Objects: A Study in Kantian Formal Epistemology

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Abstract   We propose a formal representation of *objects*, those being mathematical or empirical objects. The powerful framework inside which we represent them in a unique and coherent way is grounded, on the formal side, in a logical approach with a direct mathematical semantics in the well-established field of constructive topology, and, on the philosophical side, in a neo-Kantian perspective emphasizing the knowing subject's role, which is constructive for the mathematical objects and constitutive for the empirical ones.

1 Introduction

Knowledge representation is one of the most interesting and challenging topics in the recent philosophical scene both from an epistemological and a formal point of view with immediate applications also in information technology. Its story can be traced back to the first attempts by R. Carnap (see the seminal work [6]) and other German-speaking neopositivists to use formal techniques to face philosophical problems. The same approach was later also carried on by some American postpositivists. Nevertheless, after their efforts there was a quiescence in this tradition, also because it was considered partially failed, at least by the advocates of the so-called "new philosophy of science" (for a compendium, see Brown [5]). However, we are now witnessing its renaissance. Among the many attempts (see, e.g., Horsten and Douven [12]), in particular two schools of thought are emerging: *formal ontology* and *formal epistemology*.

This is not the right place to deal with formal ontology or to compare it with formal epistemology. However, we must spend some words on the latter or, at least, on our specific approach to it. By *formal epistemology* we mean a formal approach to the epistemological questions, that is, to the questions concerning how something is known and how knowledge is structured. There are several avenues of proposing

formal epistemology (see Hendricks [11]). Our particular view is based on two main features: a philosophical one and a formal one.

From a philosophical point of view, we put ourselves into the neo-Kantian mainstream (see Boniolo [3], Bitbol, Kerszberg, and Petitot [2]). This means both that

1. we are interested in the conditions which permit and validate knowledge;
2. we assume that the main role in any cognitive process is played by the knowing subject's conceptual framework.

These two claims imply accepting the well-known Kant's "Copernican revolution," according to which the knowing subject (from now on, KS) *constitutes* the empirical world; that is, the KS renders cognitively significant the empirical world by imposing his conceptual set upon it.

At this point, it might be useful recalling that *constituting* the empirical world has nothing to do with any form of idealism or constructivism with respect to the world itself, as Kant [9] himself highlighted in the 1787 second edition of his *Critique of Pure Reason*. Differently, in the neo-Kantian approach, synthesized in Cassirer's jargon (see [7]), *concepts do not construct empirical objects* at all; instead *concepts constitute empirical objects as knowable objects by giving them cognitive significance*.

We have just emphasized that the KS does not construct the empirical world but constitutes it in a cognitively significant way. Actually, and continuing to be neo-Kantianly correct, we can use the term "construct," but only when dealing with mathematics and "pure" physics. Indeed, both in the "transcendental esthetics" and in the "transcendental doctrine of method" of the *Critique of Pure Reason* (see [9]) and in the *Metaphysical Foundations of Natural Science* (see [10]), Kant clearly writes that the two mentioned disciplines are based on the *construction of concepts*. For in them we can (rather, we are forced to) construct objects since in these two cases we do not have an empirical intuition of the objects in question, or, said differently, they are independent of experience.

Therefore, being neo-Kantian implies, at minimum, accepting two theses.

1. *The constitutivist thesis*: empirical objects can be known only by imposing on the empirical world the KS's conceptual framework that renders them cognitively significant.
2. *The constructivist thesis*: objects which are independent of experience can be known by conceptually constructing them.

These cursory notes should be already sufficient to make our philosophical perspective to formal epistemology perspicuous, and meanwhile they should have clarified that our approach is totally different from the (neo-)Aristotelian approaches which do not start from the KS and his or her concepts but from an alleged "metaphysical reality."

Some remarks may be useful at this point. First, adopting a neo-Kantian perspective in doing formal epistemology does not necessarily mean starting from Kant's "transcendental deduction" or "system of principle" and trying to cope with them formally.[1] But, it necessarily means assuming at least the primary cognitive role of the KS and of his or her conceptual framework, as the Marburg school taught us. Second, it is surely not necessary to recall that there is a huge philosophical literature and debate, starting from the ancient Greek philosophers, on the topic of "objects."[2] Nevertheless, our perspective is rather peculiar, both from the philosophical side and

from the formal one, and this peculiarity could be of some interest also for those who do not share our philosophical or mathematical point of view.

Let us now move to the formal side of the matter, and let us stress that our way of interpreting knowledge representation (and in particular formal epistemology) involves the use of a logical system tailored after a specific constructivist approach to topology. This means that we will exploit the results of the well-established field of *formal topology*.[3]

Using formal topology means adopting a constructivist perspective in the field of the foundations of mathematics, that is, a predicativist and intuitionist position. Consequently we will construct step by step what will be necessary in order to realize the formalization that we need. In this way, we will have a total knowledge (and control) of what we are using and how we have provided it.

Our proposal to formally represent objects by using formal topology from a neo-Kantian perspective will be presented by the following steps. We first give a prima facie representation of the objects as suitable subsets of properties. Then we introduce the *conceptual objects* by characterizing the subsets of properties which can represent some object (see Section 2). Later we introduce a logical language and a deductive system to provide a logic for the KS (see Section 3). Finally, we show how some mathematical objects are constructed and how some empirical objects are constituted (see Section 4).

By reading what follows it will be clear how our neo-Kantian approach, both in its constitutivist and in its constructivist side, permeates all of our proposal. For we will start from concepts possessed by the KS, and then, after having *constructed* (to be intended in the foundational sense) the formal representation of the objects and having made explicit his or her logic, we arrive at *constructing* (to be intended in the epistemological sense) the mathematical objects and at *constituting* (to be intended in the epistemological sense) the empirical objects. In particular, we exemplify the former showing natural numbers, real numbers, and functions, and the latter presenting what we call *Newtonian objects*. This does not mean, of course, that we claim to say something new on natural numbers or real numbers, on functions, or on the so-called Newtonian objects. In fact, these are only a few examples showing how our approach works. But we believe that if someone is interested in pursuing our path, maybe some genuine novelty could be found.

2 Representing Objects

What does it mean to know an object? What does it mean to have a representation of an object? We will propose a formal and philosophical answer to these questions by providing a single, coherent, and constructive representation for both the empirical objects (like chairs, tables, electrons, molecules, etc.) and the mathematical ones (like numbers, functions, etc.).

2.1 Objects as sets of bunches of properties To achieve our result, let us start by supposing to have a collection X of objects and a set P of properties which can be enjoyed by them. To formalize the fact that an object $x \in X$ enjoys a property $p \in P$ we introduce the relation $\Vdash$ such that $x \Vdash p$ holds whenever x enjoys p.

For instance, supposing that the objects that we want to represent are the natural numbers, we can consider the set $\mathcal{N}^+$ constituted by all the properties q_n, for any natural number n, such that the relation $x \Vdash q_n$ holds if and only if $n \leq x$. (We

will use this example throughout the paper as a guiding example to illustrate our definitions and theorems.)[4]

It is worth emphasizing immediately that we are assuming to have a collection X of objects only in order to understand how we can "get rid" of it, and in what follows, at a certain moment, we will stop speaking in terms of the elements of X and we will begin speaking in terms of suitable subsets (of finite subsets) of P only. This means that we are not begging the question and speaking of objects before having offered a representation of them but that we do this exactly in order to get such a representation. Thus, with the statements above we have simply said that there is something and that this something can enjoy certain properties; that is, we have made explicit one of our primary intuitive insights on the elements of the world that we are speaking of.

Note that with this claim we have not reached any knowledge on the objects; we have only given, or posed, them. Under this move there is the well-known Kantian difference between "knowing an object" and "giving an object" or, using the Kantian original jargon, between *Objekt* (the object known) and *Gegenstand* (the object given). However, to avoid semantical ambiguities, let us call *concrete objects* the elements of the collection X.

Let us begin our formalization from this intuitive insight. We can think of the properties belonging to the set P as the basic tokens of information that the KS can express on objects. Therefore the KS cognitively works by means of clusters or bunches of these properties. Thus, we are naturally led to the following definition.

Definition 2.1 (Bunch of properties)   A *bunch of properties* (or simply a "bunch") over P is a finite subset of P.

