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Abstract

A sequence of operations may be validly reordered, provided that only pairs of independent operations are commuted. Focusing on a program scheme, idealized as a local finite automaton, we consider the problem of checking whether a given string is a valid permutation of a word recognized by the automaton. Within the framework of trace theory, this is the word membership problem for rational trace languages. Existing general algorithms, although time-polynomial, have unbounded degree related to some properties of the dependence graph. Here we present two original linear-time solutions. A straightforward algorithm is suitable for any local finite automaton such that any two successors of an operation are dependent or not mutually reachable. The second approach is currently restricted to nested repeat-until loops. Using integer compositions to represent loop iterations, the algorithm constructs the loop nesting syntax tree by exploiting newly introduced functions on integer compositions. The result may be relevant for checking dependencies of rescheduled programs on parallel processors.

1 Introduction

A sequence of operations may be validly reordered, provided that only pairs of independent operations are commuted. For instance, a computer program can be idealized as a deterministic finite automaton (DFA), recognizing a set of strings from an alphabet of abstract instructions, some of which are mutually dependent. At a coarser granularity of operations, the same problem occurs when concurrent accesses to a database are serialized. Formally, the problem we address is the following: Is a given string a valid permutation of a word recognized by the automaton?

A motivation comes from the areas of compiler and processor design. Modern compilers [9] and processors [13] reorder ("reschedule") machine instructions, with respect to the original sequential program ordering, with the goal of minimizing program completion time by taking advantage of available hardware parallelism. This task involves the capability to check that instruction dependencies are not violated. We present two very efficient algorithms under different assumptions.

Trace theory is a convenient framework for stating and analyzing dependencies checking problems. We recall that trace languages were introduced in the '70s as a tool for the study of concurrent systems [12] and a comprehensive treatment of their properties and related theory is presented in [8].

Using concepts from formal language theory, the program scheme is the state-transition graph of a DFA, and since in a program each instruction differs from any other by its memory address, the DFA can be assumed to be a local one [2]. Then the previous problem is the word membership problem for the trace language represented by a local DFA.

In the past a few algorithms [5, 3, 1] have been proposed for *rational* trace languages, i.e., partially commutative languages represented by a regular string language, and also for trace languages represented by context-free languages [14, 5]. Such algorithms examine the prefixes of a trace, also using sophisticated data structures for more efficient analysis. Their time complexity, although polynomial, has an unbounded degree, which is related to some properties of the independence graph. More precisely the degree is linearly related to the size of the largest clique of the independence relation, a value which is likely to be too large for realistic application. Here we present two efficient linear-time solutions of the word membership problem for rational trace languages represented by local regular languages.

After the basic definitions of Section 2, we present in Section 2.1 a straightforward algorithm, suitable for any local DFA satisfying the following hypothesis: if a state has two successors, then either they are dependent, or one is not reachable from the other on the DFA graph. These conditions correspond to rather realistic assumptions for frequent program patterns.

The second approach is much more involved and takes the rest of the paper. The problem definition and some of the initial ideas and properties come from [7, 16], but the mathematical setting based on integer compositions (Section 4) and the problem solving strategy are new. This approach is currently restricted to nested repeat-until loops, a popular class of computationally intensive program structures, defined in Section 3. Using integer compositions to represent loop iterations, a syntax tree of an unrolled nested repeat-until expression is precisely represented by labelled integer compositions.

Section 5 focuses on the word membership problem for trace languages represented by such repeat-until string languages, and proves a theorem relating the existence of a syntax tree and of certain labelled compositions, which can be derived by observing the runs of dependent letters. Then a simple *closure assumption* is introduced which concerns the dependencies between nested loops.

Based on that, Section 6 outlines, defines and analyzes an efficient algorithm, which repeatedly applies the previously introduced product and quotient operations on the integer compositions. Under the closure assumption this algorithm solves the problem in linear time.

At last we discuss the meaning and the limits of such a closure hypothesis, and in the conclusion we

hint to possible generalizations and alternative assumptions.

2 Basic notions

Given a finite alphabet Σ and a word $x \in \Sigma^*$, $|x|$ represents the length of x while, for each $y \in \Sigma^+$, $|x|_y$ denotes the number of occurrences of y in x. Moreover, given a subset $A \subseteq \Sigma$, $\pi_A(x)$ is the projection of x over A. Further, if x is not the empty word ε, $P(x)$ and $U(x)$ denote, respectively, the first and the last symbol of x, while $S_1(x)$ is the suffix of x of length $|x|-1$.

Given a word $x \in \{a,b\}^*$, a *run* of a in x is an occurrence of a maximal factor of x included in $\{a\}^+$. An analogous definition holds for b. For instance, the word *aaabbabbbbaaa* has 3 runs of a and 2 runs of b (*aaa*, *a*, *aaa* and *bb*, *bbb*, respectively). Clearly, two words $x,y \in \{a,b\}^+$ are equal if they have the same sequence of runs of a, the same sequence of runs of b and $P(x) = P(y)$.

There is a natural relationship between runs of a letter in binary words and compositions of integers. A *composition* of an integer $n \geq 1$ is a nonempty finite sequence $(i_1, i_2, \ldots, i_h)$ of integers such that $i_1 + i_2 + \cdots + i_h = n$ and $i_j \geq 1$ for every $j = 1, \ldots, h$. Integer compositions are classical combinatorial structures, widely studied in the literature; for instance it is known that there are 2^{n-1} compositions of any integer $n \geq 1$ [10]. In our context they are used to represent projections of words on pairs of letters. In particular, every word $x \in \{a,b\}^+$, where $a \neq b$, $|x|_a \geq 1$ and $|x|_b \geq 1$, defines two compositions γ_a and γ_b determined, respectively, by the runs of a and the runs of b in x. More precisely, $\gamma_a = (i_1, i_2, \ldots, i_h)$ is a composition of $|x|_a$, where h is the number of runs of a in x and each i_j is the length of the j-th run. Analogously, γ_b is a composition of $|x|_b$ defined in a similar way. We also say that γ_a (resp., γ_b) is the composition *generated* by x on a (resp., b).

