A Sharp Trudinger-Moser Type Inequality for Unbounded Domains in $\mathbb{R}^n$

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ABSTRACT. The Trudinger-Moser inequality states that for functions $u \in H_0^{1,n}(\Omega)$ ($\Omega \subset \mathbb{R}^n$ a bounded domain) with $\int_\Omega |\nabla u|^n \,\mathrm{d}x \le 1$], one has

$$\int_\Omega (e^{\alpha_n |u|^{n/(n-1)}} - 1)\,\mathrm{d}x \le c|\Omega|,$$

with c independent of u. Recently, the second author has shown that for $n = 2$ the bound $c|\Omega|$ may be replaced by a uniform constant d independent of Ω if the Dirichlet norm is replaced by the Sobolev norm, i.e., requiring

$$\int_\Omega (|\nabla u|^n + |u|^n)\,\mathrm{d}x \le 1.$$

We extend here this result to arbitrary dimensions $n > 2$. Also, we prove that for $\Omega = \mathbb{R}^n$ the supremum of $\int_{\mathbb{R}^n} (e^{\alpha_n |u|^{n/(n-1)}} - 1)\,\mathrm{d}x$ over all such functions is attained. The proof is based on a blow-up procedure.

1. INTRODUCTION

Let $H_0^{1,p}(\Omega)$, $\Omega \subseteq \mathbb{R}^n$, be the usual Sobolev space, i.e., the completion of $C_0^\infty(\Omega)$ with the norm

$$\|u\|_{H^{1,p}(\Omega)} = \left(\int_\Omega (|\nabla u|^p + |u|^p)\,\mathrm{d}x \right)^{1/p}.$$

It is well known that

$$H_0^{1,p}(\Omega) \subset L^{pn/(n-p)}(\Omega) \quad \text{if } 1 \le p < n,$$
$$H_0^{1,p}(\Omega) \subset L^\infty(\Omega) \quad \text{if } n < p.$$

The case $p = n$ is the limit case of these embeddings and it is known that

$$H_0^{1,n}(\Omega) \subset L^q(\Omega) \quad \text{for } n \le q < +\infty.$$

When Ω is a bounded domain, we usually use the Dirichlet norm

$$\|u\|_D = \left(\int |\nabla u|^n \,\mathrm{d}x\right)^{1/n}$$

in place of $\|\cdot\|_{H^{1,n}}$. In this case, we have the famous Trudinger-Moser inequality (see [16], [18], [15]) for the limit case $p = n$ which states that

$$\sup_{\|u\|_D \le 1} \int_\Omega (e^{\alpha|u|^{n/(n-1)}} - 1)\,\mathrm{d}x = c(\Omega,\alpha) \begin{cases} < +\infty & \text{when } \alpha \le \alpha_n, \\ = +\infty & \text{when } \alpha > \alpha_n, \end{cases} \tag{1.1}$$

where $\alpha_n = n\omega_{n-1}^{1/(n-1)}$, and ω_{n-1} is the measure of the unit sphere in $\mathbb{R}^n$. The Trudinger-Moser result has been extended to Sobolev spaces of higher order and Sobolev spaces over compact manifolds (see [2], [9]). Moreover, for any bounded Ω, the constant $c(\Omega,\alpha_n)$ can be attained. For the attainability, we refer to [5], [8], [14], [10], [11], [6], [12].

Another interesting extension of (1.1) is to construct Trudinger-Moser type inequalities on unbounded domains. When $n = 2$, this has been done by B. Ruf in [17]. On the other hand, for an unbounded domain in $\mathbb{R}^n$, S. Adachi and K. Tanaka ([1]) get a weaker result. Let

$$\Phi(t) = e^t - \sum_{j=1}^{n-2} \frac{t^j}{j!}.$$

The following result was proved by S. Adachi and K. Tanaka:

Theorem A. *For any $\alpha \in (0,\alpha_n)$ there is a constant $C(\alpha)$ such that*

$$\int_{\mathbb{R}^n} \Phi\left(\alpha\left(\frac{|u|}{\|\nabla u\|_{L^n(\mathbb{R}^n)}}\right)^{n/(n-1)}\right)\mathrm{d}x \le C(\alpha)\frac{\|u\|^n_{L^n(\mathbb{R}^n)}}{\|\nabla u\|^n_{L^n(\mathbb{R}^n)}} \tag{1.2}$$

$$\text{for } u \in H^{1,n}(\mathbb{R}^n) \setminus \{0\}.$$

In this paper, we shall discuss the critical case $\alpha = \alpha_n$. More precisely, we prove the following result.

Theorem 1.1. *There exists a constant $d > 0$, s.t. for any domain $\Omega \subset \mathbb{R}^n$,*

$$\sup_{u \in H^{1,n}(\Omega),\, \|u\|_{H^{1,n}(\Omega)} \le 1} \int_\Omega \Phi(\alpha_n |u|^{n/(n-1)})\,\mathrm{d}x \le d. \tag{1.3}$$

The inequality is sharp: for any $\alpha > \alpha_n$, the supremum is $+\infty$.

We set

$$S = \sup_{u \in H^{1,n}(\mathbb{R}^n),\, \|u\|_{H^{1,n}(\mathbb{R}^n)} \le 1} \int_{\mathbb{R}^n} \Phi(\alpha_n |u|^{n/(n-1)})\, dx.$$

Further, we will prove the following result.

Theorem 1.2. *S is attained. In other words, we can find a function $u \in H^{1,n}(\mathbb{R}^n)$, with $\|u\|_{H^{1,n}(\mathbb{R}^n)} = 1$, s.t.*

$$S = \int_{\mathbb{R}^n} \Phi(\alpha_n |u|^{n/(n-1)})\, dx.$$

The second part of Theorem 1.1 is trivial: Given any fixed $\alpha > \alpha_n$, we take $\beta \in (\alpha_n, \alpha)$. By (1.1) we can find a positive sequence $\{u_k\}$ in

$$\left\{ u \in H_0^{1,n}(B_1) \mid \int_{B_1} |\nabla u|^n \, dx = 1 \right\},$$

such that

$$\lim_{k \to +\infty} \int_{B_1} e^{\beta u_k^{n/(n-1)}} = +\infty.$$

By Lion's Lemma, we get $u_k \rightharpoonup 0$. Then by the compact embedding theorem, we may assume $\|u_k\|_{L^p(B_1)} \to 0$ for any $p > 1$. Then,

$$\int_{\mathbb{R}^n} (|\nabla u_k|^n + |u_k|^n)\, dx \to 1$$

and

$$\alpha \left(\frac{u_k}{\|u_k\|_{H^{1,n}}} \right)^{n/(n-1)} > \beta u_k^{n/(n-1)}$$

when k is sufficiently large. So, we get

$$\lim_{k \to +\infty} \int_{\mathbb{R}^n} \Phi\left(\alpha \left(\frac{u_k}{\|u_k\|_{H^{1,n}}} \right)^{n/(n-1)} \right) dx \ge \lim_{k \to +\infty} \int_{B_1} (e^{\beta u_k^{n/(n-1)}} - 1)\, dx = +\infty.$$

The first part of Theorem 1.1 and Theorem 1.2 will be proved by blow up analysis. We will use the ideas from [10] and [11] (see also [4] and [3]). However, in the unbounded case we do not obtain the strong convergence of u_k in $L^n(\mathbb{R}^n)$, and so we need more techniques.

Concretely, we will find positive and symmetric functions $u_k \in H_0^{1,n}(B_{R_k})$ which satisfy

$$\int_{B_{R_k}} (|\nabla u_k|^n + |u_k|^n)\, dx = 1$$

and

$$\int_{B_{R_k}} \Phi(\beta_k u_k^{n/(n-1)})\,\mathrm{d}x = \sup_{\int_{B_{R_k}}(|\nabla v|^n+|v|^n)=1,\, v\in H_0^{1,n}(B_{R_k})} \int_{B_{R_k}} \Phi(\beta_k |v|^{n/(n-1)})\,\mathrm{d}x.$$

Here, β_k is an increasing sequence tending to α_n, and R_k is an increasing sequence tending to $+\infty$.

Furthermore, u_k satisfies the following equation:

$$-\operatorname{div}|\nabla u_k|^{n-2}\nabla u_k + u_k^{n-1} = \frac{u_k^{1/(n-1)}\Phi'(\beta_k u_k^{n/(n-1)})}{\lambda_k},$$

where λ_k is a Lagrange multiplier.

Then, there are two possibilities. If $c_k = \max u_k$ is bounded from above, then it is easy to see that

$$\lim_{k\to+\infty}\int_{\mathbb{R}^n}\left(\Phi(\beta_k u_k^{n/(n-1)}) - \frac{\beta_k^{n-1}u_k^n}{(n-1)!}\right)\mathrm{d}x = \int_{\mathbb{R}^n}\left(\Phi(\alpha_n u^{n/(n-1)}) - \frac{\alpha_n^{n-1}u^n}{(n-1)!}\right)\mathrm{d}x,$$

where u is the weak limit of u_k. It then follows that either

$$\int_{\mathbb{R}^n}\Phi(\beta_k u_k^{n/(n-1)})\,\mathrm{d}x \quad \text{converges to} \quad \int_{\mathbb{R}^n}\Phi(\alpha_n u^{n/(n-1)})\,\mathrm{d}x$$

or

$$S \le \frac{\alpha_n^{n-1}}{(n-1)!}.$$

If c_k is not bounded, the key point of the proof is to show that

$$\frac{n}{n-1}\beta_k c_k^{1/(n-1)}(u_k(r_k x) - c_k) \to -n\log(1 + c_n r^{n/(n-1)}),$$

locally for a suitably chosen sequence r_k and with

$$c_n = \left(\frac{\omega_{n-1}}{n}\right)^{1/(n-1)},$$

and that

$$c_k^{1/(n-1)}u_k \to G\,,$$

on any $\Omega \subset\subset \mathbb{R}^n \setminus \{0\}$, where G is some Green function. This will be done in Section 3.

Then, we will get in Section 4 the following result.

