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3-D viscous Cahn–Hilliard equation with memory

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SUMMARY

We deal with the memory relaxation of the viscous Cahn–Hilliard equation in 3-D, covering the well-known hyperbolic version of the model. We study the long-term dynamic of the system in dependence of the scaling parameter of the memory kernel ε and of the viscosity coefficient δ. In particular we construct a family of exponential attractors, which is robust as both ε and δ go to zero, provided that ε is linearly controlled by δ. Copyright © 2008 John Wiley & Sons, Ltd.

KEY WORDS: Cahn–Hilliard equations; memory relaxation; singular limit; robust exponential attractors

1. INTRODUCTION

In the description of phase separation phenomena in materials science, a basic role is played by the celebrated Cahn–Hilliard equation, proposed in the sixties by Cahn and Hilliard [1]. If $u(x,t):\Omega\times[0,\infty)\to\mathbb{R}$ represents the relative concentration of one component of a binary system in a given domain Ω at time t, then the evolution of u is governed by the parabolic equation

$$\partial_t u-\Delta(-\Delta u+\phi(u))=0 \tag{1}$$

where ϕ is the derivative of a double-well potential describing the free energy of the system. In recent years it has been shown (see, for instance, [2, 3]) that some separation phenomena generated by deep supercooling are better described by the hyperbolic relaxation of the model, namely

$$\varepsilon\partial_{tt}u+\partial_t u-\Delta(-\Delta u+\phi(u)+\delta\partial_t u)=0 \tag{2}$$

where $\varepsilon>0$ is the relaxation time and $\delta\geqslant 0$ a viscosity coefficient describing the action of internal microforces (see [4, 5]). The mathematical analysis of (2) has been carried out in several papers, mainly in the viscous case $\delta>0$ and in space dimensions $N=1,2$. Many results are nowadays

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available, from the well-posedness of the problem within various boundary conditions to the asymptotic analysis of the solutions. We do not discuss them here, referring the interested reader to the rich bibliography in [6] and to the very recent [7], where the non-viscous case is dealt with, in space dimension two.

In space dimension $N=3$, the viscous hyperbolic case of (2) with $\varepsilon>0$ and $\delta>0$ has been first studied in [6], with particular attention to the stability of the model with respect to the parameters ε and δ. In this paper we go further in that direction by considering an integro-differential modification of the Cahn–Hilliard equation, of the form

$$\partial_t u - \int_0^\infty k_\varepsilon(s)\Delta(-\Delta u(t-s)+\phi(u(t-s))+\delta\partial_t u(t-s))\,\mathrm{d}s=0 \tag{3}$$

where now $u:\Omega\times\mathbb{R}\to\mathbb{R}$. Here $\varepsilon>0$ is the scaling parameter and

$$k_\varepsilon(s)=\frac{1}{\varepsilon}k\left(\frac{s}{\varepsilon}\right)$$

is the rescaling of a continuous positive decreasing convex kernel k. In the special case of exponential kernels

$$k_\varepsilon(s)=\frac{1}{\varepsilon}\mathrm{e}^{-s/\varepsilon}$$

a derivation with respect to time formally leads to the hyperbolic equation (2): if u solves (3) for any $t\in\mathbb{R}$, then u is a solution to (2) for any $t>0$. Vice versa, given a solution of the hyperbolic equation on $t>0$, it is possible to construct a solution to (3) extending u backward in time: in this sense we can interpret (3) as a generalization of the hyperbolic model.

The general idea of considering memory relaxation of abstract evolution equations of first order has been presented in [8], where the non-viscous version of (3) is also discussed in dimension $N=1$. In the present work, we concentrate on the 3-D situation. Precisely, we consider a bounded domain $\Omega\subset\mathbb{R}^3$ with smooth boundary $\partial\Omega$, and we study (3) supplemented with the initial and boundary value conditions

$$\begin{cases} u(x,t)=u_0(x,t), & x\in\Omega,\quad t\leqslant 0\\ u(x,t)=\Delta u(x,t)=0, & x\in\partial\Omega,\quad t\in\mathbb{R}\end{cases} \tag{4}$$

Note that u is supposed to be known (hence a given datum) for all $t\leqslant 0$, since the whole *past history* of u is needed to compute the convolution integral. Other interesting boundary conditions like no-flux boundary conditions could be considered: here we focus on the Dirichlet case, referring, for instance, to [6] for a discussion on the additional constraints needed to treat the Neumann case.

We first reformulate our problem along the line of [8–10] by introducing the so-called *integrated past history*. Then, for any fixed $\varepsilon>0$ and $\delta>0$, we prove that the new (but equivalent) problem generates a strongly continuous semigroup $S_{\varepsilon,\delta}(t)$ in an appropriate two-component phase space $\mathscr{H}^0_\varepsilon$, constructed using the past history as an additional variable of the system. It is important to stress that when $\varepsilon>0$ and $\delta=0$ we do not have a well-posedness result, due to the possible lack of uniqueness of solutions in space dimension 3, see Remark 3.2. Then we need to assume that ε is controlled by δ

$$\delta\geqslant\tau\varepsilon \quad \text{for some } \tau>0$$

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see (7). In a second step we discuss the dissipativity of the semigroup, providing in particular the existence of the global attractor. This is obtained by means of suitable energy estimates, where we overcome the major difficulty of handling the nonlinearity in dimension $N=3$ with the help of appropriate new functionals. Our major goal is then to analyze the sensitivity of the model with respect to the parameters ε and δ, when they both vanish to zero. It is apparent that since the kernel k_ε converges (in the sense of distribution) to the Dirac mass concentrated at zero as $\varepsilon \to 0$, then the limit $(\varepsilon, \delta) \to (0,0)$ formally leads to the original parabolic Cahn–Hilliard equation (1). It is well known that (1) supplemented with the Dirichlet boundary and initial conditions admits a unique solution, which can be expressed in terms of a strongly continuous semigroup $S_{0,0}(t)$; hence, we focus on the relations between $S_{\varepsilon,\delta}(t)$ and the limiting semigroup, with the aim of establishing in a rigorous way the convergence of the relaxed equation (3) to the parabolic one (1). First, comparing the trajectories of the two semigroups, we prove a quantitative estimate of their closeness on finite time intervals. Concerning the asymptotic analysis, we succeed in constructing a family of *exponential attractors* $\mathscr{E}_{\varepsilon,\delta}$, whose basin of attraction is the whole phase space and which is uniform and *robust* with respect to the physical parameters, provided that ε is linearly controlled by δ. This means that the fractal dimension of $\mathscr{E}_{\varepsilon,\delta}$ and the rate of attraction of bounded sets of data are both independent of ε and δ; besides $\mathscr{E}_{\varepsilon,\delta}$ is close to $\mathscr{E}_{0,0}$ in the symmetric Hausdorff distance, namely

$$\lim_{(\varepsilon,\delta)\to(0,0)} \operatorname{dist}^{\mathrm{sym}}_{\mathscr{H}^0_\varepsilon}(\mathscr{E}_{\varepsilon,\delta}, \mathscr{E}_{0,0}) = 0$$

This kind of analysis is based on an abstract result developed in [9] to control the singular limit $\varepsilon \to 0$, and then successfully applied to a large class of models with memory, see for instance [11–13]. As our model includes a further (non-singular) perturbation given by the viscosity term $\delta \partial_t u$, we have to recast the suitable abstract theorem, as stated in the Appendix of the present paper. With this strategy we finally recover (indeed generalize) most of the known results concerning the hyperbolic Cahn–Hilliard equation in the viscous case.

1.1. *Plan of the paper*

In the following section, we rigorously define the functional setting and the basic assumptions. In Section 3, we reformulate our problem in the history phase space and we introduce the strongly continuous semigroup $S_{\varepsilon,\delta}(t)$ describing its solutions. The subsequent Section 4 is devoted to study the dissipativity of the semigroup and it is completed in Section 5, where we show the existence of compact attracting sets, and, consequently, of the global attractor. The comparison between the trajectories of $S_{\varepsilon,\delta}(t)$ and of the limit semigroup on finite time intervals is performed in Section 6. In Section 7, by exploiting the abstract result formulated in the Appendix, we state and prove our main results concerned with exponential attractors.

2. PRELIMINARY TOOLS

This section is devoted to describe the functional setting which will be used to formulate rigorously our problem.

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2.1. Function spaces and operators

Let H be the real Hilbert space $L^2(\Omega)$ of the measurable functions, which are square summable on Ω, endowed with the usual scalar product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$. We define

$$A:\mathscr{D}(A)\to H \quad \text{with } \mathscr{D}(A)=H_0^1(\Omega)\cap H^2(\Omega) \quad \text{and} \quad A=-\Delta$$

As A is a strictly positive operator, we can set

$$V^\sigma=\mathscr{D}(A^{(\sigma+1)/2}) \quad \forall\sigma\in\mathbb{R}$$

Then we introduce the so-called *past history spaces*, suitable to handle systems with memory effects. The history approach is nowadays well established, and following [8, 9] we put $k'(s)=-\mu(s)$, and we assume $\mu\in C^1((0,\infty))\cap L^1(0,\infty)$, with $\mu\geqslant 0$ satisfying

(K1) $\displaystyle\int_0^\infty \mu(s)\,\mathrm{d}s=1$

(K2) $\mu'(s)+\lambda\mu(s)\leqslant 0 \quad \forall s\in(0,\infty) \quad$ for some $\lambda>0$

(K3) $\displaystyle\lim_{s\to 0}\mu(s)=\mu(0)<\infty$

Besides, for $\varepsilon\in(0,1]$, we set

$$\mu_\varepsilon(s)=\frac{1}{\varepsilon^2}\mu\left(\frac{s}{\varepsilon}\right)$$

As $\mu_\varepsilon=-k_\varepsilon'$, assumption (K2) now reads

$$\mu_\varepsilon'(s)+\frac{\lambda}{\varepsilon}\mu_\varepsilon(s)\leqslant 0 \quad \forall s\in(0,\infty)$$

In addition, we have

$$\int_0^\infty \mu_\varepsilon(s)\,\mathrm{d}s=\frac{1}{\varepsilon}, \qquad \int_0^\infty s\mu_\varepsilon(s)\,\mathrm{d}s=1 \tag{5}$$

We can now introduce the family of history spaces

$$\mathscr{M}_\varepsilon^\sigma=L^2_{\mu_\varepsilon}(0,\infty;V^{\sigma-2})$$

endowed with the inner products

$$\langle\eta_1,\eta_2\rangle_{\mathscr{M}_\varepsilon^\sigma}=\int_0^\infty \mu_\varepsilon(s)\langle\eta_1(s),\eta_2(s)\rangle_{V^{\sigma-2}}\,\mathrm{d}s \quad \forall\eta_1,\eta_2\in\mathscr{M}_\varepsilon^\sigma$$

If $\varepsilon=0$, we agree to denote $\mathscr{M}_0^\sigma=\{0\}$. We shall also make use of the linear operator T_ε on $\mathscr{M}_\varepsilon^0$ with domain

$$\mathscr{D}(T_\varepsilon)=\{\eta\in\mathscr{M}_\varepsilon^0:\partial_s\eta\in\mathscr{M}_\varepsilon^0,\ \eta(0)=0\}$$

defined as $T_\varepsilon\eta=-\partial_s\eta$, where $\partial_s\eta$ is the distributional derivative of η with respect to the internal variable s. We recall some of its properties in Theorem A.1 in the Appendix.
The functional setup of our investigation will finally consist of the product Banach spaces

$$\mathscr{H}_\varepsilon^\sigma=V^\sigma\times\mathscr{M}_\varepsilon^\sigma$$

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2.2. *Assumptions on ϕ*

In order to state our results, we need to make some structural assumptions on the nonlinearity. We shall assume $\phi \in C^2(\mathbb{R})$ with $\phi(0)=0$ satisfying

$$\text{(H1)} \quad |\phi''(r)| \leqslant c(1+|r|) \quad \forall r \in \mathbb{R} \quad \text{for some } c \geqslant 0$$

$$\text{(H2)} \quad \liminf_{|r|\to\infty} \phi'(r) > -\lambda_1$$

where $\lambda_1 > 0$ is the first eigenvalue of A. In particular ϕ is bounded below and there exists $\ell > 0$ such that

$$\phi'(r) \geqslant -\ell \quad \forall r \in \mathbb{R} \tag{6}$$

This includes the case of the derivative of a double-well potential $\phi(r)=r^3-r$.

