Operations on Concavoconvex Type-2 Fuzzy Sets

Hooman Tahayori and Giovanni Degli Antoni

Abstract

Concavoconvex fuzzy set is the result of the combination of the concepts of convex and concave fuzzy sets. This paper investigates concavoconvex type-2 fuzzy sets. Basic operations, union, intersection and complement on concavoconvex type-2 fuzzy sets using min and product t-norm and max t-conorm are studied and some of their algebraic properties are explored.

Keywords: Concave fuzzy set, Convex fuzzy set, Concavoconvex fuzzy set, Type-2 fuzzy set, Join, Meet.

1. Introduction

Zadeh in [41] proposed the concept of type-2 fuzzy sets as an extension to the ordinary fuzzy sets (type-1 fuzzy sets) after examining the association between the concept of linguistic truth with truth-values and fuzzy sets with linguistic grades of membership. It is argued that type-2 fuzzy sets are specifically useful in circumstances where determining exact membership function for a fuzzy set is difficult; it also permits modeling and minimizing the effects of uncertainties in rule based fuzzy logic systems [13,22,27,37].

Once the membership function of a type-1 fuzzy set is determined, which fully describes its underlying fuzzy set, then it is certain and does not convey uncertainty, however the term fuzzy has the connotation of uncertainty. This is counter-intuitive, i.e. how is it possible to represent something that is uncertain with something that is certain or, more specifically, how is it possible to determine an exact membership function of a fuzzy set that is to represent an uncertain concept [22]. This is recognized by many researchers. Klir and Folger in [17] have mentioned, “The accuracy of any MF is necessarily limited. In addition, it may seem problematical, if not paradoxical, that a representation of fuzziness is made using membership grades that are themselves precise real numbers.” Pedrycz in [32] has indicated, “… a membership grade indicates an extent to which a given point in the universe of discourse belongs to a concept we are about to represent. Once the membership function has been established (estimated or defined), the concept is described very precisely as the membership values are exact numerical quantities. This seems to raise a certain dilemma of excessive precision in describing imprecise phenomena. In fact, this concern has already sparked a lot of debates starting from the very inception of fuzzy sets.”

In reality, there are situations in which the grade of membership itself is frequently ill-defined [29] and may not be determined precisely. This can be explained by the fact that on one hand a crisp value, as a result of a measurement, is not a suitable representative for a membership value [18] and, on the other hand, many researchers believe that assigning an exact number to an experts’ opinion is too restrictive [10]. Type-2 fuzzy set enable capturing the uncertainty on membership functions of fuzzy sets through relating one or more crisp numbers as membership values to an entity and that with not necessarily equal strengths. This will introduce the third dimension in type-2 fuzzy sets. Although once the membership function of a type-2 fuzzy set is chosen it is totally precise, the additional dimension of type-2 fuzzy sets provides a further degree of freedom in handling uncertainties in membership degrees. Of course, this in turn may raise debates on the deficiencies of type-2 fuzzy sets and promote the use of higher-order fuzzy sets and eventually lead to the idea of type-∞ fuzzy set, which is not practically possible. We have to stick to a finite-type fuzzy set; the type-2 fuzzy set constitutes a sensible trade-off between computational complexity and ability to handle uncertainty. Mendel [22] argued: “The original F[uzzy]L[ogic], funded by Lotfi Zadeh … is unable to handle uncertainties. By handle, I mean to model and minimize the effect of. … The expanded FL—type-2 FL—is able to handle uncertainties because it can model them and minimize their effect.” John and Coupland [12] further noted that “fuzzy logic, as it is commonly used, is essentially precise in nature and that for many applications it is unable to model knowledge from an expert adequately. We argued that the modeling of imprecision can be enhanced by the use of type-2 fuzzy sets – providing a higher level of imprecision. ... The use of type-2 fuzzy sets allows for a better representation of uncertainty and imprecision in particular applications and domains. The more imprecise or vague the data are, then type-2 fuzzy sets offer a significant

Corresponding Author: Hooman Tahayori, Universita degli Studi di Milano, Dipartimento di Scienze dell'Informazione,Via Comelico 39/41, 20135, Milano, Italy.
E-mail: hooman.tahayori@unimi.it

improvement on type-1 fuzzy sets."

Before 1990s, only a few researches were carried on type-2 fuzzy sets. Amongst the first endeavors on type-2 fuzzy sets, Mizumoto and Tanaka in [29,30] investigated some algebraic properties of fuzzy grades under various algebraic and fuzzy operations. Join and meet operations on fuzzy numbers using minimum t-norm are studied in [9,16]. Nieminen [31] investigated in more detail the type-2 fuzzy set algebraic structures. In [7,8,9] fuzzy valued logic is discussed and a formula for the composition of type-2 fuzzy relations for minimum t-norm is also proposed. A general algorithm for the extended sup-star composition of type-2 fuzzy relations later was proposed in [15] where using the algorithm, type-2 fuzzy logic system theory was extensively discussed. Practical algorithms for calculating union, intersection and complement of convex normal type-2 fuzzy sets are studied in [13]. Being inspired by computational geometry, Coupland and John in [4-6] have proposed a method for performing various type-2 fuzzy set operations specially join and meet. The concept of the centroid of a type-2 fuzzy set and the related algorithms are introduced and discussed in [14]. Reference [35] has discussed mathematical treatment of algebras of fuzzy truth values and [36] has studied the extension of ordinary t-norms on the algebra of truth values of type-2 fuzzy sets. In addition, many researches including [1,19,28,34] have been done in the field of interval type-2 fuzzy sets. A comprehensive list of publication on the subject could be found in (www.Type2fuzzylogic.org) however, Mendel's book, [22] accompanied with [24] also provide valuable bibliographies in the area.

The main bottleneck of type-2 fuzzy sets backs to the computational complexity of exploiting type-2 fuzzy set operations [27]. There has been some researches in this regard that are mainly focused on convex type-2 fuzzy sets, e.g. [6,13]. This paper, however, elaborates on providing simple and efficient algorithms for calculating union and intersection of concavoconvex type-2 fuzzy sets which actually reduces to the calculation of join and meet of concavoconvex fuzzy grades using min and product t-norm and max t-conorm. We have also explored some algebraic properties of such type-2 fuzzy sets. Concavoconvex fuzzy set proposed by Sarkar in [33] is the result of the combination of convex and concave fuzzy sets. Concavoconvex fuzzy sets are argued to be extensively used in modeling linguistic modifiers like "true", "very true", "more or less true", "false", "very false", "more or less false" and so on [11]. Recently fuzzy quantifiers have known to play basic roles in uncertain system modeling and computing with words theory [21]. However, the basis of all relative quantifiers is said to be regular increasing monotone (RIM) quantifiers [38,39] e.g. "at least half", "more than 0.4", etc., which, in specific, are very important in many fields like decision analysis, database query, and computing with words theory [20]. Such modifiers and quantifiers are analogous to the fuzzy grades in concavoconvex type-2 fuzzy sets.

The paper is organized as follows. Section 2 provides basic notions of type-2 fuzzy sets. The concept of concavoconvex fuzzy sets will be reviewed in the section 3 where also some related properties are studied. Algorithms for performing union, intersection and complement on concavoconvex type-2 fuzzy sets under min and product t-norm and max t-conorm are explored in section 4, where also some algebraic properties of concavoconvex fuzzy grades under join, meet and negation are discussed. Section 5 concludes the paper.