Proposition 1.3. *If S cannot be attained, then*

$$S \le \min\left\{\frac{\alpha_n^{n-1}}{(n-1)!}, \frac{\omega_{n-1}}{n} e^{\alpha_n A + 1 + 1/2 + \cdots + 1/(n-1)}\right\},$$

where $A = \lim_{r\to 0}(G(r) + (1/\alpha_n)\log r^n)$.

So, to prove the attainability, we only need to show that

$$S > \min\left\{\frac{\alpha_n^{n-1}}{(n-1)!}, \frac{\omega_{n-1}}{n} e^{\alpha_n A + 1 + 1/2 + \cdots + 1/(n-1)}\right\}.$$

In Section 5, we will construct a function sequence u_ε such that

$$\int_{\mathbb{R}^n} \Phi(\alpha_n u_\varepsilon^{n/(n-1)})\, dx > \frac{\omega_{n-1}}{n} e^{\alpha_n A + 1 + 1/2 + \cdots + 1/(n-1)}$$

when ε is sufficiently small. And in the last section we will construct, for each $n > 2$, a function sequence u_ε such that for ε sufficiently small

$$\int_{\mathbb{R}^n} \Phi(\alpha_n u_\varepsilon^{n/(n-1)})\, dx > \frac{\alpha_n^{n-1}}{(n-1)!}.$$

Thus, together with Ruf's result of attainability in [17] for the case $n = 2$, we will get Theorem 1.2.

2. The Maximizing Sequence

Let $\{R_k\}$ be an increasing sequence which diverges to infinity, and $\{\beta_k\}$ an increasing sequence which converges to α_n. By compactness, we can find positive functions $u_k \in H_0^{1,n}(B_{R_k})$, with $\int_{B_{R_k}} (|\nabla u_k|^n + u_k^n)\, dx = 1$, such that

$$\int_{B_{R_k}} \Phi(\beta_k u_k^{n/(n-1)})\, dx = \sup_{\int_{B_{R_k}}(|\nabla v|^n + |v|^n) = 1,\, v \in H_0^{1,n}(B_{R_k})} \int_{B_{R_k}} \Phi(\beta_k |v|^{n/(n-1)})\, dx.$$

Moreover, we may assume that

$$\int_{\mathbb{R}^n} \Phi(\beta_k u_k^{n/(n-1)})\, dx = \int_{B_{R_k}} \Phi(\beta_k u_k^{n/(n-1)})\, dx$$

is increasing.

Lemma 2.1. *Let u_k as above. Then*

(a) *u_k is a maximizing sequence for S;*

(b) *u_k may be chosen to be radially symmetric and decreasing.*

Proof. **(a)** Let η be a cut-off function which is 1 on B_1 and 0 on $\mathbb{R}^n \setminus B_2$. Then given any $\varphi \in H^{1,n}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} (|\nabla\varphi|^n + |\varphi|^n)\,\mathrm{d}x = 1$, we have

$$\tau^n(L) := \int_{\mathbb{R}^n} \left(\left| \nabla\eta\left(\frac{x}{L}\right)\varphi \right|^n + \left| \eta\left(\frac{x}{L}\right)\varphi \right|^n \right) \mathrm{d}x \to 1, \quad \text{as } L \to +\infty.$$

Hence for a fixed L and $R_k > 2L$

$$\begin{aligned}\int_{B_L} \Phi\left(\beta_k \left| \frac{\varphi}{\tau(L)} \right|^{n/(n-1)} \right) \mathrm{d}x &\le \int_{B_{2L}} \Phi\left(\beta_k \left| \frac{\eta(x/L)\varphi}{\tau(L)} \right|^{n/(n-1)} \right) \mathrm{d}x \\ &\le \int_{B_{R_k}} \Phi(\beta_k u_k^{n/(n-1)})\,\mathrm{d}x.\end{aligned}$$

By the Levi Lemma, we then have

$$\int_{B_L} \Phi\left(\alpha_n \left| \frac{\varphi}{\tau(L)} \right|^{n/(n-1)} \right) \mathrm{d}x \le \lim_{k\to+\infty} \int_{\mathbb{R}^n} \Phi(\beta_k u_k^{n/(n-1)})\,\mathrm{d}x.$$

Then, letting $L \to +\infty$, we get

$$\int_{\mathbb{R}^n} \Phi(\alpha_n |\varphi|^{n/(n-1)})\,\mathrm{d}x \le \lim_{k\to+\infty} \int_{\mathbb{R}^n} \Phi(\beta_k u_k^{n/(n-1)})\,\mathrm{d}x.$$

Hence, we get

$$\begin{aligned}&\lim_{k\to+\infty} \int_{\mathbb{R}^n} \Phi(\beta_k u_k^{n/(n-1)})\,\mathrm{d}x \\ &\quad = \sup_{\int_{\mathbb{R}^n}(|\nabla v|^n + |v|^n)=1,\, v\in H^{1,n}(\mathbb{R}^n)} \int_{\mathbb{R}^n} \Phi(\alpha_n |v|^{n/(n-1)})\,\mathrm{d}x.\end{aligned}$$

(b) Let u_k^* be the radial rearrangement of u_k; then we have

$$\tau_k^n := \int_{B_{R_k}} (|\nabla u_k^*|^n + u_k^{*n})\,\mathrm{d}x \le \int_{B_{R_k}} (|\nabla u_k|^n + u_k^n)\,\mathrm{d}x = 1.$$

It is well known that $\tau_k = 1$ iff u_k is radial. Since

$$\int_{B_{R_k}} \Phi(\beta_k u_k^{*n/(n-1)})\,\mathrm{d}x = \int_{B_{R_k}} \Phi(\beta_k u_k^{n/(n-1)})\,\mathrm{d}x,$$

we have

$$\int_{B_{R_k}} \Phi\left(\beta_k\left(\frac{u_k^*}{\tau_k}\right)^{n/(n-1)}\right) \mathrm{d}x \geq \int_{B_{R_k}} \Phi(\beta_k u_k^{n/(n-1)}) \,\mathrm{d}x.$$

Hence $\tau_k = 1$ and

$$\int_{B_{R_k}} \Phi(\beta_k {u_k^*}^{n/(n-1)}) \,\mathrm{d}x = \sup_{\int_{B_{R_k}}(|\nabla v|^n+|v|^n)=1,\, v\in H_0^{1,n}(B_{R_k})} \int_{B_{R_k}} \Phi(\beta_k |v|^{n/(n-1)}) \,\mathrm{d}x.$$

So, we can assume $u_k = u_k(|x|)$, and $u_k(r)$ is decreasing. □

Assume now $u_k \rightharpoonup u$. Then, to prove Theorems 1.1 and 1.2, we only need to show that

$$\lim_{k\to+\infty} \int_{\mathbb{R}^n} \Phi(\beta_k u_k^{n/(n-1)}) \,\mathrm{d}x = \int_{\mathbb{R}^n} \Phi(\alpha_n u^{n/(n-1)}) \,\mathrm{d}x.$$

3. Blow up Analysis

By the definition of u_k we have the equation

$$-\operatorname{div}|\nabla u_k|^{n-2}\nabla u_k + u_k^{n-1} = \frac{u_k^{1/(n-1)}\Phi'(\beta_k u_k^{n/(n-1)})}{\lambda_k}, \tag{3.1}$$

where λ_k is the constant satisfying

$$\lambda_k = \int_{B_{R_k}} u_k^{n/(n-1)}\Phi'(\beta_k u_k^{n/(n-1)}) \,\mathrm{d}x.$$

First, we need to prove the following result.

Lemma 3.1. $\inf_k \lambda_k > 0$.

Proof. Assume $\lambda_k \to 0$. Then

$$\int_{\mathbb{R}^n} u_k^n \,\mathrm{d}x \leq C\int_{\mathbb{R}^n} u_k^{n/(n-1)}\Phi'(\beta_k u_k^{n/(n-1)}) \,\mathrm{d}x \leq C\lambda_k \to 0.$$

Since $u_k(|x|)$ is decreasing, we have $u_k^n(L)|B_L| \leq \int_{B_L} u_k^n \leq 1$, and then

$$u_k(L) \leq \frac{n}{\omega_n L^n}. \tag{3.2}$$

Set $\varepsilon = n/(\omega_n L^n)$. Then $u_k(x) \le \varepsilon$ for any $x \notin B_L$, and hence we have, using the form of Φ, that

$$\int_{\mathbb{R}^n \setminus B_L} \Phi(\beta_k u_k^{n/(n-1)})\,\mathrm{d}x \le C \int_{\mathbb{R}^n \setminus B_L} u_k^n\,\mathrm{d}x \le C\lambda_k \to 0.$$

And on B_L, since $u_k \to 0$ in $L^q(B_L)$ for any $q > 1$, we have by Lebesgue

$$\begin{aligned}
&\lim_{k\to+\infty} \int_{B_L} \Phi(\beta_k u_k^{n/(n-1)})\,\mathrm{d}x \\
&\le \lim_{k\to+\infty} \left[\int_{B_L} C u_k^{n/(n-1)} \Phi'(\beta_k u_k^{n/(n-1)})\,\mathrm{d}x + \int_{\{x\in B_L | u_k(x)\le 1\}} \Phi(\beta_k u_k^{n/(n-1)})\,\mathrm{d}x \right] \\
&\le \lim_{k\to+\infty} C\lambda_k + \int_{B_L} \Phi(0)\,\mathrm{d}x = 0.
\end{aligned}$$

This is impossible. □

We denote $c_k = \max u_k = u_k(0)$. Then we have the following result.

