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SOME ANALYTIC AND GEOMETRIC ASPECTS OF THE p-LAPLACIAN ON RIEMANNIAN MANIFOLDS

MAT/03

Candidato	Relatore
Giona Veronelli	**Stefano Pigola**
Matricola	Coordinatore del Dottorato
R07466	**Prof. Marco Peloso**

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Contents

Introduction

In the mid 1960's, J. Eells and J.H. Sampson extended the notion of harmonicity from real-valued functions to manifold-valued maps, [ES]. The topological relevance of harmonic maps, already visible in the seminal paper [ES], became clear in the works by P. Hartman, L. Lemaire, R. Hamilton and others authors. See e.g. [Har], [EL], [Ham]. Notably, R. Schoen and S.T. Yau developed the theory and topological consequences of harmonic maps with finite energy from a non-compact domain, [SY2], [SY3], [SY1].
A natural extension of the concept of a harmonic map is that of a p-harmonic map. To this end, a great deal of work has been done by B. White, [Wh], R. Hardt and F.-H. Lin, [HL], and S.W. Wei, [We2], [We1]. In particular, it is known that p-harmonic maps give information on the higher homotopy groups, and on the homotopy class of higher energy maps between Riemannian manifolds, [We2]. In view of these topological links we are led to understand which of the well known results holding in the harmonic case can be extended to the non-linear setting. In this thesis, I study some problems related to the existence, uniqueness and triviality of the p-harmonic representative in the homotopy class of a map.
First, we recall that a C^1 map $u : (M, \langle,\rangle_M) \to (N, \langle,\rangle_N)$ between Riemannian manifolds is said to be p-harmonic, $p > 1$, if its p-tension field $\tau_p u$ vanishes everywhere, i.e. if u satisfies the non-linear system

$$\tau_p u = \operatorname{div}\left(|du|^{p-2}\, du\right) = 0. \tag{1}$$

Here, $du \in T^*M \otimes u^{-1}TN$ denotes the differential of u and the bundle $T^*M \otimes u^{-1}TN$ is endowed with its Hilbert-Schmidt scalar product $\langle,\rangle_{HS}$. Moreover, $-\operatorname{div}$ stands for the formal adjoint of the exterior differential d with respect to the standard L^2 inner product on vector-valued 1-forms. Observe that, when $N = \mathbb{R}$, τ_p coincides with the standard p-laplace operator Δ_p. Clearly, in general equality (1) has to be considered in the weak sense, i.e.

$$\int_M \left\langle |du|^{p-2} du, d\eta \right\rangle_{HS} = 0$$

for every smooth compactly supported $\eta \in \Gamma(u^{-1}TN)$. In case $p = 2$, the non-linear factor $|du|^{p-2}$ disappears and the 2-harmonic map is simply called harmonic.

The starting point of this work are the following celebrated results due to Schoen and Yau dating to the 1970s, [SY2], [SY3].

Theorem I (Schoen-Yau). *Let M be a complete manifold with non-negative Ricci curvature and N be a compact manifold with non-positive curvature. Let f be any smooth map from M to N with finite energy $|df|^2 \in L^1(M)$. Then f is homotopic to a constant on each compact set.*

Theorem II (Schoen-Yau). *Let M and N be complete Riemannian manifolds with* $\operatorname{Vol} M < \infty$.

i) Let $u : M \to N$ be a harmonic map of finite energy. If N has negative sectional curvature, there's no other harmonic map of finite energy homotopic to u unless $u(M)$ is contained in a geodesic of N.

ii) If N has non-positive sectional curvature and $u, v : M \to N$ are homotopic harmonic maps of finite energy, then there is a smooth one-parameter family $u_t : M \to N$, of harmonic maps with $u_0 = u$ and $u_1 = v$. Moreover, for each $x \in M$, the curve $\{u_t(x) : t \in \mathbb{R}\}$ is a constant (independent of x) speed parametrization of a geodesic.

Recently, S. Pigola, M. Rigoli and A.G. Setti extended both these results. First, they generalized Theorem I to the case where the domain manifold presents an amount of negative Ricci curvature, say

$$^{M}\operatorname{Ric} \geq -k(x), \tag{2}$$

provided $k(x) \geq 0$ is small in a suitable spectral sense, i.e. the Schrödinger operator $L = -\Delta - k(x)$ has non-negative spectral radius $\lambda_1(-\Delta - k(x)) \geq 0$, [PRS2]. The strategy in Schoen-Yau proof, as well as in the subsequent extension alluded to above, consists of two main steps: (a) an existence result for a (smooth) harmonic map with finite energy in the homotopy class of f and (b) a Liouville type theorem for finite energy harmonic maps. As shown by S.W. Wei, [We1], step (a) can be generalized to maps u with finite p-energy $|du|^p \in L^1(M)$ up to using $C^{1,\alpha}$ p-harmonic representatives with finite p-energy. The first main result of the thesis is a Liouville-type theorem for finite q-energy, p-harmonic maps under spectral assumptions, which generalizes to the non-linear case step (b) under spectral assumptions. The $p > 2$ case of the following result is contained in [PV].

Theorem A. *Let $u : M \to N$ be a continuous map from a complete manifold $(M, \langle , \rangle)$ with Ricci curvature satisfying (2) into a compact manifold of non-positive sectional curvature. Assume that u has finite p-energy $|du|^p \in L^1(M)$, with $p \geq 2$. If the Schrödinger operator $^{H}L = -\Delta - Hk(x)$ satisfies*

$$\lambda_1\left(^{H}L\right) \geq 0,$$

for some

$$H > \begin{cases} p^2/4(p-1) & \text{if } p > 2 \\ (m-1)/(m) & \text{if } p = 2, \end{cases}$$

then u is homotopic to a constant.

With respect to the harmonic case, difficulties arise since, for $p \neq 2$, standard tools of harmonic maps theory do not hold (e.g. refined Kato inequality, smooth regularity, properties of composition with convex functions) or do not work so well (e.g. Bochner-Weitzenböck identity). Then, it is necessary to do weak computations, combined with some approximation procedures. By the way, as it is clear by the statement, Theorem A in a sense improves also the linear case, by permitting values of H lower than 1. This weakens Pigola, Rigoli and Setti's extension of Theorem I and permits the application to minimal immersions which are less then stable (in some sense to be specified), slightly extending also the original work of Schoen and Yau, [SY2]. After a brief introductory Chapter 1, these topics will be dealt with in Chapter 2.

According to Wei's existence theorem cited above, p-harmonic maps can be considered as "canonical" representatives of homotopy class of maps with finite p-energy. Hence, one is led to investigate such a space, in particular inquiring how many p-harmonic representatives can be found in a given homotopy class. This will be the content of Chapter 3. A first uniqueness result in this direction was obtained by Wei, [We2], for smooth p-harmonic maps defined on compact M, generalizing a previous result for $p = 2$ due to Hartman, [Har]. An interesting task is then to detect a similar result for complete non-compact manifolds. In the harmonic setting, the most important result is represented by Theorem II, subsequently extended in [PRS3] replacing $\operatorname{Vol} M < \infty$ with the parabolicity of M. As a matter of fact, an inspection of Schoen and Yau's proof shows that they strongly use the property of (2-)harmonic maps to be (2-)subharmonic, once composed with a convex function. It turns out that, in general, this is not true if $p \neq 2$. By assuming some rotational symmetry on manifolds and functions, we find an example of a p-harmonic map between Riemannian manifolds $F : M \to N$ and a convex function $H : N \to \mathbb{R}$, whose composition $H \circ F$ is not p-subharmonic for some $p \neq 2$. This is the second main result of the thesis, contained in [V], which answers in the negative an open question arisen in the 2006 Midwest Geometry Conference paper by Lin and Wei, [LW].

Theorem B. *Consider two rotationally symmetric $(n+1)$-dimensional manifolds*

$$M_g = ([0, +\infty) \times \mathbb{S}^n, ds^2 + g^2(s)d\theta^2)$$
$$N_j = ([0, +\infty) \times \mathbb{S}^n, dt^2 + j^2(t)d\theta^2).$$

Suppose that $(n+1) > p > \max\{2, n\}$ and assume that the warping functions $g, j \in C^2([0, +\infty))$ have the form

$$g(s) = (s + \delta^{-\frac{1}{\delta-1}})^\delta - \delta^{-\frac{\delta}{\delta-1}}, \qquad j(t) = (t + \sigma^{\frac{1}{1-\sigma}})^\sigma - \sigma^{\frac{\sigma}{1-\sigma}},$$

where $\delta > (p-n)^{-1} > 1$ and $0 < \sigma < 1$. Then, there exist a C^2 rotationally symmetric p-harmonic map $F : M_g \to N_j$ and a sequence $\{s_k\}_{k=1}^\infty \to +\infty$, such that

$$\Delta_p(H \circ F)(s_k, \theta) < 0$$

for every rotationally symmetric convex function $H : N_j \to \mathbb{R}$, provided the corresponding $h \in C^2([0, +\infty))$ satisfies $h'(t) > 0$ for $t > 0$.

Hence, one is led to follow different paths in order to deal with the non-linear analogous of Schoen and Yau's result.
In this direction, some progresses in the special situation of a single map homotopic to a constant has been made in [PRS3], where the authors introduced a special composed vector field which permits to deduce a vanishing result for the gradient of the distance of the map from a fixed origin in N, without informations on the p-subharmonicity of the distance function. This is achieved through the application of a global form of the divergence theorem in non-compact settings due to V. Gol'dshtein and M. Troyanov which goes under the name of Kelvin-Nevanlinna-Royden criterion, [GT2]. Let us focus our attention on the case $N = \mathbb{R}^n$. According to [PRS3], if M is p-parabolic, then every p-harmonic map $u : M \to \mathbb{R}^n$ with finite p-energy $|du| \in L^p(M)$ must be constant. However, using the very special structure of $\mathbb{R}^n$, we are able to extend this conclusion, [HPV].

Theorem C. *Suppose that $(M,\langle,\rangle)$ is a p-parabolic manifold, $p>1$. Let $u,v:M\to\mathbb{R}^n$ be $C^0\cap W^{1,p}_{loc}(M)$ maps satisfying*

$$\begin{cases}\tau_p u=\tau_p v, & \text{if } n>1,\\ \Delta_p u\geq \Delta_p v & \text{if } n=1,\end{cases}$$

in the sense of distributions on M and

$$|du|\,,|dv|\in L^p\left(M\right).$$

Then $u-v$ is constant.

Since $\mathbb{R}^n$ is contractible, in this situation all maps are trivially homotopic. Consequently, Theorem C can be seen as a special case in the comprehension of general comparison results for homotopic p-harmonic maps. However, the proof is based on the good special structure of $\mathbb{R}^n$, which permits to compare in a standard way (i.e. considering their difference) vectors with different base points. Hence, though the procedure is non trivial due to the non linearity of τ_p, the problem is somehow reduced to that of a single map. A fundamental ingredient in the proof of Theorem C is a version for the p-Laplacian of a classical inequality for the mean-curvature operator. By the way, this permits also to obtain a similar comparison for real valued functions when M is not necessarily p-parabolic and u,v and $\nabla u,\nabla v$ satisfy some integral decay assumptions.
In the last part of this thesis, combining the techniques introduced by [PRS3] with those used in the proof of Theorem II and Theorem C, we finally manage to prove the desired general comparison for homotopic p-harmonic maps.

Theorem D. *Let M and N be complete Riemannian manifolds and assume that M is p-parabolic, $p\geq 2$.*

i) Let $u:M\to N$ be a $C^{1,\alpha}$ p-harmonic map of finite p-energy. If ${}^N\mathrm{Sect}<0$, there's no other p-harmonic map of finite p-energy homotopic to u unless $u(M)$ is contained in a geodesic of N.

ii) If ${}^N\mathrm{Sect}\leq 0$ and $u,v:M\to N$ are homotopic $C^{1,\alpha}$ p-harmonic maps of finite p-energy, then there is a continuous one-parameter family of maps $u_t:M\to N$ with $u_0=u$ and $u_1=v$ such that the p-energy of u_t is constant (independent of t) and for each $q\in M$ the curve $t\mapsto u_t(q)$, $t\in[0,1]$, is a constant (independent of q) speed parametrization of a geodesic. Moreover, if N is compact, u_t is a p-harmonic maps for each $t\in[0,1]$.

All the results stated above require that the domain manifold M is p-parabolic in order to allow us to apply the global form of the divergence theorem in non-compact settings given by Kelvin-Nevanlinna-Royden criterion. This fact suggests that the p-parabolicity assumption could be dropped once we give conditions, on both M and the maps, which ensure the validity of some Stokes' type results. To this end, in Appendix A we report a possible approach suggested in [VV] which links the volume growth of the geodesic spheres on M with the behaviour at infinity of the (p-)energy functional.
Finally, in Appendix B we discuss some open problems arisen throughout this thesis.

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Chapter 1

p-harmonicity and p-parabolicity

Throughout all this thesis, $(M, \langle , \rangle_M)$ and $(N, \langle , \rangle_N)$ are smooth Riemannian manifolds of dimensions m and n respectively, endowed with the Riemannian metrics $\langle , \rangle_M$ and $\langle , \rangle_N$. We consider local chart $\{x^i\}_{i=1}^m$ on M and $\{y^A\}_{A=1}^n$ on N. Moreover lower cases indexes $i, j, k, \dots$ and capitol indexes $A, B, C, \dots$ refer to objects on M and N respectively. In particular, $\langle , \rangle_M$ and $\langle , \rangle_N$ have coordinates expression

$$(\langle , \rangle_M)_{ij} = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle_M, \qquad (\langle , \rangle_N)_{AB} = \left\langle \frac{\partial}{\partial y^A}, \frac{\partial}{\partial y^B} \right\rangle_N,$$

while $(\langle , \rangle_M)^{ij}$ and $(\langle , \rangle_N)^{AB}$ denote the components of the inverse metric matrices. The metric tensors of M induces a unique Levi-Civita connection ${}^M\nabla : TM \times TM \to TM$ such that, for each pair of vector fields X, Y on M

$$\left({}^M\nabla_{X_M} Y_M\right)^k = X_M^i \frac{\partial Y_M^k}{\partial x^i} + X_M^i Y_M^j \, {}^M\Gamma_{ij}^k$$

where ${}^M\Gamma_{ij}^k$ are the Christhoffel symbols on M. Finally we introduce the Riemann, sectional and Ricci curvature tensors related to ${}^M\nabla$ defined for all $X, Y, Z, W \in TM$ as

$$\begin{aligned}
&{}^M\operatorname{Riem}(X,Y,Z,W) = \left\langle {}^M R(X,Y)Z, W \right\rangle_M, \text{ where}\\
&{}^M R(X,Y)Z = {}^M\nabla_X {}^M\nabla_Y Z - {}^M\nabla_Y {}^M\nabla_X Z - {}^M\nabla_{[X,Y]} Z;\\
&{}^M\operatorname{Sect}(X \wedge Y) = \frac{{}^M\operatorname{Riem}(X,Y,Y,X)}{\langle X,X\rangle_M \langle Y,Y\rangle_M - \langle X,Y\rangle_M^2};\\
&{}^M\operatorname{Ric}(X,Y) = {}^M\operatorname{tr} {}^M\operatorname{Riem}(\cdot, X, Y, \cdot).
\end{aligned}$$

On N, the Levi-Civita connection ${}^N\nabla$ and the curvature tensors ${}^N\operatorname{Riem}$, ${}^N R$, ${}^N\operatorname{Sect}$ and ${}^N\operatorname{Ric}$ are defined in analogous way.
Suppose we have fixed a reference origin $o \in M$. We set $r_M(q) = \operatorname{dist}_M(q, o)$ and we denote by B_t^M and ∂B_t^M the geodesic ball and sphere of radius $t > 0$ centered at o. Finally, the symbol $\operatorname{Vol}\left(B_t^M\right)$ stands for the volume of B_t^M in

the Riemannian volume measure dV_M (see Section 1.1 for the definition), while $\mathcal{A}\left(\partial B_t^M\right)$ stands for the $(m-1)$-dimensional Hausdorff measure $\mathcal{H}^{m-1}$ of ∂B_t^M. We write B_t instead of B_t^M when it is clear from the contest which manifold we are dealing with.

1.1 p-harmonic maps

Consider an isometric immersion $i : N \hookrightarrow \mathbb{R}^q$ of N into some Euclidean space $\mathbb{R}^q$, $q \geq n$. For $p > 1$, we denote by $W^{1,p}_{loc}(M, \mathbb{R}^q)$ (resp. $W^{1,p}(M, \mathbb{R}^q)$) the Sobolev space of maps $v : M \to \mathbb{R}^q$ whose component functions and their first weak derivatives are in $L^p_{loc}(M)$ (resp. in $L^p(M)$). Moreover we define

$$\begin{aligned} W^{1,p}_{loc}(M, N) &:= \{v \in W^{1,p}_{loc}(M, \mathbb{R}^q) : v(x) \in N \text{ for a.e. } x \in M\}, \\ W^{1,p}(M, N) &:= \{v \in W^{1,p}(M, \mathbb{R}^q) : v(x) \in N \text{ for a.e. } x \in M\}. \end{aligned}$$

Let $u : M \to N$ be a C^1 map and fix a point $x \in M$. The p-energy density $e_p(u) : M \to \mathbb{R}$ is the nonnegative function defined on M as

$$e_p(u)(x) = \frac{1}{p}|du|^p_{HS}(x).$$

Here the differential du is considered as a section of the $(1,1)$-tensor bundle along the map u, i.e. $du \in \Gamma(T^*M \times u^{-1}TN)$ is a vector valued differential 1-form. Moreover $T^*M \times u^{-1}TN$ is endowed with its Hilbert-Schmidt scalar product $\langle , \rangle_{HS}$ defined, for every $W_1, W_2 \in T^*M \times u^{-1}TN$ as

$$\begin{aligned} \langle W_1, W_2 \rangle_{HS}(x) :&= {}^M\mathrm{tr}\left[\langle W_1, W_2 \rangle_N (u(x))\right] \\ &= (W_1)_i^A(x)(W_2)_j^B(x)(\langle , \rangle_M)^{ij}(x)(\langle , \rangle_N)_{AB}(u(x)). \end{aligned}$$

With standard notation, we denote $|W_1|^2_{HS} := \langle W_1, W_1 \rangle_{HS}$. When the meaning is clear, we possibly omit the subscript HS on the norms.
If $\Omega \subset M$ is a compact domain, we use the canonical volume measure

$$dV_M := \sqrt{\det\left(\langle , \rangle_M\right)_{ij}}\, dx^1 \wedge \cdots \wedge dx^m$$

associated to $\langle , \rangle_M$ to define the p-energy of $u|_\Omega : (\Omega, \langle , \rangle_M) \to (N, \langle , \rangle_N)$ by

$$E_p^\Omega(u) = \int_\Omega e_p(u) dV_M.$$

The map $u : (M, \langle , \rangle_M) \to (N, \langle , \rangle_N)$ is said to be p-harmonic if, for each compact domain $\Omega \subset M$, it is a stationary point of the p-energy functional $E_p^\Omega : C^1(M, N) \to \mathbb{R}$ with respect to the variations which preserve u on $\partial\Omega$.
A vector field X along u, i.e., a section of the bundle $u^{-1}TN$ determines a variation u_t of u by setting

$$u_t(x) = {}^N\exp_{u(x)} tX(x).$$

Suppose for the moment that $u \in C^2(M, N)$. If X has support in a compact domain $\Omega \subset M$, then

$$\left.\frac{d}{dt}\right|_{t=0} E_p^\Omega(u_t) = -\int_M \langle \tau_p u(x), X(x) \rangle_N \, dV_M, \tag{1.1}$$

where the Euler-Lagrange operator $\tau_p u \in \Gamma(u^{-1}TN)$, called the p-tension field (or simply p-laplacian) of u, is given by

$$\tau_p u = {}^M \operatorname{div}(|du|_{HS}^{p-2} du). \tag{1.2}$$

As a consequence, $\tau_p(u) \in \Gamma(u^{-1}TN)$ and u is p-harmonic if and only if it satisfies the nonlinear system

$$\tau_p(u) = 0 \qquad \text{on } M. \tag{1.3}$$

Here and in what follows, $-{}^M \operatorname{div} = {}^M\delta$ is the formal adjoint of the exterior differential d, with respect to the standard L^2 inner product on vector-valued differential 1-forms on M. Moreover, for the ease of notation, sometimes we will denote

$$\mathcal{K}_p(u) := |du|_{HS}^{p-2} du.$$

Computing the divergence in (1.2), $\tau_p u$ is given by

$$\tau_p u = |du|^{p-2}\tau_2 u + i\left(\nabla |du|^{p-2}\right) du,$$

where i denotes the interior product on 1-forms. Omitting the index 2, the operator τ_2 is usually denoted by τ and is called simply tension field or Laplace operator. In local coordinates τ takes the expression

$$(\tau u)^A = (\langle , \rangle_M)^{ij}\left(\frac{\partial^2 u^A}{\partial x^i \partial x^j} - {}^M\Gamma_{ij}^k \frac{\partial u^A}{\partial x^k} + {}^N\Gamma_{BC}^A \frac{\partial u^B}{\partial x^i}\frac{\partial u^C}{\partial x^j}\right)$$

Clearly, for $u \in C^1(M,N) \setminus C^2(M,N)$, equality (1.3) has to be understood in the weak sense, namely

$$\int_M \left\langle |du|^{p-2} du, d\eta \right\rangle_{HS} dV_M = 0$$

for every smooth compactly supported $\eta \in \Gamma\left(u^{-1}TN\right)$.
We recall that a 2-harmonic map is usually called a harmonic map. Moreover, whenever $N = \mathbb{R}$, τ_p is denoted by Δ_p (or simply Δ when $p = 2$) and corresponds to the classical p-Laplace operator on the underlying manifold. In this case, given a real function $\varphi : M \to \mathbb{R}$, its p-laplacian $\Delta_p\varphi$ takes values in $\mathbb{R}$ and it makes sense to say that φ is p-subharmonic (resp. p-superharmonic) provided

$$\Delta_p \varphi \geq 0 \text{ (resp. } \leq 0) \qquad \text{on } M. \tag{1.4}$$

As before, for $\varphi \in C^1(M) \setminus C^2(M)$, equality (1.4) has to be understood in the weak sense, namely

$$\int_M \left\langle |\nabla\varphi|^{p-2}\nabla\varphi, \nabla\xi \right\rangle_M dV_M \leq 0 \text{ (resp. } \geq 0),$$

for every smooth compactly supported function $\xi \in C_c^\infty(M)$.

1.2 p-parabolicity and related properties

A Riemannian manifold $(M, \langle , \rangle_M)$ is said to be p-parabolic, $p > 1$, if for some (hence every) compact set $K \subset M$ with non empty interior the p-capacity of K is null, i.e.

$$\mathrm{Cap}_p(K) := \inf\left\{\int_M |\nabla\varphi|^p dV_M \;:\; \varphi \in W_0^{1,p}(M) \cap C_c^0(M),\; \varphi|_K \geq 1\right\} = 0,$$

where the Sobolev space $W_0^{1,p}(M)$ is the closure of $C_c^1(M)$, the space of compactly supported C^1 functions, with respect to the Sobolev norm

$$\|\varphi\|_{1,p} := \|\varphi\|_{L^p} + \|\nabla\varphi\|_{L^p}.$$

Note that, by standard density argument, we have

$$\mathrm{Cap}_p(K) = \inf\left\{\int_M |\nabla\varphi|^p dV_M \;:\; \varphi \in C_c^\infty(M),\; \varphi|_K \geq 1\right\}.$$

It is well known that this is just one of the several equivalent definitions of p-parabolicity; see [Ho1, Tr, PST]. For instance, and in view of future purposes, we recall the next

Proposition 1.1. *Let $(M, \langle , \rangle_M)$ be a complete Riemannian manifold. The following conditions are equivalent.*

(i) *M is p-parabolic.*

(ii) *If $\varphi \in C^0(M) \cap W_{loc}^{1,p}(M)$ is a bounded above weak solution of $\Delta_p\varphi \geq 0$, i.e.*

$$\int_M |\nabla\varphi|^{p-2} \langle \nabla\varphi, \nabla\eta\rangle_M \leq 0, \qquad \forall 0 \leq \eta \in C_c^\infty(M),$$

then φ is constant.

(iii) *For every domain $\Omega \subset M$ and for every $\psi \in C(\bar{\Omega}) \cap W_{loc}^{1,p}(M)$ which is bounded above and satisfies $\Delta_p\psi \geq 0$ weakly on Ω, $\sup_\Omega \psi = \sup_{\partial\Omega} \psi$.*

(iv) *There is no positive Green function for the p-Laplacian Δ_p on M.*

(v) *Every vector field X on M such that*

(a) $|X| \in L^{\frac{p}{p-1}}(M)$

(b) $\operatorname{div} X \in L_{loc}^1(M)$ *and* $\min(\operatorname{div} X, 0) =: (\operatorname{div} X)_- \in L^1(M)$

satisfies necessarily $0 \geq \int_M \operatorname{div} X dV_M$.

Proof. (i)⇔(ii) and (i)⇔(iv) in [Ho1].
(i)⇔(iii) in [PST].
(i)⇔(v) in [GT2]. The (⇒) part is also a consequence of the Proposition 1.2 below. □

We focus the attention on item (v). This very useful characterization of p-parabolicity goes under the name of Kelvin-Nevanlinna-Royden criterion. In the linear setting $p = 2$ it was proved in a paper by T. Lyons and D. Sullivan, [LS]. See also Theorem 7.27 in [PRS4]. It is worth pointing out that, even if X has low regularity and $\operatorname{div} X$ is not a function, we can obtain a similar conclusion as shown in the next

Proposition 1.2. *Let $(M, \langle , \rangle)$ be a p-parabolic Riemannian manifold, $p > 1$. Let X be a vector field satisfying $|X| \in L^{\frac{p}{p-1}}(M)$ and*

$$\operatorname{div} X \geq f$$

in the sense of distributions, for some $f \in L^1_{loc}(M)$ such that $f_- \in L^1(M)$. Then

$$\int_M f \leq 0.$$

Proof. Let $\{\Omega_j\}_{j=0}^{\infty}$ be an increasing sequence of precompact open sets with smooth boundaries such that $\Omega_j \nearrow M$. Let φ_j be the p-equilibrium potential of the condenser $C(\Omega_j, \overline{\Omega_0})$, namely

$$\int_M |\nabla \varphi_j|^p = \min \int_M |\nabla \varphi|^p \tag{1.5}$$

where the minimum is taken over all smooth φ compactly supported in Ω_j and satisfying $\varphi = 1$ on $\overline{\Omega_0}$, [GT3]. Then, φ_j solves the Dirichlet problem

$$\begin{cases} \Delta_p \varphi_j = 0 & \Omega_j \setminus \overline{\Omega_0} \\ \varphi_j = 1 & \text{on } \overline{\Omega_0} \\ \varphi_j = 0 & \text{on } \partial\Omega_j \end{cases}$$

and we have

$$\begin{aligned} 0 \leq \int_M \varphi_j f &\leq (\operatorname{div} X, \varphi_j) \\ &= -\int_M \langle X, \nabla \varphi_j \rangle \\ &\leq \left(\int_M |X|^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \left(\int_M |\nabla \varphi_j|^p \right)^{\frac{1}{p}}. \end{aligned} \tag{1.6}$$

Note that, by (1.5) and the p-parabolicity of M,

$$\int_M |\nabla \varphi_j|^p \to 0, \qquad \text{as } j \to \infty, \tag{1.7}$$

which implies that the RHS of (1.6) vanishes as $j \to \infty$. Moreover, by the comparison principle on precompact domains it follows that $0 \leq \varphi_j \leq 1$ is a non-decreasing sequence of functions pointwise converging to some $\varphi > 0$. Using an Ahlfors type characterization of p-parabolicity in terms of a boundary maximum principle for p-harmonic functions on generic domains we see that, in fact, $\varphi \equiv 1$, [AS], [PST]. Hence, taking limits as $j \to \infty$ and applying monotone and dominated convergence, we have

$$\lim_{j\to\infty} \int_M \varphi_j f = \lim_{j\to\infty} \int_M \varphi_j f_+ - \lim_{j\to\infty} \int_M \varphi_j f_- = \int_M f_+ - \int_M f_- = \int_M f.$$

Taking limits as $j \to \infty$ in inequality (1.6), assumption (1.7) and $|X| \in L^{\frac{p}{p-1}}(M)$ finally give

$$\int_M f \leq 0.$$

□

A first application of Proposition 1.1 is the following result due to Gol'dshtein and Troyanov, [GT1], and Pigola, Rigoli and Setti. We refer to [PRS3] for a direct proof and to Theorem 3.15 below for a more general formulation.

Corollary 1.3. *Let $(M, \langle , \rangle_M)$ be a p-parabolic manifold, $p > 1$. If $\varphi \in W^{1,p}_{loc}(M) \cap C^0(M)$ satisfies $\Delta_p \varphi \geq 0$ and $|\nabla\varphi| \in L^p(M)$, then φ is constant.*

It is known that p-parabolicity is related to volume growth properties of the underlying manifold. Accordingly, M is p-parabolic provided, for some origin $o \in M$,

$$\left(\frac{r}{\mathrm{Vol}(B_t)}\right)^{\frac{1}{p-1}} \notin L^1(+\infty). \tag{1.8}$$

Thus, for instance, the standard Euclidean space $\mathbb{R}^m$ is p-parabolic if $m \leq p$ and manifolds with finite volume are p-parabolic for all $p > 1$. Condition (1.8) is quite natural in that it shares the quasi-isometry invariance of p-parabolicity. Moreover, it turns out that there are geometric situations where (1.8) is also necessary for M to be p-parabolic; see [Ho2], [HK] and references therein. On the other hand, it was established in [Tr], [RS] and [Ho3] that the most general volume growth condition ensuring p-parabolicity is that, for some origin $o \in M$,

$$\left(\frac{1}{\mathcal{A}(\partial B_t)}\right)^{\frac{1}{p-1}} \notin L^1(+\infty). \tag{1.9}$$

It is important to point out that, in general, condition (1.9) is not a necessary condition for p-parabolicity, [G2]. Nevertheless, in case M is a model manifold (see Subsection 3.2.1 for the definition) also this reverse condition is verified, [Tr].