2.3. *Conditions on ε and δ*

Throughout the paper we assume that

$$(\varepsilon, \delta) \in \mathscr{T} = \{\varepsilon, \delta \in [0,1] \, : \, \delta \geqslant \tau\varepsilon \text{ for some } \tau > 0\} \tag{7}$$

Hence, we can consider Equation (3) with *any* ε and δ strictly positive, but if the viscosity coefficient is null only $\varepsilon=0$ is allowed, meaning that we cannot treat the hyperbolic non-viscous case. Besides, when performing the asymptotic analysis as $(\varepsilon, \delta) \to (0,0)$, with (7) we are prescribing that ε is linearly controlled by δ.

Remark 2.1
Throughout the paper we always assume that conditions (H1), (H2), (K1)–(K3) and (7) hold true. All the constants that will appear are understood to be *independent of ε and δ*. In particular the symbol c and $\mathscr{Q}$ will stand for a generic constant and a generic positive increasing function. With $B_{\mathscr{H}^\sigma_\varepsilon}(R)$ we shall denote the ball $\{z \in \mathscr{H}^\sigma_\varepsilon : \|z\|_{\mathscr{H}^\sigma_\varepsilon} \leqslant R\}$.

3. THE DYNAMICAL SYSTEM

Following the history approach in the line of [8], we introduce the *past history* $\eta : [0,\infty) \times (0,\infty) \to H$ defined as

$$\eta^t(s) = \int_0^s A(Au(t-y) + \phi(u(t-y)) + \delta\partial_t u(t-y))\,\mathrm{d}y$$

By a formal integration by parts one obtains

$$\int_0^\infty \mu_\varepsilon(s)\eta^t(s)\,\mathrm{d}s = [k_\varepsilon(s)\eta^t(s)]_0^\infty + \int_0^\infty k_\varepsilon(y) A(Au(t-y) + \phi(u(t-y)) + \delta\partial_t u(t-y))\,\mathrm{d}y$$

where the first contribution is null on account of the decay of k at infinity. In addition, by (formal) derivation of η with respect to t and s we obtain

$$\partial_t \eta = -\partial_s \eta + A(Au + \phi(u) + \delta\partial_t u)$$

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In this way the problem of finding a solution to the relaxed Cahn–Hilliard equation (3) can be replaced by the following problem.

Problem $\mathbf{P}_{\epsilon,\delta}$
For any $T>0$ and any $(u_0,\eta_0)\in\mathscr{H}^0_\varepsilon$, find $z=(u,\eta)\in C([0,T];\mathscr{H}^0_\varepsilon)$ satisfying the equations

$$\begin{cases} \partial_t u+\displaystyle\int_0^\infty \mu_\varepsilon(s)\eta(s)\,\mathrm{d}s=0 & (8)\\ \partial_t\eta=T_\varepsilon\eta+A(Au+\phi(u)+\delta\partial_t u) & (9)\\ (u(0),\eta^0)=(u_0,\eta_0) & (10)\end{cases}$$

As a matter of fact (3) endowed with conditions (4) and problem $\mathbf{P}_{\epsilon,\delta}$ are completely equivalent, see [8, 14] for the details. Observe that in the case $\varepsilon=0$, by formally identifying $\int_0^\infty \mu_\varepsilon(s)\eta(s)\,\mathrm{d}s$ with $A(Au+\phi(u)+\delta\partial_t u)$, we recover the limiting parabolic case (1).

By constructing a suitable approximating Faedo–Galerkin scheme adapted to systems with memory (see e.g. [15]), on account of the subsequent Lemmas 4.2 and 4.4, it is possible to prove the well-posedness of problem $\mathbf{P}_{\epsilon,\delta}$.

Theorem 3.1
For every $(\varepsilon,\delta)\in\mathscr{T}$, problem $\mathbf{P}_{\epsilon,\delta}$ generates a strongly continuous semigroup acting on $\mathscr{H}^0_\varepsilon$.

Remark 3.2
It is important to point out that, if $\varepsilon>0$ and $\delta=0$, we can still prove existence but not uniqueness of solutions in the phase space $\mathscr{H}^0_\varepsilon$, see the continuous dependence estimate Lemma 4.4. It is interesting to note that when working in a higher regularity space, with $u\in H^2(\Omega)\cap H^1_0(\Omega)$, one has the opposite situation: for $\varepsilon>0$ and $\delta=0$ there is uniqueness but not existence (see [6, Lemmas 2.2, 3.2]). On the contrary, when working in the one-dimensional case, the continuous embedding $H^1(\Omega)\subset C(\overline{\Omega})$ allows to recover the whole well-posedness result, as in [8].

In the following, given $z=\big(u_0,\eta_0\big)\in\mathscr{H}^0_\varepsilon$ we agree to denote the semigroup solution as

$$(u(t),\eta^t)=S_{\varepsilon,\delta}(t)z\in\mathscr{H}^0_\varepsilon$$

if $\varepsilon>0$, while $S_{0,0}(t)z=(u(t),0)$.

3.1. Some technical lemmas

We collect here some technical results that we will exploit several times in the course of our investigation. A crucial ingredient consists in a suitable class of functionals, whose role is to recover the dissipation terms hidden in the past history integrals.

Lemma 3.3
Let $\sigma\geqslant -1$ and $(u(t),\eta^t)\in\mathscr{H}^\sigma_\varepsilon$. Then the functional

$$L_\sigma(t)=-\langle u(t),\eta^t\rangle_{\mathscr{M}^{\sigma+1}_\varepsilon}$$

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fulfills the estimates

$$|L_\sigma(t)| \leqslant \|u(t)\|^2_{V^\sigma} + \frac{1}{\varepsilon}\|\eta^t\|^2_{\mathscr{M}^\sigma_\varepsilon} \tag{11}$$

Besides,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\varepsilon L_\sigma(t) + \frac{\delta}{2}\|u(t)\|^2_{V^\sigma}\right] = \varepsilon\|\partial_t u(t)\|^2_{V^{\sigma-1}} - \|u(t)\|^2_{V^{\sigma+1}} - \langle \phi(u(t)), u(t)\rangle_{V^\sigma}$$
$$-\varepsilon\int_0^\infty \mu'_\varepsilon(s)\langle u(t), \eta^t(s)\rangle_{V^{\sigma-1}}\,\mathrm{d}s \tag{12}$$

Proof
The first inequality follows by recalling (5) and the straightforward calculation

$$|L_\sigma| \leqslant \int_0^\infty \mu_\varepsilon(s)\|u\|_{V^\sigma}\|\eta(s)\|_{V^{\sigma-2}}\,\mathrm{d}s \leqslant \|u\|_{V^\sigma}\left(\int_0^\infty \mu_\varepsilon(s)\,\mathrm{d}s\right)^{1/2}\|\eta\|_{\mathscr{M}^\sigma_\varepsilon}$$
$$\leqslant \|u\|^2_{V^\sigma} + \frac{1}{\varepsilon}\|\eta\|^2_{\mathscr{M}^\sigma_\varepsilon}$$

Now a (formal) differentiation of $L_\sigma(t)$ with respect to the time variable, in light of Equation (9) and of the identity

$$\int_0^\infty \mu_\varepsilon(s)\langle \partial_t u, \eta(s)\rangle_{V^{\sigma-1}}\,\mathrm{d}s = -\|\partial_t u\|^2_{V^{\sigma-1}} \tag{13}$$

leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}L_\sigma = -\int_0^\infty \mu_\varepsilon(s)\langle \partial_t u, \eta(s)\rangle_{V^{\sigma-1}}\,\mathrm{d}s - \int_0^\infty \mu_\varepsilon(s)\langle u, \partial_t \eta(s)\rangle_{V^{\sigma-1}}\,\mathrm{d}s$$
$$= \|\partial_t u\|^2_{V^{\sigma-1}} - \int_0^\infty \mu_\varepsilon(s)\langle u, T_\varepsilon\eta(s)\rangle_{V^{\sigma-1}}\,\mathrm{d}s - \frac{1}{\varepsilon}\|u\|^2_{V^{\sigma+1}} - \frac{1}{\varepsilon}\langle u, \phi(u)\rangle_{V^\sigma} - \frac{\delta}{\varepsilon}\langle u, \partial_t u\rangle_{V^\sigma}$$

A formal integration by parts in light of (K2) yields (cf. [14])

$$\int_0^\infty \mu_\varepsilon(s)\langle u, T_\varepsilon\eta(s)\rangle_{V^{\sigma-1}}\,\mathrm{d}s = \int_0^\infty \mu'_\varepsilon(s)\langle u, \eta(s)\rangle_{V^{\sigma-1}}\,\mathrm{d}s$$

and inequality (12) is proven. □

Lemma 3.4
Let $\sigma \geqslant 0$ and $(u(t), \eta^t) \in \mathscr{H}^\sigma_\varepsilon$. Then, the following differential inequality holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}[\|u(t)\|^2_{V^\sigma} + \|\eta^t\|^2_{\mathscr{M}^\sigma_\varepsilon}] + 2\delta\|\partial_t u(t)\|^2_{V^{\sigma-1}} + \frac{\lambda}{\varepsilon}\|\eta^t\|^2_{\mathscr{M}^\sigma_\varepsilon}$$
$$+2\langle \partial_t u(t), A^\sigma\phi(u(t))\rangle - \int_0^\infty \mu'_\varepsilon(s)\|\eta^t(s)\|^2_{V^{\sigma-2}}\,\mathrm{d}s \leqslant 0 \tag{14}$$

Proof
By multiplying Equation (8) by $A^{\sigma}(Au+\phi(u)+\delta\partial_t u)$ in H, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{V^{\sigma}}+\langle\partial_t u, A^{\sigma}\phi(u)\rangle+\delta\|\partial_t u\|^2_{V^{\sigma-1}}+\int_0^{\infty}\mu_{\varepsilon}(s)\langle\eta(s), A^{\sigma}(Au+\phi(u)+\delta\partial_t u)\rangle\,\mathrm{d}s=0$$