Lemma 3.2. *If* $\sup_k c_k < +\infty$, *then*

(i) *Theorem 1.1 holds;*

(ii) *if S is not attained, then*

$$S \le \frac{\alpha_n^{n-1}}{(n-1)!}.$$

Proof. If $\sup_k c_k < +\infty$, then $u_k \to u$ in $C^1_{\mathrm{loc}}(\mathbb{R}^n)$. By (3.2), we are able to find L such that $u_k(x) \le \varepsilon$ for $x \notin B_L$. Then

$$\begin{aligned}
\int_{\mathbb{R}^n \setminus B_L} \left(\Phi(\beta_k u_k^{n/(n-1)}) - \frac{\beta_k^{n-1} u_k^n}{(n-1)!} \right) \mathrm{d}x &\le C \int_{\mathbb{R}^n \setminus B_L} u_k^{n^2/(n-1)}\,\mathrm{d}x \\
&\le C\varepsilon^{n^2/(n-1)-n} \int_{\mathbb{R}^n} u_k^n\,\mathrm{d}x \\
&\le C\varepsilon^{n^2/(n-1)-n}.
\end{aligned}$$

Letting $\varepsilon \to 0$, we get

$$\begin{aligned}
&\lim_{k\to+\infty} \int_{\mathbb{R}^n} \left(\Phi(\beta_k u_k^{n/(n-1)}) - \frac{\beta_k^{n-1} u_k^n}{(n-1)!} \right) \mathrm{d}x \\
&\qquad = \int_{\mathbb{R}^n} \left(\Phi(\alpha_n u^{n/(n-1)}) - \frac{\alpha_n^{n-1} u^n}{(n-1)!} \right) \mathrm{d}x.
\end{aligned}$$

Hence

$$\lim_{k\to+\infty}\int_{\mathbb{R}^n}\Phi(\beta_k u_k^{n/(n-1)}) = \int_{\mathbb{R}^n}\Phi(\alpha_n u^{n/(n-1)})\,\mathrm{d}x + \frac{\alpha_n^{n-1}}{(n-1)!}\lim_{k\to+\infty}\int_{\mathbb{R}^n}(u_k^n - u^n)\,\mathrm{d}x. \tag{3.3}$$

When $u = 0$, we can deduce from (3.3) that

$$S \le \frac{\alpha_n^{n-1}}{(n-1)!}.$$

Now, we assume $u \ne 0$. Set

$$\tau^n = \lim_{k\to+\infty}\frac{\int_{\mathbb{R}^n} u_k^n\,\mathrm{d}x}{\int_{\mathbb{R}^n} u^n\,\mathrm{d}x}.$$

By the Levi Lemma, we have $\tau \ge 1$.

Let $\tilde{u} = u(x/\tau)$. Then, we have

$$\int_{\mathbb{R}^n}|\nabla\tilde{u}|^n\,\mathrm{d}x = \int_{\mathbb{R}^n}|\nabla u|^n\,\mathrm{d}x \le \lim_{k\to+\infty}\int_{\mathbb{R}^n}|\nabla u_k|^n\,\mathrm{d}x,$$

$$\int_{\mathbb{R}^n}\tilde{u}^n\,\mathrm{d}x = \tau^n\int_{\mathbb{R}^n}u^n\,\mathrm{d}x = \lim_{k\to+\infty}\int_{\mathbb{R}^n}u_k^n\,\mathrm{d}x.$$

Then

$$\int_{\mathbb{R}^n}(|\nabla\tilde{u}|^n + \tilde{u}^n)\,\mathrm{d}x \le \lim_{k\to+\infty}\int_{\mathbb{R}^n}(|\nabla u_k|^n + u_k^n)\,\mathrm{d}x = 1.$$

Hence, we have by (3.3)

$$\begin{aligned}
S &\ge \int_{\mathbb{R}^n}\Phi(\alpha_n\tilde{u}^{n/(n-1)})\,\mathrm{d}x \\
&= \tau^n\int_{\mathbb{R}^n}\Phi(\alpha_n u^{n/(n-1)})\,\mathrm{d}x \\
&= \left[\int_{\mathbb{R}^n}\Phi(\alpha_n u^{n/(n-1)})\,\mathrm{d}x + (\tau^n - 1)\int_{\mathbb{R}^n}\frac{\alpha_n^{n-1}}{(n-1)!}u^n\,\mathrm{d}x\right] \\
&\qquad + (\tau^n - 1)\int_{\mathbb{R}^n}\left(\Phi(\alpha_n u^{n/(n-1)}) - \frac{\alpha_n^{n-1}}{(n-1)!}u^n\right)\mathrm{d}x \\
&= \lim_{k\to+\infty}\int_{\mathbb{R}^n}\Phi(\beta_k u_k^{n/(n-1)})\,\mathrm{d}x \\
&\qquad + (\tau^n - 1)\int_{\mathbb{R}^n}\left(\Phi(\alpha_n u^{n/(n-1)}) - \frac{\alpha_n^{n-1}}{(n-1)!}u^n\right)\mathrm{d}x \quad =
\end{aligned}$$

$$= S + (\tau^n - 1) \int_{\mathbb{R}^n} \left(\Phi(\alpha_n u^{n/(n-1)}) - \frac{\alpha_n^{n-1}}{(n-1)!} u^n \right) \mathrm{d}x.$$

Since $\Phi(\alpha_n u^{n/(n-1)}) - (\alpha_n^{n-1}/(n-1)!)u^n > 0$, we have $\tau = 1$, and then

$$S = \int_{\mathbb{R}^n} \Phi(\alpha_n u^{n/(n-1)}) \, \mathrm{d}x.$$

So, u is an extremal function. □

From now on, we assume $c_k \to +\infty$. We perform a blow-up procedure: We define

$$r_k^n = \frac{\lambda_k}{c_k^{n/(n-1)} e^{\beta_k c_k^{n/(n-1)}}}.$$

By (3.2) we can find a sufficiently large L such that $u_k \le 1$ on $\mathbb{R}^n \setminus B_L$. Then

$$\int_{B_L} |\nabla (u_k - u_k(L))^+|^n \, \mathrm{d}x \le 1$$

and hence, by (1.1), we have

$$\int_{B_L} e^{\alpha_n [(u_k - u_k(L))^+]^{n/(n-1)}} \le C(L).$$

Clearly, for any $p < \alpha_n$ we can find a constant $C(p)$, such that

$$p u_k^{n/(n-1)} \le \alpha_n [(u_k - u_k(L))^+]^{n/(n-1)} + C(p),$$

and then we get

$$\int_{B_L} e^{p u_k^{n/(n-1)}} \, \mathrm{d}x < C = C(L, p).$$

Hence,

$$\begin{aligned}
\lambda_k e^{-(\beta_k/2) c_k^{n/(n-1)}} &= e^{-(\beta_k/2) c_k^{n/(n-1)}} \\
&\times \left[\int_{\mathbb{R}^n \setminus B_L} u_k^{n/(n-1)} \Phi'(\beta_k u_k^{n/(n-1)}) \, \mathrm{d}x + \int_{B_L} u_k^{n/(n-1)} \Phi'(\beta_k u_k^{n/(n-1)}) \, \mathrm{d}x \right] \\
&\le C \int_{\mathbb{R}^n \setminus B_L} u_k^n \, \mathrm{d}x \, e^{-(\beta_k/2) c_k^{n/(n-1)}} + \int_{B_L} e^{(\beta_k/2) u_k^{n/(n-1)}} u_k^{n/(n-1)} \, \mathrm{d}x \, .
\end{aligned}$$

Since u_k converges strongly in $L^q(B_L)$ for any $q > 1$, we get

$$\lambda_k \le C e^{(\beta_k/2) c_k^{n/(n-1)}},$$

and hence

$$r_k^n \leq Ce^{-(\beta_k/2)c_k^{n/(n-1)}}.$$

Now, we set

$$v_k(x) = u_k(r_k x),$$
$$w_k(x) = \frac{n}{n-1}\beta_k c_k^{1/(n-1)}(v_k - c_k),$$

where v_k and w_k are defined on $\Omega_k = \{x \in \mathbb{R}^n \mid r_k x \in B_1\}$. Using the definition of r_k^n and (3.1) we have

$$-\operatorname{div}|\nabla w_k|^{n-2}\nabla w_k = \frac{v_k^{1/(n-1)}}{c_k^{1/(n-1)}}\left(\frac{n}{n-1}\beta_k\right)^{n-1} e^{\beta_k(v_k^{n/(n-1)}-c_k^{n/(n-1)})} + O(r_k^n c_k^n).$$

By Theorem 7 in [19], we know that $\operatorname{osc}_{B_R}\omega_k \leq C(R)$ for any $R > 0$. Then from the result in [18] (or [7]), it follows that $\|w_k\|_{C^{1,\delta}(B_R)} < C(R)$. Therefore w_k converges in C^1_{loc} and $v_k - c_k \to 0$ in C^1_{loc}.

Since

$$\begin{aligned} v_k^{n/(n-1)} &= c_k^{n/(n-1)}\left(1 + \frac{v_k - c_k}{c_k}\right)^{n/(n-1)} \\ &= c_k^{n/(n-1)}\left(1 + \frac{n}{n-1}\frac{v_k - c_k}{c_k} + O\left(\frac{1}{c_k^2}\right)\right), \end{aligned}$$

we get $\beta_k(v_k^{n/(n-1)} - c_k^{n/(n-1)}) \to w$ in C^0_{loc}, and so we have

$$-\operatorname{div}|\nabla w|^{n-2}\nabla w = \left(\frac{n\alpha_n}{n-1}\right)^{n-1} e^w, \tag{3.4}$$

with

$$w(0) = 0 = \max w.$$

Since ω is radially symmetric and decreasing, it is easy to see that (3.4) has only one solution. We can check that

$$w(x) = -n\log(1 + c_n|x|^{n/(n-1)}) \quad \text{and} \quad \int_{\mathbb{R}^n} e^w \, \mathrm{d}x = 1,$$

where $c_n = (\omega_{n-1}/n)^{1/(n-1)}$. Then,

$$\lim_{L\to+\infty}\lim_{k\to+\infty}\int_{B_{Lr_k}} \frac{u_k^{n/(n-1)}}{\lambda_k} e^{\beta_k u_k^{n/(n-1)}} \, \mathrm{d}x = \lim_{L\to+\infty}\int_{B_L} e^w \, \mathrm{d}x = 1. \tag{3.5}$$

For $A > 1$, let $u_k^A = \min\{u_k, c_k/A\}$. We have the following result.