Chapter 2

Maps with finite p-energy

It is well known from classical work by R. Schoen and S.T. Yau, [SY2] , that a smooth map $f : M \to N$ from a complete manifold $(M, \langle , \rangle)$ with non-negative Ricci curvature into a compact manifold $(N, (,))$ with non-positive sectional curvature is homotopic to a constant, provided f has finite energy $|df|^2 \in L^1(M)$. As a matter of fact, Schoen and Yau's original result states that f is homotopic to a constant on each compact subset of M. The version reported here is a consequence of a topological theorem quoted in a paper by F. Burstall and attributed to V.L. Hansen; see Theorem 2.5 below.
Recently, Schoen and Yau's result has been extended to the case where the domain manifold presents an amount of negative Ricci curvature, say

$$ {}^M\mathrm{Ric} \geq -k(x), \tag{2.1} $$

with $k(x) \geq 0$ a continuous function. In fact, in [PRS2], [PRS4], the authors are able to deduce that the finite energy smooth map f is homotopic to a constant provided $k(x)$ is small in a suitable spectral sense. Namely, they obtain the following theorem. See also Theorem 2.6 below.

Theorem 2.1 (Corollary 6.23 in [PRS2]). *Let $(M, \langle , \rangle_M)$ be a complete Riemannian manifolds whose Ricci tensor satisfies* (2.1) *and*

$$ 0 \leq \lambda_1(-\Delta - k(x)) := \inf\left\{ \frac{\int |\nabla\varphi|^2 - k(x)\varphi^2}{\int \varphi^2} : \varphi \in C_c^\infty(M) \setminus \{0\} \right\}. \tag{2.2} $$

Let $(N, \langle , \rangle_N)$ be a compact manifold of nonpositive sectional curvature ${}^N\mathrm{Sect} \leq 0$. Then, any smooth map $f : M \to N$ of finite energy $|df|^2 \in L^1(M)$ is homotopic to a constant on each compact subset of M.

The main achievement of this section is to extend Theorem 2.1 to continuous maps with finite p-energy, $p > 2$. The techniques introduced to this purpose permit, also in case $p = 2$, to refine Theorem 2.1 by weakening assumption (2.2). Namely we obtain the following

Theorem 2.2 (Theorem 1 in [PV]). *Let $f : M \to N$ be a continuous map from a complete manifold $(M, \langle , \rangle)$ with Ricci curvature satisfying (2.1) into a compact manifold of non-positive sectional curvature. Assume that f has finite p-energy*

$|df|^p \in L^1(M)$, with $p \geq 2$. If the Schrödinger operator ${}^H L = -\Delta - Hk(x)$ satisfies

$$\lambda_1\left({}^H L\right) \geq 0, \tag{2.3}$$

for some

$$H > H_p,$$

with

$$\begin{aligned} H_p &= p^2/4\,(p-1) \qquad \textit{if } p > 2, \\ H_2 &= (m-1)/(m), \end{aligned}$$

then f is homotopic to a constant.

Remark 2.3. *We remark that $H_p \to 1$ as $p \to 2$. On the other hand $H_2 < 1$. This gap in the lower bound for H is due to the different Kato-type inequalities used in the proof. While for a general map f one has a standard Kato inequality $|Ddf|^2 \geq |\nabla|df||^2$, for a harmonic map u and only in case $p = 2$, the stronger refined Kato inequality $|Ddu|^2 \geq \frac{m}{m-1}|\nabla|du||^2$ permits to improve the computations; see Lemma 2.8 below.*

2.1 Strategy of the proof and previous results

The strategy for the proofs of both Theorem 2.1 and Theorem 2.2 dates back to [SY2] and consists of two main steps:

(a) An existence result for a (p-)harmonic map u with finite (p-)energy in the homotopy class of f.

(b) A Liouville type theorem for finite (p-)energy (p-)harmonic maps.

With regard to step (a), in case $p = 2$ the existence of a smooth harmonic map is guaranteed by Theorem 1 in [SY2]. As shown by S.W. Wei, this can be generalized to maps f with finite p-energy $|df|^p \in L^1(M)$ up to using $C^{1,\alpha}$ p-harmonic representatives $u : M \to N$ with finite p-energy. Namely we have the following result. Since the arguments in [We1] are only sketched, and since the result plays a key role in the development of the thesis, we provide a detailed proof. To this end, we adapt to the case $p \geq 2$ the proof given by Burstall for $p = 2$, [Bu].

Theorem 2.4 (Theorem 2.2 and Corollary 2.4 in [We1])**.** *Let M be a complete Riemannian m-dimensional manifold and N a compact manifold with nonpositive sectional curvature ${}^N\mathrm{Sect} \leq 0$. Then any continuous (or more generally $W^{1,p}$) map $f : M \to N$ of finite p-energy, $2 \leq p < \infty$, can be deformed to a $C^{1,\alpha}$ p-harmonic map u minimizing p-energy in the homotopy class.*

Proof. For the ease of notation, throughout all the proof we will keep the same set of indeces each time we will extract a subsequence from a given sequence. Consider an exhaustion $\{M_k\}_{k=1}^{\infty}$ of M, i.e. a sequence such that, for each k, M_k is a compact manifold with boundary, $M_k \subset\subset M_{k+1}$ and $\cup_{k=1}^{\infty} M_k = M$.

Define $\mathcal{H}_f$ as the space of $W^{1,p}_{loc}(M,N)$ maps v such that $v|_{M_k}$ and $f|_{M_k}$ have the same 1-homotopy type, i.e.

$$\mathcal{H}_f := \{v \in W^{1,p}_{loc}(M,N) : \forall k \geq 1,\ (v|_{M_k})_\sharp \text{ is conjugated to } (f|_{M_k})_\sharp\}.$$

First, we point out that $\mathcal{H}_f$ is well defined since any map $g \in W^{1,p}(M',N)$ defined on a compact m-dimensional manifold M' induces a homomorphism $g_\sharp : \pi_1(M',*) \to \pi_1(N,*)$ as follows. Given a generator γ for $\pi_1(M',*)$, we consider a tubular neighborhood $T \subset M'$ of γ in M such that $\psi : \mathbb{S}^1 \times I^{m-1} \to T$ is a smooth immersion, where I^{m-1} is the unit $(m-1)$-cell, and define $\gamma^s : \mathbb{S}^1 \to N$ as $\gamma^s(\cdot) := \psi(\cdot, s)$. Since M' is compact and $p \geq 2$, by Hölder inequality $g \in W^{1,2}(M',N)$ and Proposition 2.3 in [Bu] ensures that there exists $I^{m-1}_g \subseteq I^{m-1}$ such that $I^{m-1} \setminus I^{m-1}_g$ has measure zero and, for all $s, s' \in I^{m-1}_g$, g is continuous on γ^s and $g(\gamma^s)$ is homotopic to $g(\gamma^{s'})$. Consequently, for each $\gamma \in \pi_1(M',*)$ and $s_0 \in I^{m-1}_g$ we can set

$$g_\sharp[\gamma^{s_0}] = [g(\gamma^{s_0})]$$

on the generators, and extend $g_\sharp$ so that it is a group homomorphism. By the above considerations, $g_\sharp$ does not depends on the choice of $s_0 \in I^{m-1}_g$, while by Proposition 2.4 in [Bu] $g_\sharp$ is also independent of the choice of the generator. Since $f \in \mathcal{H}_f$, $\mathcal{H}_f$ is non-empty and

$$\mathcal{I}_f := \inf_{v \in \mathcal{H}_f} E_p(v) < +\infty.$$

Let $\{v_j\}_{j=1}^\infty \subset \mathcal{H}_f$ be a sequence minimizing p-energy in $\mathcal{H}_f$, i.e. $E_p(v_j) \to \mathcal{I}_f$ as $j \to \infty$. Choosing a subsequence if necessary, we can suppose $E_p(v_j) < 2\mathcal{I}_f$ for all j. Fix $k \in \mathbb{N}$ and consider the sequence $\{v_j|_{M_k}\}_{j=1}^\infty$. Let $i : N \hookrightarrow \mathbb{R}^q$ be an isometric immersion of N into some Euclidean space. Since $i(N) \subset \mathbb{R}^q$ is compact and $\{E_p(v_j)\}_{j=1}^\infty$ is bounded, $\{v_j|_{M_k}\}_{j=1}^\infty$ is bounded in $W^{1,p}(M_k, \mathbb{R}^q)$ and, up to choosing a subsequence, $v_j|_{M_k}$ converges to some $v^{(k)} \in W^{1,p}(M_k, N)$ weakly in $W^{1,p}$, strongly in L^p and pointwise almost everywhere. This implies $v^{(k)} \in W^{1,p}(M_k, N)$. By the lower semicontinuity of E_p we have

$$E_p(v^{(k)}) \leq \liminf_{j\to\infty} E_p(v_j|_{M_k}). \tag{2.4}$$

We want to show that the homomorphism on fundamental groups induced by $v_j|_{M_k}$ is preserved in the limit $v_j|_{M_k} \to v^{(k)}$. As above, choose a generator γ for some fixed class in $\pi_1(M_k,*)$ and consider the relative tubular neighborhood $\psi : \mathbb{S}^1 \times I^{m-1} \to T$ and set $I^{m-1}_{v_j}$. For a.e. $s \in I^{m-1}_{v_j}$ there exists a number K_s such that

$$\int_{\mathbb{S}^1} |dv_j(t,s)|^2 dt \leq K_s \tag{2.5}$$

for infinitely many j. In fact, if by contradiction we assume there exists a set $I' \subset I^{m-1}_{v_j}$ of positive measure such that

$$\int_{\mathbb{S}^1} |dv_j(t,s')|^2 dt \to \infty, \qquad \forall s' \in I',$$

then by Hölder inequality, Fubini's theorem and Fatou's lemma we would have

$$
\begin{aligned}
(2\mathcal{I}_f)^{\frac{2}{p}} \left(\mathrm{Vol}\left(\mathbb{S}^1 \times I^{m-1}\right)\right)^{\frac{p-2}{p}} &\geq \liminf_{j\to\infty} \int_{\mathbb{S}^1 \times I^{m-1}} |dv_j(t,s)|^2 dV_{\mathbb{S}^1\times I^{m-1}} \\
&\geq \liminf_{j\to\infty} \int_{I'} \int_{\mathbb{S}^1} |dv_j(t,s)|^2 dt ds \\
&\geq \int_{I'} \left(\liminf_{j\to\infty} \int_{\mathbb{S}^1} |dv_j(t,s)|^2 dt \right) ds = +\infty.
\end{aligned}
$$

So (2.5) is proven. The one dimensional Sobolev and Kondrachov's embedding theorems (e.g. [Au], p.53) states that there is a compact immersion $W^{1,2}(\mathbb{S}^1, N) \hookrightarrow C^0(\mathbb{S}^1, N)$. By (2.5) and the compactness of N, $v_j(\cdot, s)$ is uniformly bounded in $W^{1,2}(\mathbb{S}^1, N)$ so that for a.e. $s \in I^{m-1}$ we get a subsequence v_j converging uniformly on γ^s. Thus, for a.e. $s \in I^{m-1}$ there is j such that $v_j(\gamma^s)$ is uniformly close to, and hence homotopic to, $v^{(k)}(\gamma^s)$. Hence

$$
\left(v^{(k)}\right)_\sharp = (f|_{M_k})_\sharp . \tag{2.6}
$$

Using standard diagonal arguments we can choose a subsequence of v_j which, for all k, converges to $v^{(k)} \in W^{1,p}(M_k, N)$ weakly in $W^{1,p}$, strongly in L^p and pointwise almost everywhere. The map $v_0 : M \to N$ which, on M_k, takes values $v_0|_{M_k} = v^{(k)}$ is well defined. Indeed, by pointwise convergence, $v^{(k)}$ and $v^{(k+1)}$ agree almost everywhere on M_k. Now, $v_0 \in W^{1,p}_{loc}(M, N)$ and by (2.6) we get $v_0 \in \mathcal{H}_f$. It follows from (2.4) and the uniform boundedness of $E_p(v_j|_{M_k})$ that

$$
\begin{aligned}
\mathcal{I}_f \leq E_p(v_0) &= \lim_{k\to\infty} E_p(v_0|_{M_k}) = \lim_{k\to\infty} E_p(v^{(k)}) \\
&\leq \lim_{k\to\infty} \liminf_{j\to\infty} E_p(v_j|_{M_k}) \leq \liminf_{j\to\infty} E_p(v_j) = \mathcal{I}_f,
\end{aligned}
$$

so that $E_p(v_0) = \mathcal{I}_f$, i.e. v_0 minimize the energy in $\mathcal{H}_f$.
We are going to show that $v_0 \in C^{1,\alpha}$ and, hence, a p-harmonic map. To this end, we recall some definitions. A map $v \in W^{1,p}(M, N)$ is said to be *p-minimizing on ϵ-balls* if $E_p(v) \leq E_p(w)$ for any $w \in W^{1,p}(M, N)$ which agrees with v outside some ball B_r of radius $r < \epsilon$, that is, if $v = w$ on $M \setminus B_r$ and $(v - w)|_{B_r} \in W^{1,2}_0(B_r, N)$. Moreover, a map $\psi : \mathbb{S}^l \to N$ is said to be a *p-minimizing tangent map* of $\mathbb{S}^l$ if its homogeneous extension $\bar{\psi}$ to $\mathbb{R}^{l+1}$ given by

$$
\bar{\psi}(x) := \psi\left(\frac{x}{|x|}\right), \quad \forall x \neq 0,
$$

minimizes the p-energy on every compact subset of $\mathbb{R}^{l+1}$. As observed in [SU] for $p = 2$, when restricting to $\mathbb{S}^l$ the p-energy of $\bar{\psi}$ splits in one component tangential to $\mathbb{S}^l$ and one component normal to $\mathbb{S}^l$ which vanishes by homogeneity. Then $\bar{\psi}$ is p-harmonic in $\mathbb{R}^{l+1}$ if and only if ψ is p-harmonic on $\mathbb{S}^l$. Since ψ is a p-harmonic map defined on $\mathbb{S}^l$ with values in N, which is compact and non-negatively curved, Theorem 1.7 in [We2] implies ψ is constant, thus proving that N admits no non-trivial p-minimizing tangent maps of l sphere for every $l \geq 1$. On the other hand, choose ϵ to be less than half the width of the tubular neighborhhoods about the generating curves of various $\pi_1(M, *)$ (ϵ may vary according to the element of $\pi_1(M)$ considered, but this is not important due to the local nature of the regularity results). Then, if $w \in W^{1,p}(M_k, N)$ agrees with $v_0|_{M_k}$ outside some ϵ-ball, we can extend w to $\bar{w} \in W^{1,p}_{loc}(M, N)$ by setting $\bar{w} = v_0$ on $M \setminus M_k$, and it is clear that $\bar{w} \in \mathcal{H}_f$. Thus

$$E_p(v_0|_{M_k}) + E_p(v_0|_{M\setminus M_k}) = E_p(v_0) \leq E_p(\bar{w}) = E_p(w) + E_p(v_0|_{M\setminus M_k}),$$

giving that v_0 is p-minimizing on ϵ-balls. At this point we can apply a regularity result by Hardt and Lin (see Theorem 4.5 in [HL]) which gives that v_0 is $C^{1,\alpha}$ on M_k for each k, so $v_0 \in C^{1,\alpha}(M, N)$ and is therefore p-harmonic since locally p-minimizing.
It remains to prove that v_0 is homotopic to f. Since N has non-positive sectional curvatures, N is $K(\pi, 1)$, i.e. each homotopy group $\pi_k(N)$ of N is trivial for $k > 1$. A standard result says that, in this case, for every compact manifold M' the conjugacy classes of homomorphisms from $\pi_1(M')$ to $\pi_1(N)$ are in bijective correspondence with the homotopy classes of maps from M' to N (see e.g. [S] p.428). Thus, the continuous elements of $\mathcal{H}_f$, and in particular v_0, are all homotopic to f on compacta. Finally, to conclude that v_0 and f are homotopic as maps from M to N, we use the following result attributed to V.L. Hansen.

Theorem 2.5 (Theorem 5.1 in [Bu]). *Let M, N be connected $C-W$ complexes with M countable and N a $K(\pi, 1)$. Let $f, g : M \to N$ be maps that are homotopic on compacta. Then f, g are homotopic as maps from M to N.*

□

According to Theorem 2.4, Theorem 2.1 follows from a Liouville-type result for harmonic maps with finite energy, under the spectral assumption (2.3). This is an immediate consequence of the next vanishing result, due to [PRS2].

Theorem 2.6 (Theorem 6.1 in [PRS4]). *Let $(M, \langle,\rangle_M)$ be a complete manifold whose Ricci tensor satisfies (2.1) for some continuous function $k(x)$. Having fixed $H > \frac{m-2}{m-1}$, assume (2.2) holds. Let $(N, \langle,\rangle_N)$ be a manifold of non-positive sectional curvature ${}^N\mathrm{Sect} \leq 0$. Then, any harmonic map $u : M \to N$ with energy density satisfying*

$$|du|^q \in L^1(M), \tag{2.7}$$

for some $\frac{m-2}{m-1} \leq \frac{q}{2} \leq H$, is constant.

For completeness, and for comparison with the non-linear case, we report here the proof of Theorem 2.6 given in [PRS4].

Proof. We begin by recalling the following lemmas, for which we refer respectively to [EL] and [Br],[CGH].

Lemma 2.7 (Bochner-Weitzenböck identity). *Let $f : (M, \langle,\rangle_M) \to (N, \langle,\rangle_N)$ be a smooth map. Then*

$$\begin{aligned} \frac{1}{2}\Delta|df|^2 = |Ddf|^2 - \langle df, d\delta df\rangle_{HS} + \sum_{i=1}^{m} \left\langle df\left({}^M\mathrm{Ric}(e_i, \cdot)^\sharp\right), df(e_i)\right\rangle_N \\ - \sum_{i,j=1}^{m} \left\langle {}^N\mathrm{Riem}\left(df(e_i), df(e_j)\right) df(e_j), df(e_i)\right\rangle_N, \end{aligned} \tag{2.8}$$

where $\{e_i\}_{i=1}^m$ is a local ortho-normal frame on M.

Lemma 2.8 (Refined Kato inequality). *Let $v : M \to N$ be a harmonic map. Then*

$$|Ddv|^2 - |\nabla|dv||^2 \geq \frac{1}{(m-1)}|\nabla|dv||^2 \tag{2.9}$$

pointwise on the open, dense subset $\{x \in M : |dv|(x) \neq 0\}$ and weakly on all of M.

Applying Lemma 2.7 to the smooth harmonic map u, since $\delta du \equiv 0$ and by the curvature assumption we get

$$\Delta|du|^2 \geq 2|Ddu|^2 - 2k(x)|du|^2 \text{ on } M.$$

Set $\psi := |du| \in Lip_{loc}(M)$. Computing $\Delta\psi^2$ and applying Lemma 2.8, we obtain that

$$\psi\Delta\psi + k(x)\psi^2 - \frac{1}{m-1}|\nabla\psi|^2 \geq 0 \tag{2.10}$$

holds weakly on M. By a result of Moss and Piepenbrink, [MP], and Fisher-Colbrie and Schoen, [FCS], the spectral assumption (2.2) implies that there exists a positive function $\varphi \in C^1(M)$ satisfying

$$\Delta\varphi + Hk(x)\varphi = 0 \tag{2.11}$$

weakly on M. Since $\varphi > 0$, we can define the nonnegative function

$$\zeta := \varphi^{-q/(2H)}\psi^{q/2} \in Lip_{loc}.$$

We proceed by steps. First, we will prove that

$$\zeta \operatorname{div}\left(\varphi^{q/H}\nabla\zeta\right) \geq 0 \tag{2.12}$$

weakly on M. Then we will show that (2.12) togheter with the integrability assumption (2.7) implies that ζ is constant. Finally we will deduce the thesis.
Step a. Observe that (2.12) has the weak expression

$$\int_M \left[\left\langle \varphi^{q/H}\zeta\nabla\zeta, \nabla\rho\right\rangle_M + \rho\varphi^{q/H}|\nabla\zeta|^2\right] \leq 0, \tag{2.13}$$

for every non-negative compactly supported function $\rho \in L^\infty(M) \cap W^{1,2}(M)$. Computing $\nabla\zeta$, this latter is equivalent to the validity of

(2.14)
$$\begin{aligned} 0 \geq \frac{q}{2}\int \psi^{q-1}\left\langle\nabla\psi, \nabla\rho\right\rangle_M - \frac{q}{2H}\int\psi^q\left\langle\frac{\nabla\varphi}{\varphi}, \nabla\rho\right\rangle_M \\ + \frac{q^2}{4H^2}\int\rho\psi^q\frac{|\nabla\varphi|^2}{\varphi^2} + \frac{q^2}{4}\int\rho\psi^{q-2}|\nabla\psi|^2 - 2\frac{q^2}{4H}\int\rho\psi^{q-1}\left\langle\frac{\nabla\varphi}{\varphi}, \nabla\psi\right\rangle_M. \end{aligned}$$

We first consider the first integral on the right-hand side, and assume that $q < 2$, the other case being easier. Since ψ satisfies (2.10) and $\frac{q}{2} \geq \frac{m-2}{m-1} \geq \frac{m-2}{2(m-1)}$, we can apply the density results of [PRS4], Lemma 4.12 and Lemma 4.13, to deduce that

$$\psi^{q/2} \in L^2_{loc}(M), \quad \text{and} \quad \psi^{q/2-1}\nabla\psi \in L^2_{loc}(M). \tag{2.15}$$

Let $\varepsilon > 0$. By (2.15) and Cauchy-Schwarz inequality, we have

$$|(\psi+\varepsilon)^{q-2}\psi\nabla\psi| \leq |\psi^{q-1}\nabla\psi| = |\psi|^{q/2}\left(|\psi^{q/2-1}\nabla\psi|\right) \in L^1_{loc}(M). \tag{2.16}$$

According to (2.10), for every non-negative, compactly supported $\sigma \in W^{1,2}(M)$,

$$\int \langle \nabla\psi, \nabla(\sigma\psi)\rangle_M \leq \int (k(x)\psi^2 - \frac{1}{m-1}|\nabla\psi|^2)\sigma.$$

Applying the above inequality with $\sigma := \rho(\psi+\varepsilon)^{q/2-1}$, we deduce

$$\begin{aligned}\int (\psi+\varepsilon)^{q-2}\psi\,\langle\nabla\psi,\nabla\rho\rangle_M + \int \rho|\nabla\psi|^2(\psi+\varepsilon)^{q-2}\left(\frac{(q-1)\psi+\varepsilon}{\psi+\varepsilon}\right)\\ \leq \int k(x)\rho\psi^2(\psi+\varepsilon)^{q-2} - \frac{1}{m-1}\int \rho(\psi+\varepsilon)^{q-2}|\nabla\psi|^2.\end{aligned} \tag{2.17}$$

Applying dominated convergence, by (2.16) we have that

$$\lim_{\varepsilon\to 0}\int (\psi+\varepsilon)^{q-2}\psi\,\langle\nabla\psi,\nabla\rho\rangle_M = \int \psi^{q-1}\,\langle\nabla\psi,\nabla\rho\rangle_M\,,$$

while, since $0 \leq q \leq 2$ implies

$$\left|(\psi+\varepsilon)^{q-2}\left(\frac{(q-1)\psi+\varepsilon}{\psi+\varepsilon}\right)\right| \leq \psi^{q-2},$$

it holds

$$\lim_{\varepsilon\to 0}\int \rho|\nabla\psi|^2(\psi+\varepsilon)^{q-2}\left(\frac{(q-1)\psi+\varepsilon}{\psi+\varepsilon}\right) = \int (q-1)\rho|\nabla\psi|^2(\psi)^{q-2}.$$

Moreover, we can apply monotone convergence to the two integrals at RHS of (2.17), thus obtaining

$$\begin{aligned}\int \psi^{q-1}\,\langle\nabla\psi,\nabla\rho\rangle_M &\leq \int k(x)\rho\psi^q\\ &\quad - \frac{1}{m-1}\int \rho\psi^{q-2}|\nabla\psi^2| - (q-1)\int \rho\psi^{q-2}|\nabla\psi^2|\\ &= \int k(x)\rho\psi^q + \left(\frac{m-2}{m-1} - q\right)\int \rho\psi^{q-2}|\nabla\psi^2|.\end{aligned} \tag{2.18}$$

Similarly, consider the second integral at RHS of (2.14). According to (2.11), for every non-negative, compactly supported $\hat{\sigma} \in W^{1,2}(M)$,

$$\int \langle\nabla\varphi,\nabla\hat{\sigma}\rangle_M \geq \int Hk(x)\varphi\hat{\sigma},$$

and choosing $\hat{\sigma} := \psi^q \varphi^{-1}\rho$ we get

$$\int \psi^q \left\langle \frac{\nabla\varphi}{\varphi}, \nabla\rho \right\rangle_M \geq \int Hk(x)\rho\psi^q - q\int \rho\psi^{q-1}\varphi^{-1}\langle\nabla\varphi,\nabla\psi\rangle_M + \int \rho\psi^q \frac{|\nabla\varphi|^2}{\varphi^2}. \tag{2.19}$$

Inserting (2.18) and (2.19), we obtain that (2.14) is implied by the stronger condition

$$0 \geq \frac{q}{2H^2}(\frac{q}{2} - H)\int \rho\psi^q \frac{|\nabla\varphi|^2}{\varphi^2} + \frac{q}{2}\left[\frac{m-2}{m-1} - \frac{q}{2}\right]\int \rho\psi^{q-2}|\nabla\psi|^2,$$

which is always verified by the choice of parameters. This proves (2.13).

Step b. For fixed $\delta, t > 0$ let θ_δ be the Lipschitz function defined by

$$\theta_\delta(x) := \begin{cases} 1 & \text{if } r(x) \leq t, \\ \frac{t+\delta-r(x)}{\delta} & \text{if } t < r(x) < t+\delta, \\ 0 & \text{if } r(x) \geq t+\delta. \end{cases}$$

Let μ be a non-negative compactly supported Lipschitz function. Since $0 \leq \mu\theta_\delta \in L^\infty(M) \cap W^{1,2}(M)$ has compact support, we can set $\rho := \mu\theta_\delta$ in (2.13) obtaining

$$\begin{aligned} -\int \varphi^{q/H}\zeta\theta_\delta\langle\nabla\zeta,\nabla\mu\rangle_M &\geq \int \mu\varphi^{q/H}\zeta\langle\nabla\zeta,\nabla\theta_\delta\rangle_M + \int \mu\varphi^{q/H}\theta_\delta|\nabla\zeta|^2 \\ &\geq \int \mu\varphi^{q/H}\theta_\delta|\nabla\zeta|^2 - \frac{1}{\delta}\int_{B_{t+\delta}\setminus B_t} \mu\varphi^{q/H}\zeta|\nabla\zeta|. \end{aligned}$$

Choosing μ in such a way that $\mu \equiv 1$ on $\overline{B_{t+\delta}}$ the integral on the left-most side vanishes, and applying the Cauchy-Schwarz inequality to the second integral on the right-most side we deduce that

$$\int_{B_t} \varphi^{q/H}|\nabla\zeta|^2 \leq \left(\frac{1}{\delta}\int_{B_{t+\delta}\setminus B_t} \varphi^{q/H}\zeta^2\right)^{1/2}\left(\frac{1}{\delta}\int_{B_{t+\delta}\setminus B_t} \varphi^{q/H}|\nabla\zeta|^2\right)^{1/2}. \tag{2.20}$$

Setting

$$W(t) := \int_{B_t} \varphi^{q/H}|\nabla\zeta|^2,$$

it follows by the co-area formula (see Theorem 3.2.12 in [F]) that

$$W'(t) = \lim_{\delta\to 0}\frac{1}{\delta}\int_{B_{t+\delta}\setminus B_t} \varphi^{q/H}|\nabla\zeta|^2 = \int_{\partial B_t} \varphi^{q/H}|\nabla\zeta|^2 d\mathcal{H}^{m-1}$$

and

$$\lim_{\delta\to 0}\frac{1}{\delta}\int_{B_{t+\delta}\setminus B_t} \varphi^{q/H}\zeta^2 = \int_{\partial B_t} \varphi^{q/H}\zeta^2 d\mathcal{H}^{m-1}.$$

Hence, letting $\delta \to 0$ in (2.20), we get

$$W(t)^2 \leq \left(\int_{\partial B_t} \varphi^{q/H}\zeta^2 d\mathcal{H}^{m-1}\right) W'(t).$$

By contradiction, suppose ζ is not constant. Then, there exists $R_0 > 0$ such that $W(t) \geq W_0 > 0$ for all $t \geq R_0$. Integrating the last inequality on $[r, R]$, with $R > r \geq R_0$, and by the definition of ζ, we get

$$
\begin{aligned}
\text{(2.21)}\quad W_0^{-1} \geq W(r)^{-1} \geq W(r)^{-1} - W(R)^{-1} &\geq \int_r^R \left(\int_{\partial B_t} \varphi^{q/H} \zeta^2 d\mathcal{H}^{m-1} \right)^{-1} \\
&= \int_r^R \left(\int_{\partial B_t} \psi^q d\mathcal{H}^{m-1} \right)^{-1}.
\end{aligned}
$$

On the other hand, setting $U(r) := \int_{B_r} |du|^q dV_M$, by the co-area formula we have that U has strictly positive derivative $U'(r) = \int_{\partial B_r} |du|^q d\mathcal{H}^{m-1}$. Hence, by Cauchy-Schwarz inequality,

$$
R - r = \int_r^R 1 dt = \int_r^R \left(U'(t)\right)^{1/2} \left(U'(t)\right)^{-1/2} dt \leq U(R)^{1/2} \left(\int_r^R \frac{dt}{U'(t)} \right)^{1/2}.
$$

Combining this latter with (2.21), we deduce that

$$
\frac{(R-r)^2}{U(R)} \leq W_0^{-1} < +\infty
$$

and we obtain a contradiction letting $R \to +\infty$, since, by assumption (2.7), $U(R) \leq \int_M |du|^q < +\infty$ for all $R > 0$.