The product of Equation (9) by η in $\mathscr{M}^{\sigma}_{\varepsilon}$ leads to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\eta\|^2_{\mathscr{M}^{\sigma}_{\varepsilon}}=\int_0^{\infty}\mu_{\varepsilon}(s)\langle T_{\varepsilon}\eta(s),\eta(s)\rangle_{V^{\sigma-2}}\,\mathrm{d}s+\int_0^{\infty}\mu_{\varepsilon}(s)\langle\eta(s), A^{\sigma}(Au+\phi(u)+\delta\partial_t u)\rangle\,\mathrm{d}s$$

Hence, adding the results, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}[\|u\|^2_{V^{\sigma}}+\|\eta\|^2_{\mathscr{M}^{\sigma}_{\varepsilon}}]+2\langle\partial_t u, A^{\sigma}\phi(u)\rangle+2\delta\|\partial_t u\|^2_{V^{\sigma-1}}=2\int_0^{\infty}\mu_{\varepsilon}(s)\langle T_{\varepsilon}\eta(s),\eta(s)\rangle_{V^{\sigma-2}}\,\mathrm{d}s$$

As in the proof of the previous lemma, thanks to (K2) we have

$$2\int_0^{\infty}\mu_{\varepsilon}(s)\langle T_{\varepsilon}\eta(s),\eta(s)\rangle_{V^{\sigma-2}}\,\mathrm{d}s\leqslant-\frac{\lambda}{\varepsilon}\|\eta\|^2_{\mathscr{M}^{\sigma}_{\varepsilon}}-\int_0^{\infty}\mu'_{\varepsilon}(s)\|\eta(s)\|^2_{V^{\sigma-2}}\,\mathrm{d}s$$

proving (14). □

In the following sections we shall also make use of the inequality

$$\|\partial_t u\|^2_{V^{\sigma-2}}\leqslant\frac{1}{\varepsilon}\|\eta\|^2_{\mathscr{M}^{\sigma}_{\varepsilon}} \tag{15}$$

holding if $\eta\in\mathscr{M}^{\sigma}_{\varepsilon}$. It can be easily recovered by multiplying Equation (8) by $A^{\sigma+1}\partial_t u$ and exploiting (5). We finally report a generalized version of the Gronwall lemma that we shall exploit several times in the following (see for instance [16, Lemma 2.2]).

Lemma 3.5
Let $\Lambda:[0,\infty)\to[0,\infty)$ be an absolutely continuous function satisfying

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda(t)+2\nu\Lambda(t)\leqslant h(t)\Lambda(t)+k$$

where $\nu>0, k\geqslant 0$ and $\int_s^t h(\tau)\,\mathrm{d}\tau\leqslant\nu(t-s)+m$, for all $t\geqslant s\geqslant 0$ and some $m\geqslant 0$. Then

$$\Lambda(t)\leqslant\Lambda(0)\mathrm{e}^{m}\mathrm{e}^{-\nu t}+\frac{k\mathrm{e}^{m}}{\nu}$$

4. DISSIPATIVITY

In this section we analyze the dissipative character of the semigroup $S_{\varepsilon,\delta}(t)$, showing the existence of a bounded absorbing set in $\mathscr{H}^0_{\varepsilon}$.

Proposition 4.1
For every $(\varepsilon,\delta)\in\mathscr{T}$, the semigroup $S_{\varepsilon,\delta}(t)$ possesses an absorbing set $\mathscr{B}^0_{\varepsilon}\subset\mathscr{H}^0_{\varepsilon}$, which is bounded in $\mathscr{H}^0_{\varepsilon}$ with a bound independent of ε and δ. Precisely, for any bounded set $\mathscr{B}\subset\mathscr{H}^0_{\varepsilon}$, there exists

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a time $t_0\geqslant 0$, depending on $\mathscr{B}$ but independent of ε and δ, such that

$$S_{\varepsilon,\delta}(t)\mathscr{B}\subset\mathscr{B}^0_\varepsilon \quad \forall t\geqslant t_0$$

This fact is straightforward consequence of the following lemma.

Lemma 4.2
There exists $C_0\geqslant 0$ such that

$$\|S_{\varepsilon,\delta}(t)z\|_{\mathscr{H}^0_\varepsilon}\leqslant\mathscr{Q}(\|z\|_{\mathscr{H}^0_\varepsilon})\mathrm{e}^{-t}+C_0$$

Besides,

$$\int_0^\infty \|A^{-1/2}\partial_t u(y)\|^2\,\mathrm{d}y\leqslant\mathscr{Q}(\|z\|_{\mathscr{H}^0_\varepsilon}) \tag{16}$$

and

$$\sup_{t\geqslant 0}\int_t^{t+1}\|Au(y)\|^2\,\mathrm{d}y\leqslant\mathscr{Q}(\|z\|_{\mathscr{H}^0_\varepsilon}) \tag{17}$$

Proof
Set $\Phi(r)=\int_0^r\phi(s)\,\mathrm{d}s$ and consider the functional

$$\Lambda_0(t)=\|u(t)\|^2_{V^0}+\|\eta^t\|^2_{\mathscr{M}^0_\varepsilon}+2v\varepsilon L_{-1}(t)+v\delta\|u(t)\|^2+2\langle\Phi(u(t)),1\rangle$$

where $v\geqslant 0$ will be chosen small enough in order to guarantee the validity of all the estimates below. In light of (H1), there exists $0<\vartheta<\frac{1}{2}$ such that

$$\|u\|^2_{V^0}+2\langle\Phi(u),1\rangle\geqslant 2\vartheta\|u\|^2_{V^0}-c$$
$$\|u\|^2_{V^0}+\langle\phi(u),u\rangle\geqslant\vartheta\|u\|^2_{V^0}-c$$

Hence, using (H2) and (11),

$$2\vartheta\|S_{\varepsilon,\delta}(t)z\|^2_{\mathscr{H}^0_\varepsilon}-c\leqslant\Lambda_0(t)\leqslant\mathscr{Q}(\|S_{\varepsilon,\delta}(t)z\|_{\mathscr{H}^0_\varepsilon})+c$$

Adding together (14) for $\sigma=0$ and v times (12) for $\sigma=-1$, we obtain the differential inequality

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_0+\frac{\lambda}{\varepsilon}\|\eta\|^2_{\mathscr{M}^0_\varepsilon}-\int_0^\infty\mu'_\varepsilon(s)\|\eta(s)\|^2_{V^{-2}}\,\mathrm{d}s+2\delta\|\partial_t u\|^2-v\varepsilon\|A^{-1/2}\partial_t u\|^2\\&\quad\leqslant-2v\|u\|^2_{V^0}-2v\langle\phi(u),u\rangle-2v\varepsilon\int_0^\infty\mu'_\varepsilon(s)\langle u,\eta(s)\rangle_{V^{-2}}\,\mathrm{d}s\\&\quad\leqslant-2v\vartheta\|u\|^2_{V^0}-2v\varepsilon\int_0^\infty\mu'_\varepsilon(s)\langle u,\eta(s)\rangle_{V^{-2}}\,\mathrm{d}s+c\end{aligned}$$

Letting $v=0$ we learn that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_0+\frac{\lambda}{\varepsilon}\|\eta\|^2_{\mathscr{M}^0_\varepsilon}\leqslant 0$$

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An integration on $[0,t]$ leads to

$$\|S_{\varepsilon,\delta}(t)z\|_{\mathscr{H}^0_\varepsilon}\leqslant\mathscr{Q}(\|z\|_{\mathscr{H}^0_\varepsilon})\quad\forall t\geqslant 0 \tag{18}$$

together with the boundedness of $\varepsilon^{-1}\int_0^\infty\|\eta^t(y)\|^2_{\mathscr{M}^0_\varepsilon}\,\mathrm{d}y$. Then the required integral control for $\|A^{-1/2}\partial_t u\|^2$ follows by recalling (15). If $v>0$ is chosen small enough, in light of (K3) we can control the right-hand side of the differential inequality as

$$-2v\varepsilon\int_0^\infty\mu_\varepsilon'(s)\langle u,\eta(s)\rangle_{V^{-2}}\,\mathrm{d}s\leqslant\frac{v\vartheta}{2}\|u\|^2_{V^0}-\frac{1}{2}\int_0^\infty\mu_\varepsilon'(s)\|\eta(s)\|^2_{V^{-3}}\,\mathrm{d}s$$

and in light of (15)

$$v\varepsilon\|A^{-1/2}\partial_t u(t)\|^2\leqslant v\|\eta^t\|^2_{\mathscr{M}^0_\varepsilon}$$

We thus end up with the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_0(t)+v\|S_{\varepsilon,\delta}(t)z\|^2_{\mathscr{H}^0_\varepsilon}\leqslant c$$

A generalized version of the Gronwall lemma (see [17, Lemma 2.7]) provides the existence of $C_0\geqslant 0$ and a time $t_0=t_0(\|z\|_{\mathscr{H}^0_\varepsilon})>0$ such that

$$\|S_{\varepsilon,\delta}(t)z\|_{\mathscr{H}^0_\varepsilon}\leqslant C_0\quad\forall t\geqslant t_0$$

Combining this estimate with (18), the conclusion follows. We are left to show the validity of the integral control (17). To this aim we modify the functional Λ_0 by using L_0 instead of L_1, namely we define

$$\Theta(t)=\|u(t)\|^2_{V^0}+\|\eta^t\|^2_{\mathscr{M}^0_\varepsilon}+2v\varepsilon L_0(t)+v\delta\|u(t)\|^2+2\langle\Phi(u(t)),1\rangle$$

with $v>0$ to be chosen. It is immediate to check that

$$\|\Theta(t)\|_{\mathscr{H}^0_\varepsilon}\leqslant c,\quad t\geqslant 0 \tag{19}$$

Besides, using inequalities (14) and v times (12) both for $\sigma=0$, the following differential inequality holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Theta+2v\|u\|^2_{V^1}+\frac{\lambda}{\varepsilon}\|\eta\|^2_{\mathscr{M}^0_\varepsilon}-\int_0^\infty\mu_\varepsilon'(s)\|\eta(s)\|^2_{V^{-2}}\,\mathrm{d}s+2(\delta-v\varepsilon)\|\partial_t u\|^2$$
$$\leqslant-2v\langle\phi(u),u\rangle_{V^0}-2v\varepsilon\int_0^\infty\mu_\varepsilon'(s)\langle u,\eta(s)\rangle_{V^{-1}}\,\mathrm{d}s$$

Choosing v small enough, in light of (K3) and $\|u\|_{V^0}\leqslant c$, the right-hand side can be controlled as

$$-2v\langle\phi(u),u\rangle_{V^0}-2v\varepsilon\int_0^\infty\mu_\varepsilon'(s)\langle u,\eta(s)\rangle_{V^{-2}}\,\mathrm{d}s$$
$$\leqslant 2v\|u\|_{V^1}\|\phi(u)\|-\|u\|_{V^0}\int_0^\infty\mu_\varepsilon'(s)\|\eta(s)\|_{V^{-2}}\,\mathrm{d}s$$
$$\leqslant v\|u\|^2_{V^1}-\frac{1}{2}\int_0^\infty\mu_\varepsilon'(s)\|\eta(s)\|^2_{V^{-2}}\,\mathrm{d}s+c$$

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Finally, since $(\varepsilon,\delta)\in\mathscr{T}$, we can choose v sufficiently small to ensure $\delta-v\varepsilon\geqslant 0$ and we end up with

$$\frac{d}{dt}\Theta+v\|u\|^2_{V^1}\leqslant c$$

An application of the Gronwall lemma, in light of (19), immediately provides (17). □

Remark 4.3
It is worth noting that a recent cutoff technique introduced in [18] allows in many cases to handle singular kernels in the origin; thus, removing condition (K3). Unfortunately, this idea is not applicable to the present work, due to the crucial role played by (13) in Lemma 3.3.

