A Simple Method for Performing Type-2 Fuzzy Set Operations Based on Highest Degree of Intersection Hyperplane

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Abstract **- Regarding the three dimensional nature of type-2 fuzzy sets their related operations are costly, in terms of time and computation, which is one of the main burdens among the popularity of type-2 fuzzy sets. In this paper we will provide a novel method for performing type-2 fuzzy set operations. We will prove that the operations would be simply performed based on the hyperplane of the highest degree of intersection of two type-1 fuzzy sets.**

I. Introduction

Dr. Mendel has mentioned in [3], "*The original FL, funded by Lotfi Zadeh,*[…] *is unable to handle uncertainties. By handle, I mean to model and minimize the effect of. … The expanded FL -type-2 FL- is able to handle uncertainties because it can model them and minimize their effect*". Moreover in [4] he has demonstrated that using type-1 fuzzy sets to model words is scientifically incorrect. By the way type-2 fuzzy set theory has not received the attention it deserves from researchers, given that it provides an enhanced framework to understand and replicate the dynamics of the human decision making processes. Furthermore, type-2 fuzzy set theory is not used as it should by practitioners; perhaps this is due to the intrinsic complexity of the two dimensional space of type-1 fuzzy set theory being extended to three dimensions. However, most probably the reason lies in the fact that little effort has been spent by researchers in developing its theory, as it is demonstrated by the fact that proposed operations are not so efficient and comprehensible as they should be in order to be adopted in real-world applications.

To be sure, type-2 fuzzy sets provide more power and functionality; consequently, it is reasonable that practitioners should expect to pay for that additional power. To-date type-2 fuzzy sets are not that used in fuzzy logic systems mainly due to their underlying computational complexity [9], rather interval type-2 fuzzy sets are used. But Interval type-2 fuzzy sets may handle first-order uncertainties while general type-2 fuzzy sets can handle first and second order uncertainties [6].

In [11] we have discussed approximating binary operations on type-2 fuzzy sets based on the highest degree of separation and using type-1 intersection and algebraic difference. However in this paper we will provide algorithms that binary type-2 fuzzy set operations union and intersection on two type-2 fuzzy sets would be efficiently performed based on the hyperplane of highest degree of intersection of two type-1 fuzzy sets, given the type-2 fuzzy sets are convex and not essentially normal.

II. Preliminaries and Notations

A fuzzy set A in universe of discourse U is characterized by a membership function $\mu_A : U \to [0,1]$, and would be denoted as $A = \sum_{u\in U} \frac{\mu_A(u)}{u}$ or $A = \int_{u\in U} \frac{\mu_A(u)}{u}$ when U is discrete or continuous respectively. The *support* of fuzzy set A is defined to be $\{u \in U \mid \mu_A(u) > 0\}$ while its *height* is $Sup_{u\in U}\mu_A(u)$. If the *height* of fuzzy set A is 1, then A is a *normal* fuzzy set, otherwise it is called *subnormal*. The *core* of normal fuzzy set A is defined as $\{u \in U \mid \mu_A(u) = 1\}$. Any fuzzy set A can be represented as a (infinite) union of all of its α-level sets. The α-level set of a given fuzzy set A is defined as $A_\alpha = \{u \in U \mid \mu_A(u) \ge \alpha\}$ where $0 \le \alpha \le 1$, hence fuzzy set A would be defined as $A = \bigcup_{0\le\alpha\le1} \alpha A_\alpha$. Fuzzy set A is said to be *empty* if $Sup_{u\in U}\mu_A(u) = 0$ or equivalently $Height(A) = 0$. On the other hand, fuzzy set A is said to be *less than or equal to* – or fuzzy subset of – fuzzy set B, if $\mu_A(u) \le \mu_B(u), \forall u \in U$.

The complement of a fuzzy set A, when the universe of discourse is discrete, is defined to be $\bar{A} = \sum_{u\in U} \frac{1-\mu_A(u)}{u}$. The union and intersection of two fuzzy sets A and B, in discrete universe of discourse is define as $A \bigcup B = \sum_{u\in U} \frac{\mu_A(u) \vee \mu_B(u)}{u}$ and $A \bigcap B = \sum_{u\in U} \frac{\mu_A(u) \wedge \mu_B(u)}{u}$ respectively, such that $\vee$ must satisfy t-conorm conditions -we mainly use *Max*- and $\wedge$ must satisfy t-norm conditions -we mainly use *Min*.