Step c. So far, we have proven that

$$
\zeta^{2/q} = \varphi^{-1/H} \psi \equiv c \tag{2.22}
$$

for some constant $c \geq 0$. By contradiction, suppose $c > 0$. Then, by assumption (2.7),

$$
\mathrm{Vol}(M) < +\infty. \tag{2.23}
$$

Moreover, we have $0 < c^H \varphi = \psi^H$. By (2.11) we get

$$
H\psi^{H-2} \left[\psi \Delta \psi + (H-1)|\nabla \psi|^2 + k(x)\psi^2 \right] \leq 0
$$

and combining with (2.10) we deduce

$$
H \left[H - \frac{m-2}{m-1} \right] \psi^{H-2} |\nabla \psi|^2 \leq 0.
$$

By the assumptions on H, this latter forces ψ to be constant. Then by (2.22) also φ is constant and (2.11) reduces to ${}^M\mathrm{Ric} \geq 0 \equiv k(x)$. A well known result by Yau, [Y], and E. Calabi, [C], shows that if M has non-negative Ricci curvature, then it has at least a linear volume growth. This fact contradicts (2.23), thus concluding the proof. $\square$

Remark 2.9. *As it is clear from the proof, assumption (2.7) of Theorem 2.6 can be replaced by the sharper condition*

$$
\left(\int_{\partial B_R} |du|^q d\mathcal{H}^{m-1} \right)^{-1} \notin L^1(+\infty).
$$

This latter is verified, for instance, if

$$
\int_{B_R} |du|^q dV_M = o(R).
$$

2.2 A Caccioppoli-type theorem

In order to prove Theorem 2.2, one is led to investigate the validity of nonlinear extensions of Theorem 2.6. However, we point out that the technique used in the proof reported above, cannot be adapted to the present situation. This was already remarked in [PRS3], where the authors prove an L^q-Liouville theorem for a smooth p-harmonic map provided the domain supports a global Poincaré-Sobolev inequality, and $k(x)$ is small (in a suitable integral sense) with respect to the Poincarè-Sobolev constant.
In fact, as it is clear from the proof of Theorem 2.6, a fundamental tool is the Bochner-Weitzenbök formula (2.8), which well behaves with harmonic (hence smooth) maps, while some difficulties arise when dealing with $p \neq 2$. First, the term $\langle du, d\delta du\rangle_{HS}$ in general does not vanish for a p-harmonic map u with $p \neq 2$. Moreover, the p-harmonic representatives in homotopy class predicted in Theorem 2.4 are $C^{1,\alpha}$ instead of C^∞. We overcome these problems by approximating the $C^{1,\alpha}$ p-harmonic map u via a sequence of smooth maps u_k, applying (2.8) to u_k and finally manipulating in integal form the terms which remain due to the non-harmonicity of u_k. Note also that, for values of p near to 2, from the weak formulation of (2.8) negative powers of $|du|$ appear. To face this further problem, we shall use an approximation procedure introduced by F. Duzaar and M. Fuchs, [DF], which enables us to extend to all of M the computations performed on $M_+ := \{x : |du|(x) > 0\}$.
As observed before, the techniques introduced in this section well adapt also to the linear case. We thus obtain another proof of Theorem 2.1, with a different assumption on parameters H and q.

Theorem 2.10 (Theorem 2 in [PV]). *Let $(M, \langle,\rangle_M)$ be a complete manifold with Ricci curvature satisfying*

$$ {}^M Ric \geq -k(x), \tag{2.24}$$

for some continuous function $k(x) \geq 0$, and let $(N, \langle,\rangle_N)$ be a manifold of non-positive sectional curvature ${}^N\mathrm{Sect} \leq 0$. Let $u : M \to N$ be a C^1 p-harmonic map, $p \geq 2$, such that

$$\int_{B_R(o)} |du|^q = o(R), \qquad \text{as } R \to +\infty, \tag{2.25}$$

for some $q \geq p$, where $B_R(o)$ denotes the geodesic ball centered at some fixed origin $o \in M$ and of radius $R > 0$. Moreover, assume that the Schrödinger operator ${}^H L = -\Delta - Hk(x)$ satisfies condition (2.3) for some

$$H > \begin{cases} \frac{q^2}{4(q-1)} & \text{if } p > 2 \\ \frac{q^2}{4\left(q-\frac{m-2}{m-1}\right)} & \text{if } p = 2. \end{cases} \tag{2.26}$$

Then u is constant.

By direct comparison with our result we recall that, in case ${}^M Ric \geq 0$, Liouville-type properties of p-harmonic maps from complete manifolds M into non-positively curved targets N are well understood. In this respect, we quote the paper [N] by N. Nakauchi where the author considers p-harmonic maps

u of class C^1 satisfying the energy condition $|du| \in L^p(M)$, and [Ta] by K. Takegoshi, where the Liouville conclusion is reached assuming that u is C^∞ and $|du| \in L^q$ for some $q > p-1$. Comparing Theorem 2.10 with Takegoshi result, [Ta], one observes that Theorem 2.10 requires the stronger condition $q \geq p$. As the proof will show, this is crucial in order to deal with C^1 maps and to use the Duzaar-Fuchs approximating procedure, [DF]. Therefore, the request $q \geq p$ is technical. We do not know whether our C^1 Liouville theorem extends to values of q below p. However, we emphasize that, even in the smooth situation, Takegoshi's λ-cut off argument seems to be not applicable in the Ricci curvature assumptions of Theorem 2.10.

Proof (of Theorem 2.10). Let $p > 2$. We will use the Duzaar and Fuchs' approximation procedure. Define

$$M_+ = \{x \in M : |du|(x) > 0\}.$$

From [DF] we know that both $|du|^{p/2-1} du \in W^{1,2}_{loc}$ on M and $du \in W^{1,2}_{loc}$ on M_+. Then we can consider a sequence $\{u_k\}_{k=1}^{\infty}$ of smooth maps such that

- $u_k \to u$ in $C^1_{loc}(M, N)$;
- $u_k \to u$ in $W^{2,2}_{loc}(M_+, N)$.

In particular, since $du_k \to du$ uniformly on compact sets, we have that $|du_k| \neq 0$ for k large enough on each compact $C \subset M_+$. Applying the Weitzenböck formula to the approximating map u_k, we obtain the following

$$\begin{aligned}\frac{1}{2}\Delta |du_k|^2 &= |Ddu_k|^2 - \langle du_k, (d\delta + \delta d)\, du_k\rangle_{HS} \\ &\quad + \sum_{i=1}^{m} \left\langle du_k\left({}^M \operatorname{Ric}(e_i, \cdot)^\sharp\right), du_k(e_i)\right\rangle_N \\ &\quad - \sum_{i,j=1}^{m} \left\langle {}^N \operatorname{Riem}(du_k(e_i), du_k(e_j))\, du_k(e_j), du_k(e_i)\right\rangle_N,\end{aligned}$$

where D denotes covariant differentiation and $\{e_i\}_{i=1}^m$ is a local orthonormal frame for M. By the curvature assumptions, noting also that $ddu_k = 0$, we get

$$\frac{1}{2}\Delta |du_k|^2 \geq |Ddu_k|^2 - \langle du_k, d\delta du_k\rangle - k(x)\,|du_k|^2$$

pointwise on M. Let $\phi = \rho^2 |du_k|^{q-2}$ with $\rho \in C_c^\infty(M_+) \subset C_c^\infty(M)$ to be chosen later. Then, multiplying both sides of the latter by ϕ and integrating over M, we obtain

$$\begin{aligned}(2.27)\qquad &\frac{1}{2}\int \rho^2 |du_k|^{q-2}\,\Delta |du_k|^2 + \int \rho^2 |du_k|^{q-2}\langle du_k, d\delta du_k\rangle \\ &\geq \int \rho^2 |du_k|^{q-2}\,|Ddu_k|^2 - \int \rho^2 |du_k|^q\, k(x).\end{aligned}$$

Integrating by parts the first term, we have

(2.28)

$$
\begin{aligned}
&\frac{1}{2}\int_{M_+} \rho^2 |du_k|^{q-2} \Delta |du_k|^2 \\
&= -\frac{1}{2}\int_{M_+} \rho^2 \left\langle \nabla |du_k|^{q-2}, \nabla |du_k|^2 \right\rangle - \frac{1}{2}\int_{M_+} |du_k|^{q-2} \left\langle \nabla \rho^2, \nabla |du_k|^2 \right\rangle \\
&= -(q-2)\int_{M_+} \rho^2 |du_k|^{q-2} |\nabla |du_k||^2 - 2\int_{M_+} \rho |du_k|^{q-1} \langle \nabla \rho, \nabla |du_k| \rangle \\
&\le -(q-2-\alpha)\int_{M_+} \rho^2 |du_k|^{q-2} |\nabla |du_k||^2 + \alpha^{-1}\int_{M_+} |du_k|^q |\nabla \rho|^2
\end{aligned}
$$

for any fixed $\alpha > 0$. As for the second term on the LHS of (2.27) we note that

$$
\begin{aligned}
\delta\left(\rho^2 |du_k|^{q-2} du_k\right) &= \delta\left(\rho^2 |du_k|^{q-p} |du_k|^{p-2} du_k\right) \\
&= \rho^2 |du_k|^{q-p} \delta(|du_k|^{p-2} du_k) - |du_k|^{q-2} i\left(\nabla \rho^2\right) du_k \\
&\quad - |du_k|^{p-2} \rho^2 i\left(\nabla |du_k|^{q-p}\right) du_k
\end{aligned}
$$

and

$$
\begin{aligned}
\delta du_k &= \delta(|du_k|^{2-p}|du_k|^{p-2}du_k) \\
&= |du_k|^{2-p}\delta(|du_k|^{p-2}du_k) - |du_k|^{p-2} i\left(\nabla |du_k|^{2-p}\right) du_k \\
&= |du_k|^{2-p}\delta(|du_k|^{p-2}du_k) + (p-2)|du_k|^{-1} i\left(\nabla |du_k|\right) du_k,
\end{aligned}
$$

where i denotes the interior product on 1-forms. Using the last two relations, together with the inequality $|\delta du_k| \le \sqrt{m}\,|Ddu_k|$, we obtain

(2.29)

$$
\begin{aligned}
&\int_{M_+} \rho^2 |du_k|^{q-2} \langle du_k, d\delta du_k \rangle = \int_{M_+} \left\langle \rho^2 |du_k|^{q-2} du_k, d\delta du_k \right\rangle \\
&= \int_{M_+} \left\langle \delta\left(\rho^2 |du_k|^{q-2} du_k\right), \delta du_k \right\rangle \\
&= \int_{M_+} \rho^2 |du_k|^{q-p} \left\langle \delta(|du_k|^{p-2} du_k), \delta du_k \right\rangle - \int_{M_+} |du_k|^{q-2} \left\langle i\left(\nabla \rho^2\right) du_k, \delta du_k \right\rangle \\
&- \int_{M_+} \rho^2 |du_k|^{p-2} \left\langle i\left(\nabla |du_k|^{q-p}\right) du_k, \delta du_k \right\rangle \\
&\le 2\int_{M_+} \sqrt{m}\rho |du_k|^{q-1} |\nabla \rho| |Ddu_k| + F(u_k) \\
&- (q-p)\int_{M_+} \rho^2 |du_k|^{q-3} \left\langle i\left(\nabla |du_k|\right) du_k, (p-2)|du_k|^{-1} i\left(\nabla |du_k|\right) du_k \right\rangle \\
&\le \beta \int_{M_+} \rho^2 |du_k|^{q-2} |Ddu_k|^2 + \beta^{-1} m \int_{M_+} |du_k|^q |\nabla \rho|^2 \\
&- (q-p)(p-2)\int_{M_+} \rho^2 |du_k|^{q-4} |i\left(\nabla |du_k|\right) du_k|^2 + F(u_k) \\
&\le \beta \int_{M_+} \rho^2 |du_k|^{q-2} |Ddu_k|^2 + \beta^{-1} m \int_{M_+} |du_k|^q |\nabla \rho|^2 + F(u_k)
\end{aligned}
$$

for any fixed $\beta > 0$, where we have set

$$F(u_k) = -\,(q-p)\int_{M_+}\rho^2\,|du_k|^{q-p-1}\left\langle i\left(\nabla\,|du_k|\right)du_k,\delta(|du_k|^{p-2}du_k)\right\rangle$$
$$+\int_{M_+}\rho^2\,|du_k|^{q-p}\left\langle \delta du_k,\delta(|du_k|^{p-2}\,du_k)\right\rangle.$$

Finally, in order to deal with the term containing $k(x)$ in (2.27), we now use the spectral assumption. In fact, by (2.3), for every $\xi \in C_c^\infty(M)$, we have

$$\int |\nabla\xi|^2 - Hk\,(x)\,\xi^2 \geq 0. \tag{2.30}$$

Choose $\xi = |du_k|^{q/2}\,\rho$, so that, for any fixed $\gamma > 0$,

$$|\nabla\xi|^2 \leq (1+\gamma)\,\frac{q^2}{4}\rho^2\,|du_k|^{q-2}\,|\nabla\,|du_k||^2 + \left(1+\gamma^{-1}\right)|du_k|^q\,|\nabla\rho|^2.$$

Then from (2.30) we deduce

$$\int_{M_+} k\,(x)\,\rho^2\,|du_k|^q \leq (1+\gamma)\,\frac{q^2}{4}H^{-1}\int_{M_+}\rho^2\,|du_k|^{q-2}\,|\nabla\,|du_k||^2$$
$$+\left(1+\gamma^{-1}\right)H^{-1}\int_{M_+}|du_k|^q\,|\nabla\rho|^2. \tag{2.31}$$

Inserting (2.28), (2.29) and (2.31) into (2.27) we get

$$\left\{(q-2-\alpha)-(1+\gamma)\,\frac{q^2}{4}H^{-1}\right\}\int_{M_+}\rho^2\,|du_k|^{q-2}\,|\nabla\,|du_k||^2$$
$$+\{1-\beta\}\int_{M_+}\rho^2\,|du_k|^{q-2}\,|Ddu_k|^2$$
$$\leq \left\{\alpha^{-1}+m\beta^{-1}+\left(1+\gamma^{-1}\right)H^{-1}\right\}\int_{M_+}|du_k|^q\,|\nabla\rho|^2 + F(u_k).$$

If we choose $\beta \leq 1$, recalling that

$$|\nabla\,|du_k|| \leq |Ddu_k|\,, \tag{2.32}$$

we finally obtain

$$A\int_{M_+}\rho^2\,|du_k|^{q-2}\,|\nabla\,|du_k||^2 \leq B\int_{M_+}|du_k|^q\,|\nabla\rho|^2 + F(u_k), \tag{2.33}$$

where

$$A(q,H,\alpha,\gamma,\beta) = \left\{q-1-\alpha-(1+\gamma)\,\frac{q^2}{4}H^{-1}-\beta\right\}$$
$$B(H,\alpha,\gamma,\beta,m) = \left\{\alpha^{-1}+m\beta^{-1}+\left(1+\gamma^{-1}\right)H^{-1}\right\}.$$

Note that assumption (2.26) ensures that $A > 0$ for a suitable choice of the parameters $0 < \alpha,\beta,\gamma \ll 1$.

Note also that, from the convergence properties of $\{u_k\}_{k=1}^{\infty}$ and the p-harmonicity condition $\delta(|du|^{p-2}du) = 0$ we have, on M_+:

$$\begin{aligned}
|du_k|^t &\to |du|^t\,,\ \forall t \in\ \mathbb{R} && \textit{in } C^0_{loc} \textit{ as } k \to \infty;\\
|\nabla|du_k||^2 &\to |\nabla|du||^2 && \textit{in } L^1_{loc} \textit{ as } k \to \infty;\\
i_{\nabla|du_k|}(du_k) &\to i_{\nabla|du|}(du) && \textit{in } L^2_{loc} \textit{ as } k \to \infty;\\
\delta(|du_k|^{p-2}du_k) &\to 0 && \textit{in } L^2_{loc} \textit{ as } k \to \infty;\\
\delta du_k &\to \delta du && \textit{in } L^2_{loc} \textit{ as } k \to \infty.
\end{aligned}$$

Therefore, taking limits as $k \to \infty$ in (2.33) we get

$$A \int_{M_+} \rho^2 \,|du|^{q-2} \,|\nabla\,|du||^2 \le B \int_{M_+} |du|^q \,|\nabla \rho|^2\,, \qquad \forall \rho \in C^\infty_c(M_+). \tag{2.34}$$

Using a variation of the Duzaar-Fuchs cut-off trick, we now want to extend (2.34) to every $\rho \in C^\infty_c(M)$. Define

$$\varphi_\varepsilon = \min\left\{\frac{|du|^{q/2}}{\varepsilon}, 1\right\}$$

for $\epsilon > 0$ and set $\xi = \varphi_\varepsilon \eta^2$ for any $\eta \in\ C^\infty_c(M)$. Note that $\xi \in W^{1,2}_0(M_+)$. Indeed ξ is continuous, compactly supported and $\xi = 0$ on $M \setminus M_+$. Moreover, since $|du|^{p/2} = ||du|^{p/2-1}du|$, by Kato inequality and the fact that $|du|^{p/2-1}du \in W^{1,2}_{loc}$ on M we have

$$\left|\nabla\,|du|^{p/2}\right| = \left|\nabla||du|^{p/2-1}du|\right| \le \left|D(|du|^{p/2-1}du)\right| \in L^2_{loc},$$

proving that $|du|^{p/2} \in W^{1,2}_{loc}(M)$. Since $|du|^{q/2} = (|du|^{p/2})^{q/p}$, with $q/p \ge 1$, we conclude that $|du|^{q/2} \in W^{1,2}_{loc}(M)$. Hence we can find a sequence $\{\rho_j\}_{j=1}^{\infty} \subset C^\infty_c(M_+)$ such that $\rho_j \to \xi$ in $W^{1,2}_0(M)$. Substituting $\rho = \rho_j$ into (2.34), taking the liminf as $j \to \infty$, and using Fatou Lemma on the left hand side, we get

$$\begin{aligned}
&A \int_{M_+} (\varphi_\varepsilon)^2 \eta^4 \,|du|^{q-2} \,|\nabla\,|du||^2\\
\le&6B \int_{M_+} |du|^q \,|\nabla \eta|^2 \,\eta^2 (\varphi_\varepsilon)^2 + 3B \int_{M_+} |du|^q \,|\nabla \varphi_\varepsilon|^2 \,\eta^4.
\end{aligned} \tag{2.35}$$

Finally, we let $\varepsilon \to 0$. Note that $\varphi_\varepsilon \to 1$ pointwise in M_+. Moreover

$$\begin{aligned}
\int_{M_+} |du|^q \,|\nabla \varphi_\varepsilon|^2 \,\eta^4 &= \int_{M_+} |du|^q \,\frac{\left|\nabla\,|du|^{q/2}\right|^2}{\varepsilon^2} \eta^4 \chi_{\{|du|^q < \varepsilon^2\}}\\
&\le \int_{M_+} \left|\nabla\,|du|^{q/2}\right|^2 \eta^4 \chi_{\{|du|^q < \varepsilon^2\}}
\end{aligned}$$

and the last term vanishes by dominated convergence as $\varepsilon \to 0$, because

$$\nabla\,|du|^{q/2} \in L^2_{loc}(M)$$

as observed before. Therefore, letting $\varepsilon \to 0$, and applying also Fatou Lemma to the integral on the left hand side and dominated convergence to the first integral in the right hand side of (2.35), we have the Caccioppoli inequality

$$\int_{M_+} \eta^4 |du|^{q-2} |\nabla |du||^2 \leq C \int_{M_+} |du|^q |\nabla \eta|^2 \eta^2, \qquad \forall \eta \in C_c^\infty(M), \tag{2.36}$$

with $C = 6B/A$.
Now, by contradiction, suppose u is non-constant. For any fixed $R > 0$, we choose $\eta(x) = \eta_R(x)$ so to satisfy

$$\begin{array}{ll} \text{(a) } 0 \leq \eta(x) \leq 1, & \text{(b) } \eta(x) = 1 \text{ on } B_R(o), \\ \text{(c) } \eta(x) = 0 \text{ off } B_{2R}(o), & \text{(d) } |\nabla \eta| \leq 2/R \text{ on } M. \end{array} \tag{2.37}$$

Whence, we deduce

$$\int_{B_R(o) \cap M_+} |du|^{q-2} |\nabla |du||^2 \leq \frac{4C}{R^2} \int_{B_{2R}(o) \cap M_+} |du|^q, \tag{2.38}$$

for some computable positive constant C, and letting $R \to +\infty$ we conclude

$$\int_{M_+} |du|^{q-2} |\nabla |du||^2 = 0,$$

proving that $|du| = const.$ on every connected component of M_+. This easily implies that $M_+ = M$. Indeed, if $M_+ \subsetneq M$, since $|du| = 0$ on ∂M_+, we would obtain $|du| \equiv 0$ on every connected component of M_+. Therefore $|du| \equiv 0$ on M and u is constant. Contradiction. Hence $M_+ = M$ and $|du| = const. \neq 0$. By assumption (2.25) we deduce that

$$\operatorname{vol} B_R = o(R) \qquad as\ R \to +\infty. \tag{2.39}$$

Using this information together with the spectral assumption and choosing $\eta = \eta_R$ to be the cut-off functions defined in (2.37), we get

$$\begin{aligned} 0 &\leq \lim_{R \to +\infty} \int_{B_{2R}(o)} \left\{ H^{-1} |\nabla \eta|^2 - k(x) \eta^2 \right\} \\ &\leq \lim_{R \to +\infty} \left\{ \frac{4 \ \operatorname{vol} B_{2R}(o)}{HR^2} - \int_{B_R(o)} k(x) \right\} \\ &= -\int_M k(x) \leq 0, \end{aligned}$$

proving that $k(x) = 0$, i.e., ${}^M Ric \geq 0$. A well known result by Yau, [Y], and E. Calabi, [C], now shows that M has at least a linear volume growth, contradicting (2.39).
In case $p = 2$, the proof is simpler. Since u is harmonic (hence smooth), we don't need to approximate u via smooth maps, i.e. we can choose $u_k \equiv u$. Then, inequality (2.27) reduces to

$$\frac{1}{2} \int \rho^2 |du|^{q-2} \Delta |du|^2 \geq \int \rho^2 |du|^{q-2} |Ddu|^2 - \int \rho^2 |du|^q k(x),$$

since $\delta du \equiv 0$ by harmonicity. Moreover, by Lemma 2.8, we can replace (2.32) with the refined relation

$$\sqrt{\frac{m}{m-1}}\,|\nabla\,|du_k|| \leq |Ddu_k|\,,$$

and (2.34) holds true with

$$\begin{aligned} A(q,H,\alpha,\gamma) &= \left\{q - \frac{m-2}{m-1} - \alpha - (1+\gamma)\,\frac{q^2}{4}H^{-1}\right\} \\ B(H,\alpha,\gamma,m) &= \left\{\alpha^{-1} + \left(1+\gamma^{-1}\right)H^{-1}\right\}. \end{aligned}$$

Since assumption (2.26) ensures that $A > 0$ for a suitable choice of the parameters $0 < \alpha, \gamma \ll 1$, from now on we can repeat the proof of the $p > 2$ case to conclude. □

2.3 Applications in the harmonic case

In [SY2], Schoen and Yau used harmonic maps techniques to study the fundamental group of manifolds with non-negative Ricci curvature and of stable minimal hypersurfaces immersed into non-positively curved ambient spaces. First, given a complete, m-dimensional manifold with ${}^M\operatorname{Ric} \geq 0$ and any compact domain $D \subset M$ with smooth, simply connected boundary, they obtained that there is no non-trivial homomorphism of $\pi_1(D)$ into the fundamental group of a compact manifold with non-positive sectional curvature. Using Theorem 2.1, their result has been generalized in [PRS2] by replacing assumption ${}^M\operatorname{Ric} \geq 0$ with the spectral assumption (2.2) on the Ricci curvature. Applying Theorem 2.2 instead of Theorem 2.1, and procceding exactly as in [PRS2], we can slightly weaken the spectral assumption.

Corollary 2.11. *Let M be a complete, m-dimensional manifold satisfying* (2.1) *and* (2.3) *with $H > (m-1)/m$. If $D \subset M$ is a compact domain in M with smooth, simply connected boundary, then there is no non-trivial homomorphism of $\pi_1(D)$ into the fundamental group of a compact manifold with non-positive sectional curvature.*

Now, suppose that $(M, \langle,\rangle_M)$ is isometrically immersed as complete, stable, minimal hypersurface into a space Q with ${}^Q\operatorname{Sect} \geq 0$. According to Gauss equations, ${}^M\operatorname{Ric} \geq -|\mathbf{II}|^2$, where $|\mathbf{II}|$ denotes the length of the second fundamental tensor of the immersion. Moreover, the stability assumption implies the fact that the operator $L = -\Delta - |\mathbf{II}|^2$ satisfies $\lambda_1(L) \geq 0$. Hence, we are precisely in the assumption of Theorem 2.1 and information on $\pi_1(M)$ can be deduced. As above, our improved Theorem 2.2 permits to sharpen the spectral assumption, which in this case is related to the stability of the hypersurface. Namely, Schoen and Yau's result holds true provided the hypersurface is "almost stable" in the following sense. A minimal immersion is said to be δ-stable if

$$\int_M \left[|\nabla\varphi|^2 - (1-\delta)|\mathbf{II}|^2\varphi^2\right] \geq 0$$

for all $\varphi \in C_c^\infty(M)$, [CM], [TZ], [CZ], [FL]. As it is clear, setting $\delta = 1 - H$, the δ-stability corresponds to the spectral assumption (2.3) adapted to the contest of minimal immersion. Hence, we have the following

Corollary 2.12. *Let $(M, \langle , \rangle_M)$ be a complete non-compact minimally immersed hypersurface in a manifold of non-negative sectional curvature. Suppose that the immersion is $(1-H)$-stable, for some $H > (m-1)/m$. If $D \subset M$ is a compact domain in M with smooth, simply connected boundary, then there is no non-trivial homomorphism of $\pi_1(D)$ into the fundamental group of a compact manifold with non-positive sectional curvature.*

As shown in [PRS4], Corollary 7.12, a further application of Theorem 2.1 permits to obtain information on the number of non-parabolic ends of manifolds with controlled Ricci curvature. We recall that, given a compact set K in M, an end $E(K)$ with respect to K is an unbounded connected component of $M \setminus K$. By a compactness argument, it is readily seen that the number of ends with respect to K is finite and increases as the compact K enlarges. We say that M has a finite number of ends if there exists a constant C such that for every compact K the number of ends with respect to K is bounded above by C. In this case, there exists a compact K_0 and a number N_e such that for every compact $K \supset K_0$ the number of ends with respect to K is exactly N_e, and we say that M has N_e ends. Moreover, an end E will be said to be parabolic if the Riemannian double of E (see Section 8.3 in [PRS4]) is a parabolic manifold. In analogy with Proposition 1.1 (ii), this is equivalent to ask that every positive superharmonic function u satisfying $\partial u/\partial\nu \geq 0$ on ∂E, ν being the unit outward normal to ∂E, is constant.
It's well known that the number of non-parabolic ends of a manifold M is bounded above by the dimension of the space of bounded harmonic functions with finite energy, see [LT], [G1]. On the other hand, given a manifold M as in Theorem 2.2, we easily deduce that any harmonic function on M with finite energy is necessarily constant, i.e. the space of (bounded) harmonic functions with finite Dirichlet integral is 1-dimensional.

Corollary 2.13. *Let M be a Riemannian manifold as in Corollary 2.11. Then M has at most one non-parabolic hand.*

As observed before, Theorem 2.10 with $p = 2$ recovers the conclusion of Theorem 2.6, but with a different assumption on the range of parameters. Indeed, Theorem 2.6 requires

$$H > \frac{m-2}{m-1}, \qquad \frac{m-2}{m-1} \leq \frac{q}{2} \leq H \tag{2.40}$$

while for Theorem 2.10 we impose

$$q \geq 2, \qquad H > \frac{q^2}{4\left(q - \frac{m-2}{m-1}\right)} \tag{2.40'}$$

Note that, on the one hand, assumption (2.40) is stronger than (2.40') and lower values of q, i.e. $2\frac{m-2}{m-1} \leq q < 2$, are allowed. Nevertheless, for $q \geq 2$, it holds $q/2 > q^2\left[4\left(q - \frac{m-2}{m-1}\right)\right]^{-1}$, and hence the relation between H and q given in (2.40') is less restrictive. So far we have used this fact to extend previous results by weakening the spectral assumption (2.3) permitting $H < 1$ when $q = p = 2$. In a different direction, this improvement turns useful under classical assumption

(2.2) (i.e. with $H = 1$), since in this case we can choose values of q greater than 2,

$$2 \leq q \leq 2 + \sqrt{\frac{4}{m-1}}.$$

As an example of this latter approach, we recall a result by Bérard, [Be], which states that an m-dimensional oriented stable complete minimal hypersurface M in $\mathbb{R}^{m+1}$ satisfying $|\mathbf{II}| \in L^m(M)$ is an affine m-plane if $m \leq 5$. The proof given by Bérard is very similar to the $p = 2$ case of the proof of Theorem 2.10. Namely he considers the norm of second fundamental form $|\mathbf{II}|$ instead of $|du|$. Then, Simons' inequality and the refined Kato inequality for minimal immersion give (compare with (2.10))

$$|\mathbf{II}|^M\Delta|\mathbf{II}| + |\mathbf{II}|^4 \geq \frac{2}{m}|\nabla|\mathbf{II}||^2.$$

which, in turn, implies (compare with (2.27))

$$\frac{1}{2}\int \rho^2|\mathbf{II}|^{q-2}\Delta|\mathbf{II}|^2 \geq \frac{m+2}{m}\int \rho^2|\mathbf{II}|^{q-2}\left|\nabla|\mathbf{II}|\right|^2 - \int \rho^2|\mathbf{II}|^q k\,(x)\,,$$

where we have set $k(x) = |\mathbf{II}|^2$. Moreover, the stability of the immersion implies that $\lambda_1^L \geq 0$ for the operator $L = -^M\Delta - |\mathbf{II}|^2 = -^M\Delta - k$. Hence we can proceed as in the proof of Theorem 2.10, obtaining Bérard's result for $m \leq 3$. In these assumptions of finite total curvature an estimate due to M. Anderson gives $\operatorname{Vol} B_R^M \leq CR^m$ for some positive constant C, [An1], [An2]. This permits to apply Hölder inequality to RHS of (2.38), enlarging the admissible value of q, i.e. $q = m \leq 5$.

