axioms

Article

Semi-Conformally Flat Singly Warped Product Manifolds and Applications

Samesh Shenawy [1], Alaa Rabie [2], Uday Chand De [3], Carlo Mantica [4] and Nasser Bin Turki [5,*]

[1] Basic Science Department, Modern Academy for Engineering and Technology, Maadi 11571, Egypt; drssshenawy@eng.modern-academy.edu.eg
[2] Department of Mathematics, Faculty of Science, Fayoum University, Fayoum 63541, Egypt; ara15@fayoum.edu.eg
[3] Department of Pure Mathematics, University of Calcutta, Ballygaunge Circular Road, Kolkata 700019, India; ucdpm@caluniv.ac.in
[4] Physics Department Aldo Pontremoli, Universita degli Studi di Milano and I.N.F.N. Sezione di Milano, Via Celoria 16, 20133 Milan, Italy; carlo.mantica@mi.infn.it
[5] Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia
* Correspondence: nassert@ksu.edu.sa

Abstract: This paper investigates singly warped product manifolds admitting semi-conformal curvature tensors. The form of the Riemann tensor and Ricci tensor of the base and fiber manifolds of a semi-conformally flat singly warped product manifold are provided. It is demonstrated that the fiber manifold of a semi-conformally flat warped product manifold has a constant curvature. Sufficient requirements on the warping function to ensure that the base manifold is a quasi-Einstein or an Einstein manifold are provided.

Keywords: warped product; semi-conformal curvature tensor; perfect fluid space-times

MSC: 53C15; 53C20

Citation: Shenawy, S.; Rabie, A.; De, U.C.; Mantica, C.; Bin Turki, N. Semi-Conformally Flat Singly Warped Product Manifolds and Applications. *Axioms* **2023**, *12*, 1078. https://doi.org/10.3390/axioms12121078

Academic Editors: Zhigang Wang, Yanlin Li, Juan De Dios Pérez and Emil Saucan

Received: 20 September 2023
Revised: 9 November 2023
Accepted: 20 November 2023
Published: 24 November 2023

1. Introduction

The study of specific curvature-like tensors on Riemannian manifolds offers a comprehensive grasp of the geometry of the underlying manifold. There are far too many fascinating curvature-like tensors investigated in the literature. For instance, Pokhariyal and Mishra proposed and investigated a family of symmetric and skew-symmetric $(0,4)$ tensors that include the Riemann tensor as well as a combination of the Ricci and metric tensors [1–3]. Later, many generalizations and relativistic applications were extensively investigated in the literature for a long time (see [4–10] and references therein).

Some of the curvature tensors arose as invariants of a particular transformation. For example, the concircular curvature tensor is invariant under the concircular transformation that transforms geodesic circles to geodesic circles [11]. This tensor represents the deviation of a Riemannian manifold from being of a constant curvature. As a result, numerous fascinating investigations of this tensor in various contexts have been conducted (see [12–16]).

One of the most common and significant transformations is the conformal transformation. Let (V,g) be a Riemannian manifold with metric g and a positive function σ on M. A conformal transformation is a diffeomorphism $\omega:(V,g)\to(V,e^{\sigma}g)$. Under conformal transformations, the Weyl conformal curvature tensor $\mathcal{C}$ on V defined by

$$\begin{aligned}\mathcal{C}^i_{jkl} &= R^i_{jkl}+\frac{1}{n-2}(\delta^i_k R_{jl}-\delta^i_l R_{jk}+g_{jl}R^i_k-g_{jk}R^i_l)\\ &+\frac{R}{(n-1)(n-2)}(\delta^i_l g_{jk}-\delta^i_k g_{jl})\end{aligned}$$

remains invariant in dimensions greater than two [17,18] where R^i_{jkl} and R_{jl} are the Riemannian and Ricci curvature tensors, respectively, and g_{ik} is the metric tensor. A function f is called harmonic if its Laplacian Δf vanishes. It should be emphasized that conformal transformation does not preserve function harmonicity; that is, a harmonic function does not transform into another harmonic function under conformal transformation. Conharmonic transformations are a special type of conformal transformation that converts harmonic functions to harmonic functions. The conharmonic curvature tensor $\mathcal{L}$ defined by

$$\mathcal{L}^i_{jkl} = R^i_{jkl} + \frac{1}{n-2}(\delta^i_k R_{jl} - \delta^i_l R_{jk} + g_{jl}R^i_k - g_{jk}R^i_l)$$

was first considered in [19] as an invariant of conharmonic transformation. J. Kim created a novel curvature-like tensor that also stays invariant by conharmonic transformations along the same line of the conharmonic curvature tensor [20]. This curvature-like tensor is known as the semi-conformal curvature tensor and is defined by

$$\mathcal{P}^i_{jkl} = -(n-2)BC^i_{jkl} + [A + (n-2)B]\mathcal{L}^i_{jkl},$$

where A and B are constants. The semi-conformal curvature tensor reduces to a conformal curvature tensor if $A = 1$ and $B = \frac{-1}{n-2}$ and to a conharmonic curvature tensor if $A = 1$ and $B = 0$. In simple terms, conharmonic transformations are conformal transformations that preserve harmonic functions. Under conharmonic transformation, the conharmonic curvature tensor and semi-conformal curvature tensor remain invariant. The invariance of the semi-conformal curvature tensor under conharmonic transformation indicates that the semi-conformal curvature tensor evolves in a precise way to keep its physical meaning unchanged. For instance, the conharmonic transformation preserves the property $\mathcal{P}^i_{jkl} = 0$. The flatness of the semi-conformal curvature tensor is sufficient for a manifold to be conformally flat or of a constant scalar curvature. In addition, a semi-conformal curvature tensor of type $(0,4)$ satisfies the symmetries and skew symmetries of the Riemann curvature tensor. Also, the semi-conformal curvature tensor has the cyclic property of the Riemann tensor. Such tensors are called generalized curvature tensors and were first introduced and studied by Kobayashi and Nomizu [21,22]. Consequently, the semi-conformal curvature tensor is a generalized curvature tensor.

