$-symmetry of the $\mathcal{N}=2$ formalism). A way of verifying the existence of the enhanced $SU(4)$ symmetry is to derive the whole scalar potential of the theory and verify it there. We will take these steps in what follows.

This $\mathcal{N}=6$ Chern-Simons matter theory is called ABJ theory for $N\neq M$ and ABJM theory for $N=M$. For completion we write down its full $\mathcal{N}=2$ superspace action which is given by $S^{\mathcal{N}=6}=S_{CS}+S_{mat}+S_{sup}$, where

$$S_{CS}=\frac{k}{4\pi}\int d^3x d^4\theta \int_0^1 dt\ \mathrm{tr}\left[V\bar{D}^\alpha\left(e^{-tV}D_\alpha e^{tV}\right)-\hat{V}\bar{D}^\alpha\left(e^{-t\hat{V}}D_\alpha e^{t\hat{V}}\right)\right] \tag{3.2.6}$$

$$S_{mat}=\int d^3x d^4\theta\ \mathrm{tr}\left(\bar{W}^A e^{\hat{V}}W_A e^{-V}+\bar{Z}_A e^V Z^A e^{-\hat{V}}\right) \tag{3.2.7}$$

$$S_{sup}=\int d^3x d^2\theta\ \mathcal{W}[W,Z]+\int d^3x d^2\bar{\theta}\ \bar{\mathcal{W}}[\bar{W},\bar{Z}], \tag{3.2.8}$$

with

$$\mathcal{W}=i\frac{2\pi}{k}\,\epsilon_{AC}\epsilon^{BD}\mathrm{tr}(Z^A W_B Z^C W_D),\qquad \bar{\mathcal{W}}=i\frac{2\pi}{k}\,\epsilon^{AC}\epsilon_{BD}\mathrm{tr}(\bar{Z}_A\bar{W}^B\bar{Z}_C\bar{W}^D). \tag{3.2.9}$$

Since we will work with this theory in all the remaining part of the work, we summarize its content fields setting up the notation. The chiral superfields Z^A and W_A (where ${\scriptstyle A,B,C,D=1,2}$) transform in the $(\mathbf{2},\mathbf{1})$ and $(\mathbf{1},\mathbf{2})$ of the global $SU(2)\times SU(2)$ that we have just established. Moreover, they transform in the $(\mathbf{M},\bar{\mathbf{N}})$ and $(\bar{\mathbf{M}},\mathbf{N})$ of the gauge group $U_k(M)\times U_{-k}(N)$, such that if explicit gauge group labeling is needed, the chiral superfields are $Z^a_{\hat{a}}$, $\bar{Z}^{\hat{a}}_a$, $W^{\hat{a}}_a$, $\bar{W}^a_{\hat{a}}$, with $a,b,\hat{a},\hat{b}=1,..,N$. The gauge vector superfields V

and $\hat{V}$ are in the adjoint representation of the groups $U(M)$ and $U(N)$ respectively and may be written either as $V = T^I V^I$ with $I, J, K = 0, \ldots, N^2 - 1$ or with matrix labeling $V_b{}^a$, $\hat{V}_{\hat{a}}^{\hat{b}}$. In appendix B we give useful properties of $U(N)$ structure constants and in appendix D we provide the Feynman rules for ABJ theory. We denote the component fields of the matter fields as

$$\begin{aligned} W_A(x_L, \theta) &= w_A(x_L) + \theta^\alpha \psi_{A\alpha}(x_L) - \theta^2 F_A(x_L), \\ Z^A(x_L, \theta) &= z^A(x_L) + \theta^\alpha \eta_\alpha{}^A(x_L) - \theta^2 G^A(x_L). \end{aligned} \tag{3.2.10}$$

As any gauge theory with matter in the adjoint or bifundamental, this theory admits a large N, M expansion such that the perturbative expansion is dominated by planar diagrams. Since the perturbative expansion is given in terms of $1/k$, we may take k large such that perturbation theory is valid, and N and M large such that only planar diagrams are considered, as long as we keep $\lambda = N/k$, $\hat{\lambda} = M/k$ fixed and small. In fact, when one is interested only in the planar limit, one defines

$$\bar{\lambda} = \sqrt{\lambda \hat{\lambda}}, \qquad \sigma = \frac{\lambda - \hat{\lambda}}{\bar{\lambda}}. \tag{3.2.11}$$

Perturbation theory is then given in powers of $\bar{\lambda}$ while σ measures how far we are from ABJM theory.

In order to verify the larger $SU(4)_R$ symmetry of the theory we derive the scalar sextic potential. For this, we first project (3.2.6) in the Wess-Zumino gauge

$$\begin{aligned} S_{CS} = \frac{k}{4\pi} \int d^3x \, \Big[\mathrm{tr} \Big(&-i\epsilon^{\mu\nu\rho} (A_\mu \partial_\nu A_\rho + i\tfrac{2}{3} A_\mu A_\nu A_\rho - \hat{A}_\mu \partial_\nu \hat{A}_\rho - i\tfrac{2}{3} \hat{A}_\mu \hat{A}_\nu \hat{A}_\rho) \\ &-2D\sigma - \bar{\chi}^\beta \chi_\beta + 2\hat{D}\hat{\sigma} + \hat{\bar{\chi}}^\beta \hat{\chi}_\beta \Big) \Big] . \end{aligned} \tag{3.2.12}$$

We also need the matter action (3.2.7) D-terms to quadratic order in V and $\hat{V}$ (in the Wess-Zumino gauge, the subsequent orders are vanishing)

$$\begin{aligned}
S_{\rm mat} = &\int d^3x d^4\theta \Big[\operatorname{tr} \bar{Z}_A \Big(Z^A + VZ^A - Z^A\hat{V} + \frac{1}{2}(V^2Z^A + Z^A\hat{V}^2) - VZ^A\hat{V} \Big) \; + \ldots \\
&+ \operatorname{tr} \bar{W}^A \Big(W_A + \hat{V}W_A - W_AV + \frac{1}{2}(\hat{V}^2W_A + W_AV^2) - \hat{V}W_AV \Big) + \ldots \Big] = \\
&\int d^3x \operatorname{tr} \Big(-(\mathcal{D}^\mu w)^{\dagger A}(\mathcal{D}_\mu w)_A - (\mathcal{D}^\mu z)^\dagger_A(\mathcal{D}_\mu z)^A + G^A\bar{G}_A + F_A\bar{F}^A \\
&+ \bar{w}^A(\hat{D}w_A - w_AD) + \bar{z}_A(Dz^A - z^A\hat{D}) + \bar{w}^A(-\hat{\sigma}^2 w_A - w_A\sigma^2 + 2\hat{\sigma}w_A\sigma) \\
&+ \bar{z}_A(-\sigma^2 z^A - z^A\hat{\sigma}^2 + 2\sigma w_A\hat{\sigma}) \Big) + \text{‘fermions’}, \qquad (3.2.13)
\end{aligned}$$

with gauge covariant derivatives defined by

$$\mathcal{D}_\mu z^A = \partial_\mu z^A + iA_\mu\, z^A - iz^A\, \hat{A}_\mu, \qquad \mathcal{D}_\mu w_A = \partial_\mu w_A + i\hat{A}_\mu\, w_A - iw_A\, A_\mu. \qquad (3.2.14)$$

We have only retained the bosonic terms since we are interested in deriving the scalar potential. The equation of motion for the auxiliary fields of the gauge-vector permits us to solve for fields $D, \hat{D}, \sigma, \hat{\sigma}$

$$\begin{aligned}
\sigma &= \frac{2\pi}{k}(z^A\bar{z}_A - \bar{w}^A w_A), \quad D = \frac{2\pi}{k}(\bar{w}^A M_A + \bar{M}^A w_A - N^A\bar{z}_A - z^A\bar{N}_A) + \text{‘ferm.’}, \\
\hat{\sigma} &= \frac{2\pi}{k}(\bar{z}_A z^A - w_A\bar{w}^A), \quad \hat{D} = \frac{2\pi}{k}(w_A\bar{M}^A + M_A\bar{w}^A - \bar{N}_A z^A - \bar{z}_A N^A) + \text{‘ferm.’},
\end{aligned} \qquad (3.2.15)$$

where we used the definitions

$$N^A = \sigma z^A - z^A\hat{\sigma} \qquad \text{and} \qquad M_A = \hat{\sigma}w_A - w_A\sigma \qquad (3.2.16)$$

On the other hand, projecting the θ-integrals in the (anti)-chiral part of the action (3.2.8) we obtain

$$S_{sup} = \int d^3x \operatorname{tr} \left(\frac{\partial\mathcal{W}}{\partial Z^A}\bigg| G^A + \frac{\partial\mathcal{W}}{\partial W_A}\bigg| F_A + \text{h.c.} \right) + \text{‘ferm.’}, \qquad (3.2.17)$$

from which among with the F and G dependent terms in (3.2.13) we may derive the equation of motion for auxiliary matter fields

$$F_A = -\frac{4\pi i}{k}\epsilon^{BD}\epsilon_{AC}\bar{z}_D\bar{w}^C\bar{z}_B, \quad G^A = -\frac{4\pi i}{k}\epsilon^{AC}\epsilon_{BD}\bar{w}^B\bar{z}_C\bar{w}^D. \tag{3.2.18}$$

After substituting the equations of motion for the auxiliary fields back in the action we obtain the ABJ component action

$$\begin{aligned} S^{\mathcal{N}=6} = \int d^3x\, \mathrm{tr}\Big[&-\frac{ik}{4\pi}\epsilon^{\mu\nu\rho}(A_\mu\partial_\nu A_\rho + ig\frac{2}{3}A_\mu A_\nu A_\rho - \hat{A}_\mu\partial_\nu\hat{A}_\rho + i\frac{2}{3}\hat{A}_\mu\hat{A}_\nu\hat{A}_\rho) \\ &-(\mathcal{D}^\mu w)^{\dagger A}(\mathcal{D}_\mu w)_A + i\bar{\psi}^A \not{\mathcal{D}}\psi_A - (\mathcal{D}^\mu z)^\dagger_A(\mathcal{D}_\mu z)^A + i\bar{\eta}_A\not{\mathcal{D}}\eta^A\Big] - V_{bos} - V_{ferm}. \end{aligned} \tag{3.2.19}$$

The scalar potential V_{bos} will be given by the sum of the F-terms contribution (F and G auxiliary fields) and the D-terms contribution (σ, D, $\hat{\sigma}$ and $\hat{D}$ auxiliary fields). After a lengthy calculation where the Chern-Simons contribution ($2\sigma D$) combines with the linear in D and quadratic in σ substituted pieces of the action, we obtain

$$V^D_{bos} = \mathrm{tr}\left(\bar{M}^A M_A + \bar{N}_A N^A\right), \quad \text{and} \quad V^F_{bos} = \mathrm{tr}\left(\bar{F}_A F^A + \bar{G}^A G_A\right), \tag{3.2.20}$$

making evident the fact that it vanishes only when F_A, G^A, N_A and M^A vanish (this happens for example in the abelian $N = M = 1$ case). Substituting the expressions we had obtained for the F_A, G^A, N_A and M^A fields we get

$$\begin{aligned} V^F_{bos} = \frac{16\pi^2}{k^2}\mathrm{tr}\big(&\bar{z}_A\,\bar{w}^B\,\bar{z}_C\,z^C\,w_B\,z^A - \bar{z}_A\,\bar{w}^B\,\bar{z}_C\,z^A\,w_B\,z^C \\ &+\bar{w}^A\,\bar{z}_B\,\bar{w}^C\,w_C\,z^B\,w_A - \bar{w}^A\,\bar{z}_B\,\bar{w}^C\,w_A\,z^B\,w_C\big), \end{aligned} \tag{3.2.21}$$

and

$$\begin{aligned} V^D_{bos} = \frac{4\pi^2}{k^2}\mathrm{tr}\big[&(z^A\,\bar{z}_A + \bar{w}^A\,w_A)(z^B\,\bar{z}_B - \bar{w}^B\,w_B)(z^C\,\bar{z}_C - \bar{w}^C\,w_C) \\ &+ (w_A\,\bar{w}^A + \bar{z}_A\,z^A)(w_B\,\bar{w}^B - \bar{z}_B\,z^B)(w_C\,\bar{w}^C - \bar{z}_C\,z^C) \\ &+ 2w_A\,(z^B\,\bar{z}_B - \bar{w}^B\,w_B)\bar{w}^A\,(w_C\,\bar{w}^C - \bar{z}_C\,z^C) \\ &+2z^A\,(w_B\,\bar{w}^B - \bar{z}_B\,z^B)\bar{z}_A\,(z^C\,\bar{z}_C - \bar{w}^C\,w_C)\big]. \end{aligned} \tag{3.2.22}$$

To write this scalar potential in a $SU(4)_R$ invariant way we define

$$Y^{\mathcal{A}} = (z^1, z^2, \bar{w}^1, \bar{w}^2), \qquad \bar{Y}_{\mathcal{A}} = (\bar{z}_1, \bar{z}_2, w_1, w_2), \qquad \mathcal{A} = 1, \ldots, 4, \tag{3.2.23}$$

such that they transform in the **4** and $\bar{\mathbf{4}}$ of $SU(4)_R$ respectively. After reverse engineering the construction of the potential [5] one arrives to

$$V_{bos} = \frac{4\pi^2}{k^2} \operatorname{tr} \left(Y^{\mathcal{A}} \bar{Y}_{[\mathcal{A}} Y^{\mathcal{B}} \bar{Y}_{\mathcal{C}]} Y^{\mathcal{C}} \bar{Y}_{\mathcal{B}} - \frac{1}{3} Y^{\mathcal{A}} \bar{Y}_{[\mathcal{A}} Y^{\mathcal{B}} \bar{Y}_{\mathcal{B}} Y^{\mathcal{C}} \bar{Y}_{\mathcal{C}]} \right), \tag{3.2.24}$$

thus showing explicitly the larger $SU(4)$ symmetry of it [3]. A similar construction can be made for the 'Yukawa-like' interaction V_{ferm} between scalars and spinors [16].

3.3. AdS_4/CFT_3

The original motivation in constructing such a highly supersymmetric gauge theory for a generic group (of the type $U(N)$ for any N) that admits a 't Hooft expansion is of course related to the AdS/CFT conjecture. To motivate the corresponding conjecture we devote a few paragraphs on explaining the moduli of ABJM theory as an evidence of the conjecture. For full details see [5].

The moduli space is given by the space of values of the scalar fields such that the superpotential vanishes up to gauge transformations. In the abelian case, as we explained in the last section, both the component potentials and the superpotential vanishes for any value of the scalar fields. Naively one would think that the moduli space in this case is then $\mathbb{C}^4$ by gauge fixing both abelian fields to zero.

However, we still have the freedom of making constant parameter $(\Lambda_1,\ \Lambda_2)$ gauge transformations while leaving $A = \hat{A} = 0$. In our derivation of gauge invariance of the Chern-Simons action in Chapter 2, we had omitted [4] a discussion by dropping boundary terms after applying Stokes theorem that appeared when studying the variation of the action under gauge transformations. The remaining boundary term cannot be dropped for those gauge transformations that are everywhere constant. In fact, we had derived

[3]We anti-symmetrize indices without symmetry factors; *e.g.* $Y_{[\mathcal{A}} Y_{\mathcal{B}]} = Y_{\mathcal{A}} Y_{\mathcal{B}} - Y_{\mathcal{B}} Y_{\mathcal{A}}$.

[4]See subsection 2.2.1.

there (*c.f.* equation (2.2.6)), and we adapt here to our abelian two Chern-Simons case, that under a gauge transformation, the Chern-Simons action varies by

$$\delta S_{cs} = i\frac{k}{4\pi}\int\limits_{\partial M} \Lambda_2 \hat{F} - i\frac{k}{4\pi}\int\limits_{\partial M} \Lambda_1 F. \tag{3.3.1}$$

Since the gauge field strengths are quantized $\frac{1}{2\pi}\int F \in \mathbb{Z}$, the gauge parameters should be $\Lambda = \frac{2\pi n}{k}$ so that the Gaussian measure in the quantum path integral is invariant ($n \in \mathbb{Z}$). This residual gauge symmetry acts on the $SU(4)_R$ quadruplets we defined in the last section as $Y^{\mathcal{A}} \to e^{2i\pi n/k} Y^{\mathcal{A}}$. We thus find that the moduli space in the abelian case is $\mathbb{C}^4/\mathbb{Z}_k$ [5].

By making a similar reasoning, it turns out that in the non-abelian case, all diagonal matter configurations produce the vanishing of the superpotential, while all non-diagonal ones may be interpreted as masses and do not expand the moduli space. The quantization of the residual gauge symmetry still holds and the moduli turns out to be $(\mathbb{C}^4/\mathbb{Z}_k)^N/S_N$, where S_N permutes the diagonal elements.

Since the conformal field theory under study is a three-dimensional Chern-Simons theory, the natural brane dynamic one has to look to find gravity duals are $M2$-branes in M-theory. In fact, the moduli space of N $M2$-branes probing a $\mathcal{C}^4/\mathbb{Z}_k$ singularity is given by the moduli space of ABJM theory. This orbifold preserves as much supersymmetry as the amount in ABJM theory. This motivates the conjecture that the infrared limit of the theory of N $M2$-branes probing such a singularity in M-theory is given by ABJM theory. The gravity dual for this theory is that of M-theory in a $AdS_4 \times S^7/\mathbb{Z}_k$ background.

In particular, the transverse space S^7 can be seen as an S^1 Hopf fibration over $\mathbb{C}P_3$, such that when k is large, the circle becomes small and the gravity dual proposed for ABJM theory becomes in this limit type IIA string theory on $AdS_4 \times \mathbb{C}P_3$.

The AdS radius is proportional to $(\lambda)^{1/4} = (N/k)^{1/4}$ such that we may trust the string theory description when $k, N \to \infty$ by keeping N/k fixed. The supergravity approximation of string theory is valid when λ is finite but large. On the other hand, this is the opposite of the perturbative regime of the quantum field theory, which admits a planar expansion for $k, N \to \infty$ with N/k fixed but small.

The authors of [6] generalized this conjecture to the case of M-theory in an $AdS_4 \times S^7/\mathbb{Z}_k$ background in presence of a torsion flux as the gravity dual of ABJM theory but when the two gauge groups are different ($N \neq M$).

Chapter 4.

Infrared divergencies in the $\mathcal{N} = 2$ formalism

The perturbative expansion of the off-shell amplitudes for a supersymmetric gauge theory may be plagued by infrared divergences for a number of different motivations. One possible source of infinities is given by the presence of positive mass-dimension couplings associated to massless fields in superrenormalizable theories. Canonical Yang-Mills theory coupled to massless matter in three dimensions turns out to be a good playground to study these phenomena [18]. In this case, going high in the order of the dimensionful coupling in the expansion of a given amplitude, for dimensional reasons one obtains high powers of external momenta in the denominator. If the fields are massless, upon inserting those amplitudes in higher order graphs, IR divergences inevitably show up in the Euclidean integrals.

Considering theories with only dimensionless couplings greatly improves the situation. A direct power counting argument shows that in four-dimensions classical marginality is a sufficient condition to exclude the presence of IR infinities in amplitudes with generic external momenta [10]. Even if this argument may be generalized to marginal three dimensional theories, it only applies to component field formulations and cannot be applied to superspace formulations as we will show throughout this work. The non applicability of the Poggio-Quinn theorem has to do with the fact that the power-counting of a final Feynman diagram cannot in many cases be known a priori in superspace formulations. This is because super Feynman diagrams involve two steps in formalisms such as $d = 4$, $\mathcal{N} = 1$ and $d = 3$, $\mathcal{N} = 2$: the $\mathcal{D}$-algebra and the Feynman integral calculus. While the naive power-counting we made in Chapter 2 gives an idea of the scaling of diagrams and sub-diagrams, we also explained in that context, that the process of $\mathcal{D}$-algebra may meliorate the ultraviolet behavior of final Feynman integrals and at the same time worsen its infrared behavior.

In particular, a potential source of IR divergences has to be considered as soon as the computations are performed using supergraph techniques in supersymmetric gauge theories. While proving to be an efficient method to compute perturbative corrections, superspace algebra comes with additional infrared issues due to the peculiar nature of the gauge superfield propagator. As it is clearly described in [38] in the case of four-dimensional Super-Yang-Mills theories, the appearance of infrared infinities can be ascribed to the presence of (corrected) vector lines in loop diagrams. With a canonically gauge-fixed action and omitting the color structure, the gauge superfield propagator in momentum space can be written as

$$\frac{1 + (\alpha - 1)\mathcal{P}_0}{p^2}\, \delta^4(\theta - \theta') \tag{4.0.1}$$

where α is the gauge-fixing parameter and $\mathcal{P}_0 = -\frac{1}{p^2}(D^2\bar{D}^2 + \bar{D}^2 D^2)$ is the superspin zero projector. Recall that this operator satisfies $\mathcal{P}_0 + \mathcal{P}_{1/2} = 1$, where $\mathcal{P}_{1/2} = \frac{1}{p^2} D^\alpha \bar{D}^2 D_\alpha$ is the superspin $1/2$ projector [1]. It is clear that, already at the one-loop level, the Fermi-Feynman gauge $\alpha = 1$ is the only infrared safe choice. On the other hand radiative corrections, being governed by Slavnov-Taylor identities, come with the transverse structure $\mathcal{P}_{1/2}$, thus reintroducing the infrared dangerous part in the propagator. Therefore, unless a way is found to perturbatively maintain the Fermi-Feynman form of the propagator, IR divergences will show up again starting from two-loop order.

An explicit prescription to cure the IR divergences in the case of four dimensional Super-Yang-Mills theory has been given in [11]. The main idea is to introduce a non-local gauge fixing term and renormalize the gauge fixing parameter to preserve the tree level structure of the Fermi-Feynman propagator. The prescription presented in [11] is strongly based on the nature of the model (gauge sector of Yang-Mills type) and on the space-time dimension.

[1]See Appendix A for a list of useful properties of these operators

The aim of this Chapter is to study the infrared behavior of supergraph amplitudes in the case of marginal Chern-Simons-matter systems in three dimensions described in the $\mathcal{N} = 2$ superspace formalism. These models can be treated in strong analogy with 4d Yang-Mills theories while exhibiting a completely different gauge structure. We start our analysis in the specific case of ABJM theory [5]. By directly computing Green's functions up to two-loop order, we show that infrared divergences appear in the amplitudes but can be seen as a gauge artefact of the formalism. At first, using the gauge fixing introduced in chapter 2 (α-gauge), we show that, in analogy with the four-dimensional SYM case, IR infinities show up when a corrected gauge vector propagator is inserted in loop amplitudes. By direct inspection of the dependence of the infrared singularities on the gauge fixing parameter α, we conclude that there is no suitable choice for the latter that both eliminate the divergences and preserve hermiticity of the action.

To solve this problem, we slightly revise the prescription of [11] introducing a set of non-canonical gauge fixing terms (η-gauge). This new set of gauges has the virtue that it can be used, by perturbatively fine tuning the parameter η, to complete loop by loop the transverse structure of the gauge vector propagator with the longitudinal part, thus improving its behavior in the infrared. We will show how the infinities are canceled in this way by direct perturbative computations. The η-gauge can hence be considered as a tool to consistently study the perturbative expansion of the amplitudes of the model without the presence of IR divergences. It's important to stress that infrared divergences, being an artefact of the superspace formalism, do not manifest themselves in physical gauge invariant quantities. In this case the α- and η-gauges produce coincident results.

As a byproduct of our analysis, we explicitly compute the finite expression in a general gauge for the two-loop propagator of the chiral superfield in ABJM. Moreover, we study a special vanishing external momenta limit of the two-loop vertex function of ABJM theory showing that it produces a finite result. Finally we comment on the extension of our results to a general perturbative order and to any classically marginal Chern-Simons-matter system.

4.1. Gauge-fixing alternatives

To address the problem of IR divergences in three-dimensional Chern-Simons theories we restrict ourself to the specific case of ABJM model. As described in chapter 3, this theory possesses remarkable properties such as extended supersymmetry and exact conformal invariance which will simplify the analysis of the infrared behavior. We extend our results to more general CS theories in section 4.4.