Lemma 3.3. *For any $A > 1$, there holds*

$$\limsup_{k\to+\infty}\int_{\mathbb{R}^n}(|\nabla u_k^A|^n + |u_k^A|^n)\,\mathrm{d}x \le \frac{1}{A}. \tag{3.6}$$

Proof. Since $|\{x \mid u_k \ge c_k/A\}|\,|c_k/A|^n \le \int_{\{u_k\ge c_k/A\}} u_k^n \le 1$, we can find a sequence $\rho_k \to 0$ such that

$$\left\{x \mid u_k \ge \frac{c_k}{A}\right\} \subset B_{\rho_k}.$$

Since u_k converges in $L^p(B_1)$ for any $p > 1$, we have

$$\lim_{k\to+\infty}\int_{\{u_k>c_k/A\}} |u_k^A|^p\,\mathrm{d}x \le \lim_{k\to+\infty}\int_{\{u_k>c_k/A\}} u_k^p\,\mathrm{d}x = 0,$$

and

$$\lim_{k\to+\infty}\int_{\mathbb{R}^n}\left(u_k - \frac{c_k}{A}\right)^+ u_k^p\,\mathrm{d}x = 0$$

for any $p > 0$.

Hence, testing Equation (3.1) with $(u_k - c_k/A)^+$, we have

$$\begin{aligned}
\int_{\mathbb{R}^n}&\left(\left|\nabla\left(u_k - \frac{c_k}{A}\right)^+\right|^n + \left(u_k - \frac{c_k}{A}\right)^+ u_k^{n-1}\right)\mathrm{d}x\\
&= \int_{\mathbb{R}^n}\left(u_k - \frac{c_k}{A}\right)^+ \frac{u_k^{1/(n-1)}}{\lambda_k} e^{\beta_k u_k^{n/(n-1)}}\,\mathrm{d}x + o(1)\\
&\ge \int_{B_{Lr_k}}\left(u_k - \frac{c_k}{A}\right)^+ \frac{u_k^{1/(n-1)}}{\lambda_k} e^{\beta_k u_k^{n/(n-1)}}\,\mathrm{d}x + o(1)\\
&= \int_{B_L}\frac{v_k - c_k/A}{c_k}\left(\frac{v_k - c_k}{c_k} + 1\right)^{1/(n-1)} e^{w_k + o(1)}\,\mathrm{d}x + o(1).
\end{aligned}$$

Hence

$$\liminf_{k\to+\infty}\int_{\mathbb{R}^n}\left(\left|\nabla\left(u_k - \frac{c_k}{A}\right)^+\right|^n + \left(u_k - \frac{c_k}{A}\right)^+ u_k^{n-1}\right)\mathrm{d}x \ge \frac{A-1}{A}\int_{B_L} e^w\,\mathrm{d}x.$$

Letting $L \to +\infty$, we get

$$\liminf_{k\to+\infty}\int_{\mathbb{R}^n}\left(\left|\nabla\left(u_k - \frac{c_k}{A}\right)^+\right|^n + \left(u_k - \frac{c_k}{A}\right)^+ u_k^{n-1}\right)\mathrm{d}x \ge \frac{A-1}{A}.$$

Now observe that

$$\begin{aligned}
&\int_{\mathbb{R}^n} (|\nabla u_k^A|^n + |u_k^A|^n)\,\mathrm{d}x \\
&\quad = 1 - \int_{\mathbb{R}^n} \left(\left| \nabla \left(u_k - \frac{c_k}{A} \right)^+ \right|^n + \left(u_k - \frac{c_k}{A} \right)^+ u_k^{n-1} \right) \mathrm{d}x \\
&\qquad + \int_{\mathbb{R}^n} \left(u_k - \frac{c_k}{A} \right)^+ u_k^{n-1}\,\mathrm{d}x - \int_{\{u_k > c_k/A\}} u_k^n\,\mathrm{d}x + \int_{\{u_k > c_k/A\}} |u_k^A|^n\,\mathrm{d}x \\
&\quad \le 1 - \left(1 - \frac{1}{A} \right) + o(1).
\end{aligned}$$

Hence, we get this lemma. □

Corollary 3.4. *We have*

$$\lim_{k\to+\infty} \int_{\mathbb{R}^n \setminus B_\delta} (|\nabla u_k|^n + u_k^n)\,\mathrm{d}x = 0,$$

for any $\delta > 0$, and then $u = 0$.

Proof. Letting $A \to +\infty$, then for any constant c, we have

$$\int_{\{u_k \le c\}} (|\nabla u_k|^n + u_k^n)\,\mathrm{d}x \to 0.$$

So we get this corollary. □

Lemma 3.5. *We have*

$$\begin{aligned}
(3.7)\quad &\lim_{k\to+\infty} \int_{\mathbb{R}^n} \Phi(\beta_k u_k^{n/(n-1)})\,\mathrm{d}x \\
&\qquad \le \lim_{L\to+\infty} \lim_{k\to+\infty} \int_{B_{Lr_k}} (e^{\beta_k u_k^{n/(n-1)}} - 1)\,\mathrm{d}x = \limsup_{k\to\infty} \frac{\lambda_k}{c_k^{n/(n-1)}},
\end{aligned}$$

and consequently

$$(3.8)\qquad \frac{\lambda_k}{c_k} \to +\infty \quad \text{and} \quad \sup_k \frac{c_k^{n/(n-1)}}{\lambda_k} < +\infty.$$

Proof. We have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Phi(\beta_k u_k^{n/(n-1)})\,\mathrm{d}x \\
&\le \int_{\{u_k \le c_k/A\}} \Phi(\beta_k u_k^{n/(n-1)})\,\mathrm{d}x + \int_{\{u_k > c_k/A\}} \Phi'(\beta_k u_k^{n/(n-1)})\,\mathrm{d}x \\
&\le \int_{\mathbb{R}^n} \Phi(\beta_k (u_k^A)^{n/(n-1)})\,\mathrm{d}x + A^{n/(n-1)} \frac{\lambda_k}{c_k^{n/(n-1)}} \int_{\mathbb{R}^n} \frac{u_k^{n/(n-1)}}{\lambda_k} \Phi'(\beta_k u_k^{n/(n-1)})\,\mathrm{d}x.
\end{aligned}$$

Applying (3.2), we can find L such that $u_k \le 1$ on $\mathbb{R}^n \setminus B_L$. Then by Corollary 3.4 and the form of Φ, we have

$$\lim_{k\to+\infty} \int_{\mathbb{R}^n\setminus B_L} \Phi(p\beta_k (u_k^A)^{n/(n-1)})\,\mathrm{d}x \le \lim_{k\to\infty} C(p) \int_{\mathbb{R}^n\setminus B_L} u_k^n\,\mathrm{d}x = 0 \tag{3.9}$$

for any $p > 0$.

Since by Lemma 3.3 $\limsup_{k\to+\infty} \int_{\mathbb{R}^n} (|\nabla u_k^A|^n + |u_k^A|^n)\,\mathrm{d}x \le 1/A < 1$ when $A > 1$, it follows from (1.1) that

$$\sup_k \int_{B_L} e^{p'\beta_k ((u_k^A - u_k(L))^+)^{n/(n-1)}}\,\mathrm{d}x < +\infty$$

for any $p' < A^{1/(n-1)}$. Since for any $p < p'$

$$p(u_k^A)^{n/(n-1)} \le p'((u_k^A - u_k(L))^+)^{n/(n-1)} + C(p,p'),$$

we have

$$\sup_k \int_{B_L} \Phi(p\beta_k (u_k^A)^{n/(n-1)})\,\mathrm{d}x < +\infty \tag{3.10}$$

for any $p < A^{1/(n-1)}$. Then on B_L, by the weak compactness of Banach spaces, we get

$$\lim_{k\to+\infty} \int_{B_L} \Phi(\beta_k (u_k^A)^{n/(n-1)})\,\mathrm{d}x = \int_{B_L} \Phi(0)\,\mathrm{d}x = 0.$$

Hence we have

$$\begin{aligned}
&\lim_{k\to+\infty} \int_{\mathbb{R}^n} \Phi(\beta_k u_k^{n/(n-1)})\,\mathrm{d}x \\
&\qquad \le \lim_{L\to\infty} \lim_{k\to+\infty} A^{n/(n-1)} \frac{\lambda_k}{c_k^{n/(n-1)}} \int_{B_L} \frac{u_k^{n/(n-1)}}{\lambda_k} \Phi'(\beta_k u_k^{n/(n-1)})\,\mathrm{d}x + C\varepsilon \\
&\qquad = \lim_{k\to+\infty} A^{n/(n-1)} \frac{\lambda_k}{c_k^{n/(n-1)}} + C\varepsilon.
\end{aligned}$$

As $A \to 1$ and $\varepsilon \to 0$, we obtain (3.7).

