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THE GOLDBACH-LINNIK PROBLEM: Some conditional results

MAT/05

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Contents

Introduction

0.1 The Goldbach problem

Goldbach stated in 1742 that every even integer can be written as a sum of two primes. This problem has been studied by many mathematicians such as Hardy and Littlewood in the 1920's, Vinogradov in the 1930's, Montgomery and Vaughan in the 1970's, to quote just a few.

In 1923 Hardy and Littlewood [12] studied the size of the exceptional set for Goldbach's problem, that is

$$E(N) = |\{n < N \text{ even integer} : n \neq p_1 + p_2\}| \tag{0.1.1}$$

where p_1, p_2 are two primes. They proved that, assuming the Generalized Riemann Hypothesis (GRH for short, see §0.4), almost all even numbers are sums of two primes: in other words that $E(N) = o(N)$ as $N \to +\infty$. Then in 1937 Vinogradov [48] was able to remove the dependence on the GRH and he gave an unconditional proof of the above conclusion. More precisely, Hardy and Littlewood [12] proved that:

$$E(N) \ll N^{1/2+\epsilon}$$

for every $\epsilon > 0$ and large N.

In 1952 Linnik [29] proved that, if the Riemann Hypothesis (RH for short, see §0.4) is true, then for large N the interval $[N, N + \log^{3+\epsilon} N]$ contains a sum of two primes.

In 1975 Montgomery and Vaughan [36] unconditionally proved that there is a positive (effectively computable) constant δ such that, for all large N,

$$E(N) \ll N^{1-\delta}.$$

This result has been improved by Pintz in 2006: in fact, he announced in [40] that one can take $\delta = 1/3$, but the proof has not been published yet.

Assuming GRH, Goldston [8] obtained in 1992 that

$$E(N) \ll N^{1/2} L^4$$

where $L = \log_2(N)$ and $\log_2$ denotes the base 2 logarithm. Then he improved it to

$$E(N) \ll N^{1/2} L^3.$$

We will also look for exceptions in a so-called *short interval*, that is, an interval of the type $[N, N+H]$ with $N \to +\infty$ and $H = o(N)$. When we study this kind of problem, we give conditions on H such that the size of the exceptional set is $o(H)$. The best unconditional results have been obtained in 1993 by Perelli and Pintz in [39], where they proved that almost all even numbers in the interval $[N, N + N^{7/36+\epsilon}]$ are sums of two primes, provided that $0 < \epsilon < 2/3$. Actually, they proved something even stronger.

In the same year Kaczorowski, Perelli and Pintz [17] proved that, assuming the Generalized Riemann Hypothesis and supposing that $HL^{-10} \to \infty$, all even integers in any interval of the form $[N, N+H]$ with at most $O(H^{1/2}L^5)$ exceptions are sums of two primes.

0.2 The Goldbach-Linnik problem

The Goldbach-Linnik problem is a variation of Goldbach's one: here the goal is to prove that all even integers can be written as a sum of two primes and k powers of 2, where k is a fixed positive integer. In some sense this problem is an approximation of the Goldbach one and is, in fact, simpler.

Assuming GRH, in 1951 Linnik [28] proved that there exists a constant $k > 0$ such that every sufficiently large even integer has a representation as the sum of two primes and k powers of 2. Then in 1953 [30] he proved the same thing unconditionally and later in 1975 Gallagher [5] simplified his proofs.

From 1998 many mathematicians studied, both conditionally and unconditionally, the problem of finding a value k_0 such that every sufficiently large even integer has a representation as a sum of two primes and k_0 powers of 2. In 1998 Liu, Liu and Wang [31] found that $k_0 = 770$ is acceptable under GRH and in the same year they found $k_0 = 54000$ unconditionally. The subsequent unconditional results are $k_0 = 25000$ due to Li [26], $k_0 = 2250$ due to Wang [49], $k_0 = 1906$ due to Li [27], $k_0 = 13$ due to Heath-Brown and Puchta [16], $k_0 = 12$ due to Elsholtz (unpublished) and $k_0 = 8$ due to Pintz and Ruzsa [41]. Assuming GRH, the improvements were $k_0 = 200$ due to Liu, Liu and Wang [33], $k_0 = 160$ due to Wang [49] and then $k_0 = 7$ proved independently by Pintz and Ruzsa [41] and Heath-Brown and Puchta [16].

In Chapter 5 we improve conditionally the last result of Pintz and Ruzsa. To do this we will introduce an appropriate version of the Generalized Montgomery Conjecture (GMC(θ)), described in (0.4.5): in fact we will try and find the values of $\theta \in [1/4, 1/2]$, such that, assuming GRH and GMC(θ), we can prove that all the even integers larger than N_0, where N_0 is a computable constant, are sum of two primes and k powers of 2, with $k \in \{3, 4, 5, 6\}$.

The idea is to use GMC(θ) to give an estimate for a suitable exponential sum, see (0.5.11), instead of Vaughan's one in (10.6) of [41].

Another kind of problem has been studied by Languasco, Pintz and Zaccagnini in [24]: they gave an asymptotic formula for the number of representation of n as the sum of two primes and k powers of 2, valid for almost all even integers, that is, that the asymptotic formula fails for at most $o(N)$ values of $n \in [N, 2N]$.

In Chapter 1 and Chapter 2 we give an estimate for the size of the exceptional set of Goldbach-Linnik in short intervals. In our case we consider intervals of the form $[N, N + H]$, where $H = N^\gamma$ with $0 < \gamma \leq 1$. To reach our aim, first we consider the case with a single power of 2 (Ch.1) and then we extend the result to the case with k powers of 2 (Ch.2). We use together the technique of Kaczorowski, Perelli and Pintz [17], the technique of Pintz and Ruzsa [41] and the one of Languasco, Pintz and Zaccagnini [24] to prove that the size of the exceptional set of Goldbach-Linnik in short intervals is $16c(\gamma)^{2k}(1 + o(1))/17$ times the size of the exceptional set of Goldbach in short intervals, where $c(\gamma)$ is a computable constant such that $c(\gamma) \to 1$ as $\gamma \to 0$ and $c(1) = 0.7163435444776661$.

Furthermore in Chapter 4 we use another technique involving series instead of finite sums to estimate the size of the exceptional set of Goldbach-Linnik in long intervals and we prove that this is c^{2k} times the size of the exceptional set of Goldbach in long intervals, where c is defined as above.

0.3 Diophantine problem with powers of two and two primes

It is of interest to examine what happens if we analyze the real analogous of Goldbach-Linnik problem, that is concerning the numbers of the form

$$\lambda_1 p_1 + \lambda_2 p_2 + \mu_1 2^{\nu_1} + \cdots + \mu_s 2^{\nu_s} \tag{0.3.1}$$

with $\lambda_1, \lambda_2, \mu_1, \ldots, \mu_s$ real numbers not all equal to one.

This kind of problem has been studied in 1974-1976 by Vaughan [44], [45], [46], then in the nineties by Brüdern, Cook and Perelli [1], by Harman [14], and in 2006 by Cook and Harman [2]. In particular in 2003 Parsell [38] proved, under certain conditions, that the values of the form (0.3.1) approximate any real number to arbitrary accuracy as s increases, more precisely he found an upper bound to the number s of power of two involved. Then in 2007 Languasco and Zaccagnini [25], under certain conditions, improved this result and in Chapter 6 of this work we will find a better upper bound introducing GRH.

0.4 Description of the main hypotheses

Let $s = \sigma + it$ and $\sigma > 1$. Let us define the Riemann zeta-function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}. \tag{0.4.1}$$

The ζ function can be continued analytically over $\mathbb{C} \setminus \{1\}$ and then is meromorphic. It has only a simple pole in $s = 1$ with residue 1; it has also *trivial zeros* in $-2n$ with $n \geq 1$, $n \in \mathbb{N}$ and we know that the zeros for $\sigma \geq 0$ are located in the critical strip $\sigma \in (0, 1)$.

1. *Riemann Hypothesis (RH):* *All the non-trivial zeros of $\zeta(s)$ are on the line $\sigma = 1/2$.*

For more details we refer to Davenport's book [3], Chapters 1, 8 and 13.

We recall that a Dirichlet primitive character χ to the modulus q is a function of the integer variable n, which is periodic with period q and is also completely multiplicative. For more details we refer to Davenport's book [3], Chapters 1, 4 and 5.

Now let $q \geq 3$ be an integer, $s = \sigma + it$ with $\sigma > 1$ and χ be a primitive character modulo q; then we define the Dirichlet L function associated to χ

$$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}. \tag{0.4.2}$$

The L function can be continued analytically over the whole complex plane and then is holomorphic. It has trivial zeros in $-2n$ with $n \geq 0$, $n \in \mathbb{N}$, if $\chi(-1) = 1$ and in $-2n + 1$ with $n \geq 1$, $n \in \mathbb{N}$, if $\chi(-1) = -1$. Furthermore we know that for $\sigma \geq 0$ the zeros are in the critical strip $\sigma \in (0, 1)$.

2. *Generalized Riemann Hypothesis (GRH):* *All the non-trivial zeros of $L(s, \chi)$ are on the line $\sigma = 1/2$, for χ primitive.*

Again for more details we refer to Davenport's book [3], Chapters 4, 5, 9 and 14.

3. *Montgomery's Conjecture (MC):* *Assume RH and let*

$$F(N,T) = 4 \sum_{0<\gamma_1,\gamma_2\le T} \frac{N^{i(\gamma_1-\gamma_2)}}{4+(\gamma_1-\gamma_2)^2}$$

where γ_1, γ_2 run over the imaginary part of the non-trivial zeros of the Riemann zeta-function $\zeta(s)$. This is the so called ***pair-correlation function*** *and gives a relationship between zeros on the critical line. Then*

$$F(N,T) \sim \frac{1}{2\pi} T \log T \qquad \text{for } N \to \infty \tag{0.4.3}$$

uniformly for $N^\epsilon \le T \le N$, for every fixed $\epsilon > 0$.

For a detailed description see Montgomery's article [35]. He introduced MC while he was studying the vertical distribution of the zeros of the Riemann zeta function and it is now well known that MC is strongly connected with the distribution of primes and related problems, see for example Gallagher and Mueller [6], Heath-Brown [15], Goldston and Montgomery [9] and Goldston [7].

4. *Generalized Montgomery's Conjecture (GMC(θ)):* *For $(a,q)=1$ write*

$$F(N,T;q,a) = 4 \sum_{\chi_1,\chi_2(q)} \chi_1(a)\overline{\chi}_2(a)\tau(\overline{\chi}_1)\tau(\chi_2) \sum_{|\gamma_1|,|\gamma_2|\le T} \frac{N^{i(\gamma_1-\gamma_2)}}{4+(\gamma_1-\gamma_2)^2} \tag{0.4.4}$$

where $\tau(\chi)$ denotes the Gauss sum, see (A.3.1)*, and γ_1, γ_2 run over the imaginary part of the non-trivial zeros of $L(s,\chi_j)$ with $j=1,2$ and χ_j is a primitive character. Now assume GRH and let $\theta \in (0,1/2]$ be fixed and $V = N^{1-\theta}/q$. Then we assume that for every $\epsilon > 0$*

$$F(N,T;q,a) \ll_\epsilon q^2 T N^\epsilon \tag{0.4.5}$$

uniformly for $V \le T \le N$ and $q \le N^\theta$.

For a detailed description see page 350 of the article of Languasco and Perelli [23]. They have introduced GMC(θ) in order to study conditionally the exceptional set in Goldbach's problem.

0.5 Global definitions

In this section we will give the definitions that will be useful in all this work. Let $N \to +\infty, H = H(N) = o(N)$; first we have to define the functions related to Goldbach's problem, that are:

$$r''(n) = |\{(p_1, p_2) \in \mathfrak{P}^2 : n = p_1 + p_2\}|, \tag{0.5.1}$$

$$R''(n) = \sum_{h_1+h_2=n} \Lambda(h_1)\Lambda(h_2), \tag{0.5.2}$$

where $R''(n)$ is strictly linked to $r''(n)$, $\mathfrak{P}$ is the set of all prime numbers and $\Lambda(h)$ is the von Mangoldt function, see (A.1.1).

The problem here is to study the size of the exceptional set, that is:

$$E(N, H) = |\{N \le n \le N + H, \text{ with } n \text{ even } : n \ne p_1 + p_2\}|, \tag{0.5.3}$$

where $p_1, p_2 \in \mathfrak{P}$. If $H = N$ we will write $E(N)$ instead of $E(N, N)$.

The asymptotic formula that we expect is linked to the singular series, defined as

$$\mathfrak{S}(n) = \begin{cases} 2\prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right)\prod_{\substack{p|n \\ p>2}}\left(\frac{p-1}{p-2}\right) & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \tag{0.5.4}$$

The expected asymptotic formula for even n is

$$R''(n) \sim n\mathfrak{S}(n). \tag{0.5.5}$$

A heuristic explanation for this can be found in the article of Hardy and Littlewood [11].

Now for brevity, let $L = \log_2 N$, $\boldsymbol{\nu} = (\nu_1, \dots, \nu_k) \in [1, L]^k$ and $s(\boldsymbol{\nu}) = s(\nu_1, \dots, \nu_k) = 2^{\nu_1} + \ldots + 2^{\nu_k}$, where $k \ge 1$ is fixed. Then we have the functions related to the Goldbach-Linnik problem:

$$r''_k(n) = |\{(p_1, p_2, \boldsymbol{\nu}) \in \mathfrak{P}^2 \times [1, L]^k : n = p_1 + p_2 + s(\boldsymbol{\nu})\}|, \tag{0.5.6}$$

$$R''_k(n) = R''_k(n, N) = \sum_{\substack{\boldsymbol{\nu} \in [1,L]^k \\ h_1+h_2+s(\boldsymbol{\nu})=n}} \Lambda(h_1)\Lambda(h_2). \tag{0.5.7}$$

We want to study the size of the exceptional set, that is:

$$E_k(N, H) = |\{N \le n \le N + H, \text{ with } n \text{ even } : n \ne p_1 + p_2 + s(\boldsymbol{\nu})\}|, \tag{0.5.8}$$

where $p_1, p_2 \in \mathfrak{P}$ and $\boldsymbol{\nu} \in [1, L]^k$. If $H = N$ we will write $E_k(N)$ instead of $E_k(N, N)$.

If we let

$$M_k(n) = \sum_{1 \le \nu_1 \le L} \cdots \sum_{1 \le \nu_k \le L} (n - s(\boldsymbol{\nu}))\mathfrak{S}(n - s(\boldsymbol{\nu})), \tag{0.5.9}$$

then the asymptotic formula that we expect is

$$R_k''(n) \sim M_k(n), \tag{0.5.10}$$

as we will see from the main term of (1.2.13) in the case $k = 1$ and from the main term of (2.2.6) in the general case.

Furthermore for every real α let

$$e(\alpha) = e^{2\pi i \alpha}$$

and, for brevity, for integer q and for real α let

$$e_q(\alpha) = e\left(\frac{\alpha}{q}\right).$$

Now we introduce the relevant exponential sums. Let

$$S(\alpha) = \sum_{n \le N} \Lambda(n) e(n\alpha) \tag{0.5.11}$$

and

$$G(\alpha) = \sum_{1 \le \nu \le L} e(2^\nu \alpha). \tag{0.5.12}$$

Notice that by the orthogonality of the complex exponential functions

$$R_k''(n) = \int_0^1 S(\alpha)^2 G(\alpha)^k e(-n\alpha) d\alpha, \tag{0.5.13}$$

which is the starting point of the circle method. The idea is to split the interval $[0, 1]$ by means of Farey's dissection of order Q, see §A.4. In Goldbach's problem we want to study

$$R''(n) = \int_0^1 S(\alpha)^2 e(-n\alpha) d\alpha,$$

and the circle method leads us to write

$$R''(n) = \int_{\mathfrak{M}} S(\alpha)^2 e(-n\alpha) d\alpha + \int_{\mathfrak{m}} S(\alpha)^2 e(-n\alpha) d\alpha, \tag{0.5.14}$$

where $\mathfrak{M}$ and $\mathfrak{m}$ are called respectively major and minor arcs and are defined as in (1.2.1) and (1.2.2). Since classically we know that S has peaks near rationals a/q when q is small, then we expect that, using this method, the main term arises from a part of the integration on the major arcs; furthermore we have an error term that collects error terms arising from major and minor arcs and from the tail of the singular series.

We are going to consider the Goldbach-Linnik problem so we have to study (0.5.13), but in this case the circle method is not enough because, while the position of the peaks of the exponential sum $S(\alpha)$ is well known, the same thing is not true for G. In fact, the sum (0.5.12) is large for those α for which the first L coefficients in the expansion

$$\alpha \equiv \sum_{\mu=1}^{\infty} a_\mu 2^{-\mu} \bmod 1 \quad (a_\mu \in \{0,1\}).$$

are mostly a few strings of 0 and a few of 1. Linnik [28] proved the following Lemma (for the general case see Lemma 1 of Gallagher [5]):

Lemma 0.5.1. *Let $|G(\alpha)| \geq (1-\eta)L$, then among the coefficients $a_1, \dots, a_L$ there are $\ll \eta L$ changes from 0 to 1 or from 1 to 0; here the constant is uniform for $0 \leq \eta \leq 1$, $L \geq 2$.*

This characterization is not quite useful in practice. Linnik himself proved that the set where $|G(\alpha)|$ is large has comparatively small measure. This is the basis for the method developed by Pintz and Ruzsa in [41]. They consider a set $\mathcal{E}$ of small measure such that, for $\alpha \in \mathcal{E}$, $|G(\alpha)| \geq (1-\eta)L$ and $|G(\alpha)| \leq (1-\eta)L$ for $\alpha \in C(\mathcal{E}) = [0,1]\backslash\mathcal{E}$, where $1-\eta = c$. Then we can split the integral on the minor arcs in the following way

$$\int_{\mathfrak{m}\cap\mathcal{E}} S(\alpha)^2 G(\alpha)^k e(-n\alpha)d\alpha + \int_{\mathfrak{m}\cap C(\mathcal{E})} S(\alpha)^2 G(\alpha)^k e(-n\alpha)d\alpha.$$

Here the integral on $\mathfrak{m} \cap \mathcal{E}$ will be $o(NL^k)$ and the other one will give the constant c^{2k} that we want.

In Chapter 4 we will use a different method that implies the use of series instead of finite sums, then we will consider

$$\tilde{S}(\alpha) = \sum_{n=1}^{\infty} \Lambda(n) e^{-n/N} e(n\alpha), \tag{0.5.15}$$

furthermore we will replace $G(\alpha)$, see (0.5.12), with

$$\tilde{G}(\alpha) = \sum_{1\leq\nu\leq L} e^{-2^\nu/N} e(2^\nu\alpha). \tag{0.5.16}$$

Then we want to study

$$\tilde{R}_k''(n) = e^{-n/N} R_k''(n) = \int_0^1 \tilde{S}(\alpha)^2 \tilde{G}(\alpha)^k e(-n\alpha) d\alpha \tag{0.5.17}$$

and the asymptotic formula that we expect is

$$\tilde{R}_k''(n) \sim e^{-n/N} M_k(n) \tag{0.5.18}$$

as we will see from the main term of (4.2.9).

0.6 Results

The problem we are interested in is to study the size of the exceptional set for the Goldbach-Linnik problem in short intervals, so we want to estimate the quantity in (0.5.8).

In Chapter 1 we study the case with a single power of 2: to do this we start from the article of Kaczorowski, Perelli and Pintz [17] and we observe that, using their technique to estimate $E_1(N, H)$, we do not have any improvement with respect to the estimate they have found for the size of the exceptional set for Goldbach problem unless we also use some others techniques. Then we introduce the method, used by Languasco, Pintz and Zaccagnini to prove Lemma 6.2 in [24], to estimate the tail of the singular series; furthermore we use the one, used by Pintz and Ruzsa to prove Lemma 13 in [41], to give an estimate on the minor arcs.

We set

$$\Sigma_0(n, N, H) = \left| R''(n) - n\mathfrak{S}(n) + F_0(n, N, H) \right|, \tag{0.6.1}$$

where $F_0(n, N, H)$ is defined in (1.1.6) and

$$\Sigma_k(n, N, H) = \left| R_k''(n) - M_k(n) + F_k(n, N, H) \right|, \tag{0.6.2}$$

where $F_k(n, N, H)$ is a function that collects the error terms arising from the major and the minor arcs and from the tail of the singular series. Furthermore let $M = \log H(\log\log H)$, then from (1.2.38) and (2.2.14) we will see that

$$F_k(n, N, H) \ll H^{-1/8} N (M L^{2(1+k)})^{1/2} + o(NL^k).$$

First we recall the Theorem proved by Kaczorowski, Perelli and Pintz in [17], which is

Theorem 0.6.1 (Kaczorowski-Perelli-Pintz). *Assume GRH. Then*

$$\sum_{N \le n \le N+H} \Sigma_0(n, N, H)^2 \le f(N, H) N^2,$$

where $f(N, H) = c_0 H^{1/2} L^5$, c_0 *is a real positive constant and*

$$F_0(n, N, H) \ll N H^{-1/8} (L^2 M)^{1/2}.$$

Then using the methods we quoted above we reach the following results:

Theorem 0.6.2. *Assume GRH, let* $0 < \gamma \le 1$ *be fixed and* $H = N^\gamma$, *then with the same notation as in the statement of Theorem 0.6.1 we have that there exists an effectively computable constant* $c(\gamma) < 1$ *such that*

$$\sum_{N \le n \le N+H} \Sigma_1(n, N, H)^2 \le \frac{16 c(\gamma)^2}{17} (1 + o(1)) L^2 f(N, H) N^2.$$

In particular $c(1) = 0.7163435444776661$ *and* $c(\gamma) \to 1$ *as* $\gamma \to 0$.

From this we can deduce that $E_1(N, H)$ is $16c(\gamma)^2(1+o(1))/17$ times the size of the exceptional set of Goldbach's problem in short intervals.

In Chapter 2 we extend the previous results to the case with k powers of 2 and we prove the following Theorem.

Theorem 0.6.3. *Assume GRH, let* $0 < \gamma \le 1$ *be fixed and* $H = N^\gamma$, *then with the same notation as in the statement of Theorem 0.6.1 we have that there exists an effectively computable constant* $c(\gamma) < 1$ *such that*

$$\sum_{N \le n \le N+H} \Sigma_k(n, N, H)^2 \le \frac{16 c(\gamma)^{2k}}{17} (1 + o(1)) L^{2k} f(N, H) N^2,$$

where $\Sigma_0(n, N, H)$ *is defined in* (0.6.1).
In particular $c(1) = 0.7163435444776661$ *and* $c(\gamma) \to 1$ *as* $\gamma \to 0$.

From this we can prove that $E_k(N, H)$ is $16c(\gamma)^{2k}(1+o(1))/17$ times the size of the exceptional set of Goldbach in short intervals.

We recall that Pintz and Ruzsa [41] proved the following Theorem

Theorem 0.6.4. *Assume GRH. Let* k *be a fixed natural number,* $k \ge 7$. *Then*

$$r_k''(N) > 0 \qquad \text{if } 2 \mid N, N > N_0(k)$$

where $N_0(k)$ *is an explicit constant, depending on* k.

This result implies that the exceptional set for $k \geq 7$ is empty: then in Chapter 3 we will give a detailed explanation of the method used by Pintz and Ruzsa to reach this result and we will compare that with the method that we use in Chapter 1 and Chapter 2.

In Chapter 4 we try to estimate $E_k(N)$, see (0.5.8), using a different technique involving series instead of finite sums. In this case we call

$$\tilde{\Sigma}_0(n, N) = \left| \tilde{R}''(n) - e^{-n/N} n \mathfrak{S}(n) + \tilde{F}_0(n, N) \right|, \tag{0.6.3}$$

where $\tilde{F}_0(n, N)$ satisfied (4.1.6) and

$$\tilde{\Sigma}_k(n, N) = \left| \tilde{R}''_k(n) - e^{-n/N} M_k(n) + \tilde{F}_k(n, N) \right|, \tag{0.6.4}$$

where $\tilde{F}_k(n, N)$ is a function like $F_k(n, N)$, that satisfies

$$\tilde{F}_k(n, N) \ll N^{7/8} L^{1/2+k} \log\log N + o(NL).$$

In this case Theorem 0.6.1 becomes

Theorem 0.6.5 (Kaczorowski-Perelli-Pintz)**.** *Assume GRH. Then*

$$\sum_{N \leq n \leq 2N} \tilde{\Sigma}_0(n, N)^2 \leq \tilde{f}(N) N^2,$$

where $\tilde{f}(N) = c_2(1 + o(1)) N^{1/2} L^3$ *and* c_2 *is a real positive constant.*

Then we prove

Theorem 0.6.6. *Assume GRH. With the same notation as in the statement of Theorem 0.6.5 we have:*

$$\sum_{N \leq n \leq 2N} \tilde{\Sigma}_1(n, N)^2 \leq c^{2k} L^{2k} \tilde{f}(N) N^2,$$

with $c = 0.7163435444776661$.

This result allows us to say that the size of the exceptional set is c^{2k} times the size of the exceptional set in the Goldbach problem in long intervals.