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Fuzzy set A is said to be convex if all its α-level sets are convex sets or, equivalently, $\forall u_1, u_2 \in U, \lambda \in [0,1]$, $\mu_A(\lambda u_1 + (1-\lambda)u_2) \geq Min[\mu_A(u_1), \mu_A(u_2)]$.

Ordinary convex sets A and B are said to be disjoint if there exists hyperplane H such that A is on one side of H and B is on the other side. But since the condition is too restrictive in the case of fuzzy sets, Zadeh has extended the separation theorem to convex fuzzy sets in [12] as the *highest degree of separation* of two convex fuzzy sets A and B in E^n that can be achieved with a hyperplane in E^n is $1 - Sup_{u \in U} \mu_{A \cap B}(u)$ [12]; whilst their *highest degree of intersection* is $Sup_{u \in U} \mu_{A \cap B}(u)$ which can also be achieved with a hyperplane in E^n that is not necessarily equal to the highest degree of separation hyperplane.

One of the most important results in the field of fuzzy set theory is the extension principle that allows one to *fuzzify* any mathematical theory. In brief, let f be a mapping from U to V and A be a fuzzy subset of U defined as $A = \sum_{u \in U} \frac{\mu_A(u)}{u}$. The extension principle states that $f(A) = \sum_{u \in U} \frac{\underset{v \in f^{-1}(u)}{Sup} \mu_A(v)}{f(u)}$, which is a fuzzy subset in V.

III. TYPE-2 FUZZY SET

Zadeh in [13] has defined a fuzzy set to be of type-n, n=2,3,…,n if its membership function ranges over fuzzy set of type n-1, however the membership function of type-1 ranges over [0, 1]. Consequently, a fuzzy set of type-2 $\tilde{A}$ in a set U is the fuzzy set which is characterized by a fuzzy membership function $\mu_{\tilde{A}} : U \rightarrow [0,1]^J$, with the value $\mu_{\tilde{A}}(u)$ being called a fuzzy grade and being a fuzzy set in [0,1](or in the subset J of [0,1]) [10].

Formally, a discrete type-2 fuzzy set $\tilde{A}$ can be denoted extensively as $\tilde{A} = \sum_{u \in U} \mu_{\tilde{A}}(u)/u = \sum_{u \in U} \left[\sum_{\mu_i^{(u)} \in J_u} \frac{S_i^{(u)}}{\mu_i^{(u)}} \right] \Big/ u$, where $J_u \subseteq [0,1]$ and $0 \leq S_i^{(u)} \leq 1$ consistent with the fact that $\mu_{\tilde{A}}(u)$ is a fuzzy membership grade. In this notation $\mu_i^{(u)}$ is the i^th^ membership value of u with the strength $S_i^{(u)}$. $\sum_{\mu_i^{(u)} \in J_u} \mu_i^{(u)}$ constitutes the primary membership and $\sum_{\mu_i^{(u)} \in J_u} \frac{S_i^{(u)}}{\mu_i^{(u)}}$ denotes the fuzzy grade or secondary membership function of the element *u*. The amplitude of the secondary membership function is also called secondary grade. In other words, $S_i^{(u)}$ is the strength of the i$^{\text{th}}$ membership value of *u*, i.e. $\mu_i^{(u)}$. Clearly, $\left[\sum_{\mu_i^{(u)} \in J_u} \frac{S_i^{(u)}}{\mu_i^{(u)}} \right]$ for any given $u \in U$ is a special type-1 fuzzy set that describes the imprecise membership value –fuzzy grade- of u in $\tilde{A}$.