2. Type-2 Fuzzy Set Notions

Type-2 fuzzy set is a fuzzy set with fuzzy membership function; "A fuzzy set is of type n, n=2,3,…,n if its membership function ranges over fuzzy sets of type n-1. The membership function of a fuzzy set of type 1 ranges over the interval [0,1]" [41]. Putting in more formal form, a fuzzy set of type-2 $\tilde{A}$ in a universe of discourse X is the fuzzy set which is characterized by a fuzzy membership function $\mu_{\tilde{A}} : X \to [0,1]^J$, with $\mu_{\tilde{A}}(x)$ being a fuzzy set in [0,1] (or in the subset J of [0,1]) denoting the fuzzy grade of membership of x in $\tilde{A}$ [29]. However, we adopt the notions and term set used in [26,27], i.e.,

$$\tilde{A} = \int_{x\in X} \int_{u\in J_x} \mu_{\tilde{A}}(x,u)/(x,u) \;\left(0 \le \mu_{\tilde{A}}(x,u) \le 1 \,,\, u \in J_x \subseteq [0,1]\right)$$

$$= \int_{x\in X} \left[\int_{u\in J_x} f_x(u)/u \right] / x \quad \left(f_x(u) \in [0,1] \,,\, u \in J_x \subseteq [0,1]\right) \quad (1)$$

denotes a type-2 fuzzy set $\tilde{A}$ over X. Here, x is *primary variable* and J_x represents the *primary membership* of x. In this regard $\mu_{\tilde{A}}(x) \triangleq \left[\int_{u\in J_x} f_x(u)/u \right]$ which is a type-1 fuzzy set that denotes the fuzzy grade of membership of x in $\tilde{A}$, is called *secondary membership function* or *secondary set,* however throughout the paper we simply refer to it as *fuzzy grade*; $f_x(u)$ is named *secondary grade*. A comprehensive notion of type-2 fuzzy set and the philosophy behind it would be found in the literature for example in [15, 22-27].

To perform operations such as complementation, union and intersection on type-2 fuzzy sets, naturally, extension principle takes place [41]. As defined by Zadeh in [41], extending the binary operation * defined in U on two type-1 fuzzy sets $A = \int_{u\in U} f(u)/u$ and

$B = \int_{w \in U} g(w)/w$ — that will result in a new type-1 fuzzy set $C = \int_{v \in U} h(v)/v$ — would be,

$$A * B = \left(\int_{u \in U} f(u)/u \right) * \left(\int_{w \in U} g(w)/w \right) \\ = \int_{u,w \in U} (f(u) \bigstar g(w))/(u * w) = \int_{v \in U} h(v)/v = C \tag{2}$$

where equivalently

$$h(v) = \underset{u*w=v}{Sup} \, (f(u) \bigstar g(w)) \tag{3}$$

that $\bigstar$ denotes a t-norm.

It must be noticed that a type-2 fuzzy set is concavoconvex, given all of its fuzzy grades i.e. secondary membership functions, are concavoconvex type-1 fuzzy sets. Being normal, also follows the same discipline.

In what preceded, all notions were with regard to the continuous domain. However the similar discussion is valid for discrete domain, wherein, $\int$ would be replaced by Σ. For example, in a fully discrete domain, type-2 fuzzy set $\tilde{A}$ would be represented as

$$\tilde{A} = \sum_{x \in X} \sum_{u \in J_x} \mu_{\tilde{A}}(x,u) / (x,u) = \sum_{x \in X} \left[\sum_{u \in J_x} f_x(u)/u \right] / x \, , \\ 0 \le \mu_{\tilde{A}}(x,u) \le 1 , \quad f_x(u) \in [0,1] \quad , \quad u \in J_x \subseteq [0,1] \tag{4}$$

3. Concavoconvex Fuzzy Set

The idea of concavoconvex fuzzy set is based on the combination of the concepts of convex fuzzy sets with concave fuzzy sets. Convex fuzzy sets introduced by Zadeh in his seminal paper [40], are well studied by researchers. Formally, a fuzzy set $F = \int_{u \in U} f(u)/u$ is defined to be convex [40], if

$$\forall u_1, u_2 \in U \, , \forall \lambda \in [0,1] \, , \\ f(\lambda u_1 + (1-\lambda) u_2) \ge Min\big(f(u_1), f(u_2) \big) \tag{5}$$

On the other hand, concave fuzzy set, as a complementary concept to convex fuzzy set was first discussed in [2] and [3]. Chaudhuri in [3] has proposed some related concepts such as concave hull, concave containment and concavity tree. He has argued that concave fuzzy sets would be used for decomposing or approximating fuzzy sets and for developing fuzzy geometry of space. A fuzzy set $F = \int_{u \in U} f(u)/u$ is defined to be concave [3], if

$$\forall u_1, u_2 \in U \, , \ \forall \lambda \in [0,1] \, , \text{ then} \\ f(\lambda u_1 + (1-\lambda) u_2) \le Max\big(f(u_1), f(u_2) \big) \tag{6}$$

Although Chaudhuri in [3] had mentioned that there exist fuzzy sets which are both convex and concave, Sarkar in [33] formally defined concavoconvex fuzzy sets. $F = \int_{u \in U} f(u)/u$ is concavoconvex fuzzy set, if

$$\forall u_1, u_2 \in U \, , \ \forall \lambda \in [0,1] \, , \text{ then} \\ Min\big(f(u_1), f(u_2) \big) \le f(\lambda u_1 + (1-\lambda) u_2) \le Max\big(f(u_1), f(u_2) \big) \tag{7}$$

In [33] it is proved that if F is a concavoconvex fuzzy set, then its complement, $\bar{F}$, is also a concavoconvex fuzzy set. It is also shown - Theorem 1 - that if the characteristic function of a fuzzy set is monotonic, then the fuzzy set is concavoconvex. Moreover the membership functions of all concavoconvex fuzzy sets are monotone functions. However, it can be easily deduced that if the characteristic function of a concavoconvex fuzzy set is monotonically increasing then the characteristic function of its complement is monotonically decreasing and vice versa. We assume that concavoconvex fuzzy sets are defined on a closed interval of $U = [u^-, u^+]$ of $\mathbb{R}$, in which they attain their height and plinth.

Theorem 1 [33]: Let $F = \int_{u \in U} f(u)/u$ be a concavoconvex fuzzy set, then f is a monotone function and vice versa.

Theorem 2 [33]: Let $F = \int_{u \in U} f(u)/u$ and $G = \int_{w \in U} g(w)/w$ be concavoconvex fuzzy sets, then their union, $F \bigcup G$, using max t-conorm, is a concavoconvex or concave fuzzy set.

Theorem 3 [33]: Let $F = \int_{u \in U} f(u)/u$ and $G = \int_{w \in U} g(w)/w$ be concavoconvex fuzzy sets, then their intersection, $F \bigcap G$, using min t-norm, is a concavoconvex or convex fuzzy set.

Corollary 1: Let $F = \int_{u \in U} f(u)/u$ be a concavoconvex fuzzy set, then $F \bigcup \bar{F}$ is a concave fuzzy set and $F \bigcap \bar{F}$ is a convex fuzzy set.

Corollary 2: Let $F = \int_{u \in U} f(u)/u$ be a concavoconvex fuzzy set and $G = \int_{w \in U} g(w)/w$ be a convex fuzzy set, then $F \bigcap G$, using min t-norm, is a convex fuzzy set.

Corollary 3: Let $F = \int_{u \in U} f(u)/u$ be a concavoconvex fuzzy set and $G = \int_{w \in U} g(w)/w$ be a concave fuzzy set, then $F \bigcup G$, using max t-conorm, is a concave fuzzy set.

Theorem 4: Let $F = \int_{u \in U} f(u)/u$ and $G = \int_{w \in U} g(w)/w$ be respectively increasing and decreasing concavoconvex fuzzy sets defined in the U. There always exists a

$v \in U$ such that $\forall u < v$, $f(u) \le g(u)$ and $\forall u > v$, $f(u) \ge g(u)$.