In Chapter 5 we consider the result due to Pintz and Ruzsa [41], that is, that under GRH, all even integers larger than N_0, where $N_0 > 0$ is a constant, have a representation as a sum of two primes and k powers of 2, with $k \geq 7$. Adding a condition, that is GMC(θ), we prove the following result:

Theorem 0.6.7. *Assume GRH. Let k be a fixed natural number $k \in \{3, 4, 5, 6\}$. Then we can find a $\theta \in [1/4, 1/2]$ such that, assuming GMC(θ),*

$$r''_k(N) > 0 \qquad \text{if} \qquad 2 \mid N, N > N_0(k)$$

where $N_0(k)$ is an explicit constant depending on k.

Here there are the value of θ that we have found for each $3 \leq k \leq 6$.

k	θ
3	0.47169811315754716981132
4	0.37389380525973451327434
5	0.30357142852142857142857
6	0.25490196073431372549020

We obtain this result after numerical computations made using a program of Alessandro Languasco, that we thank. The details of this computations are in §A.7 together with all the results that we obtain by the use of this program.

Finally in Chapter 6 we study a different kind of problem, concerning the study of the sum of two primes and s powers of 2, but with real coefficients. As we have just said, this is the real analogous of Goldbach-Linnik problem. Then we prove the following Theorem

Theorem 0.6.8. *Assume RH. Let λ_1, λ_2 be real numbers such that $\boldsymbol{\lambda} = \lambda_1/\lambda_2$ is negative and irrational with $\lambda_1 > 1, \lambda_2 < -1$ and $|\lambda_1/\lambda_2| \geq 1$. Further suppose that $\mu_1, \dots, \mu_s$ are nonzero real numbers such that $\lambda_i/\mu_i \in \mathbb{Q}$, for $i = 1, 2$, and denote by a_i/q_i their reduced representations as rational numbers. Let moreover η be a sufficiently small positive constant such that $\eta < \min(\lambda_1/a_1; |\lambda_2/a_2|)$. Finally let*

$$s_0 = 2 + \left[\frac{\log \left(C(q_1, q_2)\lambda_1\right) - \log \eta}{-\log 0.7163435444776661} \right], \tag{0.6.5}$$

where

$$C(q_1, q_2) = (\log 2 + C \cdot \mathfrak{S}'(q_1))^{1/2} (\log 2 + C \cdot \mathfrak{S}'(q_2))^{1/2},$$

with $C = 10.0219168340$ and

$$\mathfrak{S}'(n) = \prod_{\substack{p \mid n \\ p > 2}} \frac{p-1}{p-2}.$$

Then for every real number γ and every $s \geq s_0$ the inequality

$$|\lambda_1 p_1 + \lambda_2 p_2 + \mu_1 2^{\nu_1} + \cdots + \mu_s 2^{\nu_s} + \gamma| < \eta \qquad (0.6.6)$$

has infinitely many solutions in primes p_1, p_2 and positive integers $\nu_1, \ldots, \nu_s$.

We notice that the number 0.7163435444776661 has been obtained using Corollary A.6.4 and the computer program made by Languasco, that we have already used in Chapter 5.

Chapter 1

Upper bound for the size of the exceptional set for the Goldbach-Linnik problem with a single power of 2

1.1 Introduction

We consider the Goldbach-Linnik conjecture that asserts that: All even integers can be written as a sum of two primes and k powers of 2 for every fixed $k \geq 1$.

In order to show some of the characteristic features of this problem, we begin to treat the case $k = 1$, so the aim of this first part of our work is to give, under hypothesis, an estimate for the size of the exceptional set $E_1(N, H)$, as defined in (0.5.8).

We consider $N \leq n \leq N + H, L = \log N, M = \log H(\log\log H), c_q(-n)$ the Ramanujan sum, see §A.2, $R_1''(n)$ as in (0.5.7) and $\mathfrak{S}(n)$ as in (0.5.4).

Now we take $M_1(n)$ as defined in (4.2.9) and we write

$$\Sigma_1(n, N, H) = |R_1''(n) - M_1(n) + F_1(n, N, H)| \tag{1.1.1}$$

where $F_1(n, N, H)$ will be a function that collects some of the error terms arising from major and minor arcs and from the tail of the singular series: in fact we will see that they are $o(NL)$ and $F_1(n, N, H)$ satisfies (1.2.38). Then we will follow a part of the procedure used by Kaczorowski, Perelli and Pintz [17] to estimate

$$\sum_{N \leq n \leq N+H} \Sigma_1(n, N, H)^2. \tag{1.1.2}$$

We set

$$\Sigma_0(n, N, H) = |R''(n) - n\mathfrak{S}(n) + F_0(n, N, H)|\,. \tag{1.1.3}$$

and

$$f(N, H) = c_0 H^{1/2} L^5, \tag{1.1.4}$$

where c_0 is a real positive constant, then we recall that in [17] and in [18] Kaczorowski, Perelli and Pintz proved the following theorem:

Theorem 1.1.1 (Kaczorowski, Perelli and Pintz)**.** *Assume GRH. Then:*

$$\sum_{N \le n \le N+H} \Sigma_0(n, N, H)^2 \le f(N, H) N^2, \tag{1.1.5}$$

where $f(N, H)$ is defined by (1.1.4) *and $F_0(n, N, H)$ is a certain function that satisfies*

$$F_0(n, N, H) \ll N H^{-1/8} (L^2 M)^{1/2}. \tag{1.1.6}$$

From this they obtained:

Corollary 1.1.2. *Suppose the truth of GRH and let $HL^{-10} \to \infty$; then all even integers in $[N, N+H]$ are sums of two primes with at most $O(H^{1/2}L^5)$ exceptions.*

This implies that

$$E(N, H) = |\{n \in [N, N+H] : n \ne p_1 + p_2\}| \le f(N, H) \tag{1.1.7}$$

with $(p_1, p_2) \in \mathfrak{P}^2$.

The first aim of our work is to prove that adding a power of 2 we obtain:

Theorem 1.1.3. *Assume GRH, let $0 < \gamma \le 1$ be fixed and $H = N^\gamma$, then with the same notation as in the statement of Theorem 1.1.1 we have that there exists an effectively computable constant $c(\gamma) < 1$ such that*

$$\sum_{N \le n \le N+H} \Sigma_1(n, N, H)^2 \le \frac{16 c(\gamma)^2}{17} (1 + o(1)) L^2 f(N, H) N^2.$$

In particular $c(1) = 0.7163435444776661$ and $c(\gamma) \to 1$ as $\gamma \to 0$.

The constant $c(\gamma)$ is linked with the constant $d(\beta)$ in Corollary A.6.4: in fact $c(\gamma) = d(\beta)$, with $\beta = 1 - \gamma/2$.

Corollary 1.1.4. *Assume GRH, let $0 < \gamma \le 1$ be fixed and $H = N^\gamma$, then there exists an effectively computable constant $c(\gamma) < 1$ such that*

$$E_1(N, H) \le \frac{16 c(\gamma)^2}{17} (1 + o(1)) f(N, H) \tag{1.1.8}$$

where $f(N, H)$ is defined in (1.1.4)*.*

1.2 Proof of Theorem 1.1.3

In order to estimate (1.1.2), we first have to consider the Farey dissection of order Q of $I = [1/Q, 1+1/Q]$, as described in §A.4; we will eventually choose $Q = H^{1/2}$, see (1.2.28). Call the arc relative to a/q

$$I_{q,a} = \left\{ \alpha = \frac{a}{q} + \eta : \eta \in \xi_{q,a} \right\} \qquad \text{with} \qquad \xi_{q,a} \subset \left(-\frac{1}{qQ}, \frac{1}{qQ} \right).$$

We observe that, if $a_1/q_1, a/q, a_2/q_2 \in \mathfrak{F}_Q$ are consecutive, then $q, q_1, q_2 \leq Q$ and, by (A.4.1), the arc from $P_{\mu_1} = (a + a_1)/(q + q_1)$ to a/q and the one from a/q to $P_{\mu_2} = (a + a_2)/(q + q_2)$ have length respectively

$$\frac{a}{q} - \frac{a + a_1}{q + q_1} = \frac{aq_1 - a_1 q}{q(q + q_1)} = \frac{1}{q(q + q_1)}$$

and

$$\frac{a + a_2}{q + q_2} - \frac{a}{q} = \frac{a_2 q - a q_2}{q(q + q_2)} = \frac{1}{q(q + q_2)}.$$

Now we have that $q + q_1 < 2Q$, $q + q_2 < 2Q$ so that

$$\left(-\frac{1}{2qQ}, \frac{1}{2qQ} \right) \subset \xi_{q,a} \subset \left(-\frac{1}{qQ}, \frac{1}{qQ} \right).$$

Let

$$\mathfrak{M} = \bigcup_{q \leq P} \bigcup_{a=1}^{q}{}^{*} I_{q,a} \tag{1.2.1}$$

be the set of the so-called major arcs, where $\bigcup^{*}$ means that we make the union on the a such that $(a, q) = 1$, and let

$$\mathfrak{m} = I \backslash \mathfrak{M} \tag{1.2.2}$$

be the minor arcs.

Moreover consider $S(\alpha)$ as in (0.5.11) and $G(\alpha)$ as in (0.5.12): then we can write

$$\begin{aligned} R_1''(n) &= \int_0^1 S(\alpha)^2 G(\alpha) e(-n\alpha) d\alpha \\ &= \int_{\mathfrak{M}} S(\alpha)^2 G(\alpha) e(-n\alpha) d\alpha + \int_{\mathfrak{m}} S(\alpha)^2 G(\alpha) e(-n\alpha) d\alpha \end{aligned} \tag{1.2.3}$$

$$=R''_{\mathfrak{M}}(n) + R''_{\mathfrak{m}}(n),$$

say.

Now let $\alpha = a/q + \eta \in \mathfrak{M}$ and

$$T(\eta) = \sum_{n \le N} e(n\eta). \tag{1.2.4}$$

Then by Prime Number Theorem for Arithmetic Progressions we expect that, for q small

$$S(\alpha) \approx \frac{\mu(q)}{\varphi(q)} T(\eta).$$

Hence we write

$$S\left(\frac{a}{q} + \eta\right) = \frac{\mu(q)}{\varphi(q)} T(\eta) + R(\eta; q, a), \tag{1.2.5}$$

where $R(\eta; q, a)$ is the error term arising from the approximation, and we know from computations in Chapter 26, pages 146-147, of Davenport's book [3] that $R(\eta; q, a)$ is "small".

Finally we recall that

$$|T(\eta)| \ll \min\left(N, \frac{1}{\|\eta\|}\right), \tag{1.2.6}$$

since $T(\eta)$ is essentially the sum of a geometric progression.

1.2.1 Estimate on the major arcs

Consider P such that $P \cdot Q \le H$. Now using (1.2.5) we can do the following computation on the major arcs:

$$\begin{aligned}
R''_{\mathfrak{M}}(n) =& \sum_{q \le P} \sideset{}{^*}\sum_{a=1}^{q} \int_{\xi_{q,a}} S\left(\frac{a}{q} + \eta\right)^2 G\left(\frac{a}{q} + \eta\right) e_q(-na) e(-n\eta) d\eta \\
=& \sum_{1 \le \nu \le L} \sum_{q \le P} \frac{\mu(q)^2}{\varphi(q)^2} \sideset{}{^*}\sum_{a=1}^{q} e_q(-(n - 2^\nu)a) \int_{\xi_{q,a}} T(\eta)^2 e\left(2^\nu \eta\right) e(-n\eta) d\eta \\
&+ \sum_{1 \le \nu \le L} \sum_{q \le P} \sideset{}{^*}\sum_{a=1}^{q} e_q(-(n - 2^\nu)a) \int_{\xi_{q,a}} e\left(2^\nu \eta\right) R(\eta; q, a)^2 e(-n\eta) d\eta \\
&+ 2 \sum_{1 \le \nu \le L} \sum_{q \le P} \frac{\mu(q)}{\varphi(q)} \sideset{}{^*}\sum_{a=1}^{q} e_q(-(n - 2^\nu)a)
\end{aligned}$$

$$
\begin{aligned}
&\cdot \int_{\xi_{q,a}} T(\eta)R(\eta;q,a)e\left(2^\nu\eta\right)e(-n\eta)d\eta\\
=&\Sigma_1(n)+\Sigma_2(n)+\Sigma_3(n),
\end{aligned}
$$

say, where $\Sigma_1(n)$ is the main term, while $\Sigma_2(n), \Sigma_3(n)$ are error terms; we recall that $\sum^*$ is a sum on the a such that $(a,q)=1$. Now we have:

$$
\Sigma_1(n)=\sum_{1\le\nu\le L}\sum_{q\le P}\frac{\mu(q)^2}{\varphi(q)^2}\sum_{a=1}^{q}{}^{*} e_q(-(n-2^\nu)a)\int_{\xi_{q,a}} T(\eta)^2 e(-(n-2^\nu)\eta)d\eta.
$$

We can extend the integral to $[0,1]$ with an error term that is

$$
O\left(QL\sum_{q\le P}\frac{\mu(q)^2 q}{\varphi(q)}\right);
$$

in fact, by (1.2.6), $|T(\eta)|\ll\|\eta\|^{-1}$ in the interval $[1/2qQ, 1-1/2qQ]$, so that

$$
\int_{1/2qQ}^{1-1/2qQ}|T(\eta)|^2 d\eta\ll qQ.
$$

Then $\Sigma_1(n)$ becomes

$$
\begin{aligned}
\Sigma_1(n)=&\sum_{1\le\nu\le L}\sum_{q\le P}\frac{\mu(q)^2}{\varphi(q)^2}\sum_{a=1}^{q}{}^{*} e_q(-(n-2^\nu)a)\int_0^1 T(\eta)^2 e(-(n-2^\nu)\eta)d\eta\\
&\quad + O\left(QL\sum_{q\le P}\frac{\mu(q)^2 q}{\varphi(q)}\right)\\
=&\sum_{1\le\nu\le L}\sum_{q\le P}\frac{\mu(q)^2}{\varphi(q)^2}\sum_{a=1}^{q}{}^{*} e_q(-(n-2^\nu)a) \qquad (1.2.7)\\
&\cdot\int_0^1\sum_{n_1\le N}\sum_{n_2\le N} e(-(n-2^\nu-n_1-n_2)\eta)d\eta+O\left(QL\sum_{q\le P}\frac{\mu(q)^2 q}{\varphi(q)}\right)\\
=&\sum_{1\le\nu\le L}(n-2^\nu)\sum_{q\le P}\frac{\mu(q)^2}{\varphi(q)^2}c_q(-(n-2^\nu))+O(PQL), \qquad (1.2.8)
\end{aligned}
$$

where we use Lemma 2 of Goldston [8], which is Lemma A.6.2 in §A.6, to obtain the bound for the error term. Moreover by the triangle inequality

$$
\Sigma_2(n)=\left|\sum_{q\le P}\sum_{1\le\nu\le L}\sum_{a=1}^{q}{}^{*} e_q(-(n-2^\nu)a)\int_{\xi_{q,a}} R(\eta;q,a)^2 e(-(n-2^\nu)\eta)d\eta\right|
$$

$$\ll L \sum_{q\le P} \sideset{}{^*}\sum_{a=1}^{q} \int_{-\frac{1}{qQ}}^{\frac{1}{qQ}} |R(\eta;q,a)|^2 d\eta$$

and by the Cauchy-Schwarz inequality and the trivial one (1.2.6) it follows that

$$\Sigma_3(n) \ll N^{1/2} L \sum_{q\le P} \frac{\mu(q)^2}{\varphi(q)^{1/2}} \left(\sideset{}{^*}\sum_{a=1}^{q} \int_{-\frac{1}{qQ}}^{\frac{1}{qQ}} |R(\eta;q,a)|^2 d\eta \right)^{1/2}.$$

Now thanks to Lemma 1 of Kaczorowski, Perelli and Pintz [17], see Lemma A.6.1, we have

$$\sideset{}{^*}\sum_{a=1}^{q} \int_{-\frac{1}{qQ}}^{\frac{1}{qQ}} |R(\eta;q,a)|^2 d\eta \ll \frac{NL^4}{Q}. \tag{1.2.9}$$

Furthermore, from Lemma A.6.2, we have that

$$\sum_{q\le P} \frac{\mu(q)^2}{\varphi(q)^{1/2}} \ll P^{1/2},$$

hence we obtain:

$$\begin{aligned}\Sigma_2(n) + \Sigma_3(n) \ll & L \sum_{q\le P} \frac{NL^4}{Q} + N^{1/2} L \sum_{q\le P} \frac{\mu(q)^2}{\varphi(q)^{1/2}} \frac{N^{1/2}L^2}{Q^{1/2}} \\ \ll & \frac{PNL^5}{Q} + \left(\frac{P}{Q}\right)^{1/2} NL^3 \ll \left(\frac{P}{Q}\right)^{1/2} NL^3 \end{aligned} \tag{1.2.10}$$

provided that

$$P \ll QL^{-4}. \tag{1.2.11}$$

Collecting (1.2.8) and (1.2.10) we have

$$\begin{aligned}\int_{\mathfrak{M}} S(\alpha)^2 G(\alpha) e(-n\alpha) d\alpha = & \sum_{1\le\nu\le L} (n-2^\nu) \sum_{q\le P} \frac{\mu(q)^2}{\varphi(q)^2} c_q(-(n-2^\nu)) \\ & + O(PQL) + O\left(\left(\frac{P}{Q}\right)^{1/2} NL^3 \right). \end{aligned} \tag{1.2.12}$$

Now we complete the previous sum to obtain the singular series defined in (0.5.4) so that:

$$\int_{\mathfrak{M}} S(\alpha)^2 G(\alpha) e(-n\alpha) d\alpha = M_1(n)$$

$$- \sum_{1\le\nu\le L}(n-2^\nu)\sum_{q>P}\frac{\mu(q)^2}{\varphi(q)^2}c_q(-(n-2^\nu)) + O\left(PQL + \left(\frac{P}{Q}\right)^{1/2} NL^3\right), \tag{1.2.13}$$

where $M_1(n)$ is the main term while we are going to prove that the tail of the singular series is $o(NL)$.

1.2.2 Estimate of the tail of the singular series

Now we have to estimate the tail of the singular series: we want to prove that it is $o(NL)$. We start with the trivial inequality

$$\left|\sum_{1\le\nu\le L}(n-2^\nu)\sum_{q>P}\frac{\mu(q)^2}{\varphi(q)^2}c_q(-(n-2^\nu))\right| \le N\sum_{1\le\nu\le L}\left|\sum_{q>P}\frac{\mu(q)^2}{\varphi(q)^2}c_q(-(n-2^\nu))\right|,$$

and then we need the following lemma:

Lemma 1.2.1. *If $P\to+\infty$ then*

$$\sum_{1\le\nu\le L}\left|\sum_{q>P}\frac{\mu(q)^2}{\varphi(q)^2}c_q(-(n-2^\nu))\right| = o(L).$$

Proof of Lemma 1.2.1. Thanks to Theorem A.2.1 we have:

$$c_q(-(n-2^\nu)) = \mu(q_1)\frac{\varphi(q)}{\varphi(q_1)},$$

where $q_1 = q/(q, n-2^\nu)$. Now we take $d = (q, n-2^\nu)$, $q = dl$ and we observe that, thanks to the fact that this is a sum on square-free values of q, then q_1 and d are also square-free. We also have $\varphi(dl)\ge\varphi(d)\varphi(l)$ and so:

$$\begin{aligned}\left|\sum_{q>P}\frac{\mu(q)^2}{\varphi(q)^2}c_q(-(n-2^\nu))\right| &\le \sideset{}{'}\sum_{d\mid(n-2^\nu)}\frac{1}{\varphi(d)}\sideset{}{'}\sum_{l>P/d}\frac{1}{\varphi(l)^2}\\ &\ll \sideset{}{'}\sum_{\substack{d\mid(n-2^\nu)\\ d>P}}\frac{1}{\varphi(d)} + \frac{1}{P}\sideset{}{'}\sum_{\substack{d\mid(n-2^\nu)\\ d\le P}}\frac{d}{\varphi(d)} = A_1 + A_2,\end{aligned}$$

where $\sum'$ denotes a sum on square-free values of d. As in (8.10) and (8.13) of Pintz and Ruzsa [41] and Lemma 6.2 of Languasco, Pintz and Zaccagnini [24], let

$$f(d) = \prod_{\substack{p\mid d\\ p>2}}\frac{1}{p-2}$$

for all integers d, furthermore let $f(1) = 1$ and $f(d) = 0$ if d is even. Then let

$$\xi(d) = \min\{\mu \in \mathbb{N}^* : 2^\mu \equiv 1 \bmod d\};$$

we observe that for odd d

$$\varphi(d) \gg 1/f(d)$$

so that

$$1/\varphi(d) \ll f(d). \tag{1.2.14}$$

Also let

$$S(n,d) = \begin{cases} 1 & \text{if there exists } \nu \in [1,L] \text{ such that } n \equiv 2^\nu \bmod d \\ 0 & \text{otherwise.} \end{cases}$$

Consider first the sum on ν of A_2:

$$\begin{aligned}\sum_{1\le\nu\le L} \frac{1}{P} \sideset{}{'}\sum_{\substack{d|(n-2^\nu)\\ d\le P}} \frac{d}{\varphi(d)} =& \frac{1}{P} \sideset{}{'}\sum_{d\le P} \frac{d}{\varphi(d)} \sum_{\substack{1\le\nu\le L\\ n\equiv 2^\nu (d)}} 1 \\ \ll& \frac{1}{P} \sideset{}{'}\sum_{d\le P} \frac{d}{\varphi(d)} \left(\frac{L}{\xi(d)} + S(n,d) \right);\end{aligned}$$

now we recall that from Theorem A.6.5

$$\varphi(d) \gg d(\log\log d)^{-1} \tag{1.2.15}$$

so that the previous sum becomes

$$\ll \frac{1}{P} \log\log P \sideset{}{'}\sum_{d\le P} \frac{L}{\log d} + \frac{1}{P} \sideset{}{'}\sum_{d\le P} \frac{d}{\varphi(d)} S(n,d)$$

and by partial summation we have

$$\ll \frac{L}{P} \log\log P \cdot \frac{P}{\log P} + \frac{1}{P} \left(P \sideset{}{'}\sum_{d\le P} f(d)S(n,d) - \int_1^P \sideset{}{'}\sum_{d\le t} f(d)S(n,d)dt \right).$$

Now by the computations in the proof of Lemma 6.2 of [24] the last term above is

$$\ll \frac{L\log\log P}{\log P} + \log L = o(L) \qquad \text{when} \qquad P \to +\infty. \tag{1.2.16}$$

Now we evaluate the sum on ν of A_1:

$$\sum_{1\le\nu\le L}\sideset{}{'}\sum_{\substack{d|(n-2^\nu)\\ d>P}}\frac{1}{\varphi(d)}=\sideset{}{'}\sum_{d>P}\frac{1}{\varphi(d)}\sum_{\substack{1\le\nu\le L\\ n\equiv 2^\nu(d)}}1\ll\sideset{}{'}\sum_{d>P}\frac{1}{\varphi(d)}\left(\frac{L}{\xi(d)}+S(n,d)\right)$$
$$\ll L\sideset{}{'}\sum_{d>P}\frac{1}{\varphi(d)\xi(d)}+\sideset{}{'}\sum_{d>P}\frac{1}{\varphi(d)}S(n,d),$$

by (1.2.14) we have that this is

$$\ll L\sideset{}{'}\sum_{d>P}\frac{f(d)}{\xi(d)}+\sideset{}{'}\sum_{d>P}\frac{1}{\varphi(d)}S(n,d).$$

Again thanks to the computation in the proof of Lemma 6.2 of [24] we have that for all $\epsilon>0$ the previous sum becomes:

$$\ll L\frac{\epsilon}{4}+\sideset{}{'}\sum_{d>P}f(d)S(n,d)\ll L\frac{\epsilon}{4}+\log L=o(L)\qquad\text{when}\qquad P\to+\infty, \tag{1.2.17}$$

and Lemma 1.2.1 is proved. $\square$

Summing up we have:

$$\left|\sum_{1\le\nu\le L}(n-2^\nu)\sum_{q>P}\frac{\mu(q)^2}{\varphi(q)^2}c_q(-(n-2^\nu))\right|=o(NL). \tag{1.2.18}$$

From (1.2.3), (1.2.13) and (1.2.18) we have

$$\sum_{N\le n\le N+H}\left|R_1''(n)-M_1(n)+F_1(n,N,H)\right|^2$$
$$\ll\sum_{N\le n\le N+H}\left|\int_{\mathfrak{m}}S(\alpha)^2G(\alpha)e(-n\alpha)d\alpha\right|^2, \tag{1.2.19}$$

where we recall that $F_1(n,N,H)$ collects error terms arising from (1.2.13), (1.2.18) and then it will collect also a part of the error term arising from the minor arcs, which is $o(NL)$ as we will prove in §1.2.3.