The set $\{ \sum_{u \in U} \sum_{i \in J_u} \mu_i^{(u)} / u \quad , \quad S_i^{(u)} > 0 \}$ is called *Domain of Uncertainty* (DOU) [7,9]. The DOU identifies the region of primary membership degrees but it does not say anything about the strength of each membership degree. However if the type-2 fuzzy set is continuous with naturally ordered primary variable then the region is called *Footprint of Uncertainty* (FOU). However we use DOU and FOU interchangeably. Before continuing, we observe that type-1 fuzzy sets are a special case of type-2 fuzzy sets, where, for all $u \in U$, the set of primary membership degrees, namely $\sum_{i \in J_u} \mu_i^{(u)}$, is a singleton with the strength of unity.

Example 1: Given $U = \{0,1,2,...,10\}$, fuzzy set $\tilde{A}$ in *U,* defined as below is of type-2.

$$
\begin{aligned}
\tilde{A} = & \left[1/0 + .9/.1 + .8/.2 \right] / 1 + \\
& \left[1/0 + .8/.1 + .6/.2 + .4/.3 + .2/.4 \right] / 2 + \\
& \left[.3/.2 + .6/.3 + .8/.4 + 1/.5 + .8/.6 + .6/.7 + .3/.8 \right] / 3 + \\
& \left[.2/.6 + .4/.7 + .6/.8 + .8/.9 + 1/1 \right] / 4 + \\
& \left[.2/.8 + .6/.9 + 1/1 \right] / 5 + \\
& \left[.2/.6 + .4/.7 + .6/.8 + .8/.9 + 1/1 \right] / 6 + \\
& \left[.3/.2 + .6/.3 + .8/.4 + 1/.5 + .8/.6 + .6/.7 + .3/.8 \right] / 7 + \\
& \left[1/0 + .8/.1 + .6/.2 + .4/.3 + .2/.4 \right] / 8 + \\
& \left[1/0 + .9/.1 + .8/.2 \right] / 9
\end{aligned}
$$

Table 1 shows the type-2 fuzzy set $\tilde{A}$ more clearly and fig. 1 shows the three dimensional shape of $\tilde{A}$ - $\mu_{\tilde{A}}(u)$ in the example are discrete but for the sake of simplicity are shown as continuous type-1 fuzzy sets. Be aware that 0 and 10 are not member of $\tilde{A}$ i.e. their membership grade in $\tilde{A}$ is $\left[1/0 \right]$. The shaded cells of the Table 1 constitute the FOU. For instance, primary membership of $u = 6$ is $J_6 = \{0.6, 0.7, 0.8, 0.9, 1\}$ while its secondary membership function is $\left[.2/.6 + .4/.7 + .6/.8 + .8/.9 + 1/1 \right]$. At each $u = u'$ the two dimensional plane with the axes $\mu^{(u)}$ and $S^{(u)}$ is called a vertical slice hence, $\tilde{A}$ has nine vertical slices. ■

$\mu_{\tilde{A}}(u)$											
1	0	0	0	0	1	1	1	0	0	0	0
0.9	0	0	0	0	0.8	0.6	0.8	0	0	0	0
0.8	0	0	0	0.3	0.6	0.2	0.6	0.3	0	0	0
0.7	0	0	0	0.6	0.4	0	0.4	0.6	0	0	0
0.6	0	0	0	0.8	0.2	0	0.2	0.8	0	0	0
0.5	0	0	0	1	0	0	0	1	0	0	0
0.4	0	0	0.2	0.8	0	0	0	0.8	0.2	0	0
0.3	0	0	0.4	0.6	0	0	0	0.6	0.4	0	0
0.2	0	0.8	0.6	0.3	0	0	0	0.3	0.6	0.8	0
0.1	0	0.9	0.8	0	0	0	0	0	0.8	0.9	0
0	1	1	1	0	0	0	0	0	1	1	1
	0	1	2	3	4	5	6	7	8	9	10

Table 1: Type-2 fuzzy set of Example 1 shown in a table.