Now, let us recall some basic definitions on traces. An independence relation I on Σ is a symmetric and irreflexive relation $I \subseteq \Sigma \times \Sigma$. For every $a,b \in \Sigma$ we say that a and b are independent if $(a,b) \in I$ and in this case we also write aIb. The dependence relation D is the complement of I, that is $D = \{(a,b) \in \Sigma \times \Sigma \mid (a,b) \notin I\}$. We say that a and b are dependent if $(a,b) \in D$ (aDb for brevity). The pair (Σ, I) is called independence alphabet and it is usually represented by an undirected graph where Σ is the set of nodes and I the set of edges. An independence relation I establishes an equivalence relation $\equiv_I$ on Σ^* as the reflexive and transitive closure of the relation $\sim_I$ defined by

$$xaby \sim_I xbay \qquad \forall x,y \in \Sigma^*,\ \forall (a,b) \in I.$$

The relation $\equiv_I$ is a congruence over Σ^*, i.e. an equivalence relation preserving concatenation between words. For every $x \in \Sigma^*$ the equivalence class $[x] = \{y \in \Sigma^* \mid y \equiv_I x\}$ is called *trace*, the quotient monoid $\Sigma^* / \equiv_I$ is called trace monoid and usually denoted by $M(\Sigma, I)$. A subset $T \subseteq M(\Sigma, I)$ is called trace language and, for every $L \subseteq \Sigma^*$, we define $[L] = \{[x] \in M(\Sigma, I) \mid x \in L\}$ as the trace language represented by L. A trace language is called *rational* if it is represented by a regular language. Moreover, the rational operations on trace languages (union, product and Kleene closure) are defined as in the free monoids. It is known that the class of rational trace languages is the smallest family of trace languages including the finite sets in $M(\Sigma, I)$ and closed under the rational operations. We also recall that a rational trace language $T \subseteq M(\Sigma, I)$ is called *unambiguous* if $T = [L]$ for a regular language $L \subseteq \Sigma^*$ such that, for every $t \in T$, there is exactly one string $x \in L$ belonging to t. The class of unambiguous rational trace languages can be characterized by the so-called unambiguous rational operations and it coincides with the class of all rational trace languages if and only if the independence relation I is transitive [3, 4, 15].

In the present work we are interested in the recognition problem of rational trace languages. Given independence alphabet (Σ, I) and a regular language $L \subseteq \Sigma^*$, such a problem consists of deciding, for

an input $x \in \Sigma^*$, whether $[x]$ belongs to $[L]$ (i.e. whether $[x] \cap L$ is empty). Our purpose is to study the recognition problem for rational trace languages represented by local (string) languages [2].

2.1 Recognition of local languages with dependent successors

In this section we present a linear time algorithm for the recognition of rational trace languages that admit a local representative language satisfying a further special condition. To formalize this case, consider a (deterministic) finite state automaton $\mathcal{A} = (Q, q_0, \delta, F)$ over an alphabet Σ, where Q is the set of states, q_0 is the initial state, $\delta : Q \times \Sigma \to Q \cup \{\perp\}$ is the (partially defined) transition function and $F \subseteq Q$ is the set of final states. Such an automaton is said to be *local* if there exists a function $f : \Sigma \to Q$ such that, for all $q \in Q$ and all $a \in \Sigma$, $\delta(q,a) \neq \perp$ implies $\delta(q,a) = f(a)$. In this case Q can be reduced to the set $\{f(a) \mid a \in \Sigma\} \cup \{q_0\}$. Moreover, we can define the successors of a state $q \in Q$ as the set $Suc(q) = \{a \in \Sigma \mid \delta(q,a) \neq \perp\}$. A regular language $L \subseteq \Sigma^*$ is a *local* language if it is accepted by a local finite state automaton [2].

Now, given an independence alphabet (Σ, I) with dependent relation D, we say that the above local automaton $\mathcal{A} = (Q, q_0, \delta, F)$ has *dependent successors* if, for every $q \in Q$ and every $a, b \in Suc(q)$, we have aDb. In this case, if $L \subseteq \Sigma^*$ is the language accepted by $\mathcal{A}$, then the trace language $T = [L]$ can be recognized in linear time.

To describe the algorithm, let us represent by Δ the set of all pairs and singletons forming a covering of the dependence graph (Σ, D):

$$\Delta \;=\; \{\{a,b\} \mid a,b \in \Sigma,\ aDb\} \;\cup\; \{\{b\} \mid b \in \Sigma,\ \forall a \in \Sigma\ a \neq b \Rightarrow aIb\}$$

Recall that, for every $x, y \in \Sigma^*$ we have $x \equiv_I y$ if and only if $\pi_\ell(x) = \pi_\ell(y)$ for all $\ell \in \Delta$. Moreover, for every $a \in \Sigma$, we denote by $\Delta(a)$ the set

$$\Delta(a) := \{\ell \in \Delta \mid a \in \ell\}\ .$$

It is clear that the sets Δ and $\Delta(a)$, $a \in \Sigma$, only depend on the independence alphabet (Σ, I) and can be computed in a preprocessing phase.

For a given input $w \in \Sigma^+$ the procedure computes a word $z \equiv_I w$ accepted by $\mathcal{A}$, if any, otherwise it returns 0. Note that such a z also represents an accepting computation of the automaton. The procedure maintains a family of strings $\{y_\ell : \ell \in \Delta\}$, where at the beginning $y_\ell = \pi_\ell(w)$ for every ℓ. A state $q \in Q$ is also updated which represents the current state of the computation. In the main iteration one looks for a letter $a \in Suc(q)$ that occurs as first symbol in all y_ℓ with $a \in \ell$: by the hypothesis of dependence of successors, there is at most one a satisfying that condition; in this case $f(a)$ becomes the new current state and its first occurrences are erased from all y_ℓ's such that $a \in \ell$. This process is iterated until either all projections y_ℓ are empty or no new symbol can be found among the successors of the current state. The input is accepted if and only if, at the end of the computation, all y_ℓ's are empty and the current state is final.

```
begin
    for ℓ ∈ Δ do y_ℓ := π_ℓ(w)
    q := q_0
    z := ε
    out := 0
    while out = 0 ∧ ∃ℓ ∈ Δ such that |y_ℓ| > 0 do
```

```
            begin
                  if ∃a ∈ Suc(q) such that a = P(y_ℓ) for all ℓ ∈ Δ(a)
                                          then { q := δ(q,a)
                                               { z := za
                                               { for ℓ ∈ Δ(a) do y_ℓ := S_1(y_ℓ)
                                          else out := 1
            end
      if out = 0 ∧ q ∈ F
            then return z
            else return 0
end
```

Note that, if $[w] \in T$ then the procedure returns a word $z \equiv_I w$ accepted by $\mathcal{A}$. Otherwise the procedure returns 0.

For an input w of length n the algorithm works in $O(n)$ time since the cardinality of Δ only depends on the dependence relation and hence the main iteration, which is repeated at most n times, requires $O(1)$ time.