For completeness we recall that, using the convergence theory for minimal surfaces, Shen and Zhu, [SZ], were able to extend Bérard's theorem to the case of general dimension $m \geq 2$.

Chapter 3

Homotopic p-harmonic maps

In Chapter 2 we studied the homotopy class of higher energy maps. The fundamental ingredient was an existence result for the finite p-energy p-harmonic representative in homotopy class, see Theorem 2.4, combined with a Liouville type theorem for finite p-energy p-harmonic maps, see Theorem 2.10. Notably, we recall that Wei's existence Theorem 2.4 states that, given manifolds $(M, \langle,\rangle_M)$ complete and $(N, \langle,\rangle_N)$ compact and a finite p-energy continuous map $f : M \to N$, there exists a finite p-energy $C^{1,\alpha}$ p-harmonic map homotopic to f. This implies that, in the above assumptions on the manifolds, the space of p-harmonic maps from M to N with finite p-energy homotopic to f is non-empty. Hence, one is led to investigate such a space, in particular inquiring how many p-harmonic representatives can be found in a given homotopy class. First results in this direction were obtained by Wei for general p and smooth p-harmonic maps defined on compact M, see Theorem 3.1 and Theorem 3.2 below. This generalizes a previous work due to Hartman, [Har], holding for $p = 2$.

Theorem 3.1 (Theorem 8.1 in [We2]). *Every smooth p-harmonic map u defined on a compact Riemannian manifold M into a manifold N that supports a strictly convex function is constant.*

Theorem 3.2 (Theorem 8.5 in [We2]). *Let M and N be compact manifolds with ${}^N\mathrm{Sect} \leq 0$. If u_0 and u_1 are homotopic p-harmonic maps from M into N, then they are homotopic through p-harmonic maps $u_s(\cdot)$ and the p-energy is constant on any arcwise connected set of p-harmonic maps, i.e. $E_p(u_s) = E_p(u_0) = E_p(u_1)$ for all $s \in [0,1]$. Furthermore, each path $s \mapsto u_s(q)$ is a geodesic segment with length independent of $q \in M$. In particular*

i) Every homotopy class of maps from M to N that agree on $\partial M \neq \emptyset$ contains a unique p-harmonic map.

ii) Let $u_0 : M \to N$ be a p-harmonic map with $\partial M = \emptyset$. Assume that there is some point of $u_0(M)$ at which ${}^N\mathrm{Sect} < 0$. Then u_0 is unique in its homotopy class unless it is constant or maps M onto a closed geodesic σ in N. In the latter case, we have uniqueness up to rotations of σ.

As we will report below, Hartman's result for $p = 2$ has been generalized to non-compact manifolds M and N with $\operatorname{Vol} M < +\infty$ by Schoen and Yau, [SY3].

Subsequently, it has been proven that in the harmonic setting the assumption of finite volume can be replaced by the weaker condition of parabolicity of M, [PRS3]. Nevertheless, the case of two non-trivial homotopic p-harmonic maps between complete manifolds remained an open problem when $p \neq 2$.
We start this chapter by recalling Schoen and Yau's result for $p = 2$ and reporting their proof. This will permit us to highlight which are the obstructions to generalize Schoen and Yau's proof to the p-harmonic setting and which different techniques have been so far developed or can be developed to overcome these difficulties. We thus obtain partial related results of independent interest. Finally we will prove the following $p \neq 2$ analogue of Schoen and Yau's result.

Theorem 3.3. *Suppose M is p-parabolic, $p \geq 2$, and N is complete.*

i) Let $u : M \to N$ be a $C^{1,\alpha}$ p-harmonic map of finite p-energy. If ${}^N\mathrm{Sect} < 0$, there's no other p-harmonic map of finite p-energy homotopic to u unless $u(M)$ is contained in a geodesic of N.

ii) If $u, v : M \to N$ are homotopic $C^{1,\alpha}$ p-harmonic maps of finite p-energy and ${}^N\mathrm{Sect} \leq 0$, then there is a continuous one-parameter family of maps $u_t : M \to N$ with $u_0 = u$ and $u_1 = v$ such that the p-energy of u_t is constant (independent of t) and for each $q \in M$ the curve $t \mapsto u_t(q)$, $t \in [0, 1]$, is a constant (independent of q) speed parametrization of a geodesic. Moreover, if N is compact, u_t is a p-harmonic maps for each $t \in [0, 1]$.

3.1 Uniqueness of harmonic maps in free homotopy class

In this section, following Schoen and Yau, we give a description of the space of (2-)harmonic maps from M to N which are homotopic to a given one.

Theorem 3.4 (Theorem 1 and Theorem 2 in [SY3])**.** *Suppose M is parabolic and N is complete.*

i) Let $u : M \to N$ be a harmonic map of finite energy. If ${}^N\mathrm{Sect} < 0$, there's no other harmonic map of finite energy homotopic to u unless $u(M)$ is contained in a geodesic of N.

ii) If ${}^N\mathrm{Sect} \leq 0$ and $u, v : M \to N$ are homotopic harmonic maps of finite energy, then there is a smooth one-parameter family $u_t : M \to N$, of harmonic maps with $u_0 = u$ and $u_1 = v$. Moreover, for each $q \in M$, the curve $t \mapsto u_t(q)$, $t \in [0, 1]$, is a constant (independent of q) speed parametrization of a geodesic. Also the map $M \times \mathbb{R} \to N$ given by $(q, t) \mapsto u_t(q)$ is harmonic with respect to the product metric on $M \times \mathbb{R}$. Moreover, $x \mapsto du_t(\partial/\partial t)$ is a parallel section of $u^{-1}TN$, the pullback of the tangent bundle of N with pulled back connection.

Remark 3.5. *Theorem 3.4, as stated in the original paper [SY3], required* $\mathrm{Vol}(M) < +\infty$. *The improved version reported here is due to [PRS3], Remark 4.*

Here we outline how Schoen and Yau proved Theorem 3.4 (we refer to [SY1] and to Section 3.4 below for further details). In particular we try to underline the steps of their proof more significant in view of a p-harmonic generalization.

Proof (of Theorem 3.4).
Step a. Let u and v be two smooth harmonic maps from M to N which are freely homotopic, and such that $|du|, |dv| \in L^2(M)$. Let $P_M : \tilde{M} \to M$ and $P_N : \tilde{N} \to N$ be the universal Riemannian covers respectively of M and N. Then $\pi_1(M,*)$ and $\pi_1(N,*)$ act as groups of isometries on $\tilde{M}$ and $\tilde{N}$ respectively so that $M = \tilde{M}/\pi_1(M,*)$ and $N = \tilde{N}/\pi_1(N,*)$. Let $\mathrm{dist}_{\tilde{N}} : \tilde{N} \times \tilde{N} \to \mathbb{R}$ be the distance function on $\tilde{N}$. Since ${}^{\tilde{N}}\mathrm{Sect} \leq 0$, we know that $\mathrm{dist}_{\tilde{N}}$ is smooth on $\tilde{N} \times \tilde{N} \setminus \tilde{D}$, where $\tilde{D}$ is the diagonal set $\{(\tilde{x}, \tilde{x}) : \tilde{x} \in \tilde{N}\}$, and $\mathrm{dist}^2_{\tilde{N}}$ is smooth on $\tilde{N} \times \tilde{N}$. Now $\pi_1(N,*)$ acts on $\tilde{N} \times \tilde{N}$ as a group of isometries by

$$\beta(\tilde{x}, \tilde{y}) = (\beta(\tilde{x}), \beta(\tilde{y})) \qquad \text{for } \beta \in \pi_1(N,*).$$

Thus $\mathrm{dist}^2_{\tilde{N}}$ induces a smooth function

$$\tilde{r}^2 : \tilde{N}_{\times/} \to \mathbb{R},$$

where we have defined

$$\tilde{N}_{\times/} := \tilde{N} \times \tilde{N}/\pi_1(N,*).$$

Let $U : M \times [0,1] \to N$ be a homotopy of u with v so that $U(q,0) = u(q)$ and $U(q,1) = v(q)$ for all $q \in M$. We choose a lifting $\tilde{U} : \tilde{M} \times [0,1] \to \tilde{N}$, and call $\tilde{U}(\tilde{q},0) =: \tilde{u}(\tilde{q})$ and $\tilde{U}(\tilde{q},1) =: \tilde{v}(\tilde{q})$ for all $\tilde{q} \in \tilde{M}$. This defines liftings $\tilde{u}, \tilde{v}$ of u, v and, since Riemannian coverings are local isometries, $\tilde{u}$ and $\tilde{v}$ are p-harmonic maps and

$$|d\tilde{u}|(\tilde{q}) = |du|(P_M(\tilde{q})), \qquad |d\tilde{v}|(\tilde{q}) = |dv|(P_M(\tilde{q})).$$

Now, $\pi_1(M,*)$ acts as a group of isometries on $\tilde{M}$ and we have
(3.1)
$$\tilde{u}(\gamma(\tilde{q})) = u_\sharp(\gamma)\tilde{u}(\tilde{q}), \qquad \tilde{v}(\gamma(\tilde{q})) = v_\sharp(\gamma)\tilde{v}(\tilde{q}), \qquad \forall \tilde{q} \in \tilde{M}, \gamma \in \pi_1(M,*),$$

where $u_\sharp, v_\sharp : \pi_1(M,*) \to \pi_1(N,*)$ are the induced homomorphism and $u_\sharp \equiv v_\sharp$ since u is homotopic to v.
Thus, the map

$$\tilde{j} : \tilde{M} \to \tilde{N} \times \tilde{N} \qquad \text{defined by} \qquad \tilde{j}(\tilde{x}) := (\tilde{u}(\tilde{x}), \tilde{v}(\tilde{x}))$$

induces by (3.1) a map

$$j : M \to \tilde{N}_{\times/}.$$

Using again the fact that Riemannian coverings are local isometries and by the good behaviour of the tension field with respect to the Riemannian product, j is a harmonic map.
Step b. Using the second variation of arc length, [CE], Schoen and Yau proved that

Lemma 3.6 (Proposition 1 in [SY3])**.** *Given two points $x_1, x_2 \in Q$ in a simply-connected Riemannian manifold Q with ${}^{Q}\mathrm{Sect} \leq 0$, and given*

$$X = X_1 + X_2 \in T_{x_1}Q \oplus T_{x_2}Q \equiv T_{(x_1,x_2)}(Q \times Q),$$

then

$$ {}^{Q\times Q}\operatorname{Hess}\operatorname{dist}_Q\big|_{(x_1,x_2)}(X,X)\geq 0, \qquad \textit{where } x_1\neq x_2, \tag{3.2}$$

$$ {}^{Q\times Q}\operatorname{Hess}\operatorname{dist}^2_Q\big|_{(x_1,x_2)}(X,X)\geq 0, \qquad \textit{on } Q\times Q, \tag{3.3}$$

and equality holds in (3.3) *if and only if there is a parallel vector field Z, defined along the unique geodesic γ_{x_1,x_2} from x_1 to x_2, such that $Z(x_1)=X_1$, $Z(x_2)=X_2$ and $\left\langle {}^Q R(Z,T)T,Z\right\rangle_Q\equiv 0$ on γ_{x_1,x_2}. In particular, if ${}^Q\operatorname{Sect}<0$, Z is proportional to T.*

We apply Lemma 3.6 with $Q=\tilde N$ and, with an abuse of notation, $X=d\tilde j$. Since $\tilde N\times\tilde N$ is the universal cover of $\tilde N_{\times/}$ and $\tilde j$ projects on j, this in particular gives

$$ {}^{\tilde N_{\times/}}\operatorname{Hess}\tilde r|_{j(q)}(dj,dj)\geq 0 \tag{3.4}$$

and

$$ {}^{\tilde N_{\times/}}\operatorname{Hess}\tilde r^2|_{j(q)}(dj,dj)\geq 0. \tag{3.5}$$

Step c. Define the real function $\rho: M\to\mathbb{R}$ as

$$\rho:=\tilde r\circ j$$

and observe that $\rho^2\in C^\infty(M)$. By the composition law of tension fields we see that the harmonicity of j, (3.4) and (3.5) imply

$$\begin{aligned} {}^M\Delta\rho &= {}^M\operatorname{tr}\,{}^{\tilde N_{\times/}}\operatorname{Hess}\tilde r\Big|_j(dj,dj)+d\tilde r|_j(\tau j)\\ &= {}^M\operatorname{tr}\,{}^{\tilde N_{\times/}}\operatorname{Hess}\tilde r\Big|_j(dj,dj)\geq 0 \end{aligned}\tag{3.6}$$

and

$$\begin{aligned} {}^M\Delta\rho^2 &= {}^M\operatorname{tr}\,{}^{\tilde N_{\times/}}\operatorname{Hess}\tilde r^2\Big|_j(dj,dj)+d\tilde r^2|_j(\tau j)\\ &= {}^M\operatorname{tr}\,{}^{\tilde N_{\times/}}\operatorname{Hess}\tilde r^2\Big|_j(dj,dj)\geq 0. \end{aligned}\tag{3.7}$$

Define the smooth function $\phi: M\to\mathbb{R}$ as $\phi:=\sqrt{\rho^2+1}$. Then ϕ satisfies

$$|\nabla\phi|=\frac{\rho}{\sqrt{\rho^2+1}}|\nabla\rho|\leq|\nabla\rho|\leq(|du|+|dv|)\in L^2(M)$$

and, by (3.6),

$${}^M\Delta\phi={}^M\operatorname{div}\frac{\rho^M\nabla\rho}{\sqrt{\rho^2+1}}=\frac{\rho}{\sqrt{\rho^2+1}}{}^M\Delta\rho+\frac{|{}^M\nabla\rho|^2}{(\rho^2+1)^{3/2}}\geq 0.$$

Hence ϕ is a finite-energy subharmonic function on a parabolic manifold, and applying Corollary 1.3 we obtain that ϕ is necessarily constant. Thus also ρ must be constant and (3.7) implies

$$ {}^M\operatorname{tr}\,{}^{N\times N}\operatorname{Hess}{}^N\tilde r^2\big|_{(u,v)}((du,dv),(du,dv))=0. \tag{3.8}$$

Step d. Consider a local orthonormal frame $\{\tilde{E}_j\}_{j=1}^m$ on $\tilde{M}$ and define vector fields $\tilde{X}_j$, $j = 1, \ldots, m$, on $\tilde{N} \times \tilde{N}$ as

$$\tilde{X}_j := (d\tilde{u}(\tilde{E}_j), d\tilde{v}(\tilde{E}_j)).$$

Then, when lifted to the universal covers, (3.8) gives

$${}^{\tilde{N}\times\tilde{N}}\operatorname{Hess}\operatorname{dist}^2_{\tilde{N}}(\tilde{X}_j, \tilde{X}_j) = 0,$$

for all $j = 1, \ldots, m$. At this point, applying again Lemma 3.6 implies

$$d\left(\operatorname{dist}_{\tilde{N}}\right)\left(d\tilde{u}(\tilde{E}_i), d\tilde{v}(\tilde{E}_i)\right) = d\left(\operatorname{dist}_{\tilde{N}} \circ (\tilde{u}, \tilde{v})\right)(\tilde{E}_i) \equiv 0$$

and, since $\{\tilde{E}_i\}_{i=1}^m$ span all $T_{\tilde{q}}\tilde{M}$, we get that $(\operatorname{dist}_{\tilde{N}} \circ (\tilde{u}, \tilde{v}))$ is constant on $\tilde{M}$. Accordingly, for each $\tilde{q} \in \tilde{M}$ the unique geodesic $\tilde{\gamma}_{\tilde{q}}$ from $\tilde{u}(\tilde{q})$ to $\tilde{v}(\tilde{q})$ can be parametrized on $[0, 1]$ proportional (independent of $\tilde{q}$) to arclength. We define a one-parameter family of maps $\tilde{u}_t : \tilde{M} \to \tilde{N}$ by letting $\tilde{u}_t(\tilde{q}) := \tilde{\gamma}_{\tilde{q}}(t)$. Then we see that $\tilde{u}_0 = \tilde{u}$ and $\tilde{u}_1 = \tilde{v}$. Lemma 3.6 states also that for each $i = 1, \ldots, m$ there exists a parallel vector field Z_i, defined along $\tilde{\gamma}_{\tilde{q}}$ in $\tilde{N}$, such that $Z_i(\tilde{u}(\tilde{q})) = d\tilde{u}|_{\tilde{q}}(\tilde{E}_i)$, $Z_i(\tilde{v}(\tilde{q})) = d\tilde{v}|_{\tilde{q}}(\tilde{E}_i)$ and $\left\langle {}^N R(Z_i, \dot{\tilde{\gamma}}_{\tilde{q}})\dot{\tilde{\gamma}}_{\tilde{q}}, Z_i \right\rangle_N \equiv 0$ along $\tilde{\gamma}_{\tilde{q}}$. Actually, one can prove that

$$Z_i \equiv d\tilde{u}_t(\tilde{E}_i). \tag{3.9}$$

In the special situation ${}^N\operatorname{Sect} < 0$, for all $\tilde{q} \in \tilde{M}$ the parallel vector field Z along $\gamma_{\tilde{q}}$ has to be proportional to $\dot{\gamma}_{\tilde{q}}$. Hence $\tilde{u}(\tilde{M})$ and $\tilde{v}(\tilde{M})$ have to be contained in a geodesic of $\tilde{N}$ and projecting on M we get the proof of case $i)$ of Theorem 3.4.

In general, by the homotopy between u and v, the family $\tilde{u}_t$ induces maps $u_t : M \to N$ for $t \in [0, 1]$ such that $u_0 \equiv u$ and $u_1 \equiv v$. Let $\gamma_q(t)$ be the geodesic from $u(q)$ to $v(q)$ in M obtained by projection from $\tilde{\gamma}_{\tilde{q}}$. Projecting Z_i, (3.9) implies that du_t is a parallel vector field along γ_q. Therefore

$$e(u_t)(q) = \sum_{i=1}^m |du_t(E_i)|^2$$

is constant along γ_q for each $q \in M$ and, consequently,

$$E(u) = E(u_t) = E(v), \qquad \forall t \in [0, 1], \tag{3.10}$$

that is, every harmonic map of finite energy homotopic to u has the same energy as u.

Step e. We prove that u_t is harmonic for all t. Assume M is non-compact, the other case being easier; see [SY1].

By contradiction, suppose u_t is not harmonic. Then, by (1.1) and subsequent considerations we know that there exist a compact set $\Omega \subset M$ and a smooth map $\hat{u}_t$ such that

$$\begin{aligned} &\hat{u}_t \equiv u_t \qquad \text{on } M \setminus \Omega, \\ &E(\hat{u}_t) < E(u_t) = E(u), \\ &\hat{u}_t \text{ and } u \text{ are homotopic maps.} \end{aligned} \tag{3.11}$$

Consider an exhaustion $\{M_k\}_{k=1}^{\infty}$ of M. Up to choose a subsequence of the exhaustion, we can suppose that $\Omega \subset\subset M_1$. Following Hamilton, [Ham], for each k we use the heat flow to solve the boundary value problem on M_k with initial value $\hat{u}_t|_{M_k}$. Indeed, by the results in [SY1], Section IX.8, the heat flow is well defined for all time and converges to a harmonic map. Accordingly, we get a harmonic map $u_{t,k} : M_k \to N$ homotopic to $u_t|_{M_k}$ with

$$u_{t,k} \equiv \hat{u}_t \equiv u_t \qquad \text{on } \partial M_k$$

and

$$E\left(u_{t,k}\right) \leq E\left(\hat{u}_t|_{M_k}\right). \tag{3.12}$$

Since $\tau u = \tau u_{t,k} = 0$, reasoning as in (3.6) the function $\mathrm{dist}_N(u_{t,k}, u) : M_k \to \mathbb{R}$ is subharmonic on M_k. Moreover,

$$\mathrm{dist}_N(u_{t,k}, u)|_{\partial M_k} = \mathrm{dist}_N(u_t, u)|_{\partial M_k} \leq C,$$

where the constant $C = \mathrm{dist}_{\tilde{N}}(\tilde{u}, \tilde{u}_q)$ does not depend on k. By the maximum principle, we deduce that for each $l \geq 1$ there exist a compact set $K_l \subseteq N$ independent of k (namely, we can choose K_l as a neighborhood of $u(M_l)$) such that $u_{t,k}(M_l) \subseteq K_l$. Let $i : N \hookrightarrow \mathbb{R}^q$ be an isometric immersion of N into some Euclidean space. Since $\{E(u_{t,k})\}_{k=1}^{\infty}$ is bounded, $i(u_{t,k}(M_l)) \subset i(K_l)$ and $i(K_l) \subset \mathbb{R}^q$ is compact, the sequence of harmonic maps $\{u_{t,k}|_{M_l}\}_{k=l}^{\infty}$ is bounded in $W^{1,2}(M_l, \mathbb{R}^q)$ and, up to choose a subsequence, $u_{t,k}|_{M_l}$ converges to some harmonic map $u_t^{(l)} \in W^{1,2}(M_l, \mathbb{R}^q)$ weakly in $W^{1,2}$, strongly in L^2 and pointwise almost everywhere, which implies $u_t^{(l)} \in W^{1,2}(M_l, N)$. Because of the convergence properties of $u_{t,k}|_{M_l}$ and the uniform boundedness of $u_{t,k}(M_l)$, reasoning as in the proof of Theorem 2.4 we know that the 1-homotopy type of $u_{t,k}|_{M_l}$ is preserved under the limit procedure. Since N is $K(\pi, 1)$, also the homotopy class is preserved and so $u_t^{(l)}$ is homotopic to $u_t|_{M_l}$. Now, using standard diagonal arguments we can choose a subsequence of $u_{t,k}$, still denoted $u_{t,k}$, such that, for all l, $u_{t,k}|_{M_l}$ converges to $u_t^{(l)} \in W^{1,2}(M_l, N)$ weakly in $W^{1,2}$, strongly in L^2 and pointwise almost everywhere. Since by pointwise convergence $u_t^{(l)}$ and $u_t^{(l+1)}$ agree almost everywhere on M_l, the map $u_{t,\infty} : M \to N$ which takes values $u_{t,\infty}|_{M_l} = u_t^{(l)}$ is well defined and harmonic. Moreover, by Theorem 2.5, $u_{t,\infty}$ is homotopic to u_t since they are homotopic on compact sets. Applying (3.10) with $v = u_{t,\infty}$ we have

$$E(u_{t,\infty}) = E(u) = E(u_t). \tag{3.13}$$

On the other hand, (3.12) and the lower semicontinuity of E give

$$E(u_{t,\infty}|_{M_l}) = E(u_t^{(l)}) \leq \liminf_{k\to\infty} E(u_{t,k}|_{M_l}) \leq E(u_{t,l}) \leq E(\hat{u}_t|_{M_l}),$$

which, by (3.11), gives

$$E(u_{t,\infty}) \leq E(\hat{u}_t) < E(u_t) = E(u).$$

This contradicts (3.13) and concludes the proof. $\square$

As announced in the introduction, the extension of Schoen and Yau's result to the non-linear setting seems not obvious, since their strategy apparently doesn't work. In particular difficulties arise in adapting steps **b**, **c** and **e**. Troughout this Chapter 3 we will analyze these problems pointing out, when possible, how to overcome them in order to prove Theorem 3.3.

3.2 The composition of p-harmonic maps and convex functions

A crucial point in the proof of Theorem 3.4 is the subharmonicity of ρ. If N is simply connected, hence Cartan-Hadamard, it represents the distance function on N between the harmonic maps u and v. As it is clear by inequality (3.6), the subharmonicity of ρ is obtained since ρ is the composition of the harmonic map $(f,g) : M \to N \times N$ with dist_N, which is convex due to Lemma 3.6. Actually, it is a general well known fact that the composition $H \circ F$ of a harmonic map $F : M \to N$ with a convex function $H : N \to \mathbb{R}$ is subharmonic, regardless of any curvature assumption on the manifolds. This is easily implied by the composition law of tension fields

$$^{M}\Delta(H \circ F)(x) = {}^{M}\operatorname{tr}{}^{N}\operatorname{Hess} H\big|_{F(x)}(dF, dF)(x) + dH|_{F(x)} \circ {}^{M}\tau F(x).$$

As a matter of fact this property can be used to characterize the harmonicity of F, see Theorem 3.4 in [I]. Namely, T. Ishihara proved that the map $F : M \to N$ is harmonic if and only if, for any open subset $U \subseteq N$, it pulls back any convex function defined on U to a subharmonic function on $F^{-1}(U)$. This is extremely useful since, for example, Liouville type theorems for harmonic maps into targets supporting a convex function can be obtained directly from results in linear potential theory of real valued functions. Such Liouville conclusions, in turn, have topological consequences, as shown in the previous section.
In this respect, one is led to inquire whether the composition of a p-harmonic map with a convex function is p-subharmonic and, therefore, if the non-linear potential theory of real-valued functions suffices to get the desired conclusions. By the way, this problem was pointed out by Lin and Wei among a list of open questions in geometry; see Problem 7 in [LW]. In this section we show a counterexaple, i.e., we describe a p-harmonic map and a convex function whose composition is not p-subharmonic. It was folklore that, in general, the subharmonicity of the composition fails, so that one is forced to follow different paths to obtain the results alluded to above; see e.g. [CL], [Kaw], [PRS3] and the results in Section 3.3 and Section 3.4. However, to the best of our knowledge, previous counterexamples are not available in the literature.

3.2.1 A counterexample

Through all this Section 3.2, let M_g and N_j be $(n+1)$-dimensional model manifold in the sense of Greene and Wu, [GW], that is $(n+1)$-dimensional Riemannian manifolds with rotationally symmetric metrics defined as

$$M_g = ([0, +\infty) \times \mathbb{S}^n, ds^2 + g^2(s)d\theta^2)$$
$$N_j = ([0, +\infty) \times \mathbb{S}^n, dt^2 + j^2(t)d\theta^2),$$

where $g, j \in C^2([0, +\infty))$ satisfy

(3.14) $\quad g(0) = j(0) = 0, \qquad g'(0) = j'(0) = 1, \qquad g(s), j(t) > 0$ for $s, t > 0$,

and $(\mathbb{S}^n, d\theta^2)$ is the Euclidean n-sphere with its standard metric. We say that the C^2 map $F : M_g \to N_j$ is rotationally symmetric if

$$F(s, \theta) = (f(s), \theta) \qquad \forall s > 0, \theta \in \mathbb{S}^n,$$

for some function $f \in C^2([0, \infty))$. Similarly, by a C^2 rotationally symmetric real valued function on N_j we mean a function $H : N_j \to \mathbb{R}$ of the form

$$H(t, \theta) = h(t) \qquad \forall t > 0, \theta \in \mathbb{S}^n,$$

for some $h \in C^2([0, \infty))$.

We shall prove the following

Theorem 3.7 (Theorem 1 in [V]). *Consider two rotationally symmetric $(n+1)$-dimensional manifolds M_g, N_j. Suppose that $(n + 1) > p > \max\{2, n\}$ and assume that the warping functions $g, j \in C^2([0, +\infty))$ have the form*

$$g(s) = (s + \delta^{-\frac{1}{\delta-1}})^{\delta} - \delta^{-\frac{\delta}{\delta-1}}, \qquad j(t) = (t + \sigma^{\frac{1}{1-\sigma}})^{\sigma} - \sigma^{\frac{\sigma}{1-\sigma}},$$

where $\delta > (p - n)^{-1} > 1$ and $0 < \sigma < 1$. Then, there exist a C^2 rotationally symmetric p-harmonic map $F : M_g \to N_j$ and a sequence $\{s_k\}_{k=1}^{\infty} \to +\infty$, such that

$$\Delta_p(H \circ F)(s_k, \theta) < 0$$

for every rotationally symmetric convex function $H : N_j \to \mathbb{R}$, provided the corresponding $h \in C^2([0, +\infty))$ satisfies $h'(t) > 0$ for $t > 0$.

Remark 3.8. *As announced above, the counterexample proposed in Theorem 3.7 shows that, in general, the composition of a p-harmonic map with a convex function is not p-subharmonic. Hence, the proof of Theorem 3.4 can not be trivially adapted to the non-linear case. Nevertheless, it has to be pointed out that*

i) According to standard computations, we have that

$$^{N_j}\,\mathrm{Sect}_{rad}(t) = -\frac{j''(t)}{j(t)} \geq 0.$$

ii) Since M_g is a model manifold and, by the assumption on the parameters,

$$A(\partial B_s^{M_g})^{-\frac{1}{p-1}} \sim s^{-\frac{n\delta}{p-1}} \in L^1(+\infty),$$

we deduce (see Section 1.2) that M_g is not p-parabolic.

In order to proof Theorem 3.3, we should assume that both the domain manifold is p-parabolic and the target manifold has non-positive sectional curvature. Actually, in this case, so far we have not been able neither to find a suitable counterexample nor to estabilish that the composition well behaves under these restrictive conditions. See also Appendix B below.