We quantize the theory using the standard gauge fixing procedure explained in chapter 2 and an alternative one that will ensure the cancelation of the IR divergences in loop amplitudes.

The gauge fixing functions are chosen to be $F = \bar{D}^2 V$, $\bar{F} = D^2 V$ in both gauge-fixing procedures. We saw in Chapter 2 (*c.f.* equation (2.2.35) and equations therein), that the standard gauge fixing is produced by a gauge averaging given by gaussian weights with chiral integrals of the form $\sim e^{\int ff} e^{\int \bar{f}\bar{f}}$, while the standard gauge averaging in $d = 4$, $\mathcal{N} = 1$ superspace is performed with a non-chiral (whole superspace) gaussian weight of the form $e^{\int \bar{f} f}$. The average produces the canonical quadratic gauge fixed action which in the ABJM case is

$$\mathcal{S}_{gf}^{(\alpha)} = \frac{1}{2} \int d^3x d^4\theta \ \mathrm{tr} \left[V \left(\bar{D}^\gamma D_\gamma + \frac{1}{\bar{\alpha}} D^2 + \frac{1}{\alpha} \bar{D}^2 \right) V \right] - \mathrm{tr} \left[\hat{V} \left(\bar{D}^\gamma D_\gamma + \frac{1}{\bar{\alpha}} D^2 + \frac{1}{\alpha} \bar{D}^2 \right) \hat{V} \right], \tag{4.1.1}$$

and after inverting the quadratic operators we obtain the gauge field propagators in momentum space for this theory

$$V^a{}_b \sim\!\sim\!\sim V^c{}_d = \frac{1}{p^2} \left(\bar{D}^\alpha D_\alpha + \alpha D^2 + \bar{\alpha} \bar{D}^2 \right) \delta^4_{(\theta,\theta')} \delta^b_c \delta^d_a,$$

$$\hat{V}^{\hat{a}}{}_{\hat{b}} \sim\!\sim\!\sim \hat{V}^{\hat{c}}{}_{\hat{d}} = -\frac{1}{p^2} \left(\bar{D}^\alpha D_\alpha + \alpha D^2 + \bar{\alpha} \bar{D}^2 \right) \delta^4_{(\theta,\theta')} \delta^{\hat{b}}_{\hat{c}} \delta^{\hat{d}}_{\hat{a}}. \tag{4.1.2}$$

In the next Section we will see that, if we want to preserve the hermiticity of the gauge fixed action considering α and $\bar{\alpha}$ as complex conjugates, then the infrared divergences cannot be canceled by a simple fine tuning of the gauge fixing parameter.

To solve this problem, in analogy with [11], we propose a different gauge averaging procedure. We choose the same gauge fixing functions as before, but this time we introduce the following term in the functional integral:

$$\det\hat{\mathcal{M}} \int \mathcal{D}f\,\mathcal{D}\bar{f}\,\Delta(V)\Delta^{-1}(V)\exp\left(\int d^3x\,d^4\theta\,\mathrm{tr}\left(\bar{f}\hat{\mathcal{M}}f\right)\right). \tag{4.1.3}$$

In this way we allow for a non-trivial gauge averaging by the insertion of the operator $\hat{\mathcal{M}}$. It is important to stress that as long as $\hat{\mathcal{M}}$ is field independent, the $\det\hat{\mathcal{M}}$ factor appearing in the functional integral is irrelevant and there is no need to introduce Nielsen-Kallosh ghosts [39, 40] in the action. In $d=4$ the $\hat{\mathcal{M}}$ operator is dimensionless and one can simply choose $\hat{\mathcal{M}}=$ constant. In three dimensions it has dimensions of length so that we may choose either a dimensionful constant or a non-local gauge fixing term. Our choice of this operator in momentum space is $\hat{\mathcal{M}}(p)=\frac{1}{\eta^{\epsilon}_{(p)}\,|p|}$, where $\eta^{\epsilon}_{(p)}$ is a dimensionless function that contains ϵ powers of p [2]. The early introduction of the $\epsilon=\frac{3}{2}-\frac{d}{2}$ regulator parameter has to be understood formally in the sense of dimensional reduction, that is, we will still perform D-algebra calculations in three dimensions and only at the end we will regularize Feynman integrals. More specifically, we will define η as an odd power series in the 't Hoof coupling $\lambda=\frac{N}{k}$ with coefficients that we will conveniently choose. By choosing the same Gaussian measure (4.1.3) for both gauge sectors we obtain the gauge fixed action in momentum space

$$\begin{aligned}\mathcal{S}^{(\eta)}_{gf} = \frac{1}{2}\int\frac{d^3p}{(2\pi)^3}d^4\theta\ \mathrm{tr}&\left[V(-p)\left(\bar{D}^{\gamma}D_{\gamma}-\frac{|p|}{\eta^{\epsilon}_{(p)}}\mathcal{P}_0\right)V(p)\right]\\ &-\mathrm{tr}\left[\hat{V}(-p)\left(\bar{D}^{\gamma}D_{\gamma}+\frac{|p|}{\eta^{\epsilon}_{(p)}}\mathcal{P}_0\right)\hat{V}(p)\right].\end{aligned} \tag{4.1.4}$$

Inverting the operators of the quadratic part of the gauge fixed action we obtain the gauge field propagators

$$V^{a}_{\ b}\ \sim\!\sim\!\sim\ V^{c}_{\ d} \quad = \left(\frac{\bar{D}^{\alpha}D_{\alpha}}{p^2}+\frac{\eta^{\epsilon}_{(p)}}{|p|}\mathcal{P}_0\right)\delta^4_{(\theta,\theta')}\delta^b_c\delta^d_a,$$

[2] This choice would introduce Nielsen-Kallosh ghosts in the action if the computations were performed using the background field method as in [11].

$$\hat{V}^{\hat{a}}_{\hat{b}} \;\sim\!\sim\!\sim\; \hat{V}^{\hat{c}}_{\hat{d}} \quad = \left(-\frac{\bar{D}^\alpha D_\alpha}{p^2} + \frac{\eta^\epsilon_{(p)}}{|p|}\mathcal{P}_0 \right) \delta^4_{(\theta,\theta')}\delta^b_c \delta^d_a. \tag{4.1.5}$$

We shall call this gauge fixing procedure as the "η-gauge". We will show that allowing η to be corrected order by order in the 't Hooft coupling $\frac{N}{k}$, we will be able to cancel the infrared divergent parts in the amplitudes.

To complete the gauge fixing procedure, we notice that since the gauge transformation properties of the gauge vectors do not change from the α to the η gauge, in both cases we have the same Fadeev-Popov b-c and $\hat{b}$-$\hat{c}$ system of Grassmanian chiral superfield ghosts. Therefore, in both of our gauge fixing choices the ghost action for ABJM reads[3]

$$\begin{aligned}\mathcal{S}_{fp} = \int d^3x d^4\theta \; \mathrm{tr}\Big[&\bar{b}\, c + \bar{c}\, b + \frac{1}{2} g(b + \bar{b})[V, c + \bar{c}] \\ &- \bar{\hat{b}}\,\hat{c} - \bar{\hat{c}}\,\hat{b} - \frac{1}{2} g(\hat{b} + \bar{\hat{b}})[\hat{V}, \hat{c} + \bar{\hat{c}}] + \mathcal{O}(g^2)\Big].\end{aligned} \tag{4.1.6}$$

In Appendix B we detail some of the relevant Feynman rules for ABJ(M) theory.

4.2. Infrared behavior of amplitudes in ABJM theory

We would like now to understand the origin of the IR divergences in the perturbative expansion of the off-shell amplitudes. In order to do so, we compute the one-loop correction to the vector gauge superfield V in the α- and η-gauges. Performing a direct two-loop computation of the (finite) corrections to the self-energy of the matter superfield and to the superpotential, we will see that IR infinities only arise when the one-loop corrected gauge propagator is inserted in loop diagrams. A suitable choice of the η gauge fixing parameter will then cancel the divergences. Our explicit examples will be completed with an all loop analysis in Section 4.4.

4.2.1. One-loop vector propagator

The one loop corrected gauge vector field receives contributions from matter, ghost and gauge vector fields as we show in Figure 4.1. In the α-gauge these evaluate

[3]Recall that we name $g = \sqrt{\frac{4\pi}{k}}$.

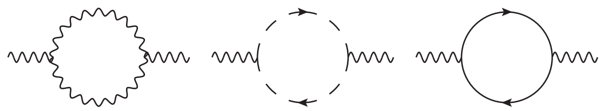


Figure 4.1.: One-loop self-energy corrections of the gauge vector superfields.

$$\begin{aligned}
\Delta^{(1)}_{V-gauge} &= \frac{\pi}{k}(\delta^b_c\delta^d_a N - \delta^b_a\delta^d_c)\int \frac{d^3k\, d^4\theta}{(2\pi)^3} G(1,1)(k^2)^{1/2-\epsilon}\, V^a_{\ b}(-k)\left(\mathcal{P}_0 + \alpha\bar{\alpha}\,\mathcal{P}_{1/2}\right) V^c_{\ d}(k)\\
\Delta^{(1)}_{V-ghost} &= -\frac{\pi}{k}(\delta^b_c\delta^d_a N - \delta^b_a\delta^d_c)\int \frac{d^3k\, d^4\theta}{(2\pi)^3} G(1,1)(k^2)^{1/2-\epsilon}\, V^a_{\ b}(-k)\left(\mathcal{P}_0 + \mathcal{P}_{1/2}\right) V^c_{\ d}(k)\\
\Delta^{(1)}_{V-matter} &= \frac{4\pi}{k}\delta^b_c\delta^d_a N \int \frac{d^3k\, d^4\theta}{(2\pi)^3} G(1,1)(k^2)^{1/2-\epsilon}\, V^a_{\ b}(-k)\mathcal{P}_{1/2} V^c_{\ d}(k)\\
\Delta^{(1)}_{mixed} &= -\frac{4\pi}{k}\delta^b_c\delta^{\hat{d}}_{\hat{a}} \int \frac{d^3k\, d^4\theta}{(2\pi)^3} G(1,1)(k^2)^{1/2-\epsilon}\, \hat{V}^{\hat{a}}_{\ \hat{d}}(-k)\mathcal{P}_{1/2} V^c_{\ b}(k)\ . \qquad (4.2.1)
\end{aligned}$$

Here we are displaying the corrections to the $V-V$ propagator as well as to the mixed $V-\hat{V}$ propagator (last line). The latter receives contributions only from the matter diagram of Figure 4.1. The corrections to the $\hat{V}-\hat{V}$ propagator can be easily read from the $V-V$ case. The definition of the $G(a,b)$ functions can be found in Appendix C. Notice that the mixed $V-\hat{V}$ contribution is subleading in N. Consistently with the Slavnov-Taylor identities, the complete one-loop correction to $V-V$ contains only the spin 1/2 projection of the gauge field and is given by

$$\Delta^{(1)}_V = \frac{\pi}{k}\left((3+\bar{\alpha}\alpha)\delta^b_c\delta^d_a N + (1-\bar{\alpha}\alpha)\delta^b_a\delta^d_c\right)\int \frac{d^3k\, d^4\theta}{(2\pi)^3} G(1,1)(k^2)^{1/2-\epsilon}\, V^a_{\ b}(-k)\mathcal{P}_{1/2} V^c_{\ d}(k). \qquad (4.2.2)$$

With the particular choice $\alpha = 0$ (Landau gauge) we exactly reproduce the results of [25, 26, 30]. We notice that, looking at the leading part of the correction, the effect of working in a general α-gauge simply results in a positive constant shift $|\alpha|^2$. Therefore we conclude that, if we want to preserve hermiticity of the action, there is no way to fully eliminate the one-loop correction by fine tuning the gauge-fixing parameter [4].

[4]Leaving ABJM aside, more in general, there is only one exception to this which is in the case where there is no matter of any type. In that case the mixed contribution would obviously not exist and the V-V and $\hat{V}$-$\hat{V}$ contributions would have been proportional to $\alpha\bar{\alpha}-1$. A choice of α with $|\alpha| = 1$ would have produced the complete cancelation of the correction.

Now we work out the correction in the η-gauge restricting the analysis to leading[5] order in N. As anticipated, the idea is to make a perturbative expansion of the gauge-fixing parameter in the 't Hooft coupling

$$\eta^{\epsilon}_{(k)} = {}^1\eta^{\epsilon}_{(k)}\lambda + O(\lambda^3), \tag{4.2.3}$$

such that to order λ we obtain the total correction of the propagator:

$$\lambda\,\frac{(-6\pi G(1,1)(k^2)^{-\epsilon}\mathcal{P}_{1/2} + {}^1\eta^{\epsilon}_{(k)}\mathcal{P}_0)}{(k^2)^{1/2}}\delta_{(\theta,\theta')}. \tag{4.2.4}$$

It's easy to see that if we choose ${}^1\eta^{\epsilon}_{(k)} = -6\pi G(1,1)(k^2)^{-\epsilon}$ we exactly complete the transverse structure $\mathcal{P}_{1/2}$ with the longitudinal part $\mathcal{P}_0$ to obtain

$$-6\pi\, G(1,1)\lambda\frac{\delta_{(\theta,\theta')}}{(k^2)^{1/2+\epsilon}}. \tag{4.2.5}$$

In the next Section we compute two-loop Green's functions and show that the improved IR behavior of the η-gauge propagator in (4.2.5) is enough to cure the problem of IR infinities.

4.2.2. Matter self-energy at two-loop order

Landau gauge

Working in the Landau gauge simplifies greatly the calculation since many diagrams can be discarded due to the form of the gauge vector propagator. It is easy to see that all one-loop corrections vanish with standard gauge averaging. In Figure 4.2 we display all non-vanishing self energy two loop quantum corrections of matter fields in this gauge. The blob in diagrams (c) and (d) represents the insertion of the full one loop correction to the gauge propagator. Any other potentially contributing diagram is zero either by D-algebra or by color symmetry.

[5] The following ideas may be also worked out at subleading order in N but we would have to add a mixed $\bar{V}-V$ gauge fixing term.

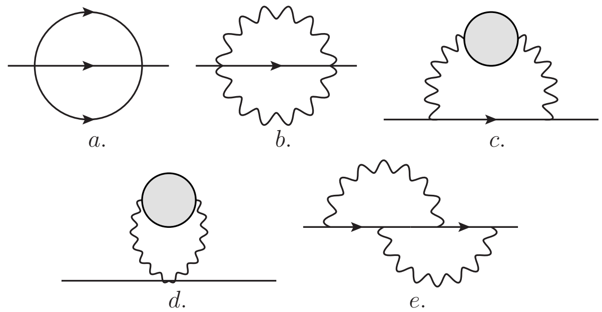


Figure 4.2.: Two loop self-energy quantum corrections.

Let us for example calculate with detail diagram b of Figure 4.2. Taking into account the possibility of having V-$\hat{V}$, V-V and $\hat{V}$-$\hat{V}$ internal lines and using color vertex factors as (D.0.4), it evaluates to

$$\Pi_b = -\frac{1}{2}\left(N^2-1\right)\left(\frac{4\pi}{k}\right)^2 \int d^4\theta\; d^4\theta' \int \frac{d^3p}{(2\pi)^3}\;\mathrm{tr}\left(\bar{Z}_A(-p)Z^A(p)\right)\;\mathcal{D}_b(\theta,\theta') \tag{4.2.6}$$

with

$$\mathcal{D}_b(\theta,\theta') = \int \frac{d^3k\, d^3l}{(2\pi)^3(2\pi)^3}\; \frac{\bar{D}^\alpha D_\alpha \delta^4_{(\theta,\theta')} \bar{D}^2 D^2 \delta^4_{(\theta,\theta')} \bar{D}^\beta D_\beta \delta^4_{(\theta,\theta')}}{l^2\, k^2\, (k+l+p)^2}, \tag{4.2.7}$$

the D-algebra factor of the supergraph. As we mentioned before, we perform all D-algebra manipulations in three dimensions and we calculate the final Feynman integral in d dimensions. After the usual integration by parts we obtain an ultraviolet divergent contribution

$$\Pi_b = \left(N^2-1\right)\left(\frac{4\pi}{k}\right)^2 \int \frac{d^3p\, d^4\theta}{(2\pi)^3}\;\mathrm{tr}\left(\bar{Z}_A(-p)Z^A(p)\right)\; G(1,1)G(1,1/2+\epsilon)\,(p^2)^{-2\epsilon}. \tag{4.2.8}$$

To obtain the contribution a from Figure 4.2 we need vertex factors from (D.0.5). We get the UV divergent contribution $\Pi_a = 2\Pi_b$.

Using the corrected vector propagator we obtain for graph d of Figure 4.2 an UV/IR divergent tadpole

$$\Pi_d = -3\left(N^2-1\right)\left(\frac{4\pi}{k}\right)^2 \int \frac{d^3p\, d^4\theta}{(2\pi)^3}\ \mathrm{tr}\left(\bar{Z}_A(-p)Z^A(p)\right)\ G(1,1)\int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2)^{\frac{3}{2}+\epsilon}}. \tag{4.2.9}$$

and for graph c an infrared divergent contribution

$$\Pi_c = 3\left(N^2-1\right)\left(\frac{4\pi}{k}\right)^2 \int \frac{d^3p\, d^4\theta}{(2\pi)^3}\ \mathrm{tr}\left(\bar{Z}_A(-p)Z^A(p)\right)\ G(1,1)\int \frac{d^dk}{(2\pi)^d}\frac{2p.(p+k)}{(k^2)^{\frac{3}{2}+\epsilon}(k+p)^2}. \tag{4.2.10}$$

Summing up a, b, c and d we obtain the cancelation of all UV divergent contributions and we are left with an IR divergent piece

$$\Pi_{a+b+c+d} = 3\left(N^2-1\right)\left(\frac{4\pi}{k}\right)^2 \int \frac{d^3p\, d^4\theta}{(2\pi)^3}\ \mathrm{tr}\left(\bar{Z}_A(-p)Z^A(p)\right)\ I_{2\mathrm{IR}}(p), \tag{4.2.11}$$

with $I_{2\mathrm{IR}}(p)$ as given in C.2.2. Finally, diagram e produces a finite correction

$$\Pi_e = -2\left(N^2-1\right)\left(\frac{4\pi}{k}\right)^2 \int \frac{d^3p\, d^4\theta}{(2\pi)^3}\ \mathrm{tr}\left(\bar{Z}_A(-p)Z^A(p)\right)\ I_e \tag{4.2.12}$$

where I_e is the factor obtained after closing the D-Algebra:

$$I_e = \int \frac{d^3k\, d^3l}{(2\pi)^3(2\pi)^3}\ \frac{(k+p)^2(l+p)^2 - k^2l^2 + p^2(k+l+p)^2}{k^2\,(k+p)^2\,(k+l+p)^2\,l^2\,(l+p)^2} = \frac{1}{64}. \tag{4.2.13}$$

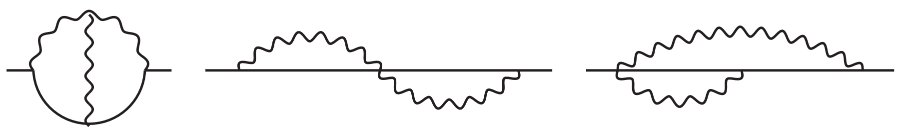
To conclude, the sum of all contributions gives a finite and an infrared divergent piece

$$\Pi = \left(N^2-1\right)\left(\frac{4\pi}{k}\right)^2 \int \frac{d^3p\, d^4\theta}{(2\pi)^3}\ \mathrm{tr}\left(\bar{Z}_A(-p)Z^A(p)\right)\ \left(3I_{2\mathrm{IR}}(p) - \frac{1}{32}\right). \tag{4.2.14}$$

Working in the Landau gauge, we explicitly see that infrared infinities are only given by graphs c and d, which correspond to insertion of the 1-loop corrected vector propagator.

α-gauge

We now take the more general case $\alpha \neq 0$. Once again there are no one-loop matter corrections. The list of two-loop self energy contributions gets larger. Apart from those already displayed in Figure 4.2, which are modified by the more general α-dependent propagator, we have some additional contributions displayed in Figure 4.3. The new



f. *g.* *h.*

Figure 4.3.: Additional quantum corrections in the α-gauge.

contributions produce additional UV, IR divergences and finite pieces. The sum of the original diagrams we had, with the modified α-dependent propagator gives

$$\Pi^{\alpha}_{a+\cdots+e} = \left(N^2-1\right)\left(\frac{4\pi}{k}\right)^2 \int \frac{d^3p\, d^4\theta}{(2\pi)^3}\, \mathrm{tr}\left(\bar{Z}_A(-p)Z^A(p)\right) \times \left(-4\alpha\bar{\alpha}\, G(1,1)G(1,1/2+\epsilon)\,(p^2)^{-2\epsilon} + (3+\alpha\bar{\alpha})I_{2\mathrm{IR}}(p) - \frac{1}{32}(1+\alpha\bar{\alpha})\right). \tag{4.2.15}$$

And the contributions from the additional diagrams of Figure 4.3

$$\Pi^{\alpha}_{f+g+h} = \left(N^2-1\right)\left(\frac{4\pi}{k}\right)^2 \int \frac{d^3p\, d^4\theta}{(2\pi)^3}\, \mathrm{tr}\left(\bar{Z}_A(-p)Z^A(p)\right) \times \left(4\alpha\bar{\alpha}\, G(1,1)G(1,1/2+\epsilon)\,(p^2)^{-2\epsilon} + \frac{1}{32}\alpha\bar{\alpha}\right). \tag{4.2.16}$$

By summing up all the contributions, we find as expected that all UV α-dependent divergences cancel out. The finite piece we had already encountered in the Landau gauge is not modified, and the IR divergent piece gets shifted:

$$\Pi^{\alpha} = \left(N^2-1\right)\left(\frac{4\pi}{k}\right)^2 \int \frac{d^3p\, d^4\theta}{(2\pi)^3}\, \mathrm{tr}\left(\bar{Z}_A(-p)Z^A(p)\right) \left((3+\alpha\bar{\alpha})I_{2\mathrm{IR}}(p) - \frac{1}{32}\right). \tag{4.2.17}$$

From this we may conclude that if we only allow a hermitian gauge-fixed action, such that $\bar{\alpha}$ is literally the complex conjugate of α, then it is not possible to choose a value of the gauge fixing parameter α such that the self-energy corrections are infra-red safe. This is a direct consequence of the fact that IR divergences are eventually produced only by corrected vector propagators. It's also important to notice that the finite correction to the propagator turns out to be gauge independent, even if the propagator itself is not a physical quantity.

η-gauge

In the η-gauge the vector superfield propagator is written as:

$$\langle V_d^c(-p)\, V_b^a(p)\rangle = \left(\frac{\bar{D}^\alpha D_\alpha}{p^2} + \frac{\eta^\epsilon_{(p)}}{|p|}\mathcal{P}_0\right)\delta^4_{(\theta,\theta')}\delta^b_c\delta^d_a. \tag{4.2.18}$$

where $\eta^\epsilon_{(p)}$ is expanded as in (4.2.3). The first piece of the propagator gives rise to matter self energy diagrams starting from two loops with the same contributions as in the Landau gauge (see fig. 4.2) such that, for $N \gg 1$, it gives a finite and an infrared divergent piece of order $(\frac{N}{k})^2$ given by

$$\Pi^\eta_{a+\cdots+e} = \left(\frac{4\pi N}{k}\right)^2 \int \frac{d^3p\, d^4\theta}{(2\pi)^3}\, \mathrm{tr}\left(\bar{Z}_A(-p)Z^A(p)\right)\left(3I_{2\mathrm{IR}}(p) - \frac{1}{32}\right). \tag{4.2.19}$$

The second part of the propagator produces one-loop corrections to matter self energy such that, if the gauge parameter is of order $\frac{N}{k}$, the contribution is of order $(\frac{N}{k})^2$. In this case, two loop and higher corrections will contribute beyond $(\frac{N}{k})^2$ so we do not consider them. The one-loop contributions are displayed in Figure 4.4.