Proposition 1 *Given an independence alphabet (Σ, I), let $L \subseteq \Sigma^*$ be accepted by a local automaton with dependent successors. Then, the trace language $T = [L]$ can be recognized in time $O(n)$.*

A first consequence of the previous algorithm is given by the following

Proposition 2 *For any independence alphabet (Σ, I), if $L \subseteq \Sigma^*$ is recognized by a local automaton with dependent successors then the rational trace language $[L]$ is unambiguous.*

Proof. In fact, let $\mathcal{A} = (Q, q_0, \delta, F)$ be a local automaton with dependent successors recognizing L. Assume there are two words $u, v \in L$ such that $[u] = [v]$. Consider the longest common prefix x of u and v. Then $u = xaz$ and $v = xbw$ for some aIb and $z, w \in \Sigma^*$. We have that $a, b \in Suc(q)$ where $q = \delta(q_0, x)$ and hence $\mathcal{A}$ has a pair of independent successors, which is contradiction. □

Another condition on $\mathcal{A}$ and D that allows us to design a linear time algorithm to recognize $[L]$ is the following: for every $q \in Q$ and every pair of independent symbols b, c in $Suc(q)$, in $\mathcal{A}$ either state $f(b)$ or state $f(c)$ is not reachable from the other one. Thus, if q is the current state and both b and c appear as first symbol in the corresponding projections (since bIc they do not share a common projection), then the procedure is forced to choose between $f(b)$ and $f(c)$, that state able to reach the other one. Note that the previous condition is milder than the hypothesis of dependence of successors.

The hypotheses of dependent successors and non-mutual reachability we consider above occur frequently in program schemes. For instance, the first condition states that the true and false successors of a conditional instruction are dependent on each other. This also happens in the common case when the successors are instructions assigning different values to the same variable, a case of write-after-write data-dependence. Also the situation where one of the successors does not reach the other is typical of rather frequent program patterns: for instance, it occurs in the case of a conditional jump raising an exception, such that the normal execution is abandoned if the exception is verified.

3 Repeat-until languages

In this section we define a family of expressions representing a program scheme consisting of nested repeat-until cycles (or loops).

Given a finite alphabet Σ, let N be the set of all regular expressions over Σ defined as follows:

i) Every $a \in \Sigma$ belongs to N,

ii) If $\alpha, \beta \in N$ then $\alpha \cdot \beta \in N$ (often represented by $\alpha\beta$),

iii) If α is a symbol in Σ or an expression $\beta \cdot \gamma$, for some $\beta, \gamma \in N$, then $(\alpha)^+ \in N$.

An element $\alpha \in N$ is called *repeat-until expression* over Σ if it contains just one occurrence of a for every $a \in \Sigma$. Thus, $\pi_\Sigma(\alpha)$ defines a linear order over Σ and, for every $a, b \in \Sigma$, we write $a < b$ if a occurs before b in $\pi_\Sigma(\alpha)$. The set of all repeat-until expressions over Σ will be denoted by $\mathrm{RUE}(\Sigma)$, or simply by RUE when Σ is understood.

For every $\alpha \in \mathrm{RUE}$, let $L(\alpha)$ be the language represented by α. Clearly, for every $x \in L(\alpha)$ and every $a, b \in \Sigma$, we have

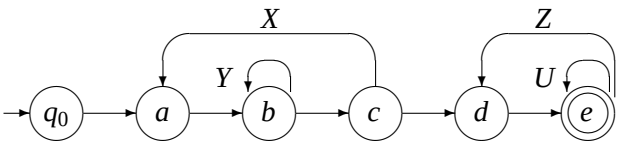
$$a < b \text{ implies } \pi_{a,b}(x) \in a\{a,b\}^* b \tag{1}$$

Moreover, we define a *cycle* of α as a subexpression $(\beta)^+$ of α such that $\beta \in N$ (note that β is not of the form $(\gamma)^+$). The string $\pi_\Sigma(\beta)$ is the *body* of the cycle, $P(\pi_\Sigma(\beta))$ and $U(\pi_\Sigma(\beta))$ are its *header* and *exit*, respectively. For instance, $((ac)^+bd(e)^+)^+$ is a cycle of $\alpha = h((ac)^+bd(e)^+)^+fg$, with header a, exit e.

Note that in every $x \in L(\alpha)$ the body of any cycle appears at least once, possibly as a subword consisting of several nonoverlapping factors. This justifies our definition: any $\alpha \in \mathrm{RUE}$ represents a program scheme of nested repeat-until cycles and every $x \in L(\alpha)$ represents an execution of the program.

Clearly, for any $\alpha \in \mathrm{RUE}$, $L(\alpha)$ is a local language and the corresponding finite automaton is obtained by a standard construction [2], where there is an initial state q_0 and a state for each symbol in Σ. Here we avoid the easy formal definition and describe such automaton by an example. For our subsequent discussion, in the diagram of these automata it is convenient to represent cycles by capital letters.

Example 1 *Consider the repeat-until expression $\alpha = (a(b)^+c)^+(d(e)^+)^+$. Then, the corresponding local automaton $\mathcal{A}(\alpha)$ is defined by the set of states $Q = \{q_0, a, b, c, d, e\}$ together with following transition diagram:*



where X, Y, Z, U represent the cycles $(a(b)^+c)^+$, $(b)^+$, $(d(e)^+)^+$ and $(e)^+$, respectively.

Observe that if the family of successors $Suc(q) = \{a \in \Sigma \mid \delta(q,a) \neq \perp\}$ is a clique of a dependence graph for any $q \in Q$, then the trace language $[L(\alpha)]$ is recognizable in $O(n)$ time by the algorithm presented in Section 2.1.

3.1 Hierarchical trees

Here we describe a tree representation of a repeat-until expression. Let us first recall that a plane tree is a rooted tree where the sons of every internal node are totally ordered (usually drawn from left to right). This clearly induces a natural total order on the leaves of the tree. Now, given $\alpha \in \mathrm{RUE}(\Sigma)$, let $\mathcal{C}$ be the family of all cycles of α (denoted by capital letters) together with a special symbol S, which will

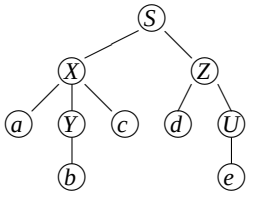
represent the root of the tree. For every $X, Y \in \mathcal{C}$, we define $X \trianglelefteq Y$ if X is nested into Y or $X = Y$. We also set $X \trianglelefteq S$ for every $X \in \mathcal{C}$. Moreover, we write $X \triangleleft Y$ if $X \trianglelefteq Y$ and $X \neq Y$.

Then we define the *hierarchical tree* of α as the plane tree $T(\alpha)$ with root S, satisfying the following properties:

1. $\mathcal{C}$ is the set of internal nodes and $\pi_\Sigma(\alpha) = a_1 a_2 \cdots a_m$ is the ordered list of leaves;
2. For any $X, Y \in \mathcal{C}$, X is son of Y if $X \triangleleft Y$ and X is immediately nested in Y, i.e. there is no $Z \in \mathcal{C}$ such that $X \triangleleft Z \triangleleft Y$;
3. A leaf $a \in \Sigma$ is son of a node $X \in \mathcal{C}$ if X is the smallest cycle of α including a. If a is not included in any cycle then a is son of S;
4. For every node $X \in \mathcal{C}$ and every two sons u, v of X we set $u < v$ if u (either as a cycle or as a letter in Σ) occurs before v in α.