It should be noted that, in the paper [I] cited above, the author also considers a special category of harmonic maps, the harmonic morphisms, which pull back germs of harmonic functions on the target to harmonic functions in the domain. It is proved that harmonic morphisms are characterized by a weakly horizontal conformality condition. Recently, [Lo], such a characterization has been extended to the p-harmonic setting, $p > 2$. It turns out that the p-tension field of the composition of a p-harmonic morphism with a generic function enjoys a very special decomposition. Moreover it's proven that C^2 convex functions are p-subharmonic; see [WLW1] and [WLW2]. Accordingly one has that p-harmonic morphisms pull back p-subharmonic functions (and hence pull-back convex functions) to p-subharmonic functions. Such a special decomposition, however, fails to be true in general for a p-harmonic map, and the rotationally symmetric realm provides concrete examples.

The proof of Theorem 3.7 relies on a number of preliminary facts on rotationally symmetric p-harmonic maps, ranging from explicit formulas up to existence results and companion asymptotic estimates. In all that follows, notations are those introduced in Theorem 3.7.

3.2.2 Rotationally symmetric p-harmonic maps

The p-tension field of the map F, on the subset of M_g where $|dF| \neq 0$, writes as

$$\begin{aligned}\tau_p(F) &= \operatorname{div}(|dF|^{p-2}\, dF) \\ &= |dF|^{p-2}\left\{\tau(F) + i(\nabla(\ln |dF|^{p-2}))dF\right\},\end{aligned} \tag{3.15}$$

where i denotes the interior product on 1-forms. Using the rotational symmetry condition we have

$$|dF|^2(s) = \left\{(f'(s))^2 + n\frac{j^2(f(s))}{g^2(s)}\right\}.$$

Furthermore, the energy of F is (up to a constant factor)

$$E(F) = \int_0^\infty \left\{(f'(s))^2 + n\frac{j^2(f(s))}{g^2(s)}\right\} g^n(s)ds,$$

and the tension field of F that is involved in the Euler-Lagrange equation of the energy functional takes the expression

$$\tau(F) = \left\{f''(s) + \frac{n}{g^2(s)}\left[g(s)g'(s)f'(s) - j(f(s))j'(f(s))\right]\right\} \left.\frac{\partial}{\partial t}\right|_{f(s)}, \tag{3.16}$$

Substituting (3.16) into (3.15), we have

$$\begin{aligned}\tau_p(F) = |dF|^{p-2}(s)\Bigg\{&\left[f''(s) + \frac{n}{g^2(s)}\left(g(s)g'(s)f'(s) - j(f(s))j'(f(s))\right)\right] \\ &+ (p-2)\,|dF|^{-2}(s)f'(s)\Big[f'(s)f''(s) \\ &+n\frac{j(f(s))}{g^3(s)}\left(j'(f(s))f'(s)g(s) - j(f(s))g'(s)\right)\Big]\Bigg\} \left.\frac{\partial}{\partial t}\right|_{f(s)} = 0,\end{aligned} \tag{3.17}$$

provided F is p-harmonic. Now, we want to compute the p-laplacian of the composition $H \circ F$. Using (3.15) with F replaced by $H \circ F$, and setting

$$K(s) = |d(H \circ F)|^{p-2} = |h'(f(s))f'(s)|^{p-2}$$

we conclude

$$\begin{aligned}\Delta_p(H \circ F) = K(s)\, \{&h'(f(s))f''(s) + h''(f(s))(f'(s))^2 \\ &+ng^{-1}(s)g'(s)f'(s)h'(f(s)) \\ &+ (p-2)\left[h'(f(s))f''(s) + h''(f(s))(f'(s))^2\right]\}\,, \end{aligned} \tag{3.18}$$

on the subset

$$M_+ = \{(s,\theta) : h'(f(s))f'(s) > 0\} \subseteq M_g.$$

Finally, we recall that for vector fields V and W on N, $(\mathrm{Hess}(H)(t,\theta))(V,W) = (W(VH) - dH(D_V W))(t,\theta)$, and for vertical vector fields V and W on N, the projection of $D_V W$ onto its horizontal subspace is given by the formula

$$(D_V W)^{\perp} = -\left(\frac{\langle V, W\rangle}{f}\right)\nabla f$$

where D is the Riemannian connection on N, $\langle , \rangle$ is the warped metric, and f is the warping function, see p.206, Proposition 35(3) in [O]. Hence

$$\mathrm{Hess}\,(H)\,(t,\theta) = h''\,(t)\,dt^2 + j'\,(t)\,j\,(t)\,h'\,(t)\,d\theta^2.$$

Since the function $j(t)$ defined in Theorem 3.7 is positive and strictly increasing, the above expression gives us that the convexity of H is equivalent to the set of conditions

$$\begin{cases} h''(t) \geq 0 \\ h'(t) \geq 0, \end{cases} \qquad \forall t > 0. \tag{3.19}$$

3.2.3 Existence results and asymptotic estimates

The existence of rotationally symmetric p-harmonic maps has been investigated by several authors. Here, we recall the following theorem which encloses in a single statement Lemma 2.5, Theorem 2.11, Proposition 3.1 and Theorem 3.2 in [CLLM](see also Corollary 3.22 in [Le]). From now on, given real functions $g_1(s)$ and $g_2(s)$, we say that $g_1 \asymp g_2$ for large s if there exists positive constants k_1 and k_2 such that $k_1 g_2(s) \leq g_1(s) \leq k_2 g_2(s)$ for all large s, while we say that $g_1(s) \sim g_2(s)$ if $\lim_{s\to\infty} \frac{g_1(s)}{g_2(s)} = 1$.

Theorem 3.9 (Lemma 2.5, Theorem 2.11, Proposition 3.1 and Theorem 3.2 in [CLLM]). *Suppose that $p > 2$ and assume that there exist constants $a > 0$ and $\delta > 1$ with $n\delta > p-1$ such that $g, j \in C^2(0,\infty)$,*

$$j(t) > 0, \qquad 0 \leq j'(t) \leq a \qquad \forall t > 0,$$

and

$$g(s) \asymp s^{\delta}, \qquad g'(s) > 0 \qquad \text{for large } s,$$

where g and j satisfy the conditions in (3.14). Then, for any $\alpha > 0$, there is a bounded solution $f \in C^2[0,+\infty)$ to equation (3.17) such that $f(0) = 0$, $f'(0) = \alpha$ and $f(s), f'(s) > 0$ for all $s > 0$.

Remark 3.10. *Note that Theorem 3.9 and the assumption $h'(t) > 0$ imply that $(s,\theta) \in M_+$ and $K(s) \neq 0$ for every $s > 0$.*

We now want to obtain an asymptotic estimate for $f'(s)$. The following lemma, which is modeled on Corollary 3.13 in [CLLM], will play a crucial role.

Lemma 3.11 (Lemma 1 in [V]). *Suppose that $(n+1) > p > \max\{2, n\}$ and assume that there exist constants $a > 0$ and $\delta > (p-n)^{-1} > 1$, such that $g, j \in C^1(0,\infty)$,*

$$j(t) > 0, \qquad 0 < j'(t) \leq a \qquad \forall t > 0,$$

and

$$g(s) \sim C_1 s^{\delta}, \qquad g'(s) > 0 \qquad \text{for large } s,\ C_1 > 0,$$

where g and j satisfy the conditions in (3.14). Then all positive solutions to equation (3.17) satisfy

$$f'(s) \sim D s^{-\delta(n-(p-2))}, \qquad \text{as } s \to +\infty, \tag{3.20}$$

for some positive constant D.

Proof. Let us begin by recalling the following estimate which will be useful later (see (3.7) in [CLLM])

$$g^n(s)|dF|^{p-2}(s)f'(s) \leq \tilde{C}\left(\int_{s_0}^{s} r^{(n-2)\delta}|dF|^{p-2}(r)f(r)dr + 1\right), \tag{3.21}$$

for $s \geq s_0$, where s_0 is some positive constant. From equations (3.15) and (3.16), we get

$$f'(s)(|dF|^{p-2}(s))' = |dF|^{p-2}(s)\left[\frac{nj(f(s))j'(f(s))}{g^2(s)} - f''(s) - \frac{ng'(s)f'(s)}{g(s)}\right],$$

from which we obtain that

$$(g^n|dF|^{p-2}f')'(s) = n|dF|^{p-2}(s)g^{n-2}(s)j(f(s))j'(f(s)) \geq 0, \qquad \forall s > 0.$$

Hence $(g^n|dF|^{p-2}f')$ is non-decreasing and the following limit holds

$$(g^n|dF|^{p-2}f')(s) \to P \in (0, +\infty], \qquad \text{for } s \to +\infty.$$

We claim that the limit P is finite. By contradiction suppose $P = +\infty$, then there exists a sequence $\{S_N\}_{N=1}^{\infty}$ such that

$$S_N \to +\infty \qquad \text{and} \qquad (g^n|dF|^{p-2}f')(S_N) = N,$$

which implies, for all $s \leq S_N$,

$$g^n(s)(f'(s))^{p-1} \leq g^n(s)|dF|^{p-2}(s)f'(s) \leq N$$

and

$$f'(s) \leq N^{\frac{1}{p-1}} g^{-\frac{n}{p-1}}(s) \leq C N^{\frac{1}{p-1}} s^{-\frac{n\delta}{p-1}}.$$

Moreover, since $p < n+1$ and $\delta > (p-n)^{-1}$ imply $n\delta > (p-1)$, we can apply Theorem 3.9 to deduce that $f'(s) > 0$ and $f(s)$ is bounded. Thus

$$f(s) \to \hat{c} > 0 \ \text{ as } s \to +\infty, \tag{3.22}$$

$f(s) \leq \hat{c}$ for all s and $f(s) > \hat{c}/2$ for s large enough. Now,

$$
\begin{aligned}
|dF|^2(s) &\leq C N^{\frac{2}{p-1}} s^{-\frac{2n\delta}{p-1}} + n\frac{a^2 f^2(s)}{g^2(s)} \\
&\leq C \max\left\{ N^{\frac{2}{p-1}} s^{-\frac{2n\delta}{p-1}}; s^{-2\delta} \right\} \leq C N^{\frac{2}{p-1}} s^{-2\delta}, \qquad (3.23)
\end{aligned}
$$

since $n > (p-1)$. Hence, from (3.21), (3.22) and (3.23), we get

$$
\begin{aligned}
N = (g^n |dF|^{p-2} f')(S_N) &\leq \tilde{C} \left(\int_{s_0}^{S_N} r^{(n-2)\delta} |dF|^{p-2}(r) f(r) dr + 1 \right) \\
&\leq C \left(N^{\frac{p-2}{p-1}} \int_{s_0}^{S_N} r^{-\delta(p-n)} dr + 1 \right) = o(N), \qquad \text{as } s \to +\infty,
\end{aligned}
$$

since $p > n$ and $\delta(p-n) > 1$. Contradiction. Then

$$
f'(s) \sim P |dF|^{2-p}(s) g^{-n}(s), \qquad (3.24)
$$

as $s \to \infty$, for some positive constant $P < \infty$.
Now, we need an asymptotic estimate for $|dF|$. Note that

$$
0 \leq \frac{(f'(s))^2 g^2(s)}{nj^2(f(s))} \leq \frac{C s^{-\frac{2n\delta}{p-1}} s^{2\delta}}{nj^2(\frac{\hat{c}}{2})} = C s^{2\delta(1-\frac{n}{p-1})} \to 0, \qquad \text{as } s \to +\infty,
$$

since $p - 1 < n$. Therefore

$$
\lim_{s \to +\infty} \frac{|dF|^2(s)}{n\frac{j^2(f(s))}{g^2(s)}} = \lim_{s \to +\infty} \frac{(f'(s))^2 g^2(s)}{nj^2(f(s))} + 1 = 1, \qquad (3.25)
$$

proving that

$$
|dF|^2(s) \sim n\frac{j^2(f(s))}{g^2(s)} \sim \frac{nj^2(\hat{c})}{C_1^2} s^{-2\delta}, \text{ as } s \to +\infty.
$$

Using this information into (3.24) we conclude

$$
f'(s) \sim D s^{-\delta(n-(p-2))}, \qquad \text{with } D := P C_1^{-n} \left(\frac{C_1^2}{nj^2(\hat{c})} \right)^{\frac{p-2}{2}} > 0,
$$

where $n > p - 1 > p - 2$. □

3.2.4 Proof of Theorem 3.7

Observe that the warping functions g and j defined as in Theorem 3.7 satisfy the assumptions of Theorem 3.9 and Lemma 3.11. Then, there exists a rotationally symmetric p-harmonic map $F(s,\theta) = (f(s),\theta) : M_g \to N_j$ where $f(s)$ is a positive, bounded, increasing function which satisfies (3.17) and the asymptotic estimates (3.20) and (3.22).

Now, multiplying (3.17) by $h'(f(s))$, we get

$$
\begin{aligned}
& h'(f(s)) f''(s) + n g^{-1}(s) h'(f(s)) g'(s) f'(s) \\
&= n g^{-2}(s) h'(f(s)) j(f(s)) j'(f(s)) - (p-2) |dF|^{-2} \left[h'(f(s)) f''(s) (f'(s))^2 \right. \\
&\left. + n g^{-2}(s) j(f(s)) j'(f(s)) h'(f(s)) (f'(s))^2 - n g^{-3}(s) j^2(f(s)) g'(s) h'(f(s)) f'(s) \right].
\end{aligned}
$$

and inserting the latter into (3.18) we obtain

$$\Delta_p(H \circ F) = K(s)\tilde{K}(s)\left\{A_1(s) + A_2(s) + A_3(s)\right\}, \tag{3.26}$$

where we have set

$$\begin{aligned}
K(s) &= |h'(f(s))f'(s)|^{p-2} > 0, \qquad \forall s > 0;\\
\tilde{K}(s) &:= \frac{nj(f(s))h'(f(s))}{|dF|^2(s)g^2(s)} > 0, \qquad \forall s > 0;\\
A_1(s) &:= j'(f(s))\left[(3-p)(f'(s))^2 + n\frac{j^2(f(s))}{g^2(s)}\right];\\
A_2(s) &:= (p-2)j(f(s))\left[\frac{g'(s)f'(s)}{g(s)} + f''(s)\right];\\
A_3(s) &:= (p-1)(f'(s))^2h''(f(s))\frac{|dF|^2(s)g^2(s)}{nj(f(s))h'(f(s))}.
\end{aligned}$$

Remark 3.12. *In the harmonic case $p = 2$, (3.26) reduces to*

$$\Delta(H \circ F) = (f'(s))^2h''(f(s)) + \frac{n}{g^2(s)}j(f(s))j'(f(s))h'(f(s))$$

which is always nonegative when H is convex, as we observed at the beginning of this section.

Reasoning as in the proof of (3.25) above, we obtain

$$A_1(s) \sim nj'(\hat{c})j^2(\hat{c})s^{-2\delta},$$

and

$$A_3(s) \sim \frac{(p-1)D^2h''(\hat{c})}{h'(\hat{c})}j(\hat{c})s^{-2\delta(n-(p-2))},$$

as $s \to +\infty$. Moreover, according to l'Hôpital rule we have

$$1 = \limsup_{s\to+\infty}\frac{f'(s)}{Ds^{-\delta(n-(p-2))}} \le \limsup_{s\to+\infty}\frac{f''(s)}{-\delta(n-(p-2))Ds^{-\delta(n-(p-2))-1}}.$$

Thus, for every $\epsilon > 0$ there exists a sequence $\{s_k\}_{k=1}^{\infty}$ such that $s_k \to +\infty$ and

$$f''(s_k) \le -\delta(n-(p-2))Ds_k^{-\delta(n-(p-2))-1}(1-\epsilon).$$

Since

$$\frac{g'(s)f'(s)}{g(s)} \sim \delta Ds^{-\delta(n-(p-2))-1}, \qquad \text{as } s \to +\infty,$$

we have

$$\begin{aligned}
A_2(s_k) \le (p-2)j(\hat{c})\Big\{&(1+\epsilon)\delta Ds_k^{-\delta(n-(p-2))-1}\\
&-(1-\epsilon)\delta(n-(p-2))Ds_k^{-\delta(n-(p-2))-1}\Big\}.
\end{aligned}$$

for k large enough. Now recall that, by the assumptions on n and p, it holds

$$D\delta(1-(n-(p-2))) < 0.$$

Therefore, we can choose

$$0 < \epsilon < (n+1-p)/(n+3-p)$$

in order to ensure that, for every k large enough,

$$A_2(s_k) < 0.$$

Finally note that, as $s_k \to +\infty$, $A_1(s_k)$ and $A_3(s_k)$ decay faster than $A_2(s_k)$ because, by the assumptions on δ, n and p,

$$-2\delta(n-(p-2)) < -1-\delta(n-(p-2)), \qquad -2\delta < -1-\delta(n-(p-2)).$$

According to (3.26), this shows that, for k large enough, $\Delta_p(H \circ F)(s_k) < 0$, as requested.

3.3 Global comparisons

In Section 3.2, we identified the first great obstruction to adapt the proof of Schoen and Yau's Theorem 3.4 to $p \neq 2$. Namely, we showed in Theorem 3.7 how apparently one can not take advantage of the good properties of the composition of harmonic maps with convex functions. Thus, one is led to look for different techniques. Some first steps in this direction have been taken.
The first result is due to Pigola, Rigoli and Setti, [PRS3] which studied the case of a single finite p-energy p-harmonic map $u : M \to N$ homotopic to a constant.

Theorem 3.13 (Theorem 1 in [PRS3])**.** *Let M and N be complete Riemannian manifolds. Assume that M is p-parabolic and that ${}^N\mathrm{Sect} \leq 0$. If $u : M \to N$ is a p-harmonic map homotopic to a constant and with finite p-energy $|du|^p \in L^1(M)$, then u is a constant map.*

To prove Theorem 3.13, the authors, reasoning in a way similar to the proof of Theorem 3.4, step **a**, introduced a special $\pi_1(M)$-equivariant vector field

$$\tilde{X} := [dh|_{\tilde{u}} \circ (|d\tilde{u}|^{p-2} d\tilde{u})]^\sharp \tag{3.27}$$

on $\tilde{M}$, where $\sharp$ denotes the isomorphism defined, by using the Riemannian metric, as $\langle \omega^\sharp, V \rangle = \omega(V)$ for all differential 1-forms ω and vector fields V. Here, $\tilde{u}$ it the lifting of u to the universal covers $\tilde{M}$ and $\tilde{N}$. Furthermore, having fixed $\tilde{q}_0 \in \tilde{M}$, $h : \tilde{N} \to \mathbb{R}$ is a C^2 function, strictly convex in a neighborhood Ω of $\tilde{u}(\tilde{q}_0)$ and weakly convex on all of $\tilde{N}$, defined by $h(\cdot) = k(\mathrm{dist}_{\tilde{N}}(\tilde{u}(\tilde{q}_0), \cdot))$, where $k \in C^2([0, +\infty))$ is such that $k' \geq 0$, $k'' \geq 0$ and

$$k(t) := \begin{cases} At^2 + B, & t \leq 1, \\ t, & t > 1, \end{cases}$$

for suitable constants $A, B > 0$. Due to the equivariant property, $\tilde{X}$ projects to a well defined vector field X on M. Then

$${}^M\mathrm{div}\, X = |d\tilde{u}|^{p-2} \operatorname{tr}_{\tilde{M}} {}^{\tilde{N}}\mathrm{Hess}\, h(d\tilde{u}, d\tilde{u}) + dh \circ \tau_p(\tilde{u}) \geq 0.$$

Since $|X| \in L^{p/(p-1)}$, Proposition 1.2 can be applied, and they got $\operatorname{div}_M X = 0$. By curvature assumptions, the Hessian comparison theorem thus gives

$$0 = \operatorname{tr}_{\tilde{M}} {}^{\tilde{N}}\operatorname{Hess} h(d\tilde{u}, d\tilde{u}) \geq 2A|d\tilde{u}|^2 \geq 0,$$

i.e., $|d\tilde{u}| \equiv 0$ on $\tilde{u}^{-1}(\Omega)$. Since

$$|d\tilde{u}|(\tilde{q}_0) = |du|(P_M(\tilde{q}_0)),$$

letting q_0 vary in M, they could conclude.

In this section, we focus our attention on the case $N = \mathbb{R}^n$. According to Theorem 3.13, it is clear that, if M is p-parabolic, then every p-harmonic map $u : M \to \mathbb{R}^n$ with finite p-energy $|du| \in L^p(M)$ must be constant. However, using the very special structure of $\mathbb{R}^n$ and some special vector fields inspired to Theorem 3.13, we are able to extend this conclusion, thus establishing a comparison principle for maps $u, v : M \to \mathbb{R}^n$ having the same p-Laplacian.

Theorem 3.14 (Theorem 3 in [HPV] and Theorem 15 in [VV])**.** *Suppose that $(M, \langle,\rangle)$ is a complete non-compact Riemannian manifold. For $p > 1$, let $u, v : M \to \mathbb{R}^n$ be $C^0 \cap W^{1,p}_{loc}(M, \mathbb{R}^n)$ maps satisfying*

$$\tau_p u = \tau_p v \text{ on } M, \tag{3.28}$$

in the sense of distributions on M and

$$|du|, |dv| \in L^p(M).$$

Suppose M is p-parabolic. Then $u - v$ is constant.

In case $n = 1$, in the assumption (3.28) the real valued laplacians $\tau_p u = \Delta_p u$ and $\tau_p v = \Delta_p v$ have sign and so (3.28) can be relaxed by assuming $\Delta_p u \geq \Delta_p v$.

Theorem 3.15 (Theorem 1 [HPV])**.** *Let $(M, \langle,\rangle)$ be a connected, possibly incomplete, p-parabolic Riemannian manifold, with $p > 1$. Assume that $u, v \in W^{1,p}_{loc}(M) \cap C^0(M)$ satisfy*

$$\Delta_p u \geq \Delta_p v \text{ weakly on } M,$$

and

$$|\nabla u|, |\nabla v| \in L^p(M).$$

Then, $u - v$ is constant.

Simple examples show that both the p-parabolicity of M and the L^p integrability of $|\nabla u|$ or $|\nabla v|$ are needed above. Indeed, let M be, for instance, the open unit ball in $\mathbb{R}^m$, u a constant function, and v a non-constant p-harmonic function in M (i.e. a continuous weak solution to $\Delta_p v = 0$), with $|\nabla v| \in L^p(M)$. Then M is non-p-parabolic for all $p > 1$ and the conclusion of Theorem 3.15 clearly fails. On the other hand, let M be the infinite cylinder $\mathbb{R} \times \mathbb{S}^{m-1}$ equipped with the product metric $ds^2 = dr^2 + d\vartheta^2$, where $d\vartheta^2$ is the standard metric of the sphere $\mathbb{S}^{m-1}$. Furthermore, let u be a constant function and $v(t, \vartheta) = t$. Now M is p-parabolic for all $p > 1$, u and v are p-harmonic in M, but the conclusion of Theorem 3.15 again fails.

3.3.1 A key inequality

In the proofs of Theorems 3.14 and 3.15 we will use two main ingredients. The first is the weak version of the Kelvin-Nevanlinna-Royden criterion stated in Proposition 1.2. The second one is a version for the p-Laplacian of a classical inequality for the mean-curvature operator. By the way this inequality will be also used in Subsection 3.3.3 to prove a further comparison whithout the p-parabolicity assumption; see Theorem 3.20.
The following basic inequality was discovered by Lindqvist, [Li].

Lemma 3.16 (Lemma 4.2 in [Li]). *Let $(V, \langle , \rangle)$ be a finite dimensional, real vector space endowed with a positive definite scalar product and let $p > 1$. Then, for every $x, y \in V$ it holds*

$$|x|^p + (p-1)\,|y|^p - p\,|y|^{p-2}\,\langle x, y\rangle \geq C(p)\Psi(x, y),$$

where

$$\Psi(x, y) := \begin{cases} |x-y|^p & p \geq 2 \\ \frac{|x-y|^2}{(|x|+|y|)^{2-p}} & 1 < p < 2, \end{cases}$$

and $C(p)$ is a positive constant depending only on p.

As a consequence, we deduce the validity of the next

Corollary 3.17. *In the above assumptions, for every $x, y \in V$, it holds*

$$\left\langle |x|^{p-2}\,x - |y|^{p-2}\,y, x-y \right\rangle \geq 2C(p)\Psi(x, y). \tag{3.29}$$

Proof. We start computing

$$\left\langle |x|^{p-2}\,x - |y|^{p-2}\,y, x-y \right\rangle = |x|^p + |y|^p - \langle x, y\rangle \left(|x|^{p-2} + |y|^{p-2}\right).$$

On the other hand, applying twice Lindqvist inequality with the role of x and y interchanged we get

$$p\left(|x|^p + |y|^p\right) \geq p\left(|x|^{p-2} + |y|^{p-2}\right)\langle x, y\rangle + 2C(p)\Psi(x, y).$$

Inserting into the above completes the proof. □

Remark 3.18. *Inequality (3.29) can be considered as a version for the p-Laplacian of the classical Miklyukov-Hwang-Collin-Krust inequality; [Mi], [Hw], [CK]. This latter states that, for every $x, y \in V$,*

$$\left\langle \frac{x}{\sqrt{1+|x|^2}} - \frac{y}{\sqrt{1+|y|^2}}, x-y \right\rangle \geq \frac{\sqrt{1+|x|^2}+\sqrt{1+|y|^2}}{2}\left|\frac{x}{\sqrt{1+|x|^2}} - \frac{y}{\sqrt{1+|y|^2}}\right|^2,$$

equality holding if and only if $x = y$. This analogy suggests the validity of global comparison results, without any p-parabolicity asssumption, in the spirit of [PRS1], as exemplified in Subsection 3.3.3.

3.3.2 Proofs of the finite-energy comparison principles

We are now in the position to prove the main results.

Proof (of Theorem 3.15). Fix any $x_0 \in M$, let $A = u(x_0) - v(x_0)$ and define Ω_A to be the connected component of the open set

$$\{x \in M : A - 1 < u(x) - v(x) < A + 1\}$$

which contains x_0. By standard topological arguments, $\Omega_A \neq \emptyset$ is a (connected) open set. Let $\alpha : \mathbb{R} \to [0, +\infty)$ be the piece-wise linear function defined by

$$\alpha(t) = \begin{cases} 0 & t \leq A - 1 \\ (t - A + 1)/2 & A - 1 \leq t \leq A + 1 \\ 1 & t \geq A + 1. \end{cases}$$

Consider the vector field

$$X = \alpha \circ (u - v) \left\{ |\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right\},$$

and note that, for a suitable constant $C > 0$,

$$|X|^{\frac{p}{p-1}} \leq C \left(|\nabla u|^p + |\nabla v|^p \right) \in L^1(M).$$

From now on we abbreviate $\alpha(u - v) = \alpha \circ (u - v)$, $\alpha'(u - v) = \alpha' \circ (u - v)$, etc. Since $\alpha(u - v) \in W^{1,p}_{\mathrm{loc}}(M)$ then, by assumption, for all functions $0 \leq \varphi \in C^\infty_c(M)$ we have

$$\begin{aligned} 0 &\geq \int \left\langle \nabla(\varphi\alpha(u - v)), |\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right\rangle \\ &= \int \left\langle \nabla\varphi, \alpha(u - v) \left\{ |\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right\} \right\rangle \\ &+ \int \varphi\alpha'(u - v) \left\langle \nabla u - \nabla v, |\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right\rangle \\ &\geq -(\operatorname{div} X, \varphi) + 2C(p) \int \varphi\alpha'(u - v) \Psi(x, y) \end{aligned}$$

where in the last inequality we have used Corollary 3.17 and the fact that $\alpha' \geq 0$. Then

$$\operatorname{div} X \geq 2C(p)\alpha'(u - v) \Psi(x, y) \geq 0$$

in the sense of distributions and Proposition 1.2 yields

$$\alpha'(u - v) |\nabla u - \nabla v| = 0.$$

Since $\alpha'(u - v) \neq 0$ on Ω_A, we deduce

$$u - v \equiv A, \text{ on } \Omega_A.$$

It follows that the open set Ω_A is also closed. Since M is connected we must conclude that $\Omega_A = M$ and $u - v = A$ on M. $\square$

Remark 3.19. *In the above proof, inequality (3.29) is not used in its full strength. What we really need is that*

$$\left\langle |\nabla u|^{p-2}\,\nabla u - |\nabla v|^{p-2}\,\nabla v, \nabla u - \nabla v\right\rangle > 0$$

whenever $\nabla u \neq \nabla v$. According to this observation, the same proof works with minor changes for more general operators such as the $\mathcal{A}$-Laplacian of [HKM] or the φ-Laplacian of [RS]. In this latter case, $\varphi(t)$ is required to be increasing.