Proof: With respect to the property of bounded fuzzy sets and the separation theorem of fuzzy sets [40], given $M = \underset{u \in U}{Sup}\left(Min(f(u), g(u))\right)$, there exists a $v \in U$ such that $f(u) \le M$ for all u located on one side of v and $g(u) \le M$ for all u on the other side of v. Since F and G are increasing and decreasing concavoconvex fuzzy sets respectively, so $\forall u < v$, $f(u) \le M$ and $\forall u' > v$ $g(u') \le M$, moreover since F and G are concavoconvex, $\forall u < v, u' > v$, $Min(f(u), f(u')) \le f(v) \le Max(f(u), f(u'))$ that is $f(u) \le f(v) \le f(u')$ and similarly $g(u') \le g(v) \le g(u)$. Since $Min(f(v), g(v)) = M$ so $M \le f(v)$ and $M \le g(v)$ hence $\forall u < v$, $M \le g(v) \le g(u)$ and $\forall u' > v$, $M \le f(v) \le f(u')$. Putting all together, $\forall u < v$, $f(u) \le M$ and $M \le g(u)$, and on the other hand $\forall u > v$, $M \le f(u)$ and $g(u) \le M$. ■

4. Operations on Concavoconvex Type-2 Fuzzy Sets

Using Zadeh's extension principle to calculate union, intersection and complement of type-2 fuzzy sets $\tilde{A} = \int_{x \in X} \left[\int_{u \in J_x^{\tilde{A}}} f_x(u)/u \right] / x = \int_{x \in X} \mu_{\tilde{A}}(x)/x$ and $\tilde{B} = \int_{x \in X} \left[\int_{w \in J_x^{\tilde{B}}} g_x(w)/w \right] / x = \int_{x \in X} \mu_{\tilde{B}}(x)/x$ that are defined in the universe of discourse X, the membership grades of the results are defined [13,29] to be:

$$\tilde{A} \cup \tilde{B} \Leftrightarrow \mu_{\tilde{A} \cup \tilde{B}}(x) = \mu_{\tilde{A}}(x) \sqcup \mu_{\tilde{B}}(x)$$
$$= \left(\int_{u \in J_x^{\tilde{A}}} f_x(u)/u \right) \sqcup \left(\int_{w \in J_x^{\tilde{B}}} g_x(w)/w \right) = \int_{u,w} \frac{(f_x(u) \star g_x(w))}{(u \vee w)} \tag{8}$$

$$\tilde{A} \cap \tilde{B} \Leftrightarrow \mu_{\tilde{A} \cap \tilde{B}}(x) = \mu_{\tilde{A}}(x) \sqcap \mu_{\tilde{B}}(x) =$$
$$= \left(\int_{u \in J_x^{\tilde{A}}} f_x(u)/u \right) \sqcap \left(\int_{w \in J_x^{\tilde{B}}} g_x(w)/w \right) = \int_{u,w} \frac{(f_x(u) \star g_x(w))}{(u \star w)} \tag{9}$$

$$\bar{\tilde{A}} \Leftrightarrow \mu_{\bar{\tilde{A}}}(x) = \neg \mu_{\tilde{A}}(x) = \int_{u \in J_x^{\tilde{A}}} f_x(u)/(1-u) \tag{10}$$

where $\sqcup$, $\sqcap$ and $\neg$ denote the so-called *join*, *meet* and *negation* operation respectively [29]; $\star$ stands for a t-norm and $\vee$ represents maximum t-conorm. In (8) and (9) it can be clearly seen that calculating the union and intersection of two type-2 fuzzy sets reduces to the calculation of join and meet of the fuzzy grades – secondary sets - of the corresponding elements in the universe. In sections *4.A* and *4.B* we will study the operations under min and product t-norm respectively. Toward exploring the operations, in the following by concavoconvex fuzzy set we refer to concavoconvex fuzzy grades that are type-1 concavoconvex fuzzy sets defined in the unit interval $U = [u^-, u^+] = [0,1]$.

A. *Join and Meet Under min t-norm and max t-conorm*

Mendel and John in [27] have proposed a novel representation theorem based on which union, intersection and complementation of type-2 fuzzy sets would be calculated without applying the extension principle, using the so called *wavy-slices*. However since it results in an enormous amount of redundancy, is radically inefficient and from the computational complexity standpoint is not recommended. Focusing on Zedeh's extension principle through using the definitions in (8) and (9) we will examine in detail the operations join and meet on concavoconvex fuzzy sets under min t-norm and max t-conorm to drive computationally efficient algorithms.

Theorem 5: Let $F = \int_{u \in U} f(u)/u$ and $G = \int_{w \in U} g(w)/w$ be concavoconvex fuzzy grades, then using min t-norm and max t-conorm

$$F \sqcup G \Leftrightarrow \mu_{F \sqcup G}(\theta)$$
$$= (f(\theta) \vee g(\theta)) \wedge (f(\theta) \vee f(u^-)) \wedge (g(\theta) \vee g(u^-)) \tag{11}$$

$$F \sqcap G \Leftrightarrow \mu_{F \sqcap G}(\theta)$$
$$= (f(\theta) \vee g(\theta)) \wedge (f(\theta) \vee f(u^+)) \wedge (g(\theta) \vee g(u^+)) \tag{12}$$

Proof: We will prove (11); (12) would be proved similarly. Through substituting $\star$ in (8) with min t-norm, the join of F and G is:

$$F \sqcup G = \left(\int_u f(u)/u \right) \sqcup \left(\int_w g(w)/w \right) = \int_{u,w} \frac{(f(u) \wedge g(w))}{(u \vee w)} \tag{13}$$

The membership grade of $\theta \in U$ in $F \sqcup G$, hence would be:

$$\mu_{F \sqcup G}(\theta) = \underset{u \vee w = \theta}{Sup}(f(u) \wedge g(w)) \tag{14}$$

By the way, u and w which satisfy $u \vee w = \theta$ would be considered as $(u = \theta, w \le \theta)$ or $(u \le \theta, w = \theta)$. Thus, (14) would be rewritten as:

$$\mu_{F \sqcup G}(\theta) = \underset{\substack{u=\theta \\ w \le \theta}}{Sup}(f(u) \wedge g(w)) \vee \underset{\substack{u \le \theta \\ w=\theta}}{Sup}(f(u) \wedge g(w))$$
$$= \left(f(\theta) \wedge \underset{w \le \theta}{Sup}(g(w)) \right) \vee \left(g(\theta) \wedge \underset{u \le \theta}{Sup}(f(u)) \right) \tag{15}$$

Since F and G are concavoconvex fuzzy sets, so $\forall u \le \theta$, $f(\theta) \wedge f(u^-) \le f(u) \le f(\theta) \vee f(u^-)$, and

$g(\theta)\wedge g(u^-)\le g(w)\le g(\theta)\vee g(u^-)$, consequently (15) would be

$$\mu_{F\sqcup G}(\theta)=\Big(f(\theta)\wedge\big(g(u^-)\vee g(\theta)\big)\Big)\vee\Big(g(\theta)\wedge\big(f(u^-)\vee f(\theta)\big)\Big)$$
$$=\big(f(\theta)\vee g(\theta)\big)\wedge\Big(\big(g(u^-)\vee g(\theta)\big)\vee g(\theta)\Big)\wedge$$
$$\Big(f(\theta)\vee\big(f(u^-)\vee f(\theta)\big)\Big)\wedge\Big(\big(g(u^-)\vee g(\theta)\big)\vee\big(f(u^-)\vee f(\theta)\big)\Big)$$
$$=(f(\theta)\vee g(\theta))\wedge(f(\theta)\vee f(u^-))\wedge(g(\theta)\vee g(u^-))\ \blacksquare \tag{16}$$