1.2.3 Estimate on the minor arcs

First we do an estimate of the minor arcs with the method used in Kaczorowski, Perelli and Pintz in [17] and in its corrigendum [18]: we take

$$t(s) = \frac{1}{H}\max(H - |s|, 0)$$

and from the properties of the Fejér kernel

$$K(\alpha) = \sum_{s=-\infty}^{+\infty} t(s)e(-s\alpha) = \frac{1}{H}\left|\sum_{s=-H}^{H} e(-s\alpha)\right|^2 \ll \frac{1}{H}\min\left(H^2, \frac{1}{\|\alpha\|^2}\right) \tag{1.2.20}$$

and we have

$$\begin{aligned}
&\sum_{N\le n\le N+H}\left|\int_{\mathfrak{m}} S(\alpha)^2 G(\alpha)e(-n\alpha)d\alpha\right|^2\\
&\ll \sum_n t(n-N)\left|\int_{\mathfrak{m}}\sum_{1\le\nu\le L} S(\alpha)^2 e(2^{\nu}\alpha)e(-n\alpha)d\alpha\right|^2\\
&= \sum_n t(n-N)\left(\int_{\mathfrak{m}}\sum_{1\le\nu_1\le L}\overline{S(\xi)}^2 e(-2^{\nu_1}\xi)e(n\xi)d\xi\right)\cdot\\
&\quad\left(\int_{\mathfrak{m}}\sum_{1\le\nu_2\le L} S(\alpha)^2 e(2^{\nu_2}\alpha)e(-n\alpha)d\alpha\right)\\
&= \sum_n t(n-N)\cdot\\
&\quad\sum_{1\le\nu_1\le L}\sum_{1\le\nu_2\le L}\int_{\mathfrak{m}}\overline{S(\xi)}^2\int_{\mathfrak{m}} S(\alpha)^2 e(-n(\alpha-\xi))e(-2^{\nu_1}\xi)e(2^{\nu_2}\alpha)d\xi d\alpha\\
&= \sum_{1\le\nu_1\le L}\sum_{1\le\nu_2\le L}\int_{\mathfrak{m}}\overline{S(\xi)}^2\int_{\mathfrak{m}} S(\alpha)^2 K(\alpha-\xi)e(-N(\alpha-\xi)-2^{\nu_1}\xi+2^{\nu_2}\alpha)d\xi d\alpha\\
&\ll \frac{L^2}{H}\int_{\mathfrak{m}}|S(\xi)|^2\int_{\mathfrak{m}}|S(\alpha)|^2\min\left\{H^2, \frac{1}{\|\alpha-\xi\|^2}\right\}d\alpha d\xi.
\end{aligned}$$

Now we recall that by Prime Number Theorem

$$\int_0^1 |S(\alpha)|^2 d\alpha = \sum_{n\le N}\Lambda^2(n) \ll NL \tag{1.2.21}$$

and we use this to give an estimate of $\int_{\mathfrak{m}} |S(\xi)|^2 d\xi$.

Now for $\alpha \in \{\xi - 1/H, \xi + 1/H\}$ we have that $\min\{H^2, 1/\|\xi-\alpha\|^2\} = H^2$, furthermore we can split the interval $[0,1]$ into $H/2$ subintervals so that we have

$$\alpha \in \bigcup_{t=-H}^{H} \left(\xi - \frac{t}{H} - \frac{1}{H}, \xi - \frac{t}{H} + \frac{1}{H}\right)$$

that is the union of all the intervals of length $2/H$ translated toward left and right from ξ, where $\xi \in [0,1]$. If α belongs to the t-th interval, then $1/\|\xi-\alpha\|^2 \leq H^2/(t^2+1)$ and

$$\min\left\{H^2, 1/\|\xi-\alpha\|^2\right\} \leq \frac{H^2}{t^2+1}.$$

From this and by (1.2.21) we have

$$\begin{aligned} &\frac{L^2}{H}\int_{\mathfrak{m}} |S(\xi)|^2 \int_{\mathfrak{m}} |S(\alpha)|^2 \min\left\{H^2, \frac{1}{\|\alpha-\xi\|^2}\right\} d\alpha d\xi \\ &\ll NL^3 \max_{\xi\in[0,1]} \sum_{t=-H}^{H} \frac{H}{t^2+1} \int_{(\xi-t/H-(1/H),\xi-t/H+(1/H))\cap\mathfrak{m}} |S(\alpha)|^2 d\alpha. \end{aligned} \tag{1.2.22}$$

Now we observe that

$$\begin{aligned} &\max_{\xi\in[0,1]} \int_{(\xi-(1/H),\xi+(1/H))\cap\mathfrak{m}} |S(\alpha)|^2 d\alpha \\ &= \max_{\xi\in[0,1]} \int_{(\xi-t/H-(1/H),\xi-t/H+(1/H))\cap\mathfrak{m}} |S(\alpha)|^2 d\alpha \end{aligned}$$

for each value of $-H \leq t \leq H$ then, since $\sum_{t=-H}^{H} 1/(t^2+1)$ is a convergent series, (1.2.22) becomes

$$\begin{aligned} &\ll HNL^3 \max_{\xi\in[0,1]} \int_{(\xi-(1/H),\xi+(1/H))\cap\mathfrak{m}} |S(\alpha)|^2 d\alpha \\ &\ll HNL^3 \max_{\substack{P<q\leq Q\\(a,q)=1}} \int_{-1/qQ}^{1/qQ} \left|S\left(\frac{a}{q}+\eta\right)\right|^2 d\eta. \end{aligned} \tag{1.2.23}$$

The fact that $\alpha \in \mathfrak{m}$ allows us to say that $1/H \leq 1/(qQ)$ and this gives the following condition on Q:

$$Q^2 \leq H. \tag{1.2.24}$$

Now from (1.2.9) and for $P < q \le Q$ we have:

$$\int_{-1/qQ}^{1/qQ} \left| S\left(\frac{a}{q}+\eta\right)\right|^2 d\eta \ll \frac{1}{\varphi(q)^2}\int_{-1/qQ}^{1/qQ} |T(\eta)|^2 d\eta + \int_{-1/qQ}^{1/qQ} |R(\eta;q,a)|^2 d\eta$$
$$\ll \frac{N}{\varphi(q)^2} + \frac{NL^4}{Q} \ll \frac{N\log^2 P}{P^2} + \frac{NL^4}{Q}. \qquad (1.2.25)$$

Then combining (1.2.23) and (1.2.25) we obtain:

$$\sum_{N\le n\le N+H} \left| \int_{\mathfrak{m}} S(\alpha)^2 G(\alpha) e(-n\alpha)d\alpha \right|^2 \ll \frac{HN^2L^3\log^2 P}{P^2} + \frac{HN^2L^7}{Q}. \quad (1.2.26)$$

Now we write formula (19) of Kaczorowski, Perelli and Pintz [17] calling c_1 the implicit constant in the $O(.)$ notation:

$$\sum_{N\le n\le N+H} \left| \int_{\mathfrak{m}} S(\alpha)^2 G(\alpha) e(-n\alpha)d\alpha \right|^2 \le c_1 \left(\frac{HN^2L\log^2 P}{P^2} + \frac{HN^2L^5}{Q} \right). \quad (1.2.27)$$

We notice that, if we make their choice of M, that is $M = \log H$, and we choose

$$Q = H^{1/2} \qquad \text{and} \qquad P = \frac{H^{1/4}M}{L^2}, \qquad (1.2.28)$$

then, from (1.2.26), we have

$$\sum_{N\le n\le N+H} \left| \int_{\mathfrak{m}} S(\alpha)^2 G(\alpha) e(-n\alpha)d\alpha \right|^2 \le c_0 H^{1/2} N^2 L^7,$$

with c_0, defined in (1.1.4), that is $17c_1/16$. Then the size of the exceptional set is $E_1(N,H) \le c_0 H^{1/2} L^5$ and, if we compare this result with that in Corollary 1.1.2, we see that we have found the same thing and we do not have any improvement.

We can also observe that if we take $M = \log H(\log\log H)$, as we stated in the introduction, and parameters P and Q as in (1.2.28), we have that

$$\frac{HN^2L\log^2 P}{P^2} = o\left(\frac{HN^2L^5}{Q}\right) \qquad (1.2.29)$$

and this allows us to say that $E_1(n,H) \le 16(1+o(1))f(N,H)/17$, with H that has to be chosen a little bit larger according to $H \gg (ML^2)^4$.

Now we try to gain a small constant applying the technique used by Pintz and Ruzsa to prove Lemma 13 of [41]. The idea is to consider a set $\mathcal{E}$, which has small measure, such that, if $\alpha \in \mathcal{E}$, $G(\alpha)$ is large and, if $\alpha \in C(\mathcal{E}) = [0,1]\backslash\mathcal{E}$, $G(\alpha)$ is small. Then we can split the integral on minor arcs in two parts: one on $\mathfrak{m} \cap \mathcal{E}$ and the other on $\mathfrak{m} \cap C(\mathcal{E})$.

We consider Vaughan's estimate of $S(\alpha)$ under GRH, see Lemma 2 of Pintz and Ruzsa [41], that is

$$S(\alpha) \ll \left(\frac{N}{P} + \sqrt{NQ} + \frac{N}{\sqrt{Q}}\right) L^2; \tag{1.2.30}$$

with our choice of P and Q we have

$$|S(\alpha)| \leq C^{1/2} \frac{NL^4}{H^{1/4}M}, \tag{1.2.31}$$

where C is a real positive constant such that the inequality is verified. Then we have

$$\begin{aligned}\int_{\mathfrak{m}\cap\mathcal{E}} |S(\alpha)^2 G(\alpha) e(-n\alpha)| d\alpha \leq & C \int_{\mathfrak{m}\cap\mathcal{E}} \left(\frac{NL^4}{H^{1/4}M}\right)^2 |G(\alpha)| d\alpha \\ \leq & C \frac{N^2 L^9}{H^{1/2}M^2} |\mathcal{E}|. \end{aligned}\tag{1.2.32}$$

We want that

$$\int_{\mathfrak{m}\cap\mathcal{E}} |S(\alpha)^2 G(\alpha) e(-n\alpha)| d\alpha = o(NL), \tag{1.2.33}$$

then we have to choose the set $\mathcal{E}$ such that

$$|\mathcal{E}| = O\left(\frac{H^{1/2}M^2}{NL^8}\right). \tag{1.2.34}$$

By hypothesis $H = N^\gamma$, with $0 < \gamma \leq 1$, then (1.2.34) becomes

$$|\mathcal{E}| = O(N^{-(1-\gamma/2)} L^{-4}), \tag{1.2.35}$$

now we can use Corollary A.6.4 taking $\beta = 1 - \gamma/2$, that is, that we can find an effectively computable constant $d(1-\gamma/2) = c(\gamma) < 1$ such that

$$|G(\alpha)| \leq c(\gamma) L \tag{1.2.36}$$

with $\alpha \in [0,1]\backslash\mathcal{E}$. We can find the constant $c(\gamma)$ for each value of $0 < \gamma \leq 1$ using a computer program made by Alessandro Languasco: in particular we find that $c(1) = 0.7163435444776661$ and $c(\gamma) \to 1$ as $\gamma \to 0$.

Remark 1. *Collecting* (1.2.13), (1.2.18) *and* (1.2.33) *we have*

$$F_1(n, N, H) \ll PQL + (P/Q)^{1/2} NL^3 + o(NL) \tag{1.2.37}$$

when $P \to +\infty$. *We know that* $PQL = o(NL)$, *then with our choice of parameters* P *and* Q *in* (1.2.28), *that is* $P = H^{1/4}M/L^2$ *and* $Q = H^{1/2}$, *we have*

$$F_1(n, N, H) \ll H^{-1/8} N (ML^4)^{1/2} + o(NL); \tag{1.2.38}$$

furthermore we see that (1.2.11) *is satisfied for*

$$H \gg (ML^2)^4 \tag{1.2.39}$$

and this fits with our choice of H.

Now we consider the integral on $\mathfrak{m} \cap C(\mathcal{E})$, that will give the error term:

$$\begin{aligned}
&\sum_{N \le n \le N+H} \left| \int_{\mathfrak{m} \cap C(\mathcal{E})} S(\alpha)^2 G(\alpha) e(-n\alpha) d\alpha \right|^2 \\
=& \sum_{N \le n \le N+H} \left(\int_{\mathfrak{m} \cap C(\mathcal{E})} S(\xi)^2 G(\xi) e(-n\xi) d\xi \right) \left(\int_{\mathfrak{m} \cap C(\mathcal{E})} \overline{S(\alpha)}^2 \overline{G(\alpha)} e(n\alpha) d\alpha \right) \\
=& \sum_{N \le n \le N+H} \int_{\mathfrak{m} \cap C(\mathcal{E})} S(\xi)^2 \int_{\mathfrak{m} \cap C(\mathcal{E})} \overline{S(\alpha)}^2 e(-n(\xi - \alpha)) G(\xi) \overline{G(\alpha)} d\xi d\alpha
\end{aligned}$$

and by (1.2.36) we have

$$\le c(\gamma)^2 L^2 \int_{\mathfrak{m} \cap C(\mathcal{E})} |S(\xi)|^2 \int_{\mathfrak{m} \cap C(\mathcal{E})} |S(\alpha)|^2 |T(\xi - \alpha)| d\xi d\alpha.$$

Then we can estimate the integrals as Kaczorowski, Perelli and Pintz do in §3 of [17] obtaining:

$$\begin{aligned}
&\sum_{N \le n \le N+H} \left| \int_{\mathfrak{m} \cap C(\mathcal{E})} S(\alpha)^2 G(\alpha) e(-n\alpha) d\alpha \right|^2 \\
\le& c(\gamma)^2 L^2 \int_{\mathfrak{m}} |S(\xi)|^2 \int_{\mathfrak{m}} |S(\alpha)|^2 |T(\xi - \alpha)| d\xi d\alpha \\
\le& c(\gamma)^2 L^2 c_1 \left(\frac{HN^2 L \log^2 P}{P^2} + \frac{HN^2 L^5}{Q} \right),
\end{aligned} \tag{1.2.40}$$

where c_1 is defined in (1.2.27).

Now from (1.2.40) we obtain :

$$\sum_{N\le n\le N+H}\left|R_1''(n)-M_1(n)+F_1(n,N,H)\right|^2 \le \frac{c(\gamma)^2c_1HN^2L^3\log^2 P}{P^2}+\frac{c(\gamma)^2c_1HN^2L^7}{Q}. \tag{1.2.41}$$

With our choice of P and Q in (1.2.28) are satisfied conditions (1.2.11), (1.2.24) and (1.2.29), then we can conclude that:

$$\sum_{N\le n\le N+H}\left|R_1''(n)-M_1(n)+F_1(n,N,H)\right|^2 \le c(\gamma)^2\frac{16c_0}{17}(1+o(1))H^{1/2}N^2L^7,$$

where we noticed that c_0, defined in (1.1.4), is $17c_1/16$.

This proves Theorem 1.1.3.

Proof of Corollary 1.1.4. Let $\Sigma_1(n,N,H)$ be as defined in (0.6.2); we want to evaluate the size of the exceptional set, that is:

$$E_1(N,H)=|\{N\le n\le N+H:\Sigma_1(n,N,H)\gg NL\}|,$$

then

$$\begin{aligned}&\sum_{N\le n\le N+H}\left|R_1''(n)-M_1(n)+F_1(n,N,H)\right|^2\\ &\ge \sum_{\substack{N\le n\le N+H\\ \Sigma_1(n,N,H)\gg NL}}\left|R_1''(n)-M_1(n)+F_1(n,N,H)\right|^2\\ &\ge \sum_{\substack{N\le n\le N+H\\ \Sigma_1(n,N,H)\gg NL}} N^2L^2 \ge N^2L^2E_1(N,H)\end{aligned}$$

and so

$$E_1(N,H)\le c(\gamma)^2\frac{16c_0}{17}(1+o(1))\frac{H^{1/2}N^2L^7}{N^2L^2}\le\frac{16c(\gamma)^2}{17}(1+o(1))f(N,H). \tag{1.2.42}$$

□

Chapter 2

Estimate of the exceptional set for the Goldbach-Linnik problem with k powers of 2

2.1 Introduction

In this Chapter we want to extend Theorem 1.1.3 when $k \geq 1$: to do this we consider $N \leq n \leq N + H$, $L = \log N$, $M = \log H \log\log H$, $c_q(-n)$ the Ramanujan sum, see §A.2, $R''_k(n)$ as in (0.5.7), $\mathfrak{S}(n)$ as in (0.5.4) and $M_k(n)$ as in (0.5.9).

We recall that we let $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_k)$, $s(\boldsymbol{\nu}) = s(\nu_1, \ldots, \nu_k) = 2^{\nu_1} + \ldots + 2^{\nu_k}$ for brevity and

$$\Sigma_k(n, N, H) = \left| R''_k(n) - M_k(n) + F_k(n, N, H) \right|, \tag{2.1.1}$$

where $F_k(n, N, H)$ will be a function that collects some of the error terms arising from major and minor arcs and from the tail of the singular series: in fact we will see that all of these are $o(NL^k)$ and $F_k(n, N, H)$ satisfies (2.2.14).

To obtain our result we will act as in Chapter 1 extending the computation of the case with a single power of 2: in fact the most difficult part of the proof is the case $k = 1$.

In this chapter we will prove the following theorem

Theorem 2.1.1. *Assume GRH, let $0 < \gamma \leq 1$ be fixed and $H = N^\gamma$, then with the same notation as in the statement of Theorem 1.1.1 we have that there exists an effectively computable constant $c(\gamma) < 1$ such that*

$$\sum_{N \leq n \leq N+H} \Sigma_k(n, N, H)^2 \leq \frac{16 c(\gamma)^{2k}}{17} (1 + o(1)) L^{2k} f(N, H) N^2,$$

where $\Sigma_0(n, N, H)$ *is defined in* (0.6.1).
In particular $c(1) = 0.7163435444776661$ *and* $c(\gamma) \to 1$ *as* $\gamma \to 0$.

The constant $c(\gamma)$ is linked with the constant $d(\beta)$ in Corollary A.6.4: in fact $c(\gamma) = d(\beta)$, with $\beta = 1 - \gamma/2$.

Corollary 2.1.2. *Assume GRH, let* $0 < \gamma \leq 1$ *be fixed and* $H = N^\gamma$, *then there exists an effectively computable constant* $c(\gamma) < 1$ *such that*

$$E'_k(N, H) \leq \frac{16c(\gamma)^{2k}}{17}(1 + o(1))f(N, H) \tag{2.1.2}$$

where $f(N, H)$ *is defined in* (1.1.4).

From this Corollary we can deduce that, adding k power of two, we have a size of the exceptional set that is smaller than Goldbach's one.

2.2 Proof of Theorem 2.1.1

To prove Theorem 2.1.1 we will use the Farey dissection of order Q described in § 1.2; we will eventually choose $Q = H^{1/2}$, see (1.2.28) . From this we can write:

$$\begin{aligned} R''_k(n) =& \int_0^1 S(\alpha)^2 G^k(\alpha) e(-n\alpha) d\alpha \\ =& \int_{\mathfrak{M}} S(\alpha)^2 G^k(\alpha) e(-n\alpha) d\alpha + \int_{\mathfrak{m}} S(\alpha)^2 G^k(\alpha) e(-n\alpha) d\alpha \\ =& R''_{\mathfrak{M}}(n) + R''_{\mathfrak{m}}(n), \end{aligned} \tag{2.2.1}$$

say.

2.2.1 Estimate on major arcs

Considering P such that $P \cdot Q \leq H$ and using (1.2.5) we can do the following computation on the major arcs:

$$\begin{aligned} R''_{\mathfrak{M}}(n) =& \sum_{q \leq P} \sum_{a=1}^{q}{}^{*} \int_{\xi_{q,a}} S\left(\frac{a}{q} + \eta\right)^2 G^k\left(\frac{a}{q} + \eta\right) e_q(-na) e(-n\eta) d\eta \\ =& \sum_{q \leq P} \frac{\mu(q)^2}{\varphi(q)^2} \sum_{a=1}^{q}{}^{*} e_q(-na) \int_{\xi_{q,a}} T(\eta)^2 G^k\left(\frac{a}{q} + \eta\right) e(-n\eta) d\eta \end{aligned}$$

$$
\begin{aligned}
&+\sum_{q\le P}\sum_{a=1}^{q}{}^{*} e_q(-na)\int_{\xi_{q,a}} G^k\left(\frac{a}{q}+\eta\right) R(\eta;q,a)^2 e(-n\eta)d\eta\\
&+2\sum_{q\le P}\frac{\mu(q)}{\varphi(q)}\sum_{a=1}^{q}{}^{*} e_q(-na)\int_{\xi_{q,a}} T(\eta)R(\eta;q,a)G^k\left(\frac{a}{q}+\eta\right) e(-n\eta)d\eta\\
&=\Sigma_1(n)+\Sigma_2(n)+\Sigma_3(n),
\end{aligned}
$$

say. $\Sigma_1(n)$ is the main term, while $\Sigma_2(n), \Sigma_3(n)$ are error terms. Now we have:

$$
\begin{aligned}
\Sigma_1(n) = \sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L}\sum_{q\le P}\frac{\mu(q)^2}{\varphi(q)^2}\sum_{a=1}^{q}{}^{*} e_q(-(n-s(\boldsymbol{\nu}))a)\cdot\\
\int_{\xi_{q,a}} T(\eta)^2 e(-(n-s(\boldsymbol{\nu}))\eta)d\eta.
\end{aligned}
$$

We can extend the integral to $[0,1]$ with an error term that is

$$
O\left(QL^k\sum_{q\le P}\frac{\mu(q)^2 q}{\varphi(q)}\right);
$$

in fact, by (1.2.6), $|T(\eta)|\le\|\eta\|^{-1}$ in the interval $[1/2qQ, 1-1/2qQ]$, so that

$$
\int_{1/2qQ}^{1-1/2qQ} |T(\eta)|^2 d\eta \ll qQ.
$$

Then $\Sigma_1(n)$ becomes

$$
\begin{aligned}
\Sigma_1(n) =& \sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L}\sum_{q\le P}\frac{\mu(q)^2}{\varphi(q)^2}\sum_{a=1}^{q}{}^{*} e_q(-(n-s(\boldsymbol{\nu}))a)\cdot\\
&\int_0^1 T(\eta)^2 e(-(n-s(\boldsymbol{\nu}))\eta)d\eta + O\left(QL^k\sum_{q\le P}\frac{\mu(q)^2 q}{\varphi(q)}\right)\\
=& \sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L}(n-s(\boldsymbol{\nu}))\sum_{q\le P}\frac{\mu(q)^2}{\varphi(q)^2}c_q(-(n-s(\boldsymbol{\nu})))\\
&+O(PQL^k), \qquad (2.2.2)
\end{aligned}
$$

where we used Lemma 2 of Goldston [8], see Lemma A.6.2, to obtain

$$
\sum_{q\le P}\frac{\mu(q)^2 q}{\varphi(q)} \ll P.
$$

Moreover by the triangle inequality

$$\begin{aligned}\Sigma_2(n) =&\Bigg| \sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L}\sum_{q\le P}\sideset{}{^*}\sum_{a=1}^{q} e_q(-(n-s(\boldsymbol{\nu}))a)\cdot \\ &\int_{\xi_{q,a}} R(\eta,q,a)^2 e(-(n-s(\boldsymbol{\nu}))\eta)d\eta\Bigg| \\ \ll& L^k\sum_{q\le P}\sideset{}{^*}\sum_{a=1}^{q}\int_{-\frac{1}{qQ}}^{\frac{1}{qQ}}|R(\eta;q,a)|^2 d\eta\end{aligned}$$

and by the Cauchy-Schwarz inequality and the trivial one (1.2.6) it follows

$$\Sigma_3(n)\ll N^{1/2}L^k\sum_{q\le P}\varphi(q)^{-1/2}\left(\sideset{}{^*}\sum_{a=1}^{q}\int_{-\frac{1}{qQ}}^{\frac{1}{qQ}}|R(\eta;q,a)|^2 d\eta\right)^{1/2}.$$

Now by (1.2.9), we have:

$$\sideset{}{^*}\sum_{a=1}^{q}\int_{-\frac{1}{qQ}}^{\frac{1}{qQ}}|R(\eta;q,a)|^2 d\eta \ll \frac{NL^4}{Q};$$

furthermore, from Lemma 2 of Goldston [8], see Lemma A.6.2, we have that

$$\sum_{q\le P}\frac{\mu(q)^2}{\varphi(q)^{1/2}}\ll P^{1/2},$$

hence we have

$$\begin{aligned}\Sigma_2(n)+\Sigma_3(n)\ll&\sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L}\sum_{q\le P}\frac{NL^4}{Q} \\ &+N^{1/2}\sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L}\sum_{q\le P}\frac{\mu(q)^2}{\varphi(q)^{1/2}}\frac{N^{1/2}L^2}{Q^{1/2}} \\ \ll&\frac{PNL^{4+k}}{Q}+\left(\frac{P}{Q}\right)^{1/2}NL^{2+k}\ll\left(\frac{P}{Q}\right)^{1/2}NL^{2+k} \qquad (2.2.3)\end{aligned}$$

provided that

$$P\ll QL^{-4}. \qquad (2.2.4)$$

Collecting (2.2.2) and (2.2.3) we have

$$\begin{aligned}
&\int_{\mathfrak{M}} S(\alpha)^2 G^k(\alpha) e(-n\alpha) d\alpha \\
&= \sum_{1\le\nu_1\le L} \cdots \sum_{1\le\nu_k\le L} (n - s(\boldsymbol{\nu})) \sum_{q\le P} \frac{\mu(q)^2}{\varphi(q)^2} c_q(-(n - s(\boldsymbol{\nu}))) \\
&\qquad + O(PQL^k) + O\left(\left(\frac{P}{Q}\right)^{1/2} NL^{2+k}\right). \qquad (2.2.5)
\end{aligned}$$

Now we complete the previous sum to obtain the singular series defined in (0.5.4), and we have

$$\begin{aligned}
\int_{\mathfrak{M}} S(\alpha)^2 G^k(\alpha) e(-n\alpha) d\alpha = M_k(n) \\
&- \sum_{1\le\nu_1\le L} \cdots \sum_{1\le\nu_k\le L} (n - s(\boldsymbol{\nu})) \\
&\quad \cdot \sum_{q>P} \frac{\mu(q)^2}{\varphi(q)^2} c_q(-(n - s(\boldsymbol{\nu}))) \\
&+ O\left(PQL^k + \left(\frac{P}{Q}\right)^{1/2} NL^{2+k}\right), \qquad (2.2.6)
\end{aligned}$$

where $M_k(n)$ is the main term while we are going to prove that the tail of the singular series is $o(NL^k)$.