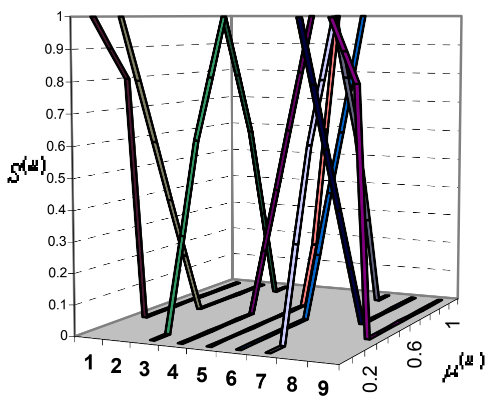

Fig.1: Three-dimensional shape of the type-2 fuzzy set of the Example 1

III. Type-2 Fuzzy Set Operations

Given two type-2 fuzzy sets $\tilde{A}$ and $\tilde{B}$ be defined in U as $\tilde{A} = \sum_{x\in X} \mu_{\tilde{A}}(x)/x$ and $\tilde{B} = \sum_{x\in X} \mu_{\tilde{B}}(x)/x$ where $\mu_{\tilde{A}}(x) = \sum_{u\in J_x^A} f_x(u)/u$ and $\mu_{\tilde{B}}(x) = \sum_{w\in J_x^B} g_x(w)/w$. Based on Zadeh's extension principle the membership grades for union and intersection of $\tilde{A}$ and $\tilde{B}$ would be:

$$\mu_{\tilde{A}\cup\tilde{B}}(x) = \sum_{u\in J_x^A}\sum_{w\in J_x^B} (f_x(u)\wedge g_x(w))/(u\vee w) \equiv \mu_{\tilde{A}}(x) \sqcup \mu_{\tilde{B}}(x)$$

$$\mu_{\tilde{A}\cap\tilde{B}}(x) = \sum_{u\in J_x^A}\sum_{w\in J_x^B} (f_x(u)\wedge g_x(w))/(u\wedge w) \equiv \mu_{\tilde{A}}(x) \sqcap \mu_{\tilde{B}}(x)$$

where $\sqcup$ denotes join and $\sqcap$ indicated meet [8] moreover throughout this paper, $\vee$ represent max t-conorm and $\wedge$ indicated min t-norm. Be reminded that $\mu_{\tilde{A}}(x)$ and $\mu_{\tilde{B}}(x)$ are type-1 fuzzy sets. It is clear that these operations are too costly and our goal is to provide efficient algorithms to compute them.

Theorem 1: Given $F_1 = \sum_{u\in U} f(u)/u$ and $F_2 = \sum_{w\in U} g(w)/w$ be two convex fuzzy sets defined over U such that their highest degree of intersection is attained at H i.e. $\underset{x}{Sup}(f(x)\wedge g(x)) = M$ attained at $x = H$, and $\underset{x}{Sup}(f(x)) = h_f$ attained at $x = v_f$, $\underset{x}{Sup}(g(x)) = h_g$ attained at $x = v_g$, then H is between v_f and v_g i.e., either $v_g \le H \le v_f$ or $v_f \le H \le v_g$.

Proof: Suppose both v_f and v_g are on the same side of H i.e. $v_g < H$, $v_f < H$ and without loss of generality imagine $v_f \le v_g$ and $H = \underset{f(\theta)\wedge g(\theta)=M}{Inf(\theta)}$. We can say $\exists\, \lambda, \lambda' \in [0,1] \mid \lambda v_f + (1-\lambda)H = \lambda' v_g + (1-\lambda')H = \theta \in (v_g, H)$ so $f(\theta) \ge Min(f(v_f), f(H)) \to f(\theta) \ge M$ and $g(\theta) \ge Min(g(v_g), g(H)) \to g(\theta) \ge M$. This means that $f(\theta)\wedge g(\theta) \ge M$ which is contradiction. So we conclude that H is located between v_f and v_g i.e. either $v_g \le H \le v_f$ or $v_f \le H \le v_g$. ■

Theorem 2: Given F_1 and F_2 be defined as in theorem 1 and $v_f \le H \le v_g$ then $\forall \theta, v_f \le \theta < H, g(\theta) \le f(\theta)$, in other words $g(\theta)\wedge f(\theta) = g(\theta),\ v_f \le \theta < H$.