There have been numerous investigations of the semi-conformal curvature tensor in different settings. In [23], a study of pseudo semi-conformally symmetric manifolds was carried out. On such manifolds, the semiconformal curvature tensor is subjected to appropriate constraints to ensure that the manifold has either a constant or zero scalar curvature. This study was expanded in [24] to weakly semiconformally symmetric manifolds. Many fascinating findings were provided. The forms of the scalar curvature of the generalized weakly semi-conformally symmetric manifold, a generalization of the weakly symmetric manifold, were investigated in [25]. The authors of [26] studied the symmetries of the semi-conformal curvature tensor in semi-conformally symmetric space-time. They demonstrated that a four-dimensional space-time that admits a suitable semi-conformal symmetry is semi-conformally flat or of the Petrov type N. Additionally, a four-dimensional space-time that possesses a divergence-free semi-conformal curvature tensor was investigated. In both instances, it was discovered that the scalar curvature of the space-time vanishes if the space-time accommodates an infinitesimal semi-conformal Killing vector field. In [27], a new symmetry property of space-times is considered. The authors called it semi-conformal curvature collineation, and its relationship with other known symmetry properties was demonstrated. Both non-null and null electromagnetic fields have been explored in relation to this new symmetry of relativistic space-times. If a perfect fluid space-time's semiconformal curvature tensor is divergence-free and its energy–momentum tensor is of the Codazzi type, then the $\rho - 3p$ is constant as shown in [28], where p is the perfect fluid pressure and ρ is the energy density. A perfect fluid space-time admitting a divergence-free semiconformal curvature tensor either fulfills the vacuum-like equation of state or is a Friedmann–Lemaitre–Robertson–Walker cosmological model satisfying $\rho - 3p =$ constant [28].

The authors of [29] studied the (k, μ)-contact metric manifold with the semi-conformal curvature tensor, satisfying $\mathcal{P} \cdot R = 0$, as well as the semi-conformally flat, where $\mathcal{P} \cdot R$ means $\mathcal{P}$ acts as a derivation on R. Further, $\mathcal{P} \cdot S = 0$ was investigated, and the relation for Ricci tensor was obtained. Also, some results for a (k, μ)-contact metric manifold satisfying the condition $\mathcal{P} \cdot S = 0$ were established. In [30], the authors studied space-times which admit a semi-conformal curvature tensor. First, it was proved that the energy–momentum tensor with a vanishing semi-conformal curvature tensor, satisfying Einstein's field equations (with cosmological constant) is covariantly constant. Next, it was shown that a perfect fluid space-time with a divergence-free semi-conformal curvature tensor satisfying Einstein field equations without a cosmological constant has constant pressure and density. Finally, the perfect fluid space-time with a vanishing semi-conformal curvature tensor satisfying Einstein field equations without a cosmological constant is shown to have constant energy density and isotropic pressure, and the perfect fluid always behaves as having a cosmological constant.

The work in [31] aims to investigate a perfect fluid space-time that fulfills Einstein's field equation without the cosmological constant and is semi-conformally flat on the Riemannian manifolds. The weakly symmetric and weakly Ricci symmetric manifolds have been used to study a variety of geometric properties connected to weakly semi-conformally symmetric manifolds. In [32], examining that space-times that admit semi-conformal curvature tensors in $f(R)$ modify gravity is the main objective of this work. Analysis was performed on the semi-conformal flatness of both generic space-time and space-time with $f(R)$ gravity with a perfect fluid. They created the isotropic pressure p and energy density forms for this analysis. A few energy conditions were then taken into consideration. The divergence-free semi-conformal curvature tensor in $f(R)$ gravity in the presence of the ideal fluid is the subject of our final investigation. We underline that the resulting space-times either achieve inflation or have constant isotropic pressure and energy density because the Ricci tensor of this space-time is semi-symmetric for a recurrent or bi-recurrent energy–momentum tensor.

Because of the significance of warped product manifolds, researchers are interested in investigating curvature-like tensors on these manifolds as well as on warped space-times. Semi-conformal curvature tensor, as an invariant of the so-called conharmonic transformation, has not been investigated on warped product manifolds. We intend to fill this gap in our current research by investigating the semi-conformal curvature tensor on singly warped product manifolds. The consequences of the flatness of the semi-conformal curvature tensors on singly warped product manifolds are examined.

2. Semi-Conformal Curvature Tensor

Let R be the curvature tensor on a pseudo-Riemannian manifold V. The $(0, 4)$ conharmonic curvature tensor $\mathcal{L}$ on a pseudo-Riemannian manifold is given by

$$\mathcal{L}_{ijkl} = R_{ijkl} + \frac{1}{n-2}(g_{ik}R_{jl} - g_{il}R_{jk} + g_{jl}R_{ik} - g_{jk}R_{il})$$

such that $i, j, k, l \in \{1, ..., n\}$, R_{ijkl} and R_{jl} are the Riemannian and Ricci curvature tensors, respectively, and g_{ik} is the metric tensor [33]. Also, the $(0, 4)$ Weyl conformal curvature tensor $\mathcal{C}$ on V is given by

$$\begin{aligned}\mathcal{C}_{ijkl} &= R_{ijkl} + \frac{1}{n-2}(g_{ik}R_{jl} - g_{il}R_{jk} + g_{jl}R_{ik} - g_{jk}R_{il}) \\ &\quad + \frac{R}{(n-1)(n-2)}(g_{il}g_{jk} - g_{ik}g_{jl}).\end{aligned}$$

It is noted that one can rewrite each of the above tensors in terms of the other one as follows:

$$\mathcal{C}_{ijkl} = \mathcal{L}_{ijkl} + \frac{R}{(n-1)(n-2)}(g_{il}g_{jk} - g_{ik}g_{jl})$$
$$= \mathcal{L}_{ijkl} + \frac{R}{(n-1)(n-2)}G_{ijkl},$$

where $G_{ijkl} = (g_{il}g_{jk} - g_{ik}g_{jl})$. These curvature-like tensors make up the semi-conformal curvature tensor as

$$\mathcal{P}_{ijkl} = -(n-2)B\mathcal{C}_{ijkl} + [A + (n-2)B]\mathcal{L}_{ijkl}.$$