If λ_k/c_k were bounded or $\sup_k c_k^{n/(n-1)}/\lambda_k = +\infty$, it would follow from (3.7) that

$$\sup_{\int_{\mathbb{R}^n}(|\nabla v|^n+|v|^n)\,\mathrm{d}x=1,\, v\in H^{1,n}(\mathbb{R}^n)} \int_{\mathbb{R}^n} \Phi(\alpha_n |v|^{n/(n-1)})\,\mathrm{d}x = 0,$$

which is impossible. □

Lemma 3.6. *We have that $c_k(u_k^{1/(n-1)}/\lambda_k)\Phi'(\beta_k u_k^{n/(n-1)})$ converges to δ_0 weakly, i.e., for any $\varphi \in \mathcal{D}(\mathbb{R}^n)$ we have*

$$\lim_{k\to+\infty} \int_{\mathbb{R}^n} \varphi c_k \frac{u_k^{1/(n-1)}}{\lambda_k} \Phi'(\beta_k u_k^{n/(n-1)})\,\mathrm{d}x = \varphi(0).$$

Proof. Suppose $\operatorname{supp}\varphi \subset B_\rho$. We split the integral

$$\begin{aligned}
&\int_{B_\rho} \varphi \frac{c_k u_k^{1/(n-1)}}{\lambda_k} \Phi'(\beta_k u_k^{n/(n-1)})\,\mathrm{d}x \\
&\quad \le \int_{\{u_k\ge c_k/A\}\setminus B_{Lr_k}} \cdots + \int_{B_{Lr_k}} \cdots + \int_{\{u_k<c_k/A\}} \cdots \\
&\quad = \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3.
\end{aligned}$$

We have

$$\begin{aligned}
\mathrm{I}_1 &\le A\|\varphi\|_{C^0} \int_{\mathbb{R}^n\setminus B_{Lr_k}} \frac{u_k^{n/(n-1)}}{\lambda_k} \Phi'(\beta_k u_k^{n/(n-1)})\,\mathrm{d}x \\
&= A\|\varphi\|_{C^0}\left(1 - \int_{B_L} e^{w_k+o(1)}\,\mathrm{d}x\right),
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{I}_2 &= \int_{B_L} \varphi(r_k x) \frac{c_k(c_k + (v_k - c_k))^{1/(n-1)}}{c_k^{n/(n-1)}} e^{w_k+o(1)}\,\mathrm{d}x \\
&= \varphi(0)\int_{B_L} e^{w}\,\mathrm{d}x + o(1) = \varphi(0) + o(1).
\end{aligned}$$

By (3.9) and (3.10) we have

$$\int_{\mathbb{R}^n} \Phi(p\beta_k |u_k^A|^{n/(n-1)})\,\mathrm{d}x < C$$

for any $p < A^{1/(n-1)}$. We set $1/q + 1/p = 1$. Then we get by (3.8)

$$\begin{aligned} \mathrm{I}_3 &= \int_{\{u_k \le c_k/A\}} \varphi c_k \frac{u_k^{1/(n-1)}}{\lambda_k} \Phi'(\beta_k u_k^{n/(n-1)})\,\mathrm{d}x \\ &\le \frac{c_k}{\lambda_k} \|\varphi\|_{C^0} \|u_k^{1/(n-1)}\|_{L^q(\mathbb{R}^n)} \|e^{\beta_k |u_k^A|^{n/(n-1)}}\|_{L^p(\mathbb{R}^n)} \to 0. \end{aligned}$$

Letting $L \to +\infty$, we deduce now that

$$\lim_{k\to+\infty} \int_{\mathbb{R}^n} \varphi \frac{c_k u_k^{1/(n-1)}}{\lambda_k} \Phi'(\beta_k u_k^{n/(n-1)})\,\mathrm{d}x = \varphi(0). \qquad \square$$

Proposition 3.7. *On any $\Omega \Subset \mathbb{R}^n \setminus \{0\}$, we have that $c_k^{1/(n-1)} u_k$ converges to G in $C^1(\Omega)$, where $G \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ satisfies the following equation:*

$$-\operatorname{div} |\nabla G|^{n-2} \nabla G + G^{n-1} = \delta_0. \tag{3.11}$$

Proof. We set $U_k = c_k^{1/(n-1)} u_k$, which satisfy by (3.1) the equations:

$$-\operatorname{div} |\nabla U_k|^{n-2} \nabla U_k + U_k^{n-1} = \frac{c_k u_k^{1/(n-1)}}{\lambda_k} \Phi'(\beta_k u_k^{n/(n-1)}). \tag{3.12}$$

For our purpose, we need to prove that

$$\int_{B_R} |U_k|^q \,\mathrm{d}x \le C(q,R),$$

where $C(q,R)$ does not depend on k. We use the idea in [20] to prove this statement.

Set $\Omega_t = \{0 \le U_k \le t\}$, $U_k^t = \min\{U_k, t\}$. Then we have

$$\begin{aligned} \int_{\Omega_t} (|\nabla U_k^t|^n + |U_k^t|^n)\,\mathrm{d}x &\le \int_{\mathbb{R}^n} (-U_k^t \Delta_n U_k + U_k^t U_k^{n-1}) \\ &= \int_{\mathbb{R}^n} U_k^t \frac{c_k u_k^{1/(n-1)}}{\lambda_k} \Phi'(\beta_k u_k^{n/(n-1)})\,\mathrm{d}x \le 2t. \end{aligned}$$

Let η be a radially symmetric cut-off function which is 1 on B_R and 0 on B_{2R}^c. Then,

$$\int_{B_{2R}} |\nabla \eta U_k^t|^n \,\mathrm{d}x \le C_1(R) + C_2(R)t.$$

Then, when t is bigger than $C_1(R)/C_2(R)$, we have

$$\int_{B_{2R}} |\nabla \eta U_k^t|^n \, dx \le 2C_2(R)t.$$

Set ρ such that $U_k(\rho) = t$. Then we have

$$\inf\left\{ \int_{B_{2R}} |\nabla v|^n \, dx \mid v \in H_0^{1,n}(B_{2R}) \text{ and } v|_{B_\rho} = t \right\} \le 2C_2(R)t.$$

On the other hand, the inf is achieved by $-t \log |x|/(2R)/\log(2R/\rho)$. By a direct computation, we have

$$\frac{\omega_{n-1} t^{n-1}}{(\log(2R/\rho^{n-1})} \le 2C_2(R),$$

and hence for any $t > C_1(R)/C_2(R)$

$$|\{x \in B_{2R} \mid U_k \ge t\}| = |B_\rho| \le C_3(R)e^{-A(R)t},$$

where $A(R)$ is a constant only depending on R. Then, for any $\delta < A$,

$$\begin{aligned}\int_{B_R} e^{\delta U_k} \, dx &\le \sum_{m=0}^{\infty} \mu\Big(\{m \le U_k \le m+1\}\Big) e^{\delta(m+1)} \\ &\le \sum_{m=0}^{\infty} e^{-(A-\delta)m} e^{\delta} \le C.\end{aligned}$$

Then, testing Equation (3.12) with the function

$$\log \frac{1 + 2(U_k - U_k(R))^+}{1 + (U_k - U_k(R))^+},$$

we get

$$\begin{aligned}&\int_{B_R} \frac{|\nabla U_k|^n}{(1 + U_k - U_k(R))(1 + 2U_k - 2U_k(R))} \, dx \\ &\qquad \le \log 2 \int_{B_R} \frac{c_k u_k^{1/(n-1)}}{\lambda_k} \Phi'(\beta_k u_k^{n/(n-1)}) \, dx \\ &\qquad\qquad - \int_{B_R} U_k^{n-1} \log \frac{1 + 2(U_k - U_k(R))}{1 + (U_k - U_k(R))} \, dx \le C.\end{aligned}$$

Given $q < n$, by Young's Inequality, we have

$$\begin{aligned}
&\int_{B_R} |\nabla U_k|^q \, dx \\
&\le \int_{B_R} \left[\frac{|\nabla U_k|^n}{(1 + U_k - U_k(R))(1 + 2U_k - 2U_k(R))} + ((1 + U_k)(1 + 2U_k))^{n/(n-q)} \right] dx \\
&\le \int_{B_R} \left[\frac{|\nabla U_k|^n}{(1 + U_k - U_k(R))(1 + 2U_k - 2U_k(R))} + Ce^{\delta U_k} \right] dx.
\end{aligned}$$

Hence, we are able to assume that U_k converges to a function G weakly in $H^{1,p}(B_R)$ for any R and $p < n$. Applying Lemma 3.6, we get (3.11).

Hence U_k is bounded in $L^q(\Omega)$ for any $q > 0$. By Corollary 3.4 and Theorem A, $e^{\beta_k u_k^{n/(n-1)}}$ is also bounded in $L^q(\Omega)$ for any $q > 0$. Then, applying Theorem 2.8 in [19], and the main result in [18] (or [7]), we get $\|U_k\|_{C^{1,\alpha}(\Omega)} \le C$. So, U_k converges to G in $C^1(\Omega)$. ◻

For the Green function G we have the following results:

Lemma 3.8. *$G \in C^{1,\alpha}_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ and near 0 we can write*

$$G = -\frac{1}{\alpha_n} \log r^n + A + O(r^n \log^n r); \tag{3.13}$$

here, A is a constant. Moreover, for any $\delta > 0$, we have

$$\begin{aligned}
\lim_{k\to+\infty} \int_{\mathbb{R}^n \setminus B_\delta} (|\nabla c_k^{1/(n-1)} u_k|^n + (c_k^{1/(n-1)} u_k)^n) \, dx &= \int_{\mathbb{R}^n \setminus B_\delta} (|\nabla G|^n + |G|^n) \, dx \\
&= G(\delta)\left(1 - \int_{B_\delta} G^{n-1} \, dx\right).
\end{aligned}$$

Proof. Testing Equation (3.12) with 1, we get

$$\omega_{n-1}(-G'(r))^{n-1} r^{n-1} = \int_{\partial B_r} |\nabla G|^{n-2} \frac{\partial G}{\partial n} = 1 - \int_{B_r} G^{n-1} \, dx.$$

Noticing that $\int_{B_r} G^{n-1} \, dx = O(r^p)$ holds for any $p < n$, we get

$$G' = -\frac{n}{\alpha_n r} + O(r^{p-1}).$$

Then, we get $G = -(1/\alpha_n) \log r^n + O(1)$, and then $\int_{B_r} G^{n-1} = O(r^n \log^{n-1} r)$, hence

$$G' = -\frac{n}{\alpha_n r} + O(r^{n-1} \log^{n-1} r).$$

Then, we get (3.13).