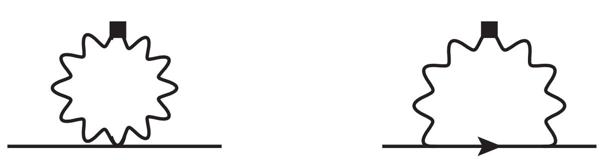


Figure 4.4.: One loop corrections in the η-gauge. The small black squares in the gauge vector propagators should be intended as the η dependent piece of the propagator.

After straightforward D-algebra, and with the choice $\eta^{\epsilon}_{(k)} = -6\pi G(1,1)(k^2)^{-\epsilon}\lambda + O(\lambda^3)$ we obtain

$$\Pi^{\eta}_{\text{1-loop}} = -\left(\frac{4\pi N}{k}\right)^2 \int \frac{d^3p\, d^4\theta}{(2\pi)^3} \,\mathrm{tr}\left(\bar{Z}_A(-p)Z^A(p)\right) 3I_{2\mathrm{IR}}(p). \tag{4.2.20}$$

From this we see that with the choice of $\eta^{\epsilon}_{(p)}$ that produces the IR improved gauge propagator, we cancel the infrared divergent part obtaining only the universal finite piece already computed in the α-gauge:

$$\Pi^{\eta} = \Pi^{\eta}_{a+\cdots+e} + \Pi^{\eta}_{\text{1-loop}} = -\frac{1}{2}\pi^2\frac{N^2}{k^2}\int d^3x\, d^4\theta \,\mathrm{tr}\left(\bar{Z}_A Z^A\right). \tag{4.2.21}$$

We therefore conclude that the improved IR behaviour of the gauge propagator is sufficient to eliminate the presence of the unwanted divergences. In the next Section we further check this assertion computing at two-loop order the matter four-point Green's function.

4.2.3. Superpotential vertex corrections

The set of all two-loop graphs which contribute to superpotential corrections to leading order in N in the Landau gauge are depicted in Figure 4.5; any other potentially contributing 2-loop graph is zero due to color symmetry, supersymmetry or particular symmetries of the Feynman integrals involved. Notice that, since in the Landau gauge the one loop correction to the vertex is exactly zero, we can discard many diagrams at 2-loops that contain the 1-loop diagram as a subdiagram.

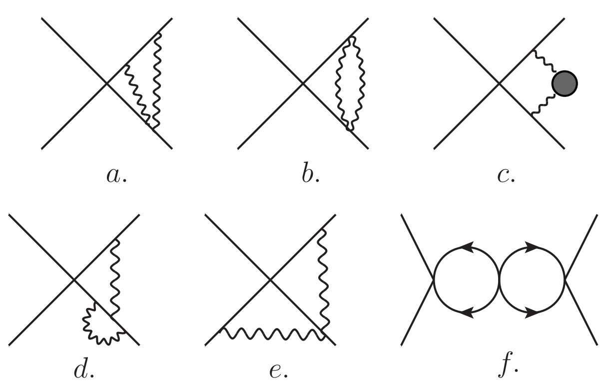


Figure 4.5.: All two loop quantum corrections of the superpotential.

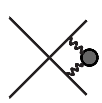
To simplify notation, whenever we put a $\mathcal{D}(\cdots)$ in front of the graph inside an equation, we mean the scalar graph with all the momenta in the numerator generated after closing the D-algebra (we put on equal foot the V and $\hat{V}$ lines and only in color/flavor vertex factors will we consider the sign difference between their propagators and couplings to matter). Else, in the absence of $\mathcal{D}$ in front of the graph, we just mean the corresponding scalar Feynman integral.

To leading order in N, all the two loop contributions produce a term proportional to the classical superpotential (no double traces are generated) given by

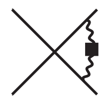
$$\Gamma_i[\mathcal{A},\mathcal{B}] = \left(\frac{4\pi N}{k}\right)^2 C_i \int d^2\theta \frac{d^3p_1}{(2\pi)^3}\cdots\frac{d^3p_4}{(2\pi)^3}\,(2\pi)^3\delta\left(p_1+p_2+p_3+p_4\right)$$
$$\frac{2\pi i}{k}\,\epsilon_{AC}\epsilon^{BD}\mathrm{tr}\left(Z^A(p_1)W_B(p_2)Z^C(p_3)W_D(p_4)\right)\,\mathcal{D}_i(p_1,\cdots,p_4) \qquad i=a,\cdots,f, \quad (4.2.22)$$

where C_i is the vertex factor of graph i. $\mathcal{D}_i(p_1,\cdots,p_4)$ is the Feynman integral which results after performing the D-algebra so as to eliminate all the $d^4\theta$ integrals except for the last one which is used to transform the D-operators applied on the fields into external momenta by using that $\int d^4\theta = \int d^2\theta\bar{D}^2(\cdots)$. The vertex factors for all graphs are $C_a=\frac{1}{2}$, $C_b=\frac{1}{4}$, $C_c=-3$, $C_d=1$, $C_e=-1$, $C_f=2$. We will always consider p_1 as the 'north-western' momentum of the graph and name the consecutive momenta counter-clockwise.

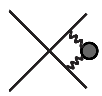
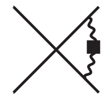
Let us start the computation of graph c of Figure 4.5 which is the one we expect to give an IR divergence. A not so straightforward calculation of the D-algebra gives

$$\mathcal{D}\Bigg($$  $$\Bigg) = \frac{G(1,1)}{2}\int\frac{d^dk}{(2\pi)^d}\frac{k^2(p_3+p_4)^2 - {p_3}^2(k+p_4)^2 - {p_4}^2(k-p_3)^2}{(k^2)^{3/2+\epsilon}(k+p_4)^2(k-p_3)^2}$$
$$= \frac{1}{2}(p_3+p_4)^2$$  $$-\frac{1}{2}I_{2\mathrm{IR}}(p_3) - \frac{1}{2}I_{2\mathrm{IR}}(p_4), \qquad (4.2.23)$$

where we have written the Feynman integral in terms of a finite scalar integral and infrared divergent contributions. Once again we obtain infrared divergences when we attach a one-loop corrected gauge vector inside a loop. We expect that IR divergences in the superpotential are canceled by the exact same choice for η we found to improve the gauge vector propagator infrared behaviour. This is in fact true: the one-loop graph with the η-dependent part of the gauge vector propagator gives

$\mathcal{D}\Bigg($  $\Bigg) = -6\pi\lambda(I_{2\mathrm{IR}}(p_3) + I_{2\mathrm{IR}}(p_4)).$ (4.2.24)

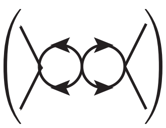
With the value of the gauge parameter we made before $\eta = -6\pi G(1,1)(p^2)^{-\epsilon}\,\lambda + O\,[\lambda^3]$, this insertion produces a superpotential correction with the same structure as in (4.2.22). In this way, the sum of this graph with graph d which was also IR divergent gives

$-3(4\pi\lambda)^2\,\mathcal{D}\Bigg($  $\Bigg) + 4\pi\lambda\,\mathcal{D}\Bigg($  $\Bigg) = -\frac{3}{2}(4\pi\lambda)^2\,(p_3+p_4)^2$  ,

(4.2.25)

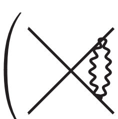
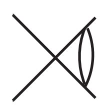
which is finite. These are the only dangerous IR graphs contributing to the superpotential; the graphs which remain to be analyzed are all finite. To show this, we list the integrals resulting from D-algebra computations.

The simplest graph is g: it has three possible channels of which two contribute to leading order in N. This factor of 2 is already taken into account in the vertex factor C_g. The D-algebra of this graph is simply

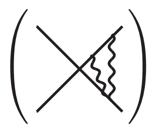
$\mathcal{D}\Bigg($  $\Bigg) = \displaystyle\int \frac{d^3k}{(2\pi)^3}\frac{d^3l}{(2\pi)^3}\frac{-(p_1+p_2)^2}{k^2\,(k+p_1+p_2)^2\,l^2\,(l-p_3-p_4)^2} = -\frac{1}{64},$ (4.2.26)

In all other diagrams, a sum over different distributions of internal lines has to be taken into account such that diagrams b, c and e appear four times with different momentum distribution, while diagrams a, d appear eight times.

A straightforward calculation shows that

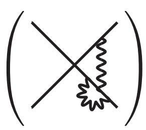
$\mathcal{D}\Bigg($  $\Bigg) = 2(p_3+p_4)^2$  . (4.2.27)

As mentioned before, this scalar integral is finite in three dimensions. For graph a we obtain the finite result

$\mathcal{D}$ 

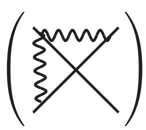
$$= \int \frac{d^d k}{(2\pi)^d} \frac{d^d l}{(2\pi)^d} \frac{Tr(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma)\, p_4^\mu \,(p_3 + p_4)^\nu \,(k + p_4)^\rho \,(l - p_4)^\sigma}{(k + p_4)^2 \,(k - p_3)^2 \,(k + l)^2 \,(l - p_4)^2 \, l^2}. \quad (4.2.28)$$

Notice that the presence of a three lined vertex is potentially dangerous, but the momenta in the numerator of the Feynman integral that we obtain through D-algebra guarantees finiteness. The same is true for graph d

$\mathcal{D}$ 

$$= \int \frac{d^d k}{(2\pi)^d} \frac{d^d l}{(2\pi)^d} \frac{-Tr(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\alpha \gamma_\beta \gamma_\gamma)\,(k + p_4)^\mu \, l^\nu \,(k + l)^\rho \,(k - p_3)^\alpha \, p_3^\beta \, p_4^\gamma}{k^2 \,(k + p_4)^2 \,(k - p_3)^2 \,(k + l)^2 \,(l + p_3)^2 \, l^2}, \quad (4.2.29)$$

and also for graph e

$\mathcal{D}$ 

$$= \int \frac{d^d k}{(2\pi)^d} \frac{d^d l}{(2\pi)^d} \frac{-Tr(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma)\, p_4^\mu \, p_2^\nu \, k^\rho \, l^\sigma}{k^2 \,(k - p_2)^2 \,(k + l + p_3)^2 \,(l - p_4)^2 \, l^2}, \quad (4.2.30)$$

which are once again finite in three dimensions.

A particular exceptional momenta configuration

By using the η gauge fixing, we showed in the last Section that it was possible to obtain an infrared safe function of the external momenta for the superpotential corrections. Moreover, it is clear that the sum of 1PI graphs plus four-legged graphs with corrected legs (using self-energy corrections we derived before), is a physical gauge invariant quantity. Having found an universal finite value for the matter propagator correction we conclude that also the correction to the superpotential is (at least at two loops) gauge independent. We would like now to compute it for a special external momenta configuration [6].

The calculation we are going to present here should be interpreted along the lines of [42–44]. In these papers, by means of direct computation of specific diagrams in four-

[6] See [41] for the calculation of the effective action on a vector superfield background.

dimensional supersymmetric models, it was shown that finite contributions may survive the limit of vanishing external momenta for the 1PI vertex function as soon as massless particles were present. Moreover, these contributions could break holomorphy in the coupling constants or supersymmetries of the action if they were to be interpreted as a "finite renormalization" of the superpotential. It then became clear (see [45] for a review and references therein) that the correct interpretation of these contributions was to consider them as IR singular D-terms in superspace, which are absent for instance in the more suitable Wilsonian definition of the effective superpotential. In what follows we would like to show that also in the case of ABJM theories does exist a special limit of vanishing external momenta for the vertex function which gives rise to a finite result.

The vertex function, with the IR safe gauge choice, is guaranteed to be finite as long as the momenta are non-exceptional. By exceptional we mean when there exists at least one equation of the form $\sum_i \rho_i \, p_i = 0$ with ρ_i either 0 or 1 and not all 0 nor all 1. In our case, it is easy to see that many exceptional configurations produce spurious IR divergences, for example if we choose any of the four momenta, say p_1 to be zero.

If we were interested in finding an exceptional configuration which is IR safe and which leads to a constant, we would need at least two supplementary "exceptional" equations. In fact, we found that modulo equivalent choices, there is only one such choice of exceptional momenta which is IR finite. This is given by choosing $p_1 + p_2 = 0$ and $p_1 + p_4 = 0$. We proceed to evaluate the graphs for this choice.

For graph b we obtain

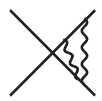
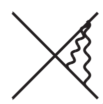
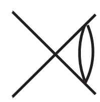
$$\mathcal{D}\left(\quad \right) = 2(p_3 + p_4)^2 \quad \longrightarrow 0. \tag{4.2.31}$$

The reader might be worried that we put this graph to zero in the exceptional configuration because of the $(p_3 + p_4)^2 = (p_1 + p_2)^2$ numerator without taking into account that the integral multiplying it is infrared divergent when $p_3 + p_4 = 0$. A careful power expansion in $|p_3 + p_4|$ gives

$$2(p_3 + p_4)^2 \quad = \left(\frac{1}{16\pi} \frac{\mathcal{K}\left[\sqrt{1 - \frac{p_3^2}{p_4^2}}\right]}{|p_4|} \right) |p_3 + p_4| + \left(\frac{1}{16\pi^2} \frac{\log(\frac{p_3^2}{p_4^2})}{p_3^2 - p_4^2} \right) (p_3 + p_4)^2 + \cdots \, , \tag{4.2.32}$$

where $\mathcal{K}(z)$ is the complete elliptic integral of the first kind[7] and the ellipsis are for higher orders in $|p_3 + p_4|$. From this equation we see that $p_3 + p_4 \to 0$ is well defined and zero since all the coefficients in the expansion are finite in this limit.

Graphs a may be represented in terms of elementary and Mellin-Barnes integral functions of the Lorentz invariants $x = \frac{p_3^2}{(p_3+p_4)^2}$ and $y = \frac{p_4^2}{(p_3+p_4)^2}$ given by [8]

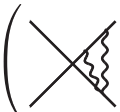
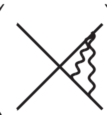
$\mathcal{D}\Bigg($  $\Bigg) + \mathcal{D}\Bigg($  $\Bigg) = -(p_3 + p_4)^2$ 

$$+ \frac{1}{64\pi^2}\left(\tfrac{2}{3}\pi^2 - Li_2(1-x) - Li_2(1-y) - \log(x)\log(y)\right)$$
$$+ \frac{\sqrt{\pi}}{64\pi^3}\frac{1}{(2\pi i)^2}\int\limits_{-i\infty}^{i\infty} ds\, dt\, \Gamma^*(-s)\Gamma\left(\tfrac{1}{2} - s\right)\Gamma(-t)^2\Gamma(t+s)\Gamma(1+t+s)(x^s\, y^t + x^t\, y^s). \tag{4.2.33}$$

This expression admits a well defined limit for $(p_3 + p_4)^2 \to 0$ given by

$$\to \frac{1}{32\pi^2}\left[\arccos^2\left(\frac{|p_3|}{|p_4|}\right) + \arccos^2\left(\frac{|p_4|}{|p_3|}\right) + \frac{1}{4}\log^2\left(\frac{p_3^2}{p_4^2}\right)\right]. \tag{4.2.34}$$

If we consider more in particular that $p_3 + p_4 = 0$, then not only $(p_3 + p_4)^2 = 0$ but also $p_3^2 = p_4^2$, we find

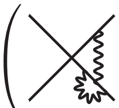
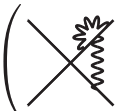
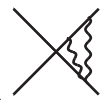
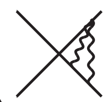
$\mathcal{D}\Bigg($  $\Bigg) + \mathcal{D}\Bigg($  $\Bigg) \longrightarrow 0.$ (4.2.35)

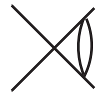
Now we move on to graphs d. By expanding the products of momenta in the numerator of the integrals and properly completing squares (see trace properties of γ^μ matrices in the appendix A), one can compare the resulting expression with the squared-completed

[7]Notice that the coefficient in front of $|p_3 + p_4|$ is implicitly symmetric under $p_3 \leftrightarrow p_4$ due to the property $\mathcal{K}\left[\sqrt{1 - \frac{p_3^2}{p_4^2}}\right] = \frac{|p_4|}{|p_3|}\mathcal{K}\left[\sqrt{1 - \frac{p_4^2}{p_3^2}}\right]$. In fact, all the coefficients of the expansion have this symmetry.

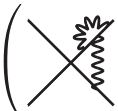
[8]Definitions, properties and relevant references of Mellin-Barnes representation are given in the Appendix C.

expression of graphs a to conclude that

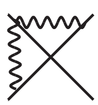
$\mathcal{D}\Bigg($  $\Bigg) + \mathcal{D}\Bigg($  $\Bigg) = \mathcal{D}\Bigg($  $\Bigg) + \mathcal{D}\Bigg($  $\Bigg)$

$+\, 2(p_3 + p_4)^2$  $- \dfrac{1}{32}.$ (4.2.36)

Thus, according to the analysis we made before, in the limit $p_1 + p_2 = -p_3 - p_4 \to 0$ we obtain

$\mathcal{D}\Bigg($  $\Bigg) + \mathcal{D}\Bigg($  $\Bigg) \longrightarrow -\dfrac{1}{32}.$ (4.2.37)

Finally, it is possible to calculate the Feynman integral of graph e when $p_1 + p_2 = 0$ and $p_1 + p_4 = 0$ by substituting $p_1 = p$, $p_2 = -p$, $p_4 = -p$ and from momentum conservation $p_3 = p$ to obtain

$\mathcal{D}\Bigg($  $\Bigg) \longrightarrow \displaystyle\int \frac{d^dk}{(2\pi)^d}\frac{d^dl}{(2\pi)^d}\frac{-2p^2\, k.l}{k^2\,(k+p)^2\,(k+l+p)^2\,(l+p)^2\,l^2} = \frac{1}{8\pi^2} - \frac{1}{64}.$

(4.2.38)

With all these elements we may make the sum to find the finite two-loop contribution

$$\Gamma^{(2)}[\mathcal{A},\mathcal{B}] = \lambda^2(-8 - \tfrac{3}{2}\pi^2)\int d^2\theta\, d^3x\, \frac{2\pi i}{k}\,\epsilon_{AC}\epsilon^{BD}\mathrm{tr}\left(Z^A W_B Z^C W_D\right). \qquad (4.2.39)$$

As mentioned before we expect this to be a well defined and gauge invariant result. Nevertheless, it' s easy to show that if such contribution had to be interpreted as a "finite renormalization" of the superpotential it would inevitably break extended supersymmetry.

4.3. Renormalization of the gauge sector

At this point, after having calculated the two- and four-point matter Green functions at two loops, it seems pretty clear that, at least at two loops, the only source of infrared divergences is given by the insertion of gauge-vector corrected lines in any diagram (in the next chapter we will need to insert this correction in other diagrams and the situation is the same). In the same way, it seems obvious that the addition of the η dependent diagrams corrects this problem at two loops since they complete the transverse structure of the gauge vector propagator for any given insertion. What is not at all obvious is if in fact this non-local gauge fixing we have introduced is really a simple alternative gauge fixing or something else. Do the gauge invariant questions we can ask to the theory depend on this fixing? If that would be the case, the method would clearly be inconsistent.

In fact, in the last two sections, we have already 'asked' two gauge invariant questions which turned out not trivially to be independent of the gauge fixing. The observations we can make from these questions are the following:

- Since the four point functions cannot produce ultraviolet divergent Feynman Integrals, the study of the renormalization flow of the superpotential coupling is reduced to the study of the matter fields anomalous dimension. This is, the beta function of the superpotential constant is proportional to the anomalous dimensions of matter fields and therefore, this anomalous dimension is gauge invariant. In the theory in question, we found in the Landau gauge, in the α gauge and in the η gauge, that the sum of ultraviolet divergent contributions was vanishing. This is a consequence of the quantum conformal symmetry of ABJM theory. They vanish for any gauge, supporting the consistency of the alternative η gauge.

- In the Landau gauge, consider for a moment the sum of all two loop 1PI four-point functions we have calculated in the previous section and add the sum of the four two-loop self energy corrected legs. As we mentioned at the end of the last section, this sum is gauge invariant and the only effect of this addition is the cancelation of all infrared divergencies and the addition of a constant. From this, it is clear that redoing the same calculation in the η gauge leads to exactly the same result as it should since the four point function with corrected legs is a gauge invariant function. This, once again gives supporting evidence of the consistency of the method.

The last check we would like to analyze before moving on is in the renormalization properties of the gauge sector. In the same way we did in chapter 2, we may consider the anomalous dimension of the gauge-vector superfield V of its corresponding ghosts b-c and the renormalization constant of their interaction vertex Z_g in the Landau gauge and in the η gauge. We may adapt the results of chapter 2 to the ABJM case (*c.f.* formula (2.2.81) and formulas therein), such that in the Landau gauge the value of the anomalous dimensions and the renormalization constant is[9]

$$\gamma_v = \frac{2}{3}\lambda^2, \qquad \gamma_{bc} = -\frac{1}{3}\lambda^2, \qquad Z_g = 1, \tag{4.3.1}$$

such that $\beta(g) = g(\gamma_v + 2\gamma_{bc}) = 0$. In the process of calculating these quantities we omitted considering some diagrams which were only infrared divergent but not ultraviolet divergent. Also, we found the UV/IR tadpole diagram, of which we extracted the ultraviolet divergency by reshuffling the external momenta without paying attention to the infrared divergence present on those diagrams.

The addition of the diagrams which contain the η dependent piece of the gauge vector propagator will clearly remove the infrared divergencies of this calculation. But what we really want to know is wether gauge invariant statements, such as the vanishing of the beta function, continue to vanish in our modified gauge-averaging scheme. The current analysis is more interesting than the previous one since in this case, the anomalous dimension of the gauge vector and the ghost, are not gauge invariant as in the case of matter. As such, these anomalous dimensions may be modified and so the vanishing of the beta function in the new gauge would be a stronger check of the consistency of the method.

We depict all infrared divergent diagrams of the gauge-vector self-energy and ghosts self-energy in figure 4.6. Diagrams a, b, c and d have to be considered in both the Landau gauge and in the η- gauge while diagrams e, f, g and h are specific of the η gauge since they contain the η dependent piece of the propagator. Diagrams a and c also contain ultraviolet divergences that were already taken into account in the calculation

[9] In chapter 2 we had calculated the influence of fundamental instead of bifundamental matter in the gauge sector renormalization properties. Going from one to the other only requires a few steps.

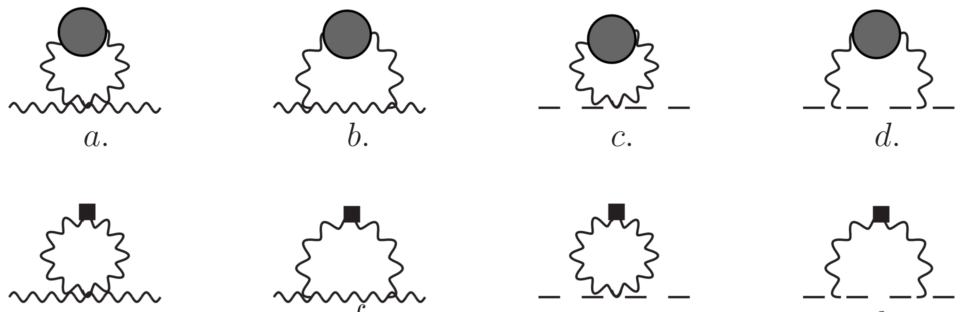


Figure 4.6.: Infrared divergent ghost and gauge-vector self-energy diagrams. The first line of diagrams contributes to the η and the Landau gauge while the second line is specific of the η gauge. The blobs represent one-loop matter, ghosts and vector corrections as in figure 4.1. The small squares are the η-dependent pieces of the propagator.

of γ_v and γ_{bc} in (4.3.1). The second line of diagrams in figure 4.6, which contain the η dependent pieces of the propagator, contain new ultraviolet divergencies which may modify the anomalous dimensions of the ghost and gauge-vector fields.