Proof: Given $\exists \theta'$, $v_f \le \theta' < H$, such that $g(\theta') > f(\theta')$. We can say
$\exists \lambda \in [0,1]$ $s.t.$ $\lambda v_f + (1-\lambda)H = \theta' \to f(\theta') \ge Min(f(v_f), f(H))$
Since $f(H) \ge M$ and $f(v_f) = h_f$ so $Min(f(v_f), f(H)) \ge M \to f(\theta') \ge M$. We assumed that $g(\theta') > f(\theta')$ so $g(\theta')\wedge f(\theta') \ge M$ which is contradiction. ■

Theorem 3: Given F_1 and F_2 be defined as in theorem 1 and $v_f \le H \le v_g$ then $\forall \theta, H < \theta \le v_g, g(\theta) \ge f(\theta)$, in other words $g(\theta)\vee f(\theta) = g(\theta),\ H < \theta \le v_f$.

Proof: Given $\exists \theta'$, $H < \theta' \le v_g$, such that $g(\theta') < f(\theta')$. So $\exists \lambda \in [0,1] \mid \lambda v_g + \bar{\lambda} H = \theta' \to g(\theta') \ge Min(g(v_g), g(H))$.
Since $g(H) \ge M$ and $g(v_g) = h_g \ge M$ then $Min(g(v_g), g(H)) \ge M \to g(\theta') \ge M$. Since $g(\theta') < f(\theta')$ then $g(\theta')\wedge f(\theta') \ge M$ which is contradiction to the fact that highest degree of intersection is attained at H. ■

Theorem 4: Given F_1 and F_2 be defined as in theorem 1, then using max t-conorm and min t-norm, the join of F_1 and F_2 is

$$\mu_{F_1\sqcup F_2}(\theta) = \begin{cases} f(\theta)\wedge g(\theta) & \theta \le H \\ (f(\theta)\vee g(\theta))\wedge(h_f \wedge h_g) & \theta > H \end{cases}$$

Proof: Regarding Theorem1, H is located between v_f and v_g. However without loss of generality we imagine that $v_f \le H \le v_g$. Regarding the extension principle, join of F_1 and F_2 is

$$F_1 \sqcup F_2 = \sum_{u,w} \frac{f(u) \wedge g(w)}{u \vee w} \quad \rightarrow$$

$$\mu_{F_1 \sqcup F_2}(\theta) = \underset{u \vee w = \theta}{Sup}(f(u) \wedge g(w)) = \phi(\theta)$$

We can rewrite $\phi(\theta)$ as follows:

$$\phi(\theta) = \begin{cases} \underset{w \le \theta, u=\theta}{Sup}\ (f(\theta) \wedge g(w)) = f(\theta) \wedge \underset{w \le \theta}{Sup}(g(w)) = \phi_1(\theta) \\ \vee \\ \underset{u \le \theta, w=\theta}{Sup}\ (f(u) \wedge g(\theta)) = g(\theta) \wedge \underset{u \le \theta}{Sup}(f(u)) = \phi_2(\theta) \end{cases}$$

That is

$$\phi(\theta) = \phi_1(\theta) \vee \phi_2(\theta)$$

Case I: $\theta \le H$ where $g(w)$ is non-decreasing. So

$$\phi_1(\theta) = f(\theta) \wedge \underset{w \le \theta}{Sup}(g(w)) = f(\theta) \wedge g(\theta)$$

$$\phi_2(\theta) = \underset{u \le \theta}{Sup}(f(u) \wedge g(\theta)) = g(\theta) \wedge \underset{u \le \theta}{Sup}(f(u))$$

$$= \begin{cases} g(\theta) \wedge f(\theta) & \theta < v_f \\ g(\theta) \wedge h_f & v_f \le \theta \end{cases}$$

Based on the results of Theorem 2, we may rewrite $\phi_2(\theta)$ as:

$$\phi_2(\theta) = \begin{cases} g(\theta) \wedge f(\theta) & \theta < v_f \\ g(\theta) \wedge h_f = (g(\theta) \wedge f(\theta)) \wedge h_f & v_f \le \theta \end{cases}$$

$$= \begin{cases} g(\theta) \wedge f(\theta) & \theta < v_f \\ g(\theta) \wedge (f(\theta) \wedge h_f) = g(\theta) \wedge f(\theta) & v_f \le \theta \end{cases}$$