Note that $X \trianglelefteq Y$ holds if and only if X is descendant of Y in $T(\alpha)$. Moreover, for every $X \in \mathcal{C}$ different from S, we denote by $F(X)$ the father of X in $T(\alpha)$.

Example 2 *The hierarchical tree of the repeat-until expression α defined in Example 1 is described by the following picture.*



For every $a \in \Sigma$, let $C(a)$ be the father of a in $T(\alpha)$: thus $C(a)$ either is the smallest cycle of α containing a or $C(a) = S$ if a is not included in any cycle. Analogously, for every $a, b \in \Sigma$, $a \neq b$, let $C(a,b)$ be the root of the smallest subtree of $T(\alpha)$ including both a and b. The following proposition states that all cycles are of the form $C(a)$ or $C(a,b)$ for some $a, b \in \Sigma$.

Proposition 3 *Let $\alpha \in RUE(\Sigma)$ and let $X \in \mathcal{C}$ be a symbol different from S. Then, $X = C(a)$ for some $a \in \Sigma$ or $X = C(a,b)$ for some distinct $a, b \in \Sigma$.*

The proof easily follows by induction on the height of the node X in the hierarchical tree $T(\alpha)$.

3.2 Syntactic trees

Now, given $\alpha \in \mathrm{RUE}(\Sigma)$, let $\mathcal{C}$ and S be defined as in the previous section. Consider the context-free grammar with regular right parts $G(\alpha)$ defined by the tuple $(\mathcal{C}, \Sigma, S, P)$, where $\mathcal{C}$ is the set of nonterminals, S is the initial symbol, Σ is the set of terminals and P is the family of productions given by

$$\begin{aligned} P \quad = \quad & \{(X \to \gamma) \mid X \in \mathcal{C},\ \gamma \text{ is obtained from the list of sons of } X \text{ in } T(\alpha) \\ & \text{by replacing each nonterminal } Y \in \mathcal{C} \text{ by } Y^+\} \end{aligned}$$

Example 3 *If α is defined as in Example 2 then*

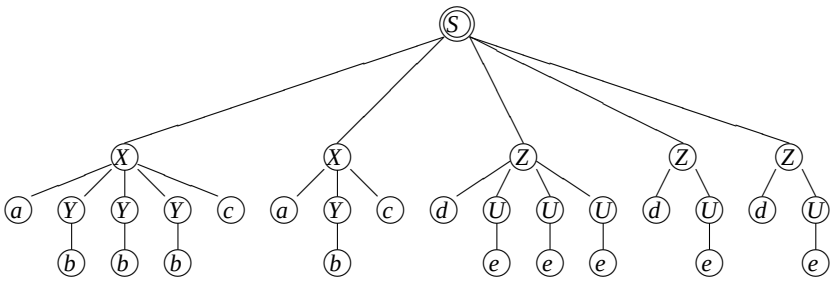
$$P = \{(S \to X^+ Z^+), (X \to aY^+ c), (Y \to b), (Z \to dU^+), (U \to e)\}$$

It is clear that $G(\alpha)$ generates $L(\alpha)$ in the usual way (see for instance [11]). Thus, for any $x \in L(\alpha)$ we define the *syntactic tree* of x as the derivation tree of x in $G(\alpha)$.

Example 4 *Let α be the repeat-until expression defined in Example 1 and let x be the string*

$$x = abbbcabcdeeedede$$

Then $x \in L(\alpha)$ and its syntactic tree is given by the following picture:



Proposition 4 *A word $x \in \Sigma^*$ belongs to $L(\alpha)$ if and only if there exists a syntactic tree T that generates x.*

Clearly any syntactic tree T is a plane tree. Its root is S and, for every $u \in \Sigma \cup \mathcal{C}$, T contains at least one node (may be more) labelled by u: for the sake of brevity, it will be called *u-node*. Clearly, all u-nodes are at the same distance from the root and they can be identified by their (left-to-right) ordering.

Other properties of the syntactic tree T of a word $w \in L(\alpha)$ are the following:

1. For every $a \in \Sigma$, $|w|_a$ equals the number of nodes of T labelled by $C(a)$;
2. For every $a, b \in \Sigma$ with $a < b$, $|\pi_{a,b}(w)|_{ab}$ equals the number of nodes of T labelled by $C(a,b)$;
3. For every $a, b \in \Sigma$, if $(a_1, a_2, \ldots, a_h)$ is the composition generated by $\pi_{a,b}(w)$ on a, then in T there are h nodes labelled by $C(a,b)$ and, for each $i = 1, \ldots, h$, there are a_i nodes of label $C(a)$ that are descendants of the i-th node of label $C(a,b)$. Moreover, an analogous statement holds for the composition generated by $\pi_{a,b}(w)$ on b.

The last property suggests to use integer compositions for representing syntactic trees. Such a representation allows us to construct a syntactic tree. To this end, in the next section we introduce properties and operations on integer compositions that are useful in our context.

4 Integer compositions for tree representation

For the sake of brevity we often represent a composition $\alpha = (a_1, a_2, \ldots, a_h)$ of an integer n in the form $\alpha = (a_i)_h$. The integer h is the *length* of α, while n is also called the *sum* of α. They will be also denoted by ℓ_α and n_α, respectively. First, for any pair of compositions $\alpha = (a_i)_h$ and $\beta = (b_i)_h$ of equal length, the relation $\alpha \leq \beta$ means that $a_i \leq b_i$ for every $i = 1, \ldots, h$.

Another natural notion is the inclusion relation $\preceq$ among compositions of the same integer.

Definition 1 *Given two compositions* $\alpha = (a_i)_h$ *and* $\beta = (b_i)_k$ *of an integer* $n \geq 1$*, we say that* α *is* finer *than* β *and write*

$$\alpha \preceq \beta$$

if $h \geq k$ *and there are* k *indices* $j_1, j_2, \ldots, j_k$ *such that* $1 \leq j_1 < j_2 < \cdots < j_k = h$ *and*

$$b_1 = \sum_{i=1}^{j_1} a_i \ , \quad b_2 = \sum_{i=j_1+1}^{j_2} a_i \ , \ \ldots \ , \quad b_k = \sum_{i=j_{k-1}+1}^{j_k} a_i$$

Note that if $\alpha \preceq \beta$ then there exists a unique k-tuple of indices $j_1, \ldots, j_k$ satisfying the previous property. Moreover, $\preceq$ is a partial order relation on the family of all compositions of n, where $(1,1,\ldots,1)$ is the smallest element and (n) the largest one (which for convenience will be also denoted by $(1,1,\ldots,1)_n$ and $(n)_1$, respectively).

Clearly, there is a $O(n)$-time algorithm that on input α, β verifies whether $\alpha \preceq \beta$ and, in the affirmative case, computes the corresponding sequence $j_1, \ldots, j_k$ defined above.