We will call CF (for *conceptual framework*) the set of all the bunches of the given KS. Of course, each property $p \in P$ can be identified with a bunch in CF, namely, the bunch $v_p \equiv \{p\}$.

Each bunch $v \equiv \{p_1, \ldots, p_n\}$ determines a subset $\mathrm{ext}(v)$ of concrete objects that we will call the *extension* of such a bunch. Indeed, it is immediate to lift the relation $\Vdash$ from properties to bunches by setting

$$x \Vdash v \equiv (\forall p \in v)\, x \Vdash p,$$

and hence we can set

$$\mathrm{ext}(v) \equiv \{x \in X \mid x \Vdash v\}.$$

We are, therefore, entitled to claim that the bunch $v \equiv \{p_1, \ldots, p_n\}$ singles out all the concrete objects enjoying all the properties $p_1, \ldots, p_n$.

For instance, continuing with our guiding example, we get that the bunch $v = \{q_{i_0}, \ldots, q_{i_n}\}$ singles out all the natural numbers that are greater than or equal to the maximum among the numbers $i_0, \ldots, i_n$, namely, $\mathrm{ext}(v) \equiv \{x \in \mathsf{N} \mid \max(i_0, \ldots, i_n) \leq x\}$.

It is worth observing that with the definition of extension of a bunch above we get both that x enjoys the property p if and only if x is in the extension of $\{p\}$ and that the empty bunch $\varnothing$ extends over all the objects.

It is trivial to observe that any concrete object $x \in X$ can be identified with the collection $\{Y \subseteq X \mid x \in Y\}$ of the subsets of X that contain x, since we have that $\{x\} = \bigcap_{x \in Y} Y$, where Y runs on subsets of X. On the other hand, we cannot require that, for each subset Y of X, the KS has a bunch whose extension coincides with Y, since the bunches in CF depend on the set of particular properties that a KS has

and they can be too few to discriminate among all the subsets of X. However, from the KS's point of view, the best thing that the KS can do is still represent an object x by using the collection of all the bunches whose extension contains x. Hence our proposal is representing a concrete object $x \in X$ using the subset of bunches $\alpha_x = \{\nu \in \mathsf{CF} \mid x \in \text{ext}(\nu)\}$.

So, in our example above, the natural number x is represented by the (finite) set of all the bunches $\nu = \{q_{i_0}, \ldots, q_{i_n}\}$ such that $i_0, \ldots, i_n$ are all less than or equal to x. Thus, even if we do not have a set of bunches to represent each subset of the set of the natural numbers (in fact, only very few of them can be represented), we still have a reasonable representation for each natural number which is able to distinguish among different natural numbers.

2.2 The conditions for a *good* representation In this way we have already arrived at a suitable subset of bunches α_x which represents the concrete object x. Unfortunately, to define it we need both the object x and the relation $\Vdash$. Actually, we wish to take the opposite perspective; that is, we want to start from the properties, and their bunches, belonging to the KS's conceptual framework and to understand which subsets of bunches are representing some object. That is, we have to understand what conditions have to be satisfied by a subset of bunches in order to be a *good* representation of an object, be it mathematical or empirical. Of course, in looking for such conditions we have to pay attention (1) to choose only those ones that hold for all the subsets of bunches α_x which come from a concrete object x, and (2) to avoid missing some relevant condition which allows us to distinguish a generic subset of CF from a subset of CF which is a *good* representation.

At this point, one should consider that shifting from objects to their representations can give rise to some problems. For instance, while it is trivially true that any object has a representation, there is no reason to think that different objects are going to produce different representations. For the representations depend on the set of properties that the KS possesses, and hence, if such a set is poor, there is the possibility of having two different objects x and y such that there is no bunch whose extension contains one of them and excludes the other one; in this case α_x is going to be equal to α_y.[5] Of course, this could be a problem from the point of view of a (neo-)Aristotelian ontology, but it is not at all a problem from our point of view stressing the cognitive role of the KS. Indeed, if the KS's conceptual framework makes it impossible to distinguish the two objects, it simply means that, for the KS, there is actually only one object.

Let us come back, however, to the core question, which is: What are the conditions that guarantee that a subset of bunches is in fact a *good* representation of an object?

It is not difficult to set the first condition that one has to require in order to have any hope of representing an object. This condition is independent from any language: a representation should be not-empty. Indeed, we have already observed that the empty bunch $\varnothing$ extends over all the objects, and hence it has to belong to any representation.

Let us see now which reasonable conditions can be added to arrive at a plausible result. We can begin by defining a binary operation of conjunction between bunches.

Definition 2.2 (Conjunction of bunches)   Let ν_1 and ν_2 be two bunches in CF. Then the *conjunction* of ν_1 and ν_2 is the bunch

$$\nu_1 \wedge \nu_2 \equiv \nu_1 \cup \nu_2,$$

that is, the union of all the properties in ν_1 and ν_2.

The presence of a finite conjunction allows us to state a second condition that a set of bunches has to enjoy in order to represent an object. For it is immediate to check that an object belongs to the intersection of the extensions of the bunches ν_1 and ν_2 if and only if it belongs to the extension of the bunch $\nu_1 \wedge \nu_2$. Hence, if a subset of bunches α has to represent an object, and $\nu_1 \in \alpha$ and $\nu_2 \in \alpha$, then also $\nu_1 \wedge \nu_2$ has to be an element of α.

Thus, if we write $\alpha \wedge \beta$ to mean the subset $\{\nu_1 \wedge \nu_2 \in \mathsf{CF} \mid \nu_1 \in \alpha, \nu_2 \in \beta\}$, we get $\alpha \wedge \alpha \subseteq \alpha$. But, note that, for any bunch ν, $\nu \wedge \nu = \nu$, and hence we also have $\alpha \subseteq \alpha \wedge \alpha$. Therefore, it follows that a condition that a subset α has to satisfy in order to be the representation of an object is $\alpha \wedge \alpha = \alpha$.

It should be observed that it is possible that a finite quantity of bunches is sufficient to determine a single object (and in this case a single bunch is going to be sufficient, namely, the finite conjunction of all the necessary bunches). For instance, this is the case in our example concerning natural numbers where a natural number x is represented by the subset of bunches $\alpha_x = \{\{q_{i_0}, \ldots, q_{i_n}\} \in \mathsf{CF} \mid i_0, \ldots, i_n \leq x\}$ and it is completely determined by the bunch $\{q_0, q_1, \ldots, q_x\}$ alone.

But it is also possible that the subset α, which represents the object, is not finite and hence no single bunch is going to be sufficient, since no (finitary) operation is going to be sufficient to get a single bunch out of the infinite quantity. (This is the case when an object carries an infinite amount of information like, for instance, a real number or a function from natural numbers to natural numbers; see Sections 4.1.1, 4.1.2.)

We can arrive at setting another desirable condition by considering the fact that usually we are not interested in objects in a general sense but in objects that we can speak about thanks to a given "conceptual vocabulary," that is, thanks to a particular subset of bunches in the KS's conceptual framework. Let us call *conceptual focus*, or simply *focus*, such a subset. To illustrate this point, let us observe that whenever we speak about the north star, actually we are speaking about it thanks to the "conceptual vocabulary" of astronomy, or navigation, or poetry, and so on.

An object x is "spoken about" by the conceptual vocabulary of the focus F if all the bunches that are necessary to represent x are in fact within F, that is, if α_x is a subset of F. Thus the natural definition of the collection of concrete objects which rest within the focus F is setting

$$\mathsf{Win}(F) \equiv \{x \in X \mid \alpha_x \subseteq F\},$$

where we read $\mathsf{Win}(F)$ as "within (the focus) F."

We have finally arrived at the moment in which we can state our notion of *good* representation in a way that requires no reference to the concrete objects in X. But in order to do this we still need to determine the subset of the bunches that are really useful in determining the objects within the focus F; that is, we have to single out from F the smallest subset of bunches $\ltimes(F)$ such that $\mathsf{Win}(F) = \mathsf{Win}(\ltimes(F))$. In order to get this result let us write Ω to mean the collection of all the *good* representations that we are trying to individuate. Then a bunch of properties in F is useful in determining $\mathsf{Win}(F)$ only if it is necessary in order to define at least one *good* representation α contained in F; thus we have to set $\ltimes(F) = \bigcup \{\alpha \in \Omega \mid \alpha \subseteq F\}$.