In light of Lemma 4.2, we learn that any ball $B_{\mathscr{H}^0_\varepsilon}(R)$ of large enough radius is a bounded absorbing set for $S_{\varepsilon,\delta}(t)$ in $\mathscr{H}^0_\varepsilon$. A closer look at the proof above shows that this still holds true for $\varepsilon>0$ and $\delta=0$. On the contrary the assumption $\delta\geqslant\tau\varepsilon$ is needed to obtain the further integral control (17), which constitutes a crucial ingredient in the proof of the continuous dependence estimate.

Lemma 4.4
For any initial data $z_1,z_2\in\mathscr{H}^0_\varepsilon$, the inequality

$$\|S_{\varepsilon,\delta}(t)z_1-S_{\varepsilon,\delta}(t)z_2\|_{\mathscr{H}^0_\varepsilon}\leqslant c\mathrm{e}^{ct}\|z_1-z_2\|_{\mathscr{H}^0_\varepsilon} \tag{20}$$

holds for some $c\geqslant 0$ only depending on $\|z_i\|_{\mathscr{H}^0_\varepsilon}$.

Proof
Given two solutions $(u^1(t),\eta^{1t})$ and $(u^2(t),\eta^{2t})$ corresponding to different data z_1 and z_2, the difference $(\bar{u}(t),\bar{\eta}^t)$ fulfills the system

$$\begin{cases}\partial_t\bar{u}+\displaystyle\int_0^\infty\mu_\varepsilon(s)\bar{\eta}(s)\,ds=0\\ \partial_t\bar{\eta}=T_\varepsilon\bar{\eta}+A[A\bar{u}+\delta\partial_t\bar{u}]+A\phi(u^1)-A\phi(u^2)\\ (\bar{u}(0),\bar{\eta}^0)=z_1-z_2\end{cases}$$

We multiply the first equation by $A\bar{u}+\delta\partial_t\bar{u}$ and the second one by $\bar{\eta}$ in $\mathscr{M}^0_\varepsilon$; hence, obtaining

$$\frac{d}{dt}[\|\bar{u}\|^2_{V^0}+\|\bar{\eta}\|^2_{\mathscr{M}^0_\varepsilon}]+\frac{\lambda}{\varepsilon}\|\bar{\eta}\|^2_{\mathscr{M}^0_\varepsilon}+\delta\|\partial_t\bar{u}\|^2\leqslant -2\langle A\phi(u^1)-A\phi(u^2),\bar{\eta}\rangle_{\mathscr{M}^0_\varepsilon}$$

$$\leqslant\frac{\lambda}{2\varepsilon}\|\bar{\eta}\|^2_{\mathscr{M}^0_\varepsilon}+c\|\phi(u^1)-\phi(u^2)\|^2_{V^0}$$

From the local Lipschitz continuity of ϕ' and taking into account Lemma 4.2 we have

$$\|\phi(u^1)-\phi(u^2)\|_{V^0}\leqslant c(\|Au^1\|^2+\|Au^2\|^2)\|\bar{u}\|_{V^0}$$

for some c depending on $\|z_i\|_{\mathscr{H}^0_\varepsilon}$. Hence, we end up with the inequality

$$\frac{d}{dt}[\|\bar{u}(t)\|^2_{V^0}+\|\bar{\eta}^t\|^2_{\mathscr{M}^0_\varepsilon}]\leqslant f(t)\|\bar{u}(t)\|^2_{V^0}$$

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with $f(t)=c(\|Au^1(t)\|^2+\|Au^2(t)\|^2)$: on account of (17), the Gronwall lemma provides the thesis. $\square$

4.1. Higher-order dissipativity

Lemma 4.5
There exists $k_1>0$ such that, for all $(\varepsilon,\delta)\in\mathscr{T}$,

$$\|S_{\varepsilon,\delta}(t)z\|_{\mathscr{H}^1_\varepsilon}\leqslant\mathscr{Q}(\|z\|_{\mathscr{H}^1_\varepsilon})\mathrm{e}^{-k_1 t}+\mathscr{Q}(R),\quad t\in[0,\infty)$$

whenever $z\in\mathscr{H}^1_\varepsilon$ with $\|z\|_{\mathscr{H}^0_\varepsilon}\leqslant R$.

Proof
In the proof, the generic constant c may depend (increasingly) on R. Let $z\in\mathscr{H}^1_\varepsilon$ be such that $\|z\|_{\mathscr{H}^0_\varepsilon}\leqslant R$. In particular, from Lemma 4.2, we know that $\|S_{\varepsilon,\delta}(t)z\|_{\mathscr{H}^0_\varepsilon}\leqslant C$. For $v>0$ to be chosen small enough so that all the estimates below are satisfied, we consider the functional

$$\Lambda_1(t)=\|u(t)\|^2_{V^1}+\|\eta^t\|^2_{\mathscr{M}^1_\varepsilon}+2v\varepsilon L_1(t)+v\delta\|u(t)\|^2_{V^0}+2\langle u(t),A\phi(u(t))\rangle$$

Thanks to (11) for $\sigma=1$ and by the growth assumption (H1), we easily see that Λ_1 fulfills the inequalities

$$k\|S_{\varepsilon,\delta}(t)z\|^2_{\mathscr{H}^1_\varepsilon}-c\leqslant\Lambda_1(t)\leqslant K\|S_{\varepsilon,\delta}(t)z\|^2_{\mathscr{H}^1_\varepsilon}-c$$

for v small enough and c large enough. Adding together inequalities (14) and v by (12) with $\sigma=1$, we obtain the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_1+2v\|u\|^2_{V^2}+\frac{\lambda}{\varepsilon}\|\eta\|^2_{\mathscr{M}^1_\varepsilon}+2(\delta-v\varepsilon)\|\partial_t u\|^2_{V^0}-\int_0^\infty\mu'_\varepsilon(s)\|\eta(s)\|^2_{V^{-1}}\,\mathrm{d}s$$
$$\leqslant-2v\langle\phi(u),u\rangle_{V^1}-2\langle\phi'(u)\partial_t u,Au\rangle-2v\varepsilon\int_0^\infty\mu'_\varepsilon(s)\langle u,\eta(s)\rangle_{V^0}\,\mathrm{d}s$$

We have the control

$$-2v\varepsilon\int_0^\infty\mu'_\varepsilon(s)\langle u,\eta(s)\rangle_{V^0}\,\mathrm{d}s\leqslant-\frac{1}{2}\int_0^\infty\mu'_\varepsilon(s)\|\eta(s)\|^2_{V^{-1}}\,\mathrm{d}s+\frac{v}{2}\|u\|^2_{V^2}$$

Besides, by the standard Sobolev embeddings and Agmon inequality, the nonlinear terms can be estimated as

$$\begin{aligned}2\langle\phi'(u)\partial_t u,Au\rangle&\leqslant2\|A^{-1/2}\partial_t u\|(\|\phi''(u)\nabla uAu\|+\|\phi'(u)\nabla Au\|)\\&\leqslant2\|A^{-1/2}\partial_t u\|(\|\phi''(u)\|_{L^6}\|\nabla u\|_{L^6}\|Au\|_{L^6}+\|\phi'(u)\|_{L^\infty}\|A^{3/2}u\|)\\&\leqslant c(1+\|Au\|)\|A^{-1/2}\partial_t u\|\|A^{3/2}u\|\\&\leqslant\frac{v}{2}\|A^{3/2}u\|^2+c\|A^{-1/2}\partial_t u\|^2\|Au\|^2+c\|A^{-1/2}\partial_t u\|^2\end{aligned}$$

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and

$$-2v\langle\phi(u),u\rangle_{V^1}\leqslant 2v\|u\|_{V^2}\|\phi'(u)A^{1/2}u\|\leqslant cv\|u\|_{V^2}(1+\|Au\|)\leqslant\frac{v}{2}\|u\|_{V^2}^2+c$$

Hence, up to further reducing v in order to ensure $2(\delta-v\varepsilon)\geqslant\delta$, we end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_1+v\Lambda_1+\delta\|\partial_t u\|_{V^0}^2\leqslant\|A^{-1/2}\partial_t u\|^2\Lambda_1+\|A^{-1/2}\partial_t u\|^2+c \tag{21}$$

The dissipation integral (16) allows the application of the Gronwall lemma 3.5, from which the thesis follows. □

Remark 4.6
Observe that an integration of (21) on $(0,t)$ leads to the integral control

$$\delta\int_0^t\|A^{1/2}\partial_t u(y)\|^2\,\mathrm{d}y\leqslant c(1+t)\quad\forall t>0 \tag{22}$$

for some $c\geqslant 0$ only depending on $\|(u_0,\eta_0)\|_{\mathscr{H}_\varepsilon^1}$.

Lemma 4.7
There exists $k_2>0$ such that, for all $(\varepsilon,\delta)\in\mathscr{T}$,

$$\|S_{\varepsilon,\delta}(t)z\|_{\mathscr{H}_\varepsilon^2}\leqslant\mathscr{Q}(\|z\|_{\mathscr{H}_\varepsilon^2})\mathrm{e}^{-k_2 t}+\mathscr{Q}(R),\quad t\in[0,\infty)$$

whenever $z\in\mathscr{H}_\varepsilon^2$ with $\|z\|_{\mathscr{H}_\varepsilon^1}\leqslant R$.

Proof
In the proof, the generic constant c may depend (increasingly) on R. Let $z\in\mathscr{H}_\varepsilon^2$ be such that $\|z\|_{\mathscr{H}_\varepsilon^1}\leqslant R$; then from Lemma 4.5 and the Agmon inequality we know that $\|u\|_{L^\infty}\leqslant c$. In particular by the growth assumptions on ϕ we learn that $\sum_{i=0}^2\|\phi^{(i)}(u)\|_{L^\infty}\leqslant c$.