Comment 1: Theorem 5 provides a general algorithm for dealing with all concavoconvex fuzzy grades, however, it can be easily proved that in particular cases the join and meet of concavoconvex fuzzy grades, (11) and (12), would be simplified as follows:

I) f increasing , g increasing:

$$\mu_{F\sqcup G}(\theta)=f(\theta)\wedge g(\theta) \tag{17}$$
$$\mu_{F\sqcap G}(\theta)=\big(f(\theta)\vee g(\theta)\big)\wedge f(u^+)\wedge g(u^+) \tag{18}$$

II) f increasing; g decreasing:

$$\mu_{F\sqcup G}(\theta)=f(\theta)\wedge g(u^-) \tag{19}$$
$$\mu_{F\sqcap G}(\theta)=f(u^+)\wedge g(\theta) \tag{20}$$

III) f decreasing; g increasing

$$\mu_{F\sqcup G}(\theta)=g(\theta)\wedge f(u^-) \tag{21}$$
$$\mu_{F\sqcap G}(\theta)=f(\theta)\wedge g(u^+) \tag{22}$$

IV) f decreasing; g decreasing:

$$\mu_{F\sqcup G}(\theta)=\big(f(\theta)\vee g(\theta)\big)\wedge f(u^-)\wedge g(u^-) \tag{23}$$
$$\mu_{F\sqcap G}(\theta)=f(\theta)\wedge g(\theta) \tag{24}$$

Figure 1, shows two examples of performing the operations join and meet on relatively two increasing and increasing-decreasing concavoconvex fuzzy grades using Theorem 5 and/or Comment 1.

Theorem 6: Let $F=\int_{u\in U} f(u)/u$ and $G=\int_{w\in U} g(w)/w$ be concavoconvex fuzzy grades, then their join and meet, $F\sqcup G$ and $F\sqcap G$ respectively, are concavoconvex fuzzy sets.

Proof: We will provide the proof for the join operation; concavoconvexity of the meet would be proved similarly.

I) Given f and g be both increasing, with Respect to (17), $\mu_{F\sqcup G}(\theta)=f(\theta)\wedge g(\theta)$. Given $u_1,u_2\in U$, $u_1<u_2$, then, $f(u_1)\le f(u_2)$ and $g(u_1)\le g(u_2)$, consequently $f(u_1)\wedge g(u_1)\le f(u_2)\wedge g(u_2)$ that is $\mu_{F\sqcup G}(u_1)\le\mu_{F\sqcup G}(u_2)$ which indicates that the membership function of $F\sqcup G$ is monotonically increasing that in accordance with the Theorem 1, signifies $F\sqcup G$ to be concavoconvex.

II) In the case of increasing f and decreasing g, considering (19), $\mu_{F\sqcup G}(\theta)=f(\theta)\wedge g(u^-)$. Given $u_1,u_2\in U$, $u_1<u_2$, then $f(u_1)\le f(u_2)$ and consequently $f(u_1)\wedge g(u^-)\le f(u_2)\wedge g(u^-)$ i.e. $\mu_{F\sqcup G}(u_1)\le\mu_{F\sqcup G}(u_2)$ which indicates that the membership function of $F\sqcup G$ is monotonically increasing that in accordance with the Theorem 1, signifies $F\sqcup G$ to be concavo-convex.

III) f decreasing; g increasing: Similar to the Case II, it can be shown that $F\sqcup G$ is concavoconvex.

IV) When f and g are both decreasing with respect to (23), $\mu_{F\sqcup G}(\theta)=\big(f(\theta)\vee g(\theta)\big)\wedge f(u^-)\wedge g(u^-)$. Given $u_1,u_2\in U$, $u_1<u_2$, then, $f(u_1)\ge f(u_2)$ and $g(u_1)\ge g(u_2)$, consequently $f(u_1)\vee g(u_1)\ge f(u_2)\vee g(u_2)$ hence $\Big(\big(f(u_1)\vee g(u_1)\big)\wedge\big(f(u^-)\wedge g(u^-)\big)\Big)\ge\Big(\big(f(u_2)\vee g(u_2)\big)\wedge\big(f(u^-)\wedge g(u^-)\big)\Big)$ that is $\mu_{F\sqcup G}(u_1)\ge\mu_{F\sqcup G}(u_2)$ which indicates that the membership function of $F\sqcup G$ is monotonically decreasing that in accordance with the Theorem 1, signifies $F\sqcup G$ to be concavoconvex. ■

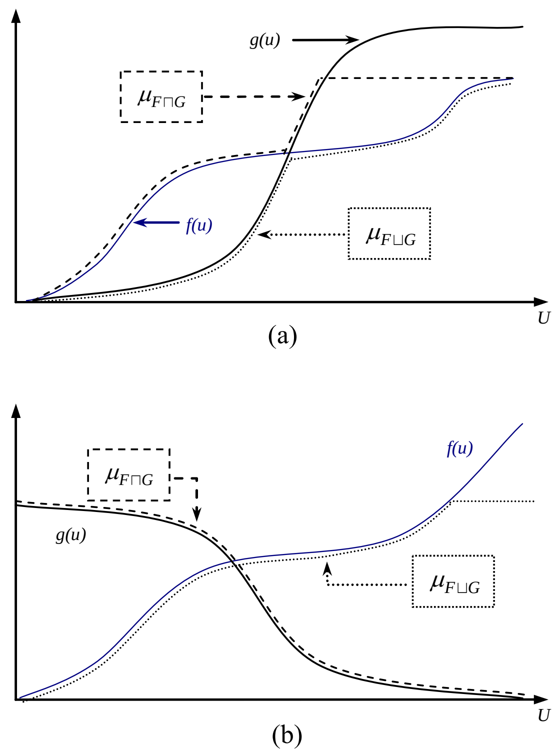

Figure 1. Calculating Join and Meet of (a) two increasing concavoconvex fuzzy grades, (b) increasing and decreasing concavoconvex fuzzy grades. The results are shown in dashed lines.

Table 1. Result type of join operation on Increasing/Decreasing concavoconvex fuzzy sets.

Join	**Increasing**	**Decreasing**
Increasing	Increasing	Increasing
Decreasing	Increasing	Decreasing

Table 2. Result type of meet operation on Increasing/Decreasing concavoconvex fuzzy sets.

Meet	**Increasing**	**Decreasing**
Increasing	Increasing	Decreasing
Decreasing	Decreasing	Decreasing

Tables 1 and 2 that are based on the Comment 1 and Theorem 6, depicts the type of resulting concavoconvex fuzzy grades with respect to the type of involving concavoconvex fuzzy grades in join and meet operations.

Theorem 7: Let $F=\int_{u\in U} f(u)/u$ be an increasing (decreasing) concavoconvex fuzzy grade, then $\neg F=\int_{u\in U} f(u)/(1-u)$ is a decreasing (increasing) concavoconvex fuzzy grade.

Proof: Obvious. ∎

Theorem 8: Let $F_i=\int_{u\in U} f_i(u)/u$, $i=1,\ldots,n$ be n concavoconvex fuzzy grades, then using min t-norm and max t-conorm,

$$F_1\sqcup\ldots\sqcup F_n \Leftrightarrow \mu_{F_1\sqcup\ldots\sqcup F_n}(\theta)=\left(\bigvee_{i=1}^{n} f_i(\theta)\right)\wedge\left(\bigwedge_{i=1}^{n}\left(f_i(\theta)\vee f_i(u^-)\right)\right) \tag{25}$$

$$F_1\sqcap\ldots\sqcap F_n \Leftrightarrow \mu_{F_1\sqcap\ldots\sqcap F_n}(\theta)=\left(\bigvee_{i=1}^{n} f_i(\theta)\right)\wedge\left(\bigwedge_{i=1}^{n}\left(f_i(\theta)\vee f_i(u^+)\right)\right) \tag{26}$$

Proof: We will prove (25) for the case of $n=3$, it can be easily extended to the case of $n>3$. Proof for the meet operation (26) is similar.