Even if we do not know what set of bunches $\ltimes(F)$ is, since in its definition we are using the collection Ω that we do not know yet, we can observe that, after the very

definition, it enjoys the following conditions:

$$(\ltimes\text{-reflexivity}) \ltimes (F) \subseteq F, \quad (\ltimes\text{-transitivity}) \; \frac{\ltimes(F) \subseteq G}{\ltimes(F) \subseteq \ltimes(G)},$$

which immediately yield

$$(\ltimes\text{-monotonicity}) \; \frac{F \subseteq G}{\ltimes(F) \subseteq \ltimes(G)}.$$

Moreover, if α is any element in Ω, then we immediately get that the following condition also holds:

$$(\ltimes\text{-positivity}) \; \frac{\alpha \subseteq F}{\alpha \subseteq \ltimes(F)}.$$

However, the following theorem shows that this condition is equivalent to a new one that can be expressed with no reference to the collection of all the subsets of CF, by using only the operator $\ltimes$.

Theorem 2.3 *The condition $\ltimes$-positivity is equivalent to*

$$(\ltimes\text{-fixed point}) \; \alpha = \ltimes(\alpha).$$

Proof $\alpha \subseteq \alpha$ clearly holds, and hence $\ltimes$-*positivity* yields $\alpha \subseteq \ltimes(\alpha)$. But $\ltimes(\alpha) \subseteq \alpha$ holds by $\ltimes$-*reflexivity*, and hence we have proved that $\ltimes$-*positivity* yields $\ltimes$-*fixed point*. On the other hand, if $\alpha \subseteq F$, then $\ltimes(\alpha) \subseteq \ltimes(F)$ follows by $\ltimes$-*monotonicity* and hence we get $\alpha \subseteq \ltimes(F)$ because, by $\ltimes$-*fixed point*, $\ltimes(\alpha) = \alpha$. $\square$

At this point, we can finally state our main definition, namely, the definition of what the elements of the collection Ω of the *good* representations are. We decided to name them *conceptual objects* to recall that they are a formal way to deal with the concrete objects.

Definition 2.4 (Conceptual object) A *conceptual object* is a subset α of CF such that $\varnothing \in \alpha$, $\alpha = \alpha \wedge \alpha$, and $\alpha = \ltimes(\alpha)$.

Of course, this definition is still meaningless since we are not able to provide an explicit definition for the operator $\ltimes$. Indeed, we have shown that the operator $\ltimes$ can be explicitly defined if we have the collection Ω of the conceptual objects, and the definition of conceptual object above indicates what the elements of the collection Ω are only if we are able to define the operator $\ltimes$.

Thus, in order to get a meaningful definition we have to show that the following two equations, where Ω and $\ltimes$ are unknown entities, are solvable at the same time:

$$\alpha \in \Omega \text{ if and only if } \varnothing \in \alpha, \; \alpha \wedge \alpha = \alpha, \text{ and } \alpha = \ltimes(\alpha), \tag{1}$$

$$\ltimes(F) = \bigcup \{\alpha \in \Omega \mid \alpha \subseteq F\}. \tag{2}$$

In fact, it is not too difficult to provide an impredicative proof that this system of two equations admits a solution; that is, it is possible to find out a collection Ω of subsets of CF and a map $\ltimes$ from subsets of CF to subsets of CF which satisfy the conditions above.[6] But an impredicative solution would not give us a clear knowledge of what the elements of Ω are; namely, it does not help us to understand what the conceptual objects are. However, in Section 3 we will introduce an infinitary logical system that will allow us to give an explicit predicative definition for the operator $\ltimes$ which does not require us to know the collection Ω in advance. Hence the use of

impredicative reasoning will be avoided, and a predicative definition of the elements of the collection Ω will be available.

Summing up, we found three formal conditions on the subsets of the KS's conceptual framework that obviously hold for all the subsets α_x coming from some concrete object $x \in X$. Then we arrived at the notion of conceptual object by imposing these conditions to subsets of bunches. Certainly, we have no way of showing that they are sufficient to define in a precise way the conceptual counterpart of the concrete objects, even if on the basis of our constructivist and transcendental approach it looks rather plausible that this is the case. Indeed the conceptual objects that we defined are the only ones that we can get by proceeding constructively from the conditions that we have required. On the other hand, we can know only objects that we can construct as such by means of our conceptual framework. Of course there could be objects that we cannot successfully deal with, but maybe, at least for our transcendental and constructive approach, they simply do not exist. (A mathematical example is for instance any object whose definition requires us to use the *actual infinite*.)

3   Formal Epistemology by a Logical Approach

In Section 2 we showed how objects can be represented. Moreover, we characterized the *good* representations by emphasizing the role of the KS's conceptual framework. Unfortunately, our proposal was only impredicative, and we could not find a direct characterization of the conceptual objects since we were not able to provide a direct definition for the operator $\ltimes$. Here, we show how we can provide such a direct definition by making explicit how the knowing subject "reasons" by using the bunches in his or her conceptual framework; that is, we present the *KS's logic*.

3.1 From $\ltimes$ to the cover operator $\lhd$ It is generally accepted that to construct a logical system one needs to define a deduction relation $\Gamma \vdash a$ stating that *a is deducible from* Γ between some abstract element a and a set Γ of abstract elements, that enjoys axioms like *if $a \in \Gamma$, then $\Gamma \vdash a$* and some form of the cut rule like *if $\Gamma \vdash a$ and, for all $\gamma \in \Gamma$, $\Delta \vdash \gamma$, then $\Delta \vdash a$*.

Unluckily, we are not able to define for the operator $\ltimes$ conditions similar to the ones for $\vdash$ above.[7] But, luckily, we can find a new condition that is equivalent to the fundamental $\ltimes$-*fixed point* condition, that can be expressed in terms of a new operator $\lhd$, that we call the *cover* operator, and that generalizes a deduction relation like $\vdash$.

In order to do this move, let us write $\alpha_x \between U$ to mean the existence in $U \subseteq \mathsf{CF}$ of a certain bunch ν such that $\nu \in \alpha_x$, namely, such that $x \in \mathrm{ext}(\nu)$. Then we can prove the following theorem.

Theorem 3.1   *Let U be any subset of* CF*, and let x be any object in X. Then $x \in \bigcup_{\nu \in U} \mathrm{ext}(\nu)$ if and only if $\alpha_x \between U$.*

Proof   We need only to unfold the definitions: $x \in \bigcup_{\nu \in U} \mathrm{ext}(\nu)$ if and only if there exists $\nu \in U$ such that $x \in \mathrm{ext}(\nu)$ if and only if there exists $\nu \in U$ such that $\nu \in \alpha_x$ if and only if $\alpha_x \between U$. $\square$

We write $\mathsf{Ext}(U)$ to mean the subset $\bigcup_{\nu \in U} \mathrm{ext}(\nu)$ of X, and, coherently with our previous definitions, we call it the *extension* of U. Thus the previous theorem states that x is in the extension of U if and only if $\alpha_x \between U$.

We can now introduce the cover operator $\lhd_F(U)$ by stating that, for any bunch ν,

$$\nu \in \lhd_F(U) \text{ if and only if, for all } \alpha \in \Omega, \text{ if } \nu \in \alpha \text{ and } \alpha \subseteq F, \text{ then } \alpha \between U.$$

Let us now show how the operator $\lhd_F(U)$ can be used to express a condition equivalent to $\ltimes$-*positivity* and hence to $\ltimes$-*fixed point*.

The first step is introducing the subset $\nu^{+F} \equiv \{\nu\} \cap \ltimes(F)$ and noting that it is empty or equal to $\{\nu\}$ according to the membership of ν to $\ltimes(F)$.

Thus, we can state our new condition

$$(\lhd\text{-positivity}) \quad \frac{\nu \in \alpha \quad \alpha \subseteq F}{\alpha \between \nu^{+F}}$$

and prove the following theorem.

Theorem 3.2 $\lhd$-*positivity is equivalent to* $\ltimes$-*positivity.*

Proof   If $\alpha \subseteq F$, then $\alpha \subseteq \ltimes(F)$ follows by $\ltimes$-*positivity*; hence, if $\nu \in \alpha$ we get that $\nu \in \ltimes(F)$, namely, $\nu \in \nu^{+F}$ holds and it yields $\alpha \between \nu^{+F}$ by using again the hypothesis $\nu \in \alpha$; thus we proved that $\ltimes$-*positivity* yields $\lhd$-*positivity*. On the other hand, if $\alpha \subseteq F$ and $\nu \in \alpha$, then $\alpha \between \nu^{+F}$ follows by $\lhd$-*positivity* and it yields $\nu \in \ltimes(F)$ by logic. $\square$

Then, after the very condition defining the cover operator, we can express $\lhd$-*positivity* with no reference to the conceptual objects by saying that, for any bunch of properties ν, $\nu \in \lhd_F(\nu^{+F})$.