It is easy to see that diagram e, which gives a contribution

$$v_e^{AB} = 8\pi^2\lambda^2(\delta^{AB} - \delta^{A0}\delta^{B0})G(1,1)\int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2)^{3/2+\epsilon}}, \tag{4.3.2}$$

exactly cancels diagram a. This is, not only it removes its infrared divergence but also its ultraviolet one in this case. Diagram g

$$h_g^{AB} = 8\pi^2\lambda^2(\delta^{AB} - \delta^{A0}\delta^{B0})G(1,1)\int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2)^{3/2+\epsilon}}, \tag{4.3.3}$$

in the exact same way, cancels exactly diagram c, removing the infrared and the ultraviolet divergence. Diagram b produces the infrared divergent contribution

$$v_b^{AB} = -24\pi^2\lambda^2(\delta^{AB} - \delta^{A0}\delta^{B0})G(1,1)\int \frac{d^dk}{(2\pi)^d}\frac{2p.(p+k)}{(k^2)^{3/2+\epsilon}(k+p)^2}, \tag{4.3.4}$$

and diagram f, an ultraviolet and infrared divergent one

$$v_f^{AB} = 24\pi^2\lambda^2(\delta^{AB} - \delta^{A0}\delta^{B0})G(1,1)\int \frac{d^dk}{(2\pi)^d}\frac{2p^2 + 2p.k + k^2}{(k^2)^{3/2+\epsilon}(k+p)^2}, \tag{4.3.5}$$

such that their sum

$$v_b^{AB} + v_f^{AB} = 24\pi^2\lambda^2(\delta^{AB} - \delta^{A0}\delta^{B0})G(1,1)\int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2)^{1/2+\epsilon}(k+p)^2}$$
$$= \frac{3}{8}\lambda^2(\delta^{AB} - \delta^{A0}\delta^{B0})\frac{1}{\epsilon} \tag{4.3.6}$$

is purely UV divergent. A similar thing happens between diagrams d and h. We have the infrared divergent contribution of d

$$h_d^{AB} = 12\pi^2\lambda^2(\delta^{AB} - \delta^{A0}\delta^{B0})G(1,1)\int \frac{d^dk}{(2\pi)^d}\frac{2p.(p+k)}{(k^2)^{3/2+\epsilon}(k+p)^2}, \tag{4.3.7}$$

and the ultraviolet and infrared divergent from h

$$h_h^{AB} = -12\pi^2\lambda^2(\delta^{AB} - \delta^{A0}\delta^{B0})G(1,1)\int \frac{d^dk}{(2\pi)^d}\frac{2p^2+2p.k+2k^2}{(k^2)^{3/2+\epsilon}(k+p)^2}. \tag{4.3.8}$$

Their sum cancels the infrared divergence and leave us with only an ultraviolet divergence

$$h_d^{AB} + h_h^{AB} = -24\pi^2\lambda^2(\delta^{AB} - \delta^{A0}\delta^{B0})G(1,1)\int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2)^{1/2+\epsilon}(k+p)^2}$$
$$= -\frac{3}{8}\lambda^2(\delta^{AB} - \delta^{A0}\delta^{B0})\frac{1}{\epsilon}. \tag{4.3.9}$$

Having checked the cancelation of all infrared divergences we now calculate the new anomalous dimensions of the ghosts and the gauge-vector in the η gauge, by adding to the already known contributions (those that were already present in the Landau gauge) those of e, f, g and h which we have already discussed. We finally find that the anomalous dimensions in the infrared-safe η gauge is

$$\gamma_v^\eta = \frac{5}{3}\lambda^2, \qquad \gamma_{bc}^\eta = -\frac{5}{6}\lambda^2. \tag{4.3.10}$$

Since the sum of new (η-dependent) contributions to the gauge/ghost vertex cancels we also have that $Z_g^\eta = 1$. We thus find that

$$\beta(g) = g(\gamma_v^\eta + 2\gamma_{bc}^\eta) = 0. \tag{4.3.11}$$

That is, the beta function of the Chern-Simons coupling vanishes consistently in the η gauge. By comparing (4.3.1) and (4.3.10) we effectively notice that the anomalous dimensions of the fields were modified but in a precise way such that the beta function keeps being zero. This is a non trivial check of the statement that our infrared-removing prescription is just a gauge choice.

The last check we perform of this method in this work is given in chapter 5 where we calculate the anomalous dimension of long chain operators. Since this anomalous dimension is gauge invariant, the use of the η gauge should produce the same result as in any gauge. In fact, in that calculation we will still find the presence of annoying infrared divergencies which can be removed by the gauge fixing choice proposed on this chapter such that the anomalous dimension of this long chain operators is not modified.

4.4. General analysis

We would like now to make some comments on the generality of the results of this chapter. In a three dimensional theory there are two sources of infrared divergences in Feynman integrals. On the one hand we have the insertion of self energy corrected lines which may produce high powers of the propagators $\frac{1}{(k^2)^a}$ with $a \geq \frac{3}{2}$. On the other hand the presence of a three-lined vertex interaction with no external legs is potentially dangerous since, if there are only scalar propagators attached to it (no momenta in the numerator), an IR divergence is produced after loop integration.

In general $\mathcal{N} = 2$ Chern-Simons-Matter theories there are three-lined vertexes that couple chiral fields with the gauge vector and there is also the three gluon vertex. Consider the matter-vector coupling as shown in figure (4.7).

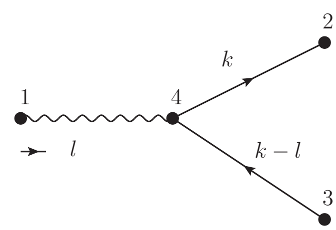


Figure 4.7.: Matter-Gluon coupling

If we integrate by parts on vertex 4 at least one of the D-operators of the gluon propagator, we get

$$\sim \frac{\bar{D}^\alpha D_\alpha}{l^2}\delta_{(1,4)}\,\frac{D^2\bar{D}^2}{k^2}\delta_{(4,2)}\,\frac{\bar{D}^2D^2}{(k-l)^2}\delta_{(4,3)} = \frac{k^{\alpha\beta}}{k^2l^2(k-l)^2}\,D_\alpha\delta_{(1,4)}\,D_\beta\bar{D}^2\delta_{(4,2)}\,\bar{D}^2D^2\delta_{(4,3)}. \tag{4.4.1}$$

The appearance in the numerator of one of the momenta carried by the lines eliminates the IR threat as long as no self energy corrections are involved in the full graph (we deal with them in what follows). A similar analysis can be done for the three gluon vertex.

It is quite obvious that the insertion of self-energy matter corrected lines inside any given graph, does not lower the scaling of the propagator thus not leading to IR issues. Then we conclude that IR problems are only generated by the insertion of self-energy corrected gluon lines: with the aid of the modified propagator we proposed at the beginning of this chapter, it seems plausible that IR divergences can in principle be cured to all loop orders.

To leading order we have shown that the key in the elimination of IR divergences was the completion of the 1-loop corrected gauge vector by adding the longitudinal part with the η piece of the propagator. Having understood this mechanism that improves the IR behaviour of the gauge propagator correction at 1-loop, we may generalize this notion to all orders in λ. Due to gauge invariance and parity we know [32] that the all order 1PI vector self-energy calculated with the ordinary piece of the propagator is given by

$$\Delta_V = \frac{1}{2}\int\frac{d^3k\,d^4\theta}{(2\pi)^3}\,\mathrm{Tr}\left(V(-k)\left(\sum_{l=1}A_l^\epsilon(k)\lambda^{2l}\bar{D}^\alpha D_\alpha + \sum_{l=0}B_l^\epsilon(k)\lambda^{2l+1}\,|k|\mathcal{P}_{1/2}\right)V(k)\right), \tag{4.4.2}$$

where the coefficients $A_l^\epsilon(k)$ and $B_l^\epsilon(k)$ are functions that contain ϵ-powers of the momentum. That is, odd loop corrections contain the superspin 1/2 projector, and even loop corrections reproduce the original structure of the action. Any odd-loop correction from (4.4.2), when attached inside a graph produces a propagator given by

$$-\sum_{l=0}B_l^\epsilon(k)\lambda^{2l+1}\,\frac{\mathcal{P}_{1/2}}{|k|}\delta_{(\theta,\theta')}, \tag{4.4.3}$$

which, as noted before, will produce an IR divergence. On the other hand even-loop corrections, when attached inside a graph, produce a term $\sim \bar{D}^\alpha D_\alpha / k^2$ which behaves in the same way as the basic propagator, thus not leading to IR issues. These formulas can be readily derived using (A.0.9).

Having understood the effect of corrected vector propagator insertions in graphs, we can now proceed to fix the η parameter perturbatively as an odd power series in λ. After fixing it to order one, $\eta^\epsilon_{(p)} = -6\pi G(1,1)(p^2)^{-\epsilon}\lambda + O(\lambda^3)$, one calculates every connected (not only 1PI) self energy vector correction at order λ^3, including lower loop $O(\lambda^3)$ corrections with the ${}^1\eta^\epsilon_{(p)}$ piece of the propagator. With this result we fix the next coefficient ${}^3\eta^\epsilon_{(p)}$ such that we complete the transverse projector with the longitudinal one effectively removing the source of infrared divergence at order λ^3. This process may be continued recursively thus improving the IR behavior of the propagator to all loops.

In this way, if one considers a given graph which contains an L-loop-dressed gauge vector, then if L is odd there will always be a complementary graph in which we substitute that dressed line with the η piece of the propagator at the corresponding order in λ, such that the whole line will behave as $\sim \frac{\delta_{(\theta,\theta')}}{|k|}$; instead, when L is even, the line behaves as the ordinary propagator $\sim \frac{\bar{D}^\alpha D_\alpha \delta_{(\theta,\theta')}}{k^2}$ and needs no modifications. In both cases the graph will be IR safe.

4.5. Summary

In this chapter we studied the infrared behavior of the off-shell amplitudes in three-dimensional Chern-Simons-matter theories with specific attention to the ABJM model. In $\mathcal{N} = 2$ superspace IR divergences show up in a very similar way as in four-dimensional Super-Yang-Mills theory, being related to the corrected vector superfield propagator insertions. At first, we showed that if the theory is gauge fixed in a standard fashion there is no way to get rid of the divergent integrals without losing the hermiticity of the action. Then we introduced a non-local gauge fixing procedure which leads to divergences cancelation without spoiling the renormalizability of the theory. In order to do so, the gauge-fixing parameter had to be perturbatively fine tuned. Moreover, we found in our computations that infrared infinities seem to be always associated to gauge dependent parts in the amplitudes, thus not affecting the physical quantities of the theory. As a non-trivial output of our calculations we provided the two-loop finite correction to the effective superpotential for ABJM theory in equation (4.2.39).

It would be interesting to address the same problems in the case of Chern-Simons-matter theories described in $\mathcal{N} = 1$ superspace, starting for instance from the formulation of BLG theory [3, 4, 46] given in [15]. In this case the analogy with the four-dimensional case is lost and one might expect a different infrared behaviour of superspace propagators. It would be also interesting to perform a similar analysis in the $\mathcal{N} = 3$ harmonic superspace formulation of [47, 48].

Chapter 5.

Four-Loop spectrum of ABJ(M) theory

The ABJM model is similar in many respects to its cousin $\mathcal{N} = 4$ super Yang-Mills in four dimensions. Similarly to it, its two point functions of single trace operators map to an integrable system in the planar limit [49–52]. For $\mathcal{N} = 4$ SYM, the integrability has been used as a powerful tool to interpolate between strong and weak coupling, where one can see the perturbative behavior of the gauge theory morph into the stringy behavior expected from the AdS/CFT conjecture [53, 54].

The ABJM model has two extra features that give it a richer structure than $\mathcal{N} = 4$ SYM, at least as far as the integrability of the two point functions is concerned. The first is that the Bethe equations and the dispersion relations contain an undetermined function $h^2(\lambda)$ of the 't Hooft coupling, $\lambda = N/k$, where k is the Chern-Simons level [52]. The second is that the theory can be deformed into a $U(M) \times U(N)$ gauge theory (ABJ theory) while still maintaining the $\mathcal{N} = 6$ supersymmetry [6]. In this ABJ case there is a single function of the 't Hooft parameters $h^2(\bar{\lambda}, \sigma)$ if integrability is maintained.

The spin-chain that appears in the ABJ(M) models has $OSp(6|4)$ symmetry and is of alternating type, with the spins on the odd sites in the singleton representation of the supergroup and the spins on the even sites in the anti-singleton representation [49–51, 55, 56]. In order to find $h^2(\bar{\lambda}, \sigma)$ it is only necessary to consider the compact subgroup $SU(2) \times SU(2)$ of $OSp(6|4)$, with the spins on the odd sites transforming in the $(\mathbf{2}, \mathbf{1})$ representation and the spins on the even sites transforming in the $(\mathbf{1}, \mathbf{2})$ representation. The ground state has all spins aligned and the excitations (or magnons) are flipped spins that live on either odd or even sites. The dispersion relations for these two types of magnons are given by

$$E_{\mathrm{odd}}(p) = \sqrt{Q^2 + 4h^2(\bar{\lambda},\sigma)\sin^2\tfrac{p}{2}} - Q\,, \qquad E_{\mathrm{even}}(p) = E_{\mathrm{odd}}(p)\big|_{\sigma\to-\sigma}\,, \tag{5.0.1}$$

where $Q = 1/2$ for fundamental magnons while larger values of Q correspond to magnon bound states.

At weak coupling the function $h^2(\bar{\lambda},\sigma)$ can be computed perturbatively. The leading contribution appears at two-loop order and is relatively easy to compute, both for ABJM [49–51], and ABJ [56, 57], where one finds

$$h^2(\bar{\lambda},\sigma) = \bar{\lambda}^2 + \mathcal{O}(\bar{\lambda}^4)\,. \tag{5.0.2}$$

However, at strong coupling on the ABJM slice where $\sigma = 0$, one readily finds from the string sigma model [50, 58, 59].

$$h^2(\bar{\lambda},0) = \frac{1}{2}\,\bar{\lambda} + \mathcal{O}(1)\,. \tag{5.0.3}$$

Hence, $h^2(\bar{\lambda},\sigma)$ is an interpolating function and can be expected to have corrections at every even order of perturbation theory, with a general structure

$$h^2(\bar{\lambda},\sigma) = \bar{\lambda}^2 + \sum_{n=2}^{\infty} \bar{\lambda}^{2n} h_{2n}(\sigma)\ . \tag{5.0.4}$$

The four-loop term in (5.0.4) was computed in [60, 61], where it was originally found that[1]

$$h_4(\sigma) = -(4+\sigma^2)\zeta(2) - 16\,. \quad \text{(wrong value!)} \tag{5.0.5}$$

After the superspace revision of this calculation [13] which we present in what follows, an error was found on it which led to the now verified value

$$h_4(\sigma) = -(4+\sigma^2)\zeta(2)\,. \tag{5.0.6}$$

[1]After it became clear that those results were in conflict with the results presented here, an overall sign error was discovered for three of the Feynman graphs.

The component calculation was done using the explicit component action and involved the computation of dozens of Feynman diagrams. A straightforward extension of the methods in [60, 61] to higher loops would lead to a mind boggling number of diagrams. Moreover, one would like to verify (or disprove) that the ABJ theory is integrable, even at the four-loop order. The $SU(2) \times SU(2)$ sector is trivially integrable at four loops, so it would be necessary to go beyond this sector to find a nontrivial check of integrability at this order. But even this seemingly modest task is extremely daunting in component language.

In this chapter we present the computation $h_4(\sigma)$ in (5.0.6) using the superspace formalism. Superspace techniques have proven to be very effective in computing the dilatation operator [62] and in evaluating wrapping corrections [63, 64] in $\mathcal{N} = 4$ SYM [65, 66] and in its β-deformation [67–69].

Naturally, one would also like to apply them to the model in question for the calculation of $h_4(\sigma)$. Their main virtue is that they drastically reduce the number of Feynman diagrams that one must compute. Furthermore, one can often find cancelation patterns between different supergraphs or demonstrate finiteness theorems for classes of diagrams [62, 65]. Such generalized finiteness conditions [62] that follow from power counting arguments and some of their implications are summarized in section 5.2.1. They predict the finiteness of many diagrams and will be of great use to us in our calculations. As we will see in this chapter at the two-loop order there is only one diagram in superspace that contributes to $h^2(\bar{\lambda}, \sigma)$. At the four-loop order there are 15 (plus reflections of some of the diagrams). Contrast this to the component calculation in [60, 61], where one has many times more diagrams. Not only does this demonstrate the formalism's power, but it is also crucial in verifying that (5.0.6) is actually correct (see footnote 1).

One can also see from (5.0.6) that $h_4(\sigma)$ has uniform transcendentality two. From the component point of view this seems almost miraculous since many diagrams have rational coefficients (that is, they have transcendentality zero), others have transcendentality two, and some are mixed. When everything is combined one finds that the rational coefficients cancel. In superspace, while there are still diagrams with rational coefficients, their cancelation appears more natural due to correlations between the single and double poles.

We will also present two possible scenarios for an all-loop function for $h^2(\lambda)$, including one that might work. It reproduces the first two orders of perturbation theory as well as the leading sigma-model contribution at strong-coupling. The one-loop sigma-model contribution to $h^2(\lambda)$ depends on how a sum is carried out over an infinite number of modes. Our proposal disagrees with the more conventional prescription in [70], but agrees with the prescription in [71]. The other proposal looks for a connection with matrix models on a Lens space. These arise in the study of supersymmetric Wilson loops in ABJ(M) models [72–75]. In particular, we consider the free energy of the matrix model which is a function of λ. We will see that $h^2(\lambda)$ has a structure similar to the derivative of the matrix model free energy, both at small and large λ. But the coefficients in their respective expansions do not quite line up.

In order to complete the four-loop analysis in the $SU(2) \times SU(2)$ subsector, we will apply the superspace formalism to compute the leading wrapping corrections for a length four operator in the $\mathbf{(1,1)}$ representation of $SU(2) \times SU(2)$. Here we find that the wrapping corrections *per se* differ from those computed in component language. However, other range five interactions must be subtracted and this subtracted piece also differs from the corresponding term in the component calculation. The two effects combine to give the same four-loop anomalous dimension for this operator as was found using components.

In section 5.1 we discuss the relation of the dilatation operator to $h^2(\bar{\lambda}, \sigma)$. In section 5.2 we enumerate and compute all Feynman diagrams that contribute to the four-loop term $h_4(\sigma)$. In section 5.3 we discuss our investigation into possible all-loop functions for $h^2(\lambda)$. In section 5.4 we apply the superspace formalism to the wrapping corrections for operators of length four. In section 5.5 we explicitly verify the cancelation of all infrared divergencies in the calculation. We also comment on the application of the η gauge presented in the previous chapter. In section 5.6 we verify the consistency of the calculation by verifying the cancelation of double poles due to ultraviolet subdivergencies and we verify the decoupling of odd and even site magnons at four loops. Finally, in section 5.7 we present a summary of the chapter, which includes suggestions for further work.

5.1. The dilatation operator and $h^2(\bar{\lambda}, \sigma)$

The dilatation operator $\mathcal{D}$ is the natural tool to study the anomalous dimensions of composite operators in field theory. It can be defined as the operator that by acting on composite operators $\mathcal{O}_a$ provides the matrix of scaling dimensions

$$\mathcal{D}\mathcal{O}_a = \Delta_a{}^b(\mathcal{O})\mathcal{O}_b \; . \tag{5.1.1}$$

Note that $\Delta_a{}^b$ leads in general to the mixing between operators. As known, the matrix of dimensions, and therefore the dilatation operator, can be extracted from the perturbative renormalization of the composite operators $\mathcal{O}_a$

$$\mathcal{O}_{a,\text{ren}} = \mathcal{Z}_a{}^b\mathcal{O}_{b,\text{bare}} \; , \qquad \mathcal{Z} = \mathbb{1} + \bar{\lambda}^2\mathcal{Z}_2 + \bar{\lambda}^4\mathcal{Z}_4 + \dots \; . \tag{5.1.2}$$

The matrix $\mathcal{Z}$ is such that $\mathcal{O}_{a,\text{ren}}$ is free from perturbative quantum divergences and can be computed in perturbation theory by means of standard methods. We use dimensional reduction with the space-time dimension D given by

$$D = 3 - 2\varepsilon \; , \tag{5.1.3}$$

in order to regularize quantum divergences that show up as inverse powers of ε in the limit $\varepsilon \to 0$. By introducing the 't Hooft mass μ and the dimensionful combination $\bar{\lambda}\mu^{2\varepsilon}$ the dilatation operator is then extracted from $\mathcal{Z}$ as

$$\mathcal{D} = \mathcal{D}_{\text{classical}} + \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\ln\mathcal{Z}(\bar{\lambda}\mu^{2\varepsilon}, \varepsilon) = \mathcal{D}_{\text{classical}} + \lim_{\varepsilon\to 0}\left[2\varepsilon\bar{\lambda}\frac{\mathrm{d}}{\mathrm{d}\bar{\lambda}}\ln\mathcal{Z}(\bar{\lambda}, \varepsilon)\right] \; . \tag{5.1.4}$$

In a loop expansion of the dilatation operator, the lth loop order is then simply given by the $\bar{\lambda}^{2l}$ coefficient of the $1/\varepsilon$ pole of $\ln\mathcal{Z}$ multiplied by $2l$. The higher order poles must be absent in $\ln\mathcal{Z}$; this will be later used as a consistency check for our result.

As discussed before, in the ABJ(M) models the dilatation operator can be mapped to the long range Hamiltonian of a spin-chain system for the whole $OSp(6|4)$ symmetry group [49, 56]. We focus on the $SU(2)\times SU(2)$ subsector where the magnons propagating along the spin chain form two sectors: the ones living on the odd sites belong to the first $SU(2)$, while those on the even sites are associated with the other $SU(2)$. As we demonstrate in section 5.6.1, in our four-loop analysis the two different types of magnons can be regarded as non-interacting, since the contributions to the dilatation operator of the respective diagrams that could lead to these interactions cancel. The all-loop Bethe Ansatz [52] predicts that such interactions start at eight loops. In analogy with the $\mathcal{N}=4$ case, the spin-chain is interpreted as a quantum mechanical system in which the ground state of length $2L$ can be chosen to be

$$\Omega = \mathrm{tr}\,(W_1 Z^1)^L \ . \tag{5.1.5}$$

With a single excitation W_2 of an odd site the momentum eigenstate is defined as

$$\psi_p = \sum_{k=0}^{L-1} e^{ipk}(W_1 Z^1)^k W_2 Z^1 (W_1 Z^1)^{L-k-1} \tag{5.1.6}$$

This describes a single magnon excitation with momentum p. The main difference between the $\mathcal{N}=6$ CS and the $\mathcal{N}=4$ SYM case is the existence in the former of two different $SU(2)$ excitations corresponding to the sectors mentioned above.

Up to four loops, the dilatation operator for a chain of length $2L$ then expands as

$$\mathcal{D} = L + \bar\lambda^2(\mathcal{D}_{2,\mathrm{odd}} + \mathcal{D}_{2,\mathrm{even}}) + \bar\lambda^4(\mathcal{D}_{4,\mathrm{odd}}(\sigma) + \mathcal{D}_{4,\mathrm{even}}(\sigma)) + \mathcal{O}(\bar\lambda^6) \ , \tag{5.1.7}$$

where the individual parts act non-trivially on odd and even sites only.

In the $\mathcal{N}=4$ SYM case chiral functions have been introduced in [65] as a very convenient basis for the dilatation operator of the $SU(2)$ subsector. The chiral functions directly capture the structure of the chiral superfields in the Feynman diagrams. As in the $\mathcal{N}=4$ SYM case, also in the $\mathcal{N}=6$ CS case the elementary building block for the chiral function of the $SU(2)\times SU(2)$ subsector is constructed from the superpotential by contracting one chiral and one anti-chiral vertex with a single chiral propagator. The resulting flavor structure then yields the simplest non-trivial chiral function.