2.2.2 Estimate of the tail of the singular series

We have now to estimate the tail of the singular series: we want to prove that this is $o(NL^k)$. We start with

$$\begin{aligned}
&\left| \sum_{1\le\nu_1\le L} \cdots \sum_{1\le\nu_k\le L} (n - s(\boldsymbol{\nu})) \sum_{q>P} \frac{\mu(q)^2}{\varphi(q)^2} c_q(-(n - s(\boldsymbol{\nu}))) \right| \\
&\le N \sum_{1\le\nu_1\le L} \cdots \sum_{1\le\nu_k\le L} \left| \sum_{q>P} \frac{\mu(q)^2}{\varphi(q)^2} c_q(-(n - s(\boldsymbol{\nu}))) \right|,
\end{aligned}$$

then, to reach our aim, we need the following lemma.

Lemma 2.2.1. *If $P \to +\infty$ then*

$$\sum_{1\le\nu_1\le L} \cdots \sum_{1\le\nu_k\le L} \left| \sum_{q>P} \frac{\mu(q)^2}{\varphi(q)^2} c_q(-(n - s(\boldsymbol{\nu}))) \right| = o(L^k).$$

Proof of Lemma 2.2.1. Thanks to Theorem A.2.1 we have:

$$c_q(-(n-s(\boldsymbol{\nu}))) = \mu(q_1)\frac{\varphi(q)}{\varphi(q_1)},$$

where $q_1 = q/(q, n-s(\boldsymbol{\nu}))$. Now we take $d = (q, n-s(\boldsymbol{\nu}))$ and $q = dl$ and we observe that, since q is square-free, then q_1 and d also are square-free. We also have $\varphi(dl) \leq \varphi(d)\varphi(l)$ and so:

$$\begin{aligned}\left|\sum_{q>P}\frac{\mu(q)^2}{\varphi(q)^2}c_q(-(n-s(\boldsymbol{\nu})))\right| &\leq \sideset{}{'}\sum_{d|(n-s(\boldsymbol{\nu}))}\frac{1}{\varphi(d)}\sideset{}{'}\sum_{l>P/d}\frac{1}{\varphi(l)^2} \\ &\ll \sideset{}{'}\sum_{\substack{d|(n-s(\boldsymbol{\nu}))\\ d>P}}\frac{1}{\varphi(d)} + \frac{1}{P}\sideset{}{'}\sum_{\substack{d|(n-s(\boldsymbol{\nu}))\\ d\leq P}}\frac{d}{\varphi(d)} = A_1 + A_2.\end{aligned}$$

Consider first the sum on $\nu_1, \ldots, \nu_k$ of A_2:

$$\begin{aligned}&\sum_{1\leq\nu_1\leq L}\cdots\sum_{1\leq\nu_k\leq L}\frac{1}{P}\sideset{}{'}\sum_{\substack{d|(n-s(\boldsymbol{\nu}))\\ d\leq P}}\frac{d}{\varphi(d)} = \frac{1}{P}\sideset{}{'}\sum_{d\leq P}\frac{d}{\varphi(d)}\sum_{\substack{1\leq\nu_1,\ldots,\nu_k\leq L\\ n\equiv s(\boldsymbol{\nu})(d)}}1 \\ &= \sum_{1\leq\nu_1\leq L}\cdots\sum_{1\leq\nu_{k-1}\leq L}\frac{1}{P}\sideset{}{'}\sum_{d\leq P}\frac{d}{\varphi(d)}\sum_{\substack{1\leq\nu_k\leq L\\ n-2^{\nu_1}-\ldots-2^{\nu_{k-1}}\equiv 2^{\nu_k}(d)}}1,\end{aligned}$$

where again $\sum'$ means the sum on square-free values. Now we consider the sum on $\nu_1, \ldots, \nu_k$ of A_1:

$$\begin{aligned}&\sum_{1\leq\nu_1\leq L}\cdots\sum_{1\leq\nu_k\leq L}\sideset{}{'}\sum_{\substack{d|(n-s(\boldsymbol{\nu}))\\ d>P}}\frac{1}{\varphi(d)} = \sideset{}{'}\sum_{d>P}\frac{1}{\varphi(d)}\sum_{\substack{1\leq\nu_1,\ldots,\nu_k\leq L\\ n\equiv s(\boldsymbol{\nu})(d)}}1 \\ &= \sum_{1\leq\nu_1\leq L}\cdots\sum_{1\leq\nu_{k-1}\leq L}\sideset{}{'}\sum_{d>P}\frac{1}{\varphi(d)}\sum_{\substack{1\leq\nu_k\leq L\\ n-2^{\nu_1}-\ldots-2^{\nu_{k-1}}\equiv 2^{\nu_k}(d)}}1.\end{aligned}$$

We write $m = n - 2^{\nu_1} - \ldots - 2^{\nu_{k-1}}$: then we have

$$\sum_{1\leq\nu_1\leq L}\cdots\sum_{1\leq\nu_{k-1}\leq L}\left(\sideset{}{'}\sum_{d>P}\frac{1}{\varphi(d)}\sum_{\substack{1\leq\nu_k\leq L\\ m\equiv 2^{\nu_k}(d)}}1 + \frac{1}{P}\sideset{}{'}\sum_{d\leq P}\frac{d}{\varphi(d)}\sum_{\substack{1\leq\nu_k\leq L\\ m\equiv 2^{\nu_k}(d)}}1\right)$$

$$\sum_{1\leq\nu_1\leq L}\cdots\sum_{1\leq\nu_{k-1}\leq L}\left(\sum_{1\leq\nu_k\leq L}\left|\sum_{q>P}\frac{\mu(q)^2}{\varphi(q)^2}c_q(-(m-2^{\nu_k}))\right|\right).$$

By Lemma 1.2.1, Lemma 2.2.1 is proved. □

Summing up we have:

$$\left|\sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L}(n-s(\boldsymbol{\nu}))\sum_{q>P}\frac{\mu(q)^2}{\varphi(q)^2}c_q(-(n-s(\boldsymbol{\nu})))\right|=o(NL^k). \quad (2.2.7)$$

From (2.2.1), (2.2.6) and (2.2.7) we have

$$\begin{aligned}\sum_{N\le n\le 2N}&\left|R''_k(n)-M_k(n)+F_k(n,N,H)\right|^2\\ &=\sum_{N\le n\le 2N}\left|\int_{\mathfrak{m}}S(\alpha)^2G(\alpha)^ke(-n\alpha)d\alpha\right|^2, \quad (2.2.8)\end{aligned}$$

where we recall that $F_k(n,N,H)$ collects error terms arising from (2.2.6), (2.2.7) and it will collect also a part of the error term arising from the minor arcs, which is $o(NL)$ as we will prove in §2.2.3.

2.2.3 Estimate on minor arcs

In §1.2.3 we have seen that, if we only use the Kaczorowski, Perelli and Pintz method [17], we do not have any improvement in the estimate of the size of the exceptional set with respect to the Goldbach case. In fact in this method we use the trivial estimate $|G(\alpha)|^k \le L^k$, then following the same computations in §1.2.3 we will obtain $E_k(N,H)\le f(N,H)$ that is exactly the same thing found by Kaczorowski, Perelli and Pintz in the Goldbach case.

Again, as in §1.2.3, we will follow the technique used by Pintz and Ruzsa to prove Lemma 13 of [41]. Then we choose $\mathcal{E}$ and we split the integral on the minor arcs in two parts: one on $\mathfrak{m}\cap\mathcal{E}$ and the other on $\mathfrak{m}\cap C(\mathcal{E})$. First we consider

$$\begin{aligned}\int_{\mathfrak{m}\cap\mathcal{E}}|S(\alpha)^2G(\alpha)^ke(-n\alpha)|d\alpha &\le C\int_{\mathfrak{m}\cap\mathcal{E}}\left(\frac{NL^4}{H^{1/4}M}\right)^2|G(\alpha)|^kd\alpha\\ &\le C\frac{N^2L^{8+k}}{H^{1/2}M^2}|\mathcal{E}|,\end{aligned}$$

where we use Vaughan's estimate of $S(\alpha)$ written in (1.2.31). We want that

$$\int_{\mathfrak{m}\cap\mathcal{E}}|S(\alpha)^2G(\alpha)^ke(-n\alpha)|d\alpha=o(NL^k), \quad (2.2.9)$$

then we choose the set $\mathcal{E}$ such that

$$|\mathcal{E}| = O\left(\frac{H^{1/2}M^2}{NL^8}\right). \tag{2.2.10}$$

By hypothesis $H = N^\gamma$, with $0 < \gamma \leq 1$, then (2.2.10) becomes

$$|\mathcal{E}| = O(N^{-(1-\gamma/2)}L^{-4}), \tag{2.2.11}$$

now we can use Corollary A.6.4 taking $\beta = 1-\gamma/2$, that is, that we can find an effectively computable constant $d(1-\gamma/2) = c(\gamma) < 1$ such that

$$|G(\alpha)| \leq c(\gamma)L \tag{2.2.12}$$

with $\alpha \in [0,1]\setminus\mathcal{E}$. We can find the constant $c(\gamma)$ for each value of $0 < \gamma \leq 1$ using a computer program made by Alessandro Languasco: in particular we find that $c(1) = 0.7163435444776661$ and $c(\gamma) \to 1$ as $\gamma \to 0$.

Remark 2. *Collecting* (2.2.6)*,* (2.2.7) *and* (2.2.9) *we have*

$$F_k(n,N,H) \ll PQL^k + (P/Q)^{1/2}NL^{2+k} + o(NL^k) \tag{2.2.13}$$

when $P \to +\infty$. We know that $PQL^k = o(NL^k)$, then with our choice of parameters P and Q in (1.2.28)*, that is $P = H^{1/4}M/L^2$ and $Q = H^{1/2}$, we have*

$$F_k(n,N,H) \ll H^{-1/8}N(ML^{2(k+1)})^{1/2} + o(NL^k); \tag{2.2.14}$$

furthermore we see that (2.2.4) *is satisfied for*

$$H \gg (ML^2)^4 \tag{2.2.15}$$

and this fits with our choice of H.

We now consider the integral on $\mathfrak{m}\cap C(\mathcal{E})$, then by (2.2.12) and computations in §1.2.3, we obtain:

$$\begin{aligned}
&\sum_{N\leq n\leq N+H}\left|\int_{\mathfrak{m}\cap C(\mathcal{E})} S(\alpha)^2G(\alpha)^k e(-n\alpha)d\alpha\right|^2 \\
&\leq c(\gamma)^{2k}L^{2k}\sum_{N\leq n\leq N+H}\left|\int_{\mathfrak{m}\cap C(\mathcal{E})} S(\alpha)^2 e(-n\alpha)d\alpha\right|^2 \\
&\leq c(\gamma)^{2k}L^{2k}c_1\left(\frac{HN^2L\log^2 P}{P^2} + \frac{HN^2L^5}{Q}\right),
\end{aligned} \tag{2.2.16}$$

where c_1 is defined in (1.2.27) and we notice that c_0, defined in (1.1.4), is $17c_1/16$.

Now, since our choice of P and Q in (1.2.28) satisfies conditions (2.2.4), (2.2.15) and (1.2.29), Theorem 2.1.1 is proved by (2.2.8) and (2.2.16).

From Theorem 2.1.1 we deduce $E_k(N,H) \le 16c(\gamma)^{2k}(1+o(1))f(N,H)/17$ and Corollary 2.1.2 is proved.

Chapter 3

Pintz-Ruzsa method

As we have seen in the previous chapters, we can not prove that the exceptional set is empty for $k \geq 7$ as Pintz and Ruzsa proved in their article [41]. Now we want to explain their method in order to show the differences with the one that we have used.

First we consider $r''_k(N)$, as defined in (0.5.6), for $2 \mid N$ and

$$r'_k(N) = |\{(p, \boldsymbol{\nu}) : N = p + s(\boldsymbol{\nu})\}| \quad \text{for } 2 \nmid N, \tag{3.0.1}$$

we recall that Pintz and Ruzsa proved the following Theorem:

Theorem 3.0.2 (Pintz-Ruzsa)**.** *Suppose GRH. Let k be a fixed natural number, $k \geq 7$. Then*

$$r''_k(N) > 0 \qquad \text{if } 2|N, N > N_0(k)$$

where $N_0(k)$ is an explicit constant, depending on k.

3.1 Previous results

3.1.1 Farey's dissection

As in our case they used the circle method although they acted in a different way.

Let N be a large integer and P, Q parameters satisfying

$$2 \leq P < Q \leq N. \tag{3.1.1}$$

We denote by

$$I_{q,a} = \left\{\alpha = \frac{a}{q} + \eta : \eta \in \xi_{q,a}\right\} \qquad \text{with} \qquad \xi_{q,a} \subset \left(-\frac{1}{qQ}, \frac{1}{qQ}\right)$$

and $1 \le a \le q$, $(a, q) = 1$. Then they had

$$\mathfrak{M} = \bigcup_{q \le P} \bigcup_{a=1}^{q}{}^{*} I_{q,a} \quad \text{and} \quad \mathfrak{m} = \left[\frac{1}{qQ}, 1 + \frac{1}{qQ}\right] \backslash \mathfrak{M}.$$

Furthermore we recall that also in this case they consider $S(\alpha)$ and $G(\alpha)$, defined by (0.5.11) and (0.5.12) respectively.

3.1.2 Lemmas

Before explaining the method we need some Lemmas that Pintz and Ruzsa [41] used in their computation.

Lemma 3.1.1. *Assume GRH. Let $h \neq 0$ be an even number. Choose*

$$P = \sqrt{N} L^{-8} \qquad Q = \sqrt{N}. \tag{3.1.2}$$

Then

$$\int_{\mathfrak{M}} |S^2(\alpha)| e(h\alpha) d\alpha = \mathfrak{S}(h) \frac{\max(N - |h|, 0)}{\log^2 N} + O(NL^{-3}),$$

where $\mathfrak{S}(h)$ is defined by (0.5.4).

Lemma 3.1.2 (Vaughan's estimate)**.** *Assume GRH. If P and Q satisfy* (3.1.1) *then for $\alpha \in \mathfrak{m}$ we have*

$$S(\alpha) \ll \left(\frac{N}{Q} + \sqrt{NQ} + \frac{N}{\sqrt{Q}} \right) L^2.$$

With the choice of P and Q in (3.1.2) *we obtain*

$$S(\alpha) \ll N^{3/4} L^2.$$

Lemma 3.1.3 (Chen's Theorem)**.** *Let $h \neq 0$ be any even integer, and N sufficiently large. Then the number of solutions of the equation $h = p_1 - p_2$ with $p_j \le N$ is*

$$R(h) = |\{h = p_1 - p_2 : p_j \le N, j = 1, 2\}| < C^* \cdot \mathfrak{S}(h) \frac{N}{\log^2 N},$$

where again $\mathfrak{S}(h)$ is defined by (0.5.4) *and $C^* = 3.9171$.*

We note that Hardy-Littlewood conjectured that if h is much smaller than N, then

$$R(h) \sim \mathfrak{S}(h)\frac{N}{\log^2 N} \qquad \text{as } N \to +\infty.$$

Then they give an upper bound for $|G(\alpha)|$ in the following Corollary.

Corollary 3.1.4 (Pintz-Ruzsa). *Assume GRH. We have*

$$|G(\alpha)| = \left|\sum_{\nu=0}^{L-1} e(2^\nu \alpha)\right| \leq 0.7163435444776661L = (1-\eta)L$$

if $\alpha \in [0,1]\backslash\mathcal{E}$ *where* $|\mathcal{E}| = O(N^{-1/2}L^{-100})$.

3.1.3 On the numbers of the form $p + 2^\nu$

The starting point of this method is to consider the numbers of the form $p + 2^\nu$. We consider

$$r_1(n) = |\{(p,\nu) : n = p + 2^\nu, p \leq N, 1 \leq \nu \leq L\}|,$$

these numbers determine the important function

$$\begin{aligned} s(N) = |\{(p_1, p_2, \nu_1, \nu_2) : p_1 - p_2 =& 2^{\nu_2} - 2^{\nu_1}, \\ & p_j \leq N, 1 \leq \nu_j \leq L, j = 1,2\}| \end{aligned} \tag{3.1.3}$$

through the relation

$$s(N) = \sum_n r_1^2(n) = \int_0^1 |S(\alpha)G(\alpha)|^2 d\alpha.$$

About the function in (3.1.3) they proved the following Lemma.

Lemma 3.1.5 (Lemma 11 of Pintz and Ruzsa [41]). *Assume GRH. Then*

$$s_2(N) = \int_{\mathfrak{m}} |S(\alpha)G(\alpha)|^2 d\alpha \leq \frac{2}{\log^2 2} C_2' N$$

where $C_2' < 3.9095$ *and* $s_2(N)$ *is* $s(N)$ *restricted to the minor arcs.*

3.1.4 Sum of k powers of 2

Now we consider the following function

$$r_{k,k}(m) = |\{m = 2^{\nu_1} + \cdots + 2^{\nu_k} - 2^{\mu_1} - \cdots - 2^{\mu_k} : \nu_i, \mu_j \in [1, L]\}|,$$

and we observe that the following Lemma holds.

Lemma 3.1.6.

$$r_{k,k}(0) \leq 2L^{2k-2}.$$

A crucial role will be played by the upper estimation of

$$S_1(k, N) = \int_{\mathfrak{M}} |S^2(\alpha)G^{2k}(\alpha)| d\alpha;$$

by Lemma 3.1.1 we have

$$S_1(k, N) = \sum_{m \leq N} r_{k,k}(m)\mathfrak{S}(m)\frac{N - |m|}{\log^2 N} + O(NL^{2k-3}),$$

which can be estimated from above by

$$S_1(k, N) \leq \frac{N}{\log^2 N} S(k, L) + O(NL^{2k-3}),$$

where

$$S(k, L) = \sum_{m \in \mathbb{Z} \setminus \{0\}} r_{k,k}(m)\mathfrak{S}(m).$$

We also need the following Theorem of Khalfalah and Pintz [19].

Theorem 3.1.7. *For any given $k \geq 1$ there exists the limit*

$$A(k) := \lim_{L \to +\infty} \left(\frac{S(k, L)}{2L^{2k}} - 1 \right), \qquad (3.1.4)$$

$A(k)$ decreases strictly monotonically with k, $A(k) > 2^{-2k-1}$ for every k and $\lim_{k \to +\infty} A(k) = 0$.

3.2 Proof of Theorem 3.0.2

The key point, as we will see later in this section, is the following Lemma proved by Pintz and Ruzsa [41].

Lemma 3.2.1. *Assume GRH. Let* $\eta = 0.283656$*. For* $k \geq 1$ *and any* $\delta > 0$ *there exists* $N_{k,\delta}$*, depending on* k *and* δ *only, such that for* $N \geq N_{k,\delta}$ *we have*

$$\sum_{m \leq N} (r_k'(m))^2 \leq \frac{2NL^{2k}}{\log^2 N}\{1 + A(k) + C_2'(1-\eta)^{2k-2} + \delta\},$$

where $A(k)$ *is defined by Theorem 3.1.7, and* $C_2' = 3.9095$ *by Lemma 3.1.5.*

Also they needed the following Lemma, which is Lemma 14 of Gallagher [5].

Lemma 3.2.2 (Gallagher)**.** *For* $N \to +\infty$*,*

$$\sum_{n \leq N} r_k'(n) \sim \frac{NL^k}{\log N}$$

Then the idea is to prove that $r_k''(N) > 0$. Since we use this part of the method of Pintz and Ruzsa [41], in Chapter 5, here we will recall only the principal steps of the proof without all the details, that the reader can find in that Chapter. First they used the dispersion method and Lemma 3.2.2 to have

$$\begin{aligned} E &= \sum_{\substack{n \leq N \\ 2 \nmid n}} (r_k^{'}(N) - \lambda_k)^2 \\ &\leq \sum_{\substack{n \leq N \\ 2 \nmid n}} (r_k^{'}(N))^2 - \frac{2NL^{2k}}{\log^2 N}(1 + o(1)), \end{aligned} \tag{3.2.1}$$

where λ_k is defined by (5.1.9). Then they considered $k = i + j$, they took $r_i'(m), r_j'(n)$, defined by (3.0.1), and wrote:

$$r_k^{''}(N) = \sum_{\substack{m+n=N \\ 2 \nmid m, 2 \nmid n}} r_i^{'}(m) r_j^{'}(n)$$

where $r_i^{'}(m)$ and $r_j^{'}(n)$ are written as in (5.1.10) and (5.1.11) respectively.

Then they had:

$$\begin{aligned} r''_k(N) =& \frac{4L^{i+j}}{\log^2 N} \sum_{\substack{m+n=N \\ 2\nmid m, 2\nmid n}} 1 \\ &+ 2\left\{ \frac{L^i}{\log N} \sum_{\substack{m+n=N \\ 2\nmid m, 2\nmid n}} s_j(n) + \frac{L^j}{\log N} \sum_{\substack{m+n=N \\ 2\nmid m, 2\nmid n}} s_i(m) \right\} \\ &+ \sum_{\substack{m+n=N \\ 2\nmid m, 2\nmid n}} s_i(m) s_j(n). \end{aligned} \tag{3.2.2}$$

Now the second and the third term of the previous sum are $o(NL^{k-2})$: in fact, by Lemma 3.2.2,

$$\sum_{\substack{m\leq N \\ 2\nmid m}} s_k(m) = \sum_{\substack{m\leq N \\ 2\nmid m}} r'_k(m) - \frac{2L^k}{\log N} \cdot \frac{N}{2} = o(NL^{k-1}).$$

The fourth term can be transformed applying the Cauchy-Schwarz inequality and using (3.2.1) such that they obtained:

$$\left| \sum_{\substack{m+n=N \\ 2\nmid m, 2\nmid n}} s_i(m) s_j(n) \right| \leq \left(\sum_{\substack{m\leq N \\ 2\nmid m}} (r'_i(m))^2 - \frac{2NL^{2i}}{\log^2 N} \right)^{1/2} \left(\sum_{\substack{n\leq N \\ 2\nmid n}} (r'_j(n))^2 - \frac{2NL^{2j}}{\log^2 N} \right)^{1/2}, \tag{3.2.3}$$

where the details of the steps are in (5.1.13). Now by Lemma 3.2.1 the (3.2.3) is

$$= \frac{2NL^k}{\log^2 N} \left(\{A(i) + C'_2(1-\eta)^{2i-2} + \delta\}\{A(j) + C'_2(1-\eta)^{2j-2} + \delta\} \right)^{1/2},$$

where the values of $A(i)$ and $A(j)$, defined by (3.1.4), are taken from the article of Khalfalah and Pintz [19] and δ is a small positive constant. Then since the first term in (3.2.2) is

$$\frac{2NL^k}{\log^2 N}$$

they wanted that

$$\left(\{A(i) + C'_2(1-\eta)^{2i-2} + \delta\}\{A(j) + C'_2(1-\eta)^{2j-2} + \delta\} \right)^{1/2} < 1, \tag{3.2.4}$$

such that the error term is not larger than the main term. Now inserting the numerical values of $1-\eta$ and $A(i), A(j)$ they could check that for $k \geq 7$ the Theorem is proved.

The key point of this method is the use of the estimate of $r'_k(n)$ made in Lemma 3.2.1: in fact this estimate allowed them to prove that, for $k \geq 7$,

$$r''_k(N) = (1 + O(1))\frac{2L^k N}{\log^2 N}$$

so that the exceptional set is empty.

In the previous Chapters we start our proof using the method of Kaczorowski, Perelli and Pintz [17] and then we mixed it with other two methods: in fact we use the one of Languasco, Pintz and Zaccagnini [24] to estimate the tail of the singular series and we use a part of the method of Pintz and Ruzsa [41] to split the integral on the minor arcs. Using this methods we prove that the exceptional set for the Goldbach-Linnik problem, under GRH, is smaller than that of Goldbach, but we can not prove that it is empty for $k \geq 7$, as Pintz and Ruzsa do. The only way to reach the same result is to act exactly in the same way of Pintz and Ruzsa: in fact the key point of their method, as we have described in this Chapter, is the use of $r'_k(N)$.