Consequently we have

$$\phi(\theta) = \phi_1(\theta) \wedge \phi_2(\theta) = g(\theta) \wedge f(\theta)\ ,\ \theta \le H.$$

Case II: $\theta > H$ where $f(u)$ is non-increasing. So

$$\phi_1(\theta) = f(\theta) \wedge \underset{w \le \theta}{Sup}(g(w))$$

$$\phi_2(\theta) = g(\theta) \wedge \underset{u \le \theta}{Sup}(f(u)) = g(\theta) \wedge h_f$$

$$\phi(\theta) = \phi_1(\theta) \vee \phi_2(\theta) = (f(\theta) \wedge \underset{w \le \theta}{Sup}(g(w))) \vee (g(\theta) \wedge h_f)$$

$$= [(f(\theta) \wedge \underset{w \le \theta}{Sup}(g(w))) \vee g(\theta)] \wedge [(f(\theta) \wedge \underset{w \le \theta}{Sup}(g(w))) \vee h_f]$$

$$= (f(\theta) \vee g(\theta)) \wedge (\underset{w \le \theta}{Sup}(g(w)) \vee g(\theta)) \wedge$$

$$(f(\theta) \vee h_f) \wedge (\underset{w \le \theta}{Sup}(g(w)) \vee h_f)$$

$$= (f(\theta) \vee g(\theta)) \wedge (\underset{w \le \theta}{Sup}(g(w))) \wedge (h_f) \wedge (\underset{w \le \theta}{Sup}(g(w)) \vee h_f)$$

Applying absorption law, we have

$$\phi(\theta) = (f(\theta) \vee g(\theta)) \wedge (\underset{w \le \theta}{Sup}(g(w)) \wedge h_f)$$

$$= \begin{cases} (g(\theta) \vee f(\theta)) \wedge (h_f \wedge g(\theta)) = h_f \wedge g(\theta) & \theta < v_g \\ (g(\theta) \vee f(\theta)) \wedge (h_f \wedge h_g) & v_g \le \theta \end{cases}$$

Regarding Theorem 3 and the fact that $\forall \theta, g(\theta) \le h_g$ we can rewrite the above equation as:

$$\phi(\theta) = \begin{cases} g(\theta) \wedge h_f = g(\theta) \wedge h_f \wedge h_g & \theta < v_g \\ (g(\theta) \vee f(\theta)) \wedge (h_f \wedge h_g) & v_g \le \theta \end{cases}$$

$$= \begin{cases} (g(\theta) \vee f(\theta)) \wedge (h_f \wedge h_g) & \theta < v_g \\ (g(\theta) \vee f(\theta)) \wedge (h_f \wedge h_g) & v_g \le \theta \end{cases}$$

And hence we have

$$\phi(\theta) = (g(\theta) \vee f(\theta)) \wedge (h_f \wedge h_g)\ ,\ \theta > H.$$

Combining results obtained in cases I and II we will get

$$\mu_{F_1 \sqcup F_2}(\theta) = \begin{cases} f(\theta) \wedge g(\theta) & \theta \le H \\ (f(\theta) \vee g(\theta)) \wedge (h_f \wedge h_g) & \theta > H \end{cases}$$

It can be similarly proved that if $v_g \le H \le v_f$ then the theorem is true. ■

Theorem 5: Given F_1 and F_2 be defined as in theorem 1, then using max t-conorm and min t-norm, the meet of F_1 and F_2 is

$$\mu_{F_1 \sqcap F_2}(\theta) = \begin{cases} (f(\theta) \vee g(\theta)) \wedge (h_f \wedge h_g) & \theta \le H \\ f(\theta) \wedge g(\theta) & \theta > H \end{cases}$$

Proof: The proof is simple and similar to the proof of the theorem 4. ■

Figures 2 and 3, show two examples of an application of the theorems 4 and 5 while $M < Min(h_f, h_g)$ which rules out $F_1 \subset F_2$ or $F_2 \subset F_1$ and $M = Min(h_f, h_g)$ respectively.