We have

$$\int_{\mathbb{R}^n\setminus B_\delta} u_k^{n/(n-1)}\Phi'(\beta_k u_k^{n/(n-1)})\,\mathrm{d}x \le C\int_{\mathbb{R}^n\setminus B_\delta} u_k^n\,\mathrm{d}x \to 0. \tag{3.14}$$

Recall that $U_k \in H_0^{1,n}(B_{R_k})$. By Equation (3.12) we get

$$\int_{\mathbb{R}^n\setminus B_\delta}(|\nabla U_k|^n + U_k^n)\,\mathrm{d}x = \frac{c_k^{n/(n-1)}}{\lambda_k}\int_{\mathbb{R}^n\setminus B_\delta} u_k^{n/(n-1)}\Phi'(\beta_k u_k^{n/(n-1)})\,\mathrm{d}x - \int_{\partial B_\delta}\frac{\partial U_k}{\partial n}|\nabla U_k|^{n-2}U_k\,\mathrm{d}S.$$

By (3.14) and (3.8) we then get

$$\begin{aligned}\lim_{k\to+\infty}\int_{\mathbb{R}^n\setminus B_\delta}(|\nabla U_k|^n + U_k^n)\,\mathrm{d}x &= -\lim_{k\to+\infty}\int_{\partial B_\delta}\frac{\partial U_k}{\partial n}|\nabla U_k|^{n-2}U_k\,\mathrm{d}S\\ &= -G(\delta)\int_{\partial B_\delta}\frac{\partial G}{\partial n}|\nabla G|^{n-2}\,\mathrm{d}S\\ &= G(\delta)\Big(1-\int_{B_\delta}G^{n-1}\,\mathrm{d}x\Big).\end{aligned}$$

□

We are now in the position to complete the proof of Theorem 1.1: We have seen in (3.9) that

$$\int_{\mathbb{R}^n\setminus B_R}\Phi(\beta_k u_k^{n/(n-1)})\,\mathrm{d}x \le C.$$

So, we only need to prove on B_R,

$$\int_{B_R} e^{\beta_k u_k^{n/(n-1)}}\,\mathrm{d}x < C.$$

The classical Trudinger-Moser inequality implies that

$$\int_{B_R} e^{\beta_k((u_k-u_k(R))^+)^{n/(n-1)}}\,\mathrm{d}x < C = C(R).$$

By Proposition 3.7, $u_k(R) = O(1/c_k^{1/(n-1)})$, and hence we have

$$\begin{aligned}u_k^{n/(n-1)} &\le ((u_k-u_k(R))^+ + u_k(R))^{n/(n-1)}\\ &\le ((u_k-u_k(R))^+)^{n/(n-1)} + C_1.\end{aligned}$$

Then, we get

$$\int_{B_R} e^{\beta_k u_k^{n/(n-1)}} \le C'.$$

□

4. THE PROOF OF PROPOSITION 1.3

We will use a result of Carleson and Chang (see [5]):

Lemma 4.1. *Let B be the unit ball in $\mathbb{R}^n$. Assume that u_k is a sequence in $H_0^{1,n}(B)$ with $\int_B |\nabla u_k|^n \,\mathrm{d}x = 1$. If $u_k \rightharpoonup 0$, then*

$$\limsup_{k\to+\infty} \int_B (e^{\alpha_n |u_k|^{n/(n-1)}} - 1)\,\mathrm{d}x \le |B| e^{1+1/2+\cdots+1/(n-1)}.$$

Proof of Proposition 1.3. Set $u_k'(x) = (u_k(x) - u_k(\delta))^+ / \|\nabla u_k\|_{L^n(B_\delta)}$ which is in $H_0^{1,n}(B_\delta)$. Then by the result of Carleson and Chang, we have

$$\limsup_{k\to+\infty} \int_{B_\delta} e^{\beta_k {u_k'}^{n/(n-1)}} \le |B_\delta|(1 + e^{1+1/2+\cdots+1/(n-1)}).$$

By Lemma 3.8, we have

$$\int_{\mathbb{R}^n\setminus B_\delta} (|\nabla c_k^{1/(n-1)} u_k|^n + (c_k^{1/(n-1)} u_k)^n)\,\mathrm{d}x \to G(\delta)\left(1 - \int_{B_\delta} G^{n-1}\,\mathrm{d}x\right),$$

and therefore we get

$$\begin{aligned} \int_{B_\delta} |\nabla u_k|^n \,\mathrm{d}x &= 1 - \int_{\mathbb{R}^n\setminus B_\delta} (|\nabla u_k|^n + u_k^n)\,\mathrm{d}x - \int_{B_\delta} u_k^n\,\mathrm{d}x \\ &= 1 - \frac{G(\delta) + \varepsilon_k(\delta)}{c_k^{n/(n-1)}}, \end{aligned} \tag{4.1}$$

where $\lim_{\delta\to 0}\lim_{k\to+\infty} \varepsilon_k(\delta) = 0$.

By (3.9) in Lemma 3.5 we have

$$\lim_{L\to+\infty}\lim_{k\to+\infty} \int_{B_\rho\setminus B_{Lr_k}} e^{\beta_k u_k^{n/(n-1)}}\,\mathrm{d}x = |B_\rho|,$$

for any $\rho < \delta$. Furthermore, on B_ρ we have by (4.1)

$$\begin{aligned} (u_k')^{n/(n-1)} &\le \frac{u_k^{n/(n-1)}}{\left(\dfrac{1 - (G(\delta) + \varepsilon_k(\delta))}{c_k^{n/(n-1)}}\right)^{1/(n-1)}} \\ &= u_k^{n/(n-1)}\left(1 + \frac{1}{n-1}\frac{G(\delta) + \varepsilon_k(\delta)}{c_k^{n/(n-1)}} + O\left(\frac{1}{c_k^{2n/(n-1)}}\right)\right) \quad = \end{aligned}$$

$$= u_k^{n/(n-1)} + \frac{1}{n-1} G(\delta) \left(\frac{u_k}{c_k}\right)^{n/(n-1)} + O(c_k^{-n/(n-1)})$$
$$\le u_k^{n/(n-1)} - \frac{\log \delta^n}{(n-1)\alpha_n}.$$

Then we have

$$\lim_{L\to+\infty} \lim_{k\to+\infty} \int_{B_\rho \setminus B_{Lr_k}} e^{\beta_k {u'_k}^{n/(n-1)}} \, \mathrm{d}x$$
$$\le O(\delta^{-n}) \lim_{L\to+\infty} \lim_{k\to+\infty} \int_{B_\rho \setminus B_{Lr_k}} e^{\beta_k u_k^{n/(n-1)}} \, \mathrm{d}x \to |B_\rho| O(\delta^{-n}).$$

Since $u'_k \to 0$ on $B_\delta \setminus B_\rho$, we get $\lim_{k\to+\infty} \int_{B_\delta \setminus B_\rho} (e^{\beta_k {u'_k}^{n/(n-1)}} - 1) \, \mathrm{d}x = 0$, then

$$0 \le \lim_{L\to+\infty} \lim_{k\to+\infty} \int_{B_\delta \setminus B_{Lr_k}} (e^{\beta_k {u'_k}^{n/(n-1)}} - 1) \, \mathrm{d}x \le |B_\rho| O(\delta^{-n}).$$

Letting $\rho \to 0$, we get $\lim_{L\to+\infty} \lim_{k\to+\infty} \int_{B_\delta \setminus B_{Lr_k}} (e^{\beta_k {u'_k}^{n/(n-1)}} - 1) \, \mathrm{d}x = 0$. So, we have

$$\lim_{L\to+\infty} \lim_{k\to+\infty} \int_{B_{Lr_k}} (e^{\beta_k {u'_k}^{n/(n-1)}} - 1) \, \mathrm{d}x \le e^{1+1/2+\cdots+1/(n-1)} |B_\delta|.$$

Now, we fix an L. Then for any $x \in B_{Lr_k}$, we have

$$\beta_k u_k^{n/(n-1)} = \beta_k \left(\frac{u_k}{\|\nabla u_k\|_{L^n(B_\delta)}}\right)^{n/(n-1)} \left(\int_{B_\delta} |\nabla u_k|^n \, \mathrm{d}x\right)^{1/(n-1)}$$
$$= \beta_k \left(u'_k + \frac{u_k(\delta)}{\|\nabla u_k\|_{L^n(B_\delta)}}\right)^{n/(n-1)} \left(\int_{B_\delta} |\nabla u_k|^n \, \mathrm{d}x\right)^{1/(n-1)}$$

(using that $u_k(\delta) = O(1/c_k^{1/(n-1)})$ and $\|\nabla u_k\|_{L^n(B_\delta)} = 1 + O(1/c_k^{n/(n-1)})$)

$$= \beta_k \left(u'_k + u_k(\delta) + O\left(\frac{1}{c_k^{(n+1)/(n-1)}}\right)\right)^{n/(n-1)} \left(\int_{B_\delta} |\nabla u_k|^n \, \mathrm{d}x\right)^{1/(n-1)}$$
$$= \beta_k {u'_k}^{n/(n-1)} \left(1 + \frac{u_k(\delta)}{u'_k} + O\left(\frac{1}{c_k^{2n/(n-1)}}\right)\right)^{n/(n-1)} \left(1 - \frac{G(\delta) + \varepsilon_k(\delta)}{c_k^{n/(n-1)}}\right)^{1/(n-1)}$$
$$= \beta_k {u'_k}^{n/(n-1)} \left[1 + \frac{n}{n-1} \frac{u_k(\delta)}{u'_k} - \frac{1}{n-1} \frac{G(\delta) + \varepsilon_k(\delta)}{c_k^{n/(n-1)}} + O\left(\frac{1}{c_k^{2n/(n-1)}}\right)\right].$$

It is easy to check that

$$\frac{u_k'(r_k x)}{c_k} \to 1 \quad \text{and} \quad (u_k'(r_k x))^{1/(n-1)} u_k(\delta) \to G(\delta).$$

So, we get

$$\begin{aligned}
&\lim_{L\to+\infty} \lim_{k\to+\infty} \int_{B_{Lr_k}} (e^{\beta_k u_k^{n/(n-1)}} - 1)\,\mathrm{d}x \\
&= \lim_{L\to+\infty} \lim_{k\to+\infty} e^{\alpha_n G(\delta)} \int_{B_{Lr_k}} (e^{\beta_k u_k'^{\,n/(n-1)}} - 1)\,\mathrm{d}x \\
&\le e^{\alpha_n G(\delta)} \delta^n \frac{\omega_{n-1}}{n} e^{1+1/2+\cdots+1/(n-1)} \\
&= e^{\alpha_n(-(1/\alpha_n)\log \delta^n + A + O(\delta^n \log^n \delta))} \delta^n \frac{\omega_{n-1}}{n} e^{1+1/2+\cdots+1/(n-1)}.
\end{aligned}$$

Letting $\delta \to 0$, then the inequality above together with Lemma 3.2 imply Proposition 1.3. ∎

5. THE TEST FUNCTION 1

In this section, we will construct a function sequence $\{u_\varepsilon\} \subset H^{1,n}(\mathbb{R}^n)$ with $\|u_\varepsilon\|_{H^{1,n}} = 1$ which satisfies

$$\int_{\mathbb{R}^n} \Phi(\alpha_n |u_\varepsilon|^{n/(n-1)})\,\mathrm{d}x > \frac{\omega_{n-1}}{n} e^{\alpha_n A + 1 + 1/2 + \cdots + /1(n-1)},$$

for $\varepsilon > 0$ sufficiently small.