The chiral functions that are relevant to two loops in $\mathcal{N}=4$ SYM and to four loops in $\mathcal{N}=6$ CS theory turn out to have identical form in terms of the respective permutation structures and read

$$
\begin{aligned}
\chi(a,b) &= \{a,b\} - \{a\} - \{b\} + \{\} \ , \\
\chi(a) &= \{a\} - \{\} \ , \\
\chi() &= \{\} \ .
\end{aligned}
\tag{5.1.8}
$$

However, the permutation structures in both theories slightly differ. In the $\mathcal{N}=6$ CS case they are given by [61]

$$
\{a_1, a_2, \ldots, a_m\} = \sum_{i=0}^{L-1} \mathrm{P}_{2i+a_1\, 2i+a_1+2}\, \mathrm{P}_{2i+a_2\, 2i+a_2+2} \ldots \mathrm{P}_{2i+a_m\, 2i+a_m+2} \ , \tag{5.1.9}
$$

where we identify $L+i \equiv i$, such that the product of permutations, in which $\mathrm{P}_{a\,a+2}$ permutes the flavors at sites a and $a+2$, is inserted at every second site of the cyclic spin chain of length $2L$.[2] The insertion at each second site thereby allows for the decomposition of the dilatation operator into two separate pieces acting only on odd or even sites as in (5.1.7). The decomposition of the dilatation operator to four loops [61] in terms of chiral functions then reads

$$
\begin{aligned}
\mathcal{D}_{2,\mathrm{odd}} &= -\chi(1) \ , \\
\mathcal{D}_{2,\mathrm{even}} &= -\chi(2) \ , \\
\mathcal{D}_{4,\mathrm{odd}}(\sigma) &= -\chi(1,3) - \chi(3,1) + (2 - h_4(\sigma))\chi(1) \ , \\
\mathcal{D}_{4,\mathrm{even}}(\sigma) &= -\chi(2,4) - \chi(4,2) + (2 - h_4(-\sigma))\chi(2) \ .
\end{aligned}
\tag{5.1.11}
$$

The coefficients are thereby fixed by the magnon dispersion relation (5.0.1) in terms of the four-loop contribution $h_4(\sigma)$ of the a priori undetermined function $h^2(\bar{\lambda}, \sigma)$ in (5.0.4). As explained in [61] to obtain the above result, one just has to compare the expansion of the magnon dispersion relation to the momentum dependence when the individual terms are applied to the single magnon momentum eigenstate (5.1.6).

The function $h_4(\sigma)$ can be computed in the weak coupling limit from a direct perturbative calculation. This has been done by using component fields techniques in [61].

[2]Note that the permutation structures obey

$$
\begin{aligned}
\{\ldots, a, b, \ldots\} &= \{\ldots, b, a, \ldots\} \ , \qquad |a-b| \neq 2 \ , \\
\{a, \ldots, b\} &= \{a+2n, \ldots, b+2n\} \ .
\end{aligned}
\tag{5.1.10}
$$

Here we present its calculation by using $\mathcal{N}=2$ supergraphs. As in the component calculation [61], also here it suffices to only consider the odd part of the dilatation operator, i.e. the contributions with chiral functions that have odd integers as arguments.[3] The supergraphs computation of the full $\mathcal{D}_{4,\text{odd}}$, and in particular of $h_4(\sigma)$, is the main result of this chapter.

5.2. Feynman diagram calculation

Before starting with the explicit evaluation of Feynman diagrams we will summarize the previously mentioned finiteness conditions which allow us to disregard entire classes of diagrams.

5.2.1. Finiteness conditions

Based on power counting and structural properties of the Feynman rules, in [62] finiteness conditions for Feynman diagrams of $\mathcal{N}=4$ SYM theory in terms of $\mathcal{N}=1$ superfields and for $\mathcal{N}=6$ CS theory in terms of $\mathcal{N}=2$ superfields were derived. They hold for each diagram that contributes to the renormalization of chiral operators in the respective $SU(2)$ or $SU(2)\times SU(2)$ subsectors. In Landau gauge, such a diagram with interaction range $R\geq 2$ has no overall UV divergence, if at least one of the following criteria is matched:[4]

1. All of its chiral vertices are part of any loop.
2. One of its spinor derivative D_α is brought outside the loops.
3. The number of its spinor derivatives $\bar{\mathrm{D}}_\alpha$ brought outside loops becomes equal or bigger than twice the number of chiral vertices that are not part of any loop.

In the flavor $SU(2)\times SU(2)$ subsector, a chiral vertex that is not part of any loop always generates flavor permutations and therefore a non-trivial chiral structure of the diagram.

[3] As we mentioned before, odd and even site magnons are decoupled here, there is therefore no contribution with chiral functions with both odd and even integer arguments. We explicitly demonstrate their absence at four loops in section 5.6.1.

[4] $R\geq 2$ means, the composite operator is 1PI connected with the rest of the diagram, not including the non-interacting fields of the operator.

Analogously to the $\mathcal{N} = 4$ SYM case, the above finiteness conditions hence imply the following rule:

- All diagrams with interaction range $R \geq 2$ and trivial chiral structure $\chi()$ are finite.

Together with the conformal invariance on the quantum level, i.e. the finiteness of the chiral self energy, this implies that any diagram which does not manipulate the flavor, i.e. it has trivial chiral structure $\chi()$ defined in (5.1.8), has no overall UV divergence.

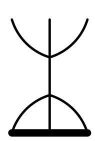
Since the propagators of the vector fields in Landau gauge carry $\mathrm{D}\,\bar{\mathrm{D}}$, the finiteness conditions imply the following statement:

- A diagram with interaction range $R \geq 2$ has no overall UV divergence, if it contains at least one cubic gauge-matter interaction with a chiral field line which is not part of any loop. In particular, if in the diagram exactly one of the chiral vertices appears outside the loops, then it also has no overall UV divergence if the anti-chiral field of at least one cubic gauge-matter interaction is not part of any loop.

According to this statement, there are no contributions to the dilatation operator that come from diagrams in which the chiral line of a cubic gauge-matter vertex is an external line. In section 5.5 we will, however, evaluate such diagrams with IR divergences explicitly to show that indeed all IR divergences cancel out in the renormalization constant $\mathcal{Z}$ in (5.1.2).

5.2.2. Two loops

Before attacking the more involved four-loop case, let us see how the two-loop result is obtained from supergraphs. There is only one non-vanishing logarithmically divergent diagram contributing. It evaluates to



$$\rightarrow \frac{(4\pi)^2}{k^2} MN\, I_2\, \chi(1) = \frac{\lambda\hat{\lambda}}{4}\frac{1}{\varepsilon}\chi(1) \ , \tag{5.2.1}$$

where the two-loop integral I_2 is given in (C.1.1). As already discussed, to obtain the contribution to the dilatation operator one has to take the coefficient of the pole $1/\varepsilon$ and multiply it by $-2l$, in this case equal to -4. Once a factor $\bar{\lambda}^2 = \lambda\hat{\lambda}$ is removed one gets

$$\mathcal{D}_2 = -\chi(1)\ . \tag{5.2.2}$$

This coincides with the results found in [49, 51, 57] in components.

5.2.3. Four loops

Now, let us move to the four-loop contributions to the dilatation operator. We will separate them according to the range of the interactions. We will explicitly present only the diagrams surviving the finiteness conditions of [62] that are summarized in section 5.2.1. It is important to note that, according to these arguments, an overall UV divergence can be present in superficially logarithmically divergent diagrams if at least one purely chiral vertex remains outside the loops. This implies that the minimum range of interaction at any loop is three. This is consistent with the fact that the minimal structure that can appear in the dilatation operator is $\chi(1)$. The range varies between three and the maximum one which at four loops is five.

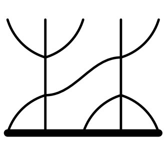
Note that together with the $1/\varepsilon$ poles we will also keep the higher order poles that display the presence of subdivergencies. Here, to four-loop order the only appearing higher order poles are double poles. In section 5.6.2 their cancelation in $\ln \mathcal{Z}$ will be explicitly demonstrated as an important consistency check of our calculation.

We note that, for the convenience of the reader, all the integrals appearing in the following are collected in the appendix C.

Range five interactions

At four loops there is only one supergraph that involves the maximum number of five neighboring fields in the interaction. It is given by

$$S_{\rm r5} =$$



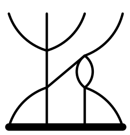
$$\to \frac{(4\pi)^4}{k^4}(MN)^2 I_4 \chi(1,3) = \frac{(\lambda\hat{\lambda})^2}{16}\Big(-\frac{1}{2\varepsilon^2}+\frac{2}{\varepsilon}\Big)\chi(1,3)\ . \tag{5.2.3}$$

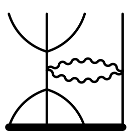
By taking into account the reflected diagram, the maximum range contribution to the renormalization constant is[5]

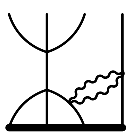
$$\mathcal{Z}_{\mathrm{r5,odd}} = -(1+\mathcal{R})S_{\mathrm{r5}} = \frac{(\lambda\hat{\lambda})^2}{16}\Big(\frac{1}{2\varepsilon^2} - \frac{2}{\varepsilon}\Big)(\chi(1,3)+\chi(3,1)) \ . \tag{5.2.4}$$

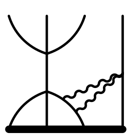
Range four interactions

There are four diagrams which have range four interactions and contribute to the structure $\chi(1)$ in the dilatation operator. According to section 5.2.1, for an overall UV divergence to be present, at least one purely chiral vertex has to remain outside the loops, and a single gauge propagator can not end up on an external leg. Therefore, the only relevant contributions turn out to be

$$S_{\mathrm{r4}} =$$

$$\to -\frac{(4\pi)^4}{k^4}\, M^3 N\, I_{4\mathbf{bbb}}\, \chi(1) = \frac{\lambda^3\hat{\lambda}}{16}\Big(-\frac{\pi^2}{2\varepsilon}\Big)\chi(1) \ ,$$

$$V_{\mathrm{r41}} =$$

$$\to \frac{(4\pi)^4}{2k^4}\, M^3 N\, I_4\, \chi(1) = \frac{\lambda^3\hat{\lambda}}{32}\Big(-\frac{1}{2\varepsilon^2} + \frac{2}{\varepsilon}\Big)\chi(1) \ ,$$

$$V_{\mathrm{r42}} =$$

$$\to \frac{(4\pi)^4}{2k^4}\, M^3 N\, I_4\, \chi(1) = \frac{\lambda^3\hat{\lambda}}{32}\Big(-\frac{1}{2\varepsilon^2} + \frac{2}{\varepsilon}\Big)\chi(1) \ , \tag{5.2.5}$$

$$V_{\mathrm{r43}} =$$

$$\to \frac{(4\pi)^4}{k^4}\, M^3 N\, I_{42\mathbf{b}\mathbf{b}d}\, \chi(1) = \frac{\lambda^3\hat{\lambda}}{16}\Big(\frac{1}{2\varepsilon^2} - \frac{1}{\varepsilon}\Big(2 - \frac{\pi^2}{4}\Big)\Big)\chi(1) \ .$$

Also in this case one has to consider the diagrams obtained by reflecting the previous ones. The total contribution to the renormalization constant is then

$$\mathcal{Z}_{\mathrm{r4,odd}} = \frac{\lambda\hat{\lambda}}{16}(\lambda^2+\hat{\lambda}^2)\frac{\pi^2}{4\varepsilon}\chi(1) \ . \tag{5.2.6}$$

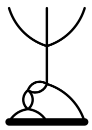
[5] By $\mathcal{R}$ we indicate the reflection of a supergraph at the vertical axis. As in [61], the operation preserves the type of chiral function, i.e. if it belongs to the odd or even sector. In case of an even number of neighbors interacting with each other the operation therefore involves a shift of the interaction by one site along the composite operator. Effectively, $\mathcal{R}$ therefore exchanges λ with $\hat{\lambda}$ and $\chi(a,b)$ with $\chi(b,a)$.

Range three interactions

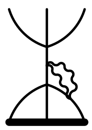
The range three interactions arise from two-loop corrections to the propagators and vertices involved in the two-loop diagram (5.2.1). It is important to note that, due to the finiteness rules of section 5.2.1, overall UV divergences can arise only from corrections to the lower vertex or one of the three lower chiral propagators. According to the analysis of section **??**, the two-loop corrections to the chiral two- and four-point functions are plagued by IR divergences even if free of UV poles. This is due to the particular structure of the gauge superfield propagator and cubic vertices in $\mathcal{N} = 2$ superspace. We stress that IR divergences do not appear in component fields [61], since in three dimensions IR dangerous cubic vertices contribute non-trivial momentum factors to the numerators of the loop integrals. In superspace, the appearance of IR divergences in intermediate steps can be cured by using a non-standard gauge fixing procedure we have introduced in chapter 4. Since we are interested only in the overall UV divergences of the diagrams, a computational strategy could be to ignore purely IR divergent diagrams and to IR-regulate diagrams that involve both UV and IR divergences in such a way as to extract the purely UV poles. For example, this is illustrated in appendix C.2 where we can regulate the IR divergences by inserting external momenta in IR divergent diagrams. However, we have decided to keep track of the IR divergences and check at the end their cancelation. Such a check is described in section 5.5 where we also comment on the consistency of the η gauge described in the previous chapter.

The interested reader should look chapter 4 for the description of two-loop corrections to the two- and four-point functions used in this section for the ABJM case. Their generalization to the ABJ case is given in appendix E.

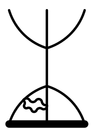
The contributions with only UV divergences are given by

$S_{\mathrm{r}3} =$ 

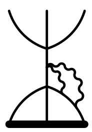
$$\to -\frac{2(4\pi)^4}{k^4}\, M^3 N\, I_{42\mathbf{bbb2}}\, \chi(1) = \frac{\lambda^3\hat{\lambda}}{16}\Big(-\frac{\pi^2}{2\varepsilon}\Big)\chi(1)\ ,$$

$V_{\mathrm{r}31a} =$ 

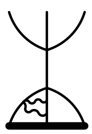
$$\to \frac{(4\pi)^4}{2k^4}\, M^3 N\, I_4\, \chi(1) = \frac{\lambda^3\hat{\lambda}}{16}\Big(-\frac{1}{4\varepsilon^2}+\frac{1}{\varepsilon}\Big)\chi(1)\ ,$$

$V_{\mathrm{r}31b} =$ 

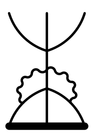
$$\to \frac{(4\pi)^4}{k^4}\, M^3 N(I_4 + I_{42\mathbf{b}bd})\chi(1) = \frac{\lambda^3\hat{\lambda}}{16}\frac{\pi^2}{4\varepsilon}\chi(1)\ ,$$

$V_{\mathrm{r}32a} =$ 

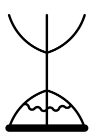
$$\to \frac{(4\pi)^4}{k^4}\, M^3 N\, I_{42\mathbf{b}bd}\, \chi(1) = \frac{\lambda^3\hat{\lambda}}{16}\Big(\frac{1}{2\varepsilon^2}+\frac{1}{\varepsilon}\Big(-2+\frac{\pi^2}{4}\Big)\Big)\chi(1)\ ,$$

$V_{\mathrm{r}32b} =$ 

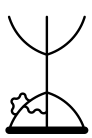
$$\to -\frac{(4\pi)^4}{2k^4}\, M^3 N\, I_{422\mathbf{q}\mathrm{tr}ABCD}\, \chi(1) = \frac{\lambda^3\hat{\lambda}}{16}\Big(-\frac{\pi^2}{6\varepsilon}\Big)\chi(1)\ , \qquad (5.2.7)$$

$V_{\mathrm{r}33a} =$ 

$$\begin{aligned}&\to \frac{(4\pi)^4}{k^4}\,(MN)^2 I_{422\mathbf{q}\mathrm{tr}ABbd}\,\chi(1)\\ &= \frac{(\lambda\hat{\lambda})^2}{16}\Big(-\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}\Big(4-\frac{2\pi^2}{3}\Big)\Big)\chi(1)\ ,\end{aligned}$$

$V_{\mathrm{r}33b} =$ 

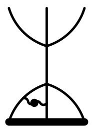
$$\to \frac{(4\pi)^4}{k^4}(MN)^2 I_{422\mathbf{q}\mathrm{tr}ABCD}\,\chi(1) = \frac{(\lambda\hat{\lambda})^2}{16}\frac{\pi^2}{3\varepsilon}\chi(1)\ ,$$

$V_{\mathrm{r}34} =$ 

$$\begin{aligned}&\to \frac{(4\pi)^4}{k^4}(MN)^2\big(2I_{42\mathbf{b}be} - I_{422\mathbf{q}\mathrm{tr}ABbd}\\ &\qquad + 2(2I_{221be} - I_{221dc})G(2-2\lambda,1)G(2-3\lambda,1)\\ &\qquad - 2(I_{42\mathbf{b}bd} + I_{42\mathbf{b}be})\big)\chi(1)\\ &= \frac{(\lambda\hat{\lambda})^2}{16}\Big(-\frac{\pi^2}{3\varepsilon}\Big)\chi(1)\ .\end{aligned}$$

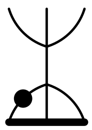
The contributions with both UV and IR divergences are given by

$$V_{\mathrm{r}35} = \quad \rightarrow -\frac{(4\pi)^4}{k^4}\big(MN(4MN-M^2)\big)\big(I_4 - I_{4\mathrm{UVIR}} + I_{42\mathbf{b}bd}\big)\chi(1)$$



$$= \frac{\lambda\hat{\lambda}}{16}(4\lambda\hat{\lambda}-\lambda^2)\Big(-\frac{1}{2\varepsilon^2}+\frac{2}{\varepsilon}\Big(-2-\frac{\pi^2}{8}+\gamma-\ln 4\pi\Big)\Big)\chi(1)\ ,$$

$$V_{\mathrm{r}36} = \quad \rightarrow \frac{(4\pi)^4}{k^4}MN\Big(2MNI_{4\mathbf{bbb}}-\frac{1}{2}\big(8MN-(M^2+N^2)\big)I_{4\mathrm{UVIR}}\Big)\chi(1)$$



$$= \frac{\lambda\hat{\lambda}}{16}\Big(\lambda\hat{\lambda}\frac{\pi^2}{\varepsilon}+\big(8\lambda\hat{\lambda}-(\lambda^2+\hat{\lambda}^2)\big)\Big(\frac{1}{4\varepsilon^2}+\frac{1}{\varepsilon}(2-\gamma+\ln 4\pi)\Big)\Big)\chi(1)\ . \tag{5.2.8}$$

Note that the expressions for the integrals that appear in the results have their UV subdivergencies subtracted. The suffix UVIR appears on integrals which due to different arrangements of their external momenta contribute both UV and IR divergences. The UV poles can be extracted by adding external momentum to the cubic vertex which causes the IR divergence, i.e. one replaces $I_{4\mathrm{UVIR}} \rightarrow I_4$. This then yields

$$\begin{aligned} V^{\mathrm{UV}}_{\mathrm{r}35} &= \frac{\lambda\hat{\lambda}}{16}(4\lambda\hat{\lambda}-\lambda^2)\Big(-\frac{1}{2\varepsilon^2}+\frac{1}{\varepsilon}\Big(2-\frac{\pi^2}{4}\Big)\Big)\chi(1)\ ,\\ V^{\mathrm{UV}}_{\mathrm{r}36} &= \frac{\lambda\hat{\lambda}}{16}\Big(\lambda\hat{\lambda}\frac{\pi^2}{\varepsilon}+(8\lambda\hat{\lambda}-(\lambda^2+\hat{\lambda}^2))\Big(\frac{1}{4\varepsilon^2}-\frac{1}{\varepsilon}\Big)\Big)\chi(1)\ , \end{aligned} \tag{5.2.9}$$

In chapter 5.5 we explicitly demonstrate that this result is also obtained if instead of choosing an IR safe momentum configuration all relevant diagrams with IR divergence are considered, i.e. the IR divergences cancel out in the final result.

The contribution of the range three interactions to the renormalization constant $\mathcal{Z}$ is then given by

$$\begin{aligned} \mathcal{Z}_{\mathrm{r}3,\mathrm{odd}} &= -(1+\mathcal{R})(S_{\mathrm{r}3}+V_{\mathrm{r}31a}+V_{\mathrm{r}31b}+V_{\mathrm{r}32a}+2V_{\mathrm{r}32b}+2V_{\mathrm{r}34}+V^{\mathrm{UV}}_{\mathrm{r}35})\\ &\quad - V_{\mathrm{r}33a}-V_{\mathrm{r}33b}-3V^{\mathrm{UV}}_{\mathrm{r}36}\\ &= \frac{\lambda\hat{\lambda}}{16}\Big(\lambda\hat{\lambda}\Big(-\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}\Big(4+\frac{2\pi^2}{3}\Big)\Big)+(\lambda^2+\hat{\lambda}^2)\frac{\pi^2}{12\varepsilon}\Big)\chi(1)\ . \end{aligned} \tag{5.2.10}$$

5.2.4. Final result

We are now ready to put together the parts of our calculations necessary to extract the four-loop dilatation operator. As discussed before the dilatation operator for odd sites is obtained by extracting the $1/\varepsilon$ pole from the renormalization constant. Summing up the contributions to the $1/\varepsilon$ pole from (5.2.4), (5.2.6) and (5.2.10), we obtain

$$\begin{aligned}\bar{\lambda}^4 \mathcal{Z}_{4,\mathrm{odd}}|_{\frac{1}{\varepsilon}} &= \big(\mathcal{Z}_{\mathrm{r5,odd}} + \mathcal{Z}_{\mathrm{r4,odd}} + \mathcal{Z}_{\mathrm{r3,odd}}\big)|_{\frac{1}{\varepsilon}} \\ &= \frac{\lambda\hat{\lambda}}{16\varepsilon}\Big[-2\lambda\hat{\lambda}(\chi(1,3)+\chi(3,1)) + \Big(\lambda\hat{\lambda}\Big(4+\frac{2\pi^2}{3}\Big) + (\lambda^2+\hat{\lambda}^2)\frac{\pi^2}{3}\Big)\chi(1)\Big] \ ,\end{aligned} \tag{5.2.11}$$

that, rewritten in terms of $\bar{\lambda}$ and σ, gives

$$\bar{\lambda}^4 \mathcal{Z}_{4,\mathrm{odd}}|_{\frac{1}{\varepsilon}} = \frac{\bar{\lambda}^4}{16\varepsilon}\Big[-2(\chi(1,3)+\chi(3,1)) + \Big(4\Big(1+\frac{\pi^2}{3}\Big) + \sigma^2\frac{\pi^2}{3}\Big)\chi(1)\Big] \ . \tag{5.2.12}$$

As already observed, in the $\ln\mathcal{Z}$ the higher order poles must be absent. This is a useful consistency check of our computation. Additional diagrams that do not contribute to the dilatation operator but have non-vanishing double poles have to be taken into account. Some of them consist of two separate two-loop interactions. Furthermore, one has to consider the diagrams that lead to interactions between magnons at odd and even sites and contribute only to the double pole when summed up. In chapter 5.6, we prove that when all these double poles are taken into account, their sum is indeed canceled by the two-loop contribution in the expansion of $\ln\mathcal{Z}$. The dilatation operator for odd sites is then obtained from (5.2.12) by multiplying the $1/\varepsilon$ pole by 8. With $\zeta(2) = \frac{\pi^2}{6}$, it reads

$$\mathcal{D}_{4,\mathrm{odd}}(\sigma) = (2 + (4+\sigma^2)\zeta(2))\chi(1) - \chi(1,3) - \chi(3,1) \ . \tag{5.2.13}$$

By comparing the previous result with equation (5.1.11) we read off the four-loop coefficient of the function $h^2(\bar{\lambda},\sigma)$

$$h_4(\sigma) = -(4+\sigma^2)\zeta(2) \ . \tag{5.2.14}$$

This result coincides with the one computed in [61]. It is interesting to note that, in contrast to the component calculation in [61], the integrals that contribute here to the dilatation operator show a correlation between the quadratic and the rational simple pole in ε: their relative coefficient is always -4 as for the simplest four-loop integral I_4 in (C.1.3). The rational term in (5.2.13) and therefore its absence in (5.2.14) is hence correlated with the quadratic pole that itself is determined by the two-loop result (5.6.5).