4.1 Product operation

Consider two compositions $\alpha = (a_i)_h$ and $\beta = (b_j)_k$, and assume $n_\alpha = \ell_\beta$, which implies $h \leq k$. Then, we define the *product* $\alpha \cdot \beta$ as the composition $\gamma = (g_l)_h$ such that

$$g_l = \sum_{j=j_{l-1}+1}^{j_l} b_j \qquad \text{for every} \quad l = 1,2\ldots,h$$

where $j_0 = 0$ and $j_l = \sum_{i=1}^{l} a_i$ for each $l = 1,2\ldots,h$.

More precisely, we have

$$\begin{array}{rcl} g_1 & = & b_1 + b_2 + \cdots + b_{a_1} \\ g_2 & = & b_{a_1+1} + b_{a_1+2} + \cdots + b_{a_1+a_2} \\ \ldots & = & \ldots \\ g_h & = & b_{a_1+\cdots+a_{h-1}+1} + b_{a_1+\cdots+a_{h-1}+2} + \cdots + b_{a_1+\cdots+a_h} \end{array}$$

Briefly, γ is obtained from β by adding consecutive elements as indexed by the composition α. Clearly, we have $\beta \preceq \gamma$, $n_\gamma = n_\beta$ and $\ell_\gamma = \ell_\alpha$.

Here is an example:

$$\alpha = (1,2,2) \qquad \beta = (1,2,1,3,2) \qquad \gamma = \alpha \cdot \beta = (1,3,5)$$

Notice that the product is associative but not commutative. Moreover, for every composition $\beta = (b_j)_k$, the following identities hold:

$$(1,1,\ldots,1)_k \cdot \beta = \beta \qquad (k)_1 \cdot \beta = \left(n_\beta\right)_1 \qquad \beta \cdot (1,1,\ldots,1)_{n_\beta} = \beta$$

The product of two compositions can be computed by scanning their elements from left to right. It is easy to design an algorithm that takes in input two compositions $\alpha = (a_i)_h$, $\beta = (b_j)_k$ such that $n_\alpha = k$ and computes their product in $O(k)$ time.

4.2 Quotient operation

In a similar way we define the quotient operation. Given two compositions $\alpha = (a_i)_h$, $\beta = (b_l)_k$ such that $\alpha \preceq \beta$ (and hence $k \leq h$), consider the sequence of indices $j_0, j_1, \ldots j_k$ such that $0 = j_0 < j_1 < \cdots < j_k = h$ and

$$b_l = \sum_{i=j_{l-1}+1}^{j_l} a_i \qquad \text{for every } l = 1,2,\ldots,k.$$

Then, the *quotient* β/α is the composition $\gamma = (g_l)_k$ of h such that

$$g_l = j_l - j_{l-1} \qquad \text{for every } l = 1,2,\ldots,k.$$

It is clear that if $\gamma = \beta/\alpha$ then $\beta = \gamma \cdot \alpha$, $\ell_\gamma = \ell_\beta$ and $n_\gamma = \ell_\alpha$.

For instance:

$$\beta = (4,2,5) \qquad \alpha = (1,3,2,1,1,3) \qquad \gamma = \beta/\alpha = (2,1,3)$$

Notice that we have the following special cases, for any composition $\alpha = (a_i)_h$:

$$\alpha/\alpha = (1,1,\ldots,1)_h \qquad (n_\alpha)_1/\alpha = (h)_1 \qquad \alpha/(1,1,\ldots,1)_{n_\alpha} = \alpha$$

Also the quotient of two compositions can be computed in linear time by scanning both operands from left to right. Then, there is an algorithm that, for an input $\alpha = (a_i)_h$, $\beta = (b_j)_k$ satisfying the relation $\alpha \preceq \beta$, computes the composition $\gamma = (g_j)_k$ such that $\gamma = \beta/\alpha$ and works in time $O(h)$.

4.3 Labelled compositions

Now, let us see how syntactic trees can be represented by integer compositions. To this end, we introduce the notion of labelled composition. Given an expression $\alpha \in$ RUE with set of cycles $\mathcal{C}$, a *labelled composition* is an integer composition equipped with two symbols $A, B \in \mathcal{C}$ such that $B \trianglelefteq A$: we denote it by an expression of the form d_B^A, for some symbol d.

Given a syntactic tree T, consider two cycles $A, B \in \mathcal{C}$ such that $B \trianglelefteq A$ and assume T has h nodes of label A and k nodes of label B. Then, define the labelled composition m_B^A by

$$m_B^A = (a_1, a_2, \ldots, a_h)$$

where, for each $i = 1,\ldots,h$, a_i is the number of B-nodes that are descendants of the i-th A-node in T. Clearly we have $k = n_{m_B^A}$, while $m_B^S = (k)$ and $m_B^B = (1,1,\ldots,1)_k$.

Thus, any syntactic tree T defines a family of labelled compositions $\{m_B^A \mid B \trianglelefteq A\}$ satisfying the following proposition, the proof of which follows from the definitions.

Proposition 5 *Given a syntactic tree T, let A, B, C be cycles in $\mathcal{C}$ such that $C \trianglelefteq B \trianglelefteq A$. Then the following properties hold:*

1. *$m_B^A \leq m_C^A$ and $m_C^B \preceq m_C^A$;*
2. *The sum of m_B^A equals the length of m_C^B and hence $m_B^A \cdot m_C^B$ is well-defined;*
3. *$m_C^A = m_B^A \cdot m_C^B$ and hence $m_B^A = m_C^A / m_C^B$.*

Thus, properties 1, 2 and 3 above are necessary conditions for a set of labelled compositions to represent a syntactic tree (with respect to a given repeat-until expression α). Actually, they are also sufficient conditions to represent a syntactic tree. However, in order to state such a property it is convenient to restrict our reasoning to the labelled compositions corresponding to pairs of father-son cycles in $T(\alpha)$.

Proposition 6 *Given a hierarchical tree $T(\alpha)$ with set of cycles $\mathcal{C}$ and initial symbol S, let $N = \{d_B^A \mid A,B \in \mathcal{C}, A = F(B)\}$ be a family of labelled compositions such that:*

1. *For every $A \in \mathcal{C}$ such that $S = F(A)$, the length of d_A^S is 1;*
2. *For every $A,B,C \in \mathcal{C}$ such that $A = F(B)$ and $B = F(C)$, the sum of d_B^A equals the length of d_C^B (and hence $d_B^A \cdot d_C^B$ is well-defined).*

Then there exists a unique syntactic tree T whose family of labelled compositions includes N.