Let

$$u_\varepsilon = \begin{cases} C - \dfrac{(n-1)\log\left(1 + c_n |x/\varepsilon|^{n/(n-1)}\right) + \Lambda_\varepsilon}{\alpha_n C^{1/(n-1)}}, & |x| \le L\varepsilon, \\[2ex] \dfrac{G(|x|)}{C^{1/(n-1)}}, & |x| > L\varepsilon, \end{cases}$$

where Λ_ε, C and L are functions of ε (which will be defined later, by (5.1), (5.2), (5.5)) which satisfy

(i) $L \to +\infty$, $C \to +\infty$, and $L\varepsilon \to 0$, as $\varepsilon \to 0$;
(ii) $C - ((n-1)\log(1 + c_n L^{n/(n-1)}) + \Lambda_\varepsilon)/\alpha_n C^{1/(n-1)} = G(L\varepsilon)/C^{1/(n-1)}$;
(iii) $\log L / C^{n/(n-1)} \to 0$, as $\varepsilon \to 0$.

We use the normalization of u_ε to obtain information on Λ_ε, C and L. We have

$$\begin{aligned}\int_{\mathbb{R}^n\setminus B_{L\varepsilon}}(|\nabla u_\varepsilon|^n+u_\varepsilon^n)\,\mathrm{d}x &= \frac{1}{C^{n/(n-1)}}\left(\int_{B_{L\varepsilon}^c}|\nabla G|^n\,\mathrm{d}x+\int_{B_{L\varepsilon}^c}G^n\,\mathrm{d}x\right)\\ &= \frac{1}{C^{n/(n-1)}}\int_{\partial B_{L\varepsilon}}G(L\varepsilon)|\nabla G|^{n-2}\frac{\partial G}{\partial n}\,\mathrm{d}S\\ &= \frac{G(L\varepsilon)-G(L\varepsilon)\int_{B_{L\varepsilon}}G^{n-1}\,\mathrm{d}x}{C^{n/(n-1)}}.\end{aligned}$$

and

$$\begin{aligned}\int_{B_{L\varepsilon}}|\nabla u_\varepsilon|^n\,\mathrm{d}x &= \frac{n-1}{\alpha_n C^{n/(n-1)}}\int_0^{c_nL^{n/(n-1)}}\frac{u^{n-1}}{(1+u)^n}\,\mathrm{d}u\\ &= \frac{n-1}{\alpha_n C^{n/(n-1)}}\int_0^{c_nL^{n/(n-1)}}\frac{((1+u)-1)^{n-1}}{(1+u)^n}\,\mathrm{d}u\\ &= \frac{n-1}{\alpha_n C^{n/(n-1)}}\sum_{k=0}^{n-2}\frac{C_{n-1}^k(-1)^{n-1-k}}{n-k-1}\\ &\quad+\frac{n-1}{\alpha_n C^{n/(n-1)}}\log(1+c_nL^{n/(n-1)})+O\left(\frac{1}{L^{n/(n-1)}C^{n/(n-1)}}\right)\\ &= -\frac{n-1}{\alpha_n C^{n/(n-1)}}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n-1}\right)\\ &\quad+\frac{n-1}{\alpha_n C^{n/(n-1)}}\log(1+c_nL^{n/(n-1)})+O\left(\frac{1}{L^{n/(n-1)}C^{n/(n-1)}}\right),\end{aligned}$$

where we used the fact

$$-\sum_{k=0}^{n-2}\frac{C_{n-1}^k(-1)^{n-1-k}}{n-k-1}=1+\frac{1}{2}+\cdots+\frac{1}{n-1}.$$

It is easy to check that

$$\int_{B_{L\varepsilon}}|u_\varepsilon|^n\,\mathrm{d}x=O((L\varepsilon)^nC^n\log L),$$

and thus we get

$$\begin{aligned}\int_{\mathbb{R}^n}(|\nabla u_\varepsilon|^n + u_\varepsilon^n)\,\mathrm{d}x \\ &= \frac{1}{\alpha_n C^{n/(n-1)}}\Big\{-(n-1)\Big(1+\frac{1}{2}+\cdots+\frac{1}{n-1}\Big)+\alpha_n A \\ &\quad + (n-1)\log(1+c_nL^{n/(n-1)}) - \log(L\varepsilon)^n + \varphi\Big\},\end{aligned}$$

where $\varphi = O\left((L\varepsilon)^n C^n \log L + (L\varepsilon)^n \log^n L\varepsilon + L^{-n/(n-1)}\right)$.

Setting $\int_{\mathbb{R}^n}(|\nabla u_\varepsilon|^n + u_\varepsilon^n)\,\mathrm{d}x = 1$, we obtain

$$\begin{aligned}(5.1)\quad \alpha_n C^{n/(n-1)} &= \\ &= -(n-1)\left(1+\frac{1}{2}+\cdots+\frac{1}{n-1}\right)+\alpha_n A+\log\frac{(1+c_nL^{n/(n-1)})^{n-1}}{L^n}-\log\varepsilon^n+\varphi \\ &= -(n-1)\left(1+\frac{1}{2}+\cdots+\frac{1}{n-1}\right)+\alpha_n A+\log\frac{\omega_{n-1}}{n}-\log\varepsilon^n+\varphi.\end{aligned}$$

By (ii) we have

$$\alpha_n C^{n/(n-1)} - (n-1)\log(1+c_nL^{n/(n-1)}) + \Lambda_\varepsilon = \alpha G(L\varepsilon)$$

and hence

$$-(n-1)\left(1+\frac{1}{2}+\cdots+\frac{1}{n-1}\right)+\alpha_n A - \log(L\varepsilon)^n + \varphi + \Lambda_\varepsilon = \alpha G(L\varepsilon);$$

this implies that

$$\Lambda_\varepsilon = -(n-1)\left(1+\frac{1}{2}+\cdots+\frac{1}{n-1}\right)+\varphi. \tag{5.2}$$

Next, we compute $\int_{B_{L\varepsilon}} e^{\alpha_n|u_\varepsilon|^{n/(n-1)}}\,\mathrm{d}x$.

Clearly, $\varphi(t) = |1-t|^{n/(n-1)} + (n/(n-1))t$ is increasing when $0 \le t \le 1$ and decreasing when $t \le 0$; then

$$|1-t|^{n/(n-1)} \ge 1 - \frac{n}{n-1}t, \quad \text{when } |t| < 1.$$

Thus we have by (ii), for any $x \in B_{L\varepsilon}$

$$\begin{aligned}
(5.3)\quad \alpha_n u_\varepsilon^{n/(n-1)} &= \\
&= \alpha_n C^{n/(n-1)} \left| 1 - \frac{(n-1)\log(1 + c_n|x/\varepsilon|^{n/(n-1)}) + \Lambda_\varepsilon}{\alpha_n C^{n/(n-1)}} \right|^{n/(n-1)} \\
&\geq \alpha_n C^{n/(n-1)} \left(1 - \frac{n}{n-1} \frac{(n-1)\log(1 + c_n|x/\varepsilon|^{n/(n-1)}) + \Lambda_\varepsilon}{\alpha_n C^{n/(n-1)}} \right).
\end{aligned}$$

Then we have

$$\begin{aligned}
&\int_{B_{L\varepsilon}} e^{\alpha_n |u_\varepsilon|^{n/(n-1)}}\,\mathrm{d}x \geq \int_{B_{L\varepsilon}} e^{\alpha_n C^{n/(n-1)} - n\log(1 + c_n|x/\varepsilon|^{n/(n-1)}) - n/(n-1)\Lambda_\varepsilon} \\
&= e^{\alpha_n C^{n/(n-1)} - (n/(n-1))\Lambda_\varepsilon} \int_{B_L} \frac{\varepsilon^n}{(1 + c_n|x|^{n/(n-1)})^n}\,\mathrm{d}x \\
&= e^{\alpha_n C^{n/(n-1)} - (n/(n-1))\Lambda_\varepsilon} (n-1)\varepsilon^n \int_0^{c_n L^{n/(n-1)}} \frac{u^{n-2}}{(1+u)^n}\,\mathrm{d}u \\
&= e^{\alpha_n C^{n/(n-1)} - (n/(n-1))\Lambda_\varepsilon} (n-1)\varepsilon^n \int_0^{c_n L^{n/(n-1)}} \frac{((u+1)-1)^{n-2}}{(1+u)^n}\,\mathrm{d}u \\
&= e^{\alpha_n C^{n/(n-1)} - (n/(n-1))\Lambda_\varepsilon} \varepsilon^n (1 + O(L^{-n/(n-1)})) \\
&= \frac{\omega_{n-1}}{n} e^{\alpha_n A + 1 + 1/2 + \cdots + 1/(n-1)} + O\Big((L\varepsilon)^n C^n \log L + L^{-n/(n-1)} + (L\varepsilon)^n \log^n L\varepsilon\Big).
\end{aligned}$$

Here, we used the fact

$$\sum_{k=0}^{m} \frac{(-1)^{m-k}}{m-k+1} C_m^k = \frac{1}{m+1}.$$

Then

$$\begin{aligned}
\int_{B_{L\varepsilon}} \Phi(\alpha_n u_\varepsilon^{n/(n-1)})\,\mathrm{d}x \geq{}& \frac{\omega_{n-1}}{n} e^{\alpha_n A + 1 + 1/2 + \cdots + 1/(n-1)} \\
&+ O\Big((L\varepsilon)^n C^n \log L + L^{-n/(n-1)} + (L\varepsilon)^n \log^n L\varepsilon\Big).
\end{aligned}$$