Chapter 4

Upper bound for the size of the exceptional set for the Goldbach-Linnik problem with k powers of 2 in long intervals

4.1 Introduction

Here we want to estimate the size of the exceptional set of Goldbach-Linnik using a different technique involving series instead of finite sums this is a technique introduced by Hardy and Littlewood in the twenties.

We consider the case of long intervals; we let $N \to +\infty$, $L = \log_2 N$ and we replace $S(\alpha)$, see (0.5.11), with

$$\tilde{S}(\alpha) = \sum_{n=1}^{\infty} \Lambda(n) e^{-n/N} e(n\alpha). \tag{4.1.1}$$

Then we use the technique described in the articles of Languasco and Perelli [22] and of Languasco [21]; in fact using this method they improved the estimate in Lemma 1 of Kaczorowski, Perelli and Pintz [17], that is Lemma A.6.1. The idea is that using $\tilde{S}(\alpha)$ instead of $S(\alpha)$ they could introduce Saffari-Vaughan's [42] technique into the machinery of the circle method and also they could avoid the use of Parseval's identity in a critical part of the unit interval. Then Languasco and Perelli [22] proved

Theorem 4.1.1 (Languasco-Perelli)**.** *Assume RH and let $z = 1/N - 2\pi i\eta$. For N sufficiently large and $0 \le \xi \le 1/2$ we have*

$$\int_{-\xi}^{\xi} \left| \tilde{S}(\eta)^2 - \frac{1}{z} \right| d\eta \ll N\xi L^2 + N\xi^{1/2} L.$$

And Languasco [21] proved

Lemma 4.1.2 (Languasco)**.** *Assume GRH and let* $\alpha = a/q + \eta, \eta \in \xi_{q,a}$ *and* $z = 1/N - 2\pi i\eta$. *Then*

$$\left|\tilde{S}(\alpha) - \frac{\mu(q)}{\varphi(q)}\frac{1}{z}\right| \ll (N(q|\eta|^{1/2}) + (qN)^{1/2}) \log qN.$$

Lemma 4.1.3 (Languasco)**.** *Assume GRH and let* $z = 1/N - 2\pi i\eta$. *Then*

$$\sum_{\substack{1\le a\le q\\(a,q)=1}} \int_{-1/qQ}^{1/qQ} \left|\tilde{S}\left(\frac{a}{q}+\eta\right) - \frac{\mu(q)}{\varphi(q)}\frac{1}{z}\right|^2 d\eta \ll \frac{N}{Q}(\log qN)^2.$$

Using this results Languasco and Perelli in [23] proved the analogue of Lemma 1 of Kaczorowski, Perelli and Pintz, see Lemma A.6.1, that is

Lemma 4.1.4. *Assume GRH. Then we have*

$$\int_{-1/qQ}^{1/qQ} \left|\sum_{n=1}^{\infty} \Lambda(n)\chi(n)e^{n/N}e(n\eta) - \frac{\delta_\chi}{z}\right|^2 d\eta \ll \frac{NL^2}{qQ},$$

uniformly for $\chi(\mathrm{mod} q)$ *with* $q \le Q \le N$.

In our case we need also to replace $G(\alpha)$, see (0.5.12), with

$$\tilde{G}(\alpha) = \sum_{1\le\nu\le L} e^{-2^\nu/N} e(2^\nu\alpha); \tag{4.1.2}$$

here is not necessary to take the series because after computations we see that for our aim is enough the finite sum. Also we have to observe that

$$\left|\tilde{G}(\alpha) - G(\alpha)\right| = O(1); \tag{4.1.3}$$

in fact we have

$$\begin{aligned}\left|\tilde{G}(\alpha) - G(\alpha)\right| &= \left|\sum_{1\le\nu\le L} e^{-2^\nu/N}e(2^\nu\alpha) - \sum_{1\le\nu\le L} e(2^\nu\alpha)\right| \\ &\le \sum_{1\le\nu\le L} \left|e^{-2^\nu/N} - 1\right| \\ &\le \sum_{1\le\nu\le L} \frac{2^\nu}{N} \le \frac{2^{L+1}-1}{N} \ll 1\end{aligned}$$

for $N \to +\infty$. Now by (0.5.15) and (0.5.16) we have that $R_1''(n)$ becomes

$$\tilde{R}_1''(n) = e^{-n/N} R_1''(n) = \int_0^1 \tilde{S}(\alpha)^2 \tilde{G}(\alpha) e(-n\alpha) d\alpha. \tag{4.1.4}$$

Consider

$$\tilde{\Sigma}_0(n, N) = \left| \tilde{R}''(n) - e^{-n/N} n\mathfrak{S}(n) + \tilde{F}_0(n, N) \right|, \tag{4.1.5}$$

where

$$\tilde{F}_0(n, N) \ll N^{7/8} L^{1/2} \log\log N \tag{4.1.6}$$

and

$$\tilde{\Sigma}_1(n, N) = \left| \tilde{R}_1''(n) - e^{-n/N} M_1(n) + \tilde{F}_1(n, N) \right|, \tag{4.1.7}$$

where $\tilde{F}_1(n, N)$ will be a function that collects some of the error terms arising from major and minor arcs and from the tail of the singular series: in fact we will see that they are $o(NL)$ and $\tilde{F}_1(n, N)$ satisfies (4.2.12).

Now let

$$\tilde{f}(N) = c_2(1 + o(1)) N^{1/2} L^3, \tag{4.1.8}$$

where c_2 is a real positive constant that satisfies (4.2.18). Then from Lemma 4.1.4 we have the analogue of Theorem 1.1.1, that is

Theorem 4.1.5. *Assume GRH. Then*

$$\sum_{N \le n \le 2N} \tilde{\Sigma}_0(n, N)^2 \le \tilde{f}(N) N^2,$$

where $\tilde{f}(N)$ is defined by (4.1.8).

And from this we have

Corollary 4.1.6. *Assume GRH, then*

$$\tilde{E}(N) \le \tilde{f}(N).$$

Then we will prove the following Theorem in the case of Goldbach-Linnik problem.

Theorem 4.1.7. *Assume GRH. With the same notation as in the statement of Theorem 4.1.5 we have*

$$\sum_{N\le n\le 2N} \tilde{\Sigma}_1(n,N)^2 \le c^2 L^2 \tilde{f}(N) N^2,$$

with $c = 0.7163435444776661$.

From this Theorem we obtain the following result:

Corollary 4.1.8. *Assume GRH, then:*

$$\tilde{E}_1(N) \le c^2 \tilde{f}(N), \tag{4.1.9}$$

where $\tilde{f}(N)$ *is defined in* (4.1.8).

4.2 Proof of Theorem 4.1.7

To prove Theorem 4.1.7, we consider the Farey dissection of order Q of $I = [1/Q, 1+1/Q]$, as described in §A.4; we will eventually choose $Q = N^{1/2}$, see (4.2.11). We call the arc relative to a/q

$$I_{q,a} = \left\{\alpha = \frac{a}{q} + \eta : \eta \in \xi_{q,a}\right\} \quad \text{with} \quad \xi_{q,a} \subset \left(-\frac{1}{qQ}, \frac{1}{qQ}\right).$$

We recall that

$$\left(-\frac{1}{2qQ}, \frac{1}{2qQ}\right) \subset \xi_{q,a} \subset \left(-\frac{1}{qQ}, \frac{1}{qQ}\right);$$

we let

$$\mathfrak{M} = \bigcup_{q\le P} \bigcup_{a=1}^{q}{}^{*} I_{q,a} \tag{4.2.1}$$

be the so-called major arcs and

$$\mathfrak{m} = I \backslash \mathfrak{M} \tag{4.2.2}$$

the minor ones.

Now (4.1.4) becomes

$$\begin{aligned}\int_0^1 \tilde{S}(\alpha)^2 \tilde{G}(\alpha) e(-n\alpha) d\alpha =& \int_{\mathfrak{M}} \tilde{S}(\alpha)^2 \tilde{G}(\alpha) e(-n\alpha) d\alpha \\ &+ \int_{\mathfrak{m}} \tilde{S}(\alpha)^2 \tilde{G}(\alpha) e(-n\alpha) d\alpha \\ =& \tilde{R}''_{\mathfrak{M}}(n) + \tilde{R}''_{\mathfrak{m}}(n),\end{aligned}$$

say.

4.2.1 Major arcs

Consider P such that $P \cdot Q \leq N$, $\alpha = a/q + \eta$ and $z = 1/N - 2\pi i \eta$, hence we can write the analogous of (1.2.5), that is

$$\tilde{S}\left(\frac{a}{q} + \eta\right) = \frac{\mu(q)}{\varphi(q)} \frac{1}{z} + \tilde{R}(\eta; q, a), \tag{4.2.3}$$

where $\tilde{R}(\eta; q, a)$ is the error term arising from the approximation and we know from Lemma 4.1.2 that it is "small". Then we can do the following computations on the major arcs:

$$\begin{aligned}
\tilde{R}''_{\mathfrak{M}}(n) =& \sum_{q \leq P} \sideset{}{^*}\sum_{a=1}^{q} \int_{\xi_{q,a}} \tilde{S}\left(\frac{a}{q} + \eta\right)^2 \tilde{G}\left(\frac{a}{q} + \eta\right) e_q(-na) e(-n\eta) d\eta \\
=& \sum_{q \leq P} \sideset{}{^*}\sum_{a=1}^{q} \sum_{1 \leq \nu \leq L} e^{-2^\nu/N} e_q(-(n - 2^\nu)a) \\
& \cdot \int_{\xi_{q,a}} \left(\frac{\mu(q)}{\varphi(q)} \frac{1}{z} + \tilde{R}(\eta; q, a)\right)^2 e(-(n - 2^\nu)\eta) d\eta \\
=& \sum_{1 \leq \nu \leq L} e^{-2^\nu/N} \sum_{q \leq P} \sideset{}{^*}\sum_{a=1}^{q} e_q(-(n - 2^\nu)a) \int_{\xi_{q,a}} \frac{\mu(q)^2}{\varphi(q)^2} \frac{1}{z^2} e(-(n - 2^\nu)\eta) d\eta \\
& + 2 \sum_{1 \leq \nu \leq L} e^{-2^\nu/N} \sum_{q \leq P} \sideset{}{^*}\sum_{a=1}^{q} e_q(-(n - 2^\nu)a) \\
& \cdot \int_{\xi_{q,a}} \frac{\mu(q)}{\varphi(q)} \frac{1}{z} \tilde{R}(\eta; q, a) e(-(n - 2^\nu)\eta) d\eta \\
& + \sum_{1 \leq \nu \leq L} e^{-2^\nu/N} \sum_{q \leq P} \sideset{}{^*}\sum_{a=1}^{q} e_q(-(n - 2^\nu)a) \\
& \cdot \int_{\xi_{q,a}} \tilde{R}(\eta; q, a)^2 e(-(n - 2^\nu)\eta) d\eta \\
=& \tilde{\Sigma}_1(n) + \tilde{\Sigma}_2(n) + \tilde{\Sigma}_3(n);
\end{aligned}$$

say. Here $\tilde{\Sigma}_1(n)$ is the main term, while $\tilde{\Sigma}_2(n)$ and $\tilde{\Sigma}_3(n)$ are the error terms. Now from Lemma 2 of Languasco [21], that is Lemma 4.1.3, and from Section 5 of Languasco and Perelli [23] we have

$$\tilde{\Sigma}_3(n) \ll \sum_{1 \leq \nu \leq L} e^{-2^\nu/N} \sum_{q \leq P} \sideset{}{^*}\sum_{a=1}^{q} \int_{\xi_{q,a}} |\tilde{R}(\eta; q, a)|^2 d\eta$$

$$
\begin{aligned}
&\ll \sum_{1\le\nu\le L} e^{-2^\nu/N} \sum_{q\le P} \frac{N}{Q}(\log qN)^2 \\
&\ll \frac{N}{Q} P L^3. && (4.2.4)
\end{aligned}
$$

Furthermore by the Cauchy-Schwarz inequality we obtain

$$
\begin{aligned}
\tilde{\Sigma}_2(n) &\ll \sum_{1\le\nu\le L} e^{-2^\nu/N} \sum_{q\le P} \sum_{a=1}^{q}{}^{*} \int_{\xi_{q,a}} \left|\frac{\mu(q)}{\varphi(q)}\frac{1}{z}\right| \cdot |\tilde{R}(\eta;q,a)| d\eta \\
&\ll \sum_{1\le\nu\le L} e^{-2^\nu/N} \sum_{q\le P} \sum_{a=1}^{q}{}^{*} \left(\int_{\xi_{q,a}} \left|\frac{\mu(q)}{\varphi(q)}\frac{1}{z}\right|^2 d\eta\right)^{1/2} \\
&\qquad \cdot \left(\int_{\xi_{q,a}} |\tilde{R}(\eta;q,a)|^2 d\eta\right)^{1/2} \\
&\ll \sum_{1\le\nu\le L} e^{-2^\nu/N} \sum_{q\le P} \left(\sum_{a=1}^{q}{}^{*} \int_{\xi_{q,a}} \left|\frac{\mu(q)}{\varphi(q)}\frac{1}{z}\right|^2 d\eta\right)^{1/2} \\
&\qquad \cdot \left(\sum_{a=1}^{q}{}^{*} \int_{\xi_{q,a}} |\tilde{R}(\eta;q,a)|^2 d\eta\right)^{1/2}.
\end{aligned}
$$

We also know that

$$
|z|^{-1} \ll \min\left(N, |\eta|^{-1}\right) \qquad (4.2.5)
$$

and again by Lemma 4.1.3

$$
\begin{aligned}
\tilde{\Sigma}_2(n) &\ll \sum_{1\le\nu\le L} e^{-2^\nu/N} \sum_{q\le P} \frac{\mu(q)^2 N^{1/2}}{\varphi(q)^{1/2}} \left(\frac{N}{Q}L^2\right)^{1/2} \\
&\ll NL^2 \left(\frac{P}{Q}\right)^{1/2}, && (4.2.6)
\end{aligned}
$$

where the last result comes from Lemma A.6.2. Now we consider $\tilde{\Sigma}_1(n)$ and we have

$$
\tilde{\Sigma}_1(n) = \sum_{1\le\nu\le L} e^{-2^\nu/N} \sum_{q\le P} \frac{\mu(q)^2}{\varphi(q)^2} \sum_{a=1}^{q}{}^{*} e_q(-(n-2^\nu)a) \int_{\xi_{q,a}} \frac{1}{z^2} e(-(n-2^\nu)\eta) d\eta.
$$

By Lemma 5 of Languasco [21]

$$
\int_{\xi_{q,a}} \frac{1}{z^2} e(-(n-2^\nu)\eta) d\eta = e^{-(n-2^\nu)/N}(n-2^\nu) + O(qQ), \qquad (4.2.7)
$$

so that

$$
\begin{aligned}
\tilde{\Sigma}_1(n) &= \sum_{1\le\nu\le L} e^{-2^\nu/N} \sum_{q\le P} \frac{\mu(q)^2}{\varphi(q)^2} c_q(-(n-2^\nu)) \left(e^{-(n-2^\nu)/N}(n-2^\nu) + O(qQ)\right) \\
&= \sum_{1\le\nu\le L} e^{-n/N}(n-2^\nu) \sum_{q\le P} \frac{\mu(q)^2}{\varphi(q)^2} c_q(-(n-2^\nu)) \\
&\quad + O\left(\sum_{1\le\nu\le L} e^{-2^\nu/N} \sum_{q\le P} \frac{\mu(q)^2 qQ}{\varphi(q)} \right),
\end{aligned}
$$

then from Lemma A.6.2 this becomes

$$
= \sum_{1\le\nu\le L} e^{-n/N}(n-2^\nu) \sum_{q\le P} \frac{\mu(q)^2}{\varphi(q)^2} c_q(-(n-2^\nu)) + O(PQL).
$$

Summing up, we have

$$
\begin{aligned}
\int_{\mathfrak{M}} \tilde{S}(\alpha)^2 \tilde{G}(\alpha) e(-n\alpha)\, d\alpha = & e^{-n/N} M_1(n) \\
& - e^{-n/N} \sum_{1\le\nu\le L} (n-2^\nu) \sum_{q>P} \frac{\mu(q)^2}{\varphi(q)^2} c_q(-(n-2^\nu)) \\
& + O\left(PQL + \left(\frac{P}{Q}\right)^{1/2} NL^2 + \left(\frac{P}{Q}\right) NL^3 \right).
\end{aligned}
$$

Now we want that $(P/Q)^{1/2}NL^2$ and $(P/Q)NL^3$ are of the same order, that happens when

$$
P \ll QL^{-2}. \tag{4.2.8}
$$

Then we obtain

$$
\begin{aligned}
\int_{\mathfrak{M}} \tilde{S}(\alpha)^2 \tilde{G}(\alpha) e(-n\alpha)\, d\alpha = & e^{-n/N} M_1(n) \\
& - e^{-n/N} \sum_{1\le\nu\le L} (n-2^\nu) \sum_{q>P} \frac{\mu(q)^2}{\varphi(q)^2} c_q(-(n-2^\nu)) \\
& + O\left(PQL + \left(\frac{P}{Q}\right)^{1/2} NL^2 \right)
\end{aligned} \tag{4.2.9}
$$

Now the tail of the singular series can be estimated by means the computations that we made in Chapter 1, §1.2.2. Then from Lemma 1.2.1 we have that the tail of the singular series is $o(NL)$.

4.2.2 Minor arcs

Now we want to estimate the integral on the minor arcs, to do this we try to use the method that Pintz and Ruzsa used to prove Lemma 13 of [41]. First we need an upper bound for $\tilde{S}(a/q+\eta)$, see (4.2.3), from Lemma 1 of Languasco [21], see Lemma 4.1.2, we have

$$|\tilde{R}(\eta;q,a)| \ll (N(q|\eta|)^{1/2} + (qN)^{1/2}) \log (qN),$$

furthermore from (6) of Languasco [21]

$$\left|\frac{\mu(q)}{\varphi(q)}\frac{1}{z}\right| \ll \min(N,|\eta|^{-1})\frac{\log\log q}{q} \ll \frac{N}{q}\log\log q,$$

then we have

$$\left|\tilde{S}\left(\frac{a}{q}+\eta\right)\right| \ll \frac{N}{q}\log\log q + (N(q|\eta|)^{1/2} + (qN)^{1/2}) \log (qN).$$

Since we are on minor arcs $P \le q \le Q$ and $1/q \le 1/P$, furthermore $|\eta| \le 1/(qQ)$ then we have

$$\begin{aligned}\left|\tilde{S}\left(\frac{a}{q}+\eta\right)\right| &\ll \frac{N}{P}\log\log P + \left(\frac{N}{Q^{1/2}} + (QN)^{1/2}\right)\log (QN) \\ &\ll N^{3/4}L \end{aligned} \tag{4.2.10}$$

with

$$Q = N^{1/2} \qquad \text{and} \qquad P = Q^{1/2}L^{-1}(\log\log N)^2. \tag{4.2.11}$$

Now we can use the method of Pintz and Ruzsa [41] then we consider a set $\mathcal{E}$ such that $|G(\alpha)| \le L$, for $\alpha \in \mathcal{E}$, and $|G(\alpha)| \le cL$, for $\alpha \in C(\mathcal{E}) = [0,1]\backslash\mathcal{E}$ and $c = 0.7163435444776661$.

First we have

$$\begin{aligned}\int_{\mathfrak{m}\cap\mathcal{E}} |\tilde{S}(\alpha)|^2|\tilde{G}(\alpha)|d\alpha \le& L\sum_{P<q\le Q}\sideset{}{^*}\sum_{a=1}^{q}\int_{[-1/qQ,1/qQ]\cap\mathcal{E}}\left|\tilde{S}\left(\frac{a}{q}+\eta\right)\right|^2 d\eta \\ \le& CN^{3/2}L^3\int_{\mathfrak{m}\cap\mathcal{E}} d\alpha \\ =& C|\mathcal{E}|N^{3/2}L^3,\end{aligned}$$

where C is the real positive constant hidden in (4.2.10). We want that this is $o(NL)$, then $|\mathcal{E}|$ has to be $O(N^{-1/2}L^{-2})$, this, together with (4.1.3), gives us the conditions to use Corollary 1 of Pintz and Ruzsa, see Corollary A.6.4.

Remark 3. *Collecting* (4.2.9)*, the estimate of the tail of the singular series and the estimate of the integral on the minor arcs, where* $|G(\alpha)| \leq L$*, we have*

$$\tilde{F}_1(n,N) \ll PQL + \left(\frac{P}{Q}\right)^{1/2} NL^2 + o(NL)$$

when $P \to +\infty$*. We know that* $PQL = o(NL)$*, then with our choice of parameters* P *and* Q *in* (4.2.11)*, that is* $Q = N^{1/2}$ *and* $P = N^{1/4}L^{-1}(\log\log N)^2$*, we have*

$$\tilde{F}_1(n,N) \ll N^{7/8}L^{3/2}\log\log N + o(NL). \tag{4.2.12}$$

Now consider

$$t(s) = \frac{1}{N}\max(N - |s|, 0)$$

and as in (1.2.20), by the properties of the Fejér kernel

$$K(\alpha) = \sum_{s=-\infty}^{+\infty} t(s)e(-s\alpha) \ll \frac{1}{N}\min\left(N^2, \frac{1}{\|\alpha\|^2}\right). \tag{4.2.13}$$

Then

$$\begin{aligned}
&\sum_{n=N}^{2N}\left|\int_{\mathfrak{m}\cap C(\mathcal{E})} \tilde{S}(\alpha)^2\tilde{G}(\alpha)e(-n\alpha)d\alpha\right|^2\\
\leq&\sum_n t(n-N)\left(\int_{\mathfrak{m}\cap C(\mathcal{E})} \overline{\tilde{S}(\xi)}^2\overline{\tilde{G}(\xi)}e(n\xi)d\xi\right)\\
&\quad\cdot\left(\int_{\mathfrak{m}\cap C(\mathcal{E})} \tilde{S}(\alpha)^2\tilde{G}(\alpha)e(-n\alpha)d\alpha\right)\\
=&\int_{\mathfrak{m}\cap C(\mathcal{E})} \overline{\tilde{S}(\xi)}^2 \int_{\mathfrak{m}\cap C(\mathcal{E})} \tilde{S}(\alpha)^2\overline{\tilde{G}(\xi)}\tilde{G}(\alpha)\sum_n t(n-N)e(-n(\alpha-\xi))d\xi d\alpha\\
=&\int_{\mathfrak{m}\cap C(\mathcal{E})} \overline{\tilde{S}(\xi)}^2 \int_{\mathfrak{m}\cap C(\mathcal{E})} \tilde{S}(\alpha)^2\overline{\tilde{G}(\xi)}\tilde{G}(\alpha)K(\alpha-\xi)d\xi d\alpha\\
\leq& c^2L^2\int_{\mathfrak{m}\cap C(\mathcal{E})} |\tilde{S}(\xi)|^2 \int_{\mathfrak{m}\cap C(\mathcal{E})} |\tilde{S}(\alpha)|^2|K(\alpha-\xi)|d\alpha d\xi
\end{aligned}$$

Now we can compute as in Kaczorowski, Perelli and Pintz [17] to estimate

$$\int_{\mathfrak{m}\cap C(\mathcal{E})} |\tilde{S}(\xi)|^2 \int_{\mathfrak{m}\cap C(\mathcal{E})} |\tilde{S}(\alpha)|^2|K(\alpha-\xi)|d\alpha d\xi. \tag{4.2.14}$$

We recall that, by the Prime Number Theorem

$$\int_0^1 |\tilde{S}(\alpha)|^2 d\alpha \ll NL \tag{4.2.15}$$

and we use this to give an estimate of $\int_{\mathfrak{m}} |\tilde{S}(\xi)|^2 d\xi$, then (4.2.14) becomes

$$\ll NL \max_{\xi\in[0,1]} \int_{\mathfrak{m}\cap C(\mathcal{E})} |\tilde{S}(\alpha)|^2 |K(\alpha-\xi)| d\alpha. \tag{4.2.16}$$

Now for $\alpha \in \{\xi - 1/N, \xi + 1/N\}$ we have that $\min\{N^2, 1/\|\xi-\alpha\|^2\} = N^2$, furthermore we can split the interval $[0,1]$ into $N/2$ subintervals so that we have

$$\alpha \in \bigcup_{t=-N}^{N} \left(\xi - \frac{t}{N} - \frac{1}{N}, \xi - \frac{t}{N} + \frac{1}{N}\right)$$

that is the union of all the intervals of length $2/N$ translated toward left and right from ξ, where $\xi \in [0,1]$. If α belongs to the t-th interval, then $1/\|\xi-\alpha\|^2 \le N^2/(t^2+1)$ and

$$\min\left\{N^2, 1/\|\xi-\alpha\|^2\right\} \le \frac{N^2}{t^2+1}$$

and (4.2.16) becomes

$$\begin{aligned}
&\ll NL \max_{\xi\in[0,1]} \sum_{t=-N}^{N} \frac{N}{t^2+1} \int_{(\xi-t/N-(1/N),\xi-t/N+(1/N))\cap\mathfrak{m}} |\tilde{S}(\alpha)|^2 d\alpha \\
&\ll N^2 L \max_{\xi\in[0,1]} \int_{(\xi-(1/N),\xi+(1/N))\cap\mathfrak{m}} |\tilde{S}(\alpha)|^2 d\alpha \\
&\ll N^2 L \max_{\substack{P<q\le Q\\(a,q)=1}} \int_{-1/qQ}^{1/qQ} \left|\tilde{S}\left(\frac{a}{q}+\eta\right)\right|^2 d\eta.
\end{aligned}$$

Now the fact that $\alpha \in \mathfrak{m}$ allows us to say that $1/N \le 1/qQ$, that is

$$Q^2 \le N. \tag{4.2.17}$$

We proceed with the computation and we have

$$N^2 L \max_{\substack{P<q\le Q\\(a,q)=1}} \int_{-1/qQ}^{1/qQ} \left|\tilde{S}\left(\frac{a}{q}+\eta\right)\right|^2 d\eta$$

$$\ll N^2 L \max_{\substack{P<q\le Q\\(a,q)=1}} \left(\int_{-1/qQ}^{1/qQ} \left| \frac{\mu(q)}{\varphi(q)} \frac{1}{z} \right|^2 d\eta + \int_{-1/qQ}^{1/qQ} \left| \tilde{R}(\eta; q, a) \right|^2 d\eta \right),$$

from Lemma 4.1.3 and estimate in page 156 of Languasco [21], we have

$$\ll N^2 L \max_{\substack{P<q\le Q\\(a,q)=1}} \left(\frac{(\log\log q)^2}{q^2} \frac{N}{\pi} \arctan \frac{2\pi N}{qQ} + \frac{N}{Q} (\log qQ)^2 \right)$$
$$\ll \frac{N^3 L (\log\log P)^2}{P^2} + \frac{N^3 L^3}{Q}.$$

Now our choice of Q and P in (4.2.11) satisfies conditions (4.2.8) and (4.2.17), then calling c_2 the implicit constant in the $O(.)$ notation, we have

$$\int_{\mathfrak{m}\cap C(\mathcal{E})} |\tilde{S}(\xi)|^2 \int_{\mathfrak{m}\cap C(\mathcal{E})} |\tilde{S}(\alpha)|^2 |K(\alpha - \xi)| d\alpha d\xi \le c_2 (1 + o(1)) N^{5/2} L^3. \tag{4.2.18}$$

Then we have

$$\sum_{n=N}^{2N} \left| \int_{\mathfrak{m}\cap C(\mathcal{E})} \tilde{S}(\alpha)^2 \tilde{G}(\alpha) e(-n\alpha) d\alpha \right|^2$$
$$\le c^2 L^2 \sum_{n=N}^{2N} \left| \int_{\mathfrak{m}\cap C(\mathcal{E})} \tilde{S}(\alpha)^2 e(-n\alpha) d\alpha \right|^2 \tag{4.2.19}$$

and from this

$$\sum_{n=N}^{2N} \left| \tilde{\Sigma}_1(n, N) \right|^2 \le c^2 L^2 \tilde{f}(N) N^2.$$

This proves Theorem 4.1.7 and Corollary 4.1.8 follows.