Proof. The syntactic tree T can be built as follows. First, T has a unique node of label S and, for every $X \in \mathcal{C}\backslash\{S\}$, it has k_X many nodes of label X, where k_X is the sum of d_X^Y with $Y = F(X)$. Second, for each $a \in \Sigma$, add an a-node as a son of each X-node such that $X = C(a)$. Then, for every $X,Y \in \mathcal{C}$ where $Y = F(X)$, consider the labelled composition $d_X^Y = (a_1, a_2, \ldots, a_h)$. By condition 2 it is easy to see that there are h nodes of label Y and $n = a_1 + \cdots + a_h$ nodes of label X: thus one can set the first a_1 nodes of label X as sons of the first Y-node, the subsequent a_2 nodes of label X as sons of the second Y-node, and so on till setting the last a_h nodes of label X as sons of the last Y-node. This defines a syntactic tree T and the ordered sequence of the labels of its leaves yields a string $x \in L(\alpha)$. □

Combining Propositions 6 and 5, we can state that a family M of labelled compositions (including at most one composition for each pair $A,B \in \mathcal{C}$ such that $B \trianglelefteq A$) defines a unique syntactic tree T if M includes the set N satisfying the hypothesis of Proposition 6 and all triples $m_B^A, m_B^C, m_C^A \in M$, for $C \trianglelefteq B \trianglelefteq A$, satisfy conditions 1, 2 and 3 of Proposition 5.

5 The membership problem for repeat-until trace languages

Now, let us consider the membership problem for trace languages defined by repeat-until expressions. Formally, given an independence alphabet (Σ, I) and an expression $\alpha \in \mathrm{RUE}(\Sigma)$, the problem consists of deciding, for an input $x \in \Sigma^+$, whether $[x] \cap L(\alpha)$ is empty. The following theorem yields an equivalent condition.

Theorem 7 *Given an independence alphabet (Σ, I) with dependence relation D and an expression $\alpha \in RUE(\Sigma)$, for any $x \in \Sigma^+$ we have $[x] \cap L(\alpha) \neq \emptyset$ if and only if there exists $w \in L(\alpha)$ having syntactic tree T such that:*

a) *For all $a \in \Sigma$, $|x|_a$ is the number of nodes in T labelled by $C(a)$;*

b) *For every $a,b \in \Sigma$ such that aDb and $a < b$, $|\pi_{a,b}(x)|_{ab}$ equals the number of nodes of T labelled by $C(a,b)$;*

c) *For any $a,b \in \Sigma$ such that aDb and $a < b$, if $X = C(a)$, $Y = C(b)$ and $Z = C(a,b)$, then the labelled compositions m_X^Z and m_Y^Z of T coincide with the compositions generated by $\pi_{a,b}(x)$ on a and b, respectively.*

Proof. First recall that a word w belongs to $[x]$ if and only if $|x|_a = |w|_a$ for every $a \in \Sigma$ and $\pi_{a,b}(x) = \pi_{a,b}(w)$ for every pair of distinct symbols $a, b \in \Sigma$ such that aDb. Therefore, if there exists $w \in [x] \cap L(\alpha)$ then w satisfies Properties 1, 2, 3 of Section 3.2. Since the projections of x and w on the pairs of (possible coincident) dependent symbols are equal, the same properties hold for x, proving conditions **a)**, **b)** and **c)**.

On the other hand, if there exists $w \in L(\alpha)$ satisfying these conditions then both x and w have the same projections on the pairs of (possible coincident) dependent symbols, proving that $w \in [x]$ and hence $[x] \cap L(\alpha) \neq \emptyset$. □

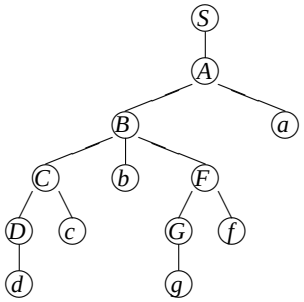
The previous theorem can be used to design a linear time algorithm for recognizing $[L(\alpha)]$ whenever α and D satisfy a further condition we introduce below. To this end, let us define the graph $G(\alpha, D)$ as the directed graph such that $\mathcal{C}$ is the set of nodes and the family of edges E is given by

$$E = \{(Y,X) \mid X,Y \in \mathcal{C}, \exists a,b \in \Sigma : aDb, X = C(a), Y = C(a,b)\}.$$

If $(Y,X) \in E$ we say that X is *adjacent* to Y. Note that in this case $X \trianglelefteq Y$. Moreover, given $U, Z \in \mathcal{C}$, we say that U is *connected* to Z through D if there is a path in $G(\alpha, D)$ from U to Z.

Definition 2 *We say that $T(\alpha)$ is* closed *with respect to D if, for every $(Y,X) \in E$, either $Y = F(X)$ or all nodes along the path from Y to X in $T(\alpha)$ are connected to X through D.*

Example 5 *Let $T(\alpha)$ be the hierarchical tree defined by the following picture:*



Then $T(\alpha)$ is closed with respect to the following dependence relations: $D = \{\{a,d\},\{b,d\},\{c,d\}\}$, $D = \{\{a,d\},\{c,f\},\{c,d\}\}$, $D = \{\{a,g\},\{b,f\},\{g,f\}\}$, $D = \{\{d,c\},\{f,g\},\{c,g\},\{a,d\}\}$. On the contrary the same $T(\alpha)$ is not closed with respect to the dependence relations defined by $D = \{\{a,c\},\{a,b\}\}$, $D = \{\{a,d\},\{b,c\}\}$, $D = \{\{c,f\},\{c,g\}\}$, $D = \{\{a,c\},\{b,d\}\}$.

6 Recognition algorithm

Now, assuming that $T(\alpha)$ is closed with respect to D, let us define an algorithm for the recognition of $[L(\alpha)]$. The key idea of the computation is to construct, for an input $x \in \Sigma^+$, the syntactic tree T of a word $w \in [x] \cap L(\alpha)$ that satisfies conditions **a)**, **b)**, **c)** of Theorem 7. Such a tree (if any) will be defined by a family of labelled compositions $\{d_B^A \mid A,B \in \mathcal{C}, A = F(B)\}$ that satisfies Proposition 6.

The algorithm consists of three phases. In the first one, by applying conditions **a)** and **b)**, we compute the number k_A of A-nodes in T, for each $A \in \mathcal{C}$. In the second phase we compute the set of all labelled computations d_B^A of T determined by the dependency relation D, i.e. those defined by condition **c)**. In the third phase we close such a set of labelled compositions with respect to the product and the quotient, checking in particular that all products are coherent. Finally, by a suitable choice, we compute explicitely the remaining undefined compositions of the form d_B^A with $A = F(B)$.

6.1 Computing the nodes of the syntactic tree

First of all, the root is the unique node labelled by S. Then, the leaves of T are determined by the occurrences of symbols of Σ in x: for every $a \in \Sigma$ one checks that $|x|_a \geq 1$ and adds $|x|_a$ leaves labelled by a in T. Moreover, by condition **a)** of Theorem 7, the number of nodes labelled by $C(a)$ has to be equal to $|x|_a$. Thus, one has to check that $|x|_b = |x|_a$ for all $b \in \Sigma$ such that $C(b) = C(a)$. Once such a condition is guaranteed, we can assign $|x|_a$ to the required number k_X of X-nodes, where $X = C(a)$. On the contrary, if $|x|_b \neq |x|_a$ for some $b \in \Sigma$ such that $X = C(b)$, then the required syntactic tree T does not exist and hence we reject the input and stop.