5.3. Possible scenarios for an all-loop function

In this section we discuss our attempts to find an all-loop function for $h^2(\bar{\lambda}, \sigma)$.

In the ABJM case where $\sigma = 0$, $h^2(\bar{\lambda}, 0) = h^2(\lambda)$, there is a surprisingly simple function that matches the weak coupling behavior up to four-loop order and also matches the leading strong coupling behavior. To this end we define $t \equiv 2\pi i \lambda$, which is a natural variable that also appears in expressions for supersymmetric ABJ(M) Wilson loops [72, 74, 75]. We then consider a rescaled function $g(t) = (2\pi)^2 \, h^2(\lambda)$. In terms of $g(t)$ the magnon dispersion relation becomes

$$\varepsilon(p) = \sqrt{\frac{1}{4} + \frac{g(t)}{\pi^2} \sin^2 \frac{p}{2}}\,, \tag{5.3.1}$$

and so has a form more in line with the $\mathcal{N} = 4$ dispersion relation where in that case $g(t)$ in (5.3.1) is replaced with λ.

In terms of $g(t)$, the proposed all-loop function is

$$g(t) = -(1-t)\log(1-t) - (1+t)\log(1+t)\,, \tag{5.3.2}$$

whose weak coupling expansion is

$$\begin{aligned} g(t) &= -\sum_{n=1}^{\infty} \frac{t^{2n}}{n(2n-1)} = -t^2 - \frac{1}{6}\,t^4 - \frac{1}{15}\,t^6 + \mathcal{O}(t^8) \\ &= (2\pi)^2 \left(\lambda^2 - 4\,\zeta(2)\,\lambda^4 + 6\,\zeta(4)\,\lambda^6 + \mathcal{O}(\bar{\lambda}^8)\right)\,. \end{aligned} \tag{5.3.3}$$

An obvious test is to compute $h^2(\lambda)$ to six-loop order, where the all-loop function in (5.3.2) predicts the value $h_6 = \frac{(2\pi)^4}{15}$. A six-loop computation is admittedly very difficult, but we believe it is manageable using the $\mathcal{N} = 2$ superspace formulation.

At strong coupling the expansion is

$$\begin{aligned} g(t) &= -i\pi\, t - 2\, \log t - 2 + \mathcal{O}(t^{-1}) \\ &= (2\pi)^2 \left(\frac{\lambda}{2} - \frac{1}{(2\pi)^2} \log(2\pi\lambda) - 2 + \mathcal{O}(\lambda^{-1}) \right) . \end{aligned} \tag{5.3.4}$$

The dominant term agrees with the leading strong coupling expansion from the string sigma-model. But also observe that the first correction corresponds to a two-loop contribution; a one-loop correction is absent. This disagrees with the prediction in [70] arising from the one-loop correction to the energy for a folded-string [71, 76–79]. In this language one would expect a $g(t)$ with leading asymptotic expansion

$$g(t) = -i\pi\, t - 2\sqrt{-i\pi t}\, \ln(2) + \ldots . \tag{5.3.5}$$

However, if one chooses a different prescription for summing over mode frequencies, where one essentially groups the modes into heavy and light [71], then $g(t)$ no longer has the $\sqrt{t}$ term, agreeing with the large t expansion (5.3.4)[6].

The function in (5.3.2) does not appear to have an easy generalization to the ABJ case where $\sigma \neq 0$. Such a function would be expected to be invariant under the transformation [6]

$$\lambda \to \hat{\lambda}\ , \qquad \hat{\lambda} \to 2\hat{\lambda} - \lambda + 1\ . \tag{5.3.6}$$

Under (5.3.6) the perturbative regime is mapped into strong coupling, making its verification difficult. Some evidence that $h^2(\bar{\lambda}, \sigma)$ is consistent with (5.3.6) was presented in [83]. One possible hint about the all-loop structure is that the four-loop contribution to $h^2(\bar{\lambda}, \sigma)$ can be rewritten as

$$\bar{\lambda}^4 (4 + \sigma^2) = \lambda\hat{\lambda}(\lambda + \hat{\lambda})^2\ . \tag{5.3.7}$$

[6]See [80] for a further discussion of this. These authors also show that the same choices of prescriptions appear in finite size corrections for giant magnons [81, 82] and lead to the same one-loop contributions to $h^2(\lambda)$.

which is zero if $\lambda = -\hat{\lambda}$ It would be interesting to see if the higher order corrections remain zero under this condition. However, it is not clear how this could square with the strong coupling behavior nor with an invariance under the transformation in (5.3.6).

Another possibility is that $h^2(\bar{\lambda}, \sigma)$ is somehow related to recent results concerning supersymmetric Wilson loops in the ABJ(M) models. In this latter case, it was found using localization [84, 85] that the Wilson loop expectation value could be reduced to a matrix model on a Lens space [72]. This matrix model is solvable in the planar limit [86, 87] and hence all-loop predictions can be extracted. In particular, for ABJM the perturbative free energy of the matrix model is [74]

$$F(t) = N^2 \left(\log(t) + \frac{1}{36} t^2 + \mathcal{O}(t^4) \right) . \tag{5.3.8}$$

It is tempting to look for a connection between $F(t)$ and $g(t)$. One might try

$$(g(t))^{1/2} = -\frac{i}{N^2} t^2 \frac{\partial F}{\partial t} = -i\, t - \frac{i}{18}\, t^3 + \mathcal{O}(t^5)\,. \tag{5.3.9}$$

The full expansion also is maximally transcendental, but here one finds that the t^3 term is off by a factor of $2/3$. At strong coupling the free energy is asymptotically [75]

$$F(t) \approx -N^2 \frac{2\pi^{3/2}}{3} \, (-it)^{-1/2} \,. \tag{5.3.10}$$

Applying the same rule as in (5.3.9) one finds

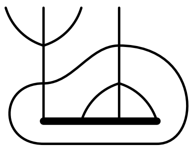
$$(g(t))^{1/2} = -\frac{i}{N^2} t^2 \frac{\partial F}{\partial t} \approx \frac{\pi}{3} (-i\pi t)^{1/2} \,, \tag{5.3.11}$$

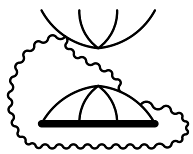
which differs by an overall factor of $\pi/3$ from the square root of the leading term in (5.3.4).

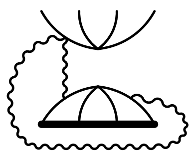
5.4. Wrapping interactions

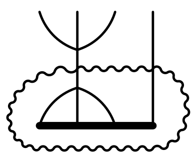
To obtain the complete four-loop spectrum of operators in the $SU(2) \times SU(2)$ subsector, we have to consider the wrapping interactions for the non-protected operators that consist of up to four elementary fields. The only non-trivial operator is in the **20** of $SU(4)$ and has $L = 2$, i.e. exactly four elementary fields.

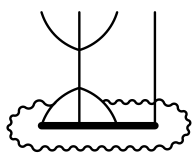
The only wrapping diagrams which according to the initially discussed finiteness theorems based on power counting can contribute to the dilatation operator are given by

$$W_1 =$$

$$\rightarrow -\frac{2(4\pi)^4}{k^4}(MN)^2 I_4\,\chi(1) = \frac{(\lambda\hat{\lambda})^4}{16}\Big(\frac{1}{\varepsilon^2}-\frac{4}{\varepsilon}\Big)\chi(1)\ ,$$

$$W_2 =$$

$$\rightarrow -\frac{2(4\pi)^4}{k^4}(MN)^2 I_{42\mathbf{bb0}cd}\,\chi(1) = \frac{(\lambda\hat{\lambda})^4}{16}\Big(-\frac{1}{2\varepsilon^2}+\frac{3}{\varepsilon}\Big)\chi(1)\ ,$$

$$W_3 =$$

$$\rightarrow \frac{(4\pi)^4}{k^4}(MN)^2 I_{422\mathbf{b}\mathrm{tr}ABcd}\,\chi(1) = \frac{(\lambda\hat{\lambda})^4}{16}\Big(\frac{1}{\varepsilon^2}-\frac{2}{\varepsilon}\Big)\chi(1)\ ,$$

$$W_4 =$$

$$\rightarrow -\frac{2(4\pi)^4}{k^4}(MN)^2 I_4\,\chi(1) = \frac{(\lambda\hat{\lambda})^4}{16}\Big(\frac{1}{\varepsilon^2}-\frac{4}{\varepsilon}\Big)\chi(1)\ ,$$

$$W_5 =$$

$$\rightarrow \frac{(4\pi)^4}{k^4}(MN)^2 I_{422\mathbf{q}\mathrm{tr}ABbd}\chi(1) = \frac{(\lambda\hat{\lambda})^4}{16}\Big(-\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}\Big(4-\frac{2}{3}\pi^2\Big)\Big)\chi(1)\ . \tag{5.4.1}$$

There are four distinct diagrams of type W_2 and two of type W_3. The sum of the wrapping diagrams is therefore given by

$$W = W_1 + 4W_2 + 2W_3 + W_4 + W_5 = \frac{(\lambda\hat{\lambda})^4}{16}\Big[\frac{1}{\varepsilon^2}+\frac{2}{\varepsilon}\Big(2-\frac{\pi^2}{3}\Big)\Big]\chi(1)\ . \tag{5.4.2}$$

Multiplying the $1/\varepsilon$ pole of W by -8, we obtain the wrapping contribution to the dilatation operator. It reads

$$\mathcal{D}^{\mathrm{w}}_{4,\mathrm{odd}} = -(2-2\zeta(2))\chi(1)\ . \tag{5.4.3}$$

Now, by subtracting from (5.2.13) the range five contribution and inserting $h_4(\sigma) = -(4+\sigma^2)\zeta(2)$, the subtracted dilatation operator becomes

$$\mathcal{D}^{\text{sub}}_{4,\text{odd}}(\sigma) = (2 - h_4(\sigma))\chi(1) = \big(2 + (4+\sigma^2)\zeta(2)\big)\chi(1) \ . \tag{5.4.4}$$

The dilatation operator for length four states then reads

$$\mathcal{D}^{\text{range 4}}_{4,\text{odd}}(\sigma) = \mathcal{D}^{\text{sub}}_{4,\text{odd}}(\sigma) + \mathcal{D}^{\text{w}}_{4,\text{odd}} = (6+\sigma^2)\zeta(2)\chi(1) \ , \tag{5.4.5}$$

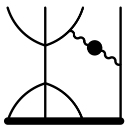
and it coincides with the results obtained in terms of component fields [60, 61].

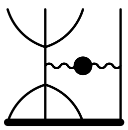
Note that the separation of the dilatation operator into wrapping and subtracted parts differs in the superfield calculation from the one obtained in component fields in [60, 61]. The sum of the two terms is, however, the same in the two calculations, and hence the resulting anomalous dimensions for operators with length $2L = 4$ agree.

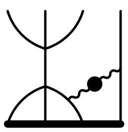
5.5. Cancelation of IR divergences

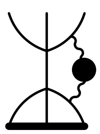
In order to check the cancelation of the IR divergences, together with the contributions having both UV and IR divergences given in section 5.2, we have to include diagrams that have pure IR poles and would have been excluded by the UV finiteness conditions of subsection 5.2.1. The cancelation of IR divergences in the combination (E.3.7) means that chiral and anti-chiral vertices with any number of legs and with external propagators are free of IR divergences from perturbative corrections. In the following we will hence attach propagators to the external fields of the diagrams that appear at four loops as quantum corrections of a chiral composite operator. This does not affect the UV poles, since the chiral self-energy is UV finite as demonstrated in section **??** and appendix E.2 for the ABJ case.

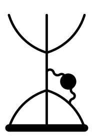
The contributions to the $\chi(1)$ structure with only an IR divergence are given by

$$V_{\mathrm{r44}} =$$

$$\begin{aligned}&\to \frac{(4\pi)^4}{k^4} MN(4MN - M^2) I_{4\mathrm{IR}}\, \chi(1)\\ &= \frac{\lambda\hat{\lambda}}{16}(4\lambda\hat{\lambda} - \lambda^2)\Big(\frac{2}{\varepsilon}(-3 + \gamma - \ln 4\pi)\chi(1)\Big)\ ,\end{aligned}$$

$$V_{\mathrm{r45}} =$$

$$\begin{aligned}&\to -\frac{(4\pi)^4}{k^4}\frac{MN}{2}(4MN - M^2)\big(I_{4\mathrm{IR}} + I_{4\mathrm{UVIR}} - I_4\big)\chi(1)\\ &= -\frac{\lambda\hat{\lambda}}{16}(4\lambda\hat{\lambda} - \lambda^2)\Big(\frac{2}{\varepsilon}(-3 + \gamma - \ln 4\pi)\chi(1)\Big)\ ,\end{aligned}$$

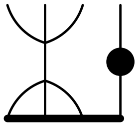
$$V_{\mathrm{r46}} =$$

$$\begin{aligned}&\to \frac{(4\pi)^4}{k^4}\frac{MN}{2}(4MN - M^2)\big(I_{4\mathrm{UVIR}} - I_4 + I_2 I_{2\mathrm{IR}} - \mathrm{K}(I_2) I_{2\mathrm{IR}}\big)\chi(1)\\ &= \frac{\lambda\hat{\lambda}}{16}(4\lambda\hat{\lambda} - \lambda^2)\Big(\frac{2}{\varepsilon}(-3 + \gamma - \ln 4\pi)\chi(1)\Big)\ ,\end{aligned}$$

$$V_{\mathrm{r37}} =$$

$$\begin{aligned}&\to -\frac{(4\pi)^4}{k^4}\frac{MN}{2}(4MN - M^2)(I_{4\mathrm{IR}} + I_2 I_{2\mathrm{IR}} - \mathrm{K}(I_2) I_{2\mathrm{IR}})\chi(1)\\ &= -\frac{\lambda\hat{\lambda}}{16}(4\lambda\hat{\lambda} - \lambda^2)\Big(\frac{2}{\varepsilon}(-3 + \gamma - \ln 4\pi)\chi(1)\Big)\ ,\end{aligned}$$

$$V_{\mathrm{r38}} =$$

$$\begin{aligned}&\to -\frac{(4\pi)^4}{k^4}\frac{MN}{2}(4MN - M^2)\big(I_4 - I_{4\mathrm{UVIR}} - I_2 I_{2\mathrm{IR}} + \mathrm{K}(I_2) I_{2\mathrm{IR}}\big)\chi(1)\\ &= \frac{\lambda\hat{\lambda}}{16}(4\lambda\hat{\lambda} - \lambda^2)\Big(\frac{2}{\varepsilon}(-3 + \gamma - \ln 4\pi)\chi(1)\Big)\ .\end{aligned} \tag{5.5.1}$$

where we have given only the IR pole terms, and the UV subdivergencies have been subtracted.

We also have to consider the correction of the chiral propagator that is a neighbor of the fields interacting via $\chi(1)$

$$V_{\mathrm{r3s}} =$$

$$\begin{aligned}&\to -\frac{(4\pi)^4}{k^4}\frac{MN}{2}\big(8MN - (M^2 + N^2)\big)\big(I_2 - \mathrm{K}(I_2)\big) I_{2\mathrm{IR}}\, \chi(1)\\ &= -\frac{\lambda\hat{\lambda}}{16}\big(8\lambda\hat{\lambda} - (\lambda^2 + \hat{\lambda}^2)\big)\Big(\frac{1}{\varepsilon}(-3 + \gamma - \ln 4\pi)\chi(1)\Big)\ .\end{aligned} \tag{5.5.2}$$

According to (E.3.7), one half of this contribution has to be taken into account, since the other half should cancel part of the IR divergence from an interaction of the isolated leg via a one-loop corrected gauge propagator with its neighbor to the right. Similar considerations hold also for the reflected diagram of V_{r3s}, such that the total contribution of these diagrams to the IR divergence is $\frac{1}{2}(1+\mathcal{R})V_{\mathrm{r3s}}$.

Further IR divergent contributions from self energy corrections of the three external and one internal line at the upper chiral vertex that forms $\chi(1)$ cancel among respective diagrams in which two of these lines are interacting via one-loop corrected gauge propagator. This is guaranteed by (E.3.7) since in the considered propagators are attached to their external lines.

At this point a simple way to check the cancelation of the IR divergences is to sum up all the contribution containing them and check that the result is the same as if from the very beginning we had omitted all IR divergent diagrams, and had only considered $V_{\mathrm{r35}}^{\mathrm{UV}}$ and $V_{\mathrm{r36}}^{\mathrm{UV}}$. In fact, the sum

$$\begin{aligned}&-(1+\mathcal{R})(V_{\mathrm{r35}}+V_{\mathrm{r44}}+V_{\mathrm{r45}}+V_{\mathrm{r46}}+V_{\mathrm{r37}}+V_{\mathrm{r38}})-3V_{\mathrm{r36}}-\frac{1}{2}(1+\mathcal{R})V_{\mathrm{r3s}}\\&=\frac{(4\pi)^4}{k^4}MN\Big(-6MNI_{4\mathbf{bbb}}+\frac{1}{2}(8MN-(M^2+N^2))(3I_4+2I_{42\mathbf{b}bd})\Big)\chi(1)\\&=\frac{\lambda\hat{\lambda}}{16}\Big(-\lambda\hat{\lambda}\frac{3\pi^2}{\varepsilon}+\big(8\lambda\hat{\lambda}-(\lambda^2+\hat{\lambda}^2)\big)\Big(-\frac{1}{4\varepsilon^2}+\frac{1}{\varepsilon}\Big(1+\frac{\pi^2}{4}\Big)\Big)\Big)\chi(1)\end{aligned}\tag{5.5.3}$$

turns out to be equal to

$$-(1+\mathcal{R})V_{\mathrm{r35}}^{\mathrm{UV}}-3V_{\mathrm{r36}}^{\mathrm{UV}}\ ,\tag{5.5.4}$$

which is the respective contribution of only the overall UV divergences from the diagrams with also an IR divergence to (5.2.10).

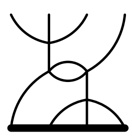
It is important to note that, besides the previously described check of the cancelation of the IR divergences, we have also performed the full computation of the range three contribution in the IR-safe η-gauge described in chapter 4. The result turns out to be the same.

5.6. Double poles

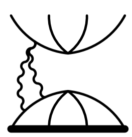
In this section we check explicitly the cancelation of the double poles in $\ln \mathcal{Z}$. For that we need to consider diagrams which are responsible for interactions between magnons at odd and even sites which are proportional to chiral functions $\chi(1,2)$ and $\chi(2,3)$. We start by computing those contributions, and then we prove the complete cancelation of the double poles.

5.6.1. Odd- and even-site magnon interactions

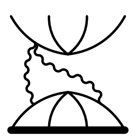
The relevant diagrams that couple the odd and even site magnons with each other are the following ones

$S_{\text{mixed}} =$ 

$$\rightarrow \frac{(4\pi)^4}{k^4}(MN)^2 I_4\, \chi(1,2) = \frac{(\lambda\hat{\lambda})^2}{16}\Big(-\frac{1}{2\varepsilon^2}+\frac{2}{\varepsilon}\Big)\chi(1,2)\ ,$$

$V_{\text{mixed1}} =$ 

$$\rightarrow \frac{(4\pi)^4}{k^4}(MN)^2 I_{42\mathbf{bb0}cd}\, \chi(1,2) = \frac{(\lambda\hat{\lambda})^2}{16}\Big(\frac{1}{4\varepsilon^2}-\frac{3}{2\varepsilon}\Big)\chi(1,2)\ ,$$

$V_{\text{mixed2}} =$ 

$$\rightarrow -\frac{(4\pi)^4}{k^4}\frac{(MN)^2}{2} I_{422\mathbf{b}\text{tr}ABcd}\, \chi(1,2) = \frac{(\lambda\hat{\lambda})^2}{16}\Big(-\frac{1}{2\varepsilon^2}+\frac{1}{\varepsilon}\Big)\chi(1,2)\ . \tag{5.6.1}$$

In the sum of all contributions one has to consider the reflected diagrams. The second contribution acquires an additional factor of two due to two distinct positions for the vector vertices which are not mapped to each other under reflection. The result for the mixed renormalization constant reads[7]

$$\mathcal{Z}_{4,\text{mixed}} = -(1+\mathcal{R})(S_{\text{mixed}} + 2V_{\text{mixed1}} + V_{\text{mixed2}}) = \frac{(\lambda\hat{\lambda})^2}{16}\frac{1}{\varepsilon^2}\chi(1,2)\ . \tag{5.6.2}$$

As expected [88], the $1/\varepsilon$ pole is canceled out such that at four loops there is no contribution to the dilatation operator that couples the magnons at odd and even sites.

[7] There is another contribution with identical prefactor that involves the chiral function $\chi(2,3)$ that we associate to the even site sector.

5.6.2. Double pole cancelation

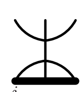
Summing up the contributions to the $1/\varepsilon^2$ poles of the odd-site sector to the four-loop $\mathcal{Z}$ from (5.2.4), (5.2.6), (5.2.10) and (5.6.2), we obtain

$$\begin{aligned}\bar{\lambda}^4(\mathcal{Z}_{4,\mathrm{odd}}+\mathcal{Z}_{4,\mathrm{mixed}})|_{\frac{1}{\varepsilon^2}} &= \big(\mathcal{Z}_{\mathrm{r5,odd}}+\mathcal{Z}_{4,\mathrm{mixed}}+\mathcal{Z}_{\mathrm{r4,odd}}+\mathcal{Z}_{\mathrm{r3,odd}}\big)|_{\frac{1}{\varepsilon^2}} \\ &= \frac{\bar{\lambda}^4}{16\varepsilon^2}\Big[\frac{1}{2}\big(\chi(1,3)+\chi(3,1)\big)+\chi(1,2)-\chi(1)\Big] \ .\end{aligned} \tag{5.6.3}$$

In the definition of the dilatation operator, the logarithm guarantees that all higher order poles in ε cancel out, such that $\ln\mathcal{Z}$ only contains simple $\frac{1}{\varepsilon}$ poles. Inserting (5.1.2), the expansion reads

$$\ln\mathcal{Z} = \bar{\lambda}^2\mathcal{Z}_2+\bar{\lambda}^4\Big(\mathcal{Z}_4-\frac{1}{2}\mathcal{Z}_2^2\Big)+\mathcal{O}(\bar{\lambda}^6) \ . \tag{5.6.4}$$

Let us now check the double pole cancelations in the $\bar{\lambda}^4$ term. The two-loop contribution to the renormalization constant for operators of length L can be written as

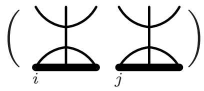
$$\bar{\lambda}^2\mathcal{Z}_2 = -\sum_{i=1}^{2L}$$  $$= -\frac{\lambda\hat{\lambda}}{4}\frac{1}{\varepsilon}(\chi(1)+\chi(2)) \ , \tag{5.6.5}$$

where we have indicated the sum over the sites explicitly. It has an obvious decomposition into two parts acting exclusively on even and on odd sites, respectively. The square of the above result can be decomposed as follows

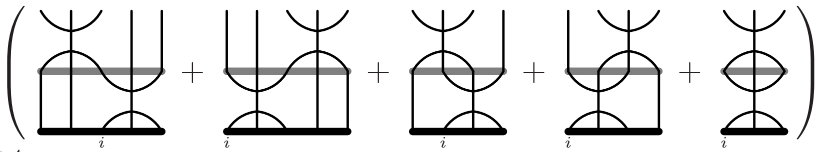
$$\frac{1}{2}\mathcal{Z}_2^2 = \mathcal{Z}_{22,\mathrm{dc}}+\mathcal{Z}_{22,S} \ . \tag{5.6.6}$$

The individual terms are given by

$$\bar{\lambda}^4 \mathcal{Z}_{22,\mathrm{dc}} = \sum_{j\geq i+3}^{2L}$$



$$\bar{\lambda}^4 \mathcal{Z}_{22,S} = \frac{1}{2}\sum_{i=1}^{2L}$$



$$\begin{aligned} &\to \frac{1}{2}\frac{(4\pi)^4}{k^4} M^2 N^2\, \mathrm{K}(I_2)^2 (\chi(1,3)+\chi(3,1)+2\chi(1,2)-2\chi(1)) \\ &= \frac{(\lambda\hat{\lambda})^2}{16}\frac{1}{2\varepsilon^2}(\chi(1,3)+\chi(3,1)+2\chi(1,2)-2\chi(1)) \ , \end{aligned} \tag{5.6.7}$$

where the arrow denotes that in the final result we have considered the chiral functions with odd indices only and $\chi(1,2)$ and neglected the ones with only even indices and $\chi(2,3)$.