Simple computations reveal that the semi-conformal curvature tensor has numerous symmetries, skew symmetries and cyclic symmetries. It is noted that

$$\mathcal{P}_{ijkl} = -(n-2)B\left(\mathcal{L}_{ijkl} + \frac{R}{(n-1)(n-2)}G_{ijkl}\right)$$
$$+ [A + (n-2)B]\mathcal{L}_{ijkl}$$
$$= -(n-2)B(\frac{R}{(n-1)(n-2)}G_{ijkl}) + A\mathcal{L}_{ijkl}$$
$$\mathcal{P}_{ijkl} = \frac{-BR}{n-1}G_{ijkl} + A\mathcal{L}_{ijkl}.$$

Likewise,

$$\mathcal{P}_{ijkl} = A\mathcal{C}_{ijkl} - \frac{R(A + (n-2)B)}{(n-1)(n-2)}G_{ijkl}. \quad (1)$$

The tensor $\mathcal{P}$ satisfies the symmetries and identities of the Riemann tensor as follows:

$$\mathcal{P}_{ijkl} = -\mathcal{P}_{jikl}$$
$$\mathcal{P}_{ijkl} = -\mathcal{P}_{ijlk}$$
$$\mathcal{P}_{ijkl} + \mathcal{P}_{jkil} + \mathcal{P}_{kijl} = 0.$$

Assume that a manifold V is semi-conformally flat, denoted by SCF, that is, the semi-conformal curvature tensor vanishes. Then

$$0 = \frac{-BR}{n-1}G_{ijkl} + A\mathcal{L}_{ijkl}$$
$$0 = \frac{-BR}{n-1}G_{ijkl} + AR_{ijkl}$$
$$+ \frac{A}{n-2}(g_{ik}R_{jl} - g_{il}R_{jk} + g_{jl}R_{ik} - g_{jk}R_{il}).$$

Two different contraction over i, l and j, k yield

$$0 = \left(B + \frac{A}{n-2}\right)Rg_{jk}$$
$$0 = ((n-2)B + A)R.$$

That is, $R = 0$ or $(n-2)B + A = 0$. Utilizing Equation (1), one obtains $\mathcal{C}_{ijkl} = 0$. Thus, a semi-conformally flat manifold V is either conformally flat, or the scalar curvature vanishes. For more details about semi-conformal curvature tensors in general relativity, see [30].

3. Singly Warped Product Manifold

Warped product manifolds were first introduced to find manifolds with a negative curvature. The choice of the warping function gives an effective tool to create such manifolds. Many interesting examples of relativistic space-times take the form of Lorentzian warped product manifolds. Let $V = \bar{V} \times_f \tilde{V}$ be a warped product manifold with its natural projections $\pi : \bar{V} \times \tilde{V} \longrightarrow \bar{V}$ and $\eta : \bar{V} \times \tilde{V} \longrightarrow \tilde{V}$ such that $(\bar{V}, \bar{g})$ and $(\tilde{V}, \tilde{g})$ are two pseudo-Riemannian manifolds. Let $\dim\bar{V} = \bar{n}$ and $\dim\tilde{V} = \tilde{n} = n - \bar{n}$. The manifolds $\bar{V}$ and $\tilde{V}$ are called the base manifold and fiber manifold with metric tensors $\bar{g}$ and $\tilde{g}$. The warped product manifold is furnished with the metric $g = \bar{g} \oplus F\tilde{g}$, where $F : V \longrightarrow (0, \infty)$ is a smooth positive function on $\bar{V}$. Let $i, j, \ldots \in \{1, \ldots, n\}$, $a, b, \ldots \in \{1, \ldots, \bar{n}\}$ and $\alpha, \beta, \ldots \in \{\bar{n}+1, \ldots, n\}$. The local components of the metric tensor g on the warped product manifold V are given by

$$g_{ij} = \begin{cases} \bar{g}_{ab}, & i = a, j = b, \\ F\tilde{g}_{\alpha\beta} & i = \alpha, j = \beta, \\ 0 & otherwise. \end{cases} \tag{2}$$

The local non-zero components of the Riemannian curvature tensor R_{ijkl} on the warped product manifold V are given by

$$R_{\alpha\beta\gamma\delta} = F\tilde{R}_{\alpha\beta\gamma\delta} - \frac{1}{4}\bar{\Delta}F\tilde{G}_{\alpha\beta\gamma\delta}, \tag{3}$$

$$R_{\alpha ab\beta} = \frac{-1}{2}T_{ab}\tilde{g}_{\alpha\beta}, \tag{4}$$

$$R_{abcd} = \bar{R}_{abcd}, \tag{5}$$

where $\tilde{G}_{\alpha\beta\gamma\delta} = g_{\alpha\delta}g_{\beta\gamma} - g_{\alpha\gamma}g_{\beta\delta}$, $\triangle\bar{F} = \bar{g}^{ab}F_aF_b$, $F_a = \partial_a F = \frac{\partial F}{\partial x^a}$ and T_{ab} is a tensor of type $(0, 2)$ defined by

$$T_{ab} = \bar{\nabla}_b F_a - \frac{1}{2F}F_aF_b,\ T_{\alpha\beta} = T_{a\alpha} = 0.$$