A similar reasoning derives from condition **b)**, which allows us to determine the number k_Z of Z-nodes for any cycle $Z \in \mathcal{C}$ such that $Z = C(a,b)$ for some $a,b \in \Sigma$ satisfying $a < b$ and aDb. This value coincides with the number of occurrences of ab in $\pi_{a,b}(x)$. Also in this case one has to verify that k_Z equals $|\pi_{a',b'}(x)|_{a'b'}$ for every pair $a',b' \in \Sigma$ satisfying the same conditions as a,b, i.e. $Z = C(a',b')$, $a' < b'$, $a'Db'$. If that is not true, the procedure rejects the input and stops.

Then, we have to compute the number of B-nodes in T for those $B \in \mathcal{C}\backslash\{S\}$ such that $B \neq C(a)$ for all $a \in \Sigma$ and $B \neq C(a,b)$ for all $a,b \in \Sigma$ satisfying aDb. Observe that every son of such a B in $T(\alpha)$ is a cycle (it is not in Σ) and any pair of symbols $a,b \in \Sigma$ that are discendent of different sons of B are independent. As a consequence, the sons of any B-node in T can be grouped consecutively according to the order defined by $T(\alpha)$. This means we can choose the minimal k_B by setting $k_B = k_A$, where $A = F(B)$. This property can be summarized by the following proposition[1].

Proposition 8 *Given $\alpha \in RUE(\Sigma)$ and a dependence relation D on Σ, let $X \in \mathcal{C}\backslash\{S\}$ be a cycle such that $X \neq C(a)$ for all $a \in \Sigma$ and $X \neq C(a,b)$ for all $a,b \in \Sigma$ satisfying aDb. Let Y be the father of X in $T(\alpha)$, i.e. $Y = F(X)$, and consider a word $w \in L(\alpha)$. Then, there exists $z \in L(\alpha) \cap [w]$ such that every Y-node in the syntactic tree of z has just one son labelled by X (and hence the number of X-nodes equals the number of Y-nodes).*

Finally we have to check that there is at least one son for each father in T, i.e. if $Y = F(X)$ then $k_X \geq k_Y$.

To define formally the computation described above, we use the subroutine `Assign`(z,v) that assigns the value of v to the variable z, checking that the previous possible value of z is not different from v (otherwise a global variable *out* is set to 0).

```
Procedure Assign(z, v)
if z = ⊥ then z := v
        else if z ≠ v then out := 0
```

Then, the computation of the nodes of the syntactic tree is given by the following procedure:

```
begin
    for X ∈ C do k_X := ⊥
    k_S := 1
    out := 1
    for a ∈ Σ do
            begin
                X := C(a)
```

[1] The proof is also given in [16, Prop. 1].

```
                    t := |x|_a
                    if t = 0 then out := 0
                    Assign(k_X, t)
                end
    for a, b ∈ Σ such that a < b ∧ aDb do
                begin
                    Z := C(a, b)
                    u := |π_{a,b}(x)|_{ab}
                    Assign(k_Z, u)
                end
    for X ∈ C\{S} (in preorder) do
                begin
                    Y := F(X)
                    if k_X = ⊥ then k_X := k_Y
                              else if  k_X < k_Y then out := 0
                end
end
```

Thus, the variable *out* is set to 0 whenever some necessary condition for computing the nodes of T does not hold. In this case the algorithm stops and rejects the input. On the contrary, if the final value of *out* is 1, then the procedure correctly computes for every $X \in \mathcal{C}$ the number k_X of X-nodes of a possible syntactic tree.

6.2 Initial labelled compositions

In the second phase we compute a set of initial labelled compositions of the required syntatic tree. They are denoted by d_B^A, where $A, B \in \mathcal{C}$ and $B \trianglelefteq A$. Clearly, those of the form d_A^A (for $A \in \mathcal{C}$) easily derive from the values k_A computed in the previous section. Other obvious compositions are those d_B^A's such that $A = F(B)$ and $k_A = k_B$; in this case, $d_B^A = (1,1,\ldots,1)_{k_A}$ and this includes all compositions d_X^Y where X and Y satisfy the hypothesis of Proposition 8.

Then, we compute the labelled compositions determined by condition **c)** of Theorem 7. Also in this case a uniqueness condition has to be verified; if a pair $A, B \in \mathcal{C}$ with $B \trianglelefteq A$ is associated with two distinct pairs of dependent symbols, the corresponding labelled compositions have to be equal, otherwise there is no syntactic tree satisfying the required conditions.

The procedure below formally defines the second phase of our algorithm. Again, we use the subroutine `Assign` and, at the end of the computation, if $out = 0$ the algorithm stops and rejects the input.

```
begin
1. labelled compositions derived from nodes
    for A, B ∈ C do d_B^A := ⊥
    for A ∈ C do d_A^A := (1,1,...,1)_{k_A}
    for A ∈ C such that S = F(A) do d_A^S := (k_A)
    for  A, B ∈ C such that A = F(B) do
        if k_A = k_B then d_B^A := (1,1,...,1)_{k_A}
2. labelled compositions derived from dependent pairs
    for a, b ∈ Σ  such that a < b ∧ aDb do
        begin
```

$X := C(a)$
$Y := C(b)$
$Z := C(a,b)$
compute the composition γ generated by $\pi_{a,b}(x)$ on a
`Assign`(d_X^Z, γ)
compute the composition δ generated by $\pi_{a,b}(x)$ on b
`Assign`(d_Y^Z, δ)

`end`

`end`

6.3 Closure operations

Once the previous phase has been completed without setting *out* to 0, we have to close the set M of labelled compositions determined so far with respect to the product and the quotient. Observe that, by the procedure of Section 6.1, for any $A,B,C \in \mathcal{C}$ such that $C \lhd B \lhd A$, if $d_B^A \neq \perp \neq d_C^B$ then the product $d_B^A \cdot d_C^B$ is well-defined because $n_{d_B^A} = k_B$ equals the length of d_C^B. Then the product $d_B^A \cdot d_C^B$ can be computed and assigned to d_C^A, checking that a unique composition is assigned to the same pair A,C.

The computation is defined by the following scheme.