Moreover, we can see that the cover operator is really working like a (generalized) deductive relation provided that we read $\nu \in \lhd_F(U)$ as ν *is a consequence of U within the focus F*.

Indeed, if $\nu \in U$, then, for all $\alpha \in \Omega$, if $\nu \in \alpha$, then $\alpha \between U$ holds and hence $\nu \lhd_F(U)$ follows independently from which subset F is; namely, $\lhd$ enjoys the following condition, which is typical for any (generalized) deductive relation:

$$(\lhd\text{-axiom}) \quad \frac{\nu \in U}{\nu \in \lhd_F(U)}.$$

Moreover, if μ is a consequence of U within the focus F, that is, if $\mu \in \lhd_F(U)$ holds, and, for all $\nu \in U$, we have that ν is a consequence of V within the focus F, that is, $\nu \in \lhd_F(V)$ holds, then $\mu \in \lhd_F(V)$ holds, that is, μ is a consequence of V within the focus F. Thus also the following condition is satisfied by $\lhd$:

$$(\lhd\text{-cut rule}) \quad \frac{\mu \in \lhd_F(U) \ (\forall \nu \in U) \ \nu \in \lhd_F(V)}{\mu \in \lhd_F(V)}.$$

Indeed, let γ be any element in Ω, and suppose that $\mu \in \gamma$ and $\gamma \subseteq F$. Then, according to the first condition, we get that $\gamma \between U$, that is, there exists some bunch ξ which is both in γ and in U. Thus, according to the second condition, $\xi \in \lhd_F(V)$ and hence $\gamma \between V$ follows since $\xi \in \gamma$ and $\gamma \subseteq F$ hold.

3.2 The logical system We have seen that to characterize the conceptual objects, the KS needs to define the operators $\ltimes$ and $\lhd$. However, we noticed that $\lhd$ behaves as a generalized deduction relation. Hence, in order to define it without referring to any conceptual object, we can use the approach which is usually adopted to present a deduction system; that is, the KS can start by stating the *axioms* on the cover operator and then go on by using suitable deduction rules. In our case, we can assume that

an axiom on the cover operator is shaped as a triple $\langle v, C, D \rangle$, where v is a bunch in CF and where C and D are subsets of bunches. Its intended meaning is that v is a consequence of C within the focus D; namely, each concrete object within the focus D and belonging to the extension of v belongs also to the extension of C.

Definition 3.3 (Axiom set)   An *axiom set* is a set of axioms

$$\{\langle v, C(v, i), D(v, i) \rangle \mid i \in I(v)\},$$

where v is a bunch in CF, $I(v)$ is a set of indexes for axioms on the bunch v, and $C(v, i)$ and $D(v, i)$ are subsets of CF. The following conditions are required to hold:[8]

- there exist two indexes i and j in $I(v \wedge \mu)$ such that $C(v \wedge \mu, i) = \{v\}$ and $D(v \wedge \mu, i) = \mathsf{CF}$ and such that $C(v \wedge \mu, j) = \{\mu\}$ and $D(v \wedge \mu, j) = \mathsf{CF}$;
- for any index $i \in I(\mu)$ there exists an index $j \in I(v \wedge \mu)$ such that $D(\mu, i) \subseteq D(v \wedge \mu, j)$ and $C(v \wedge \mu, j) \subseteq \{v \wedge y \in \mathsf{CF} \mid y \in C(\mu, i)\}$.

If we look again at our guiding example on the representation of the natural numbers we can consider, for instance, the following family of axioms:

1. if $v = \varnothing$, then set $I(v) = \{*\}$, and $C(v, *) = \mathcal{N}^+ \setminus \{\varnothing\}$ and $D(v, *) = \mathcal{N}^+$;
2. if $v \neq \varnothing$, then set $I(v) = \{*\} \cup \{\mu \in \mathcal{N}^+ \mid \mu \subseteq \overline{v}\}$ and $C(v, *) = \{\varnothing\}$ and $D(v, *) = \mathcal{N}^+$, while $C(v, \mu) = \{\mu\}$ and $D(v, \mu) = \mathcal{N}^+$.

By $\overline{v}$ we mean the bunch that *completes* v; namely, if i_n is the maximum index such that $q_{i_n} \in v$, then $\overline{v} = \{q_0, q_1, \ldots, q_{i_n}\}$.

The first axiom states that we can make more precise the information on a number x belonging to the extension of the always true bunch $\varnothing$ by stating that x is greater than or equal to a certain number. Moreover, the second axiom states that we can in any case add the always true bunch $\varnothing$, and that if a number x is greater than or equal to k, for all $q_k \in v$, then it is also greater than or equal to h, for all q_h in a bunch μ which is a subset of $\overline{v}$.

Now, our purpose is extending the basic knowledge given by the set of axioms in order to get a complete knowledge of the operators $\lhd$ and $\ltimes$. To this aim we can use a deduction system which allows us to see when $\mu \in \lhd_F (V)$ and $\mu \in \ltimes(F)$ are valid consequences of the axioms.

Definition 3.4 (Infinitary logical system)   An *infinitary logical system* is given by an axiom set, with the following rules coinductively defining an operator $\ltimes$ on subsets of CF:

$$(\ltimes\text{-reflexivity}) \quad \frac{v \in \ltimes(F)}{v \in F},$$

$$(\ltimes\text{-infinity}) \quad \frac{v \in \ltimes(F) \quad i \in I(v) \quad \ltimes(F) \subseteq D(v, i)}{(\exists y \in C(v, i))\ y \in \ltimes(F)},$$

and the following rules inductively defining an operator $\lhd$ on a couple of subsets of CF:

$$(\lhd\text{-reflexivity}) \quad \frac{v \in U}{v \in \lhd_F(U)},$$

$$(\lhd\text{-infinity}) \quad \frac{i \in I(v) \quad \ltimes(F) \subseteq D(v, i) \quad (\forall y \in C(a, i))\ y \in \lhd_F(U)}{v \in \lhd_F(U)},$$

$$(\lhd\text{-positivity}) \quad \frac{(\forall y \in \nu^{+F}) \; y \in \lhd_F(U)}{\nu \in \lhd_F(U)}.$$

For instance, we will see that in our example with the natural numbers we get $\ltimes(F) = \{\nu \in \mathsf{CF} \mid (\forall \mu \in \mathsf{CF}) \; \mu \subseteq \bar{\nu} \to \mu \in F\}$.

So, each KS has his or her own conceptual framework and axiom set, but all of them are supposed to share the same inductive rules in order to reason with them. This is the reason why in the following we will refer to an infinitary logical system by stating only its axiom set.

It can be useful to observe that, after Tarski's fixed point theorem (see [19, Theorem 1]) the rules that we proposed are in fact defining two operators since the functions $\tau_{\ltimes(F)}$, which map a subset

$$X \quad \text{into} \quad F$$
$$\cap \left\{ x \in \mathsf{CF} \mid (\forall i \in I(x)) \; X \subseteq D(x,i) \to \big(\exists y \in C(x,i)\big) \; y \in X \right\},$$

and $\tau_{\lhd_F(U)}$, which maps a subset

$$X \quad \text{into} \quad U$$
$$\cup \left\{ x \in \mathsf{CF} \mid (\exists i \in I(x)) \; \ltimes(F) \subseteq D(x,i) \to \big(\forall y \in C(x,i)\big) \; y \in X \right\}$$
$$\cup \left\{ x \in \mathsf{CF} \mid (\forall y \in x^{+F}) \; y \in X \right\},$$

are monotone.

Moreover, it is possible both to prove, by coinduction, that the operator $\ltimes$ defined above enjoys $\ltimes$-*reflexivity* and $\ltimes$-*transitivity* and to prove, by induction, that the operator $\lhd$ enjoys $\lhd$-*axiom* and $\lhd$-*cut rule*. Finally, for any bunch ν and any $i \in I(\nu)$, $\nu \in \lhd_{D(\nu,i)}(C(\nu,i))$ holds, that is, $\lhd$ satisfies all the axioms of the KS (see [20]).