Setting $\sigma=2$, we add together inequalities (14) and v by (12), for some positive v to be chosen small enough. The resulting inequality reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_2+2v\|u\|_{V^3}^2+\frac{\lambda}{\varepsilon}\|\eta\|_{\mathscr{M}_\varepsilon^2}^2+2(\delta-v\varepsilon)\|\partial_t u\|_{V^1}^2-\int_0^\infty\mu_\varepsilon'(s)\,\|\eta(s)\|_{V^0}^2\,\mathrm{d}s$$
$$\leqslant-2v\langle\phi(u),u\rangle_{V^2}-2\langle A^{3/2}u,A^{1/2}(\phi'(u)\partial_t u)\rangle-2v\varepsilon\int_0^\infty\mu_\varepsilon'(s)\langle u,\eta(s)\rangle_{V^1}\,\mathrm{d}s$$

having set

$$\Lambda_2(t)=\|u(t)\|_{V^2}^2+\|\eta^t\|_{\mathscr{M}_\varepsilon^2}^2+2v\varepsilon L_2(t)+v\delta\|u(t)\|_{V^1}^2+2\langle Au(t),A\phi(u(t))\rangle$$

Choosing v small enough we have

$$-2v\langle\phi(u),u\rangle_{V^2}-2v\varepsilon\int_0^\infty\mu_\varepsilon'(s)\langle u,\eta(s)\rangle_{V^1}\,\mathrm{d}s\leqslant-\frac{1}{2}\int_0^\infty\mu_\varepsilon'(s)\|\eta(s)\|_{V^0}^2\,\mathrm{d}s+v\|u\|_{V^3}^2+cv$$

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Note that, in light of (15), there holds

$$\|A^{1/2}(\phi'(u)\partial_t u)\| \leqslant \|\phi''(u)A^{1/2}u\partial_t u\| + \|\phi'(u)A^{1/2}\partial_t u\| \leqslant c\|A^{1/2}\partial_t u\| \leqslant \frac{c}{\sqrt{\varepsilon}}\|\eta\|_{\mathcal{M}_\varepsilon^2}$$

Hence, by interpolation we have

$$-2\langle A^{3/2}u, A^{1/2}(\phi'(u)\partial_t u)\rangle \leqslant 2\|A^{3/2}u\|\|A^{1/2}(\phi'(u)\partial_t u)\| \leqslant c\|A^{3/2}u\|^2 + \frac{\lambda}{2\varepsilon}\|\eta\|^2_{\mathcal{M}_\varepsilon^2}$$

$$\leqslant \frac{\lambda}{2\varepsilon}\|\eta\|^2_{\mathcal{M}_\varepsilon^2} + \frac{\nu}{4}\|u\|^2_{V^3} + c$$

We finally obtain the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_2 + \frac{\nu}{8}\|u\|^2_{V^3} + \frac{\lambda}{\varepsilon}\|\eta\|^2_{\mathcal{M}_\varepsilon^2} \leqslant c$$

By (11) we easily realize that Λ_2 fulfills the inequalities

$$k\|S_{\varepsilon,\delta}(t)z\|^2_{\mathcal{H}_\varepsilon^2} - c \leqslant \Lambda_2(t) \leqslant K\|S_{\varepsilon,\delta}(t)z\|^2_{\mathcal{H}_\varepsilon^2} + c$$

for some constants $0<k\leqslant K$. We finally obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_2 + \nu\Lambda_2 \leqslant c$$

for some $\nu>0$ small enough, and the desired inequality is achieved by applying the Gronwall lemma. □

Remark 4.8
Lemmas 4.5 and 4.7 imply that any ball $B_{\mathcal{H}_\varepsilon^i}(R_i)$ of radius R_i large enough is a bounded absorbing set for $S_{\varepsilon,\delta}(t)$ in $\mathcal{H}_\varepsilon^i$, for $i=1,2$.

5. REGULAR EXPONENTIALLY ATTRACTING SETS

The aim of this section is to prove that $S_{\varepsilon,\delta}(t)$ is asymptotically compact on $\mathcal{H}_\varepsilon^0$. This will be precisely stated and proved in our main Theorem 5.4: as a straightforward corollary, the classical theory of dynamical systems guarantees the existence of a compact set, which is attracting and fully invariant for the semigroup, the so-called *global attractor*.

Theorem 5.1
For every $(\varepsilon,\delta)\in\mathcal{T}$, the semigroup $S_{\varepsilon,\delta}(t)$ acting on $\mathcal{H}_\varepsilon^0$ possesses a connected global attractor $\mathcal{A}_{\varepsilon,\delta}$, which is bounded in $\mathcal{H}_\varepsilon^2$ uniformly with respect to ε and δ.

The first step to accomplish our purpose consists in proving the existence of absorbing sets in a more regular phase space, which is compactly embedded into $\mathcal{H}_\varepsilon^0$. As it is well known, the embedding $\mathcal{H}_\varepsilon^\sigma \subset \mathcal{H}_\varepsilon^0$ for $\sigma>0$ is not compact, due to the presence of the history component. A standard way to recover compactness (see [8]) is to define the family of spaces $\mathcal{L}_\varepsilon^\sigma \subset \mathcal{M}_\varepsilon^\sigma$,

$$\mathcal{L}_\varepsilon^\sigma = \{\eta\in\mathcal{M}_\varepsilon^\sigma \cap \mathcal{D}(T_\varepsilon) : \sup_{x\geqslant 1} x\mathcal{T}_\eta^\varepsilon(x) < \infty\}$$

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where the *tail* of η is given by

$$\mathscr{T}^{\varepsilon}_{\eta}(x)=\int_{(0,\varepsilon/x)\cup(\varepsilon x,\infty)}\mu_\varepsilon(s)\|A^{-1}\eta(s)\|^2\,\mathrm{d}s$$

The space $\mathscr{L}^\sigma_\varepsilon$ endowed with the norm

$$\|\eta\|^2_{\mathscr{L}^\sigma_\varepsilon}=\|\eta\|^2_{\mathscr{M}^\sigma_\varepsilon}+|||\eta|||^2_\varepsilon$$

where

$$|||\eta|||^2_\varepsilon=\varepsilon^2\|T_\varepsilon\eta\|^2_{\mathscr{M}^0_\varepsilon}+\sup_{x\geqslant 1}x\mathscr{T}^\varepsilon_\eta(x)\quad \forall\eta\in\mathscr{L}^\sigma_\varepsilon$$

is a Banach space and the embedding $\mathscr{L}^\sigma_\varepsilon\subset\mathscr{M}^\sigma_\varepsilon$ turns out to be compact. Hence, for all $\sigma>0$ the product space

$$\mathscr{Z}^\sigma_\varepsilon=V^\sigma\times\mathscr{L}^\sigma_\varepsilon$$

is compactly embedded into $\mathscr{H}^0_\varepsilon$ (see [8, Lemma 5.1]).

Lemma 5.2
Assume that for some $\Theta\geqslant 0$

$$\|Au(t)\|^2+\delta\int_0^t\|\partial_t u(y)\|^2\,\mathrm{d}y\leqslant\Theta\quad\forall t\geqslant 0$$

If η^t satisfies the differential equation (9) with initial condition $\eta_0\in\mathscr{D}(T_\varepsilon)$, then $\eta^t\in\mathscr{D}(T_\varepsilon)$ for all $t\geqslant 0$ and there exists $c\geqslant 0$ such that

$$|||\eta^t|||^2_\varepsilon\leqslant 2(t+2)\mathrm{e}^{-2\delta t}|||\eta_0|||^2_\varepsilon+c\Theta$$

Proof
Argue exactly as in [9, Lemmas 3.3 and 3.4]. Looking at those proofs, the only difference consists in the derivation of the inequality

$$\|A^{-1}\eta^t(s)\|^2\leqslant 2\Theta s^2+2\Theta s+2\|A^{-1}\eta_0(s-t)\|^2$$

which is crucial in order to get the control on the tail. But this follows by the representation formula (A1) in the Appendix,

$$\begin{aligned}\|A^{-1}\eta^t(s)\|^2&\leqslant 2\|A^{-1}\eta_0(s-t)\|^2\\&\quad+2s\int_0^s(\|Au(t-\tau)\|^2+\|\phi(u(t-\tau))\|^2+\delta^2\|\partial_t u(t-\tau)\|^2)\,\mathrm{d}\tau\\&\leqslant 2\|A^{1/2}\eta_0(s-t)\|^2+2\Theta s^2+2\delta\Theta s\end{aligned}$$

Now the proof goes exactly as in [9, Lemma 3.4]. □

In light of Lemmas 4.5 and 4.7 and applying Lemma 5.2 it turns out that any closed ball in $\mathscr{Z}^\sigma_\varepsilon$ with sufficiently large radius is an absorbing set in $\mathscr{Z}^\sigma_\varepsilon$.

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Proposition 5.3
Let $\sigma=1,2$. Then the semigroup $S_{\varepsilon,\delta}(t)$ maps $\mathscr{Z}_\varepsilon^\sigma$ into $\mathscr{Z}_\varepsilon^\sigma$. Besides, there exists $R_\sigma>0$ such that the closed ball

$$\mathscr{B}_\varepsilon^\sigma=\{z\in\mathscr{Z}_\varepsilon^\sigma : \|z\|_{\mathscr{Z}_\varepsilon^\sigma}\leqslant R_\sigma\}$$

is a bounded absorbing set for the restriction of $S_{\varepsilon,\delta}(t)$ to $\mathscr{Z}_\varepsilon^\sigma$.

As a matter of fact $\mathscr{B}_\varepsilon^2$ is exponentially attracting in $\mathscr{H}_\varepsilon^0$.

Theorem 5.4
Up to (possibly) enlarging R_2, the ball $\mathscr{B}_\varepsilon^2$ exponentially attracts bounded sets of $\mathscr{H}_\varepsilon^0$, namely, there exists $k>0$ such that

$$\mathrm{dist}_{\mathscr{H}_\varepsilon^0}(S_{\varepsilon,\delta}(t)\mathscr{B},\mathscr{B}_\varepsilon^2)\leqslant\mathscr{Q}(R)\mathrm{e}^{-\kappa t}\quad\forall t\geqslant 0$$

for every $\mathscr{B}\subset B_{\mathscr{H}_\varepsilon^0}(R)$.

In light of the continuous dependence estimate (20) and thanks to the transitivity of the exponential attraction [19, Theorem 5.1], it is sufficient to prove that the absorbing set $\mathscr{B}_\varepsilon^0$ on $\mathscr{H}_\varepsilon^0$ is exponentially attracted by an absorbing ball on $\mathscr{H}_\varepsilon^1$, which is in turn exponentially attracted by an absorbing ball on $\mathscr{H}_\varepsilon^2$.

The proof of this fact is based on a suitable decomposition of the solution semigroup, as in [8, Lemma 7.5]. For any fixed $z=(u_0,\eta_0)\in\mathscr{B}_\varepsilon^0$, we define

$$S_{\varepsilon,\delta}(t)z=D_{\varepsilon,\delta}(t)z+N_{\varepsilon,\delta}(t)z$$

with $D_{\varepsilon,\delta}(t)z=(v(t),\xi^t)$ and $N_{\varepsilon,\delta}(t)z=(w(t),\chi^t)$ are the solutions to the problems

$$\begin{cases}\partial_t w+\displaystyle\int_0^\infty \mu_\varepsilon(s)\chi(s)\,\mathrm{d}s=0\\ \partial_t\chi=T_\varepsilon\chi+A[Aw+\delta\partial_t w]+A\phi(w)+\ell^2 w-\ell^2 u\\ (w(0),\chi^0)=(0,0)\end{cases}$$

and

$$\begin{cases}\partial_t v+\displaystyle\int_0^\infty \mu_\varepsilon(s)\xi(s)\,\mathrm{d}s=0\\ \partial_t\xi=T_\varepsilon\xi+A[Av+\delta\partial_t v]+A\phi(u)-A\phi(w)+\ell^2 v\\ (v(0),\xi^0)=(u_0,\eta_0)\end{cases}$$

where $\ell>0$ is chosen as in (6). Arguing exactly as in Lemmas 4.5 and 4.7, keeping in mind that the initial conditions are null, it is immediate to realize that

Lemma 5.5
If $\|z\|_{\mathscr{H}_\varepsilon^{\sigma-1}}\leqslant R$ for $\sigma=1,2$, then

$$\|K_\varepsilon(t)z\|_{\mathscr{H}_\varepsilon^\sigma}\leqslant\mathscr{Q}(R)\quad\forall t\geqslant 0$$