The local components of the Ricci curvature tensor R_{ij} of the warped product manifold V are the following:

$$R_{ab} = \bar{R}_{ab} - \frac{\tilde{n}}{2F}T_{ab}, \tag{6}$$

$$R_{\alpha\beta} = \tilde{R}_{\alpha\beta} - \frac{1}{2}[T + \frac{\tilde{n}-1}{2F}\bar{\triangle}F]\tilde{g}_{\alpha\beta}, \tag{7}$$

$$R_{a\alpha} = 0. \tag{8}$$

The scalar curvature R of the warped product manifold is

$$R = \bar{R} + \frac{1}{F}\tilde{R} - \frac{\tilde{n}}{F}[T + \frac{\tilde{n}-1}{4F}\bar{\triangle}F] \tag{9}$$

where $T = \bar{g}^{ab}T_{ab}$ and $\bar{R}, \tilde{R}$ are the scalar curvatures of the base manifold $\bar{V}$ and the fiber manifold $\tilde{V}$. The non-vanishing components of the covariant derivative of the Riemannian curvature tensor are

$$\nabla_e R_{abcd} = \bar{\nabla}_e \bar{R}_{abcd}, \tag{10}$$

$$\nabla_\epsilon R_{\alpha\beta\gamma\delta} = F\tilde{\nabla}_\epsilon \tilde{R}_{\alpha\beta\gamma\delta}, \tag{11}$$

$$R_{\alpha ab\delta} = 0. \tag{12}$$

The covariant derivative of the Ricci curvature tensor is

$$\nabla_c R_{ab} = \bar{\nabla}_c \bar{R}_{ab} - \frac{\tilde{n}}{2}\nabla_c \frac{T_{ab}}{F}, \tag{13}$$

$$\nabla_\gamma R_{\alpha\beta} = \tilde{\nabla}_\gamma \tilde{R}_{\alpha\beta}, \tag{14}$$

$$\nabla_\gamma R_{ab} = 0. \tag{15}$$

For more details, the reader is referred to [34,35].

4. Semi-Conformal Curvature Tensor On Warped Product Manifolds

The conharmonic curvature tensor $\mathcal{L}$ and the Weyl curvature tensor $\mathcal{C}$ make up the semi-conformal curvature tensor such that

$$\mathcal{P}_{ijkl} = -(n-2)B\mathcal{C}_{ijkl} + [A + (n-2)B]\mathcal{L}_{ijkl}, \tag{16}$$

where

$$\mathcal{L}_{ijkl} = R_{ijkl} + \frac{1}{n-2}(g_{ik}R_{jl} - g_{il}R_{jk} + g_{jl}R_{ik} - g_{jk}R_{il}) \tag{17}$$

$$\begin{aligned}\mathcal{C}_{ijkl} &= R_{ijkl} + \frac{1}{n-2}(g_{ik}R_{jl} - g_{il}R_{jk} + g_{jl}R_{ik} - g_{jk}R_{il}) \\ &+ \frac{R}{(n-1)(n-2)}(g_{il}g_{jk} - g_{ik}g_{jl}) \\ &= \mathcal{L}_{ijkl} + \frac{R}{(n-1)(n-2)}G_{ijkl},\end{aligned} \tag{18}$$

where $G_{ijkl} = (g_{il}g_{jk} - g_{ik}g_{jl})$. Then

$$\begin{aligned}\mathcal{P}_{ijkl} &= -(n-2)B\left(\mathcal{L}_{ijkl} + \frac{R}{(n-1)(n-2)}G_{ijkl}\right) \\ &+ [A + (n-2)B]\mathcal{L}_{ijkl} \\ &= -(n-2)B(\frac{R}{(n-1)(n-2)}G_{ijkl}) + A\mathcal{L}_{ijkl}\end{aligned} \tag{19}$$

$$\mathcal{P}_{ijkl} = \frac{-BR}{n-1}G_{ijkl} + A\mathcal{L}_{ijkl} \tag{20}$$

such that $i, j, k, l \in \{1, \ldots, n\}$, R_{ijkl} and R_{jl} are the Riemannian and Ricci curvature tensors, respectively, and g_{ik} is the metric tensor. For more details about semi-conformal curvature tensors in general relativity, see [30]. Now we have the conversion of $\mathcal{L}_{ijkl}$, $\mathcal{C}_{ijkl}$, and $\mathcal{P}_{ijkl}$ from the Riemannian manifold to WPMs such that $i, j, k, l \in \{1, \ldots, \bar{n}\}$ then $i = a, j = b$, $k = c, l = d$.

The symmetries and anti-symmetries of the tensor $\mathcal{P}$ reduce the non-zero components to only three components out of sixteen components. By using Equations (5) and (6) in Equation (17), one obtains the following.

4.1. The Component $\mathcal{P}_{a\alpha\beta b}$

The first case $\mathcal{P}_{a\alpha\beta b}$ is computed as

$$\begin{aligned}\mathcal{P}_{a\alpha\beta b} &= -(n-2)B\mathcal{C}_{a\alpha\beta b} + [A + (n-2)B]\mathcal{L}_{a\alpha\beta b} \\ &= \frac{-BR}{n-1}G_{a\alpha\beta b} + A\mathcal{L}_{a\alpha\beta b}.\end{aligned}$$

Let us first find $\mathcal{L}_{a\alpha\beta b}$ as

$$\begin{aligned}\mathcal{L}_{a\alpha\beta b} &= R_{a\alpha\beta b} + \frac{1}{n-2}(-g_{ab}R_{\alpha\beta} - g_{\alpha\beta}R_{ab}) \\ &= R_{a\alpha\beta b} - \frac{1}{n-2}g_{ab}R_{\alpha\beta} - \frac{1}{n-2}g_{\alpha\beta}R_{ab}.\end{aligned}$$

The use of the definitions of the Riemann tensor and the Ricci tensor on warped product manifolds lead to

$$\begin{aligned}
\mathcal{L}_{a\alpha\beta b} &= \frac{-1}{2}T_{ab}\tilde{g}_{\alpha\beta} - \frac{1}{n-2}\bar{g}_{ab}\left(\tilde{R}_{\alpha\beta} - \frac{1}{2}[T + \frac{\tilde{n}-1}{2F}\bar{\triangle}F]\tilde{g}_{\alpha\beta}\right) \\
&- \frac{1}{n-2}F\tilde{g}_{\alpha\beta}\left(\bar{R}_{ab} - \frac{\tilde{n}}{2F}T_{ab}\right) \\
&= \left(\frac{-1}{2} + \frac{\tilde{n}}{2(n-2)}\right)T_{ab}\tilde{g}_{\alpha\beta} - \frac{1}{n-2}\bar{g}_{ab}\tilde{R}_{\alpha\beta} \\
&+ \frac{1}{2(n-2)}[T + \frac{\tilde{n}-1}{2F}\bar{\triangle}F]\bar{g}_{ab}\tilde{g}_{\alpha\beta} - \frac{1}{n-2}F\tilde{g}_{\alpha\beta}\bar{R}_{ab} \\
&= \left(\frac{-\bar{n}+2}{2(n-2)}\right)T_{ab}\tilde{g}_{\alpha\beta} - \frac{1}{n-2}\bar{g}_{ab}\tilde{R}_{\alpha\beta} \\
&+ \frac{1}{2(n-2)}[T + \frac{\tilde{n}-1}{2F}\bar{\triangle}F]\bar{g}_{ab}\tilde{g}_{\alpha\beta} - \frac{1}{n-2}F\tilde{g}_{\alpha\beta}\bar{R}_{ab}.
\end{aligned}$$