Proof (of Theorem 3.14). We suppose that either u or v is non-constant, for otherwise there's nothing to prove. Fix $q_0 \in M$ and set $C := u(q_0) - v(q_0) \in \mathbb{R}^n$. Up to replace v with $\tilde{v} := v + C$, we can suppose $C = 0$. Introduce the radial function $r : \mathbb{R}^n \to \mathbb{R}$ defined as $r(x) = |x|$. For $A > 1$, consider the weakly differentiable vector field X_A defined as

$$X_A(x) := \left[dh_A|_{(u-v)(x)} \circ \left(|du(x)|^{p-2}du(x) - |dv(x)|^{p-2}dv(x)\right)\right]^\sharp, \quad x \in M,$$

where $h_A \in C^\infty(\mathbb{R}^n, \mathbb{R})$ is the function

$$h_A(y) := \sqrt{A + r^2(y)}.$$

We observe that X_A is well defined since there exists a canonical identification

$$T_{(u-v)(q)}\mathbb{R}^n \cong T_{u(q)}\mathbb{R}^n \cong T_{v(q)}\mathbb{R}^n \cong \mathbb{R}^n.$$

Compute

$$dh_A = \frac{dr^2}{2\sqrt{A + r^2}}$$

and observe that, because of the special structure of $\mathbb{R}^n$, for each vector field Y on $\mathbb{R}^n$ it holds

$$(dr^2)|_{(u-v)(x)}(Y) = 2\left\langle (u-v)(x), Y\right\rangle_{\mathbb{R}^n}.$$

By definition of weak divergence, for each test function $0 \le \phi \in C_c^\infty(M)$, we have

$$\begin{aligned}
-\left(\operatorname{div} X_A, \phi\right) &= \int_M \left\langle X_A, \nabla\phi\right\rangle_M \\
&= \int_M \left\langle \left[dh_A|_{(u-v)(x)} \circ \left(|du(x)|^{p-2}du(x) - |dv(x)|^{p-2}dv(x)\right)\right]^\sharp, \nabla\phi\right\rangle_M \\
&= \int_M \left.\frac{dr^2}{2\sqrt{A + r^2}}\right|_{(u-v)(x)} \circ \left(|du(x)|^{p-2}du|_x - |dv(x)|^{p-2}dv|_x\right)(\nabla\phi) \\
&= \int_M \frac{\left\langle (u-v)(x), \left(|du(x)|^{p-2}du|_x - |dv(x)|^{p-2}dv|_x\right)(\nabla\phi(x))\right\rangle_{\mathbb{R}^n}}{\sqrt{A + r^2(u-v)(x)}}.
\end{aligned}$$

Since $u, v \in W^{1,p}_{loc}(M)$, assumption (3.28) implies that

$$
\begin{aligned}
(3.30)\qquad 0 &= \int_M \left\langle d\left(\frac{(u-v)\phi}{\sqrt{A+r^2(u-v)}}\right), |du|^{p-2}du - |dv|^{p-2}dv \right\rangle_{HS} \\
&= \int_M \frac{1}{\sqrt{A+r^2(u-v)}} \left\langle d\phi \otimes (u-v), |du|^{p-2}du - |dv|^{p-2}dv \right\rangle_{HS} \\
&+ \int_M \frac{\phi}{\sqrt{A+r^2(u-v)}} \left\langle du - dv, |du|^{p-2}du - |dv|^{p-2}dv \right\rangle_{HS} \\
&- \int_M \frac{\phi \left\langle dr^2|_{(u-v)} \circ (du-dv) \otimes (u-v), |du|^{p-2}du - |dv|^{p-2}dv \right\rangle_{HS}}{2\left(A+r^2(u-v)\right)^{3/2}} \\
&\geq -(\operatorname{div} X_A, \phi) \\
&+ \int_M \frac{2C(p)\phi}{\sqrt{A+r^2(u-v)}} \Psi(du, dv) \\
&- \int_M \frac{\phi r^2(u-v)}{(A+r^2(u-v))^{3/2}} (|du| + |dv|)(|du|^{p-1} + |dv|^{p-1}),
\end{aligned}
$$

where we have used Lemma 3.17 for the second term and Cauchy-Schwarz inequality for the third one. Setting

$$
f_A := \frac{2C(p)}{\sqrt{A+r^2(u-v)}} \Psi - \frac{2r^2(u-v)}{(A+r^2(u-v))^{3/2}} (|du|^p + |dv|^p),
$$

by Young's inequality, (3.30) gives

$$
\operatorname{div} X_A \geq f_A \tag{3.31}
$$

in the sense of distributions. Let us now compute the $L^{\frac{p}{p-1}}$-norm of X_A. Since

$$
\left| |du|^{p-2}du - |dv|^{p-2}dv \right|^{\frac{p}{p-1}} \leq \left(|du|^{p-1} + |dv|^{p-1}\right)^{\frac{p}{p-1}} \leq 2^{\frac{1}{p-1}} \left(|du|^p + |dv|^p\right),
$$

we have

$$
\begin{aligned}
|X_A|^{\frac{p}{p-1}} &= \left| \sqrt{\frac{r^2(u-v)}{A+r^2(u-v)}} \right|^{\frac{p}{p-1}} \left| |du|^{p-2}du - |dv|^{p-2}dv \right|^{\frac{p}{p-1}} \\
&\leq 2^{\frac{1}{p-1}} \left(|du|^p + |dv|^p\right) \in L^1(M).
\end{aligned}
$$

Hence X_A is a weakly differentiable vector field with $|X_A| \in L^{\frac{p}{p-1}}(M)$. To apply Proposition 1.2, it remains to show that $(f_A)_- \in L^1(M)$. To this purpose, we note that

$$
\begin{aligned}
(3.32)\qquad (f_A)_- &\leq \frac{2r^2(u-v)}{(A+r^2(u-v))^{3/2}} (|du|^p + |dv|^p) \\
&\leq \frac{r^2(u-v)}{A+r^2(u-v)} \frac{2}{\sqrt{A+r^2(u-v)}} (|du|^p + |dv|^p) \\
&\leq \frac{2}{\sqrt{A}} (|du|^p + |dv|^p) \in L^1(M).
\end{aligned}
$$

Then, the assumptions of Proposition 1.2 are satisfied and we get, for every $A > 1$,

$$
\begin{aligned}
(3.33) \qquad 0 &\geq \int_M f_A \\
&= \int_M \left[\frac{2C(p)}{\sqrt{A + r^2(u-v)}} \Psi - \frac{2r^2(u-v)}{(A + r^2(u-v))^{3/2}} (|du|^p + |dv|^p) \right].
\end{aligned}
$$

Fix $T > 0$ and define

$$
M_T := \{x \in M : r(u-v)(x) \leq T\} \quad \text{and} \quad M^T := M \setminus M_T.
$$

Then, we can write (3.33) as

(3.34)

$$
0 \geq \int_{M^T} f_A + \int_{M_T} \frac{2C(p)}{\sqrt{A + r^2(u-v)}} \Psi - \int_{M_T} \frac{2r^2(u-v)}{(A + r^2(u-v))^{3/2}} (|du|^p + |dv|^p).
$$

Note that

(3.35)

$$
\int_{M^T} f_A \geq - \int_{M^T} \frac{2}{\sqrt{A + T^2}} (|du|^p + |dv|^p) = - \frac{2}{\sqrt{A + T^2}} \int_{M^T} (|du|^p + |dv|^p).
$$

On the other hand, to deal with $\int_{M_T} f_A$, observe that

$$
(3.36) \qquad \int_{M_T} \frac{2C(p)}{\sqrt{A + r^2(u-v)}} \Psi \geq \frac{2C(p)}{\sqrt{A + T^2}} \int_{M_T} \Psi.
$$

Furthermore, the real function $t \mapsto \frac{2t}{(A+t)^{3/2}}$ has a global maximum at $t = 2A$ and is increasing in $(0, 2A)$. Hence, up to choosing $A > T^2/2$, we have also

(3.37)

$$
\int_{M_T} \frac{2r^2(u-v)}{(A + r^2(u-v))^{3/2}} (|du|^p + |dv|^p) \leq \frac{2T^2}{(A + T^2)^{3/2}} \int_{M_T} (|du|^p + |dv|^p).
$$

Inserting (3.35), (3.36) and (3.37) in (3.34), we get

$$
\begin{aligned}
\frac{2C(p)}{\sqrt{A + T^2}} \int_{M_T} \Psi \leq{}& \frac{2T^2}{(A + T^2)^{3/2}} \int_{M_T} (|du|^p + |dv|^p) \\
&+ \frac{2}{\sqrt{A + T^2}} \int_{M^T} (|du|^p + |dv|^p),
\end{aligned}
$$

which gives

$$
C(p) \int_{M_T} \Psi \leq \int_{M^T} (|du|^p + |dv|^p) + \frac{T^2}{\sqrt{A + T^2}} \int_{M_T} (|du|^p + |dv|^p),
$$

for all $A > \max\{1; T^2/2\}$. Letting $A \to +\infty$, this latter yields

$$C(p) \int_{M_T} \Psi \leq \int_{M^T} (|du|^p + |dv|^p). \tag{3.38}$$

Since $(|du|^p + |dv|^p) \in L^1(M)$, for $T \to \infty$ we can apply respectively dominated convergence on the RHS and monotone convergence on the LHS of (3.38), thus getting

$$C(p) \int_M \Psi = 0,$$

which in turn gives $|d(u-v)| \equiv 0$ on M, that is, $u - v \equiv u(q_0) - v(q_0) = C$ on M. □

3.3.3 Further comparison results without parabolicity

A basic use of Corollary 3.17 enables us to get also the next result in the spirit of [PRS1]. Note that the techniques developed in [PRS1] can be used to conclude further (e.g. L^∞) comparison results.

Theorem 3.20 (Theorem 2 in [HPV]). *Let $(M, \langle,\rangle)$ be a complete Riemannian manifold. Let $u, v \in C^\infty(M)$ be such that*

$$\Delta_p u \geq \Delta_p v \text{ on } M$$

for some $p \geq 2$. Suppose there exist $q \geq 1$ and $s > p$ such that

$$\int^{+\infty} f_{p,q,s}(r) dr = +\infty, \tag{3.39}$$

where we have set

$$f_{p,q,s}(r) = \left(\int_{\partial B_r(o)} |u - v|^{q + \frac{1}{s-1}} (|\nabla u| + |\nabla v|)^{p - \frac{s}{s-1}} \right)^{1-s}, \tag{3.40}$$

for some $o \in M$. Then either $u - v$ is constant or $u \leq v$ on M.

Remark 3.21. *Condition (3.39) could appear a little bit hard to decode due the presence of many parameters. Here we try to make it more transparent. First, note that the assumptions of Theorem 3.20 are trivially met if either $(u - v)$ or $(|\nabla u| + |\nabla v|)$ has compact support. More importantly, applying Hölder and reverse Hölder inequalities, we can see that condition* (3.39) *is implied by the stronger assumption*

$$\left(\int^R \left\| |u - v|^{q + \frac{1}{s-1}} \right\|_{t, \partial B_r}^{-\frac{s-1}{z}} dr \right)^z \left(\int^R \left\| (|\nabla u| + |\nabla v|)^{p - \frac{s}{s-1}} \right\|_{\frac{t}{t-1}, \partial B_r}^{\frac{s-1}{z-1}} dr \right)^{1-z} \nearrow \infty,$$

as $R \to \infty$, for some $t \in [1, +\infty]$ and $z \in (-\infty, 0) \cup (1, +\infty)$. Here $\|f\|_{t,\Omega}$ denotes the L^t norm of f on Ω. In particular we obtain that Theorem 3.20 holds if we replace (3.39) with either of the following set of assumptions:

i) $|\nabla u|, |\nabla v| \in L^\infty(M)$ *and* $\left[\int_{\partial B_r} |u-v|^{q+\frac{1}{s-1}}\right]^{1-s} \notin L^1(+\infty)$ *for some* $q \geq 1$ *and* $s > p$;

ii) $|u-v| \in L^\infty(M)$ *and* $\left[\int_{\partial B_r} (|\nabla u| + |\nabla v|)^{p-\frac{s}{s-1}}\right]^{1-s} \notin L^1(+\infty)$ *for some* $s > p$;

iii) $|\nabla u|, |\nabla v| \in L^{\left(p-\frac{s}{s-1}\right)t}(M)$, *for some* $s > p$ *and* $t > 1$, *and*

$$\left[\int_{\partial B_r} |u-v|^{\left(q+\frac{1}{s-1}\right)\frac{t}{t-1}}\right]^{\frac{(1-s)(t-1)}{s+t-1}} \notin L^1(+\infty)$$

for some $q \geq 1$;

iv) $|u-v| \in L^{\left(q+\frac{1}{s-1}\right)t}(M)$, *for some* $s > p$, $q \geq 1$ *and* $t > 1$, *and*

$$\left[\int_{\partial B_r} (|\nabla u| + |\nabla v|)^{\left(p-\frac{s}{s-1}\right)\frac{t}{t-1}}\right]^{\frac{(1-s)(t-1)}{s+t-1}} \notin L^1(+\infty).$$

To prove Theorem 3.20 we shall need the following lemma

Lemma 3.22. *Let* $p \geq 2$. *Then, for every* $x, y \in \mathbb{R}^n$, *it holds*

$$||x|^{p-2}x - |y|^{p-2}y| \leq (p-1)(|x|+|y|)^{p-2}|x-y|.$$

Proof. Set $k_p(x) := |x|^{p-2}x$. We start by computing

$$\begin{aligned}\left|\frac{d}{dt}k_p(tx+(1-t)y)\right| &\leq (p-1)|tx+(1-t)y|^{p-2}|x-y| \\ &\leq (p-1)(|x|+|y|)^{p-2}|x-y|,\end{aligned}$$

from which we obtain

$$\begin{aligned}|k_p(x)-k_p(y)| &= \left|\int_0^1 \frac{d}{dt}k_p(tx+(1-t)y)dt\right| \\ &\leq \int_0^1 \left|\frac{d}{dt}k_p(tx+(1-t)y)\right| dt \\ &\leq (p-1)(|x|+|y|)^{p-2}|x-y|.\end{aligned}$$

□

Proof (of Theorem 3.20). As above, for the ease of notation, we set

$$k_p(\xi) := |\xi|^{p-2}\xi, \ \xi \in TM.$$

Suppose that $u-v$ is not constant and, by contradiction, assume that there exists a point $x_0 \in M$ such that $u(x_0) > v(x_0)$. Fix a real number $0 < \epsilon < (u(x_0) - v(x_0))/2$ and define Ω_ϵ to be the connected component of the open set $\{x \in M : u(x) - v(x) > \epsilon\}$ which contains x_0. Note that, necessarily, $u-v$ is not constant on Ω_ϵ. Indeed, otherwise, by standard topological arguments we would have $\Omega_\epsilon = M$ and $u-v$ would be constant on all of M. We choose a smooth, non-decreasing function λ such that $\lambda(t) = 0$ for every $t < 2\epsilon$ and $0 < \lambda(t) \leq 1$ for every $t > 2\epsilon$ and we define the vector field

$$X := \lambda(u - v)(u - v)^q \left(k_p(\nabla u) - k_p(\nabla v)\right).$$

We write B_R for $B_R(o)$ and $\partial/\partial r$ for the radial vector field centered at o. Applying the divergence theorem, Lemma 3.22 and Hölder inequality, we get

$$\begin{aligned}
&\int_{B_R\cap\Omega_\epsilon} \operatorname{div} X \\
&= \int_{\partial B_R\cap\Omega_\epsilon} \left\langle X, \frac{\partial}{\partial r}\right\rangle \\
&\leq \int_{\partial B_R\cap\Omega_\epsilon} |k_p(\nabla u) - k_p(\nabla v)|\lambda(u-v)(u-v)^q \\
&\leq (p-1)\int_{\partial B_R\cap\Omega_\epsilon} \lambda(u-v)(|\nabla u|+|\nabla v|)^{p-2}|\nabla u - \nabla v|(u-v)^q \\
&\leq (p-1)\left(\int_{\partial B_R\cap\Omega_\epsilon} F(u,v)\right)^{\frac{1}{s}} \\
&\times \left(\int_{\partial B_R\cap\Omega_\epsilon} \lambda(u-v)(|\nabla u|+|\nabla v|)^{\frac{(p-2)s}{s-1}}|\nabla u-\nabla v|^{\left(1-\frac{p}{s}\right)\frac{s}{s-1}}(u-v)^{\frac{sq-q+1}{s-1}}\right)^{\frac{s-1}{s}} \\
&\leq (p-1)\left(\int_{\partial B_R\cap\Omega_\epsilon} F(u,v)\right)^{\frac{1}{s}} \left(\int_{\partial B_R} |u-v|^{q+\frac{1}{s-1}}(|\nabla u|+|\nabla v|)^{p-\frac{s}{s-1}}\right)^{\frac{s-1}{s}},
\end{aligned}$$

where

$$F(u,v) = \lambda(u-v)|\nabla u - \nabla v|^p(u-v)^{q-1}$$

and, we recall, $s > p$. On the other hand, computing the divergence of X we obtain

$$\begin{aligned}
\int_{B_R\cap\Omega_\epsilon} \operatorname{div} X &= \int_{B_R\cap\Omega_\epsilon} \lambda'(u-v)(u-v)^q \left\langle k_p(\nabla u) - k_p(\nabla v), \nabla u - \nabla v\right\rangle \\
&+ q\int_{B_R\cap\Omega_\epsilon} (u-v)^{q-1}\lambda(u-v)\left\langle k_p(\nabla u) - k_p(\nabla v), \nabla u - \nabla v\right\rangle \\
&+ \int_{B_R\cap\Omega_\epsilon} (\Delta_p u - \Delta_p v)\,\lambda(u-v)(u-v)^q \\
&\geq 2qC(p)\int_{B_R\cap\Omega_\epsilon} F(u,v),
\end{aligned}$$

where, in the last inequality, we have used Corollary 3.17. It follows that

$$H(R)^s \leq C'\xi(R)H'(R), \tag{3.41}$$

where we have defined

$$\begin{aligned}
H(R) &:= \int_{B_R\cap\Omega_\epsilon} F(u,v) \geq 0; \\
\xi(R) &:= \left(\int_{\partial B_R} |u-v|^{q+\frac{1}{s-1}}(|\nabla u|+|\nabla v|)^{p-\frac{s}{s-1}}\right)^{s-1} \\
C' &:= (p-1)^s\left[2qC(p)\right]^{-s}.
\end{aligned}$$

Choose $r_1 \gg 1$ such that $F(u,v)$ does not vanish identically on $B_{r_1} \cap \Omega_\epsilon$. According to (3.41) we have $\xi(R), H(R) > 0$, for every $R \geq r_1$. Therefore, we can integrate (3.41) on $[r_1, r_2]$ to obtain

$$\left(\frac{C'}{s-1}\right)\frac{1}{H(r_1)^{s-1}} \geq \left(\frac{C'}{s-1}\right)\left(-H(r_2)^{1-s} + H(r_1)^{1-s}\right) \geq \int_{r_1}^{r_2} \frac{dt}{\xi(t)}. \tag{3.42}$$

Letting $r_2 \to \infty$, the RHS of (3.42) goes to infinity by assumption, and this force $H(r_1) = 0$ for all r_1. Hence

$$\nabla(u-v) \equiv 0 \text{ on } \Omega_\epsilon$$

proving that $u - v$ is constant on Ω_ϵ. Contradiction. □

3.4 Proof of the main result

At the beginning of Section 3.3 we have presented the possible strategy proposed in [PRS3] to overcome the bad behaviour of the composition of p-harmonic maps and convex functions. Namely, to prove Theorem 3.13 they introduced the special composed vector field X shown in (3.27) which permits, through the application of Kelvin-Nevanlinna-Royden criterion, to deduce a vanishing result for $\operatorname{Hess} r^2 \geq 0$, without information on the p-subharmonicity of the distance function. Afterward, in Theorem 3.14 we showed how to adapt this technique to the case of two different vector-valued maps with the same p-tension field. Since $\mathbb{R}^n$ is contractible, this can be seen as a special case in the comprehension of Theorem 3.3. However, the proof of Theorem 3.14 is based on the good special structure of $\mathbb{R}^n$, which permits to compare in a standard way (i.e. considering their difference) tangent vectors with different base points. Hence, though the procedure is non trivial due to the non linearity of τ_p, the problem is somehow reduced to that of a single map. Nevertheless, it is worth noting that in this case the Kelvin-Nevanlinna-Royden criterion has to be used in his full power in the sense that the the vector field X introduced has a non-trivial negative part. This leads to employ a new limit procedure which turns out to be useful in the future investigation.
In order to prove Theorem 3.3, we try to combine the strategies adopted for the theorems presented so far in Chapter 3. Since we are dealing with two different maps which are non-trivially comparable, as in step **a** of the proof of Theorem 3.4 we introduce the map $j : M \to N \times N$. We recall that $j = (u,v)$ when N is simply connected. In this contest (u,v) is not p-harmonic for $p \neq 2$, but considering $J = (|du|^{p-2}du, |dv|^{p-2}dv) \in T^*M \otimes j^{-1}T_{(u,v)}N \times N$ instead of dj, it turns out that $\operatorname{div} J = 0$ due to the p-harmonicity of u and v. Hence we can proceed in a way similar to that we followed for Theorem 3.14. A further difficulty arises since, unlike step **b** in the harmonic case, we have to compute the hessian of the distance function in N evaluated along two different vector field, i.e. ${}^{N\times N}\operatorname{Hess} \operatorname{dist}_N((du,dv),(|du|^{p-2}du, |dv|^{p-2}dv))$. This will be done in Theorem 3.24.

Proof (of Theorem 3.3). Let u and v be two $C^{1,\alpha}$ p-harmonic maps from M to N which are freely homotopic, and such that $|du|, |dv| \in L^p(M)$. Proceeding exactly as in step **a** of the proof of Theorem 3.4, we define the manifold $\tilde{N}_{\times/}$ and the function $\tilde{u}, \tilde{v}, \tilde{j}, \tilde{r}$ and j. Furthermore, we can construct a vector valued 1-form $J \in T^*M \otimes j^{-1}T\tilde{N}_{\times/}$ along j by projecting via (3.1) the vector valued 1-form $\tilde{J} \in T^*\tilde{M} \otimes \tilde{j}^{-1}T\left(\tilde{N} \times \tilde{N}\right)$ along $\tilde{j}$ defined as

$$\tilde{J} := (\mathcal{K}_p(\tilde{u}), \mathcal{K}_p(\tilde{v})),$$

where, we recall, the symbol $\mathcal{K}_p(\tilde{u})$ stands for

$$\mathcal{K}_p(\tilde{u}) := |d\tilde{u}|^{p-2} d\tilde{u}.$$

Set $\hat{h}_A : [0, +\infty) \to \mathbb{R}$ as $\hat{h}_A(t) := \sqrt{A + t^2}$ for every $A > 1$ and define $h_A := \hat{h}_A(\tilde{r}) \in C^\infty(\tilde{N}_{\times/}, \mathbb{R})$. Consider the vector field on M given by

$$X_A|_q := \left[dh_A|_{j(q)} \circ J|_q\right]^\sharp . \tag{3.43}$$

Note that

$$X_A|_q := dP_M|_{\tilde{q}} \circ \left.\tilde{X}_A\right|_{\tilde{q}}, \tag{3.44}$$

where

$$\left.\tilde{X}_A\right|_{\tilde{q}} := \left[\left.d\tilde{h}_A\right|_{\tilde{j}(\tilde{q})} \circ \left.\tilde{J}\right|_{\tilde{q}}\right]^\sharp, \qquad \tilde{h}_A := \hat{h}_A \circ \left(\mathrm{dist}^2_{\tilde{N}}\right) : \tilde{N} \times \tilde{N} \to \mathbb{R}.$$

We claim that (3.44) is well defined. To this end, let $S_{\tilde{q}} \in T_{\tilde{q}}\tilde{M}$ be an arbitrary vector and let $\tilde{q}' \in P_M^{-1}(q) \subset T\tilde{M}$. If $\tilde{q}' \neq \tilde{q}$, there exists $\gamma \in \pi_1(M, *)$ such that $q' = \gamma q$. Then,

$$\tilde{J}|_{\gamma\tilde{q}}(d\gamma(S_{\tilde{q}})) = \left(d\left[u_\sharp(\gamma)\right](\mathcal{K}_p(\tilde{u})(S_{\tilde{q}})), d\left[v_\sharp(\gamma)\right](\mathcal{K}_p(\tilde{v})(S_{\tilde{q}}))\right).$$

Since u is homotopic to v, $u_\sharp = v_\sharp$. Moreover $\mathrm{dist}_{\tilde{N}}$ is equivariant with respect to the action of $\pi_1(N)$ on $\tilde{N} \times \tilde{N}$, i.e.

$$\mathrm{dist}_{\tilde{N}}(\beta\tilde{x}_1, \beta\tilde{x}_2) = \mathrm{dist}_{\tilde{N}}(\tilde{x}_1, \tilde{x}_2), \qquad \forall \beta \in \pi_1(N),\ x_1, x_2 \in \tilde{N}.$$

Then,

$$dP_M|_{\tilde{q}} \circ \left[d\left(\mathrm{dist}^2_{\tilde{N}}\right)|_{\tilde{j}(\tilde{q})} \circ \left.\tilde{J}\right|_{\tilde{q}}\right]^\sharp$$

is well defined, and consequently the same holds for $dP_M|_{\tilde{q}} \circ \tilde{X}_A|_{\tilde{q}}$.
Now, we want to compute (in the weak sense) $\operatorname{div} X_A$ on M. We start with the following result, obtained with minor changes from a lemma of Kawai, [Kaw].

Lemma 3.23. *Consider $C^{1,\alpha}$ p-harmonic maps $u, v : M \to N$ and a smooth function $h : N \times N \to \mathbb{R}$. Then the identity*

$$\begin{aligned} &{}^M\operatorname{tr}^{N\times N} \operatorname{Hess} h|_{(u,v)}\left((du, dv), (\mathcal{K}_p(u), \mathcal{K}_p(v))\right) \\ &\qquad = {}^M\operatorname{div}\left[dh|_{(u,v)} \circ (\mathcal{K}_p(u), \mathcal{K}_p(v))\right]^\sharp, \end{aligned} \tag{3.45}$$

holds weakly on M.

Proof. Consider $(u,v) : M \to N \times N$. Let $\eta \in C^{\infty}(M,\mathbb{R})$ be a compactly supported function and define a vector valued 1-form $\psi \in T^*M \otimes (u,v)^{-1}T(N \times N)$ along (u,v) as $\psi := D(\eta \nabla h|_{(u,v)})$, that is

$$\psi(V) = (d\eta(V))\ {}^{N\times N}\nabla h|_{(u,v)} + \eta\ {}^{N\times N}\nabla_{d(u,v)(V)}{}^{N\times N}\nabla h|_{(u,v)}$$

for all vector fields V on M. Since u, v are p-harmonic, by the structure of Riemannian products we have that $\operatorname{div}(\mathcal{K}_p(u), \mathcal{K}_p(v)) = 0$ weakly on M, that is

$$\int_M \langle \xi, (\mathcal{K}_p(u), \mathcal{K}_p(v)) \rangle_{HS_\times} = 0, \quad \forall \xi \in T^*M \otimes (u,v)^{-1}T(N \times N),$$

where $\langle , \rangle_{HS_\times}$ is the Hilbert-Schmidt scalar product on $T^*M \otimes (u,v)^{-1}T(N \times N)$. Hence, choosing $\xi = \psi$ in this latter, we obtain

$$\begin{aligned}
0 &= \int_M \langle \psi, (\mathcal{K}_p(u), \mathcal{K}_p(v)) \rangle_{HS_\times} \\
&= \int_M \left\langle d\eta(\cdot) \otimes {}^{N\times N}\nabla h|_{(u,v)}, (\mathcal{K}_p(u), \mathcal{K}_p(v)) \right\rangle_{HS_\times} \\
&+ \int_M \left\langle \eta\ {}^{N\times N}\nabla_{d(u,v)(\cdot)}{}^{N\times N}\nabla h|_{(u,v)}, (\mathcal{K}_p(u), \mathcal{K}_p(v)) \right\rangle_{HS_\times} \\
&= \int_M \left[dh|_{(u,v)} \circ (\mathcal{K}_p(u), \mathcal{K}_p(v)) \right] ({}^M\nabla\eta) \\
&+ \int_M \eta\ {}^M\operatorname{tr} \left\langle {}^{N\times N}\nabla_{(du,dv)}{}^{N\times N}\nabla h|_{(u,v)}, (\mathcal{K}_p(u), \mathcal{K}_p(v)) \right\rangle_{N\times N},
\end{aligned}$$

which turns to be the weak formulation of (3.45). □

According to Lemma 3.23, because of (3.43) and since $\pi_1(M,*)$ acts on $\tilde{M}$ as a group of isometries, we have that for all $q \in M$ and for any choice of $\tilde{q} \in P_M^{-1}(q)$

(3.46)
$$\begin{aligned}
{}^M\operatorname{div} X_A|_q = {}^{\tilde{M}}\operatorname{div} \tilde{X}_A|_{\tilde{q}} &= {}^{\tilde{M}}\operatorname{div}\left[d\tilde{h}_A|_{\tilde{j}(\tilde{q})} \circ \tilde{J} \right]^\sharp \\
&= {}^{\tilde{M}}\operatorname{tr}\ {}^{\tilde{N}\times\tilde{N}}\operatorname{Hess} \tilde{h}_A|_{\tilde{j}(\tilde{q})}\left(d\tilde{j}, \tilde{J} \right) = {}^M\operatorname{tr}\ {}^{\tilde{N}_{\times}/}\operatorname{Hess} h_A|_{j(q)}(dj, J)
\end{aligned}$$

holds weakly on M. Observe that

$$dh_A = \frac{d\frac{\tilde{r}^2}{2}}{\sqrt{A+\tilde{r}^2}} = \frac{\tilde{r}d\tilde{r}}{\sqrt{A+\tilde{r}^2}} \tag{3.47}$$

and

$${}^{\tilde{N}_{\times}/}\operatorname{Hess} h_A = \frac{{}^{\tilde{N}_{\times}/}\operatorname{Hess}\tilde{r}^2}{2\sqrt{A+\tilde{r}^2}} - \frac{\tilde{r}^2}{(A+\tilde{r}^2)^{3/2}} d\tilde{r} \otimes d\tilde{r}.$$

Then, in order to deal with $\operatorname{div} X_A$, we want to compute

$${}^{\tilde{N}_{\times}/}\operatorname{Hess}\tilde{r}^2|_{j(q)}(dj, J) = {}^{\tilde{N}\times\tilde{N}}\operatorname{Hess}\operatorname{dist}^2_{\tilde{N}}|_{\tilde{j}(\tilde{q})}\left(d\tilde{j}, \tilde{J} \right), \qquad \tilde{q} \in P_M^{-1}(q). \tag{3.48}$$

Theorem 3.24. *Suppose N is a simply connected Riemannian manifolds such that ${}^N\operatorname{Sect} \leq 0$ and fix points u, v in N. Let ${}^N r : N \times N \to \mathbb{R}$ be defined by ${}^N r(u,v) := {}^N \operatorname{dist}(u,v)$ and let $X = X_1 + X_2 \in T_{(u,v)}N \times N$, with $X_1 \in T_u N$ and $X_2 \in T_v N$. Then , for every $p \geq 2$,*

$$ {}^{N\times N}\operatorname{Hess}{}^N r^2\big|_{(u,v)} (X, (|X_1|^{p-2}X_1, |X_2|^{p-2}X_2)) \geq 0 $$

and the equality holds if and only if there is a parallel vector field Z, defined along the unique geodesic γ_{uv} joining u and v, such that $Z(u) = X_1$, $Z(v) = X_2$ and $\left\langle {}^N R(Z,T)T, Z\right\rangle_N \equiv 0$ along γ_{uv}. Moreover, $d({}^N r)(X) = 0$.
In particular, if ${}^N\operatorname{Sect} < 0$, Z is proportional to T.