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Besides, for some $C>0$ depending only on $\|z\|_{\mathscr{H}^0_\varepsilon}$,

$$\int_0^\infty \|A^{-1/2}\partial_t w(y)\|^2\,\mathrm{d}y\leqslant C \tag{23}$$

Lemma 5.6
There exists $\kappa>0$ and $M\geqslant 0$ such that

$$\|D_{\varepsilon,\delta}(t)z\|_{\mathscr{H}^0_\varepsilon}\leqslant M\mathrm{e}^{-\kappa t}$$

Proof
We argue in a similar way as in the proof of Lemma 4.2; hence, we only give the main steps. Set $\Phi(r)=\int_0^r \phi(s)\,\mathrm{d}s$ and consider the functional

$$\Lambda(t)=\|D_{\varepsilon,\delta}(t)z\|_{\mathscr{H}^0_\varepsilon}+2\nu\varepsilon\langle v(t),\xi^t\rangle_{\mathscr{M}^0_\varepsilon}+\nu\delta\|v(t)\|^2+2\langle\Phi(u(t))-\Phi(w(t))-\phi(w(t))v(t),1\rangle$$

where $\nu\geqslant 0$ will be chosen small enough in order to guarantee the validity of all the estimates below. Then Λ satisfies the differential inequality

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Lambda+\frac{\lambda}{\varepsilon}\|\xi\|^2_{\mathscr{M}^0_\varepsilon}-\int_0^\infty \mu_\varepsilon'(s)\|\xi(s)\|^2_{V^{-2}}\,\mathrm{d}s+2\delta\|\partial_t v\|^2-\nu\|\xi\|^2_{\mathscr{M}^0_\varepsilon}+2\nu\ell^2\|A^{-1/2}v\|^2\\ \leqslant -2\nu\|A^{1/2}v\|^2-2\nu\langle\phi(u)-\phi(w),u\rangle+2\langle\phi(u)-\phi(w)-\phi'(w)v,\partial_t w\rangle\\ -2\nu\varepsilon\int_0^\infty \mu_\varepsilon'(s)\langle v,\xi(s)\rangle_{V^{-1}}\,\mathrm{d}s\end{aligned}$$

Observe that due to (6) there holds

$$2\langle\Phi(u)-\Phi(w)-\phi(w)v,1\rangle\geqslant-\ell\|v\|^2\geqslant-\frac{1}{2}\|A^{1/2}v\|^2-\frac{\ell^2}{2}\|A^{-1/2}v\|^2$$

$$\langle\phi(u)-\phi(w),v\rangle\geqslant-\frac{1}{2}\|A^{1/2}v\|^2-\frac{\ell^2}{2}\|A^{-1/2}v\|^2$$

Therefore, $\Lambda(t)$ is controlled by $\frac{1}{2}\|D_{\varepsilon,\delta}(t)z\|^2\leqslant\Lambda(t)\leqslant c\|D_{\varepsilon,\delta}(t)z\|^2$. Estimating the right-hand side as usual, with the further control

$$2\langle\phi(u)-\phi(w)-\phi'(w)v,\partial_t w\rangle\leqslant c\|A^{-1/2}\partial_t w\|\,\|A^{1/2}v\|^2$$

we are led to the differential inequality

$$\frac{\mathrm{d}\Lambda}{\mathrm{d}t}+\nu\Lambda\leqslant c\|A^{-1/2}\partial_t w\|\Lambda$$

An application of the Gronwall lemma 3.5 allowed by (23) provides the desired inequality. □

6. COMPARISON ON FINITE TIME INTERVALS

In this section we provide a quantitative estimate of the closeness of the trajectories of $S_{\varepsilon,\delta}(t)$ and $S_{0,0}(t)$, originating from the same (smoother) initial data, on finite time intervals. Precisely, we have

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Theorem 6.1
For every $R\geqslant 0$ and $z\in B_{\mathscr{H}^2_\varepsilon}(R)$ there hold

$$\|S_{\varepsilon,\delta}(t)z-S_{0,0}(t)z\|^2_{\mathscr{H}^0_\varepsilon}\leqslant\|\eta_0\|^2_{\mathscr{M}^0_\varepsilon}\mathrm{e}^{-\lambda t/2\varepsilon}+c\mathrm{e}^{ct}(\sqrt[8]{\varepsilon}+\delta) \tag{24}$$

for all $t\geqslant 0$, for some $c\geqslant 0$ only depending on R.

Then let $(\varepsilon,\delta)\in\mathscr{T}$ be fixed. Given $z=(u_0,\eta_0)\in B_{\mathscr{H}^2_\varepsilon}(R)$, we denote

$$(\hat{u}(t),\hat{\eta}^t)=S_{\varepsilon,\delta}(t)z \quad \text{and} \quad (u(t),0)=S_{0,0}(t)z$$

Besides, let η^t be the solution at time t of the Cauchy problem in $\mathscr{M}^0_\varepsilon$

$$\begin{cases}\partial_t\eta=T_\varepsilon\eta+A[Au+\phi(u)], & t>0\\ \eta^0=\eta_0\end{cases}$$

The representation formula for η (see (A1) in the Appendix) reads

$$\eta^t(s)=\begin{cases}\displaystyle\int_0^s A[Au(t-y)+\phi(u(t-y))]\,\mathrm{d}y, & 0<s\leqslant t\\ \displaystyle\eta_0(s-t)+\int_0^t A[Au(t-y)+\phi(u(t-y))]\,\mathrm{d}y, & s>t\end{cases} \tag{25}$$

In this section $c\geqslant 0$ may depend on R, but is independent of ε and δ. Note that by Lemma 4.7

$$\sup_{\|z\|_{\mathscr{H}^2_\varepsilon}\leqslant R}\|S_{\varepsilon,\delta}(t)z\|_{\mathscr{H}^2_\varepsilon}\leqslant c \tag{26}$$

Lemma 6.2
There holds

$$\|\hat{\eta}^t\|^2_{\mathscr{M}^0_\varepsilon}\leqslant\|\eta_0\|^2_{\mathscr{M}^0_\varepsilon}\mathrm{e}^{-\lambda t/2\varepsilon}+c\varepsilon \quad \forall t\geqslant 0$$

Proof
Multiplying the equation for η by η in $\mathscr{M}^0_\varepsilon$, in light of (26) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\hat{\eta}\|^2_{\mathscr{M}^0_\varepsilon}+\frac{\delta}{\varepsilon}\|\hat{\eta}\|^2_{\mathscr{M}^0_\varepsilon}\leqslant(\|A^{3/2}\hat{u}\|+\|A^{1/2}\phi(\hat{u})\|)\int_0^\infty\mu_\varepsilon(s)\|A^{-1/2}\hat{\eta}(s)\|\,\mathrm{d}s$$
$$+\delta\int_0^\infty\mu_\varepsilon(s)\langle\partial_t\hat{u}(t),\hat{\eta}(s)\rangle\,\mathrm{d}s\leqslant\frac{c}{\sqrt{\varepsilon}}\|\hat{\eta}\|_{\mathscr{M}^0_\varepsilon}-\delta\|\partial_t\hat{u}\|^2\leqslant\frac{\delta}{2\varepsilon}\|\hat{\eta}\|^2_{\mathscr{M}^0_\varepsilon}+c$$

The assertion then follows from the Gronwall lemma. □

Lemma 6.3
There holds

$$\|\hat{u}(t)-u(t)\|^2_{V^0}\leqslant c\mathrm{e}^{ct}(\sqrt[4]{\varepsilon}+\delta) \quad \forall t\geqslant 0$$

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Proof
Set $\bar{u}(t)=\hat{u}(t)-u(t)$ and $\bar{\eta}^t=\hat{\eta}^t-\eta^t$. Then we have the system

$$\begin{cases} \partial_t\bar{u}+\int_0^\infty \mu_\varepsilon(s)\bar{\eta}(s)\,\mathrm{d}s=-A[Au+\phi(u)]-\int_0^\infty \mu_\varepsilon(s)\eta(s)\,\mathrm{d}s \\ \partial_t\bar{\eta}=T_\varepsilon\bar{\eta}+A[A\bar{u}+\delta\partial_t\bar{u}]+A\phi(\hat{u})-A\phi(u)+\delta A\partial_t u \\ (\bar{u}(0),\bar{\eta}^0)=(0,0) \end{cases} \tag{27}$$

Multiplying the first equation by $A\bar{u}+\delta\partial_t\bar{u}$ in H and the second equation for $\bar{\eta}$ in $\mathscr{M}_\varepsilon^0$ we end up with

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}[\|\bar{u}\|_{V^0}^2+\|\bar{\eta}\|_{\mathscr{M}_\varepsilon^0}^2]+\frac{\lambda}{\varepsilon}\|\bar{\eta}\|_{\mathscr{M}_\varepsilon^0}^2+\delta\|\partial_t\bar{u}\|^2 \\ &\quad\leqslant -2\langle A\phi(\hat{u})-A\phi(u),\bar{\eta}\rangle_{\mathscr{M}_\varepsilon^0}+2\delta\langle A\partial_t u,\bar{\eta}\rangle_{\mathscr{M}_\varepsilon^0}-2\langle\psi,\bar{u}\rangle_{V^0}-2\delta\langle\psi,\partial_t\bar{u}\rangle \end{aligned} \tag{28}$$

where $\psi=A[Au+\phi(u)]+\int_0^\infty \mu_\varepsilon(s)\eta(s)\,\mathrm{d}s$. Exploiting the representation formula for η, we can write $\psi=\sum_j I_j$ where

$$I_1(t)=\int_{\sqrt{\varepsilon}}^\infty s\mu_\varepsilon(s)A[Au(t)+\phi(u(t))]\,\mathrm{d}s$$

$$I_2(t)=-\int_{\sqrt{\varepsilon}}^\infty \mu_\varepsilon(s)\eta^t(s)\,\mathrm{d}s$$

$$I_3(t)=-\int_{\min\{\sqrt{\varepsilon},t\}}^{\sqrt{\varepsilon}} \mu_\varepsilon(s)\eta_0(s-t)\,\mathrm{d}s$$

$$I_4(t)=\int_{\min\{\sqrt{\varepsilon},t\}}^{\sqrt{\varepsilon}} (s-t)\mu_\varepsilon(s)A[Au(t)+\phi(u(t))]\,\mathrm{d}s$$

$$I_5(t)=\int_0^{\sqrt{\varepsilon}} \mu_\varepsilon(s)\left[\int_0^{\min\{s,t\}} A[Au(t)-Au(t-y)+\phi(u(t))-\phi(u(t-y))]\,\mathrm{d}y\right]\mathrm{d}s$$

We then control the terms involving ψ as

$$-2\langle\psi,\bar{u}\rangle_{V^0}\leqslant 2\|\psi\|_{V^{-2}}\|\bar{u}\|_{V^2}\leqslant c\sum_{i=1}^5\|I_i\|_{V^{-2}}$$

$$-2\delta\langle\psi,\partial_t\bar{u}\rangle\leqslant 2\delta\|\psi\|_{V^{-2}}\|\partial_t\bar{u}\|_{V^0}\leqslant 4\sum_{i=1}^5\|I_i\|_{V^{-2}}^2+\delta^2\|\partial_t\bar{u}\|_{V^0}^2$$

The norm of I_1–I_4 can be estimated by

$$\sum_{i=1}^4\|I_i(t)\|_{V^{-2}}\leqslant\frac{c}{\sqrt{\varepsilon}}\mathrm{e}^{-\lambda t/2\varepsilon}+c\sqrt{\varepsilon}$$

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This can be done arguing as in the proof of [9, Lemma 5.5] (see also [8]) by exploiting the exponential decay of μ_ε, the straightforward inequalities $\int_{\sqrt{\varepsilon}}^{\infty} s\mu_\varepsilon(s)\,\mathrm{d}s \leqslant c\varepsilon$, $\int_{\sqrt{\varepsilon}}^{\infty} \mu_\varepsilon(s)\,\mathrm{d}s \leqslant c\sqrt{\varepsilon}$ and Lemma 6.2.