According to (5.6.6), the square of the two-loop contribution expands as

$$\frac{1}{2}(\bar{\lambda}\mathcal{Z}_2)^2 = \frac{(\lambda\hat{\lambda})^2}{16}\frac{1}{2\varepsilon^2}(\chi(1,3)+\chi(3,1)+2\chi(1,2)-2\chi(1)) + \dots \ , \tag{5.6.8}$$

where we have neglected the chiral functions with only even arguments and $\chi(2,3)$. We have also disregarded the terms $\mathcal{Z}_{22,\mathrm{dc}}$ which trivially cancel against four-loop diagrams that only contain double poles and hence become disconnected when the composite operator is removed. We have omitted to present these diagrams.

Comparing equations (5.6.3) and (5.6.8) we finally find our desired result

$$\Big(\mathcal{Z}_4 - \frac{1}{2}\mathcal{Z}_2^2\Big)\Big|_{\frac{1}{\varepsilon^2}} = 0 \ , \tag{5.6.9}$$

where we have considered that the discussion is identical for the neglected contributions with chiral functions with even arguments and $\chi(2,3)$.

5.7. Summary

In this chapter we have computed $h_4(\sigma)$ using the $\mathcal{N} = 2$ superspace formalism. The computation is greatly simplified from the component version [60, 61] because the manifest supersymmetry in combination with finiteness conditions leads to a large reduction in the number of Feynman diagrams.

With this reduction in diagrams, it should be possible to tackle more challenging computations, including the six-loop term $h_6(\sigma)$. Six loops would give one more data point and might provide further insights into an all-loop function.

Alternatively, one could also apply the superspace formalism to four loops but beyond the $SU(2)\times SU(2)$ sector. This would not give us further information on $h^2(\bar{\lambda}, \sigma)$, but it would provide a check of higher-loop integrability in both ABJM and ABJ models. One reason that integrability in the ABJ case is not assured is because at strong coupling a nonzero σ would correspond to a nonzero θ-angle for the world-sheet, which is normally thought to destroy integrability. However, at the lowest order in perturbation theory, the spin-chain is integrable in all sectors, even when $\sigma \neq 0$ [56, 57]. It would be interesting to see how this plays out at higher loops.

Chapter 6.

Conclusions

We have studied different aspects of general $\mathcal{N} = 2$ Chern-Simons matter theories and particularly of the ABJ(M) case. Exciting new phenomena seems to arise in the AdS_4/CFT_3 conjecture which makes the details of the gauge/gravity conjecture more complicated than the canonical AdS_5/CFT_4 case.

In this thesis work we have reviewed many aspects of the $\mathcal{N} = 2$ formalism and of the formulation of Chern-Simons matter theories in it. Using this formalism, we have analyzed the influence of matter in the gauge sector renormalization and we have obtained matching results with those from component field and $\mathcal{N} = 1$ formulations. We plan to use these results in future works.

Moreover, we have studied one of the main technical disadvantages of the formalism which has to do with the presence of unphysical infrared divergencies. We have proposed a method to solve this problem in order to avoid the appearance of them and we have tested its applicability.

We have also fully exploited the power of the superspace formalism by calculating the dispersion relation of magnon excitations in long chain operators to four loops. Our result triggered the revision of the component field calculation which turned out to be wrong by a sign misinterpretation of a set of diagrams; after this revision both results turned out to match. We have also proposed a very simple all-loop interpolation function $h(\lambda)$ for this anomalous dimension which matches the known weak coupling results and the leading strong coupling one.

Having obtained such success in the four-loop calculation of this anomalous dimension, it could be feasible to go further on the number of loops and make the six-loop calculation. The number of diagrams will grow considerably, but we still believe the calculation is doable. This would provide a check or would reject our all loop proposal.

It could be also interesting to study the applicability of the method proposed in [89] where the F-equation of motion was used to derive non perturbative results on the anomalous dimension of chain operators in $\mathcal{N} = 4$ Super Yang-Mills theory. It would seem that the method could be directly applied in the $\mathcal{N} = 6$ Chern-Simons matter case, but this turned out not to be the case and some modification of the approach is needed. A successful adaptation of this method could reduce the number of loops in the calculation of the perturbative expansion of the interpolation function $h(\lambda)$.

Appendix A.

Superspace coventions

Our choice for the lorentz invariant tensor is given by

$$C_{\alpha\beta} = -C^{\alpha\beta} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{such that} \quad C_{\alpha\beta}C^{\gamma\delta} = \delta_\alpha^\gamma \delta_\beta^\delta - \delta_\alpha^\delta \delta_\beta^\gamma. \tag{A.0.1}$$

Our convention for spinor indexes contraction is always by contracting from the upper left to the lower right. With this convention the raising and lowering of indexes and the 'square' of a spinor are defined by

$$\Psi^\alpha = C^{\alpha\beta}\Psi_\beta, \quad \Psi_\alpha = \Psi^\beta C_{\beta\alpha} \quad \text{and} \quad \Psi^2 = \tfrac{1}{2}\Psi^\alpha\Psi_\alpha. \tag{A.0.2}$$

Notice that with our choice of the antisymmetric symbol, the 'squared' real spinor is hermitian (as opposed to other popular conventions).

We always work in Wick-rotated euclidean space with the diagonal metric $g_{\mu\nu} = g^{\mu\nu} = diag(1,1,1)$. The γ-matrices are defined by their algebraic properties:

$$(\gamma^\mu)^\alpha{}_\gamma(\gamma^\nu)^\gamma{}_\beta = -g^{\mu\nu}\delta^\alpha{}_\beta - \epsilon^{\mu\nu\rho}(\gamma_\rho)^\alpha{}_\beta \ . \tag{A.0.3}$$

where the Levi-Civita tensor is such that $\epsilon^{012} = 1$. When one spinor index is lowered or raised the γ-matrices are symmetric

$$(\gamma^\mu)_{\alpha\beta} = (\gamma^\mu)_\alpha{}^\delta C_{\delta\beta} = (\gamma^\mu)_{\beta\alpha} \ , \quad (\gamma^\mu)^{\alpha\beta} = C^{\alpha\delta}(\gamma^\mu)_\delta{}^\beta = (\gamma^\mu)^{\beta\alpha} \quad . \tag{A.0.4}$$

After D-Algebra manipulations of supergraphs, one arrives to a given feynman integral with traces of products of momenta in the numerator. These are related to the trace of products of γ-matrices which satisfy

$$\begin{aligned} \mathrm{tr}(\gamma^\mu\gamma^\nu) &\equiv (\gamma^\mu)^\alpha{}_\beta(\gamma^\nu)^\beta{}_\alpha = -2g^{\mu\nu} \ , \\ \mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho) &\equiv -(\gamma^\mu)^\alpha{}_\beta(\gamma^\nu)^\beta{}_\gamma(\gamma^\rho)^\gamma{}_\alpha = -2\epsilon^{\mu\nu\rho} \ , \\ \mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &\equiv (\gamma^\mu)^\alpha{}_\beta(\gamma^\nu)^\beta{}_\gamma(\gamma^\rho)^\gamma{}_\delta(\gamma^\sigma)^\delta{}_\alpha = 2(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) \ . \end{aligned} \tag{A.0.5}$$

We use the convention that the first of two contracted indices is always an upper index; this is used in the previous formulas in the definition of the trace of products of gamma matrices and it is very useful for D-algebra manipulations [14]. Using the γ-matrices we can move from vector to bi-spinor indices thanks to the following definitions

$$\begin{aligned} x^{\alpha\beta} &= \frac{1}{2}(\gamma_\mu)^{\alpha\beta}x^\mu \ , \qquad x^\mu = (\gamma^\mu)_{\alpha\beta}x^{\alpha\beta} \ , \\ p_{\alpha\beta} &= (\gamma^\mu)_{\alpha\beta}p_\mu \ , \qquad p_\mu = \frac{1}{2}(\gamma_\mu)^{\alpha\beta}p_{\alpha\beta} \ , \\ A_{\alpha\beta} &= \frac{1}{\sqrt{2}}(\gamma^\mu)_{\alpha\beta}A_\mu \ , \qquad A_\mu = \frac{1}{\sqrt{2}}(\gamma_\mu)^{\alpha\beta}A_{\alpha\beta} \ , \end{aligned} \tag{A.0.6}$$

respectively for coordinates, momenta and fields.

Defining $\square = \partial^\mu\partial_\mu = \frac{1}{2}\partial^{\alpha\beta}\partial_{\alpha\beta}$, $D^2 = \frac{1}{2}D^\alpha D_\alpha$ and $\bar{D}^2 = \frac{1}{2}\bar{D}^\alpha\bar{D}_\alpha$ the following properties hold

$$D_\alpha D^2 = 0, \quad \bar{D}_\alpha\bar{D}^2 = 0, \quad [D^\alpha, \bar{D}^2] = i\partial^{\alpha\beta}\bar{D}_\beta, \quad [\bar{D}^\beta, D^2] = i\partial^{\alpha\beta}D_\alpha$$

$$D^2\bar{D}^2D^2 = \square D^2, \quad D^\alpha D_\beta = \delta^\alpha_\beta D^2, \quad \bar{D}^\alpha\bar{D}_\beta = \delta^\alpha_\beta\bar{D}^2. \tag{A.0.7}$$

Superspin projectors are defined as

$$\mathcal{P}_0 = \frac{1}{\square}(D^2\bar{D}^2 + \bar{D}^2D^2), \quad \mathcal{P}_{1/2} = -\frac{1}{\square}D^\alpha\bar{D}^2D_\alpha, \tag{A.0.8}$$

and together with $\bar{D}^\alpha D_\alpha$ operator, they satisfy the properties

$$\mathcal{P}_0^2 = \mathcal{P}_0, \quad \mathcal{P}_{1/2}^2 = \mathcal{P}_{1/2}, \quad \mathcal{P}_0 + \mathcal{P}_{1/2} = 1, \quad \mathcal{P}_0\mathcal{P}_{1/2} = 0,$$

$$(\bar{D}^\alpha D_\alpha)^2 = \Box \mathcal{P}_{1/2}, \quad \mathcal{P}_{1/2}\bar{D}^\alpha D_\alpha = \bar{D}^\alpha D_\alpha, \quad \mathcal{P}_0 \bar{D}^\alpha D_\alpha = 0, \tag{A.0.9}$$

which turned out to be very useful in the infrared behavior analysis of the gauge sector in Chapter 4 and throughout all this work, specially in the D-algebra of the gauge sector corrections.

Appendix B.

U(N) structure constants

We define the completely anti-symmetric structure constants of the algebra $\mathfrak{u}(N)$ by $[T^I, T^J] = if^{IJK}T^K$ with $I, J, K = 0, 1, \ldots (N^2 - 1)$. Since $U(N)$ is the semi-direct product $U(1) \times SU(N)$ we choose the generators in the fundamental representation as $T^I = (T^0, T^i)$ where

$$T^0 = \frac{1}{\sqrt{N}} \mathbb{1}_{N\times N}, \qquad \text{and} \quad T^i \in \mathfrak{su}(N) \quad \text{such that} \quad \operatorname{tr} T^i = 0. \tag{B.0.1}$$

With the normalization $\operatorname{tr}(T^I T^J) = \delta^{IJ}$, which is consistent with our choice of T^0, the following set of properties hold

$$f^{0IJ} = f^{I0J} = f^{IJ0} = 0$$

$$f^{IJK} f^{LMK} + f^{LIK} f^{JMK} + f^{JLK} f^{IMK} = 0 \tag{B.0.2}$$

$$f^{IJK} f^{LJK} = 2N(\delta^{IL} - \delta^{I0}\delta^{L0})$$

$$f^{IJK} f^{KLM} f^{MJN} f^{NLP} = 2N^2(\delta^{IP} - \delta^{I0}\delta^{P0}) \tag{B.0.3}$$

$$T^I T^I = N \mathbb{1}_{N\times N}$$

$$T^I T^J T^I = \sqrt{N}\delta^{J0}\mathbb{1}_{N\times N} \tag{B.0.4}$$

$$\mathrm{tr}(T^I[T^J, T^K]) = i f^{IJK}$$

$$\mathrm{tr}(T^I[T^J, [T^K, T^L]]) = -f^{IJM} f^{KLM}. \tag{B.0.5}$$

Let $V = V^I T^I$ and $W = W^I T^I$, then

$$\mathrm{tr}(V) = V^0\sqrt{N},\ \mathrm{tr}(W) = W^0\sqrt{N}$$

$$\mathrm{tr}(VT^I)\mathrm{tr}(WT^I) = \mathrm{tr}(VW),$$

$$\mathrm{tr}(VT^I WT^I) = \mathrm{tr}\, V \,\mathrm{tr}\, W.$$

Appendix C.

Integrals

In this section we collect all the integrals we used. Many of the results of this appendix are based on the Appendices H, I, J of [61] where the reader should look to have a complete description of the notations and results that we are using.

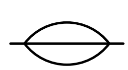
The integrals are computed by using dimensional regularization in Euclidean space with d dimensions and

$$d = 2(\lambda + 1) = 3 - 2\varepsilon \ , \qquad \lambda = \frac{1}{2} - \varepsilon \ . \tag{C.0.1}$$

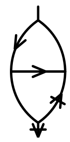
As usual we will expand the integrals in the limit $\varepsilon \to 0$ up to the order needed for our computations. The parameter λ in this appendix should not be confused with the 't Hooft coupling that appears in the main body of the work. The integrals have a simple dependence on the external momentum p_μ which we will omit. Relations between four-loop expressions are understood to hold for the pole parts up to disregarded finite contributions.

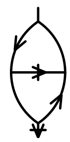
C.1. Integrals with only UV divergences

We need the following two-loop integral

$I_2 =$  $= G(1,1)G(1-\lambda,1) \ .$ (C.1.1)

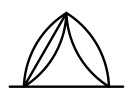
The reader can look at the appendix H of [61] for our notations in using the G-functions. Furthermore, we need the following two-loop integrals with two contracted momenta in their numerators

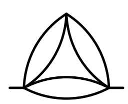
$I_{221be} =$  $= \frac{1}{2}\left(-G_1(1,1)G(1,1) - G(1,1)G_1(2-\lambda,1) + G_1(1,1)G_1(2-\lambda,1)\right)$,

$I_{221dc} =$  $= -G_1(1,1)G_1(2-\lambda,1)$.

(C.1.2)

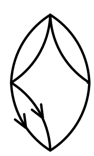
At four loops there are many integrals involved in the computations. Here we list the results for the pole parts of the UV logaritmically divergent integrals where the subdivergences have already been subtracted. Four-loop integrals with no momenta in their numerators are

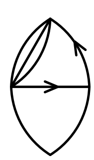
$I_4 =$  $= \frac{1}{(8\pi)^4}\left(-\frac{1}{2\varepsilon^2} + \frac{2}{\varepsilon}\right)$,

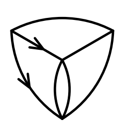
$I_{4\mathbf{bbb}} =$  $= \frac{1}{(8\pi)^4}\frac{\pi^2}{2\varepsilon}$.

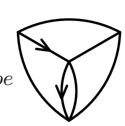
(C.1.3)

Four-loop integrals with two contracted momenta in their numerators are

$I_{42\mathbf{bbb2}} =$  $= \frac{1}{(8\pi)^4}\frac{\pi^2}{4\varepsilon}$,

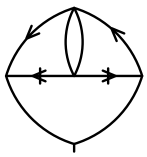
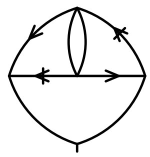
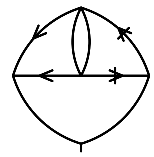
$I_{42\mathbf{bb0}cd} =$  $= \frac{1}{(8\pi)^4}\left(\frac{1}{4\varepsilon^2} - \frac{3}{2\varepsilon}\right)$,

$I_{42\mathbf{b}bd} =$  $= \frac{1}{(8\pi)^4}\left(\frac{1}{2\varepsilon^2} - \frac{1}{\varepsilon}\left(2 - \frac{\pi^2}{4}\right)\right)$,

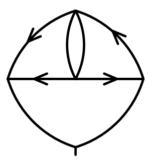
$I_{42\mathbf{b}be}$  $= \frac{1}{(8\pi)^4}\left(-\frac{1}{4\varepsilon^2}\right)$.

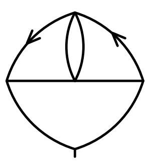
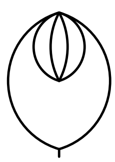
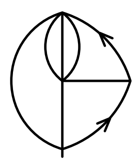
(C.1.4)

Let us consider now four-loop integrals with four pairwise contracted momenta in their numerators. The following ones

$I_{422\mathbf{b}ABcd} =$  , $I_{422\mathbf{b}AcBd} =$  , $I_{422\mathbf{b}AdBc} =$  , (C.1.5)

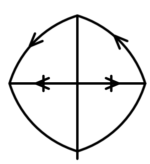
appear in a fixed combination which can be recast into the form

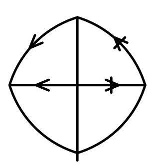
$I_{422\mathbf{b}\mathrm{tr}ABcd} = -\,\mathrm{tr}$  $= -2(I_{422\mathbf{b}ABcd} - I_{422\mathbf{b}AcBd} + I_{422\mathbf{b}AdBc})$

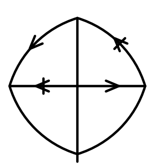
$= 2$  $+\,2$  $-\,4$  $= \dfrac{1}{(8\pi)^4}\Big(\dfrac{1}{\varepsilon^2} - \dfrac{2}{\varepsilon}\Big)$, (C.1.6)

Here we have taken the trace of γ-matrices contracted with the momenta in the integral. We thereby read off the momenta in a cycle, but keep their direction as indicated by the arrows.

We also need the integrals

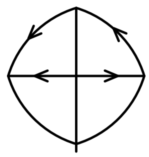
$I_{422\mathbf{q}ABbd} =$  $= \dfrac{1}{(8\pi)^4}\Big(\dfrac{1}{4\varepsilon^2} + \dfrac{1}{4\varepsilon}\Big)$,

$I_{422\mathbf{q}AdBb} =$  $= \dfrac{1}{(8\pi)^4}\Big(\dfrac{1}{2\varepsilon^2} - \dfrac{1}{\varepsilon}\Big(1 - \dfrac{\pi^2}{4}\Big)\Big)$, (C.1.7)

$I_{422\mathbf{q}AbBd} =$  $= \dfrac{1}{(8\pi)^4}\Big(\dfrac{1}{4\varepsilon^2} + \dfrac{1}{\varepsilon}\Big(\dfrac{5}{4} - \dfrac{\pi^2}{12}\Big)\Big)$.

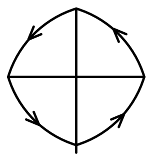
The linear combinations of integrals originating from the traces of γ-matrices read

$$I_{422\mathbf{q}\mathrm{tr}ABbd} = -\mathrm{tr}$$



$$\begin{aligned} &= -2(I_{422\mathbf{q}ABbd} - I_{422\mathbf{q}AbBd} + I_{422\mathbf{q}AdBb}) \\ &= \frac{1}{(8\pi)^4}\Big(-\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\Big(4 - \frac{2}{3}\pi^2\Big)\Big) \ , \end{aligned} \tag{C.1.8}$$

$$I_{422\mathbf{q}\mathrm{tr}ABCD} = \mathrm{tr}$$



$$= \frac{1}{(8\pi)^4}\frac{\pi^2}{3\varepsilon} \ .$$

There is an interesting relation involving the traces. It reads

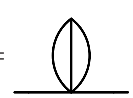
$$I_{422\mathbf{q}\mathrm{tr}ABCD} = I_{422\mathbf{q}\mathrm{tr}ABbd} + 2I_4 + 4I_{42\mathbf{b}bd} \quad = \frac{1}{(8\pi)^4}\frac{\pi^2}{3\varepsilon} \ . \tag{C.1.9}$$

C.2. Integrals with IR divergences

In this subsection we collect the integrals having poles in ε which are due to IR divergences. By suffixes IR and UVIR we thereby label integrals which have one or both IR and UV divergences.

The simplest two-loop integral with both an IR and an UV divergence is the logarithmically divergent tadpole

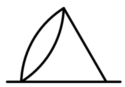
$$I_{2\mathrm{tp}} = I_{2\mathrm{UVIR}} =$$



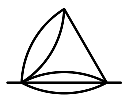
$$= 0 \ . \tag{C.2.1}$$

It is zero in dimensional regularization, i.e. the IR and the UV divergence cancel against each other. The UV divergence can be extracted by reshuffling the external momentum. In particular, the UV divergence of $I_{2\mathrm{tp}}$ is I_2 defined in (C.1.1); then the IR divergence of $I_{2\mathrm{tp}}$ is $-I_2$.

The simplest two-loop integral with only an IR divergence is given by

$$I_{2\mathrm{IR}} =$$  $$= G(1,1)G(2-\lambda,1) = \frac{1}{(8\pi)^2}\Big(-\frac{1}{\varepsilon} + 2(1+\gamma-\ln 4\pi) + O(\varepsilon)\Big) \,. \tag{C.2.2}$$

One four-loop integral with both, an IR and a UV divergence is given by[1]

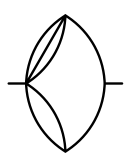
$$I_{4\mathrm{UVIR}} =$$  $$\begin{aligned} &= \mathrm{K}(G(1,1)^2 G(1-\lambda,1)G(1-2\lambda,2-\lambda)) \\ &= \frac{1}{(8\pi)^4}\Big(-\frac{1}{2\varepsilon^2} + \frac{2}{\varepsilon}(-2+\gamma-\ln 4\pi)\Big) \,. \end{aligned} \tag{C.2.3}$$

Its IR divergence is extracted as

$$I_{4\mathrm{UVIR}} - I_4 = \frac{1}{(8\pi)^4}\Big(\frac{2}{\varepsilon}(-3+\gamma-\ln 4\pi)\Big) \,, \tag{C.2.4}$$

where I_4 removes the overall UV divergence, since $I_{4\mathrm{UVIR}}$ does not have a UV subdivergence.

The simplest four-loop integral with only an IR divergence as overall divergence is given by

$$I_{4\mathrm{IR}} =$$  $$\begin{aligned} &= \mathrm{K}(G(1,1)^2 G(1-\lambda,1)G(2-2\lambda,2-\lambda)) - \mathrm{K}(I_2) I_{2\mathrm{IR}} \\ &= \frac{1}{(8\pi)^4}\Big(\frac{2}{\varepsilon}(-3+\gamma-\ln 4\pi)\Big) \,. \end{aligned} \tag{C.2.5}$$

Here we have subtracted the UV subdivergence.