IV. CONCLUSION

One of the difficulties with classical set theory, that fuzzy set theory tries to overcome, is to determine whether an element is a member of a given set or not. Fuzzy set theory "provides a natural way of dealing with problems in which the source of imprecision is the absence of sharply defined criteria of class membership"[12]. Thus fuzzy set theory tries to alleviate the problem by associating a degree of membership to the element. However, that leads to the question how membership degrees should be defined. Therefore, the problem is reduced to determining a membership degree, which is a *crisp* (i.e., exact) number, for all those cases in which, the extent to which, a given element belongs in a set would be uncertain. To solve this paradox [5], in 1975, Zadeh enhanced his theory by introducing type-2 fuzzy sets. But regarding the intrinsic complexity of type-2 fuzzy sets, the theory is not that appreciated by practitioners.

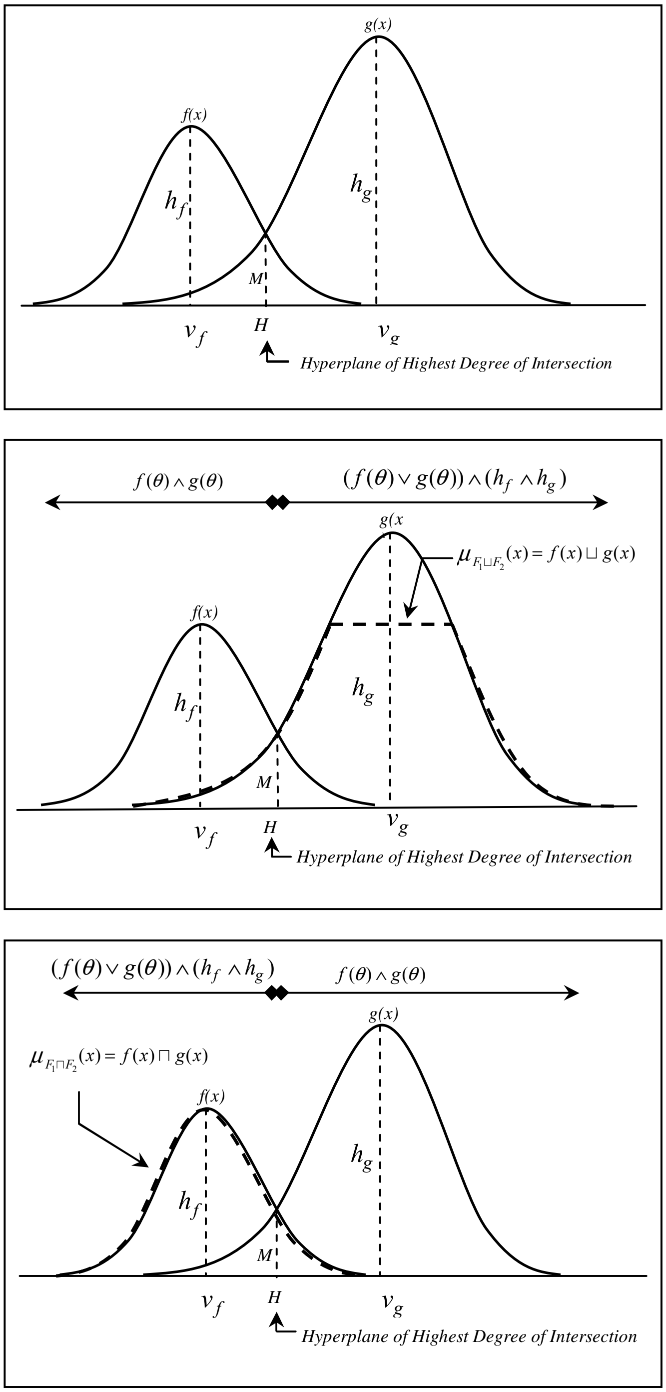

Fig.2 : An illustration of the theorems 4 and 5 when $M < Min(h_f, h_g)$. Results are shown in thick-dashed line.