4.3 Extension to the case with k powers of 2.

Now we can extend our result to the case with k powers of 2: we recall that we call $\boldsymbol{\nu} = (\nu_1, \dots, \nu_k)$ and $s(\boldsymbol{\nu}) = 2^{\nu_1} + \cdots + 2^{\nu_k}$ and we consider

$$\tilde{\Sigma}_k(n, N) = \left| \tilde{R}_k''(n) - e^{-n/N} M_k(n) + \tilde{F}_k(n, N) \right|, \tag{4.3.1}$$

where $\tilde{R}_k''(n)$ is defined in (0.5.17) and $\tilde{F}_k(n, N)$ will be a function that collects some of the error terms arising from major and minor arcs and from the tail of the singular series: in fact we will see that they are $o(NL^k)$ and $\tilde{F}_k(n, N)$ satisfies (4.3.8). Then we prove the following Theorem.

Theorem 4.3.1. *Assume GRH. With the same notation as in the statement of Theorem 4.1.5 we have*

$$\sum_{N\le n\le 2N} \tilde{\Sigma}_k(n,N)^2 \le c^{2k}L^{2k}\tilde{f}(N)N^2,$$

with $c = 0.7163435444776661$.

From this Theorem we obtain the following result:

Corollary 4.3.2. *Assume GRH, then:*

$$\tilde{E}_k(N) \le c^{2k}\tilde{f}(N), \tag{4.3.2}$$

where $\tilde{f}(N)$ *is defined in* (4.1.8).

Proof of Theorem 4.3.1. We follow the proof of Theorem 4.1.7, first we consider the Farey dissection of order Q of $[1/Q, 1+1/Q]$, as described in §A.4; we will eventually choose $Q = N^{1/2}$, see (4.2.11). Then we have

$$\begin{aligned}
R''_k(n) =& \int_0^1 \tilde{S}(\alpha)^2\tilde{G}(\alpha)^k e(-n\alpha)d\alpha \\
=& \int_{\mathfrak{M}} \tilde{S}(\alpha)^2\tilde{G}(\alpha)^k e(-n\alpha)d\alpha + \int_{\mathfrak{m}} \tilde{S}(\alpha)^2\tilde{G}(\alpha)^k e(-n\alpha)d\alpha \\
=& \tilde{R}''_{\mathfrak{M}}(n) + \tilde{R}''_{\mathfrak{m}}(n),
\end{aligned}$$

say.

Now we choose P such that $P\cdot Q \le N$ and we consider (4.2.3), then we can do the following computations on the major arcs

$$\begin{aligned}
\tilde{R}''_{\mathfrak{M}}(n) =& \sum_{q\le P}\sideset{}{^*}\sum_{a=1}^{q} \int_{\xi_{q,a}} \tilde{S}\left(\frac{a}{q}+\eta\right)^2 \tilde{G}\left(\frac{a}{q}+\eta\right)^k e_q(-na)e(-n\eta)d\eta \\
=& \sum_{q\le P}\sideset{}{^*}\sum_{a=1}^{q} \sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_1\le L} e^{-s(\boldsymbol{\nu})/N} e_q(-(n-s(\boldsymbol{\nu}))a) \\
&\cdot \int_{\xi_{q,a}} \left(\frac{\mu(q)}{\varphi(q)}\frac{1}{z} + \tilde{R}(\eta;q,a)\right)^2 e(-(n-s(\boldsymbol{\nu}))\eta)d\eta \\
=& \sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L} e^{-s(\boldsymbol{\nu})/N} \sum_{q\le P}\sideset{}{^*}\sum_{a=1}^{q} e_q(-(n-s(\boldsymbol{\nu}))a) \\
&\cdot \int_{\xi_{q,a}} \frac{\mu(q)^2}{\varphi(q)^2}\frac{1}{z^2} e(-(n-s(\boldsymbol{\nu}))\eta)d\eta
\end{aligned}$$

$$
\begin{aligned}
&+2\sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L} e^{-s(\boldsymbol{\nu})/N}\sum_{q\le P}\sideset{}{^*}\sum_{a=1}^{q} e_q(-(n-s(\boldsymbol{\nu}))a)\\
&\cdot\int_{\xi_{q,a}}\frac{\mu(q)}{\varphi(q)}\frac{1}{z}\tilde{R}(\eta;q,a)e(-(n-s(\boldsymbol{\nu}))\eta)d\eta\\
&+\sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L} e^{-s(\boldsymbol{\nu})/N}\sum_{q\le P}\sideset{}{^*}\sum_{a=1}^{q} e_q(-(n-s(\boldsymbol{\nu}))a)\\
&\cdot\int_{\xi_{q,a}}\tilde{R}(\eta;q,a)^2 e(-(n-s(\boldsymbol{\nu}))\eta)d\eta\\
=&\tilde{\Sigma}_1(n)+\tilde{\Sigma}_2(n)+\tilde{\Sigma}_3(n);
\end{aligned}
$$

say. Here again $\tilde{\Sigma}_1(n)$ is the main term, while $\tilde{\Sigma}_2(n)$ and $\tilde{\Sigma}_3(n)$ are the error terms. Now from Lemma 4.1.3 and (4.2.4) we have

$$
\tilde{\Sigma}_3(n)\ll\frac{N}{Q}PL^{2+k}; \tag{4.3.3}
$$

furthermore from Cauchy-Schwarz inequality and (4.2.6)

$$
\tilde{\Sigma}_2(n)\ll NL^{1+k}\left(\frac{P}{Q}\right)^{1/2}. \tag{4.3.4}
$$

Now from (4.2.7) we have

$$
\begin{aligned}
\tilde{\Sigma}_1(n)=&\sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L} e^{-s(\boldsymbol{\nu})/N}\sum_{q\le P}\sideset{}{^*}\sum_{a=1}^{q} e_q(-(n-s(\boldsymbol{\nu}))a)\\
&\cdot\int_{\xi_{q,a}}\frac{\mu(q)^2}{\varphi(q)^2}\frac{1}{z^2}e(-(n-s(\boldsymbol{\nu}))\eta)d\eta\\
=&\sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L} e^{-n/N}(n-s(\boldsymbol{\nu}))\sum_{q\le P}\frac{\mu(q)^2}{\varphi(q)^2}c_q(-(n-s(\boldsymbol{\nu})))\\
&+O\left(\sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L} e^{-s(\boldsymbol{\nu})/N}\sum_{q\le P}\frac{\mu(q)^2 qQ}{\varphi(q)}\right).
\end{aligned}
$$

From Lemma A.6.2 we obtain

$$
\begin{aligned}
\tilde{\Sigma}_1(n)=&\sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L} e^{-n/N}(n-s(\boldsymbol{\nu}))\sum_{q\le P}\frac{\mu(q)^2}{\varphi(q)^2}c_q(-(n-s(\boldsymbol{\nu})))\\
&+O(PQL^k);
\end{aligned} \tag{4.3.5}
$$

collecting (4.3.5), (4.3.4) and (4.3.3) we have

$$\tilde{R}''_{\mathfrak{M}}(n) = \sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L} e^{-n/N}(n-s(\boldsymbol{\nu}))\sum_{q\le P}\frac{\mu(q)^2}{\varphi(q)^2}c_q(-(n-s(\boldsymbol{\nu})))$$
$$+O\left(PQL^k + NL^{1+k}\left(\frac{P}{Q}\right)^{1/2} + \frac{N}{Q}PL^{2+k}\right)$$

and provided that

$$P \ll QL^{-2} \tag{4.3.6}$$

we reach the following result

$$\begin{aligned}\tilde{R}''_{\mathfrak{M}}(n) =& e^{-n/N}M_k(n)\\ &+ e^{-n/N}\sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L}(n-s(\boldsymbol{\nu}))\sum_{q>P}\frac{\mu(q)^2}{\varphi(q)^2}c_q(-(n-s(\boldsymbol{\nu})))\\ &+O\left(PQL^k + NL^{1+k}\left(\frac{P}{Q}\right)^{1/2}\right).\end{aligned} \tag{4.3.7}$$

From computations made in Chapter 1, §1.2.2, and from Lemma 2.2.1, we have that the tail of the singular series is $o(NL^k)$.

Now we want to see what happens on the minor arcs using the Pintz and Ruzsa method, then we consider a set $\mathcal{E}$ such that $|G(\alpha| \le L$, for $\alpha \in \mathcal{E}$, and $|G(\alpha)| \le cL$, for $\alpha \in C(\mathcal{E}) = [0,1]\backslash\mathcal{E}$ and $c = 0.7163435444776661$. Then we split the integral on the minor arcs in the following way

$$\begin{aligned}&\int_{\mathfrak{m}}\tilde{S}(\alpha)^2\tilde{G}(\alpha)^k e(-n\alpha)d\alpha\\ &=\int_{\mathfrak{m}\cap\mathcal{E}}\tilde{S}(\alpha)^2\tilde{G}(\alpha)^k e(-n\alpha)d\alpha + \int_{\mathfrak{m}\cap C(\mathcal{E})}\tilde{S}(\alpha)^2\tilde{G}(\alpha)^k e(-n\alpha)d\alpha.\end{aligned}$$

First we consider

$$\begin{aligned}\int_{\mathfrak{m}\cap\mathcal{E}}|\tilde{S}(\alpha)|^2|\tilde{G}(\alpha)|^k d\alpha \le& L^k\sum_{P<q\le Q}\sideset{}{^*}\sum_{a=1}^{q}\int_{[-1/qQ,1/qQ]\cap\mathcal{E}}\left|\tilde{S}\left(\frac{a}{q}+\eta\right)\right|^2 d\eta\\ &\le |\mathcal{E}|N^{3/2}L^{2+k},\end{aligned}$$

where we use the estimate of $|\tilde{S}(a/q+\eta)|$ given by (4.2.10), with the same choice of P and Q in (4.2.11), that is $Q = N^{1/2}$ and $P = N^{1/4}L^{-1}(\log\log N)^2$. We want that this is $o(NL^k)$, then $|\mathcal{E}|$ has to be $O(N^{-1/2}L^{-2})$. This and (4.1.3) give us the condition of Corollary 1 of Pintz and Ruzsa, see Corollary A.6.4.

Remark 4. *Collecting* (4.3.7)*, the estimate of the tail of the singular series and the estimate of the integral on the minor arcs, where* $|G(\alpha)| \le L$*, we have*

$$\tilde{F}_k(n,N) \ll PQL^k + \left(\frac{P}{Q}\right)^{1/2} NL^{1+k} + o(NL^k)$$

when $P \to +\infty$*. We know that* $PQL^k = o(NL^k)$*, then with our choice of parameters* P *and* Q *in* (4.2.11) *we have*

$$\tilde{F}_k(n,N) \ll N^{7/8} L^{1/2+k} \log\log N + o(NL). \tag{4.3.8}$$

Now from the same computations made in §4.2.2 we have

$$\begin{aligned} &\sum_{N \le n \le 2N} \left| \int_{\mathfrak{m} \cap C(\mathcal{E})} \tilde{S}(\alpha)^2 \tilde{G}(\alpha)^k e(-n\alpha) d\alpha \right|^2 \\ &\le c^{2k} L^{2k} \int_{\mathfrak{m} \cap C(\mathcal{E})} |\tilde{S}(\xi)|^2 \int_{\mathfrak{m} \cap C(\mathcal{E})} |\tilde{S}(\alpha)|^2 |K(\alpha - \xi)| d\alpha d\xi. \end{aligned} \tag{4.3.9}$$

Then from (4.2.19) we have that (4.3.9) becomes

$$\sum_{n=N}^{2N} \left| \int_{\mathfrak{m} \cap C(\mathcal{E})} \tilde{S}(\alpha)^2 \tilde{G}(\alpha)^k e(-n\alpha) d\alpha \right|^2 \tag{4.3.10}$$

$$\le c^{2k} L^{2k} \sum_{n=N}^{2N} \left| \int_{\mathfrak{m} \cap C(\mathcal{E})} \tilde{S}(\alpha)^2 e(-n\alpha) d\alpha \right|^2. \tag{4.3.11}$$

Now, since P and Q satisfy conditions (4.3.6) and (4.2.17), Theorem 4.3.1 is proved. □

The proof of Corollary 4.3.2 is immediate from Theorem 4.3.1.

Chapter 5

Conditions for the validity of the Goldbach-Linnik hypothesis for small k

5.1 Statement of the problem

In this chapter we want to study another kind of thing about the Goldbach-Linnik problem; the aim is to find, under suitable conditions, a value k_0 such that for all $k \geq k_0$ the Goldbach-Linnik hypothesis is true for large N. To do this we start from Theorem 1 of Pintz and Ruzsa [41] that is:

Theorem 5.1.1 (Pintz and Ruzsa)**.** *Assume GRH. Let k be a fixed natural number with $k \geq 7$. Then*

$$r_k''(N) > 0 \qquad \text{if} \qquad 2 \mid N, N > N_0(k)$$

where $N_0(k)$ is an explicit constant, depending on k.

This means that, if GRH is true, all large even numbers can be written as a sum of two primes and k powers of 2, for all $k \geq 7$.

Now the idea is to use an appropriate variant of Montgomery's hypothesis, discussed in §0.4, in order to estimate $|S(\alpha)|$ on the minor arcs, to prove the previous result also for some $k < 7$. We have:

Theorem 5.1.2. *Assume GRH. Let k be a fixed natural number $k \in \{3, 4, 5, 6\}$. Then we can find a $\theta \in [1/4, 1/2]$ such that, assuming GMC(θ),*

$$r_k''(N) > 0 \qquad \text{if} \qquad 2 \mid N, N > N_0(k)$$

where $N_0(k)$ is an explicit constant depending on k.

We write here the value of θ that we have found for each value of $3 \leq k \leq 6$:

k	θ
3	0.47169811315754716981132
4	0.37389380525973451327434
5	0.30357142852142857142857
6	0.25490196073431372549020

Remark 5. *We recall that in this Chapter $L = \log_e N$ instead of $\log_2 N$ as in the other Chapters.*

Remark 6. *We recall that, if $\theta = 1/4$, GMC(θ) is the same of GRH, so that we are in the same condition of Theorem 5.1.1 of Pintz and Ruzsa. Then the interesting part is to prove the thesis for $3 \leq k_0 \leq 6$ and to do this, we have found the specific values of θ that allow us to reach our goal.*

Remark 7. *We can not prove the same result for $k = 2$ as we will see in §5.2.*

To reach our aim we will follow the idea of the proof of Lemma 13 of Pintz and Ruzsa [41]. First we will consider a set $\mathcal{E}$ such that $|G(\alpha)| \leq (1-\eta)L$, for all $\alpha \in [0,1] \setminus \mathcal{E}$, where $(1-\eta)$ is a real positive constant less than 1, and $|G(\alpha)| \leq L$ for all $\alpha \in \mathcal{E}$. Then we have

$$\int_{\mathfrak{m}} |S(\alpha)G(\alpha)^k|^2 d\alpha = \int_{\mathfrak{m}\cap\mathcal{E}} |S(\alpha)G(\alpha)^k|^2 d\alpha + \int_{\mathfrak{m}\cap C(\mathcal{E})} |S(\alpha)G(\alpha)^k|^2 d\alpha.$$

Now we can consider the two integrals separately, first we take that on $\mathfrak{m} \cap C(\mathcal{E})$:

$$\int_{\mathfrak{m}\cap C(\mathcal{E})} |S(\alpha)G(\alpha)^k|^2 d\alpha = \int_{\mathfrak{m}\cap C(\mathcal{E})} |G(\alpha)^{k-1}|^2 |S(\alpha)G(\alpha)|^2 d\alpha. \tag{5.1.1}$$

From Lemma 11 of [41], see Lemma A.6.3, we have that (5.1.1) is

$$\leq ((1-\eta)L)^{2k-2} \int_{\mathfrak{m}} |S(\alpha)G(\alpha)|^2 d\alpha \leq ((1-\eta)L)^{2k-2} \cdot \frac{2}{\log^2 2} C_2' N, \tag{5.1.2}$$

where $C_2' = 3.9095$.

The second integral is

$$\int_{\mathfrak{m}\cap\mathcal{E}} |S(\alpha)G(\alpha)^k|^2 d\alpha \leq C_\epsilon \cdot N^{2(1-\theta+\epsilon)} L^{2k} |\mathcal{E}|. \tag{5.1.3}$$

where, instead of Vaughan's estimate, we use the bound $|S(\alpha)| \ll N^{1-\theta+\epsilon}$, with $\theta \in (\frac{1}{4}, \frac{1}{2}]$ and C_ϵ is the implicit constant in our estimate of $|S(\alpha)|$. This bound comes from Corollary 2 of Languasco and Perelli [23], which is the following:

Corollary 5.1.3. *Let $\theta \in (0, 1/2]$ be fixed, $Q = \frac{1}{2}N^\theta$ and assume $GMC(\theta)$. Then for every $\epsilon > 0$*

$$S\left(\frac{a}{q} + \eta\right) = \frac{\mu(q)}{\varphi(q)} T(\eta) + O_\epsilon\left(\frac{N^{1+\epsilon}}{(qQ)^{1/2}}\right)$$

uniformly for $q \le Q$, $(a, q) = 1$ and $|\eta| \le 1/qQ$.

Now we wish to have that (5.1.3) leads to a negligible error term so that we need that:

$$C \cdot N^{2(1-\theta+\epsilon)} L^{2k} |\mathcal{E}| = o(NL^{2k})$$

and hence

$$|\mathcal{E}| = o(N^{-1+2\theta-\epsilon}). \tag{5.1.4}$$

Now we want to find the values of θ such that this condition is verified; to do this we will follow the proof of Theorem 5.1.1, see page 192 of [41]. We recall that by Remark 6 we need to analyze only the cases with $k = 3, 4, 5, 6$.

First we need the following definitions

$$r'_k(n) = |\{(p, \boldsymbol{\nu}) \in \mathfrak{P} \times [1, L]^k : n = p + s(\boldsymbol{\nu})\}|, \tag{5.1.5}$$

$$r_{k,k}(m) = |\{m = 2^{\nu_1} + \ldots + 2^{\nu_k} - 2^{\mu_1} - \ldots - 2^{\mu_k} : \nu_i, \mu_j \in [1, L]\}| \tag{5.1.6}$$

and

$$S(k, L) = \sum_{m \in \mathbb{Z} \setminus \{0\}} r_{k,k}(m) \mathfrak{S}(m). \tag{5.1.7}$$

Then, for every value of k, we need $A(k)$ that is defined in the Theorem of Khalfalah and Pintz [19], that we quote in Chapter 3, see Theorem 3.1.7.

Now the idea is to prove that $r''_k(N) > 0$, first we use the dispersion method and Lemma 14 of Pintz and Ruzsa [41] to have

$$E = \sum_{\substack{n \le N \\ 2 \nmid n}} (r'_k(N) - \lambda_k)^2$$

$$\leq \sum_{\substack{n\leq N\\ 2\nmid n}} (r_k^{'}(N))^2 - \frac{2NL^{2k}}{\log^2 N}(1+o(1)), \tag{5.1.8}$$

where we define

$$\lambda_k = \frac{2L^k}{\log N} \sim \frac{2L^{k-1}}{\log 2}. \tag{5.1.9}$$

Then we consider $k = i + j$, furthermore we take $r_i'(m), r_j'(n)$ defined by (5.1.5) and we write:

$$r_k^{''}(N) = \sum_{\substack{m+n=N\\ 2\nmid m, 2\nmid n}} r_i^{'}(m) r_j^{'}(n)$$

where

$$r_i^{'}(m) = \lambda_i + s_i(m) = \frac{2L^i}{\log N} + s_i(m) \tag{5.1.10}$$

$$r_j^{'}(n) = \lambda_j + s_j(n) = \frac{2L^j}{\log N} + s_j(n). \tag{5.1.11}$$

Then we have:

$$\begin{aligned} r_k^{''}(N) = & \sum_{\substack{m+n=N\\ 2\nmid m, 2\nmid n}} \left(\frac{2L^i}{\log N} + s_i(m)\right)\left(\frac{2L^j}{\log N} + s_j(n)\right) \\ = & \frac{4L^{i+j}}{\log^2 N} \sum_{\substack{m+n=N\\ 2\nmid m, 2\nmid n}} 1 \\ & + 2\left\{\frac{L^i}{\log N} \sum_{\substack{m+n=N\\ 2\nmid m, 2\nmid n}} s_j(n) + \frac{L^j}{\log N} \sum_{\substack{m+n=N\\ 2\nmid m, 2\nmid n}} s_i(m)\right\} \\ & + \sum_{\substack{m+n=N\\ 2\nmid m, 2\nmid n}} s_i(m) s_j(n). \end{aligned} \tag{5.1.12}$$

Now by Lemma 14 of [41] the second and the third term of the previous sum are $o(NL^{k-2})$.

The fourth term can be transformed applying the Cauchy-Schwarz inequality and we obtain:

$$\left|\sum_{\substack{m+n=N\\ 2\nmid m, 2\nmid n}} s_i(m) s_j(n)\right| \leq \left(\sum_{\substack{m\leq N\\ 2\nmid m}} s_i(m)^2\right)^{1/2} \left(\sum_{\substack{n\leq N\\ 2\nmid n}} s_j(n)^2\right)^{1/2}$$

$$\leq\left(\sum_{\substack{m\leq N\\ 2\nmid m}}(r_i^{'}(m)-\lambda_i)^2\right)^{1/2}\left(\sum_{\substack{n\leq N\\ 2\nmid n}}(r_j^{'}(n)-\lambda_j)^2\right)^{1/2}. \tag{5.1.13}$$

Now by (5.1.8), (5.1.13) becomes:

$$\leq\left(\sum_{\substack{m\leq N\\ 2\nmid m}}(r_i^{'}(m))^2-\frac{2NL^{2i}}{\log^2 N}\right)^{1/2}\left(\sum_{\substack{n\leq N\\ 2\nmid n}}(r_j^{'}(n))^2-\frac{2NL^{2j}}{\log^2 N}\right)^{1/2} \tag{5.1.14}$$

and, by Lemma 13 of Pintz and Ruzsa [41], (5.1.14) is

$$=\frac{2NL^k}{\log^2 N}\left(\{A(i)+C_2^{'}(1-\eta)^{2i-2}+\delta\}\{A(j)+C_2^{'}(1-\eta)^{2j-2}+\delta\}\right)^{1/2},$$

where the values of $A(i)$ and $A(j)$, defined by (3.1.4), are taken from the article of Khalfalah and Pintz [19] and δ is a small positive constant. Since the first term in (5.1.12) is

$$\frac{2NL^k}{\log^2 N}$$

then we want that

$$\left(\{A(i)+C_2^{'}(1-\eta)^{2i-2}+\delta\}\{A(j)+C_2^{'}(1-\eta)^{2j-2}+\delta\}\right)^{1/2}<1. \tag{5.1.15}$$

Now by (5.1.15) we can compute the value of $(1-\eta)$ for each $3\leq k\leq 6$ and then the relative values of θ; in this computation we will forget δ because it can be assimilated in the rounding of numerical values.