C.3. Mellin-Barnes representation

In the computation of four-point integrals in the exceptional configuration, we found it necessary to expand Feynman integrals in powers of the kinematic invariants in order to carefully take the appropriate limits. In the end, the correct limit through this

[1]Note that, according to [61], with K() we mean the extraction of the pole parts of a function of ε.

analysis became a confirmation of the "naive" result since it coincides with the limit taken directly on the integrand. To deal with the expansions we used multiple Mellin-Barnes contour representation of vertex integrals [90, 91]. These representations are based on the identity

$$\frac{1}{(k^2+M^2)^a} = \frac{1}{(M^2)^a\Gamma(a)}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} ds\,\Gamma(-s)\Gamma(s+a)\left(\frac{k^2}{M^2}\right)^s, \tag{C.3.1}$$

where the contour is given by a straight line along the imaginary axis such that indentations are used if necessary in order to leave the series of poles $s = 0, 1, \cdots, n$ to the right of the contour and the series $s = -a, -a-1, \cdots, -a-n$ to the left of the contour. After Feynman-parametrizing a triangle integral and using (C.3.1), the following formula holds

$$\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^{2\mu_1}(k-p)^{2\mu_2}(k+q)^{2\mu_3}} = \frac{(4\pi)^{-d/2}(p+q)^{2(d/2-\sum_i \mu_i)}}{\prod_i \Gamma(\mu_i)\Gamma(d-\sum_i \mu_i)}\times$$

$$\times\left(\frac{1}{2\pi i}\right)^2\int_{-i\infty}^{i\infty} ds\,dt\,\Gamma(-s)\Gamma(-t)\Gamma(\tfrac{d}{2}-\mu_1-\mu_2-s)\Gamma(\tfrac{d}{2}-\mu_1-\mu_3-t)\times$$

$$\Gamma(\mu_1+s+t)\Gamma(\sum_i \mu_i - \tfrac{d}{2}+s+t)\left(\frac{p^2}{(p+q)^2}\right)^s\left(\frac{q^2}{(p+q)^2}\right)^t; \tag{C.3.2}$$

and for a vector-like triangle we have

$$\int \frac{d^dk}{(2\pi)^d}\frac{k^\nu}{k^{2\mu_1}(k-p)^{2\mu_2}(k+q)^{2\mu_3}} = \frac{(4\pi)^{-d/2}(p+q)^{2(d/2-\sum_i \mu_i)}}{\prod_i \Gamma(\mu_i)\Gamma(d-\sum_i \mu_i+1)}\left(\frac{1}{2\pi i}\right)^2\times$$

$$\times\int_{-i\infty}^{i\infty} ds\,dt\,\Gamma(-s)\Gamma(-t)\Gamma(\mu_1+s+t)\Gamma(\sum_i \mu_i-\tfrac{d}{2}+s+t)\left(\frac{p^2}{(p+q)^2}\right)^s\left(\frac{q^2}{(p+q)^2}\right)^t\times$$

$$\left[\Gamma(\tfrac{d}{2}-\mu_1-\mu_2-s)\Gamma(\tfrac{d}{2}-\mu_1-\mu_3-t+1)p^\nu-\Gamma(\tfrac{d}{2}-\mu_1-\mu_2-s+1)\Gamma(\tfrac{d}{2}-\mu_1-\mu_3-t)q^\nu\right], \tag{C.3.3}$$

where the multiple contours are taken using the same convention as the first definition unless otherwise indicated. It is customary to indicate with a * over the $\Gamma(z)$ function the case where one leaves a pole to the other of the conventional side of the contour. With these representations, among with Barnes 1^{st} and 2^{nd} lemmas

$$\frac{1}{2\pi i}\int ds\,\Gamma(a+s)\Gamma(b+s)\Gamma(c-s)\Gamma(d-s)=\frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}, \tag{C.3.4}$$

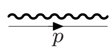
$$\begin{aligned}&\frac{1}{2\pi i}\int ds\,\frac{\Gamma(a+s)\Gamma(b+s)\Gamma(c+s)\Gamma(d-s)\Gamma(-s)}{\Gamma(e+s)}\\ &=\frac{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(a+d)\Gamma(b+d)\Gamma(c+d)}{\Gamma(e-a)\Gamma(e-b)\Gamma(e-c)},\quad \text{with}\quad e=a+b+c+d,\end{aligned} \tag{C.3.5}$$

and their multiple corollaries, we were able to carefully expand the 2-loop four-point integrals in the relevant kinematic invariants in order to take the limit of exceptional momenta.

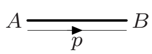
Appendix D.

Feynman rules in superspace

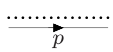
We use the Wick rotated Feynman rules, i.e. we have $\mathrm{e}^{-iS} \to \mathrm{e}^{S}$ in the path integral. The propagators in the α gauge are given by



$$= \langle V(p)V(-p)\rangle = -\langle \hat{V}(p)\hat{V}(-p)\rangle = \frac{1}{p^2}(\mathrm{D}\,\bar{\mathrm{D}} + \alpha\,\mathrm{D}^2 + \bar{\alpha}\,\bar{\mathrm{D}}^2)\delta^4(\theta_1 - \theta_2)\ ,$$



$$= \langle Z^B(p)\bar{Z}_A(-p)\rangle = \langle \bar{W}^B(p)W_A(-p)\rangle = \frac{\delta_A^B}{p^2}\delta^4(\theta_1 - \theta_2)\ ,$$

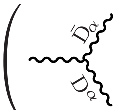
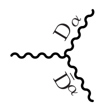


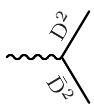
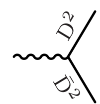
$$\begin{aligned} &= \langle \bar{c}'(p)c(-p)\rangle = -\langle c'(p)\bar{c}(-p)\rangle \\ &= -\,\langle \hat{\bar{c}}'(p)\hat{c}(-p)\rangle = \langle \hat{c}'(p)\hat{\bar{c}}(-p)\rangle = \frac{1}{p^2}\delta^4(\theta_1 - \theta_2)\ , \end{aligned} \tag{D.0.1}$$

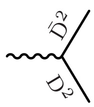
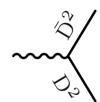
where diagonality in the gauge group indices and a factor $\frac{4\pi}{k}$ for each propagator have been suppressed. The Landau gauge corresponds to $\alpha \to 0$. In the η-gauge we introduced in Chapter 4, the gauge-vector propagators are given by

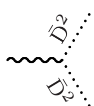
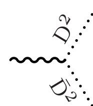
$$\begin{aligned} \langle V(p)V(-p)\rangle &= \frac{1}{p^2}\left(\mathrm{D}\,\bar{\mathrm{D}} + \frac{\eta^{\epsilon}_{(p)}}{|p|}\mathcal{P}_0\right)\delta^4(\theta_1 - \theta_2)\ , \\ \langle \hat{V}(p)\hat{V}(-p)\rangle &= \frac{1}{p^2}\left(-\,\mathrm{D}\,\bar{\mathrm{D}} + \frac{\eta^{\epsilon}_{(p)}}{|p|}\mathcal{P}_0\right)\delta^4(\theta_1 - \theta_2). \end{aligned} \tag{D.0.2}$$

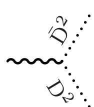
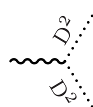
The vertices are obtained by taking the functional derivatives of the Wick rotated action (no factors of i) w.r.t. the corresponding superfields; we will give only the vertices involved in the computations of this work. The only exception is the four gluon vertex, used in chapter 2, which we omit though it can be derived following the logic of this section. When a functional derivative w.r.t. the (anti)-chiral superfields is taken, factors of $(\mathrm{D}^2)\ \bar{\mathrm{D}}^2$ are generated in the vertices. Omitting factors $\frac{k}{4\pi}$, for the three point vertices we obtain

$V_{V^3} = \Bigg($  $-$  $\Bigg) \frac{1}{2}\,\mathrm{tr}\left(T^a\left[T^b,T^c\right]\right)\ ,$

$V_{VZ^B\bar{Z}_C} =$  $\delta^C_B\,\mathrm{tr}\left(T^a B^{\underline{b}} B_{\underline{c}}\right)\ ,\qquad V_{\hat{V}W_B\bar{W}^C} =$  $\delta^B_C\,\mathrm{tr}\left(T^{\hat{a}} B_{\underline{b}} B^{\underline{c}}\right)\ ,$

$V_{\hat{V}\bar{Z}_B Z^C} =$  $(-1)\delta^B_C\,\mathrm{tr}\left(T^{\hat{a}} B_{\underline{b}} B^{\underline{c}}\right)\ ,\qquad V_{V\bar{W}^B W_C} =$  $(-1)\delta^C_B\,\mathrm{tr}\left(T^a B^{\underline{b}} B_{\underline{c}}\right)\ ,$

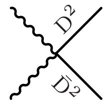
$V_{Vcc'} =$  $\frac{1}{2}\,\mathrm{tr}\left(T^a\left[T^b,T^c\right]\right)\ ,\qquad V_{Vc\bar{c}'} =$  $\frac{1}{2}\,\mathrm{tr}\left(T^a\left[T^b,T^c\right]\right)\ ,$

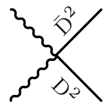
$V_{V\bar{c}c'} =$  $\frac{1}{2}\,\mathrm{tr}\left(T^a\left[T^b,T^c\right]\right)\ ,\qquad V_{V\bar{c}\bar{c}'} =$  $\frac{1}{2}\,\mathrm{tr}\left(T^a\left[T^b,T^c\right]\right)\ ,$

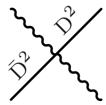
(D.0.3)

where the colour indices are labeled (a, b, c) counter clockwise starting with the leg to the left. Besides the matrices T^a and $T^{\hat{a}}$ transforming in the adjoint of the respective gauge groups $U(M)$ and $U(N)$, we have introduced matrices $B^{\underline{a}}$ and $B_{\underline{a}}$, with underlined $\underline{a} = 1, \cdots, MN$ indices that transform in the $(\mathbf{M}, \bar{\mathbf{N}})$ and $(\mathbf{N}, \bar{\mathbf{M}})$ of the gauge group $U(M) \times U(N)$. The previous notations are useful because one can effectively consider all the matrices to be the same for $M = N$ and then only at the end one can easily recover the different factors of M and N coming from the colour contractions.

Some of the quartic vertices that were used are

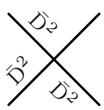
$$V_{V^2 Z^C \bar{Z}_D} =$$

$$\frac{1}{2}\delta_C^{D}\big[\,\mathrm{tr}\left(\{T^a, T^b\}B^{\underline{c}}B_{\underline{d}}\right)\big]\ ,$$

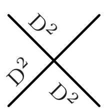
$$V_{\hat{V}^2 \bar{Z}_C Z^D} =$$

$$\frac{1}{2}\delta_D^{C}\big[\,\mathrm{tr}\left(\{T^{\hat{a}}, T^{\hat{b}}\}B_{\underline{c}}B^{\underline{d}}\right)\big]\ , \qquad \text{(D.0.4)}$$

$$V_{V Z^B \hat{V} \bar{Z}_D} =$$

$$(-1)\delta_B^{D}\,\mathrm{tr}\left(T^a B^{\underline{b}} T^{\hat{c}} B_{\underline{d}}\right)\ ,$$

where the colour indices are labeled (a, b, c, d) counter clockwise starting with the leg in the upper left corner. The vertices $V_{\hat{V}^2 W_C \bar{W}^D}$, $V_{V^2 \bar{W}^C W_D}$, $V_{\hat{V} W_B V \bar{W}^D}$ involving the W_A and $\bar{W}^A$ superfields are respectively identical to the previous three vertices up to trivial modifications in the flavour and colour structures.

The quartic superpotential vertices are

$$V_{Z^A W_B Z^C W_D} =$$

$$i\epsilon^{AC}\epsilon_{BD}\big[\,\mathrm{tr}\left(B^{\underline{a}}B_{\underline{b}}B^{\underline{c}}B_{\underline{d}}\right) - \mathrm{tr}\left(B^{\underline{c}}B_{\underline{b}}B^{\underline{a}}B_{\underline{d}}\right)\big]\ ,$$

$$V_{\bar{Z}_A \bar{W}^B \bar{Z}_C \bar{W}^D} =$$

$$i\epsilon_{AC}\epsilon^{BD}\big[\,\mathrm{tr}\left(B_{\underline{a}}B^{\underline{b}}B_{\underline{c}}B^{\underline{d}}\right) - \mathrm{tr}\left(B_{\underline{c}}B^{\underline{b}}B_{\underline{a}}B^{\underline{d}}\right)\big]\ , \qquad \text{(D.0.5)}$$

where again the colour indices are labeled (a, b, c, d) counter clockwise starting with the leg in the upper left corner. Note also that, in a standard way, one of the (D^2) $\bar{\mathrm{D}}^2$ factors has been absorbed into the (anti)chiral integration such that the integration measure of the (anti)chiral vertex is promoted to the full superspace measure.

Appendix E.

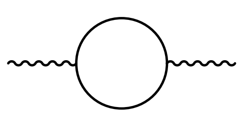
ABJ one- and two-loop subdiagrams

While studying infrared divergencies in Chapter 4, we derived the one-loop gauge-vector corrections and the two-loop two- and four-point functions of ABJM theory ($N = M$). The two-loop corrections to the chiral propagator and superpotential enter as subdiagrams in the evaluation of the dilatation operator given in section 5.1 of chapter 5. Thus, for our ABJ calculation in Chapter 5 we need to generalize the results of chapter 4 to the $N \neq M$ case. We collect the results for the planar contributions of these objects.

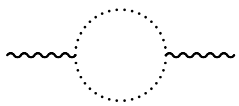
E.1. One-loop vector two-point function

For the $U(M)$ vector superfield V the one-loop two-point function gets contributions from three kind of diagrams respectively having matter, ghosts and vector superfields propagating in the one-loop bubble.

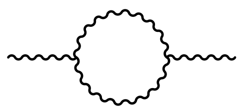
The contribution coming from the chiral matter superfields is

$$\Sigma_{V,\text{matter}} =$$

$$\to 2N\delta^{ab}G(1,1)\,\mathrm{D}^{\alpha}\,\bar{\mathrm{D}}^2\,\mathrm{D}_{\alpha}\ . \tag{E.1.1}$$

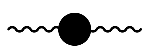
The ghosts correction is

$$\Sigma_{V,\text{ghosts}} =$$

$$\to \frac{1}{2} M \delta^{ab} G(1,1) \big(- \mathrm{D}^\alpha \, \bar{\mathrm{D}}^2 \, \mathrm{D}_\alpha + \big\{ \mathrm{D}^2 \, , \bar{\mathrm{D}}^2 \big\} \big) \ . \qquad \text{(E.1.2)}$$

The diagrams involving a loop of vectors sum up to the following contribution

$$\Sigma_{V,\text{vectors}} =$$

$$\to \frac{1}{2} M \delta^{ab} G(1,1) \big(- \big\{ \mathrm{D}^2 \, , \bar{\mathrm{D}}^2 \big\} \big) \ . \qquad \text{(E.1.3)}$$

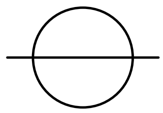
The total contribution to the two-point function for the V superfield is then

$$\Sigma_V =$$

$$\to \frac{1}{2} \delta^{ab} G(1,1)(4N - M) \, \mathrm{D}^\alpha \, \bar{\mathrm{D}}^2 \, \mathrm{D}_\alpha \ . \qquad \text{(E.1.4)}$$

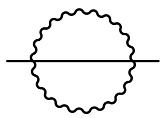
The corrections to the $U(N)$ gauge vector $\hat{V}$ two point function are clearly the same with the only difference that one has to exchange M with N in the results.

E.2. Two-loop chiral two-point function

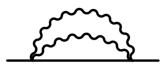
The non-vanishing contributions to the two-point function of chiral superfields can be seen to arise from the following diagrams



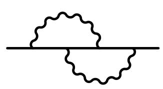
$$\to 2MNI_2 \ ,$$



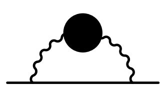
$$\to 2MNI_2 \ ,$$



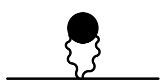
$$\to -\frac{1}{2}M^2 I_2 \ ,$$



$$\to -MN(G(1,1))^2 \ , \tag{E.2.1}$$



$$\begin{aligned}&\to (4N-M)MG(1,1)G_1(1,2-\lambda)\\ &= \frac{1}{2}(4N-M)M(I_{2\mathrm{tp}} - I_2 + I_{2\mathrm{IR}})\end{aligned}$$

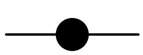


$$\to -\frac{1}{2}(4N-M)MI_{2\mathrm{tp}} \ ,$$

where, in each contribution, we have omitted a factor $\mathrm{D}^2\,\bar{\mathrm{D}}^2$ together with the colour and flavour structures. As discussed in section C.2, the tadpole integral $I_{2\mathrm{tp}}$ is zero in dimensional regularization. However, we keep track of it by splitting its UV and IR divergent parts. This is necessary for the check of the cancellation of the IR divergences performed in appendix 5.5.

Taking into account reflections of the diagrams at the vertical and horizontal axes where necessary, and summing up the contributions, the result reads

$$\Sigma_C =$$

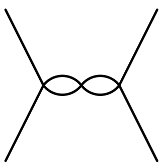
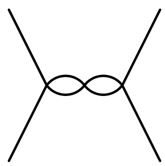


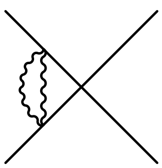
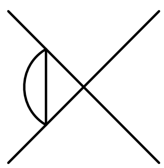
$$\to -2MN(G(1,1))^2 + \frac{1}{2}(8MN - (M^2+N^2))I_{2\mathrm{IR}} \ . \tag{E.2.2}$$

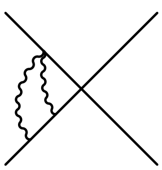
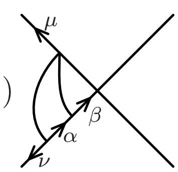
Note that the result is UV finite and it includes an IR divergent term which turns out to be gauge dependent [12] and, according to the discussion in appendix 5.5, does not contribute to the dilatation operator.

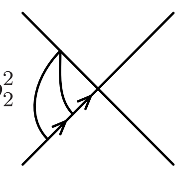
E.3. Two-loop chiral four-point function

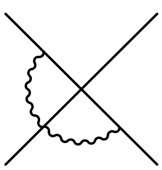
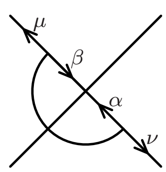
The two-loop renormalization of the superpotential was studied in chapter 4. Here we summarize the results and extend them to the ABJ $U(M) \times U(N)$ case. It holds

 $\to -(4\pi)^2\lambda^2(p_1+p_2)^2$  ,

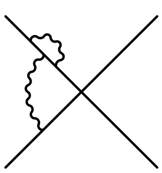
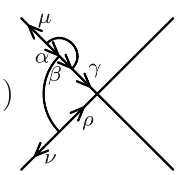
 $\to \dfrac{(4\pi)^2\lambda^2}{2}(p_1+p_2)^2$  ,

 $\to -\dfrac{(4\pi)^2\lambda^2}{2}\Bigg(\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta)$ 

$+\, 2p_2^2$  $\Bigg)$, (E.3.1)

 $\to (4\pi)^2\lambda\hat{\lambda}\,\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta)$ 

,

 $\to (4\pi)^2\lambda\hat{\lambda}\,\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\alpha\gamma^\beta\gamma^\gamma)$ 

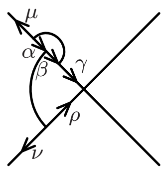
.

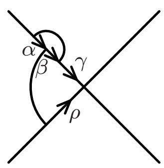
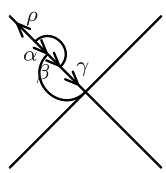
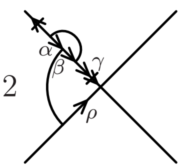
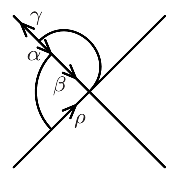
Here the external momenta $(p_1, \cdots, p_4)$ are ordered counterclockwise with p_1 the momentum of the upper-left leg.

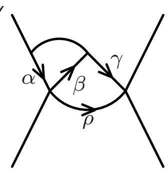
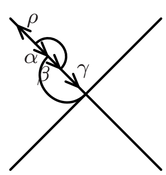
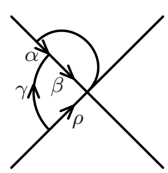
The last contribution is rather complicated. However, it can be simplified by using momentum conservation to eliminate p_2^ν in the trace and the symmetrization inside the trace as

$$
\begin{aligned}
&\frac{1}{2}(\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\alpha\gamma^\beta\gamma^\gamma)+\mathrm{tr}(\gamma^\rho\gamma^\nu\gamma^\mu\gamma^\alpha\gamma^\beta\gamma^\gamma))\\
&=-g^{\mu\nu}\,\mathrm{tr}(\gamma^\rho\gamma^\alpha\gamma^\beta\gamma^\gamma)+g^{\mu\rho}\,\mathrm{tr}(\gamma^\nu\gamma^\alpha\gamma^\beta\gamma^\gamma)-g^{\nu\rho}\,\mathrm{tr}(\gamma^\mu\gamma^\alpha\gamma^\beta\gamma^\gamma)\ .
\end{aligned}
\tag{E.3.2}
$$

One then obtains

$\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\alpha\gamma^\beta\gamma^\gamma)$ 

$= \mathrm{tr}(\gamma^\rho\gamma^\alpha\gamma^\beta\gamma^\gamma) \Bigg(p_1^2$  $+$  $+\,2$  $-$  $\Bigg)$

$= \mathrm{tr}(\gamma^\rho\gamma^\alpha\gamma^\beta\gamma^\gamma) \Bigg($  $+$  $-$  $\Bigg)$,

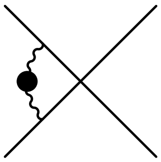
(E.3.3)

where we have used

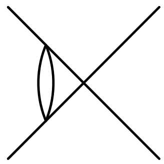
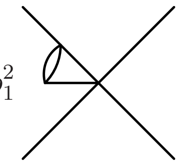
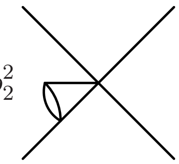
$$
p_1^2(k-p_1)^\gamma+2p_1\cdot(k-p_1)(k-p_1)^\gamma-(k-p_1)^2p_1^\gamma=k^2(k-p_1)^\gamma-(k-p_1)^2k^\gamma\,,
\tag{E.3.4}
$$

with k being one of the loop momenta.

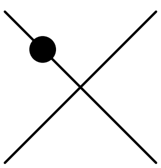
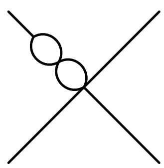
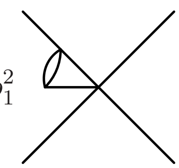
The contribution involving the one-loop vacuum polarization reads



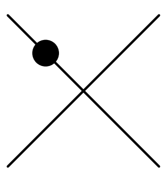
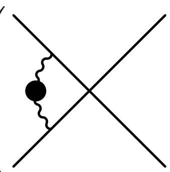
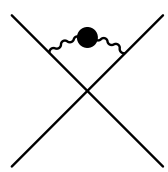
$$\to (4\pi)^2 \frac{1}{2}(4\lambda\hat{\lambda} - \lambda^2)$$

$$\Bigg(-(p_1 + p_2)^2$$

$$+\, p_1^2$$

$$+\, p_2^2$$

$$\Bigg) \, . \qquad \text{(E.3.5)}$$

Considering a factor -1 from the cancellation of the propagator connecting the chiral vertex to the two-loop self energy, we obtain from the D-algebra manipulations

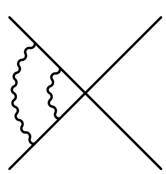
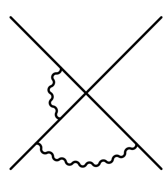

$$\to (4\pi)^2 \Bigg(2\lambda\hat{\lambda} p_1^2$$

$$- \frac{1}{2}(8\lambda\hat{\lambda} - (\lambda^2 + \hat{\lambda}^2))p_1^2$$

$$\Bigg) \, . \qquad \text{(E.3.6)}$$

Let us conclude by mentioning a useful property that was used in appendix 5.5. In the combination


$$+ \frac{1}{2} \Bigg($$

$$+$$

$$\Bigg) \qquad \text{(E.3.7)}$$

the infrared divergence from the integrals involving the first leg is cancelled out.

There are two other diagrams with non-trivial D-algebra and colour structure


$$,$$

$$. \qquad \text{(E.3.8)}$$

Interestingly, these can be seen to be proportional to the very same integrals which appear in components [61]. The two diagrams are zero due to the vanishing of the one-loop triangle subdiagrams.

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