We have seen that each KS proposes a set of axioms and then can inductively construct the operators $\ltimes$ and $\lhd$ and, thanks to them, can define the conceptual objects. Then, he or she chooses a focus and selects the conceptual objects which are relevant to that focus. Of course, given a focus F, a KS can propose different axiom sets and, thus, arrive at constructing different collections of conceptual objects all living within the same focus F since their bunches are all contained in F. To emphasize the joint epistemic role of the focus and of the set of axioms, we introduce the notion of *theoretical domain*.

Definition 3.5 (Theoretical domain) Let (I, C, D) be an axiom set, and let F be a focus. Then we call the *theoretical domain* the couple $\mathsf{TD}_F \equiv \langle F, (I, C, D) \rangle$.

Theoretical domains are the KS's very epistemic elements since thanks to the axiom set one can construct the conceptual objects and then choose the ones which are permitted by the conceptual vocabulary given by the focus.

3.2.1 Correctness of the inductive and coinductive rules In order to check that the inductive and the coinductive rules that we proposed in the previous section are sound, we have to prove the following theorem.

Theorem 3.6 (Validity theorem) *Let (I, C, D) be an infinitary logical system. Then*

    1. *if there is some conceptual object α such that $\nu \in \alpha$ and $\alpha \subseteq F$, then $\nu \in \ltimes(F)$ holds;*

    2. *if $\nu \in \lhd_F(U)$ holds, then for all the conceptual objects α, if $\nu \in \alpha$ and $\alpha \subseteq F$, then $\alpha \between U$.*

Of course, in order to prove the validity theorem above, for each KS we have to consider only those conceptual objects α which satisfy the KS's axioms, namely, those ones such that if $\nu \in \alpha$, then for any $i \in I(\nu)$, if $\alpha \subseteq D(\nu, i)$, then $\alpha \between C(\nu, i)$.

Let us first consider the coinductive rules for the operator $\ltimes$. In this case we can reason by coinduction. Indeed, the operator $\ltimes$ is defined by coinduction, namely, $\ltimes(F)$ is the greatest fixed point of the function $\tau_{\ltimes(F)}$. Hence, for any property $\mathscr{P}_F$ on bunches, we get that if the subset $\{\xi \in \mathsf{CF} \mid \mathscr{P}_F(\xi)\}$ satisfies the coinductive rules for $\ltimes$, then it is a subset of $\ltimes(F)$. So, to prove that $\nu \in \ltimes(F)$ it is sufficient to prove that $\mathscr{P}_F(\nu)$ holds.

Now, in order to prove that if there is some conceptual object α such that $\nu \in \alpha$ and $\alpha \subseteq F$ then $\nu \ltimes F$ holds, we can use the property

$$\mathscr{P}_F(\xi) \equiv (\exists \alpha \in \Omega)\, \xi \in \alpha \subseteq F.$$

Indeed, in this way we immediately get that $\mathscr{P}_F(\nu)$ holds. Hence in order to conclude we have only to show that $\{\xi \in \mathsf{CF} \mid \mathscr{P}_F(\xi)\}$ satisfies $\ltimes$-*reflexivity* and $\ltimes$-*infinity*. Now, we clearly have that if $\mu \in \{\xi \in \mathsf{CF} \mid \mathscr{P}_F(\xi)\}$, then $\mathscr{P}_F(\mu)$ holds and hence there is a conceptual object α such that $\mu \in \alpha \subseteq F$, which immediately yields $\mu \in F$. Moreover, if $\mu \in \{\xi \in \mathsf{CF} \mid \mathscr{P}_F(\xi)\}$, $i \in I(\mu)$, and $\{\xi \in \mathsf{CF} \mid \mathscr{P}_F(\xi)\} \subseteq D(\mu, i)$, then we know that there is a conceptual object α such that $\mu \in \alpha$ and $\alpha \subseteq F$, since $\mathscr{P}_F(\mu)$ is a consequence of the first assumption. Consider now any bunch $\xi \in \alpha$; then we immediately get that $\mathscr{P}_F(\xi)$ holds, and hence, from the fact that $\{\xi \in \mathsf{CF} \mid \mathscr{P}_F(\xi)\}$ is a subset of $D(\mu, i)$ it is possible to deduce that $\xi \in D(\mu, i)$. So $\alpha \subseteq D(\mu, i)$ holds. Thus there exists some bunch $\rho \in C(\mu, i)$ such that $\rho \in \alpha \subseteq F$, since α is a conceptual object satisfying the KS's axioms. Thus $\mathscr{P}_F(\rho)$ holds; that is, $\rho \in \{\xi \in \mathsf{CF} \mid \mathscr{P}_F(\xi)\}$.

Let us prove now that the inductive rules to generate $\lhd$ are sound. In this case we can reason by induction. Now, if $\nu \in \lhd_F(U)$ holds because $\nu \in U$, then if $\nu \in \alpha$ and α is included in F, then there exists some bunch in U, namely, ν itself, which belongs to α, and hence $\lhd$-*reflexivity* is trivially valid. In order to prove that $\lhd$-*infinity* is also sound, let us suppose that there is $i \in I(\nu)$ such that $\ltimes(F) \subseteq D(\nu, i)$ and, for any $\mu \in C(\nu, i)$, $\mu \in \lhd_F(U)$ holds. Now, if $\nu \in \alpha$ and α is included in F, then α is also included in $\ltimes(F)$, and hence $\alpha \subseteq D(\nu, i)$ follows since $\ltimes(F) \subseteq D(\nu, i)$. But α is assumed to be a conceptual object that respects the KS's axioms, and hence there exists $\rho \in C(\nu, i)$ such that $\rho \in \alpha$. Then $\rho \in \lhd_F(U)$ follows by logic since, for any $\mu \in C(\nu, i)$, $\mu \in \lhd_F(U)$ holds. Thus, by inductive hypothesis, we get that there exists $\xi \in U$ such that $\xi \in \alpha$; namely, $\alpha \between U$ holds. Finally, also *positivity* is sound. Indeed, let us suppose that $\nu \in \alpha$ and $\alpha \subseteq F$. Then $\alpha \subseteq \ltimes(F)$, and so $\nu \in \ltimes(F)$; thus $\nu \in \lhd_F(U)$ follows by logic since the assumption of *positivity* states that, for every $\mu \in \nu^{+F}$, $\mu \in \lhd_F(U)$ holds.

3.2.2 Sufficiency of the formalization The purpose of the operators $\ltimes$ and $\lhd$ is harmonizing into a single and coherent picture all the fragments of knowledge that the KS expresses by means of his axioms.

Moreover, the validity of the rules, which we proved in the previous section, shows that the operators that we generate inductively are in fact respecting the intended meaning. But a question has still to be addressed: Are the rules sufficient to get all the possible knowledge on $\ltimes$ and $\lhd$, which is contained in the axioms? The following theorem gives a positive answer in the case when the quantity of axioms that the KS expresses is countable (for a proof, see [22]).

Theorem 3.7 (Completeness theorem)   *Let (I, C, D) be an infinitary logical system with a countable quantity of axioms. Then*

1. *if $v \in \ltimes(F)$, then there exists a conceptual object α such that $v \in \alpha$ and $\alpha \subseteq F$, and*
2. *if, for all conceptual objects α, $v \in \alpha$ and $\alpha \subseteq F$ yield $\alpha \, \between \, U$, then $v \in \lhd_F(U)$.*

After the validity Theorem 3.6 and the completeness Theorem 3.7 we can finally prove that we solved the two equations defining the conceptual objects.

Theorem 3.8 (Equation solution)   *Let (I, C, D) be an infinitary logical system with a countable quantity of axioms. Then the operator $\ltimes$ defined by coinduction and the collection $\Omega_\ltimes$ of the $\ltimes$-conceptual objects are a solution of the equations* (1) *and* (2).