Since fuzzy grades under join satisfy associative law [29], so $F_1\sqcup F_2\sqcup F_3=\left(F_1\sqcup F_2\right)\sqcup F_3$. Using Theorem 5,

$$F\sqcup G\Leftrightarrow \mu_{F\sqcup G}(\theta) = (f(\theta)\vee g(\theta))\wedge(f(\theta)\vee f(u^-))\wedge(g(\theta)\vee g(u^-)) \tag{27}$$

that is proved in Theorem 6 to be a concavoconvex fuzzy set. So

$$\left(F_1\sqcup F_2\right)\sqcup F_3\Leftrightarrow \mu_{(F_1\sqcup F_2)\sqcup F_3}(\theta)=\left(\mu_{F_1\sqcup F_2}(\theta)\vee f_3(\theta)\right)\wedge \wedge\left(\mu_{F_1\sqcup F_2}(\theta)\vee\mu_{F_1\sqcup F_2}(u^-)\right)\wedge\left(f_3(\theta)\vee f_3(u^-)\right) \tag{28}$$

$$=\left(\left(\left(f_1(\theta)\vee f_2(\theta)\right)\wedge\left(f_1(\theta)\vee f_1(u^-)\right)\wedge\left(f_2(\theta)\vee f_2(u^-)\right)\right)\vee f_3(\theta)\right)\wedge$$

$$\begin{pmatrix}\left(\left(f_1(\theta)\vee f_2(\theta)\right)\wedge\left(f_1(\theta)\vee f_1(u^-)\right)\wedge\left(f_2(\theta)\vee f_2(u^-)\right)\right)\vee\\ \left(\left(f_1(u^-)\vee f_2(u^-)\right)\wedge\left(f_1(u^-)\vee f_1(u^-)\right)\wedge\left(f_2(u^-)\vee f_2(u^-)\right)\right)\end{pmatrix} \wedge\left(f_3(\theta)\vee f_3(u^-)\right) \tag{29}$$

$$=\left(f_1(\theta)\vee f_2(\theta)\vee f_3(\theta)\right)\wedge\left(f_1(\theta)\vee f_1(u^-)\vee f_3(\theta)\right)\wedge \left(f_2(\theta)\vee f_2(u^-)\vee f_3(\theta)\right)\wedge \begin{pmatrix}\left(\left(f_1(\theta)\vee f_2(\theta)\right)\wedge\left(f_1(\theta)\vee f_1(u^-)\right)\wedge\left(f_2(\theta)\vee f_2(u^-)\right)\right)\vee\\ \left(f_1(u^-)\wedge f_2(u^-)\right)\end{pmatrix} \wedge\left(f_3(\theta)\vee f_3(u^-)\right)\ \textit{(Absorption Law)} \tag{30}$$

$$=\left(f_1(\theta)\vee f_2(\theta)\vee f_3(\theta)\right)\wedge\left(f_1(\theta)\vee f_1(u^-)\vee f_3(\theta)\right)\wedge \left(f_2(\theta)\vee f_2(u^-)\vee f_3(\theta)\right)\wedge\left(f_1(\theta)\vee f_2(\theta)\vee f_1(u^-)\right)\wedge \left(f_1(\theta)\vee f_1(u^-)\vee f_1(u^-)\right)\wedge\left(f_2(\theta)\vee f_2(u^-)\vee f_1(u^-)\right)\wedge \left(f_1(\theta)\vee f_2(\theta)\vee f_2(u^-)\right)\wedge\left(f_1(\theta)\vee f_1(u^-)\vee f_2(u^-)\right)\wedge \left(f_2(\theta)\vee f_2(u^-)\vee f_2(u^-)\right)\wedge\left(f_3(\theta)\vee f_3(u^-)\right) \tag{31}$$

$$=\left(f_1(\theta)\vee f_2(\theta)\vee f_3(\theta)\right)\wedge\left(\left(f_1(\theta)\vee f_1(u^-)\right)\vee f_3(\theta)\right)\wedge \left(\left(f_2(\theta)\vee f_2(u^-)\right)\vee f_3(\theta)\right)\wedge\left(\left(f_1(\theta)\vee f_1(u^-)\right)\vee f_2(\theta)\right)\wedge \left(f_1(u^-)\vee\left(f_2(\theta)\vee f_2(u^-)\right)\right)\wedge\left(f_1(\theta)\vee\left(f_2(\theta)\vee f_2(u^-)\right)\right)\wedge \left(\left(f_1(\theta)\vee f_1(u^-)\right)\vee f_2(u^-)\right)\wedge\left(f_1(\theta)\vee f_1(u^-)\right)\wedge \left(f_2(\theta)\vee f_2(u^-)\right)\wedge\left(f_3(\theta)\vee f_3(u^-)\right) \tag{32}$$

$$=\left(f_1(\theta)\vee f_2(\theta)\vee f_3(\theta)\right)\wedge\left(f_1(\theta)\vee f_1(u^-)\right)\wedge \left(f_2(\theta)\vee f_2(u^-)\right)\wedge\left(f_3(\theta)\vee f_3(u^-)\right)\ \blacksquare \tag{33}$$

Theorem 9: Let $F=\int_{u\in U} f(u)/u$, $G=\int_{w\in U} g(w)/w$ and $H=\int_{z\in U} h(z)/z$ be concavoconvex fuzzy grades, then the distributive laws are satisfied, i.e.

$$F\sqcap(G\sqcup H)=(F\sqcap G)\sqcup(F\sqcap H) \tag{34}$$

$$F\sqcup(G\sqcap H)=(F\sqcup G)\sqcap(F\sqcup H) \tag{35}$$

Proof: We will investigate (34) using (17)-(24) in the following cases. Proving the other law, (35) obeys the similar method. For the sake of simplicity, in the following we will use fuzzy sets and their membership functions interchangeably.

I) f increasing; g increasing; h increasing:

$$F\sqcap(G\sqcup H)=$$
$$=F\sqcap(g(\theta)\wedge h(\theta))$$
$$=\left(f(\theta)\vee\left(g(\theta)\wedge h(\theta)\right)\right)\wedge f(u^+)\wedge g(u^+)\wedge h(u^+)$$
$$(G\sqcup H\ \ \textit{increasing})$$

$$=\big(f(\theta)\vee g(\theta)\big)\wedge\big(f(\theta)\vee h(\theta)\big)\wedge f(u^+)\wedge g(u^+)\wedge h(u^+) \tag{36}$$

$(F\sqcap G)\sqcup(F\sqcap H)=$

$$=\begin{pmatrix}\Big(\big(f(\theta)\vee g(\theta)\big)\wedge f(u^+)\wedge g(u^+)\Big)\sqcup\\ \Big(\big(f(\theta)\vee h(\theta)\big)\wedge f(u^+)\wedge h(u^+)\Big)\end{pmatrix}$$

$$=\begin{pmatrix}\Big(\big(f(\theta)\vee g(\theta)\big)\wedge f(u^+)\wedge g(u^+)\Big)\wedge\\ \Big(\big(f(\theta)\vee h(\theta)\big)\wedge f(u^+)\wedge h(u^+)\Big)\end{pmatrix}\begin{pmatrix}F\sqcap G,\ F\sqcap H\\ \textit{increasing}\end{pmatrix}$$

$$=\big(f(\theta)\vee g(\theta)\big)\wedge\big(f(\theta)\vee h(\theta)\big)\wedge f(u^+)\wedge g(u^+)\wedge h(u^+) \tag{37}$$