Remark 8. *We observe that, if $k=2$, we need $A(1)+C_2'<1$ that is a contradiction: this means that with this method we can not prove the case $k=2$.*

5.2 Computation of parameters:

Now we suppose $k=2s$ and $i=j=s$ then by (5.1.15):

$$\{A(s)+C_2^{'}(1-\eta)^{2s-2}\}<1;$$

this implies

$$(1-\eta)^{2s-2} < \frac{1-A(s)}{C_2^{'}}$$

hence

$$(1-\eta) < \left[\frac{1-A(s)}{C_2^{'}}\right]^{1/(2s-2)}. \tag{5.2.1}$$

In the same way, we suppose $k = 2s+1$, $i = s+1$ and $j = s$ then by (5.1.15):

$$\{A(s+1) + C_2^{'}(1-\eta)^{2s}\}^{1/2}\{A(s) + C_2^{'}(1-\eta)^{2s-2}\}^{1/2} < 1,$$

hence

$$\begin{aligned} &A(s+1)A(s) + C_2^{'}A(s+1)(1-\eta)^{2s-2} \\ &+ C_2^{'}A(s)(1-\eta)^{2s} + {C_2^{'}}^2(1-\eta)^{4s-2} < 1. \end{aligned} \tag{5.2.2}$$

Now from (5.2.1) and (5.2.2) we can deduce the values of $(1-\eta)$ for $k = 3, 4, 5, 6$ and then we use a computer program written by Alessandro Languasco program to compute the value of $|\mathcal{E}|$ for each value of $(1-\eta)$. After finding each value of $|\mathcal{E}|$, by (5.1.4) we will be able to find each value of θ that prove Theorem 5.1.2.

5.2.1 Derivation of the value for θ.

CASE $k = 3$

Consider $k = 3, i = 2, j = 1$, then by (5.2.2):

$$\begin{aligned} &\{A(2) + C_2^{'}(1-\eta)^2\}\{A(1) + C_2^{'}\} < 1 \\ &C_2^{'}(1-\eta)^2 < \frac{1}{A(1) + C_2^{'}} - A(2) \\ &(1-\eta) < \sqrt{\frac{1}{C_2^{'}}\left(\frac{1}{A(1) + C_2^{'}} - A(2)\right)} \\ &\quad < 0.21649955302846423742000028966 \end{aligned}$$

Now we use the Languasco program to find $|\mathcal{E}|$, after few computations, see §A.7 Table 1, we find that the best value of $|\mathcal{E}|$ is $|\mathcal{E}| < N^{-3/53-10^{-10}}$, then we obtain:

$$\begin{aligned} -1+2\theta &= -\frac{3}{53} - 10^{-10} \\ \theta &= \frac{1}{2}\left(1 - \frac{3}{53} - 10^{-10}\right) < 0.47169811315754716981132. \end{aligned} \tag{5.2.3}$$

Since this value of θ is in the interval $(1/4, 1/2)$, we can conclude that Theorem 5.1.2 is proved for $k = 3$.

CASE $k = 4$

Consider $k = 4, i = 2, j = 2$ then from (5.2.1):

$$(1-\eta) < \sqrt{\frac{1-A(2)}{C_2'}} < 0.4915221192975883608179476389882.$$

Also in this case after few computations, see §A.7 Table 2, we obtain $|\mathcal{E}| < N^{-57/226-10^{-10}}$ and then

$$\theta = 0.37389380525973451327434. \tag{5.2.4}$$

This value of θ satisfies the condition $\theta \in (1/4, 1/2)$ so we have the value of θ that proves Theorem 5.1.2 for $k = 4$.

CASE $k = 5$

Consider $k = 5, i = 3, j = 2$, then from (5.2.2)

$$\begin{aligned} &\{A(3)A(2) + C_2'A(3)(1-\eta)^2 + C_2'A(2)(1-\eta)^4 + {C_2'}^2(1-\eta)^6\} < 1 \\ &(1-\eta) < 0.6287527895 \end{aligned}$$

The computations collected in Table 3 of §A.7 show that $|\mathcal{E}| < N^{-11/28-10^{-10}}$ and

$$\theta = 0.30357142852142857142857. \tag{5.2.5}$$

This value belongs to $(1/4, 1/2)$ so we have the value of θ that proves Theorem 5.1.2 for $k = 5$.

CASE $k = 6$

Consider $k = 6, i = 3, j = 3$, then from (5.2.1) we have

$$(1-\eta) < \sqrt[4]{\frac{1-A(3)}{C_2'}} < 0.70873388432540316934557100411404$$

Again for computations in Table 4 of §A.7 we have $|\mathcal{E}| < N^{-25/51-10^{-10}}$ and

$$\theta = 0.25490196073431372549020. \tag{5.2.6}$$

Also this value lies in $(1/4, 1/2)$ so that this is the θ that proves Theorem 5.1.2 for $k = 6$.

Now collecting the value in (5.2.3), (5.2.4), (5.2.5) and (5.2.6) we have the following chart, which contains the values of θ by which Theorem 5.1.2 is proved.

k	θ
3	0.47169811315754716981132
4	0.37389380525973451327434
5	0.30357142852142857142857
6	0.25490196073431372549020

Chapter 6

A Diophantine problem with two primes and s powers of two

In this Chapter we will study a different kind of problem related to the numbers written as a sum of two primes and k powers of 2. In this case we will analyze the number of the following type

$$\lambda_1 p_1 + \lambda_2 p_2 + \mu_1 2^{\nu_1} + \cdots + \mu_s 2^{\nu_s},$$

where $\lambda_1, \lambda_1, \mu_1, \ldots, \mu_s$ are real coefficients which satisfy certain conditions that we will define later.

In this case the problem is different from the other proposed in this work: in fact here we can only study if a real number can be approximated as a sum of two primes and k powers of 2 with suitable coefficients, but we can not prove equality. To prove our result we will use the circle method but in a very different way: in fact here we have to consider the real line instead of S^1 such that the method changes.

Theorem 6.0.1. *Assume RH. Let λ_1, λ_2 be real numbers such that $\boldsymbol{\lambda} = \lambda_1/\lambda_2$ is negative and irrational with $\lambda_1 > 1, \lambda_2 < -1$ and $|\lambda_1/\lambda_2| \geq 1$. Further suppose that $\mu_1, \ldots, \mu_s$ are nonzero real numbers such that $\lambda_i/\mu_i \in \mathbb{Q}$, for $i = 1, 2$, and denote by a_i/q_i their reduced representations as rational numbers. Let moreover η be a sufficiently small positive constant such that $\eta < \min(\lambda_1/a_1; |\lambda_2/a_2|)$. Finally let*

$$s_0 = 2 + \left[\frac{\log\left(C(q_1, q_2)\lambda_1\right) - \log \eta}{-\log 0.7163435444776661} \right], \tag{6.0.1}$$

where

$$C(q_1, q_2) = (\log 2 + C \cdot \mathfrak{S}'(q_1))^{1/2} (\log 2 + C \cdot \mathfrak{S}'(q_2))^{1/2},$$

with $C = 10.0219168340$ *and*

$$\mathfrak{S}'(n) = \prod_{\substack{p|n \\ p>2}} \frac{p-1}{p-2}.$$

Then for every real number γ *and every* $s \geq s_0$ *the inequality*

$$|\lambda_1 p_1 + \lambda_2 p_2 + \mu_1 2^{\nu_1} + \cdots + \mu_s 2^{\nu_s} + \gamma| < \eta \tag{6.0.2}$$

has infinitely many solutions in primes p_1, p_2 *and positive integers* $\nu_1, \ldots, \nu_s$.

We notice that the constant 0.7163435444776661 comes from the estimate on the minor arc, see §6.1.3: in particular this constant will be obtained from the use of Corollary A.6.4.

6.1 Proof of Theorem 6.0.1

Before starting the proof of Theorem 6.0.1 we have to define some of the functions that we will use in our work.

Let ϵ be a sufficiently small positive constant, let X be a large parameter, $M = |\mu_1| + \cdots + |\mu_s|$ and $L = \log_2(\epsilon X/2M)$. Let $\mathfrak{N}(X)$ be the number of solutions of the inequality (6.0.2), with $\epsilon X \leq p_1, p_2 \leq X$ and $1 \leq \nu_1, \ldots, \nu_s \leq L$. Furthermore we take the analogous of (0.5.11), that is

$$S(\alpha) = \sum_{\epsilon X \leq p \leq X} (\log p) e(p\alpha) \tag{6.1.1}$$

and $G(\alpha)$ defined by (0.5.12). Now for $\alpha \neq 0$ we define

$$K(\alpha, \eta) = \left(\frac{\sin(\pi\eta\alpha)}{\pi\alpha} \right)^2 \tag{6.1.2}$$

and hence we know that

$$K(\alpha, \eta) \ll \min(\eta^2; \alpha^{-2}) \tag{6.1.3}$$

and

$$\hat{K}(\alpha, \eta) = \int_{\mathbb{R}} K(\alpha, \eta) e(t\alpha) d\alpha = \max(0; \eta - |t|). \tag{6.1.4}$$

Now we define

$$I(X; \mathbb{R}) = \int_{\mathbb{R}} S(\lambda_1\alpha) S(\lambda_2\alpha) G(\mu_1\alpha) \cdots G(\mu_s\alpha) e(\gamma\alpha) K(\alpha, \eta) d\alpha \tag{6.1.5}$$

and we observe that by (6.1.4) we have

$$I(X;\mathbb{R}) \ll \eta \log^2 X \cdot \mathfrak{N}(X).$$

Since we will prove that

$$I(X;\mathbb{R}) \gg_{s,\boldsymbol{\lambda},\epsilon} \eta^2 X(\log X)^s, \tag{6.1.6}$$

where $\boldsymbol{\lambda} = \lambda_1/\lambda_2$, then

$$\mathfrak{N}(X) \gg_{s,\boldsymbol{\lambda},\epsilon} \eta X(\log X)^{s-2}$$

and the Theorem is proved.

Also in this case we use the circle method but we work on $\mathbb{R}$ instead of S^1, then the method is quite different. We have to dissect the real line as follows: let P be a parameter, that we will choose later, then we take

$$\mathfrak{M} = \{\alpha \in \mathbb{R} : |\alpha| \le P/X\} \tag{6.1.7}$$

for the major arc,

$$\mathfrak{m} = \{\alpha \in \mathbb{R} : P/X < |\alpha| \le L^2\} \tag{6.1.8}$$

for the minor arcs and

$$\mathfrak{t} = \{\alpha \in \mathbb{R} : |\alpha| > L^2\} \tag{6.1.9}$$

for the trivial ones.

After this dissection (6.1.5) becomes

$$I(X;\mathbb{R}) = I(X;\mathfrak{M}) + I(X;\mathfrak{m}) + I(X;\mathfrak{t}). \tag{6.1.10}$$

6.1.1 The major arc

Let

$$\theta(x) = \sum_{p\le x} \log p, \qquad\qquad \psi(x) = \sum_{n\le x} \Lambda(n);$$

we observe that

$$\psi(x) - \theta(x) \ll x^{1/2+\epsilon}.$$

We consider the Selberg integral

$$J(X,H) = \int_{\epsilon X}^{X} (\theta(x+H) - \theta(x) - H)^2 dx$$

and we recall that in 1943 Selberg [43] proved under RH that

$$J(X,H)=\int_1^X(\psi(x+H)-\psi(x)-H)^2dx\ll XHL^2. \tag{6.1.11}$$

Then we write

$$\begin{aligned}
I(X;\mathfrak{M}) &= \int_{\mathfrak{M}} S(\lambda_1\alpha)S(\lambda_2\alpha)G(\mu_1\alpha)\dots G(\mu_s\alpha)e(\gamma\alpha)K(\alpha,\eta)d\alpha\\
&= \int_{\mathfrak{M}} T(\lambda_1\alpha)T(\lambda_2\alpha)G(\mu_1\alpha)\dots G(\mu_s\alpha)e(\gamma\alpha)K(\alpha,\eta)d\alpha\\
&\quad+\int_{\mathfrak{M}} (S(\lambda_1\alpha)-T(\lambda_1\alpha))T(\lambda_2\alpha)G(\mu_1\alpha)\dots G(\mu_s\alpha)e(\gamma\alpha)K(\alpha,\eta)d\alpha\\
&\quad+\int_{\mathfrak{M}} (S(\lambda_2\alpha)-T(\lambda_2\alpha))S(\lambda_1\alpha)G(\mu_1\alpha)\dots G(\mu_s\alpha)e(\gamma\alpha)K(\alpha,\eta)d\alpha\\
&= J_1+J_2+J_3,
\end{aligned}$$

say, where

$$T(\alpha)=\int_{\epsilon X}^X e(t\alpha)dt\ll_\epsilon \min\left(X,\frac{1}{|\alpha|}\right). \tag{6.1.12}$$

First we want to prove that $J_3=o(\eta^2XL^s)$. We let

$$U(\alpha)=\sum_{\epsilon X\le n\le X}e(\alpha n)$$

and we observe that, by the partial summation formula, we have

$$T(\alpha)-U(\alpha)\ll(1+X|\alpha|).$$

To reach our result we need the following Lemma.

Lemma 6.1.1 (Lemma 1 of Brüdern, Cook and Perelli [1])**.** *For* $1/X\le Y\le 1/2$ *we have*

$$\int_{-Y}^{Y}|S(\alpha)-U(\alpha)|^2d\alpha\ll_\epsilon\frac{\log^2X}{Y}+Y^2X+Y^2J\left(X,\frac{1}{Y}\right). \tag{6.1.13}$$

Furthermore we can see that

$$\int_{-P/X}^{P/X}|T(\lambda_2\alpha)-U(\lambda_2\alpha)||S(\lambda_1\alpha)|d\alpha\ll X\log X\int_{-P/X}^{P/X}(1+X|\lambda_2\alpha|)d\alpha$$

$$\begin{aligned}&\ll_{\lambda} X^2 \log X \cdot \frac{P^2}{X^2}\\&\ll_{\lambda} P^2 \log X. \qquad (6.1.14)\end{aligned}$$

Now by (6.1.3) and (6.1.14) we have

$$\begin{aligned}J_3 &= \int_{\mathfrak{M}} (S(\lambda_2\alpha) - T(\lambda_2\alpha))S(\lambda_1\alpha)G(\mu_1\alpha)\dots G(\mu_s\alpha)e(\gamma\alpha)K(\alpha,\eta)d\alpha\\&\ll \eta^2 L^s \int_{\mathfrak{M}} |S(\lambda_2\alpha) - U(\lambda_2\alpha)||S(\lambda_1\alpha)|d\alpha\\&\quad + \eta^2 L^s \int_{\mathfrak{M}} |T(\lambda_2\alpha) - U(\lambda_2\alpha)||S(\lambda_1\alpha)|d\alpha\\&\ll_{\boldsymbol{\lambda}} \eta^2 L^s \left(\int_{\mathfrak{M}} |S(\lambda_2\alpha) - U(\lambda_2\alpha)|^2 d\alpha\right)^{1/2} \left(\int_{\mathfrak{M}} |S(\lambda_1\alpha)|^2 d\alpha\right)^{1/2}\\&\quad + \eta^2 L^{s+1} P^2;\end{aligned}$$

now taking $Y = P/X$ and by (6.1.13) we have

$$J_3 \ll_{\boldsymbol{\lambda}} \eta^2 L^s \left(\frac{\log^2 X}{Y} + Y^2 X + Y^2 J\left(X, Y^{-1}\right)\right)^{1/2} (X \log X)^{1/2} + \eta^2 L^{s+1} P^2.$$

Then using (6.1.11) we have

$$J_3 \ll_{\lambda} \eta^2 L^s \left(\frac{\log^2 X}{Y} + Y^2 X + Y^2 \frac{X}{Y} L^2\right)^{1/2} X^{1/2} (\log X)^{1/2} + \eta^2 L^{s+1} P^2,$$

we want that $Y^2 X \leq Y X L^2$, then $P \leq X L^2$. Now

$$J_3 \ll \eta^2 L^s Y^{1/2} X L^{3/2} + \eta^2 L^{s+1} P^2,$$

this has to be $o(\eta^2 L^s X)$ then $Y = o(L^{-3})$ and $P = o(X L^{-3})$. Furthermore $\eta^2 L^{s+1} P^2 = o(\eta^2 X L^s)$, then $P = o(X^{1/2} L^{-2})$; finally we can take

$$P = X^{1/2} L^{-3} \qquad (6.1.15)$$

then $J_3 = o(\eta^2 L^s X)$ as we want.

Now we observe that under the assumption of RH we choose P as in (6.1.15) and this leads to a major arc larger than that in the article of Languasco and Zaccagnini [25]: in fact in our case we have

$$\mathfrak{M} = \{\alpha \in \mathbb{R} : |\alpha| \leq X^{-1/2} L^{-3}\}$$

instead they have

$$\mathfrak{M} = \{\alpha \in \mathbb{R} : |\alpha| \le X^{-2/3}\}.$$

Now we have to estimate J_2, the computations are quite the same of J_3 but we will use (6.1.12) instead of the Prime Number Theorem.

$$\begin{aligned}
J_2 &= \int_{\mathfrak{M}} (S(\lambda_1\alpha) - T(\lambda_1\alpha))T(\lambda_2\alpha)G(\mu_1\alpha)\dots G(\mu_s\alpha)e(\gamma\alpha)K(\alpha,\eta)d\alpha \\
&\ll \eta^2 L^s \int_{\mathfrak{M}} |S(\lambda_1\alpha) - U(\lambda_1\alpha)||T(\lambda_2\alpha)|d\alpha \\
&\quad + \eta^2 L^s \int_{\mathfrak{M}} |T(\lambda_1\alpha) - U(\lambda_1\alpha)||T(\lambda_2\alpha)|d\alpha \\
&\ll \eta^2 L^s \left(\int_{\mathfrak{M}} |S(\lambda_1\alpha) - U(\lambda_1\alpha)|^2 d\alpha\right)^{1/2} \left(\int_{\mathfrak{M}} |T(\lambda_2\alpha)|^2 d\alpha\right)^{1/2} + \eta^2 P^2 L^s \\
&\ll \eta^2 L^s (YXL^2)^{1/2} \left(X^2 \cdot \frac{P}{X}\right)^{1/2} + \eta^2 P^2 L^s \\
&\ll \eta^2 L^{s+1} \left(\frac{P}{X} \cdot X\right)^{1/2} X^{1/2} P^{1/2} + \eta^2 P^2 L^s \\
&\ll \eta^2 L^{s+1} L^{-3} X + \eta^2 P^2 L^s \\
&\ll \eta^2 L^{s-2} X + \eta^2 P^2 L^s.
\end{aligned}$$

By (6.1.15) this is $o(\eta^2 L^s X)$ as we want. Finally we have to give an estimate of J_1 then we have

$$J_1 = \sum_{1\le\nu_1\le L} \cdots \sum_{1\le\nu_s\le L} J(\mu_1 2^{\nu_1} + \cdots + \mu_s 2^{\nu_s} + \gamma, \eta) + \eta^2 L^s \int_{\mathbb{R}\backslash\mathfrak{M}} |T(\lambda_1\alpha)T(\lambda_2\alpha)|d\alpha$$

with

$$\begin{aligned}
J(u,\eta) :&= \int_{\mathbb{R}} T(\lambda_1\alpha)T(\lambda_2\alpha)e(u\alpha)K(\alpha,\eta)d\alpha \\
&= \int_{\epsilon X}^{X} \int_{\epsilon X}^{X} \hat{K}(\lambda_1 u_1 + \lambda_2 u_2 + u, \eta)du_1 du_2.
\end{aligned}$$

First we consider

$$\eta^2 L^2 \int_{\mathbb{R}\backslash\mathfrak{M}} |T(\lambda_1\alpha)T(\lambda_2\alpha)|d\alpha,$$

since $\alpha \in \mathbb{R}\backslash\mathfrak{M}$ we obtain $T(\lambda_i\alpha) \ll_\epsilon 1/|\alpha|$, then

$$\eta^2 L^s \int_{\mathbb{R}\backslash\mathfrak{M}} |T(\lambda_1\alpha)T(\lambda_2\alpha)|d\alpha$$

$$
\begin{aligned}
&\ll_\epsilon \eta^2 L^s \int_{\mathbb{R}\setminus\mathfrak{M}} \frac{1}{|\alpha|^2} d\alpha \\
&\ll_\epsilon \eta^2 L^{s+3} X^{1/2}.
\end{aligned}
$$

Now the estimate of $J(\mu_1 2^{\nu_1}+\cdots+\mu_s 2^{\nu_s}+\gamma,\eta)$ is the same made by Languasco and Zaccagnini [25], here we quote their computations for completeness. For simplicity let

$$
J_0(u,\eta) := \int_0^X \int_0^X \hat{K}(\lambda_1 u_1 + \lambda_2 u_2 + u, \eta) du_1 du_2,
$$

where $\lambda_1 > -\lambda_2 > 1, |u| \le \epsilon X, 0 < \eta \le \epsilon X$ and $\epsilon > 0$ is sufficiently small in terms of λ_1 and λ_2. Now we make the following change of variables: $y_1 = \lambda_1 u_1$ and $y_2 = -\lambda_2 u_2$, then we have

$$
\begin{aligned}
J_0(u,\eta) :&= -\frac{1}{\lambda_1\lambda_2} \int_0^{\lambda_1 X} \int_0^{-\lambda_2 X} \hat{K}(y_1 - y_2 + u, \eta) dy_1 dy_2 \\
&= -\frac{1}{\lambda_1\lambda_2} \int_0^{\lambda_1 X} dy_1 \int_0^{-\lambda_2 X} \max(0; \eta - |y_1 - y_2 + u|) dy_2.
\end{aligned}
$$

We can assume that $X \ge (\lambda_1+\lambda_2)^{-1}(\eta+|u|)$, then the lines $y_2 = y_1+u+j\eta$, for $j \in \{-1,0,1\}$, intersect the boundary of the rectangle $[0,\lambda_1 X] \times [0,-\lambda_2 X]$ on its upper horizontal side. Since the integrand vanishes outside the set $y_1 + u - \eta \le y_2 \le y_1 + u + \eta$, we can replace the condition $y_1 \in [0,\lambda_1 X]$ with $y_1 \in [0, -\lambda_2 X - \eta - u] \cup [-\lambda_2 X - \eta - u, -\lambda_2 X + \eta - u] = I_1 \cup I_2$, say. If $y_1 \in I_1$ we have

$$
\int_0^{-\lambda_2 X} \max(0; \eta - |y_1 - y_2 + u|) dy_2 = \int_{-y_1-u}^{-\lambda_2 X - y_1 - u} \max(0; \eta - |w|) dw = \eta^2,
$$

and the total contribution of I_1 to J_0 is therefore $\eta^2(-\lambda_2 X - \eta - u)$. The contribution of I_2 is non-negative and it is easily seen that it is $O(\eta^3)$.

Finally

$$
J_0(u,\eta) = -\frac{1}{\lambda_1\lambda_2}\eta^2(-\lambda_2 X - \eta - u) + O(\eta^3) \ge \frac{1}{\lambda_1} X\eta^2 + O(\epsilon X \eta^2).
$$

Now we have that $J_0(u,\eta) - J(u,\eta) \ll \epsilon X \eta^2$ and then

$$
J_1 \ge \frac{1 - c_4\epsilon}{\lambda_1} \eta^2 X L^s, \tag{6.1.16}
$$

where c_4 is some positive constant.