Proof   After the correctness of the coinductive rules for the operator $\ltimes$ that we proved in the previous section, we know that if, for some conceptual object α, $v \in \alpha$ and $\alpha \subseteq F$ hold, then $v \in \ltimes(F)$, that is, $\bigcup\{\alpha \in \Omega_\ltimes \mid \alpha \subseteq F\} \subseteq \ltimes(F)$. Moreover, the completeness Theorem 3.7 shows that if $v \in \ltimes(F)$, then there exists a conceptual object α such that $v \in \alpha$ and $\alpha \subseteq F$, and hence $\ltimes(F) \subseteq \bigcup\{\alpha \in \Omega_\ltimes \mid \alpha \subseteq F\}$. $\square$

The completeness theorem is also useful because it offers us a test to know if there are conceptual objects which model our knowledge on a specific focus F since we just need to check if $\ltimes(F)$ is inhabited, that is, there is some bunch v such that $v \in \ltimes(F)$.

Even if we are not going to repeat here the proof of the completeness theorem which is already present in the literature (see Valentini [21], [22]), one point that is contained in such a proof is relevant to our topic, namely, the characterization of the conceptual objects within a logical system. Indeed, it is possible to prove the following theorem.

Theorem 3.9 (Characterization theorem)   *Let (I, C, D) be an axiom set, and let α be a subset of CF containing $\varnothing$. Then α is a conceptual object if and only if it enjoys the following conditions:*

$$(\wedge\text{-}closure) \quad \frac{v \in \alpha \quad \mu \in \alpha}{v \wedge \mu \in \alpha}, \qquad (ax\text{-}compl.) \quad \frac{v \in \alpha \quad i \in I(v) \quad \alpha \subseteq D(v, i)}{(\exists y \in C(v, i)) \, y \in \alpha}.$$

It is worth noticing that the conditions in Theorem 3.9 above are just tools to check if a given subset of bunches is a conceptual object, and they are not at all an inductive method for generating conceptual objects.

After this theorem we can finally see what are the *conceptual natural numbers*, and, even better, we can do this without the need to use directly the very definition of conceptual object which requires us to find the fixed points of the $\ltimes$-operator, which can be difficult. Indeed, according to the characterization Theorem 3.9, a subset α of $\mathcal{N}^+$ is a conceptual object if $\alpha = \{v \in \mathcal{N}^+ \mid v \subseteq \{q_0, \dots, q_n\}\}$ for some natural number n; namely, we have a conceptual natural number in correspondence with each concrete natural number. Of course these are the conceptual natural numbers corresponding to the representations $\alpha_x = \{\{q_{i_0}, \dots, q_{i_n}\} \in \mathcal{N}^+ \mid i_0, \dots, i_n \leq x\}$, for x a natural number, that we already met at the end of Section 2.1 after the very definition of the notion of representation of a concrete object by means of a set of bunches.

But, and this may be an unexpected novelty, $\alpha = \mathcal{N}^+$ is also a conceptual natural number; namely, we added to the standard natural numbers an *ideal* one that we can consider to be *infinite* since it is greater than any *standard* natural number. Thus, in our approach we cannot only miss some concrete object when we cannot *see* it by using the KS's conceptual framework, but we can also *create* some unexpected conceptual object which corresponds to no concrete object. In our opinion this fact should not be considered as a defect of our approach but only a side effect of our choice to depend completely on the KS's conceptual framework.

Our logical systems are specified by giving a set of axioms according to the individual KS. It should be clear that in this way we can arrive at different collections of conceptual objects depending on different KSs. This conforms to the usual experience that different individuals "cognitively look" at the world in different ways. In order to escape from a complete solipsism, different "cognitive looks" have to be harmonized by means of social interactions. Now, our approach can be useful to model the possible communications between different KSs. Indeed, consider that they cannot communicate their conceptual objects, since these are in general infinite subsets of bunches of properties determined by their individual axioms; moreover, they cannot communicate how they produced them, since a conceptual object in general is not obtained inductively. However, they can communicate their axioms, since these are in a finite number or at least it should be possible to list them in some effective way. In particular, we think that our approach is extremely suitable to deal in a computer system with the representation of the objects. Indeed, it allows us to avoid completely the presence of the concrete objects, and it only requires us to manage with linguistic entities, namely, the properties, in an algorithmic way, namely, by induction and coinduction.

In principle, two different KSs can never know if they are speaking of the *same* object. But still they can continue speaking of that object by stating one after the other their axioms concerning it until they agree on the latter. Therefore, they can get all the possible consequences of the finite subset of axioms that they stated up to that moment, or they can arrive at a mismatch which will require some basic revision on what they believe.

4 Mathematical and Empirical Objects

It is time to show the efficacy and powerfulness of our constructive and transcendental approach, namely, how it can be fruitfully applied to represent mathematical and empirical objects.

4.1 Mathematical objects Since formal topology has been constructed to answer to a precise mathematical request, namely, dealing in a constructive way with topological spaces, it is not surprising that mathematics is the most studied field of application besides being the most relevant source of inspiration. Therefore, it should be not at all unexpected that our approach works in a plain way when the objects to be represented are mathematical.

4.1.1 The real numbers The construction of the real numbers is a main example of the construction of conceptual objects requiring an infinite amount of information. In this case the process is based on the use of the set of properties

$$R \equiv \{r_{(p,q)} \mid p, q \in \mathbb{Q}\},$$

where Q is the set of the rational numbers. The intended meaning is that, for any concrete real number x, we have $x \Vdash r_{(p,q)}$ if and only if x is included in the open interval (p, q), namely, if $p < x < q$.

Now we want to define an infinitary logical system on the focus $\mathcal{R}$ realized by using bunches of properties in R. To this aim, if $\nu \equiv \{r_{(p_1,q_1)}, \ldots, r_{(p_n,q_n)}\}$, let us write $\min(\nu)$ to mean $\max(p_1, \ldots, p_n)$ and $\max(\nu)$ to mean $\min(q_1, \ldots, q_n)$, and note that ν extends on the interval of real numbers $(\min(\nu), \max(\nu))$.

Then we have the following set of axioms:

1. if $\nu = \varnothing$, then $I(\nu) = \{*\}$, $C(\nu, *) = \mathcal{R} \setminus \{\varnothing\}$, and $D(\nu, *) = \mathcal{R}$;
2. if $\nu \neq \varnothing$, then

$$I(\nu) = \big\{\mu \mid \min(\mu) \leq \min(\nu) \text{ and } \max(\nu) \leq \max(\mu)\big\} \cup$$

$$\big\{\langle\mu_1, \mu_2\rangle \mid \min(\mu_1) = \min(\nu), \max(\mu_2) = \max(\nu), \min(\mu_2) < \max(\mu_1)\big\} \cup$$

$$\{*\}$$

and

$$C(\nu, \mu) = \{\mu\} \text{ and } D(\nu, \mu) = \mathcal{R},$$

$$C(\nu, \langle\mu_1, \mu_2\rangle) = \{\mu_1, \mu_2\} \text{ and } D(\nu, \langle\mu_1, \mu_2\rangle) = \mathcal{R},$$

$$C(\nu, *) = \big\{\mu \mid \min(\nu) < \min(\mu) \text{ and } \max(\mu) < \max(\nu)\big\} \text{ and } D(\nu, *) = \mathcal{R}.$$

While the meaning of the first axiom should be clear, since the always true information can be refined to a more precise information, some comments on the meaning of the second group of axioms can be useful. The first possibility states that the interval $(\min(\nu), \max(\nu))$ determined by ν is covered by the interval $(\min(\mu), \max(\mu))$ determined by μ if $\min(\mu) \leq \min(\nu)$ and $\max(\nu) \leq \max(\mu)$. Moreover, the second possibility states that the interval determined by ν is covered by the union of the two intervals determined by μ_1 and μ_2 if $\min(\mu_1) = \min(\nu)$, $\max(\mu_2) = \max(\nu)$, and $\min(\mu_2) < \max(\mu_1)$. Finally, the last possibility states that the interval determined by ν is covered by the set of all the intervals that are included within it.

Now, the characterization Theorem 3.9 allows us to recognize the subsets of bunches which are conceptual real numbers. For instance, it is not difficult to see that the subset $\{\varnothing\} \cup \{\nu \in \mathcal{R} \mid \min(\nu) < 0 \text{ and } \max(\nu)^2 > 2\} \cup \{\nu \in \mathcal{R} \mid \min(\nu)^2 < 2 \text{ and } \max(\nu)^2 > 2\}$ is a conceptual real number corresponding to the square root of 2.

4.1.2 The functions on natural numbers The collection of the functions from the set of the natural numbers into itself is another example of a collection such that each of its elements requires an infinite amount of information in order to be specified.