However, the tensor $G_{a\alpha\beta b}$ is given as

$$G_{a\alpha\beta b} = g_{ab}g_{\alpha\beta} = F\bar{g}_{ab}\tilde{g}_{\alpha\beta}.$$

Thus, the corresponding component of the tensor $\mathcal{P}$ is given by

$$\begin{aligned}
\mathcal{P}_{a\alpha\beta b} &= \frac{-BR}{n-1}G_{a\alpha\beta b} + A\mathcal{L}_{a\alpha\beta b} \\
&= \frac{-FBR}{n-1}\bar{g}_{ab}\tilde{g}_{\alpha\beta} + A\mathcal{L}_{a\alpha\beta b} \\
&= \frac{-FBR}{n-1}\bar{g}_{ab}\tilde{g}_{\alpha\beta} + A\left(\frac{-\bar{n}+2}{2(n-2)}\right)T_{ab}\tilde{g}_{\alpha\beta} - \frac{A}{n-2}\bar{g}_{ab}\tilde{R}_{\alpha\beta} \\
&+ \frac{A}{2(n-2)}[T + \frac{\tilde{n}-1}{2F}\bar{\triangle}F]\bar{g}_{ab}\tilde{g}_{\alpha\beta} - \frac{A}{n-2}F\tilde{g}_{\alpha\beta}\bar{R}_{ab}.
\end{aligned}$$

This leads us to the form of the first non-zero component of the tensor $\mathcal{P}$ on warped product manifolds. Thus, we have the following.

Theorem 1. *In a warped product manifold, the semi-conformal curvature tensor satisfies*

$$\begin{aligned}
\mathcal{P}_{a\alpha\beta b} &= \frac{-FBR}{n-1}\bar{g}_{ab}\tilde{g}_{\alpha\beta} + A\left(\frac{-\bar{n}+2}{2(n-2)}\right)T_{ab}\tilde{g}_{\alpha\beta} - \frac{A}{n-2}\bar{g}_{ab}\tilde{R}_{\alpha\beta} \\
&+ \frac{A}{2(n-2)}[T + \frac{\tilde{n}-1}{2F}\bar{\triangle}F]\bar{g}_{ab}\tilde{g}_{\alpha\beta} - \frac{A}{n-2}F\tilde{g}_{\alpha\beta}\bar{R}_{ab}.
\end{aligned}$$

In a semi-conformally flat warped product manifold, one obtains

$$\begin{aligned}
0 &= \frac{-2(n-2)FBR}{n-1}\bar{g}_{ab}\tilde{g}_{\alpha\beta} - A(\bar{n}-2)T_{ab}\tilde{g}_{\alpha\beta} - 2A\bar{g}_{ab}\tilde{R}_{\alpha\beta} \\
&+ A[T + \frac{\tilde{n}-1}{2F}\bar{\triangle}F]\bar{g}_{ab}\tilde{g}_{\alpha\beta} - 2AF\tilde{g}_{\alpha\beta}\bar{R}_{ab}.
\end{aligned}$$

Two different contractions of the above equation will give us many important implications of the flatness of the tensor $\mathcal{P}$ on warped product manifolds. The first contraction by $\tilde{g}^{\alpha\beta}$ implies

$$\begin{aligned}
0 &= \frac{-2\tilde{n}(n-2)FBR}{n-1}\bar{g}_{ab} - A\tilde{n}(\bar{n}-2)T_{ab} - 2A\tilde{R}\bar{g}_{ab} \\
&+ \tilde{n}A[T + \frac{\tilde{n}-1}{2F}\bar{\triangle}F]\bar{g}_{ab} - 2\tilde{n}AF\bar{R}_{ab}.
\end{aligned} \tag{21}$$

This equation gives

$$0 = \frac{-2\bar{n}\tilde{n}(n-2)FBR}{n-1} - A\tilde{n}(\bar{n}-2)T - 2\bar{n}A\tilde{R} + \bar{n}\tilde{n}A[T + \frac{\tilde{n}-1}{2F}\bar{\triangle}F] - 2\tilde{n}AF\bar{R}.$$
$$0 = \left[\frac{-2\tilde{n}(n-2)FBR}{n-1} - 2A\tilde{R} + \tilde{n}A[T + \frac{\tilde{n}-1}{2F}\bar{\triangle}F]\right]\bar{n} - A\tilde{n}(\bar{n}-2)T - 2\tilde{n}AF\bar{R}.$$

A direct computation infers

$$\left[\frac{-2\tilde{n}(n-2)FBR}{n-1} - 2A\tilde{R} + \tilde{n}A[T + \frac{\tilde{n}-1}{2F}\bar{\triangle}F]\right] = \frac{\tilde{n}}{\bar{n}}A[(\bar{n}-2)T + 2F\bar{R}]. \tag{22}$$

Updating Equation (21), one obtains

$$\begin{aligned} 0 &= \frac{\tilde{n}}{\bar{n}}A[(\bar{n}-2)T + 2F\bar{R}]\bar{g}_{ab} - 2\tilde{n}AF\bar{R}_{ab} - A\tilde{n}(\bar{n}-2)T_{ab} \\ 0 &= \frac{1}{\bar{n}}A[(\bar{n}-2)T + 2F\bar{R}]\bar{g}_{ab} - 2AF\bar{R}_{ab} - A(\bar{n}-2)T_{ab} \\ 0 &= -2AF\left[\bar{R}_{ab} - \frac{1}{2\bar{n}F}[(\bar{n}-2)T + 2F\bar{R}]\bar{g}_{ab} + \frac{\bar{n}-2}{2F}T_{ab}\right]. \end{aligned} \tag{23}$$

A manifold is called quasi-Einstein if the Ricci tensor satisfies $R_{ij} = ag_{ij} + bu_iu_j$ for some scalars a and b, where u is a $1-$form. Hence, we can state the following.