6.1.2 The trivial arc

Here we will only quote the result given by Languasco and Zaccagnini in section 5 of [25]: in fact the computation does not change under GRH, then is not useful to repeat the same step. The result they obtained is

$$|I(X;\mathfrak{t})| = o(XL^s). \tag{6.1.17}$$

6.1.3 The minor arc

Let

$$I(X;\mathfrak{m}) = \int_{\mathfrak{m}} S(\lambda_1\alpha)S(\lambda_2\alpha)G(\mu_1\alpha)\dots G(\mu_s\alpha)e(\gamma\alpha)K(\alpha,\eta)d\alpha,$$

$c \in (0,1), \mathfrak{m} = \{\alpha \in \mathbb{R} : P/X < |\alpha| \le L^2\} = \mathfrak{m}_1 \cup \mathfrak{m}_2$, with $\mathfrak{m}_1 \cap \mathfrak{m}_2 = \emptyset$ and $\mathfrak{m}_2 = \{\alpha \in \mathfrak{m} : |G(\alpha)| > \nu(c)L\}$. Using Corollary A.6.4 $|\mathfrak{m}_2| \ll_{M,\epsilon} sL^2X^{-c}$, then we have

$$\begin{aligned}
|I(X;\mathfrak{m}_2)| &= \left|\int_{\mathfrak{m}_2} S(\lambda_1\alpha)S(\lambda_2\alpha)G(\mu_1\alpha)\dots G(\mu_s\alpha)e(\gamma\alpha)K(\alpha,\eta)d\alpha\right| \\
&\ll L^s\left(\int_{\mathfrak{m}_2} |S(\lambda_1\alpha)S(\lambda_2\alpha)|^2K(\alpha,\eta)d\alpha\right)^{1/2}\left(\int_{\mathfrak{m}_2} K(\alpha,\eta)d\alpha\right)^{1/2} \\
&\ll \eta L^s|\mathfrak{m}_2|^{1/2}\left(\int_{\mathfrak{m}_2} |S(\lambda_1\alpha)S(\lambda_2\alpha)|^2K(\alpha,\eta)d\alpha\right)^{1/2} \\
&\ll s^{1/2}\eta L^{s+1}X^{-c/2}\left(\int_{\mathfrak{m}_2} |S(\lambda_1\alpha)S(\lambda_2\alpha)|^2K(\alpha,\eta)d\alpha\right)^{1/2}.
\end{aligned}$$

We observe that

$$\int_{\mathfrak{m}_2} |S(\lambda_1\alpha)S(\lambda_2\alpha)|^2K(\alpha,\eta)d\alpha \ll \int_{\mathfrak{m}} |S(\lambda_1\alpha)S(\lambda_2\alpha)|^2K(\alpha,\eta)d\alpha.$$

Now let $\alpha \in \mathfrak{m}$, let $Q = X^{1/4}L^{-2}$. Then there exist a_i and q_i with $1 \le q_i \le XQ^{-1}$ and $(a_i, q_i) = 1$ such that

$$|\lambda_i\alpha q_i - a_i| \le QX^{-1}.$$

Suppose that $q_1 \le Q$ and $q_2 \le Q$ then

$$\left|a_2q_1\frac{\lambda_1}{\lambda_2} - a_1q_2\right| \le 2\left(1 + \left|\frac{\lambda_1}{\lambda_2}\right|\right)Q^2X < \frac{1}{2q} \tag{6.1.18}$$

for X sufficiently large. Then, as in the article of Parsell [38], from (6.1.8) and Legendre's law of best approximation we arrive to the following contradiction

$$X^{1/2} = q^{1/2} \leq |a_2 q_1| \ll q_1 q_2 L^2 \leq X^{1/2} L^{-2}.$$

Then we have that either $q_1 \geq Q$ or $q_2 \geq Q$. Now we need the Vaughan estimate of $S(\alpha)$ under GRH that is

$$S(\alpha) \ll \left(\frac{X}{Q} + \sqrt{XP} + \frac{X}{\sqrt{P}} \right) L^2.$$

We recall that $P = X^{1/2} L^{-3}$, as defined in (6.1.15), and we have

$$S(\lambda_i \alpha) \ll \left(X^{3/4} L^2 + X^{3/4} L^{-3/2} + X^{3/4} L^{3/2} \right) L^2 \ll X^{3/4} L^{7/2}.$$

Then

$$\begin{aligned}
&\int_{\mathfrak{m}} |S(\lambda_1 \alpha) S(\lambda_2 \alpha)|^2 K(\alpha, \eta) d\alpha \\
&\ll X^{3/2} L^7 \sum_{\epsilon X \leq p, p' \leq X} \log p \log p' \int_{-\infty}^{+\infty} e((p - p')\boldsymbol{\lambda}\alpha) K(\alpha, \eta) d\alpha \\
&\ll X^{3/2} L^7 \sum_{\epsilon X \leq p, p' \leq X} \hat{K}(\alpha, \boldsymbol{\lambda}(p - p')) \log p \log p' \\
&\ll X^{3/2} L^7 \eta \sum_{\epsilon X \leq p \leq X} \log^2 p \\
&\ll \eta X^{5/2} L^8. \qquad (6.1.19)
\end{aligned}$$

Now we have

$$\begin{aligned}
|I(X, \mathfrak{m}_2)| &\ll s^{1/2} \eta L^{s+1} X^{-c/2} \eta^{1/2} X^{5/4} L^4 \\
&\ll s^{1/2} \eta^{3/2} L^{s+5} X^{-c/2 + 5/4},
\end{aligned}$$

and we want that this is $o(\eta X)$ then $c \geq 1/2$. We want to take c as small as possible according to Corollary A.6.4, then we take $c = 1/2$.

Finally we have to consider

$$\begin{aligned}
|I(X; \mathfrak{m}_1)| &= \left| \int_{\mathfrak{m}_1} S(\lambda_1 \alpha) S(\lambda_2 \alpha) G(\mu_1 \alpha) \dots G(\mu_s \alpha) e(\gamma \alpha) K(\alpha, \eta) d\alpha \right| \\
&\leq (\nu L)^{s-2} \left(\int_{\mathfrak{m}} |S(\lambda_1 \alpha) G(\mu_1 \alpha)|^2 K(\alpha, \eta) \right)^{1/2} \\
&\quad \cdot \left(\int_{\mathfrak{m}} |S(\lambda_2 \alpha) G(\mu_2 \alpha)|^2 K(\alpha, \eta) \right)^{1/2},
\end{aligned}$$

now by Lemma 4 of Languasco and Zaccagnini [25] and using Corollary A.6.4 with $\beta = c = 1/2$ we have

$$\begin{aligned}
|I(X;\mathfrak{m}_1)| &\le (\nu L)^{s-2}\left[\eta X L^2((1-\epsilon)\log 2 + C\cdot\mathfrak{S}'(q_1)) + O_{M,\epsilon}(\eta X L)\right]^{1/2}\\
&\quad\cdot\left[\eta X L^2((1-\epsilon)\log 2 + C\cdot\mathfrak{S}'(q_2)) + O_{M,\epsilon}(\eta X L)\right]^{1/2}\\
&\le \nu^{s-2}L^s\eta X(\log 2 + C\cdot\mathfrak{S}'(q_1))^{1/2}(\log 2 + C\cdot\mathfrak{S}'(q_2))^{1/2}\\
&\le \nu^{s-2}L^s\eta X C(q_1,q_2)\\
&\le (0.7163435444776661)^{s-2}L^s\eta X C(q_1,q_2). \qquad (6.1.20)
\end{aligned}$$

Then we have

$$\begin{aligned}
I(X;\mathfrak{M}) &\ge c_1\eta^2 L^s X\\
I(X;\mathfrak{t}) &= o(L^s X)\\
I(X;\mathfrak{m}) &\le c_2(s)L^s\eta X,
\end{aligned}$$

where $c_2(s) > 0$ depends on s, $c_2(s)\to 0$ as $s\to+\infty$, and $c_1 = c_1(\epsilon,\boldsymbol{\lambda}) > 0$ is a constant such that

$$c_1\eta - c_2(s) \ge c_3\eta \qquad (6.1.21)$$

for some absolute constant c_3 and $s \ge s_0$. We can find the value of s_0 from (6.1.21): in fact by (6.1.16) and (6.1.20) we have:

$$\begin{aligned}
&\frac{1-c_4\epsilon}{\lambda_1}\eta - (0.7163435444776661)^{s-2}C(q_1,q_2) \ge c_3\eta\\
&(0.7163435444776661)^{s-2} \le \left(\frac{1-c_4\epsilon}{\lambda_1} - c_3\right)\frac{\eta}{C(q_1,q_2)}\\
&s \ge 2 + \left(\frac{\log\left(C(q_1,q_2)\lambda_1\right) - \log\eta}{-\log\left(0.7163435444776661\right)}\right) := s_0,
\end{aligned}$$

say. Now Theorem 6.0.1 is proved.

Appendix A

A.1 The von Mangoldt function

The von Mangoldt function is defined in the following way

$$\Lambda(n) = \begin{cases} \log p & \text{if } n \text{ is a power of a prime } p \\ 0 & \text{otherwise.} \end{cases} \tag{A.1.1}$$

A.2 The Ramanujan sum

Theorem A.2.1. *(Ramanujan) Ramanujan's sum defined as*

$$c_q(-n) = \sum_{a=1}^{q}{}^{*} e_q(an) \tag{A.2.1}$$

is multiplicative and we have

$$c_q(-n) = \mu\left(\frac{q}{(q,n)}\right)\frac{\varphi(q)}{\varphi\left(q/(q,n)\right)}. \tag{A.2.2}$$

This is Theorem 272 of Hardy and Wright [13].

A.3 The Gaussian sum

For any character $\chi(n)$ to the modulus q, the Gaussian sum $\tau(\chi)$ is defined by

$$\tau(\chi) = \sum_{m=1}^{q} \chi(m) e_q(m), \tag{A.3.1}$$

furthermore, for a primitive character χ to the modulus q we have

$$|\tau(\chi)| = q^{1/2}. \tag{A.3.2}$$

This is formula (5) of Chapter 9 of Davenport's book [3].

A.4 The Farey dissection

We define the Farey dissection of the unit circle and we recall some theorems of Hardy and Wright [13] using our notation.

Definition A.4.1. *The Farey dissection* $\mathfrak{F}_n$ *of order* n *is the ascending series of the irreducible fractions between* 0 *and* 1 *whose denominators do not exceed* n*. Thus* a/q *belongs to* $\mathfrak{F}_n$ *if*

$$0 \leq a \leq q \leq n, \qquad (a,q) = 1$$

where the numbers 0 *and* 1 *are included in the form* $0/1, 1/1$.

This series has some important properties.

Theorem A.4.2. *If* a/q*,* a_1/q_1 *are two successive terms of* $\mathfrak{F}_n$*, then*

$$qa_1 - aq_1 = 1 \tag{A.4.1}$$

Theorem A.4.3. *If* a/q*,* a_2/q_2 *and* a_1/q_1 *are three successive terms of* $\mathfrak{F}_n$ *then*

$$\frac{a_2}{q_2} = \frac{a+a_1}{q+q_1} \tag{A.4.2}$$

Moreover there are some simpler properties of $\mathfrak{F}_n$ that will be useful in the definition of our Farey dissection of the unit circle.

Theorem A.4.4. *If* a/q*,* a_1/q_1 *are two successive terms of* $\mathfrak{F}_n$*, then*

$$q + q_1 > n \tag{A.4.3}$$

Theorem A.4.5. *If* $n > 1$*, then no two successive terms of* $\mathfrak{F}_n$ *have the same denominator.*

Now the idea is to represent the real numbers on a circle instead of on a straight line, via the map $x \mapsto e(x)$. We consider a circle C of unit circumference and a point O of the circumference as the representative of 0. Then we represent x by the point P_x whose distance from O, measured round the circumference in the counter-clockwise direction, is x. Moreover all the integers are represented by the same point O, and numbers which differ by an integer have the same representative point.

We can dissect the circumference of C in the following way: consider the mediants

$$\mu = \frac{a+a_1}{q+q_1}$$

of successive pairs $a/q, a_1/q_1$ of number of the Farey series. We recall that the mediants naturally do not belong themselves to $\mathfrak{F}_n$; we represent each of them with a point P_μ. Now the circle is divided up into arcs which we call *Farey arcs*, each bounded by two points $\mathfrak{F}_n$ and containing one *Farey point*, the representative of a term of $\mathfrak{F}_n$. The aggregate of the Farey arcs is called the *Farey dissection* of the circle.

A.5 Continued fractions

Definition A.5.1. *Let $a_0, a_1, \ldots, a_N$ be $N+1$ variables, then we call* ***finite continued fraction*** *the following function*

$$a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots}}} \tag{A.5.1}$$

Usually we will use the following two forms to write (A.5.1):

$$a_0 + \frac{1}{a_1+} \frac{1}{a_2+} \cdots \frac{1}{a_N}$$

or

$$[a_0, a_1, a_2, \ldots, a_N].$$

We call $a_0, a_1, \ldots, a_N$ the **partial quotient** of the continued fraction.

A.5.1 Convergents of a continued fraction

Definition A.5.2. *We call*

$$[a_0, a_1, \ldots, a_n] \qquad (0 \leq n \leq N)$$

the n-th ***convergent*** *to $[a_0, a_1, \ldots, a_N]$.*

We can calculate the convergents by means of the following Theorem.

Theorem A.5.3. *If p_n and q_n are defined by*

$$p_0 = a_0, \quad p_1 = a_1 a_0 + 1, \quad p_n = a_n p_{n-1} + p_{n-2} \quad (2 \leq n \leq N) \tag{A.5.2}$$

$$q_0 = 1, \quad q_1 = a_1, \quad q_n = a_n q_{n-1} + q_{n-2} \quad (2 \leq n \leq N) \tag{A.5.3}$$

then

$$[a_0, a_1, \ldots, a_n] = \frac{p_n}{q_n}. \tag{A.5.4}$$

Now we recall some Theorem on the convergents, for details we refer to the book of Hardy and Wright [13].

Theorem A.5.4. *The functions p_n and q_n satisfy*

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$$

or

$$\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^{n-1}}{q_{n-1} q_n}.$$

We consider continued fractions with positive convergents, then the following Theorems hold

Theorem A.5.5. *The even convergents x_{2n} increase strictly with n, while the odd ones x_{2n+1} decrease strictly.*

Theorem A.5.6. *The value of the continued fraction is greater than that of any of its even convergents and less than that of any of its odd convergents (except that it is equal to the last convergent, whether this be even or odd).*

A.5.2 Infinite simple continued fractions

Suppose that $a_0, a_1, a_2, \ldots$ is a sequence of integers such that $a_i > 0$ for $i \geq 1$ and

$$x_n = [a_0, a_1, \ldots, a_n]$$

is, for every n, a simple continued fraction representing a rational number x.

Theorem A.5.7. *If $a_0, a_1, a_2, \ldots$ is a sequence of integers such that $a_i > 0$ for $i \geq 1$, then $x_n = [a_0, a_1, \ldots, a_n]$ tends to a limit x when $n \to \infty$.*

Theorem A.5.8. *All infinite simple continued fractions are convergent. We write the convergents*

$$x_n = \frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n]$$

for $n \geq 1$.

Now let

$$a'_n = [a_n, a_{n+1}, \ldots]$$

be the n-th complete quotient of the continued fraction $x = [a_0, a_1, \ldots]$.

Theorem A.5.9. *Every irrational number can be expressed in just one way as an infinite simple continued fraction.*

Theorem A.5.10. *For every infinite continued fraction*

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}} < \frac{1}{q_n^2}.$$

Theorem A.5.11. *If*

$$x = \frac{P\zeta + R}{Q\zeta + S}$$

where $\zeta > 1$ and P, Q, R, S are integers such that

$$Q > S > 0, \qquad PS - QR = \pm 1,$$

then R/S and P/Q are two consecutive convergents to the simple continued fraction whose value is x. If R/S is the $(n-1)$-th convergent, and P/Q the n-th, then ζ is the $(n+1)$-th complete quotient.

A.6 Lemmas and Theorems

Lemma A.6.1 (Kaczorowski-Perelli-Pintz)**.** *Assume GRH. Let*

$$\psi'(2N, \chi, \eta) = \sum_{n \le 2N} \Lambda(n)\chi(n)e(n\eta) - \delta_\chi T(\eta), \quad \delta_\chi = \begin{cases} 1 \text{ if } \chi = \chi_0 \\ 0 \text{ if } \chi \neq \chi_0 \end{cases}$$

where $T(\eta)$ is defined in (1.2.4)*. Then for any $\chi(\text{mod} q)$*

$$\int_{-1/qQ}^{1/qQ} |\psi'(2N, \chi, \eta)|^2 d\eta \ll \frac{NL^4}{qQ}.$$

For the proof of this Lemma see the article of Kaczorowski, Perelli and Pintz [17] and its corrigendum [18].

Lemma A.6.2 (Goldston)**.** *Let*

$$G(x; a, b) = \sum_{r \le x} \frac{\mu(r)^2 r^a}{\varphi(q)^b}$$

for real numbers a, b with $a - b > -1$, then we have

$$G(x; a, b) = \frac{g(a - b + 1; a, b)}{a - b + 1} x^{a-b+1} + o_{a,b}(x^{a-b+1})$$

where

$$g(s;a,b) = \prod_p \left(1 - \frac{1 - p^{s-a+b}(1-(1-1/p)^b)}{(p-1)^b p^{2(s-a)+b}}\right).$$

For more details see §2, p.125 of [8].

Lemma A.6.3 (Pintz-Ruzsa)**.** *Assume GRH. Then*

$$\int_{\mathfrak{m}} |S(\alpha)G(\alpha)|^2 d\alpha \le \frac{2}{(\log 2)^2} C_2' N$$

where $C_2' < 3.9095$.

For more details see Lemma 13 of Pintz and Ruzsa [41].

Corollary A.6.4 (Pintz-Ruzsa)**.** *Assume GRH. Let* $1/2 \le \beta < 1$ *fixed, then there exists an effectively computable constant* $d = d(\beta) < 1$ *such that*

$$|G_L(\alpha)| = |\sum_{j=0}^{L-1} e(2^j \alpha)| \le d(\beta) L$$

if $\alpha \in [0,1] \backslash \mathcal{E}$, *where* $\mu(\mathcal{E}) = |\mathcal{E}| = O(N^{-\beta} L^{-100})$. *In particular, if* $\beta \to 1$ *then* $d(\beta) \to 1$ *and* $d(1/2) = 0.7163435444776661$.

Theorem A.6.5. *We have*

$$\liminf \frac{\varphi(n) \log\log n}{n} = e^{\gamma}$$

for $n \to +\infty$.

For more details see §18.4 of Hardy and Wright [13].

A.7 Results of computations

In this section we will collect the results of computations that we do to find the value of $|\mathcal{E}|$ that we need in Chapter 5 and we describe the method that we use to find them. Let k be the number of powers of 2 in the Goldbach-Linnik problem, $(1-\eta)$ the value found with the computations in §5.2 and $|\mathcal{E}| = N^{-s}$, then we want to find the value of s for each value of $3 \le k \le 6$. To do this we use a program, created by Alessandro Languasco, which takes as entry the value s and returns the value $(1-\eta)$; for brevity we call the program Lang(s). Furthermore we need a list of values given again by Languasco, that collects some values of s and their values of $(1-\eta)$. We give a name to all this values of s, that is s_i for $i \in \mathbb{N}$ and we call r_i the respective value of $(1-\eta)$. Then we act in the following way:

1. We choose from the list a value s_i such that r_i is near $(1-\eta)$;

2. We take from the list the value s_{i-1} and we compute the average of s_i and s_{i-1}, that is $s_j = (s_i + s_{i-1})/2$, then r_j =Lang(s_j);

3. If $r_j < (1-\eta)$ then compute $s_{j+1} = (s_j + s_{i-1})/2$ and r_{j+1} =Lang(s_{j+1})

otherwise compute $s_{j+1} = (s_j + s_i)/2$ and r_{j+1} =Lang(s_{j+1})

4. We repeat step 3 until we reach a value of r that is very close to $(1-\eta)$, then the value of s related to it is the value that we were looking for.

We can observe that, since $s_i = a_i/b_i$, if we proceed considering the average between two value of s_i, we reach very quickly values of a_i and b_i that are large, then we decide to use the same steps described over here but using mediants instead of averages. If we have $s_i = a_i/b_i$ and $s_{i-1} = a_{i-1}/b_{i-1}$ the mediants is $s_j = (a_i + a_{i-1})/(b_i - b_{i-1})$.

In the following tables we collect our computations for each value of k using both averages and mediants.

Table 1

$k=3$ $(1-\eta) < 0.216499553028464237420002896 6$			
Average		Mediants	
Entry	Result	Entry	Result
$1/18 + 10^{-10}$	0.214307436510237	$2/35 + 10^{-10}$	0.217586745040090
$35/612 + 10^{-10}$	0.217682591709440	$3/53 + 10^{-10}$	0.216477511405988
$23/408 + 10^{-10}$	0.216000330782897	$4/71 + 10^{-10}$	0.215929016789708

Table 2

$k = 4$	$(1-\eta) < 0.4915221192975883608179476389888\,2$		
Average		Mediants	
Entry	Result	Entry	Result
$1/4 + 10^{-10}$	0.489108914786800	$2/7 + 10^{-10}$	0.526732775198028
$13/48 + 10^{-10}$	0.511330238556573	$3/11 + 10^{-10}$	0.513311530593329
$25/96 + 10^{-10}$	0.500319422060185	$4/15 + 10^{-10}$	0.506949226922104
$49/192 + 10^{-10}$	0.494739780354024	$5/19 + 10^{-10}$	0.503235949176656
$97/384 + 10^{-10}$	0.491930835559473	$6/23 + 10^{-10}$	0.500802231320754
$193/768 + 10^{-10}$	0.490521508038745	$7/27 + 10^{-10}$	0.499083859917304
$193/768 + 10^{-5}$	0.490532343981749	$8/31 + 10^{-10}$	0.497805841127635
$387/1536 + 10^{-10}$	0.491226578645889	$9/35 + 10^{-10}$	0.496818126534589
		$10/39 + 10^{-10}$	0.496031887414767
		$11/43 + 10^{-10}$	0.495391181942612
		$12/47 + 10^{-10}$	0.494859024686886
		$13/51 + 10^{-10}$	0.494409984141755
		$14/55 + 10^{-10}$	0.494025997812252
		$15/59 + 10^{-10}$	0.493693883420393
		$16/63 + 10^{-10}$	0.493403794971336
		$17/67 + 10^{-10}$	0.493148230082613
		$18/71 + 10^{-10}$	0.492921371838357
		$19/75 + 10^{-10}$	0.492718640639646
		$20/79 + 10^{-10}$	0.492536381826326
		$21/83 + 10^{-10}$	0.492371643387747
		$22/87 + 10^{-10}$	0.492222014840801
		$23/91 + 10^{-10}$	0.492085508499601
		$24/95 + 10^{-10}$	0.491960470671118
		$25/99 + 10^{-10}$	0.491845514331525
		$26/103 + 10^{-10}$	0.491739467456229
		$27/107 + 10^{-10}$	0.491641332915682
		$28/111 + 10^{-10}$	0.491550257025206
		$29/115 + 10^{-10}$	0.491465504645818
		$57/226 + 10^{-10}$	0.491507132284718

Table 3

$k = 5$ $\quad (1-\eta) < 0.6287527895$			
Average		Mediants	
Entry	Result	Entry	Result
$1/3 + 10^{-10}$	0.573691564239119	$2/5 + 10^{-10}$	0.634352120490386
$5/12 + 10^{-10}$	0.648713782020717	$3/8 + 10^{-10}$	0.612231635749988
$9/24 + 10^{-10}$	0.612231635749988	$5/13 + 10^{-10}$	0.620824041974019
$17/48 + 10^{-10}$	0.593235539687793	$7/18 + 10^{-10}$	0.624608617077183
$35/96 + 10^{-10}$	0.602799786521891	$9/23 + 10^{-10}$	0.626738540141600
$19/48 + 10^{-10}$	0.630714453778923	$11/28 + 10^{-10}$	0.628104300619421
$37/96 + 10^{-10}$	0.621535240230047		
$75/192 + 10^{-10}$	0.626140165951504		
$151/384 + 10^{-10}$	0.628431111772854		

Table 4

$k=6$	$(1-\eta) < 0.7087338843254031693455710041140\,4$		
Average		Mediants	
Entry	Result	Entry	Result
$1/2 + 10^{-10}$	0.716343544554577	$2/5 + 10^{-10}$	0.634352120490386
$5/12 + 10^{-10}$	0.648713782020717	$3/7 + 10^{-10}$	0.658792099220323
$11/24 + 10^{-10}$	0.683361656536514	$4/9 + 10^{-10}$	0.672004833737808
$23/48 + 10^{-10}$	0.700052052803369	$5/11 + 10^{-10}$	0.680282894012563
$47/96 + 10^{-10}$	0.708246690420190	$6/13 + 10^{-10}$	0.685956026260663
		$7/15 + 10^{-10}$	0.690086716822662
		$8/17 + 10^{-10}$	0.693228761000133
		$9/19 + 10^{-10}$	0.695699169573014
		$10/21 + 10^{-10}$	0.697692503806505
		$11/23 + 10^{-10}$	0.699334792730406
		$12/25 + 10^{-10}$	0.700711266346573
		$13/27 + 10^{-10}$	0.701881629465817
		$14/29 + 10^{-10}$	0.702888951770892
		$15/31 + 10^{-10}$	0.703765084253517
		$16/33 + 10^{-10}$	0.704534088694507
		$17/35 + 10^{-10}$	0.705214481573457
		$18/37 + 10^{-10}$	0.705820744522825
		$19/39 + 10^{-10}$	0.706364366576179
		$20/41 + 10^{-10}$	0.706854579215513
		$21/43 + 10^{-10}$	0.707298884909854
		$22/45 + 10^{-10}$	0.707703443805272
		$23/47 + 10^{-10}$	0.708073361078919
		$24/49 + 10^{-10}$	0.708412903506130
		$25/51 + 10^{-10}$	0.708725664782302

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