To define such a collection as a collection of conceptual objects we introduce the following set of properties:

$$F \equiv \{p_{(n,m)} \mid n, m \in \mathsf{N}\}$$

such that if f is a function from the natural numbers to the natural numbers, then $f \Vdash p_{(n,m)}$ holds if and only if $f(n) = m$. Observe that a bunch of properties of F can be considered as a partial description of a (possible) graph of a function, and let us call $\mathcal{F}$ the subset of all such bunches.

Now, we want to define a set of axioms on the bunches in the focus $\mathcal{F}$. Beyond the standard axiom on the always true bunch, namely,

1. $I(\varnothing) = \{*\}$, $C(\varnothing, *) = \mathcal{F} \setminus \{\varnothing\}$, and $D(\varnothing, *) = \mathcal{F}$,

we propose the following axioms in the case of a nonempty bunch ν:

2. if v contains two couples of properties $p_{(n,m_1)}$ and $p_{(n,m_2)}$, with $m_1 \neq m_2$, then $I(v) = \{*\}$ and $C(v, *) = \emptyset$ and $D(v, *) = \mathcal{F}$;

3. otherwise, $I(v) = \{*\}$ and

$$C(v, *) = \big\{ v \cup \{p_{(n,m)}\} \in \mathcal{F} \mid \neg(\exists k \in \mathsf{N}) \ p_{(n,k)} \in v \big\} \qquad \text{and} \qquad D(v, *) = \mathcal{F}.$$

In the former case, the axiom states that if v contains two couples $\langle n, m_1 \rangle$ and $\langle n, m_2 \rangle$ such that $m_1 \neq m_2$, then it cannot be part of the graph of a function, and, in the latter case, the axiom states that we can always extend correct information on a function by adding a new couple in a free way.

It is trivial to see that the conceptual objects that we obtain in this way are representations of graphs of functions from the natural numbers to the natural numbers.

4.2 Empirical objects Let us move now to the empirical objects in order to arrive at what we have Kantianly called the constitution of cognitive significance of the empirical world. In this case we have to deal with

1. *conceptual objects*, which are the KS's conceptual constructions realized on the basis of his or her conceptual framework and his or her set of axioms;
2. *concrete empirical objects*, which are the first intuitive insight on the components of the empirical world and which cannot be known directly but only through their conceptual representations.

In the following we show that concrete empirical objects have conceptual representations in terms of conceptual objects, and we discuss the problem of understanding which conceptual objects have an empirical counterpart.

The first question concerns whether our approach is really representationally efficacious, that is, whether we are really able to construct the conceptual counterpart of the concrete empirical objects that we speak about every day or within a given scientific domain. We illustrate this point by proposing an example regarding what we call *Newtonian objects* (see Section 4.2.1).

The second question is methodologically relevant since, given a conceptual object α constructed in a theoretical way, we have first to understand what it means that α has an empirical counterpart and then find a way to see whether such an empirical counterpart really exists, that is, whether the theoretical domain that defines α really constitutes the empirical world. Of course this task cannot be successfully managed by using logic alone, and we have to resort to experiments and observations[9]. What our approach can do is simply offer a formal setup of the issue (see Section 4.2.2).

4.2.1 An example: Newtonian objects As just announced, we are going to apply our transcendental and constructivist approach to a specific case. Note, however, that in this section we do not touch the empirical level yet, that is, the level of the constitution of the cognitive significance, but we still remain at the conceptual level, that is, the level of the construction of the representations. The empirical level will be discussed in the next section.

Let us consider concrete objects whose representation is strongly empirically corroborated; that is, let us think about what we call *Newtonian objects*: chairs, tables, horses, cars, balls, toys, fishes, philosophers, mathematicians, and so on. This means that they "cognitively live" within a Newtonian theoretical domain TD_N that we define by introducing suitable bunches of properties and a suitable set of axioms. Of course, we could have exemplified the applicability and the efficacy of our approach on other fields as well. For instance, by discussing Maxwellian objects, that

is, objects, such as charged bodies, currents, and so on, "cognitively living" within a Maxwellian theoretical domain; or quantum objects, that is, objects, such as subatomic particles, "cognitively living" within a quantum theoretical domain; or special relativistic objects, that is, objects, such as charged and noncharged bodies moving with speed close or equal to the speed of light, "cognitively living" within a special relativistic theoretical domain. Of course, in all of these cases, we should have individuated the right focus and the right set of axioms.

The first step is finding the properties by which we will construct the bunches composing the focus N of the theoretical domain TD_N, that is, fixing the Newtonian conceptual vocabulary. To get this result, we propose using the following properties:

$$ST_{(x,y,z,t)} \equiv \text{being in the space-time position } (x, y, z, t),$$
$$M_{(m,x,y,z,t)} \equiv \text{having the mass } m \text{ in the space-time position } (x, y, z, t),$$
$$V_{(v_x,v_y,v_z,x,y,z,t)} \equiv \text{having the velocity } (v_x, v_y, v_z) \text{ in the space-time}$$
$$\text{position } (x, y, z, t),$$
$$A_{(a_x,a_y,a_z,x,y,z,t)} \equiv \text{having the acceleration } (a_x, a_y, a_z) \text{ in the space-time}$$
$$\text{position } (x, y, z, t).$$

Thus the focus N consists of all the bunches that we can build by using the following set of properties:

$$\begin{aligned}
&\left\{ ST_{(x,y,z,t)} \mid (x, y, z, t) \in \Re^4 \right\} \\
&\quad \cup \left\{ M_{(m,x,y,z,t)} \mid (m, x, y, z, t) \in \Re^+ \times \Re^4 \right\} \\
&\quad \cup \left\{ V_{(v_x,v_y,v_z,x,y,z,t)} \mid (v_x, v_y, v_z, x, y, z, t) \in \Re^7 \right\} \\
&\quad \cup \left\{ A_{(a_x,a_y,a_z,x,y,z,t)} \mid (a_x, a_y, a_z, x, y, z, t) \in \Re^7 \right\}.
\end{aligned}$$

In this way we have the conceptual vocabulary that allows us to speak in terms of being in a certain space-time position (x, y, z, t), having a certain mass m, a certain velocity (v_x, v_y, v_z), and a certain acceleration (a_x, a_y, a_z).

After the definition of the focus, in order to fix the theoretical domain TD_N we need to state a suitable set of axioms. For example, we can consider the following ones.

- No object is in two different space positions at the same time. We can express it formally by saying that if ν is a bunch containing the properties $ST_{(x,y,z,t)}$ and $ST_{(x',y',z',t)}$, for $(x, y, z) \neq (x', y', z')$, then there should be $* \in I(\nu)$ such that $C(\nu, *) = \emptyset$ and $D(\nu, *) = N$, which makes it impossible for ν to belong to any conceptual object.
- The mass of an object does not depend on its space-time position. This axiom can be expressed formally by saying that if μ is a bunch containing the properties $M_{(m,x,y,z,t)}$ and $M_{(m',x',y',z',t')}$, for $m \neq m'$, then there should be $* \in I(\mu)$ such that $C(\mu, *) = \emptyset$ and $D(\mu, *) = N$.
- No object has two different velocities in the same space-time position. We can express this condition formally by stating that if a bunch μ contains the properties $V_{(v_x,v_y,v_z,x,y,z,t)}$ and $V_{(v'_x,v'_y,v'_z,x,y,z,t)}$, for $(v_x, v_y, v_z) \neq (v'_x, v'_y, v'_z)$, then there should be $* \in I(\mu)$ such that $C(\mu, *) = \emptyset$ and $D(\mu, *) = N$.
- No object has two different accelerations in the same space-time position. We can express this condition formally by stating that if a bunch μ contains the properties $A_{(a_x,a_y,a_z,x,y,z,t)}$ and $A_{(a'_x,a'_y,a'_z,x,y,z,t)}$, for

$(a_x, a_y, a_z) \neq (a'_x, a'_y, a'_z)$, then there should be $* \in I(\mu)$ such that $C(\mu, *) = \emptyset$ and $D(\mu, *) = N$.