We begin with the following lemma

Lemma 3.25. *Consider a Riemannian manifold Q and a function $f \in C^2(Q,\mathbb{R})$. Let $q \in Q$ and $X, Y \in T_q Q$. Let $\eta_X : [-\epsilon, \epsilon] \to Q$ be the constant speed geodesic s.t. $\eta_X(0) = q$ and $\dot\eta_X(0) = X$. Moreover, define $Y_s \in T_{\eta_X(s)}Q$ as the vectors obtained by parallel translating $Y = Y_0$ along η_X, and let $\eta_Y^{(s)} : [-\delta, \delta] \to Q$ be the constant speed geodesic s.t. $\eta_Y^{(s)}(0) = \eta_X(s)$ and $\dot\eta_Y^{(s)}(0) = Y_s$. Then*

$$ {}^Q\operatorname{Hess} f|_q(X,Y) = \left.\frac{\partial^2}{\partial s \partial t}\right|_{s=t=0} f(\eta_Y^{(s)}(t)). $$

Proof. We have

$$ \begin{aligned} \left.\frac{\partial^2}{\partial s\partial t}\right|_{s=t=0} f(\eta_Y^{(s)}(t)) &= \left.\frac{\partial}{\partial s}\right|_{s=0} \left\langle \nabla f, \dot\eta_Y^{(s)}\right\rangle\Big|_{t=0} = \left.\frac{\partial}{\partial s}\right|_{s=0} \left\langle \nabla f|_{\eta_X(s)}, Y_s\right\rangle \\ &= \left\langle \nabla_{\dot\eta_X(s)}\nabla f, Y_s\right\rangle\big|_{s=0} + \left\langle \nabla f, \nabla_{\dot\eta_X(s)} Y_s\right\rangle\big|_{s=0} \\ &= \langle \nabla_X \nabla f, Y\rangle = {}^Q\operatorname{Hess} f|_q(X,Y), \end{aligned} $$

since $\nabla_{\dot\eta_X(s)} Y_s \equiv 0$ by construction. □

Proof (of Theorem 3.24). As above, for the ease of notation, for each vector field ξ we set $k_p(\xi) := |\xi|^{p-2}\xi$. Define the vector field $Y \in T_{(u,v)}N \times N$ as $Y = (Y_1, Y_2) = (k_p(X_1), k_p(X_2))$. Let D be the diagonal set

$$ D := \{(u_1, u_1) : u_1 \in N\} \subset N \times N $$

so that ${}^N r$ is smooth on $N \times N \setminus D$, ${}^N r^2$ is smooth on $N \times N$ and for every pair $(u,v) \in N \times N \setminus D$ there is a unique shortest geodesic from u to v. We call $\gamma_{u,v}$ such a geodesic parametrized by arc length.
Let $\sigma^X : [-\epsilon, \epsilon] \to N \times N$, be the constant speed geodesic on $N \times N$ satisfying $\sigma^X(0) = (u,v)$ and $\dot\sigma^X(0) = X = (X_1, X_2)$. We then have $\sigma^X = (\sigma_1^X, \sigma_2^X)$ where σ_1^X and σ_2^X are geodesic on N satisfying $\sigma_1^X(0) = u$, $\sigma_2^X(0) = v$ and $\dot\sigma_i^X(0) = X_i$, $i = 1, 2$. As in Lemma 3.25, let Y_s be the vector field along σ^X obtained by parallel transport of Y and let $\sigma^{(s),Y} : [-\delta, \delta] \to N \times N$ be the constant speed geodesic s.t. $\sigma^{(s),Y}(0) = \sigma^X(s)$ and $\dot\sigma^{(s),Y}(0) = Y_s$. As above we can split $\sigma^{(s),Y}$ in two geodesic of N, i.e. $\sigma^{(s),Y} = (\sigma_1^{(s),Y}, \sigma_2^{(s),Y})$.
Set $\bar R := {}^N r(u,v)$ and, for each couple of points $y_1, y_2 \in N$ let $\gamma_{y_1,y_2} : [0, \bar R] \to N$ be the (unique) constant speed geodesic joining y_1 and y_2.

At this point, we can consider a two parameters geodesic variation of $\gamma_{u,v}$ defining $\alpha : [0, \bar{R}] \times [-\epsilon, \epsilon] \times [-\delta, \delta] \to N$ as

$$\alpha(t, z, w) := \gamma_{\sigma_1^{(z),Y}(w), \sigma_2^{(z),Y}(w)}(t).$$

We now define the variational vector fields

$$\hat{Z}(t, z, w) := \frac{\partial}{\partial z}\alpha(t, z, w), \quad \hat{W}(t, z, w) := \frac{\partial}{\partial w}\alpha(t, z, w),$$
$$Z(t) := \hat{Z}(t, 0, 0), \quad W(t) := \hat{W}(t, 0, 0), \quad T(t) := \frac{\partial}{\partial t}\alpha(t, 0, 0) = \dot{\gamma}_{u,v}(t).$$

Here, we are using the notation

$$\frac{\partial}{\partial z}\alpha(t, z, w) := d\alpha|_{(t,z,w)}\left(\frac{\partial}{\partial z}\right),$$

with $\frac{\partial}{\partial z} = \left(0, \frac{\partial}{\partial z}, 0\right) \in T\left([0, \bar{R}] \times [-\epsilon, \epsilon] \times [-\delta, \delta]\right)$. Since both the one parameter variations $\alpha(t, z, 0)$ and $\alpha(t, 0, w)$ are geodesic variations, we have that Z and W are the corresponding Jacobi fields along $\gamma_{u,v}$. Then they satisfy

$$\begin{array}{ll} Z(0) = X_1, & W(0) = Y_1 = k_p(X_1) = k_p(Z(0)), \\ Z(\bar{R}) = X_2, & W(\bar{R}) = Y_2 = k_p(X_2) = k_p\left(Z(\bar{R})\right), \end{array}$$

and the Jacobi equations

$$\nabla_T\nabla_T Z + {}^N R(Z, T)T = 0 = \nabla_T\nabla_T W + {}^N R(W, T)T.$$

For each $z \in [-\epsilon, \epsilon]$ and $w \in [-\delta, \delta]$, let

$$L_\alpha(z, w) := \int_0^{\bar{R}} \left|\frac{\partial}{\partial t}\alpha(t, z, w)\right| dt$$

be the length of the geodesic curve $t \mapsto \alpha(t, z, w)$. By Lemma 3.25 we have

$$\begin{aligned} {}^{N\times N}\operatorname{Hess} {}^N r|_{(u,v)}(X, Y) &= \left.\frac{\partial^2}{\partial z\partial w}\right|_{z=w=0} {}^N r(\sigma^{(z),Y}(w)) \\ &= \left.\frac{\partial^2}{\partial z\partial w}\right|_{z=w=0} {}^N r(\sigma_1^{(z),Y}(w), \sigma_2^{(z),Y}(w)) \\ &= \left.\frac{\partial^2}{\partial z\partial w}\right|_{z=w=0} L_\alpha(z, w). \end{aligned}$$

On the other hand, by the second variation of arc length (see [CE], page 20) we have

(3.49)
$$\begin{aligned} \left.\frac{\partial^2}{\partial z\partial w}\right|_{z=w=0} L_\alpha(z, w) = &\left.\left\langle \nabla_{\hat{Z}}\hat{W}(t, 0, 0), T(t)\right\rangle_N\right|_{t=0}^{t=\bar{R}} + \int_0^{\bar{R}} \langle \nabla_T Z, \nabla_T W\rangle_N \\ &- \int_0^{\bar{R}} \left\langle {}^N R(W, T)T, Z\right\rangle_N - \int_0^{\bar{R}} T\langle Z, T\rangle_N\, T\langle W, T\rangle_N. \end{aligned}$$

We note that the vector fields $\hat{Z}$ and $\hat{W}$ are defined along the map α. Accordingly, the covariant derivative at the first term on RHS of (3.49) has the meaning

$$\nabla_{\hat{Z}}\hat{W}(t,0,0) = \nabla_{\hat{Z}}\big|_{\alpha(t,0,0)} \left(\frac{\partial}{\partial w}\alpha(t,z,0)\right).$$

First, observe that, by construction of α and due to the choice of the geodesics $\sigma_1^{(z),Y}(w)$ and $\sigma_2^{(z),Y}(w)$, we have

$$\nabla_{\hat{Z}}\hat{W}(0,0,0) = \nabla_{\hat{Z}}\hat{W}(\bar{R},0,0) = 0,$$

which implies that the first term on RHS of (3.49) vanishes. Moreover, using the Jacobi equation for Z and the values of W at $t=0$ and $t=\bar{R}$ we can compute

(3.50)

$$\begin{aligned}
&\int_0^{\bar{R}} \left\{\langle \nabla_T Z, \nabla_T W\rangle_N - \left\langle {}^N R(W,T)T, Z\right\rangle_N\right\} \\
&= \int_0^{\bar{R}} \left\{\langle \nabla_T Z, \nabla_T W\rangle_N + \langle \nabla_T\nabla_T Z, W\rangle_N\right\} \\
&= \int_0^{\bar{R}} T\,\langle \nabla_T Z, W\rangle_N \\
&= \langle \nabla_T Z, W\rangle_N\big|_{t=0}^{t=\bar{R}} \\
&= \langle \nabla_T Z, k_p(Z)\rangle_N\big|_{t=0}^{t=\bar{R}} \\
&= \int_0^{\bar{R}} T\,\langle \nabla_T Z, k_p(Z)\rangle_N \\
&= \int_0^{\bar{R}} \left\{\langle \nabla_T\nabla_T Z, k_p(Z)\rangle_N + T\left(|Z|^{p-2}\right)\langle \nabla_T Z, Z\rangle_N + |Z|^{p-2}|\nabla_T Z|^2\right\} \\
&= \int_0^{\bar{R}} \left\{-|Z|^{p-2}\left\langle {}^N R(Z,T)T, Z\right\rangle_N + \frac{1}{2}T\left(|Z|^2\right)T\left(|Z|^{p-2}\right) + |Z|^{p-2}|\nabla_T Z|^2\right\}.
\end{aligned}$$

Since T is parallel, the Jacobi equation implies

$$\begin{aligned}
TT\,\langle Z,T\rangle_N &= T\,\langle \nabla_T Z, T\rangle_N \\
&= \langle \nabla_T\nabla_T Z, T\rangle_N \\
&= \left\langle {}^N R(T,Z)T, T\right\rangle_N = 0.
\end{aligned} \tag{3.51}$$

Then

$$\begin{aligned}
(3.52) \qquad & \int_0^{\bar{R}} T \langle Z,T\rangle_N T \langle W,T\rangle_N \\
&= \int_0^{\bar{R}} \{T(\langle W,T\rangle_N T\langle Z,T\rangle_N) - \langle W,T\rangle_N TT\langle Z,T\rangle_N\} \\
&= (\langle W,T\rangle_N T\langle Z,T\rangle_N)|_{t=0}^{t=\bar{R}} \\
&= \left(\langle k_p(Z),T\rangle_N T\langle Z,T\rangle_N\right)\big|_{t=0}^{t=\bar{R}} \\
&= \int_0^{\bar{R}} T\left(\langle k_p(Z),T\rangle_N T\langle Z,T\rangle_N\right) \\
&= \int_0^{\bar{R}} \left\{T\left(|Z|^{p-2}\right)\langle Z,T\rangle_N T\langle Z,T\rangle_N + |Z|^{p-2}\left(T\langle Z,T\rangle_N\right)^2\right\}.
\end{aligned}$$

Inserting (3.50) and (3.52) in (3.49) we get

$$\begin{aligned}
(3.53) \qquad & {}^{N\times N}\operatorname{Hess}{}^N r|_{(u,v)}(X,Y) \\
&= \int_0^{\bar{R}} |Z|^{p-2}\left\{-\left\langle {}^N R(Z,T)T,Z\right\rangle_N + |\nabla_T Z|^2 - \left(T\langle Z,T\rangle_N\right)^2\right\} \\
&+ \int_0^{\bar{R}} \frac{1}{2} T\left(|Z|^{p-2}\right) T\left(|Z|^2\right) - \int_0^{\bar{R}} T\left(|Z|^{p-2}\right)\langle Z,T\rangle_N T\langle Z,T\rangle_N .
\end{aligned}$$

We consider the three integrals separately. First, since $|T\langle Z,T\rangle_N| = \left|\nabla_T Z^T\right|$, we have that the first integral at RHS of (3.53) is equal to

$$(3.54) \qquad \int_0^{\bar{R}} |Z|^{p-2}\left\{|\nabla_T Z^{\perp}|^2 - \left\langle {}^N R(Z,T)T,Z\right\rangle_N\right\},$$

where Z^T and $Z^{\perp}$ denote the components of Z respectively parallel and normal to T, and the integral is positive by the curvature assumptions on N.
As for the second integral, assume $2 < p < 4$, the other cases being easier. We have

$$(3.55) \qquad T\left(|Z|^{p-2}\right) T\left(|Z|^2\right) = \frac{2}{p-2}|Z|^{4-p}\left[T\left(|Z|^{p-2}\right)\right]^2 \geq 0.$$

Finally, recall (3.51) and note that this implies that $T\langle Z,T\rangle_N$ is constant along $\gamma_{u,v}$ and takes value

$$(3.56) \qquad T\langle Z,T\rangle_N \equiv \frac{1}{\bar{R}}\left(\langle Z,T\rangle_N|_{t=\bar{R}} - \langle Z,T\rangle_N|_{t=0}\right).$$

On the other hand, we have that

$$\begin{aligned}
(3.57) \qquad d^N r\big|_{(u,v)}(X) &= d^N r\big|_{(u,v)}((X_1,X_2)) \\
&= dr_v|_u(X_1) + dr_u|_v(X_2) \\
&= -\langle X_1, \dot{\gamma}_{u,v}(0)\rangle_N + \left\langle X_2, \dot{\gamma}_{u,v}(\bar{R})\right\rangle_N,
\end{aligned}$$

where $r_u, r_v : N \to \mathbb{R}$ are defined as $r_u(\cdot) := {}^N r(u,\cdot)$ and $r_v(\cdot) := {}^N r(\cdot, v)$. Combining (3.56) and (3.57) we get

$$T \langle Z, T\rangle_N \equiv \frac{d^N r(X)}{\bar{R}}, \tag{3.58}$$

which in turn implies

$$\begin{aligned}
&\int_0^{\bar{R}} T\left(|Z|^{p-2}\right) \langle Z, T\rangle_N T\langle Z, T\rangle_N \\
&= \int_0^{\bar{R}} T\left(|Z|^{p-2} \langle Z, T\rangle_N T\langle Z, T\rangle_N\right) - \int_0^{\bar{R}} |Z|^{p-2} \left(T\langle Z, T\rangle_N\right)^2 \\
&= \frac{d^N r(X)}{\bar{R}} \left[|Z|^{p-2} \langle Z, T\rangle_N\right]_{t=0}^{t=\bar{R}} - \left(\frac{d^N r(X)}{\bar{R}}\right)^2 \int_0^{\bar{R}} |Z|^{p-2}.
\end{aligned} \tag{3.59}$$

Moreover, reasoning as for (3.58), we compute

$$\begin{aligned}
&\frac{d^N r(X)}{\bar{R}} \left[|Z|^{p-2} \langle Z, T\rangle_N\right]_{t=0}^{t=\bar{R}} \\
&= \frac{d^N r(X)}{\bar{R}} \left[\left\langle k_p(X_2), \dot{\gamma}_{u,v}(\bar{R})\right\rangle_N - \left\langle k_p(X_1), \dot{\gamma}_{u,v}(0)\right\rangle_N\right] \\
&= \frac{d^N r(X) d^N r(Y)}{\bar{R}}.
\end{aligned} \tag{3.60}$$

Combining (3.53), (3.54), (3.59) and (3.60), we obtain

$$\begin{aligned}
&{}^{N\times N}\operatorname{Hess}{}^N r|_{(u,v)}(X,Y) \\
&= \int_0^{\bar{R}} |Z|^{p-2} \left\{|\nabla_T Z^\perp|^2 - \left\langle {}^N R(Z,T)T, Z\right\rangle_N\right\} + \frac{1}{2}\int_0^{\bar{R}} T\left(|Z|^{p-2}\right) T\left(|Z|^2\right) \\
&- \frac{d^N r(X) d^N r(Y)}{\bar{R}} + \left(\frac{d^N r(X)}{\bar{R}}\right)^2 \int_0^{\bar{R}} |Z|^{p-2}.
\end{aligned}$$

Finally, since

$$\operatorname{Hess} r^2 = 2r \operatorname{Hess} r + 2dr \otimes dr,$$

recalling also (3.55), we get

$$\begin{aligned}
&{}^{N\times N}\operatorname{Hess}{}^N r^2|_{(u,v)}(X,Y) \\
&= 2\bar{R} \int_0^{\bar{R}} |Z|^{p-2} \left\{|\nabla_T Z^\perp|^2 - \left\langle {}^N R(Z,T)T, Z\right\rangle_N\right\} \\
&+ \bar{R}\int_0^{\bar{R}} T\left(|Z|^{p-2}\right) T\left(|Z|^2\right) + 2\frac{\left(d^N r(X)\right)^2}{\bar{R}} \int_0^{\bar{R}} |Z|^{p-2} \geq 0.
\end{aligned} \tag{3.61}$$

This conclude the first part of the proof. Now, assume

$${}^{N\times N}\operatorname{Hess}{}^N r^2|_{(u,v)}(X,Y) = 0.$$

From (3.61) we get that $d^N r(X) = 0$, $\left\langle {}^N R(Z,T)T, Z\right\rangle_N \equiv 0$ along $\gamma_{u,v}$ and, using also (3.51),

$$|\nabla_T Z|^2 = |\nabla_T Z^\perp|^2 + |\nabla_T Z^T|^2 \equiv 0,$$

that is Z is parallel along $\gamma_{u,v}$. □

Recalling (3.48) and applying Theorem 3.24 with $N = \tilde{N}$ and $X = d\tilde{j}$ we get

Corollary 3.26. *With the definitions introduced above, for all $q \in M$, $E_q \in T_qM$ and for any choice of $\tilde{q} \in P_M^{-1}(q)$ and $\tilde{E}_q = [d(P_M)|_{\tilde{q}}]^{-1}(E_q)$ we have*

$$^{\tilde{N}\times/}\operatorname{Hess}\tilde{r}^2\Big|_{j(q)}(dj|_q(E_q), J|_q(E_q)) \geq 0$$

and the equality holds if and only if there is a parallel vector field Z, defined along the unique geodesic $\tilde{\gamma}_{\tilde{q}}$ in $\tilde{N}$ joining $\tilde{u}(\tilde{q})$ and $\tilde{v}(\tilde{q})$, such that $Z(\tilde{u}(\tilde{q})) = d\tilde{u}|_{\tilde{q}}(\tilde{E}_q)$, $Z(\tilde{v}(\tilde{q})) = d\tilde{v}|_{\tilde{q}}(\tilde{E}_q)$ and $\left\langle {}^N R(Z, \dot{\tilde{\gamma}}_{\tilde{q}})\dot{\tilde{\gamma}}_{\tilde{q}}, Z\right\rangle_N \equiv 0$ along $\tilde{\gamma}_{\tilde{q}}$. Moreover, $d(\operatorname{dist}_{\tilde{N}})(d\tilde{j}(\tilde{E}_q)) = 0$.
In particular, if ${}^N\operatorname{Sect} < 0$, Z is proportional to $\dot{\tilde{\gamma}}_{\tilde{q}}$.

We go back to the proof of Theorem 3.3. From (3.46), (3.48), applying Corollary 3.26 and observing that

$$\frac{t^2}{(A+t^2)^{3/2}} \leq \frac{1}{\sqrt{A+t^2}} \leq A^{-1/2}, \quad \forall t > 0,$$

we get

$$\begin{aligned}
{}^M\operatorname{div} X_A|_q &= \frac{{}^M\operatorname{tr}\,{}^{\tilde{N}\times/}\operatorname{Hess}\tilde{r}^2|_{j(q)}(dj_q, J|_q)}{2\sqrt{A+\tilde{r}^2(j(q))}} \\
&\quad - \frac{\tilde{r}^2(j(q))}{(A+\tilde{r}^2(j(q)))^{3/2}}\,{}^M\operatorname{tr}\left[d\tilde{r}|_{j(q)}(dj|_q)\, d\tilde{r}|_{j(q)}(J|_q)\right] \\
&\geq -A^{-1/2}(|du|(q)+|dv|(q))\left(|du|^{p-1}(q)+|dv|^{p-1}(q)\right) \\
&\geq -2A^{-1/2}(|du|^p(q)+|dv|^p(q)),
\end{aligned} \tag{3.62}$$

from which

$$(\operatorname{div} X_A|_q)_- \leq 2A^{-1/2}(|du|^p(q)+|dv|^p(q)) \in L^1(M). \tag{3.63}$$

Moreover, since $t/\sqrt{A+t^2} < 1$, (3.47) implies

$$\begin{aligned}
|X_A|^{\frac{p}{p-1}}(q) = &\leq \left(|du|^{p-1}(q)+|dv|^{p-1}(q)\right)^{\frac{p}{p-1}} \\
&\leq 2^{\frac{1}{p-1}}(|du|^p(q)+|dv|^p(q)) \in L^1(M).
\end{aligned} \tag{3.64}$$

For every $T > 0$, set

$$M_T = \{q \in M : \tilde{r}(j(q)) \leq T\} \qquad \text{and} \qquad M^T := M \setminus M_T.$$

From (3.63) and (3.64), we can apply Proposition 1.2 to deduce that

$$\int_M {}^M\operatorname{div} X_A \leq 0,$$

which by (3.62) gives

$$\begin{aligned}
(3.65)\qquad &\int_M \frac{\tilde r^2(j(q))}{(A+\tilde r^2(j(q)))^{3/2}}\,{}^M\mathrm{tr}\left[d\tilde r|_{j(q)}\,(dj|_q)\;d\tilde r|_{j(q)}\,(J|_q)\right]\\
&\geq \int_M \frac{{}^M\mathrm{tr}\,{}^{\tilde N\times/}\,\mathrm{Hess}\,\tilde r^2|_{j(q)}\,(dj|_q, J|_q)}{2\sqrt{A+\tilde r^2(j(q))}}\\
&\geq \int_{M_T} \frac{{}^M\mathrm{tr}\,{}^{\tilde N\times/}\,\mathrm{Hess}\,\tilde r^2|_{j(q)}\,(dj|_q, J|_q)}{2\sqrt{A+\tilde r^2(j(q))}}\\
&\geq \frac{1}{2\sqrt{A+T}}\int_{M_T} {}^M\mathrm{tr}\,{}^{\tilde N\times/}\,\mathrm{Hess}\,\tilde r^2|_{j(q)}\,(dj|_q, J|_q) \geq 0.
\end{aligned}$$

The real valued function $t\mapsto \frac{t}{(A+t)^{3/2}}$ has a global maximum at $t=2A$, is increasing in $(0,2A)$ and satisfies

$$\frac{t}{(A+t)^{3/2}} < \frac{1}{(A+t)^{1/2}}.$$

Hence, up to choosing $A>T^2/2$, we have

$$\begin{aligned}
(3.66)\quad &\int_M \frac{\tilde r^2(j(q))}{(A+\tilde r^2(j(q)))^{3/2}}\,{}^M\mathrm{tr}\left[d\tilde r|_{j(q)}\,(dj|_q)\;d\tilde r|_{j(q)}\,(J|_q)\right]\\
&\leq \frac{T^2}{(A+T^2)^{3/2}}\int_{M_T} 2\left(|du|^p+|dv|^p\right) + \frac{1}{\sqrt{A+T^2}}\int_{M^T} 2\left(|du|^p+|dv|^p\right).
\end{aligned}$$

Inserting (3.66) in (3.65) we get

$$\begin{aligned}
\int_{M_T} {}^M\mathrm{tr}\,{}^{\tilde N\times/}\,\mathrm{Hess}\,\tilde r^2|_{j(q)}\,(dj|_q, J|_q) \leq{}& \frac{4T^2}{A+T^2}\int_{M_T}\left(|du|^p+|dv|^p\right)\\
&+4\int_{M^T}\left(|du|^p+|dv|^p\right),
\end{aligned}$$

and letting $A\to+\infty$ this latter gives

$$\int_{M_T} {}^M\mathrm{tr}\,{}^{\tilde N\times/}\,\mathrm{Hess}\,\tilde r^2|_{j(q)}\,(dj|_q, J|_q) \leq 4\int_{M^T}\left(|du|^p+|dv|^p\right).$$

Since $|du|,|dv|\in L^p(M)$ we can let $T\to+\infty$, applying respectively monotone and dominated convergence to LHS and RHS integrals, thus obtaining

$$(3.67)\qquad \int_M {}^M\mathrm{tr}\,{}^{\tilde N\times/}\,\mathrm{Hess}\,\tilde r^2|_{j(q)}\,(dj|_q, J|_q) = 0.$$

Fix an orthonormal frame $\{E_i\}_{i=1}^m$ for M. Then (3.67) gives

$${}^{\tilde N\times/}\,\mathrm{Hess}\,\tilde r^2|_{j(q)}\,(dj(E_i), J(E_i)) = 0,$$

for all $i=1,\dots,m$ and $\tilde q\in M$. At this point, applying again Corollary 3.26 implies

$$d\left(\mathrm{dist}_{\tilde N}\right)\left(d\tilde u(\tilde E_i), d\tilde v(\tilde E_i)\right) = d\left(\mathrm{dist}_{\tilde N}\circ(\tilde u,\tilde v)\right)(\tilde E_i)\equiv 0$$

and, since $\{\tilde{E}_i\}_{i=1}^m$ span all $T_{\tilde{q}}\tilde{M}$, we get that $(\mathrm{dist}_{\tilde{N}} \circ (\tilde{u}, \tilde{v}))$ is constant on $\tilde{M}$. Accordingly, for each $\tilde{q} \in \tilde{M}$ the unique geodesic $\tilde{\gamma}_{\tilde{q}}$ from $\tilde{u}(\tilde{q})$ to $\tilde{v}(\tilde{q})$ can be parametrized on $[0,1]$ proportional (independent of $\tilde{q}$) to arclength. We define a one-parameter family of maps $\tilde{u}_t : \tilde{M} \to \tilde{N}$ by letting $\tilde{u}_t(\tilde{q}) := \tilde{\gamma}_{\tilde{q}}(t)$. Then we see that $\tilde{u}_0 = \tilde{u}$ and $\tilde{u}_1 = \tilde{v}$. Corollary 3.26 states also that for each $i = 1, \dots, m$ there exists a parallel vector field Z_i, defined along $\tilde{\gamma}_{\tilde{q}}$ in $\tilde{N}$, such that $Z_i(0) = d\tilde{u}|_{\tilde{q}}(\tilde{E}_i)$, $Z_i(1) = d\tilde{v}|_{\tilde{q}}(\tilde{E}_i)$ and $\left\langle {}^N R(Z_i, \dot{\tilde{\gamma}}_{\tilde{q}})\dot{\tilde{\gamma}}_{\tilde{q}}, Z_i \right\rangle_N \equiv 0$ along $\tilde{\gamma}_{\tilde{q}}$. In particular Z is a Jacobi field along $\tilde{\gamma}_{\tilde{q}}$. By the proof of Theorem 3.24 it turns out that

$$Z_i(t) \equiv d\tilde{u}_t|_{\tilde{q}}(\tilde{E}_i). \tag{3.68}$$

In fact, let $\zeta_i : (-\varepsilon, \varepsilon) \to \tilde{M}$, $\varepsilon > 0$, be a smooth curve such that $\dot{\zeta}(0) = \tilde{E}_i$. By definition of differential we have that

$$d\tilde{u}_t(\tilde{E}_i) = \left.\frac{\partial}{\partial s}\right|_{s=0} (\tilde{u}_t \circ \zeta)(s).$$

On the other hand, since $(\tilde{u}_t \circ \zeta)(s) = \gamma_{\zeta(s)}(t)$, we get that $d\tilde{u}_t(\tilde{E}_i)$ is the variational field of the geodesic variation

$$(t, s) \mapsto \gamma_{\zeta(s)}(t),$$

then $d\tilde{u}_t(\tilde{E}_i)$ is a Jacobi field along $\tilde{\gamma}_{\tilde{q}}$ and, by the uniqueness of the Jacobi fields with given boundary values, (3.68) is proved.
In the special situation ${}^N\mathrm{Sect} < 0$, for all $\tilde{q} \in \tilde{M}$ and $i = 1, \dots, m$, the parallel vector field Z_i along $\gamma_{\tilde{q}}$ has to be proportional to $\dot{\gamma}_{\tilde{q}}$. Hence $\tilde{u}(\tilde{M})$ and $\tilde{v}(\tilde{M})$ have to be contained in a geodesic of $\tilde{N}$ and projecting on M we get the proof of case $i)$ of Theorem 3.3.
In general, because of the equivariance property (3.1) and by the uniqueness of the construction above, for all $\gamma \in \pi_1(M, *)$ and $t \in [0,1]$ we have that

$$\tilde{u}_t \circ \gamma = \beta \circ \tilde{u}_t, \tag{3.69}$$

where $\beta = u_\sharp(\gamma) = v_\sharp(\gamma) \in \pi_1(N, *)$. Thus we have induced maps $u_t : M \to N$ for $t \in [0,1]$ such that $u_0 \equiv u$ and $u_1 \equiv v$.
Let $\gamma_q(t)$ be the geodesic from $u(q)$ to $v(q)$ in M obtained by projection from $\tilde{\gamma}_{\tilde{q}}$. Projecting Z_i, which is equivariant by (3.68) and (3.69), the identity (3.68) implies du_t is a parallel vector field along γ_q. Therefore, the p-energy density of u_t

$$e_p(u_t)(q) = \left(\sum_{i=1}^m |du_t(E_i)|^2\right)^{\frac{p}{2}}$$

is constant along γ_q for each $q \in M$ and, consequently, the p-energies of u_t satisfy

$$E_p(u) = E_p(u_t) = E_p(v), \qquad \forall t \in [0,1], \tag{3.70}$$

that is, every p-harmonic map of finite p-energy homotopic to u has the same p-energy as u.
Now, suppose N is compact. In case also M is compact, Corollary 7.2 in [We2]

immediately implies u_t is p-harmonic for all $t \in [0, 1]$. Otherwise, by Theorem 2.4 we know that there exists a p-harmonic map $u_{t,\infty} \in \mathcal{H}_{u_t}$ which minimizes p-energy in the homotopy class of u_t, which, by construction, is the same homotopy class of u. Applying (3.70) with $v = u_{t,\infty}$ we have

$$E_p(u_{t,\infty}) = E_p(u) = E_p(u_t).$$

On the other hand, if we assume that u_t is not p-harmonic, for each $\epsilon > 0$ there exists an ϵ-ball B_ϵ such that u_t does not minimize energy on B_ϵ. Namely, there exist a map $\hat{u}_{t,\epsilon}$ such that

$$E_p^{B_\epsilon}(\hat{u}_{t,\epsilon}|_{B_\epsilon}) < E_p^{B_\epsilon}(u_t|_{B_\epsilon}),$$

so that extending $\hat{u}_{t,\epsilon}$ to all of M as

$$u_{t,\epsilon} := \begin{cases} \hat{u}_{t,\epsilon} & \text{in } B_\epsilon, \\ u_t & \text{in } M \setminus B_\epsilon, \end{cases}$$

it turn out that

$$E_p(u_{t,\epsilon}) < E_p(u_t).$$

Moreover, for ϵ small enough $u_{t,\epsilon} \in \mathcal{H}_{u_t} = \mathcal{H}_u$ and, as it is clear from the proof of Theorem 2.4, it must be

$$E_p(u_{t,\infty}) \le E_p(u_{t,\epsilon}) < E_p(u_t) = E_p(u).$$

This contradicts (3.4) and concludes the proof. □

To conclude, we note that the p-harmonic general comparison theorem we have just proved do not recover completely the previous harmonic result due to Schoen and Yau. Notably, Theorem 3.4 hold also for a non-compact target manifold N. This is achieved taking advantage of the solution to the Dirichlet problem proposed by Hamilton. Indeed, in [Ham], using the heat flow method the author was able to give solution to the Dirichlet problem for maps from compact manifolds with boundary to manifolds with Sect ≤ 0 non necessarily compact. To the best of our knowledge, a p-harmonic analogous of Hamilton's result has not been obtained yet. This problem will be treated more extensively in Appendix B.