II) f increasing; g increasing; h decreasing:

$F\sqcap(G\sqcup H)=F\sqcap(g(\theta)\wedge h(u^-))$

$$=\Big(f(\theta)\vee\big(g(\theta)\wedge h(u^-)\big)\Big)\wedge f(u^+)\wedge g(u^+)\wedge h(u^-)$$

$(G\sqcup H\ \textit{increasing})$

$$=\big(f(\theta)\vee g(\theta)\big)\wedge\big(f(\theta)\vee h(u^-)\big)\wedge f(u^+)\wedge g(u^+)\wedge h(u^-)$$

$$=\big(f(\theta)\vee g(\theta)\big)\wedge f(u^+)\wedge g(u^+)\wedge h(u^-) \tag{38}$$

$(F\sqcap G)\sqcup(F\sqcap H)=$

$$=\Big(\big(f(\theta)\vee g(\theta)\big)\wedge f(u^+)\wedge g(u^+)\Big)\sqcup\big(f(u^+)\wedge h(\theta)\big)$$

$$=\Big(\big(f(\theta)\vee g(\theta)\big)\wedge f(u^+)\wedge g(u^+)\Big)\wedge\big(f(u^+)\wedge h(u^-)\big)$$

$(F\sqcap G\ \textit{increasing}, F\sqcap H\ \textit{decreasing})$

$$=\big(f(\theta)\vee g(\theta)\big)\wedge f(u^+)\wedge g(u^+)\wedge h(u^-) \tag{39}$$

III) f increasing; g decreasing; h increasing:

$F\sqcap(G\sqcup H)=F\sqcap(g(u^-)\wedge h(\theta))$

$$=\Big(f(\theta)\vee\big(g(u^-)\wedge h(\theta)\big)\Big)\wedge f(u^+)\wedge g(u^-)\wedge h(u^+)$$

$(G\sqcup H\ \textit{increasing})$

$$=\big(f(\theta)\vee g(u^-)\big)\wedge\big(f(\theta)\vee h(\theta)\big)\wedge f(u^+)\wedge g(u^-)\wedge h(u^+)$$

$$=\big(f(\theta)\vee h(\theta)\big)\wedge f(u^+)\wedge g(u^-)\wedge h(u^+) \tag{40}$$

$(F\sqcap G)\sqcup(F\sqcap H)=$

$$=\big(f(u^+)\wedge g(\theta)\big)\sqcup\Big(\big(f(\theta)\vee h(\theta)\big)\wedge f(u^+)\wedge h(u^+)\Big)$$

$$=\big(f(u^+)\wedge g(u^-)\big)\wedge\Big(\big(f(\theta)\vee h(\theta)\big)\wedge f(u^+)\wedge h(u^+)\Big)$$

$(F\sqcap G\ \textit{decreasing},\ F\sqcap H\ \textit{increasing})$

$$=\big(f(\theta)\vee h(\theta)\big)\wedge f(u^+)\wedge g(u^-)\wedge h(u^+) \tag{41}$$

IV) f increasing; g decreasing; h decreasing:

$F\sqcap(G\sqcup H)=F\sqcap\big((g(\theta)\vee h(\theta))\wedge g(u^-)\wedge h(u^-)\big)$

$$=(g(\theta)\vee h(\theta))\wedge f(u^+)\wedge g(u^-)\wedge h(u^-) \tag{42}$$

$(G\sqcup H\ \textit{decreasing})$

$(F\sqcap G)\sqcup(F\sqcap H)=$

$$=\big(f(u^+)\wedge g(\theta)\big)\sqcup\big(f(u^+)\wedge h(\theta)\big)$$

$$=\begin{pmatrix}\Big(\big(f(u^+)\wedge g(\theta)\big)\vee\big(f(u^+)\wedge h(\theta)\big)\Big)\wedge\\ \big(f(u^+)\wedge g(u^-)\big)\wedge\big(f(u^+)\wedge h(u^-)\big)\end{pmatrix}\begin{pmatrix}F\sqcap G, F\sqcap H\\ \textit{decreasing}\end{pmatrix}$$

$$=\big(g(\theta)\vee h(\theta)\big)\wedge f(u^+)\wedge g(u^-)\wedge h(u^-) \tag{43}$$

V) f decreasing; g increasing; h increasing:

$F\sqcap(G\sqcup H)=F\sqcap(g(\theta)\wedge h(\theta))$

$$=f(\theta)\wedge g(u^+)\wedge h(u^+)\quad(G\sqcup H\ \textit{increasing}) \tag{44}$$

$(F\sqcap G)\sqcup(F\sqcap H)=$

$$=\big(f(\theta)\wedge g(u^+)\big)\sqcup\big(f(\theta)\wedge h(u^+)\big)$$

$$=\begin{pmatrix}\Big(\big(f(\theta)\wedge g(u^+)\big)\vee\big(f(\theta)\wedge h(u^+)\big)\Big)\wedge\\ \big(f(u^-)\wedge g(u^+)\big)\wedge\big(f(u^-)\wedge h(u^+)\big)\end{pmatrix}\begin{pmatrix}F\sqcap G,\ F\sqcap H\\ \textit{decreasing}\end{pmatrix}$$

$$=f(\theta)\wedge\big(g(u^+)\vee h(u^+)\big)\wedge f(u^-)\wedge g(u^+)\wedge h(u^+)$$

$$=f(\theta)\wedge f(u^-)\wedge g(u^+)\wedge h(u^+)\quad \textit{(Absorption law)}$$

$$=f(\theta)\wedge g(u^+)\wedge h(u^+)\qquad \textit{(f decreasing)} \tag{45}$$

VI) f decreasing; g increasing; h decreasing:

$F\sqcap(G\sqcup H)=F\sqcap(g(\theta)\wedge h(u^-))$

$$=f(\theta)\wedge g(u^+)\wedge h(u^-)\quad(G\sqcup H\ \textit{increasing}) \tag{46}$$

$(F\sqcap G)\sqcup(F\sqcap H)=\big(f(\theta)\wedge g(u^+)\big)\sqcup\big(f(\theta)\wedge h(\theta)\big)$

$$=\begin{pmatrix}\Big(\big(f(\theta)\wedge g(u^+)\big)\vee\big(f(\theta)\wedge h(\theta)\big)\Big)\wedge\\ \big(f(u^-)\wedge g(u^+)\big)\wedge\big(f(u^-)\wedge h(u^-)\big)\end{pmatrix}\begin{pmatrix}F\sqcap G,\ F\sqcap H\\ \textit{decreasing}\end{pmatrix}$$

$$=f(\theta)\wedge\big(g(u^+)\vee h(\theta)\big)\wedge f(u^-)\wedge g(u^+)\wedge h(u^-)$$

$$=f(\theta)\wedge f(u^-)\wedge g(u^+)\wedge h(u^-)\quad \textit{(Absorption law)}$$

$$=f(\theta)\wedge g(u^+)\wedge h(u^-)\qquad \textit{(f decreasing)} \tag{47}$$

VII) f decreasing; g decreasing; h increasing:

$F\sqcap(G\sqcup H)=F\sqcap(g(u^-)\wedge h(\theta))$

$$=f(\theta)\wedge g(u^-)\wedge h(u^+)\ (G\sqcup H\ \textit{increasing}) \tag{48}$$