`repeat`
`for` $A,B,C \in \mathcal{C}$ such that $C \lhd B \lhd A$ `do`
`if` $d_B^A \neq \perp \neq d_C^B$ `then` $\left\{ \begin{array}{l} \gamma := d_B^A \cdot d_C^B \\ \texttt{Assign}(d_C^A, \gamma) \end{array} \right.$ (1)
`for` $A,B,C \in \mathcal{C}$ such that $C \lhd B$ and $A = F(B)$ `do`
`if` $d_C^A \neq \perp \neq d_C^B$ `then if` $d_C^B \preceq d_C^A$ `then` $\left\{ \begin{array}{l} \delta := d_C^A / d_C^B \\ \text{Assign}(d_B^A, \delta) \end{array} \right.$ (2)
`else` *out* $:= 0$
`until` *out* $= 0$ or no new assignment is executed in commands (1) and (2)

Clearly, if *out* $= 0$ the input is rejected, otherwise it is accepted. Note that there could still exist pairs of father-son cycles $A,B \in \mathcal{C}$ such that $d_B^A = \perp$. However, in this case any composition of length k_A and sum k_B can be assigned to d_B^A since, by the closure hypothesis, there is no labelled composition in M connecting an ancestor of A to a descendent of B.

`for` $A,B \in \mathcal{C}$ such that $A = F(B)$ `do`
`if` $d_B^A = \perp$ `then` $\left\{ \begin{array}{l} \text{choose a composition } \gamma \text{ of length } k_A \text{ and sum } k_B \\ d_B^A := \gamma \end{array} \right.$

Thus, in case of acceptance, d_B^A is well defined for every $A,B \in \mathcal{C}$ such that $A = F(B)$ and the syntactic tree of a word $w \in [x]$ is obtained by applying Proposition 6.

Since the product and the quotient of integer compositions can be computed in linear time, the whole algorithm works in $O(n)$ time, where $n = |x|$.

Theorem 9 *For every independence alphabet (Σ, I) and every expression $\alpha \in RUE(\Sigma)$, if $T(\alpha)$ is closed with respect to the dependence relation then the trace language $[L(\alpha)]$ can be recognized in $O(n)$ time.*

If the hierarchical tree is not closed with respect to the dependence relation, the algorithm above may fail to build a syntactic tree (even if there exists one) because some father-son connection could remain undefined. This may happen when, for an adjacent pair (Y,X) and a cycle B such that $X \lhd B \lhd Y$, B is not connected to X through D and several choices for d_B^A, where $A = F(B)$, are coherent with the compositions occurring along the path from Y to X. A simple choice of one of these is not always correct, because (without the closure assumption) the resulting d_B^A might not be coherent with the initial labelled compositions occurring in other paths including B. The following example describes in detail a situation of this kind.

Consider the hierarchical tree $T(\alpha)$ defined by the picture of Example 5 and let the dependence relation be given by the pairs $\{a,d\}$, $\{b,c\}$, $\{b,f\}$, $\{a,g\}$. Clearly $T(\alpha)$ is not closed with respect to such a relation. In particular the paths from A to D and from A to G are not closed.

Now, assume the projections of the input x over $\{a,d\}$ and $\{b,c\}$ are given by $\pi_{a,d} = ddddddddddda$ and $\pi_{b,c} = ccbccbcbcb$, respectively. Here, the number of nodes of label A, B, C and D are, respectively, $k_A = 2$, $k_B = 4$, $k_C = 5$ and $k_D = 9$, while the initial compositions defined by $\{a,d\}$ and $\{b,c\}$ are $d_D^A = (4,5)$ and $d_C^B = (2,2,1,1)$. There are two possible choices for d_C^A coherent with d_D^A and d_C^B, i.e. satisfying $d_C^A \leq d_D^A$ and $d_C^B \preceq d_C^A$; they are $d_C^A = (4,2)$ and $d_C^A = (2,4)$, which produce, by the quotient operation, the labelled compositions $d_B^A = (2,2)$ and $d_B^A = (1,3)$, respectively.

However, an analogous reasoning based on dependence pairs $\{b,f\}$ and $\{a,g\}$ may yield a partially different set of possible values for d_B^A. In fact, assume $\pi_{b,f} = bffbfbffbf$ and $\pi_{a,g} = gggggaggga$. In this case we have $k_F = 6$, $k_G = 8$, $d_F^B = (2,1,2,1)$ and $d_G^A = (5,3)$. The possible values for d_F^A are $(3,3)$ and $(5,1)$, which implies the compositions $d_B^A = (2,2)$ and $d_B^A = (3,1)$, respectively.

Thus, the only value of d_B^A that is coherent with both paths $A-D$ and $A-G$ is $d_B^A = (2,2)$. Therefore, in the general case, for computing a labelled composition d_B^A one should determine the set of admissible values for each including path and compute the intersection of all these sets. However, such a computation does not seem to be feasible as the number of compositions of given sum is exponential.

6.4 On the closure assumption

To conclude this section we spend some words to discuss the closure assumption for nested repeat-until programs. We recall that the innermost loop containing instruction a is "adjacent" to an outer loop containing instructions a and b, if the two instructions are dependent. Here the adjacency relation can be seen as a directed edge from the outer loop to the inner one. Also, an innermost loop X and an outer loop Y are "connected" if there is a path of adjacences from Y to X. The closure hypothesis says that, in any chain of nested loops, such that the outermost and the innermost one are adjacent, each intermediate loop is connected to the innermost one.

For instance, it is easy to see that the standard procedure for matrix multiplication has three nested cycles, where the header of each loop increments a control variable and is dependent on the innermost instruction.

Even if the closure assumption is not satisfied by all repeat-until expressions, however we think it covers a significant part of these languages, those for which the reconstruction of the syntactic tree (from the projections of the input string on the dependence pairs) can be done univocally by using the operations of product and quotient between compositions.

Moreover, the existence of linear time recognition algorithms under this hypothesis supports the conjecture that also for more general expressions there exist efficient procedures for the membership problem, with time complexity independent of the clique size of the independence relation. A first attempt in this direction is proposed in [6] where some procedures, working in quadratic time, are described

for specific examples of repeat-until expressions without closure assumption. That approach however is not based on a general property relating the repeat-until expression to the dependence relation and involves operations over integer compositions, other than product and quotient, which do not seem to be computable in linear time.

7 Conclusion

An important problem in program optimization and in other computer applications is the schedule checking problem. It consists of checking whether a given sequence of operations is a permutation of any sequence defined by a finite-state machine, obeying a given dependence relation. We have presented two linear-time algorithms that solve the problem under certain assumptions, which we believe to be not restrictive for certain realistic cases. This may open the way to the experimentation of our algorithms, in contrast to previous procedures for the general problem, which have too high time complexity to be practical.

Analysing general iterative computations, as we did for nested repeat-until cycles, is rather complicated. In our case we have overcome the difficulty by introducing the labelled integer compositions in this context, and we have shown that they are quite expressive and convenient mathematical structures. Their use has allowed us to clarify and improve on previous efforts to solve the schedule checking problem, determining precisely the time complexity of the algorithm in several significant cases. In our opinion, it should be possible to apply similar methods based on integer compositions to more general cases, such as programs of loops of type *while* ... *do* ... and others.

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