In order to control I_5 let us first observe that, multiplying the limit equation (1) by $A\partial_t u$, there holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\|A^{3/2}u\|^2 + 2\|A^{1/2}\partial_t u\|^2 = 2\langle A\phi(u), A\partial_t u\rangle \leqslant \|A^{3/2}\phi(u)\|\|A^{1/2}\partial_t u\| \leqslant c + \|A^{1/2}\partial_t u\|^2$$

Hence, an integration on $(0,t)$ gives

$$\int_0^t \|A^{1/2}\partial_t u(s)\|^2\,\mathrm{d}s \leqslant c(1+t) \tag{29}$$

Besides, by a derivation with respect to time of (1) we learn that

$$\|A^{-3/2}\partial_{tt}u\| \leqslant \|A^{1/2}\partial_t u\| + \|\phi'(u)\partial_t u\|_{V^0}$$

and thanks to (29) and (26) we obtain the dissipation integral

$$\int_0^t \|A^{-3/2}\partial_{tt}u(y)\|^2\,\mathrm{d}y \leqslant c(1+t)$$

Setting now $h(t) = \|A^{1/2}\partial_t u(t)\|^{1/2}\chi_{[0,\infty)}$ we can estimate as follows:

$$\begin{aligned}
&\|A[Au(t) - Au(t-y) + \phi(u(t)) - \phi(u(t-y))]\|_{V^{-2}}\\
&\quad \leqslant \|\partial_t u(t) - \partial_t u(t-y)\|_{V^0}^{1/2}\|\partial_t u(t) - \partial_t u(t-y)\|_{V^{-4}}^{1/2}\\
&\quad \leqslant [h(t) + h(t-y)]\left[\int_{t-y}^t \|A^{-3/2}\partial_{tt}u(s)\|\,\mathrm{d}s\right]^{1/2}\\
&\quad \leqslant c\sqrt{y}\,[h(t) + h(t-y)]
\end{aligned}$$

As $I_5(t) \leqslant \|\bar{u}\|_{V^2}\|A[Au(t) - Au(t-y) + \phi(u(t) - \phi(u(t-y))\|_{V^{-2}}$, we finally obtain

$$I_5(t) \leqslant c\sqrt[4]{\varepsilon}\int_0^{\sqrt{\varepsilon}} \mu_\varepsilon(s)\left[\int_0^s [h(t) - h(t-y)]\,\mathrm{d}y\right]\mathrm{d}s \leqslant c\sqrt[4]{\varepsilon}\,g(t)$$

where

$$g(t) = \int_0^{\sqrt{\varepsilon}} \mu_\varepsilon(s)\left[\int_0^s [h(t) + h(t-y)]\,\mathrm{d}y\right]\mathrm{d}s$$

Going back to the differential inequality (28), the remaining terms can be controlled as

$$-2\langle A\phi(\hat{u}) - A\phi(u), \bar{\eta}\rangle_{\mathcal{M}_\varepsilon^0} \leqslant c\|\bar{u}\|_{V^0}^2 + \frac{\lambda}{2\varepsilon}\|\bar{\eta}\|_{\mathcal{M}_\varepsilon^0}^2$$

$$2\delta\langle A\partial_t u, \bar{\eta}\rangle_{\mathcal{M}_\varepsilon^0} \leqslant c\,\delta\|\partial_t u\|_{V^0}^2 + \frac{\lambda}{2\varepsilon}\|\bar{\eta}\|_{\mathcal{M}_\varepsilon^0}^2$$

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Collecting all the above estimates we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}[\|\bar{u}(t)\|^2_{V^0}+\|\bar{\eta}^t\|^2_{\mathcal{M}^0_\varepsilon}]\leqslant c\|\bar{u}(t)\|^2_{V^0}+\frac{c}{\sqrt{\varepsilon}}\mathrm{e}^{-\lambda t/2\varepsilon}+c\sqrt{\varepsilon}+\sqrt[4]{\varepsilon}g(t)+\delta f(t)$$

where $f(t)=c(\|\partial_t u\|^2_{V^0}+\delta\|\partial_t\bar{u}\|^2_{V^0})$. Note that by force of (22) and (29) $\int_0^t(g(y)+f(y))\,\mathrm{d}y\leqslant c(1+t)$: the Gronwall lemma then leads to

$$\|\bar{u}(t)\|^2_{V^0}\leqslant c\mathrm{e}^{ct}(\sqrt[4]{\varepsilon}+\delta)$$

which concludes the proof. □

The proof of Theorem 6.1 immediately follows by combining Lemmas 6.2 and 6.3.

7. ROBUST EXPONENTIAL ATTRACTORS

In this section we finally present our main results on the asymptotic behavior of the solutions to (3).

Theorem 7.1
For every $(\varepsilon,\delta)\in\mathcal{T}$ there exists a set $\mathcal{E}_{\varepsilon,\delta}$, compact in $\mathcal{H}^0_\varepsilon$ and bounded in $\mathcal{Z}^2_\varepsilon$ with the following properties.

(i) $\mathcal{E}_{\varepsilon,\delta}$ is positively invariant for $S_{\varepsilon,\delta}(t)$.
(ii) $\mathcal{E}_{\varepsilon,\delta}$ attracts any bounded set $\mathcal{B}\in\mathcal{H}^0_\varepsilon$ with uniform exponential rate, namely

$$\mathrm{dist}_{\mathcal{H}^0_\varepsilon}(S_{\varepsilon,\delta}(t)\mathcal{B},\mathcal{E}_{\varepsilon,\delta})\leqslant M(R)\mathrm{e}^{-\kappa t}\quad\forall t\geqslant 0$$

where M is a positive increasing function of $R=\sup_{\mathcal{B}}\|u\|_{\mathcal{H}^0_\varepsilon}$ and κ a positive constant.
(iii) The fractal dimension of $\mathcal{E}_{\varepsilon,\delta}$ is uniformly bounded with respect to ε and δ,
(iv) there holds

$$\mathrm{dist}^{\mathrm{sym}}_{\mathcal{H}^0_\varepsilon}(\mathcal{E}_{\varepsilon,\delta},\mathcal{E}_{0,0})\leqslant m(\varepsilon+\delta)^a$$

for some $a\in(0,1)$ and $m\geqslant 0$. In particular, $\mathcal{E}_{\varepsilon,\delta}$ is continuous at the origin with respect to the Hausdorff symmetric distance in $\mathcal{H}^0_\varepsilon$.

Here κ, m and a are independent of ε and δ and can be explicitly computed.

Remark 7.2
As $\mathcal{E}_{\varepsilon,\delta}$ is a compact attracting set, it contains the global attractor $\mathcal{A}_{\varepsilon,\delta}$. As a consequence we can conclude that *the global attractor $\mathcal{A}_{\varepsilon,\delta}$ has finite fractal dimension*, *which is uniform with respect to ε and δ*. In addition, arguing as in [9, Section 7] with obvious changes (see also [20]), it is possible to prove that the global attractor is upper semicontinuous as $(\varepsilon,\delta)\to(0,0)$, namely

$$\lim_{(\varepsilon,\delta)\to(0,0)}\mathrm{dist}(\mathcal{A}_{\varepsilon,\delta},\mathcal{A}_{0,0})=0$$

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The proof of Theorem 7.1 is based on the abstract result in the Appendix; hence, the rest of the section is devoted to the verification of the assumptions (A1)–(A4) there. According to the notations of Theorem A.2, we set

$$\mathscr{B}_{\varepsilon,\delta} = \mathscr{B}_\varepsilon^2$$

where $\mathscr{B}_\varepsilon^2$ is the regular attracting ball provided by Theorem 5.4; we denote the entering time of $\mathscr{B}_\varepsilon^2$ into itself by t_1 and set $t^* \geqslant t_1$ to be fixed, see (31). Then assumptions (A2) and (A3) are immediately provided by (24), with $\Sigma(\varepsilon,\delta) = \sqrt[8]{\varepsilon} + \sqrt{\delta}$. Thanks to the continuous dependence estimate, condition (A4) follows by proving that

$$\|S_{\varepsilon,\delta}(t_1)z - S_{\varepsilon,\delta}(t_2)z\|_{\mathscr{H}_\varepsilon^0} \leqslant C_\varepsilon \sqrt{|t_1 - t_2|}$$

for every $t_1, t_2 \in [t^*, 2t^*]$. This is consequence a of the two estimates

$$\|A^{1/2}\partial_t u\|^2 \leqslant \frac{1}{\varepsilon}\|\eta\|^2_{\mathscr{M}_\varepsilon^2} \leqslant \frac{c}{\varepsilon}$$

(we are using Lemma 4.7 with inequality (15)) and

$$\|\partial_t \eta\|^2_{\mathscr{M}_\varepsilon^0} \leqslant \frac{c}{\varepsilon}$$

which comes by comparison in the differential equation for η.

7.1. Verification of (A1)

This goes exactly as in Lemma [8, Lemma 7.6] and it is based on an appropriate decomposition of the difference of two trajectories. We report some details for the readers' convenience.