$(F\sqcap G)\sqcup(F\sqcap H)=$

$$=\big(f(\theta)\wedge g(\theta)\big)\sqcup\big(f(\theta)\wedge h(u^+)\big)$$

$$=\begin{pmatrix}\Big(\big(f(\theta)\wedge g(\theta)\big)\vee\big(f(\theta)\wedge h(u^+)\big)\Big)\wedge\\ \big(f(u^-)\wedge g(u^-)\big)\wedge\big(f(u^-)\wedge h(u^+)\big)\end{pmatrix}\begin{pmatrix}F\sqcap G,\ F\sqcap H\\ \textit{decreasing}\end{pmatrix}$$

$$=f(\theta)\wedge\big(g(\theta)\vee h(u^+)\big)\wedge f(u^-)\wedge g(u^-)\wedge h(u^+)$$

$$=f(\theta)\wedge f(u^-)\wedge g(u^-)\wedge h(u^+)\quad \textit{(Absorption law)}$$

$$=f(\theta)\wedge g(u^-)\wedge h(u^+)\qquad \textit{(f decreasing)} \tag{49}$$

VIII) f decreasing; g decreasing; h decreasing:

$F\sqcap(G\sqcup H)=$

$$=F\sqcap\big((g(\theta)\vee h(\theta))\wedge g(u^-)\wedge h(u^-)\big)$$

$$= f(\theta)\wedge\big(g(\theta)\vee h(\theta)\big)\wedge g(u^-)\wedge h(u^-)$$

$$(G\sqcup H\ decreasing) \quad (50)$$

$$(F\sqcap G)\sqcup(F\sqcap H)=$$

$$=\big(f(\theta)\wedge g(\theta)\big)\sqcup\big(f(\theta)\wedge h(\theta)\big)$$

$$=\begin{pmatrix}\Big(\big(f(\theta)\wedge g(\theta)\big)\vee\big(f(\theta)\wedge h(\theta)\big)\Big)\wedge\\ \Big(f(u^-)\wedge g(u^-)\Big)\wedge\Big(f(u^-)\wedge h(u^-)\Big)\end{pmatrix}\begin{pmatrix}F\sqcap G,\ F\sqcap H\\ decreasing\end{pmatrix}$$

$$= f(\theta)\wedge\big(g(\theta)\vee h(\theta)\big)\wedge f(u^-)\wedge g(u^-)\wedge h(u^-)$$

$$= f(\theta)\wedge\big(g(\theta)\vee h(\theta)\big)\wedge g(u^-)\wedge h(u^-)$$

$$(f\quad decreasing)\blacksquare \quad (51)$$

Investigating the absorption laws on concavoconvex fuzzy sets, we reached the following results,

I) f increasing; g increasing;

$$F\sqcap(F\sqcup H)=F\sqcup(F\sqcap H)=f(\theta)\wedge g(u^+) \quad (52)$$

II) f increasing; g decreasing;

$$F\sqcap(F\sqcup H)=F\sqcup(F\sqcap H)=f(\theta)\wedge g(u^-) \quad (53)$$

III) f decreasing; g increasing;

$$F\sqcap(F\sqcup H)=F\sqcup(F\sqcap H)=f(\theta)\wedge g(u^+) \quad (54)$$

IV) f decreasing; g decreasing;

$$F\sqcap(F\sqcup H)=F\sqcup(F\sqcap H)=f(\theta)\wedge g(u^-) \quad (55)$$

It can be easily observed that if the maximal grade of membership in F is smaller than or equal to the maximal grade of membership in G, then absorption laws are satisfied by concavoconvex fuzzy grades. This is the same result as [29] had reached for convex fuzzy grades.

Theorem 10: Concavoconvex fuzzy grades form a commutative semiring under join and meet.

Proof: According to the Theorem 6, Concavoconvex fuzzy grades are closed under $\sqcup$ and $\sqcap$, and with respect to [29], concavoconvex fuzzy grades are also associative and commutative under $\sqcup$ and $\sqcap$. Moreover, regarding the Theorem 9, concavoconvex fuzzy grades are distributive with respect to $\sqcup$ and $\sqcap$. In [29] it is proved that identity laws are satisfied by arbitrary fuzzy grades and hence concavoconvex fuzzy grades which conclude the proof. ■

Theorem 11: Normal concavoconvex fuzzy grades form a distributed lattice under join and meet where $1/1$ and $1/0$ as greatest and least elements.

Proof: Obvious. ■

B. *Join and Meet under product t-norm and max t-conorm*

Join and meet operations under product t-norm in general, are not as straightforward as under minimum t-norm. There has been some endeavors in this regard [13,29,30,35]. In [13] although a closed form formula for join of normal convex fuzzy grades under product t-norm and max t-conorm is proposed it is also mentioned that "it is very difficult to obtain a closed-form expression for the result of the meet operation" under product t-norm [13].

In this section we will study join and meet operations on concavoconvex fuzzy grades under product t-norm and max t-conorm. In specific, we will consider meet under product-product t-norm i.e. replacing all t-norms in (8) with product, and under product-min t-norms separately. In a formal form, given $F=\int_{u\in U} f(u)/u$ and $G=\int_{w\in U} g(w)/w$ be fuzzy grades, then we define $\dot\sqcup$ (join under product t-norm and max t-conorm), $\dot\sqcap$ (meet under product-min t-norm) and $\ddot\sqcap$ (meet under product-product t-norm),

$$F\dot\sqcup G=\left(\int_{u\in U} f(u)/u\right)\dot\sqcup\left(\int_{w\in U} g(w)/w\right)=\int_{u,w}\frac{(f(u).g(w))}{(u\vee w)} \quad (56)$$

$$F\dot\sqcap G=\left(\int_{u\in U} f(u)/u\right)\dot\sqcap\left(\int_{w\in U} g(w)/w\right)=\int_{u,w}\frac{(f(u).g(w))}{(u\wedge w)} \quad (57)$$

$$F\ddot\sqcap G=\left(\int_{u\in U} f(u)/u\right)\ddot\sqcap\left(\int_{w\in U} g(w)/w\right)=\int_{u,w}\frac{(f(u).g(w))}{(u.w)} \quad (58)$$

Theorem 12: Let $F=\int_{u\in U} f(u)/u$ and $G=\int_{w\in U} g(w)/w$ be concavoconvex fuzzy grades, then

I) f increasing, g increasing

$$F\dot\sqcup G\Leftrightarrow \mu_{F\dot\sqcup G}(\theta)=f(\theta).g(\theta) \quad (59)$$

$$F\dot\sqcap G\Leftrightarrow \mu_{F\dot\sqcap G}(\theta)=(f(\theta).h_g)\vee(h_f.g(\theta)) \quad (60)$$

II) f increasing, g decreasing

$$F\dot\sqcup G\Leftrightarrow \mu_{F\dot\sqcup G}(\theta)=f(\theta).h_g \quad (61)$$

$$F\dot\sqcap G\Leftrightarrow \mu_{F\dot\sqcap G}(\theta)=h_f.g(\theta) \quad (62)$$

III) f decreasing, g decreasing

$$F\dot\sqcup G\Leftrightarrow \mu_{F\dot\sqcup G}(\theta)=(f(\theta).h_g)\vee(h_f.g(\theta)) \quad (63)$$

$$F\dot\sqcap G\Leftrightarrow \mu_{F\dot\sqcap G}(\theta)=f(\theta).g(\theta) \quad (64)$$