Theorem 2. *In a semi-conformally flat warped product manifold where $A \neq 0$, the base manifold is Einstein if and only if the tensor T_{ab} is proportional to the metric tensor.*

Theorem 3. *In a semi-conformally flat warped product manifold where $A \neq 0$, the base manifold is quasi-Einstein if the tensor T_{ab} takes the form $T_{ab} = u_au_b$ for a one-form u.*

Equation (22) needs more discussion as follows:

$$\begin{aligned} \frac{-2\tilde{n}(n-2)FBR}{n-1} - 2A\tilde{R} + \tilde{n}A[T + \frac{\tilde{n}-1}{2F}\bar{\triangle}F] &= \frac{\tilde{n}}{\bar{n}}A[(\bar{n}-2)T + 2F\bar{R}]. \\ \frac{-2\tilde{n}(n-2)FBR}{n-1} + A\left(-2\tilde{R} + \tilde{n}T + \tilde{n}\frac{\tilde{n}-1}{2F}\bar{\triangle}F\right) &= A\left[\tilde{n}T - 2\frac{\tilde{n}}{\bar{n}}T + 2\frac{\tilde{n}}{\bar{n}}F\bar{R}\right] \\ \frac{-\tilde{n}(n-2)BR}{n-1} + A\left(-\frac{\tilde{R}}{F} + \frac{\tilde{n}}{F}\frac{\tilde{n}-1}{4F}\bar{\triangle}F\right) &= A\left[-\frac{\tilde{n}}{\bar{n}}\frac{T}{F} + \frac{\tilde{n}}{\bar{n}}\bar{R}\right] \\ \frac{-\tilde{n}(n-2)BR}{n-1} - A\left(\frac{\tilde{R}}{F} - \frac{\tilde{n}}{F}\frac{\tilde{n}-1}{4F}\bar{\triangle}F\right) &= A\left[-\frac{\tilde{n}}{\bar{n}}\frac{T}{F} + \frac{\tilde{n}}{\bar{n}}\bar{R}\right]. \end{aligned}$$

However,

$$R = \bar{R} - \frac{\tilde{n}}{F}T + \frac{1}{F}\tilde{R} - \frac{\tilde{n}}{F}\frac{\tilde{n}-1}{4F}\bar{\triangle}F.$$

Thus,

$$\frac{-\tilde{n}(n-2)BR}{n-1} - A\left(R - \bar{R} + \frac{\tilde{n}}{F}T\right) = A\left[-\frac{\tilde{n}}{\bar{n}}\frac{T}{F} + \frac{\tilde{n}}{\bar{n}}\bar{R}\right].$$

Simple calculations imply

$$0 = A\left[\frac{1}{\tilde{n}}R + \left(\frac{1}{\bar{n}} - \frac{1}{\tilde{n}}\right)\bar{R} + \left(1 - \frac{1}{\bar{n}}\right)\frac{T}{F}\right] + \frac{n-2}{n-1}BR. \tag{24}$$

Thus, we can write the following.

Theorem 4. *In a semi-conformally flat warped product manifold, it is*

$$0 = A\left[\frac{1}{\tilde{n}}R + \left(\frac{1}{\bar{n}} - \frac{1}{\tilde{n}}\right)\bar{R} + \left(1 - \frac{1}{\bar{n}}\right)\frac{T}{F}\right] + \frac{n-2}{n-1}BR.$$

The second contraction by $\bar{g}^{ab}$ yields

$$0 = \frac{-2\bar{n}(n-2)FBR}{n-1}\tilde{g}_{\alpha\beta} - A(\bar{n}-2)T\tilde{g}_{\alpha\beta} - 2A\bar{n}\tilde{R}_{\alpha\beta}$$
$$+ A[T + \frac{\tilde{n}-1}{2F}\bar{\triangle}F]\bar{n}\tilde{g}_{\alpha\beta} - 2AF\tilde{g}_{\alpha\beta}\bar{R}.$$

That is,

$$A\tilde{R}_{\alpha\beta} = \left[\frac{-(n-2)FBR}{n-1} + A\left(\frac{\tilde{n}-1}{4F}\bar{\triangle}F - F\frac{\bar{R}}{\bar{n}} + \frac{T}{\bar{n}}\right)\right]\tilde{g}_{\alpha\beta}.$$

This equation leads to

$$A\tilde{R} = \left[\frac{-(n-2)FBR}{n-1} + A\left(\frac{\tilde{n}-1}{4F}\bar{\triangle}F - F\frac{\bar{R}}{\bar{n}} + \frac{T}{\bar{n}}\right)\right]\tilde{n}.$$

This gives an update of the previous equation as

$$A\left(\tilde{R}_{\alpha\beta} - \frac{\tilde{R}}{\tilde{n}}\tilde{g}_{\alpha\beta}\right) = 0.$$

Hence, we have the following.

Theorem 5. *In a semi-conformally flat warped product manifold where $A \neq 0$, the fiber manifold is Einstein.*

The Lie derivative $\mathfrak{L}_\zeta$ of the metric tensor g on warped product manifolds along the flow lines of a vector field $\zeta = \bar{\zeta} + \tilde{\zeta}$ is given by

$$\mathfrak{L}_\zeta g = \bar{\mathfrak{L}}_{\bar{\zeta}}\bar{g} + F\tilde{\mathfrak{L}}_{\tilde{\zeta}}\tilde{g} + \bar{\zeta}(F)\tilde{g},$$

where $\bar{\mathfrak{L}}_{\bar{\zeta}}\bar{g}$ and $\tilde{\mathfrak{L}}_{\tilde{\zeta}}\tilde{g}$ are the Lie derivatives of the metric tensors $\bar{g}$ and $\tilde{g}$ along the flow lines of the vector fields $\bar{\zeta}$ and $\tilde{\zeta}$, respectively. Now, assume that $\zeta = \tilde{\zeta}$, then

$$\mathfrak{L}_\zeta g = F\tilde{\mathfrak{L}}_{\tilde{\zeta}}\tilde{g}.$$

The isometries of the warped product manifolds correspond to vector fields with the identity $\mathfrak{L}_\zeta g = 0$. Every vector field with this property is called a Killing vector field. This implies that all Killing vector fields on the fiber manifold are also Killing vector fields on the warped product manifolds.