Moreover, on $\mathbb{R}^n \setminus B_{L\varepsilon}$ we have the estimate

$$\int_{\mathbb{R}^n \setminus B_{L\varepsilon}} \Phi(\alpha_n u_\varepsilon^{n/(n-1)})\,\mathrm{d}x \geq \frac{\alpha_n^{n-1}}{(n-1)!} \int_{\mathbb{R}^n \setminus B_{L\varepsilon}} \left| \frac{G(x)}{C^{1/(n-1)}} \right|^n \mathrm{d}x,$$

and thus we get

$$
\begin{aligned}
(5.4)\quad \int_{\mathbb{R}^n} \Phi(\alpha_n u_\varepsilon^{n/(n-1)})\,\mathrm{d}x \ \geq\\
&\geq \frac{\omega_{n-1}}{n} e^{\alpha_n A + 1 + 1/2 + \cdots + 1/(n-1)} + \frac{\alpha_n^{n-1}}{(n-1)} \int_{\mathbb{R}^n \setminus B_{L\varepsilon}} \left| \frac{G(x)}{C^{1/(n-1)}} \right|^n \mathrm{d}x \\
&\quad + O\Big((L\varepsilon)^n C^n \log L + L^{-n/(n-1)} + (L\varepsilon)^n \log^n L\varepsilon\Big) \\
&= \frac{\omega_{n-1}}{n} e^{\alpha_n A + 1 + 1/2 + \cdots + 1/(n-1)} \\
&\quad + \frac{\alpha_n^{n-1}}{(n-1)! C^{n/(n-1)}} \Bigg[\int_{\mathbb{R}^n \setminus B_{L\varepsilon}} |G(x)|^n \,\mathrm{d}x \\
&\quad + O\bigg((L\varepsilon)^n C^{n+n/(n-1)} \log L + \frac{C^{n/(n-1)}}{L^{n/(n-1)}} + C^{n/(n-1)} (L\varepsilon)^n \log^n L\varepsilon\bigg)\Bigg].
\end{aligned}
$$

We now set

$$
L = -\log \varepsilon; \tag{5.5}
$$

then $L\varepsilon \to 0$ as $\varepsilon \to 0$. We then need to prove that there exists a $C = C(\varepsilon)$ which solves Equation (5.1). We set

$$
\begin{aligned}
f(t) = -\alpha_n t^{n/(n-1)} - (n-1)\left(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}\right)& \\
+ \alpha_n A + \log \frac{\omega_{n-1}}{n} - \log \varepsilon^n + \varphi&,
\end{aligned}
$$

Since

$$
f\left(\left(-\frac{2}{\alpha_n} \log \varepsilon^n\right)^{n/(n-1)}\right) = \log \varepsilon^n + o(1) + \varphi < 0
$$

for ε small, and

$$
f\left(\left(-\frac{1}{2\alpha_n} \log \varepsilon^n\right)^{n/(n-1)}\right) = -\frac{1}{2} \log \varepsilon^n + o(1) + \varphi > 0
$$

for ε small, f has a zero in

$$
\left(\left(-\frac{1}{2\alpha_n} \log \varepsilon^n\right)^{(n-1)/n}, \left(-\frac{2}{\alpha_n} \log \varepsilon^n\right)^{(n-1)/n}\right).
$$

Thus, we defined C, and it satisfies $\alpha_n C^{n/(n-1)} = -\log \varepsilon^n + O(1)$. Therefore, as $\varepsilon \to 0$, we have

$$\frac{\log L}{C^{n/(n-1)}} \to 0,$$

and then

$$(L\varepsilon)^n C^{n+n/(n-1)} \log L + C^{n/(n-1)} L^{-n/(n-1)} + C^{n/(n-1)} (L\varepsilon)^n \log^n L\varepsilon \to 0.$$

Therefore, (i), (ii), (iii) hold, and we can conclude from (5.4) that for $\varepsilon > 0$ sufficiently small

$$\int_{\mathbb{R}^n} \Phi(\alpha_n u_\varepsilon^{n/(n-1)})\, \mathrm{d}x > \frac{\omega_{n-1}}{n} e^{\alpha_n A + 1 + 1/2 + \cdots + 1/(n-1)}.$$

6. The Test Function 2

In this section we construct, for $n > 2$, functions u_ε such that

$$\int_{\mathbb{R}^n} \Phi\left(\alpha_n \left(\frac{u_\varepsilon}{\|u_\varepsilon\|_{H^{1,n}}}\right)^{n/(n-1)}\right) \mathrm{d}x > \frac{\alpha_n^{n-1}}{(n-1)!},$$

for $\varepsilon > 0$ sufficiently small.

Let $\varepsilon^n = e^{-\alpha_n c^{n/(n-1)}}$, and

$$u_\varepsilon = \begin{cases} c & |x| < L\varepsilon, \\ \dfrac{-n \log(x/L)}{\alpha_n c^{1/(n-1)}} & L\varepsilon \le |x| \le L, \\ 0 & L \le |x|, \end{cases}$$

where L is a function of ε which will be defined later.

We have

$$\int_{\mathbb{R}^n} |\nabla u_\varepsilon|^n = 1,$$

and

$$\int_{\mathbb{R}^n} u_\varepsilon^n \,\mathrm{d}x = \frac{\omega_{n-1}}{n} c^n (L\varepsilon)^n + \frac{\omega_{n-1} n^n L^n}{\alpha_n^n c^{n/(n-1)}} \int_\varepsilon^1 r^{n-1} \log^n r \,\mathrm{d}r.$$

Then

$$\int_{\mathbb{R}^n} \Phi\left(\alpha_n\left(\frac{u_\varepsilon}{\|u_\varepsilon\|_{H^{1,n}}}\right)^{n/(n-1)}\right) \mathrm{d}x$$

$$\geq \frac{\alpha_n^{n-1}}{(n-1)!}\frac{\int_{\mathbb{R}^n} u_\varepsilon^n \,\mathrm{d}x}{1+\int_{\mathbb{R}^n} u_\varepsilon^n \,\mathrm{d}x} + \frac{\alpha_n^n}{n!}\frac{\int_{\mathbb{R}^n\setminus B_{L\varepsilon}} u_\varepsilon^{n^2/(n-1)} \,\mathrm{d}x}{\left(1+\int_{\mathbb{R}^n} u_\varepsilon^n \,\mathrm{d}x\right)^{n/(n-1)}}$$

$$= \frac{\alpha_n^{n-1}}{(n-1)!} - \frac{\alpha_n^{n-1}}{(n-1)!}\frac{1}{1+\frac{\omega_{n-1}}{n}c^n(L\varepsilon)^n + \frac{\omega_{n-1}n^nL^n}{\alpha_n^n c^{n/(n-1)}}\int_\varepsilon^1 r^{n-1}\log^n r\,\mathrm{d}r}$$

$$+ \frac{\alpha_n^n}{n!}\frac{\omega_{n-1}L^n/c^{n^2/(n-1)^2}\left(\frac{n}{\alpha_n}\right)^{n^2/(n-1)}\int_\varepsilon^1 r^{n-1}\log^{n^2/(n-1)} r}{\left(1+\frac{\omega_{n-1}}{n}c^n(L\varepsilon)^n + \frac{\omega_{n-1}n^nL^n}{\alpha_n^n c^{n/(n-1)}}\int_\varepsilon^1 r^{n-1}\log^n r\,\mathrm{d}r\right)^{n/(n-1)}}.$$

We now ask that L satisfies

$$\frac{c^{n/(n-1)}}{L^n} \to 0, \quad \text{as } \varepsilon \to 0. \tag{6.1}$$

Then, for sufficiently small ε, we have

$$-\frac{\alpha_n^{n-1}}{(n-1)!}\frac{1}{1+\frac{\omega_{n-1}}{n}c^n(L\varepsilon)^n + \frac{\omega_{n-1}n^nL^n}{\alpha_n^n c^{n/(n-1)}}\int_\varepsilon^1 r^{n-1}\log^n r\,\mathrm{d}r}$$

$$+ \frac{\alpha^n}{n!}\frac{\frac{\omega_{n-1}L^n}{c^{n^2/(n-1)^2}}\left(\frac{n}{\alpha_n}\right)^{n^2/(n-1)}\int_\varepsilon^1 r^{n-1}\log^{n^2/(n-1)} r}{\left(1+\frac{\omega_{n-1}}{n}c^n(L\varepsilon)^n + \frac{\omega_{n-1}n^nL^n}{\alpha_n^n c^{n/(n-1)}}\int_\varepsilon^1 r^{n-1}\log^n r\,\mathrm{d}r\right)^{n/(n-1)}}$$

$$\geq B_1 L^{n-n^2/(n-1)} - B_2\frac{c^{n/(n-1)}}{L^n}$$

$$= \frac{c^{n/(n-1)}}{L^n}\left(B_1\frac{L^{2n-n^2/(n-1)}}{c^{n/(n-1)}} - B_2\right)$$

$$= \frac{c^{n/(n-1)}}{L^n}\left(B_1\frac{L^{(n/(n-1))(n-2)}}{c^{n/(n-1)}} - B_2\right),$$

where B_1, B_2 are positive constants.

When $n > 2$, we may choose $L = bc^{1/(n-2)}$; then, for b sufficiently large, we have

$$B_1 \frac{L^{(n/(n-1))(n-2)}}{c^{n/(n-1)}} - B_2 = B_1 b^{(n/(n-1))(n-2)} - B_2 > 0,$$

and (6.1) holds. Thus, we have proved that for $\varepsilon > 0$ sufficiently small

$$\int_{\mathbb{R}^n} \Phi\left(\alpha_n \left(\frac{u_\varepsilon}{\|u_\varepsilon\|_{H^{1,n}(\mathbb{R}^n)}}\right)^{n/(n-1)}\right) \mathrm{d}x > \frac{\alpha_n^{n-1}}{(n-1)!}.$$

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