Proof: We will prove terms related to $F\dot\sqcup G$, i.e. (59),(61) and (63), however $F\dot\sqcap G$ would be proved similarly. With respect to (56), the membership of θ in $F\dot\sqcup G$ is

$$F\dot\sqcup G\Leftrightarrow \mu_{F\dot\sqcup G}(\theta)=\underset{u\vee w=\theta}{Sup}\big(f(u).g(w)\big)$$

$$=\underset{\substack{u=\theta\\ w\le\theta}}{Sup}\big(f(u).g(w)\big)\vee\underset{\substack{u\le\theta\\ w=\theta}}{Sup}\big(f(u).g(w)\big)$$

$$=\left(f(\theta).\underset{w\le\theta}{Sup}\big(g(w)\big)\right)\vee\left(g(\theta).\underset{u\le\theta}{Sup}\big(f(u)\big)\right) \quad (65)$$

I) $\mu_{F\underset{\cdot}{\sqcup}G}(\theta)=\left(f(\theta).\underset{w\le\theta}{Sup}\left(g(w)\right)\right)\vee\left(g(\theta).\underset{u\le\theta}{Sup}\left(f(u)\right)\right)$

$$=\left(f(\theta).g(\theta)\right)\vee\left(g(\theta).f(\theta)\right)=f(\theta).g(\theta) \qquad (66)$$

II) $\mu_{F\underset{\cdot}{\sqcup}G}(\theta)=\left(f(\theta).\underset{w\le\theta}{Sup}\left(g(w)\right)\right)\vee\left(g(\theta).\underset{u\le\theta}{Sup}\left(f(u)\right)\right)$

$$=\left(f(\theta).g(u^-)\right)\vee\left(g(\theta).f(\theta)\right)$$

$$=\left(f(\theta).g(u^-)\right) \qquad \left(g(u^-)\ge g(\theta)\right)$$

$$=f(\theta).h_g \qquad \left(h_g=\underset{u}{Sup}\left(g(u)\right)=g(u^-)\right) \qquad (67)$$

III) $\mu_{F\underset{\cdot}{\sqcup}G}(\theta)=\left(f(\theta).\underset{w\le\theta}{Sup}\left(g(w)\right)\right)\vee\left(g(\theta).\underset{u\le\theta}{Sup}\left(f(u)\right)\right)$

$$=\left(f(\theta).g(u^-)\right)\vee\left(g(\theta).f(u^-)\right)$$

$$=\left(f(\theta).h_g\right)\vee\left(h_f.g(\theta)\right) \quad ■ \qquad (68)$$

Theorem 13: Concavoconvex fuzzy grades are closed under $\underset{\cdot}{\sqcup}$ and $\dot{\sqcap}$.

Proof: With respect to the Theorem 12 the proof is obvious. ■

In Theorem 14, we will proved that join under product-product t-norm ($\ddot{\sqcap}$) is closed on concavoconvex fuzzy grades, however, we could not found a closed form formula for it, except when one of the involving concavoconvex fuzzy grades is increasing and the other is decreasing, more exactly, let f be increasing and g decreasing,

$$F\ddot{\sqcap}G\Leftrightarrow\mu_{F\ddot{\sqcap}G}(\theta)=\underset{u.w=\theta}{Sup}\left(f(u).g(w)\right)$$

$$=\underset{\substack{\theta\le u\le\sqrt{\theta}\\ \sqrt{\theta}\le w\le 1\\ u.w=\theta}}{Sup}\left(f(u).g(w)\right)\vee\underset{\substack{\sqrt{\theta}\le u\le 1\\ \theta\le w\le\sqrt{\theta}\\ u.w=\theta}}{Sup}\left(f(u).g(w)\right)$$

$$=\left(f(\sqrt{\theta}).g(\sqrt{\theta})\right)\vee\left(f(1).g(\theta)\right)$$

$$\left(\sqrt{\theta}\ge\theta,\ f(1)\ge f(\sqrt{\theta}),\ g(\theta)\ge g(\sqrt{\theta})\right)$$

$$=f(1).g(\theta) \qquad (69)$$

Theorem 14: Concavoconvex fuzzy grades are closed under $\ddot{\sqcap}$.

Proof: Let $F=\int_{u\in U} f(u)/u$ and $G=\int_{w\in U} g(w)/w$ be concavoconvex fuzzy grades. If f and g are increasing and decreasing respectively, with respect to (69), $\mu_{F\ddot{\sqcap}G}$ is decreasing that due to the Theorem 1, $F\ddot{\sqcap}G$ is a concavoconvex fuzzy set. What remain is while f and g are both increasing or decreasing. We will prove for f and g be both increasing, the proof of the other case is similar. Based on (58),

$$F\ddot{\sqcap}G\Leftrightarrow\mu_{F\ddot{\sqcap}G}(\theta)=\underset{u.w=\theta}{Sup}\left(f(u).g(w)\right)=\underset{\theta\le u\le 1}{Sup}\left(f(u).g(\frac{\theta}{u})\right) \qquad (70)$$

In the case of $\theta=0$ then $\mu_{F\ddot{\sqcap}G}(0)=\left(f(0).g(1)\right)\vee\left(f(1).g(0)\right)$, however, $\forall u\in(0,1]$, $f(0).g(1)\le f(u).g(1)$ and $f(1).g(0)\le f(1).g(u)$. This indicates that $\mu_{F\ddot{\sqcap}G}(0)\le\mu_{F\ddot{\sqcap}G}(u)$, $u\in(0,1]$.

For the case of $\theta\ne 0$ let $u_1,u_2\in(0,1]$, such that $u_1\le u_2$. For any $u_3\in[u_2,1]$, $\frac{u_1}{u_3}\le\frac{u_2}{u_3}$ and since g is increasing so $g\left(\frac{u_1}{u_3}\right)\le g\left(\frac{u_2}{u_3}\right)$ consequently $f(u_3).g\left(\frac{u_1}{u_3}\right)\le f(u_3).g\left(\frac{u_2}{u_3}\right)$. On the other hand, for any $u_3\in[u_1,u_2]$, then $f(u_3)\le f(u_2)$ and $g(\frac{u_1}{u_3})\le g(1)$. Consequently $f(u_3).g(\frac{u_1}{u_3})\le f(u_2).g(1)$. This shows that for any $u\in[u_1,1]$, $\exists u'\in[u_2,1]$, $f(u).g\left(\frac{u_1}{u}\right)\le f(u').g\left(\frac{u_2}{u'}\right)$ and so $\underset{u_1\le u\le 1}{Sup}\left(f(u).g(\frac{u_1}{u})\right)\le\underset{u_2\le u'\le 1}{Sup}\left(f(u').g(\frac{u_2}{u'})\right)$.

Putting all together indicates that $\forall u_1,u_2\in[0,1]$, $u_1\le u_2$ then $\mu_{F\ddot{\sqcap}G}(u_1)\le\mu_{F\ddot{\sqcap}G}(u_2)$ that signifies $\mu_{F\ddot{\sqcap}G}$ to be increasing, hence with respect to the Theorem 1, $F\ddot{\sqcap}G$ is concavoconvex fuzzy grade. ■

5. Conclusions

Basic operations on concavoconvex fuzzy grades were explored and simplified algorithms for performing join and meet on concavoconvex fuzzy grades were proposed. Regarding the fact that algebraic structures of type-2 fuzzy sets under union, intersection and complement, depend on the algebraic structures of fuzzy grades under $\sqcup$, $\sqcap$ and $\neg$, we showed that concavoconvex fuzzy grades under $\sqcup$, $\sqcap$ form a commutative semiring and also normal concavoconvex fuzzy grades form a distributed lattice. We also proposed a closed form formula for $\underset{\cdot}{\sqcup}$ and $\dot{\sqcap}$. Although we could not find a closed form formula for $\ddot{\sqcap}$ we proved that concavoconvex fuzzy grades are closed under $\underset{\cdot}{\sqcup}$, $\dot{\sqcap}$ and $\ddot{\sqcap}$.

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