Given $z_1, z_2 \in \mathscr{B}_\varepsilon^2$, we define $f(t) = \int_0^1 \phi'(su^1(t) + (1-s)u^2(t))\,\mathrm{d}s$ and $\bar{u}(t) = u^1(t) - u^2(t)$, where $(u^i(t), \eta^{it}) = S_{\varepsilon,\delta}(t)z_i$. We also define

$$D_{\varepsilon,\delta}(t)(z_1,z_2) = (\bar{v}, \bar{\xi}), \quad N_{\varepsilon,\delta}(t)(z_1,z_2) = (\bar{w}, \bar{\chi})$$

where

$$\begin{cases} \partial_t \bar{w} + \displaystyle\int_0^\infty \mu_\varepsilon(s)\bar{\chi}(s)\,\mathrm{d}s = 0 \\ \partial_t \bar{\chi} = T_\varepsilon \bar{\chi} + A[A\bar{w} + \delta\partial_t \bar{w}] + Af\bar{w} + \ell^2\bar{w} - \ell^2\bar{u} \\ N_{\varepsilon,\delta}(0)(z_1,z_2) = 0 \end{cases}$$

and

$$\begin{cases} \partial_t \bar{v} + \displaystyle\int_0^\infty \mu_\varepsilon(s)\bar{\xi}(s)\,\mathrm{d}s = 0 \\ \partial_t \bar{\xi} = T_\varepsilon \bar{\xi} + A[A\bar{v} + \delta\partial_t \bar{v}] + Af\bar{v} + \ell^2\bar{v} \\ D_{\varepsilon,\delta}(0)(z_1,z_2) = z_1 - z_2 \end{cases}$$

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With multiplications analogous to those in Lemma 4.5, and exploiting the continuous dependence estimate (20) together with (15), it is immediate to verify that the following differential inequality holds

$$\frac{\mathrm{d}}{\mathrm{d}t}[\|N_{\varepsilon,\delta}(t)(z_1,z_2)\|^2_{\mathscr{H}^1_\varepsilon}+\ell^2\|\bar{w}\|^2]+\delta\|A^{1/2}\partial_t\bar{w}\|^2\leqslant c\|N_{\varepsilon,\delta}(t)(z_1,z_2)\|^2_{\mathscr{H}^1_\varepsilon}+c\|z_1-z_2\|^2_{\mathscr{H}^0_\varepsilon}$$

Recalling that the initial datum is null, via the Gronwall lemma we obtain

$$\|N_{\varepsilon,\delta}(t)(z_1,z_2)\|^2_{\mathscr{H}^1_\varepsilon}+\delta\int_0^t\|A^{1/2}\partial_t\bar{w}(y)\|^2\,\mathrm{d}y\leqslant c\mathrm{e}^{ct}\|z_1-z_2\|^2_{\mathscr{H}^0_\varepsilon}$$

A subsequent application of Lemma 5.2, suitably reformulated for $\bar{\xi}^t$, leads to

$$|||\bar{\xi}^t|||^2_\varepsilon\leqslant c\mathrm{e}^{ct}\|z_1-z_2\|^2_{\mathscr{H}^0_\varepsilon}$$

proving that

$$\|N_{\varepsilon,\delta}(t)(z_1,z_2)\|_{\mathscr{Z}^1_\varepsilon}\leqslant c\mathrm{e}^{ct}\|z_1-z_2\|^2_{\mathscr{H}^0_\varepsilon} \tag{30}$$

The second step consists in studying the asymptotic behavior of $\|D_{\varepsilon,\delta}(t)(z_1,z_2)\|_{\mathscr{H}^0_\varepsilon}$. This can be obtained by showing that the functional

$$\Lambda(t)=\|D_{\varepsilon,\delta}(t)(z_1,z_2)\|^2_{\mathscr{H}^0_\varepsilon}+\ell^2\|A^{-1/2}\bar{v}(t)\|^2-2v\varepsilon\langle\bar{v}(t),\bar{\chi}^t\rangle_{\mathscr{M}^0_\varepsilon}+v\delta\|\bar{v}(t)\|^2+2\langle f(t)\bar{v},\bar{v}\rangle_{V^{-2}}$$

is equivalent to $\|D_{\varepsilon,\delta}(t)(z_1,z_2)\|^2_{\mathscr{H}^0_\varepsilon}$ and satisfies the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda+v\Lambda\leqslant c(\|A^{-1/2}\partial_t u_1\|^2+\|A^{-1/2}\partial_t u_2\|^2)\Lambda$$

for some $v>0$. The integral estimate (16) allows the application of the Gronwall Lemma 3.5, which proves the desired decay, with a decay rate independent of ε and δ. Hence, choosing for instance $\lambda=\frac{1}{4}$ in (A1), we can finally fix t^* large enough such that

$$\|D_{\varepsilon,\delta}(t^*)(z_1,z_2)\|_{\mathscr{H}^0_\varepsilon}\leqslant\tfrac{1}{4}\|z_1-z_2\|_{\mathscr{H}^0_\varepsilon} \tag{31}$$

As $S_{\varepsilon,\delta}(t)z_1-S_{\varepsilon,\delta}(t)z_2=D_{\varepsilon,\delta}(t)(z_1,z_2)+N_{\varepsilon,\delta}(t)(z_1,z_2)$, the validity of (A1) immediately follows by (30) and (31).

At this point, by direct application of the abstract Theorem A.2 we obtain the existence of a family of compact sets $\mathscr{E}_{\varepsilon,\delta}\subset\mathscr{B}^2_\varepsilon$ satisfying properties (i), (ii) and (iv) in Theorem 7.1. Besides, by (T2) we learn that

$$\mathrm{dist}_{\mathscr{H}^0_\varepsilon}(S_{\varepsilon,\delta}(t)\mathscr{B}^2_\varepsilon,\mathscr{E}_{\varepsilon,\delta})\leqslant M_1\mathrm{e}^{-\kappa t}\quad\forall t\geqslant 0$$

for some $M_1\geqslant 0$ and $\kappa>0$. What is left to prove is (iii), namely that $\mathscr{E}_{\varepsilon,\delta}$ exponentially attracts every bounded set in $\mathscr{H}^0_\varepsilon$. Indeed, since $\mathscr{B}^2_\varepsilon$ is exponentially attracting in $\mathscr{H}^0_\varepsilon$, we can appeal once more to the transitivity of the exponential attraction [19, Theorem 5.1] in order to reach the desired conclusion. This finally ends the proof of Theorem 7.1.

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APPENDIX A

A.1. The representation formula

We report here the representation formula for η, which plays an essential role in the history approach. The proof can be found in [14].

Theorem A.1
The operator $T_\varepsilon : \mathscr{D}(T_\varepsilon) \to \mathscr{M}_\varepsilon^0$ is the generator of the right-translation (strongly continuous) linear semigroup of operators on the space $\mathscr{M}_\varepsilon^0$. Besides, if $f \in L^1(0, T; V^{-2})$ for all $T>0$ and $\eta_0 \in \mathscr{M}_\varepsilon^0$, then the Cauchy problem

$$\begin{cases} \partial_t \eta^t = T_\varepsilon \eta^t + f, & t>0 \\ \eta^0 = \eta_0 \end{cases}$$

admits a unique solution $\eta \in C([0, T); \mathscr{M}_\varepsilon^0)$ that has the explicit representation formula

$$\eta^t(s) = \begin{cases} \displaystyle\int_0^s f(t-\tau)\,\mathrm{d}\tau, & 0<s\leqslant t \\ \displaystyle\eta_0(s-t) + \int_0^t f(t-\tau)\,\mathrm{d}\tau, & s>t \end{cases} \tag{A1}$$

A.2. The abstract theorem

The abstract result used in Section 7 to prove our main Theorem 7.1 consists of a modification of [9, Theorem A.2], suitable to deal with the double limit $\varepsilon \to 0$ and $\delta \to 0$. Note that the process is singular in ε, in the sense that when ε approaches zero the phase space collapses on the first component only; on the contrary $\delta \to 0$ does not effect the structure of the space. We do not report the proof, that can be obtained by recasting the arguments of [9].

Theorem A.2
For every $\varepsilon \in [0, 1]$, $\delta \in [0, 1]$, let $S_{\varepsilon,\delta}(t) : \mathscr{H}_\varepsilon^0 \to \mathscr{H}_\varepsilon^0$ be a strongly continuous semigroup of operators. Assume that there exists a family of closed set $\mathscr{B}_{\varepsilon,\delta} \subset B_{\mathscr{H}_\varepsilon^0}(R)$ such that

$$S_{\varepsilon,\delta}(t)\mathscr{B}_{\varepsilon,\delta} \subset \mathscr{B}_{\varepsilon,\delta} \quad \forall t \geqslant t^\star$$

where R and $t^\star$ are independent of ε and δ.

Assume that there exist $\sigma>0$, $\Lambda_j \geqslant 0$, $\lambda \in [0, \frac{1}{2})$, $\alpha \in (0, 1]$ and a continuous increasing function $\Sigma : [0, 1] \to [0, \infty)$ with $\Sigma(0) = 0$ (all independent of ε and δ) such that the following conditions hold.

(A1) The map $S_{\varepsilon,\delta} = S_{\varepsilon,\delta}(t^\star)$ satisfies, for every $z_1, z_2 \in \mathscr{B}_{\varepsilon,\delta}$,

$$S_{\varepsilon,\delta} z_1 - S_{\varepsilon,\delta} z_2 = D_{\varepsilon,\delta}(z_1, z_2) + N_{\varepsilon,\delta}(z_1, z_2)$$

where

$$\|D_{\varepsilon,\delta}(z_1, z_2)\|_{\mathscr{H}_\varepsilon^0} \leqslant \lambda \|z_1 - z_2\|_{\mathscr{H}_\varepsilon^0}$$

$$\|N_{\varepsilon,\delta}(z_1, z_2)\|_{\mathscr{Z}_\varepsilon^\sigma} \leqslant \Lambda_1 \|z_1 - z_2\|_{\mathscr{H}_\varepsilon^0}$$

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(A2) There holds

$$\|S^n_{\varepsilon,\delta}z - S^n_{0,0}z\|_{\mathscr{H}^0_\varepsilon} \leqslant \Lambda^n_2 \Sigma(\varepsilon,\delta) \quad \forall z \in \mathscr{B}_{\varepsilon,\delta} \quad \forall n \in \mathbb{N}$$

(A3) There holds

$$\|S_{\varepsilon,\delta}(t)z - S_{0,0}(t)z\|_{\mathscr{H}^0_\varepsilon} \leqslant \Lambda_3 \Sigma(\varepsilon,\delta) \quad \forall z \in \mathscr{B}_{\varepsilon,\delta} \quad \forall t \in [t^\star, 2t^\star]$$

(A4) The map

$$(t,z) \mapsto S_{\varepsilon,\delta}(t)z : [t^\star, 2t^\star] \times \mathscr{B}_{\varepsilon,\delta} \to \mathscr{B}_{\varepsilon,\delta}$$

is Hölder continuous with exponent α (with a constant that may depend on ε). Besides, $z \to S_{\varepsilon,\delta}(t)z : \mathscr{B}_{\varepsilon,\delta} \to \mathscr{B}_{\varepsilon,\delta}$ is Lipschitz continuous uniformly in ε, δ and $t \in [t^\star, 2t^\star]$. Here $\mathscr{B}_{\varepsilon,\delta}$ is endowed with the metric topology of $\mathscr{H}^0_\varepsilon$.

Then there exists a family of compact sets $\mathscr{E}_{\varepsilon,\delta} \subset \mathscr{B}_{\varepsilon,\delta}$ such that

$$S_{\varepsilon,\delta}(t)\mathscr{E}_{\varepsilon,\delta} \subset \mathscr{E}_{\varepsilon,\delta} \quad \forall t \geqslant 0$$

with the following additional properties:

(T1) $\mathscr{E}_{\varepsilon,\delta}$ attracts $\mathscr{B}_{\varepsilon,\delta}$ with a uniform exponential rate,

$$\text{dist}_{\mathscr{H}^0_\varepsilon}(S_{\varepsilon,\delta}(t)\mathscr{B}_{\varepsilon,\delta}, \mathscr{E}_{\varepsilon,\delta}) \leqslant M_1 \mathrm{e}^{-\kappa t} \quad \forall t \geqslant 0$$

(T2) the fractal dimension of $\mathscr{E}_\varepsilon$ is uniformly bounded with respect to ε and δ,
(T3) there holds

$$\text{dist}^{\text{sym}}_{\mathscr{H}^0_\varepsilon}(\mathscr{E}_{\varepsilon,\delta}, \mathscr{E}_{0,0}) \leqslant M_2[\Sigma(\varepsilon,\delta)]^\tau$$

for some positive constants κ, τ and M_j, which are independent of ε and δ, and can be explicitly calculated.

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