A vector field where the Lie derivative of the Ricci tensor vanishes is called Ricci collineation. It is known that every Killing vector field is a Ricci collineation; however, the converse is not generally true. However, in a semi-conformally flat warped product manifold, where $A \neq 0$, the fiber manifold is Einstein. On Einstein manifolds, the scalar curvature is constant, and consequently, every Killing vector field on the fiber manifold is also a Ricci collineation.

Corollary 1. *In a semi-conformally flat warped product manifold where $A \neq 0$, every Ricci collineation is a Killing vector field.*

4.2. The Component $\mathcal{P}_{abcd}$

The second non-zero case is

$$\mathcal{P}_{abcd} = -(n-2)B\mathcal{C}_{abcd} + [A + (n-2)B]\mathcal{L}_{abcd}$$

or,

$$\mathcal{P}_{abcd} = \frac{-BR}{n-1} G_{abcd} + A\mathcal{L}_{abcd}. \tag{25}$$

The tensor $\mathcal{L}_{abcd}$ is given by

$$\mathcal{L}_{abcd} = R_{abcd} + \frac{1}{n-2}(g_{ac}R_{bd} - g_{ad}R_{bc} + g_{bd}R_{ac} - g_{bc}R_{ad}).$$

Using the Riemann tensor and Ricci tensor identities on warped product manifolds, one obtains

$$\begin{aligned}
&= \bar{R}_{abcd} + \frac{1}{n-2}[\bar{g}_{ac}(\bar{R}_{bd} - \frac{\tilde{n}}{2F}T_{bd}) - \bar{g}_{ad}(\bar{R}_{bc} - \frac{\tilde{n}}{2F}T_{bc}) \\
&+ \bar{g}_{bd}(\bar{R}_{ac} - \frac{\tilde{n}}{2F}T_{ac}) - \bar{g}_{bc}(\bar{R}_{ad} - \frac{\tilde{n}}{2F}T_{ad})] \\
&= \bar{R}_{abcd} + \frac{1}{n-2}[\bar{g}_{ac}\bar{R}_{bd} - \bar{g}_{ad}\bar{R}_{bc} + \bar{g}_{bd}\bar{R}_{ac} - \bar{g}_{bc}\bar{R}_{ad}] \\
&- \frac{\tilde{n}}{2(n-2)F}[\bar{g}_{ac}T_{bd} - \bar{g}_{ad}T_{bc} + \bar{g}_{bd}T_{ac} - \bar{g}_{bc}T_{ad}].
\end{aligned}$$

The tensor $\mathcal{P}$ from (25) is given by

$$\begin{aligned}
\mathcal{P}_{abcd} = &\frac{-BR}{n-1}\bar{G}_{abcd} + A\bar{R}_{abcd} \\
&+ \frac{A}{n-2}[\bar{g}_{ac}\bar{R}_{bd} - \bar{g}_{ad}\bar{R}_{bc} + \bar{g}_{bd}\bar{R}_{ac} - \bar{g}_{bc}\bar{R}_{ad}] \\
&- \frac{\tilde{n}A}{2(n-2)F}[\bar{g}_{ac}T_{bd} - \bar{g}_{ad}T_{bc} + \bar{g}_{bd}T_{ac} - \bar{g}_{bc}T_{ad}].
\end{aligned} \tag{26}$$

Theorem 6. *In a warped product manifold, the semi-conformal curvature tensor satisfies Equation (27).*

Assume that the warped product manifold is semi-conformally flat, then

$$\begin{aligned}
0 = &\frac{-BR}{n-1}\bar{G}_{abcd} + A\bar{R}_{abcd} \\
&+ \frac{A}{n-2}[\bar{g}_{ac}\bar{R}_{bd} - \bar{g}_{ad}\bar{R}_{bc} + \bar{g}_{bd}\bar{R}_{ac} - \bar{g}_{bc}\bar{R}_{ad}] \\
&- \frac{\tilde{n}A}{2(n-2)F}[\bar{g}_{ac}T_{bd} - \bar{g}_{ad}T_{bc} + \bar{g}_{bd}T_{ac} - \bar{g}_{bc}T_{ad}].
\end{aligned} \tag{27}$$

Contracting this equation by $\bar{g}^{ad}$, one may obtain

$$\begin{aligned}
0 = &\frac{-(\bar{n}-1)BR}{n-1}\bar{g}_{bc} + A\bar{R}_{bc} \\
&+ \frac{A}{n-2}[\bar{R}_{bc} - \bar{n}\bar{R}_{bc} + \bar{R}_{bc} - \bar{g}_{bc}\bar{R}] \\
&- \frac{\tilde{n}A}{2(n-2)F}[T_{bc} - \bar{n}T_{bc} + T_{bc} - \bar{g}_{bc}T].
\end{aligned} \tag{28}$$

Then,

$$0=\frac{-(\bar{n}-1)BR}{n-1}\bar{g}_{bc}+\left(\frac{\tilde{n}T-2\bar{R}}{2(n-2)F}\right)A\bar{g}_{bc}$$
$$+\frac{\tilde{n}A}{n-2}\bar{R}_{bc}+\frac{\tilde{n}(\bar{n}-2)A}{2(n-2)F}T_{bc}.$$

Accordingly, we have

$$0=\left(\frac{-(\bar{n}-1)(n-2)BR}{\tilde{n}(n-1)}+\frac{\tilde{n}T-2\bar{R}}{2\tilde{n}F}A\right)\bar{g}_{bc}$$
$$+A\bar{R}_{bc}+\frac{(\bar{n}-2)A}{2F}T_{bc}.$$