Appendix A

Volume growth vs parabolicity

Throughout Chapter 3, the domain manifold M was assumed to be p-parabolic in order to allow us to apply a global form of the divergence theorem in non-compact settings, see Proposition 1.1 and Proposition 1.2. This fact suggests that the p-parabolicity assumption could be dropped in most results once we give conditions, on both M and the maps, which ensure the validity of some Stokes' type result. To this end, the following approach has been suggested in [VV].

Karp, [Kar], extended the famous Stokes' theorem to complete m-dimensional Riemannian manifolds M by proving that, given a vector field X on M, we have $\int_M \operatorname{div} X = 0$ provided $\operatorname{div} X \in L^1(M)$ (but in fact

$$(\operatorname{div} X)_- = \max\{-\operatorname{div} X; 0\} \in L^1(M)$$

is enough) and

$$\liminf_{R\to+\infty} \frac{1}{R} \int_{B_{2R}\setminus B_R} |X| dV_M = 0.$$

On the other hand, the Kelvin-Nevanlinna-Royden criterion implies that if M is p-parabolic and X is a vector field on M such that $|X| \in L^{\frac{p}{p-1}}(M)$, $\operatorname{div} X \in L^1_{loc}(M)$ and $(\operatorname{div} X)_- \in L^1(M))$, then $\int_M \operatorname{div} X = 0$. Hence, it is natural to ask whether there exists a p-parabolic analogue of Karp theorem, i.e. if it is possible to weaken the finite $\frac{p}{p-1}$-energy assumptions on the vector field X and still conclude that $\int_M \operatorname{div} X dV_M = 0$.

This goal is fulfilled, in case $p = 2$ or M is a model manifolds, by constructing special cut-off functions related to the Evans' potentials on the manifold. Furthermore, the explicit form these cut-offs assume on models can be used in the setting of generic manifolds.

Proposition A.1. *Let $(M, \langle, \rangle)$ be a non-compact Riemannian manifold. Let X be a vector field on M such that $\operatorname{div} X \geq f$ in the sense of distributions, for some $f \in L^1_{loc}(M)$ with $f_- \in L^1(M)$. If there exists a function $g : (0, +\infty) \to (0, +\infty)$*

such that $\varphi = |X|^{\frac{p}{p-1}}$ *satisfies condition* $\mathcal{A}_{M,p}$, *i.e.*

$$(\mathcal{A}_{M,p}) \qquad \liminf_{R\to\infty} \left(\int_{B_{(R+g(R))}\setminus B_R} \varphi \, dV_M \right) \left(\int_R^{R+g(G)} A(\partial B_s)^{-\frac{1}{p-1}} ds \right)^{-1} = 0.$$

for some $p > 1$, *then* $\int_M f \leq 0$.

Remark A.2. *It has to be noted that condition* $(\mathcal{A}_{M,p})$ *is "radially" sharp in the following sense (see Remark 9 in [VV]). Given any other function* Φ *such that assumption*

$$\liminf_{R\to\infty} \left(\int_{B_{(R+g(R))}\setminus B_R} |X|^{\frac{p}{p-1}} dV_M \right) \left(\int_R^{R+g(G)} \Phi(s) ds \right)^{-1} = 0$$

implies $\int_M f \leq 0$, *then*

$$\int_R^{R+g(G)} \Phi(s) ds \leq \int_R^{R+g(G)} A(\partial B_s)^{-\frac{1}{p-1}} ds.$$

As announced above, Theorem A.1 permits to generalize, replacing the p-parabolicity assumption on M with a condition similar to $\mathcal{A}_{M,p}$, most theorems presented above. For instance, in [VV] the following results are deduced (compare respectively Theorem 3.15, Corollary 1.3, Theorem 3.4, Theorem 3.13, Theorem 3.14 and Theorem 3.3).

Theorem A.3. *Let* $(M, \langle,\rangle)$ *be a connected, non compact Riemannian manifold. Assume that* $u, v \in W^{1,p}_{loc}(M) \cap C^0(M)$, $p > 1$, *satisfy*

$$\Delta_p u \geq \Delta_p v \text{ weakly on } M,$$

and that $|\nabla u|^p$ *and* $|\nabla v|^p$ *satisfy condition* $\mathcal{A}_{M,p}$ *on* M. *Then,* $u = v + A$ *on* M, *for some constant* $A \in \mathbb{R}$.

Corollary A.4. *Let* $(M, \langle,\rangle)$ *be a connected, non compact Riemannian manifold. Assume that* $u \in W^{1,p}_{loc}(M) \cap C^0(M)$, $p > 1$, *is a weak p-subharmonic function on* M *such that* $|\nabla u|^p$ *satisfies condition* $\mathcal{A}_{M,p}$ *on* M. *Then* u *is constant.*

Theorem A.5. *Let* M *and* N *be complete manifolds.*

1) *Suppose* ${}^N\mathrm{Sect} < 0$. *Let* $u : M \to N$ *be a harmonic map such that* $|\nabla u|^2$ *satisfies condition* $\mathcal{A}_{M,2}$ *on* M. *Then there is no other harmonic map homotopic to* u *satisfying condition* $\mathcal{A}_{M,2}$ *unless* $u(M)$ *is contained in a geodesic of* N.

2) *Suppose* ${}^N\mathrm{Sect} \leq 0$. *Let* $u, v : M \to N$ *be homotopic harmonic maps such that* $|\nabla u|^2, |\nabla v|^2$ *satisfy condition* $\mathcal{A}_{M,2}$ *on* M. *Then there is a smooth one parameter family* $u_t : M \to N$ *for* $t \in [0,1]$ *of harmonic maps with* $u_0 = u$ *and* $u_1 = v$. *Moreover, for each* $x \in M$, *the curve* $\{u_t(x) : t \in [0,1]\}$ *is a constant (independent of* x*) speed parametrization of a geodesic.*

Theorem A.6. *Let $(M, \langle , \rangle_M)$ and $(N, \langle , \rangle_N)$ be complete Riemannian manifolds. Assume that M is non-compact and that N has non-positive sectional curvatures. If $u : M \to N$ is a p-harmonic map homotopic to a constant and with energy density $|du|^p$ satisfying condition $\mathcal{A}_{M,p}$, then u is a constant map.*

Theorem A.7. *Suppose that $(M, \langle , \rangle)$ is a complete non-compact Riemannian manifold. For $p > 1$, let $u, v : M \to \mathbb{R}^n$ be $C^0 \cap W^{1,p}_{loc}(M)$ maps satisfying*

$$\tau_p u = \tau_p v \text{ on } M,$$

in the sense of distributions on M and

$$|du|, |dv| \in L^p(M).$$

Suppose $|du|^p$ and $|dv|^p$ satisfy condition $\mathcal{A}_{M,p}$ on M. Then $u = v + C$, for some constant $C \in \mathbb{R}^n$.

Theorem A.8. *Let M and N be complete manifolds.*

1) *Suppose ${}^N\mathrm{Sect} < 0$. Let $u : M \to N$ be a $C^{1,\alpha}$ p-harmonic map such that $|\nabla u|^p \in L^1(M)$ satisfies condition $\mathcal{A}_{M,p}$ on M. Then there is no other p-harmonic map homotopic to u satisfying condition $\mathcal{A}_{M,p}$ unless $u(M)$ is contained in a geodesic of N.*

2) *Suppose ${}^N\mathrm{Sect} \leq 0$. Let $u, v : M \to N$ be homotopic p-harmonic maps such that $|\nabla u|^p, |\nabla v|^p \in L^1(M)$ satisfy condition $\mathcal{A}_{M,p}$ on M. Then there is a smooth one parameter family $u_t : M \to N$ for $t \in [0, 1]$ of harmonic maps with $u_0 = u$ and $u_1 = v$. Moreover, for each $x \in M$, the curve $\{u_t(x) : t \in [0, 1]\}$ is a constant (independent of x) speed parametrization of a geodesic.*

Remark A.9. *In case of Theorem A.7 and Theorem A.8 the finiteness of the p-energy of the maps is used also to guarantee that the divergence of the vector fields involved has negative part L^1. Thus, apparently this assumption can not be dropped. Nevertheless these results are relevent when dealing with manifolds "less than parabolic", i.e. with area of the geodesic spheres large enough at infinity, and maps with p-energy decaying sufficiently fast.*

Appendix B

Open problems

In Chapter 3 we presented a strategy which permits to extend to the p-harmonic setting the general comparison results for homotopic harmonic maps due to Schoen and Yau; see Theorem 3.3. Throughout the discussion we pointed out the crucial points which can not be trivially generalized to $p \neq 2$ and how to overcome, when possible, the problems thus arising.

As observed in Section 3.2, step **c** in the proof by Schoen and Yau of Theorem 3.4 makes a strong use of the good properties of harmonic maps whose compositions with convex functions give subharmonic functions. It was folklore that, in general, this fact was not true for p-harmonic maps, $p \neq 2$, and indeed Theorem 3.7 gives a counterexample. This leads to follow different paths, such as the construction of special composed vector fields proposed in [PRS3], subsequently developed in Section 3.3 and finally used in Section 3.4. Theorem 3.7 answers in the negative a general open question, which by the way arose in the 2006 Midwest Geometry Conference paper by Lin and Wei, [LW]. However, in Remark 3.8 we pointed out that in the counterexemple both the domain manifold M is not p-parabolic and the target manifold N has non-negative sectional curvatures. On the other hand, in Theorem 3.4 we are exactly in the opposite situation, i.e.

i) the domain manifold M is p-parabolic and

ii) the target manifold N has non-positive sectional curvature,

thus Theorem 3.7 does not provide a counterexample in the specific situation we are interested in. In fact, in this case, so far we have not been able neither to find a suitable counterexample nor to estabilish that the composition well behaves under these restrictive conditions.

Problem 1. *Consider a p-harmonic map $F : M \to N$ and a convex function $H : N \to \mathbb{R}$. Assuming that either condition i) or condition ii) or both of them hold, is the composition $H \circ F$ p-subharmonic?*

Actually, unlike the $p = 2$ case, it is not clear whether a positive answer to Problem 1 would trivially lead to a direct semplification of the proof of Theorem 3.3. Because of the behaiour of τ_p on Riemannian products, assuming that $u, v : M \to N$ are p-harmonic does not imply that also $(u, v) : M \to N \times N$ is a

p-harmonic map. Anyway, it would be interesting to find an answer to Problem 1 since Liouville type theorems for harmonic maps into targets supporting a convex function could be obtained directly from results in linear potential theory of real valued functions, see e.g. [PRS3] and [Kaw].

Schoen and Yau's Theorem 3.4 extends a previous result by Hartman, [Har], by permitting both the domain manifold M and the target manifold N to be non-compact. In particular, one of the conclusions of Theorem 3.4, case *ii*), is that, given two finite energy homotopic harmonic maps u and v, there exists a homotopy via harmonic maps u_t. Analogously, in Theorem 3.3 we extended the previous result due to Wei holding for compact manifolds, see Theorem 3.2. However, in order to prove the p-harmonicity of u_t, for $p \neq 2$ we have to assume that N is compact. As it is clear by the proof, this is necessary to apply Ascoli type convergence results. In fact, when $p = 2$ and N is non-compact, the proof is more complicated. Namely, step **e** of the proof of Theorem 3.4 is based on two fundamental ingredients.

i) First, given a compact manifold with boundary $M_k \subset M$, following Hamilton, [Ham], Schoen and Yau use the heat flow to solve the boundary value problem on M_k, thus getting a harmonic map $u_{t,k} : M_k \to N$ homotopic to u_t with $u_{t,k} \equiv u_t$ on ∂M_k.

ii) Then, since $\tau u = \tau u_{t,k} = 0$ and the distance function is locally convex, they obtain that the function $\mathrm{dist}_N(u_{t,k}, u) : M_k \to \mathbb{R}$ is subharmonic on M_k. This permits to apply the maximum principle, thus deducing that for each $l \geq 1$ there exist a compact set $K_l \subseteq N$ independent of k such that $u_{t,k}(B_l^M) \subseteq K_l$.

Hence, the situation is somehow reduced to that of a compact target and the proof follows.
In [We2], Theorem 7.1, Wei proves that there exists a solution to the p-harmonic Dirichlet problem when N is compact with contractible universal cover and with no non-trivial p-minimizing tangent maps of $\mathbb{S}^l$ for $l < n$. As observed in the proof of Theorem 2.4, this is the case if ${}^N\mathrm{Sect} \leq 0$. By the way, in analogy with Schoen and Yau's strategy, Wei's result could be used to give a partially different proof of the p-harmonicity of u_t in Theorem 3.3. However, to the best of our knowledge, a p-harmonic analogue of Hamilton's result when N is non compact has not been obtained yet.

Problem 2. *Let M and N be Riemannian manifold. Assume M is compact with boundary ∂M and N satisfies ${}^N\mathrm{Sect} \leq 0$. Let $f \in Lip(\partial M, N) \cap C^0(M, N)$ (or at least $f \in C^1(M, N)$) be a map with finite p-energy. Does exist a p-harmonic map $u \in C^{1,\alpha}(M \setminus \partial M, N) \cap C^\alpha(M, N)$ with $u|_{\partial M} = f|_{\partial M}$ minimizing p-energy in the homotopy class of f?*

According to Problem 1, also *ii)* seems to admit no trivial generalizations to $p \neq 2$. Hence, a solution to Problem 2 would not lead directly to extend Theorem 3.3 to non compact targets N. Therefore, this can be seen as a further interesting independent problem.

Problem 3. *In the assumptions of Theorem 3.3 with N non-compact, are the maps u_t p-harmonic?*

Bibliography

[AS] L.V. Ahlfors, L. Sario, *Riemann surfaces.* Princeton Mathematical Series, No. **26** Princeton University Press, Princeton, N.J. 1960 xi+382 pp.

[An1] M.T. Anderson, *The compactification of a minimal submanifold in Euclidean space by the Gauss map.* Preprint.

[An2] M.T. Anderson, *Remarks on curvature integrals and minimal varieties.* Contemp. Math. **49** Amer. Math. Soc., Providence, RI (1986), pp. 11–18.

[Au] T. Aubin, *Some nonlinear problems in Riemannian geometry.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998. xviii+395 pp.

[Be] P. Bérard, *Remarques sur l'équation de J. Simons.* Differential Geommetry (A symposium in honour of M. do Carmo on his 60th birthday), Pitman Monographs Surveys Pure Appl. Math., vol. **52**, Longman Sci. Tech., Harlow, 1991, pp. 47–57.

[Br] T. Branson, *Kato constants in Riemannian geometry.* Math. Res. Lett. **7** (2000), no. 2-3, 245–261.

[Bu] F. Burstall, *Harmonic maps of finite energy from non-compact manifolds.* Jour. London Math. Soc. **30** (1984), 361–370.

[C] E. Calabi, *On manifolds with non-negative Ricci curvature II.* Notices Amer. Math. Soc. **22** (1975), A205.

[CGH] D.M.J. Calderbank, P. Gauduchon, M. Herzlich, *Refined Kato inequalities and conformal weights in Riemannian geometry.* J. Funct. Anal. **173** (2000), no. 1, 214–255.

[CE] J. Cheeger, D.G. Ebin, *Comparison theorems in Riemannian geometry.* North-Holland Mathematical Library, Vol. **9**. North-Holland Publishing Co., Amsterdam-Oxford; 1975.

[CLLM] L.-F. Cheung, C.-K. Law, M.-C. Leung, J.B. McLeod, *Entire solutions of quasilinear differential equations corresponding to p-harmonic maps.* Nonlinear Anal. **31** (1998), no. 5-6, 701–715

[CL] L.-F. Cheung, P.-F. Leung, *A remark on convex functions and p-harmonic maps.* Geom. Dedicata **56** (1995), 269–270.

[CZ] X. Cheng, D. Zhou, *Manifolds with weighted Poincaré inequality and uniqueness of minimal hypersurfaces.* Comm. Anal. Geom. **17** (2009), 139–154.

[CM] T.H. Colding, W.P. Minicozzi II, *The space of embedded minimal surfaces of fixed genus in a 3-manifold II; Multi-valued graphs in disks.* Annals of Math., **160** (2004), 69–92.

[CK] P. Collin, R. Krust. Le problème de Dirichlet pour l'equation des surfaces minimales sur des domaines non bornés. Bull. Soc. Math. France, **119** (1991), 443-458.

[DF] F. Duzaar, M. Fuchs, *On removable singularities of p-harmonic maps.* Ann. Inst. H. Poincaré Anal. Non Linéire **7** (1990), no. 5, 385–405.

[EL] J. Eells, L. Lemaire, *Selected topics in harmonic maps.* CBMS Regional Conference Series in Mathematics, **50** (1983).

[ES] J. Eells, J.H. Sampson, *Harmonic mappings of Riemannian manifolds.* Amer. J. Math. **86** 1964 109–160.

[F] H. Federer, *Geometric measure theory.* Springer Verlag, Berlin, Heidelberg, New York, (1969).

[FCS] D. Fischer–Colbrie, R. Schoen, *The structure of complete stable minimal surfaces in 3-manifolds of non-negative scalar curvature*, Comm. Pure Appl. Math. XXXIII (1980) 199–211.

[FL] H. Fu, Z. Li, *The structure of complete manifolds with weighted Poincare' inequalities and minimal hypersurfaces.* International J. Math. **21** (2010), 1–8.

[GT1] V. Gol'dshtein, M. Troyanov, *Sur la non-résolubilité du p-laplacien sur* $\mathbf{R}^n$. (French) C. R. Acad. Sci. Paris Sér. I Math. **326** (1998), no. 10, 1185–1187.

[GT2] V. Gol'dshtein, M. Troyanov, *The Kelvin-Nevanlinna-Royden criterion for p-parabolicity.* Math Z. **232** (1999), 607–619.

[GT3] V. Gol'dshtein, M. Troyanov, *Capacities in metric spaces.* Integral Equations Operator Theory **44** (2002), no. 2, 212–242.

[GW] R. Greene, H.H. Wu, *Function Theory on Manifolds Which Possess a Pole.* Lecture Notes in Math. **699** Springer Verlag, Berlin, (1979).

[G1] A, Grigor'yan, *Dimensions of spaces of harmonic functions.* (Russian) Mat. Zametki **48** (1990), no. 5, 55–61, 159; Engl. transl. in Math. Notes **48** (1990), no. 5-6, 1114–1118.

[G2] A. Grigor'yan, *Analytic and Geometric Background of Recurrence and non-explosion of the Brownian Motion on Riemannian Manifolds* Bullettin of the american mathematical society

[Ham] R. Hamilton, *Harmonic maps of manifolds with boundary.* Lecture notes, Mathematics, No. **471** Springer, Berlin, Heidelberg, New York (1975).

[HL] R. Hardt, F.-H. Lin, *Mappings minimizing the* L^p *norm of the gradient.* Comm. Pure Appl. Math. **40** (1987), no. 5, 555–588.

[Har] P. Hartman, *On homotopic harmonic maps.* Canad. J. Math. **19** 1967 673–687.

[HKM] J. Heinonen, T. Kilpeläinen, O. Martio, *Nonlinear potential theory of degenerate elliptic equations.* Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1993.

[Ho1] I. Holopainen, *Nonlinear potential theory and quasiregular mappings on Riemannian manifolds.* Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes, **74** (1990), 45 pp.

[Ho2] I. Holopainen, *Volume growth, Green's function, and parabolicity of ends.* Duke Math. J. **97** (1999), 319–346.

[Ho3] I. Holopainen, *Quasiregular mappings and the p-Laplace operator.* Contemp. Math. **338** Amer. Math. Soc., Providence, RI, 2003.

[HK] I. Holopainen, P. Koskela, *Volume growth and parabolicity.* Proc. Amer. Math. Soc. **129** (2001), 3425–3435.

[HPV] I. Holopainen, S. Pigola, G. Veronelli, *Global comparison principles for the p-Laplace operator on Riemannian manifolds.* To appear in Potential Anal.

[Hw] J.-F. Hwang, *Comparison principles and Liouville theorems for prescribed mean curvature equations in unbounded domains.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. **15** (1988), 341–355.

[I] T. Ishihara, *A mapping of Riemannian manifolds which preserves harmonic functions.* J. Math. Kyoto Univ. **19** (1979), 215–229.

[Kar] L. Karp, *On Stokes' Theorem for noncompact manifolds*, Proc. Amer. Math. Soc. **82** (1981), no. 3, 487–490.

[Kaw] S. Kawai, *p-harmonic maps and convex functions.* Geom. Dedicata **74** (1999), 261–265.

[Le] M.-C. Leung, *Positive solutions of second order quasilinear equations corresponding to p-harmonic maps.* Nonlinear Anal. **31** (1998), no. 5-6, 717–733.

[LT] P. Li, L.-F. Tam, *Harmonic functions and the structure of complete manifolds.* J. Diff. Geom. **35** (1992), no. 2, 359–383.

[LW] E.-B. Lin, S.W.Wei, Some open problems in geometry. Commun. Math. Anal. (2008), Conference 1, 35–38.

[Li] P. Lindqvist, *On the equation* $\operatorname{div}(|\nabla u|^{p-2}\,\nabla u) + \lambda\,|u|^{p-2}\,u = 0$. Proc. Amer Math. Soc. **109** (1990), 157–164.

[Lo] E. Loubeau, *p-harmonic morphisms.* Differential Geom. Appl. **12** (2000), 219–229.

[LS] T. Lyons, D. Sullivan, *Function theory, random paths and covering spaces.* J. Diff. Geom. **18** (1984), 229–323.

[Mi] V.M. Mikljukov. A new approach to Bernstein theorem and to related questions for equations of minimal surface type (in Russian). Mat. Sb. **108** (1979), 268-289; Engl. transl. in Math. USSR Sb., **36** (1980), 251-271.

[MP] W.F. Moss, J. Piepenbrink, *Positive solutions of elliptic equations*, Pacific J. Math **75** (1978) 219–226.

[Mu] J. Munkres, *Elementary differential topology.* Annals of Mathematics Studies, No. **54** Princeton University Press (1966).

[N] N. Nakauchi, *A Liouville type theorem for p-harmonic maps.* Osaka J. Math. **35** (1998), no. 2, 303–312.

[O] B. O'Neill, *Semi-Riemannian Geometry with Application to Relativity.* Academic Press, New York, 1982.

[PRS1] S. Pigola, M. Rigoli, A.G. Setti, *Some remarks on the prescribed mean curvature equation on complete manifolds.* Pacific J. Math. **206** (2002), no. 1, 195–217.

[PRS2] S. Pigola, M. Rigoli, A.G. Setti, *Vanishing theorems on Riemannian manifolds and geometric applications.* J. Funct. Anal. **229** (2005), 424–461.

[PRS3] S. Pigola, M. Rigoli, A.G. Setti, *Constancy of p-harmonic maps of finite q-energy into non-positively curved manifolds.* Math. Z. **258** (2008), no. 2, 347–362.

[PRS4] S. Pigola, M. Rigoli, A.G. Setti, *Vanishing and finiteness results in geometric analysis: a generalization of the Bochner technique.* Progress in Mathematics **266** (2008), Birkhäuser.

[PST] S. Pigola, A.G. Setti, M. Troyanov. *The topology at infinity of a manifold supporting an $L^{p,q}$-Sobolev inequality.* Preprint.

[PV] S. Pigola, G. Veronelli, *On the homotopy class of maps with finite p-energy into non-positively curved manifolds.* Geom. Dedicata **143** (2009), 109–116.

[RS] M. Rigoli, A.G. Setti, *Liouville type theorems for φ-subharmonic functions.* Rev. Mat. Iberoamer. **17** (2001), 471–450.

[SU] R. Schoen, K. Uhlenbeck, *A regularity theory for harmonic maps.* J. Differential Geom. **17** (1982), no. 2, 307–335.

[SY1] R. Schoen, S.T. Yau, *Lectures on Harmonic Maps.* Lectures Notes in Geometry and Topology, Volume II, International Press.

[SY2] R. Schoen and S.T. Yau, *Harmonic Maps and the Topology of Stable Hypersurfaces and Manifolds with Non-negative Ricci Curvature.* Comm. Math. Helv. **51** (1976), 333-341.

[SY3] R. Schoen, S.T. Yau, *Compact group actions and the topology of manifolds with nonpositive curvature.* Topology **18** (1979), 361–380.

[SZ] Y.-B. Shen, X.-H. Zhu, *On stable complete minimal hypersurfaces in* R^{n+1}. Amer. J. Math. **120** (1998), no. 1, 103–116.

[S] E.H. Spanier, *Algebraic topology.* Springer-Verlag, New York, 1966. xvi+528 pp.

[Ta] K. Takegoshi, *A maximum principle for p-harmonic maps with L^q finite energy.* Proceedings of the American Mathematical Society **126** (1998), no.12, 3749–3753

[TZ] L.-F. Tam, D. Zhou, *Stability properties for the higher dimensional catenoid in* $\mathbb{R}^{m+1}$. Proc. Amer. Math. Soc. **137** (2009), 3451–3461.

[Tr] M. Troyanov, *Parabolicity of manifolds.* Siberian Adv. Math. **9** (1999), no. 4, 125–150.

[VV] D. Valtorta, G. Veronelli, *Stokes' theorem, volume growth and parabolicity.* **arXiv:1005.2343v1**. Submitted.

[V] G. Veronelli, *On p-harmonic maps and convex functions.* Manuscripta Math. **131** (2010), no. 3-4, 537–546.

[We1] S.W. Wei, *The minima of the p-energy functional.* Elliptic and parabolic methods in geometry (Minneapolis 1994), pp. 171–203. A K Peters, Wellesley (1996).

[We2] S.W. Wei, *Representing homotopy groups and spaces of maps by p-harmonic maps.* Indiana Univ. Math. J. **47** (1998), 625–670.

[WLW1] S.W. Wei, J.F. Li, L. Wu, *Convex functions are psubharmonic functions, $p > 1$ on* $\mathbb{R}^n$. Glob. J. Pure Appl. Math. **3**, no 3, (2007), 219-225.

[WLW2] S.W. Wei, J.F. Li, L. Wu, *Generalizations of the uniformization theorem and Bochner's method in p-harmonic geometry.* Commun. Math. Anal. (2008) Conference 1, 46–68.

[WY] S.W. Wei, C.-M. Yau, *Regularity ofp-energy minimizing maps and p-superstrongly unstable indices.* J. Geom. Anal. **4** (1994), no. 2, 247–272.

[Wh] B. White, *Homotopy classes in Sobolev spaces and the existence of energy minimizing maps.* Acta Math. **160** (1988), no. 1-2, 1–17.

[Y] S.T. Yau, *Some function-theoretic properties of complete Riemannian manifold and their applications to geometry.* Indiana Univ. Math. J. **25** (1976), no. 7, 659–670.

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List of Symbols