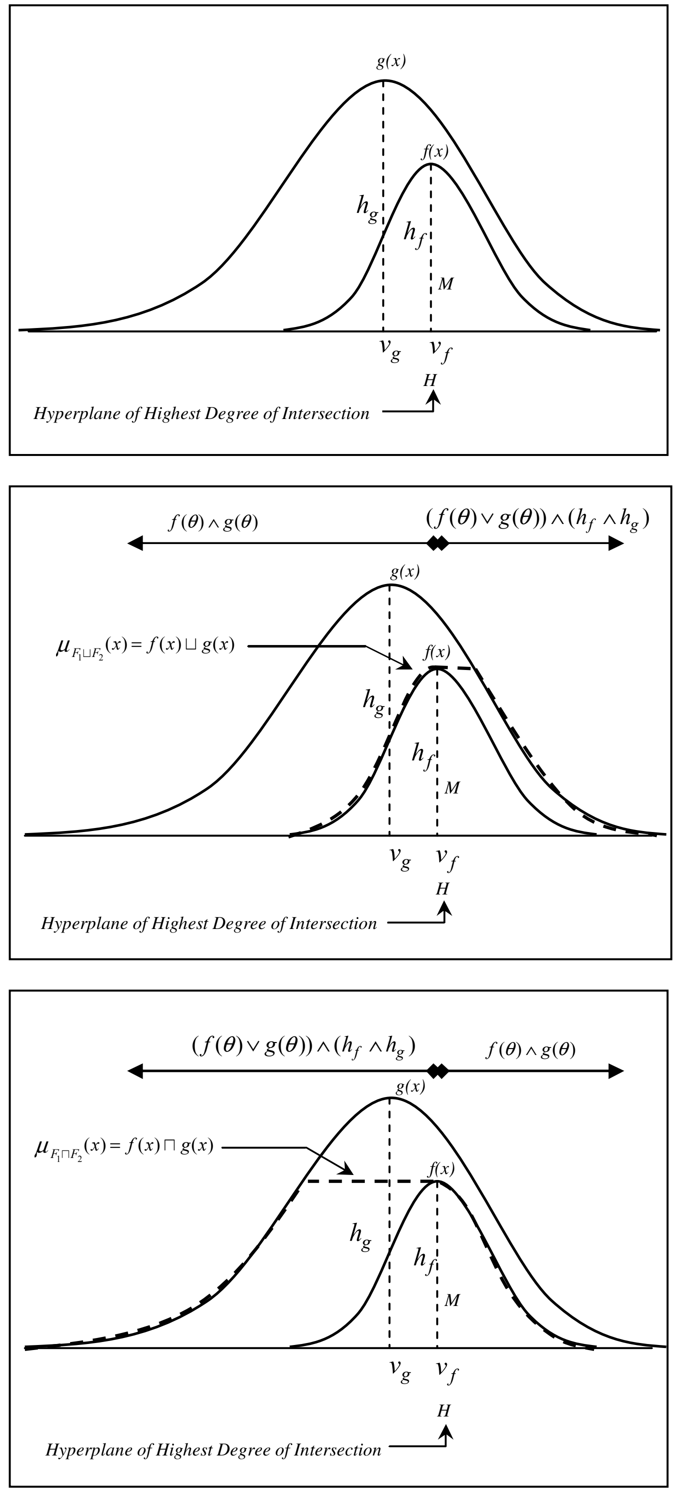

Fig.3 : An illustration of the theorems 4 and 5 when $M = Min(h_f, h_g)$. Results are shown in thick-dashed line.

In this paper first, we simply introduced the concept of type-2 fuzzy sets and by showing the (discrete) type-2 fuzzy sets in a table format the exclusive terms of type-2 fuzzy sets are made easier to understand. Then we proved that type-2 fuzzy set operations would be more efficiently calculated based on the hyperplane of highest degree of intersection of two type-1 fuzzy sets. The only restriction was the convexity of the participating type-1 fuzzy sets. Comparing the proposed algorithm with those proposed by Karnik and Mendel in [2] and Duboid and Prade [1] reveals that our algorithm is also a one pass algorithm but relies only one decision point which is at the hyperplane of highest degree of the intersection of the two participant type-1 fuzzy set. Moreover, for the case of convex fuzzy sets this method is even more efficient than the approximated method introduced in [11].

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