- If an object has a space-temporal position (or a mass, a velocity, an acceleration) then it has a mass, a velocity, and an acceleration (a space-temporal position). The possible cases are many, but we can illustrate this point by showing how to formalize the fact that if an object has a space-time position, then, in that space-time position, it has a mass. To this aim let us suppose that η is a bunch containing the property $ST_{(x,y,z,t)}$; then there should be $* \in I(\eta)$ such that $C(\eta, *) = \{\eta \cup \{M_{(m,x,y,z,t)}\} \mid m \in \Re^+\}\}$ and $D(\eta, *) = N$.

- If an object has a space position (a mass, a velocity, an acceleration) at a time t, it will have a space position (a mass, a velocity, an acceleration) at any other time. Also, in this case the possible cases are many; so let us illustrate just one of them. Suppose that η is a bunch containing the property $ST_{(x,y,z,t)}$; then there should be $* \in I(\eta)$ such that $C(\eta, *) = \{\eta \cup \{ST_{(x',y',z',t')}\} \mid (x', y', z', t') \in \Re^4\}\}$ and $D(\eta, *) = N$.

Thus, after having defined the axiom set, we can construct the Newtonian conceptual objects according to the characterization Theorem 3.9 and choose the ones within the focus N. The result is that a conceptual object is given by an infinite quantity of bunches specifying its mass, position, velocity, and acceleration in all the past, present, and future instants.

It is worth observing that the set of axioms that we have just proposed can only guarantee that it is not logically impossible that in a certain instant, in some space position there is some object having a certain mass, a certain velocity, and a certain acceleration. Unfortunately, it cannot guarantee that a conceptual object really represents a concrete empirical object. On the other hand, it would be very unexpected that a purely logical theory could guarantee the existence of an entity in the empirical world. In the next section we will deal with this problem.

4.2.2 The role of the experiments We have said that there is no reason for thinking that any conceptual object represents an empirical object. For we could construct "monsters," namely, conceptual objects which do not represent any empirical object. Therefore some kind of "sanity check" has to be considered. Luckily, when we are speaking of the empirical world such a sanity check is available, and it is given by the empirical control realized by means of suitable experiments which have to be intersubjective and repeatable.

So, let us suppose that the knowing subject wants to know whether a certain conceptual object has an empirical counterpart. After the characterization Theorem 3.9, we know that the conceptual objects are directly determined by the KS's axiom set, and hence this question is clearly related to the question of whether the theoretical domain that the KS is using constitutes, that is, gives cognitive meaning to, the empirical world. Indeed, if the KS is not able to find any empirical counterpart for his or her conceptual objects, then the KS probably has to consider the idea that his or her axiom set is not correct or at least not suitable to constitute the empirical world.

Then, the first step is deciding what it means for a conceptual object α to have an empirical counterpart. To this purpose we can consider using suitable experiments in order to check whether there is some empirical object which enjoys all of the properties in all of the bunches contained in α.

There can be some practical problems here since the theoretical properties that we have to check are *precise*, for example, "to have a mass of 5 kg," while the result of an experiment is always affected by some error, for example, "the mass is included in the interval 4.9–5.1 kg"; according to general practice, we will consider *positive* the result of an experiment if the value required from the theoretical property is included in the experimental interval.

Another problem is that each experiment is related to a specific property. For an experimental apparatus does not let us control at once if all the properties composing the conceptual object have an empirical counterpart, but only if a specific property, which the experimental apparatus has been realized for, has an empirical counterpart. Then, if the result of the experiment is negative, the KS can conclude, via modus tollens, that that conceptual object does not have any empirical counterpart and, therefore, that the theoretical domain that determined that conceptual object does not constitute the empirical world. On the other hand, if the result of the experiment is positive, the KS can conclude only that, up to that point, there is no evidence against the hypothesis that the theoretical domain can constitute the world, which instead gets corroborated. Therefore, there is no reason to abandon it for an epistemically better one. Note, however, that, in general, any finite number of independent experiments will never be sufficient to know that a conceptual object as a whole has an empirical counterpart.

5   Summary

We have proposed a formalization, via constructive topology, of a neo-Kantian epistemology concerning objects. In particular, we have shown that both the mathematical and the empirical objects can be well grasped in terms of our formal explications of the notion of *construction* and *constitution*.

In order to get this result, we have started from the set of (bunches of) properties of a knowing subject, and we have realized, step by step, a formal framework inside which we have defined the central notion of *theoretical domain*, that is, a couple composed of a *focus* and a *set of axioms*. The former is nothing but the "conceptual vocabulary" that the KS decides to use when dealing with a certain aspect of the world. The latter are the basic tokens of the KS's logic that are used to construct what we have called the *conceptual objects*, thanks to the *characterization theorem*.

Of course, different theoretical domains allow for different collections of conceptual objects. In particular, we have shown theoretical domains allowing the construction of mathematical conceptual objects like natural numbers, real numbers, and functions, and a theoretical domain allowing the construction of empirical conceptual objects like Newtonian objects.

Finally, we have also observed that in the case of empirical conceptual objects there is the experimental need to control whether they really have an empirical counterpart, that is, whether they really cognitively constitute the empirical world.

Appendix: An Impredicative Solution of the Equations

We give here some hints for an impredicate solution of the two equations (1) and (2) that characterize the operator $\ltimes$ and the collection Ω of the conceptual objects, but we invite the reader to look at [22] for a more detailed description. First of all we need to define in some way an operator Pos, mapping subsets of CF into subsets of

CF, which enjoy the following conditions:

(Pos-reflexivity) $\mathsf{Pos}(F) \subseteq F$ and (Pos-transitivity) $\dfrac{\mathsf{Pos}(F) \subseteq G}{\mathsf{Pos}(F) \subseteq \mathsf{Pos}(G)}$.

For instance, Pos can be defined by using a coinductive method as in Definition 3.4, but any other approach to define it is acceptable for an impredicative solution.

Now, we can define a notion of Pos-conceptual object by stating that a subset α of CF is a Pos-conceptual object if $\varnothing \in \alpha$, $\alpha = \alpha \wedge \alpha$, and $\alpha = \mathsf{Pos}(\alpha)$. Of course, in general Pos and the collection Ω_{Pos} of the Pos-conceptual objects are not a solution of the two equations (1) and (2).

Then, provided that we use an impredicative quantification on subsets of CF, we can define an operator $\ltimes$ by setting

$$v \in \ltimes(F) \text{ if and only if } (\exists \alpha \in \Omega_{\mathsf{Pos}})\, v \in \alpha \subseteq F.$$

Thus, $\ltimes$ chooses among the couples (v, F) such that $v \in \mathsf{Pos}(F)$ holds the ones which are in fact supported by a Pos-conceptual object.

The main observation is now stated in the following theorem (for a proof, see [22]).

Theorem A.1   *The collections Ω of the $\ltimes$-conceptual objects and Ω_{Pos} of the* Pos-*conceptual objects coincide.*

Indeed, it immediately yields the following corollary.

Corollary A.2   *The predicate $\ltimes$ defined as above and the collection Ω of the $\ltimes$-conceptual objects are solutions for the equations* (1) *and* (2) *defining the conceptual objects.*

Notes

1. In fact, this could be done, but not without serious difficulties given the severe interpretative problems involved, especially regarding the transcendental deduction. Along this line of Kantian research, however, the effort to formalize the transcendental logic pursued by Achourioti and van Lambalgen [1] should be recalled.

2. For a first glance, see Laycock [13] and Rosen [16].

3. For an account on formal topology, see Sambin [17], Coquand et al. [8], Maietti and Valentini [14], Valentini [20], and Valentini [22]; for a first philosophical application of formal topology, see Boniolo and Valentini [4]; for a general constructive framework suitable for the mathematical formalization of formal topology, see Martin-Löf [15] and Sambin and Valentini [18].

4. The reader should not be too much surprised by the fact that in order to describe the set of the properties we use the set of the natural numbers that we want to represent. At the end of the day, everybody knows the famous motto ascribed to L. Kronecker, according to which, "God made the natural numbers. Everything else is men's work."

5. This problem is clearly related to the definition of the so-called *Leibniz equality*, which is language dependent.

6. The reader can look at the appendix for some hints for such an impredicative solution.

7. Technically speaking, while ⊢ is going to be inductively defined, we will see that ⋉ is going to be coinductively defined.

8. For an explanation of these conditions, see [22].

9. There is a debate concerning whether "observation" and "experiment" involve the same epistemological act. Here we do not want to dwell upon such an issue, and, for the sake of simplicity, we speak only in terms of experiments.

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