However, Equation (23) whenever $A\neq 0$ implies

$$\bar{R}_{ab}-\frac{1}{2\bar{n}F}[(\bar{n}-2)T+2F\bar{R}]\bar{g}_{ab}+\frac{\bar{n}-2}{2F}T_{ab}=0.$$

Thus, we may update the above equation as follows:

$$0=\left(\frac{-(\bar{n}-1)(n-2)BR}{\tilde{n}(n-1)}+\frac{\tilde{n}T-2\bar{R}}{2\tilde{n}F}A\right)\bar{g}_{bc}$$
$$+A\frac{1}{2\bar{n}F}[(\bar{n}-2)T+2F\bar{R}]\bar{g}_{ab}.$$

This yields

$$\left(\frac{-(\bar{n}-1)(n-2)BR}{\tilde{n}(n-1)}+\frac{\tilde{n}T-2\bar{R}}{2\tilde{n}F}A\right)=-A\frac{1}{2\bar{n}F}[(\bar{n}-2)T+2F\bar{R}].$$

That is,

$$0=A\left(\bar{R}_{bc}-\frac{1}{2\bar{n}F}[(\bar{n}-2)T+2F\bar{R}]\bar{g}_{bc}+\frac{(\bar{n}-2)}{2F}T_{bc}\right).$$

It is clear that we obtain the same conclusion on the base manifold as in the previous case.

4.3. The Component $\mathcal{P}_{\alpha\beta\gamma\delta}$

Now if $i,j,k,l\in\{\bar{n}+1,\ldots,n\}$, first we compute $\mathcal{L}_{\alpha\beta\gamma\delta}$ in Equation (17) from Equations (2), (3) and (7) as

$$\mathcal{L}_{\alpha\beta\gamma\delta}=R_{\alpha\beta\gamma\delta}+\frac{1}{n-2}(g_{\alpha\gamma}R_{\beta\delta}-g_{\alpha\delta}R_{\beta\gamma}+g_{\beta\delta}R_{\alpha\gamma}-g_{\beta\gamma}R_{\alpha\delta}).$$

Utilizing the Riemann and Ricci tensor forms on warped product manifolds, we obtain

$$\mathcal{L}_{\alpha\beta\gamma\delta}=F\tilde{R}_{\alpha\beta\gamma\delta}-\frac{1}{4}\bar{\Delta}F\tilde{G}_{\alpha\beta\gamma\delta}$$
$$+\frac{F}{n-2}\tilde{g}_{\alpha\gamma}(\tilde{R}_{\beta\delta}-\frac{1}{2}[T+\frac{\bar{n}-1}{2F}\bar{\triangle}F]\tilde{g}_{\beta\delta})$$
$$-\frac{F}{n-2}\tilde{g}_{\alpha\delta}(\tilde{R}_{\beta\gamma}-\frac{1}{2}[T+\frac{\bar{n}-1}{2F}\bar{\triangle}F]\tilde{g}_{\beta\gamma})$$
$$+\frac{F}{n-2}\tilde{g}_{\beta\delta}(\tilde{R}_{\alpha\beta\gamma\delta}-\frac{1}{2}[T+\frac{\bar{n}-1}{2F}\bar{\triangle}F]\tilde{g}_{\alpha\gamma})$$
$$-\frac{F}{n-2}\tilde{g}_{\beta\gamma}(\tilde{R}_{\alpha\delta}-\frac{1}{2}[T+\frac{\bar{n}-1}{2F}\bar{\triangle}F]\tilde{g}_{\alpha\delta}).$$

The previous equation can be rewritten in the following form:

$$\mathcal{L}_{\alpha\beta\gamma\delta} = F\tilde{R}_{\alpha\beta\gamma\delta} + \frac{2FT + \tilde{n} - \bar{n}}{2(n-2)} \bar{\triangle} F \tilde{G}_{\alpha\beta\gamma\delta} + \frac{F}{n-2}\big(\tilde{g}_{\alpha\gamma}\tilde{R}_{\beta\delta} - \tilde{g}_{\alpha\delta}\tilde{R}_{\beta\gamma} + \tilde{g}_{\beta\delta}\tilde{R}_{\alpha\beta\gamma\delta} - \tilde{g}_{\beta\gamma}\tilde{R}_{\alpha\delta}\big).$$

Theorem 7. *In a warped product manifold, the semi-conformal curvature tensor satisfies*

$$\mathcal{L}_{\alpha\beta\gamma\delta} = F\tilde{R}_{\alpha\beta\gamma\delta} + \frac{2FT + \tilde{n} - \bar{n}}{2(n-2)} \bar{\triangle} F \tilde{G}_{\alpha\beta\gamma\delta} + \frac{F}{n-2}\big(\tilde{g}_{\alpha\gamma}\tilde{R}_{\beta\delta} - \tilde{g}_{\alpha\delta}\tilde{R}_{\beta\gamma} + \tilde{g}_{\beta\delta}\tilde{R}_{\alpha\gamma} - \tilde{g}_{\beta\gamma}\tilde{R}_{\alpha\delta}\big).$$

Assume that the warped product manifold is semi-conformally flat, then

$$0 = F\tilde{R}_{\alpha\beta\gamma\delta} + \frac{2FT + \tilde{n} - \bar{n}}{2(n-2)} \bar{\triangle} F \tilde{G}_{\alpha\beta\gamma\delta} + \frac{F}{n-2}\big(\tilde{g}_{\alpha\gamma}\tilde{R}_{\beta\delta} - \tilde{g}_{\alpha\delta}\tilde{R}_{\beta\gamma} + \tilde{g}_{\beta\delta}\tilde{R}_{\alpha\gamma} - \tilde{g}_{\beta\gamma}\tilde{R}_{\alpha\delta}\big).$$

In this case, the Riemann curvature tensor of the fiber manifold is given by

$$\tilde{R}_{\alpha\beta\gamma\delta} = -\frac{2FT + \tilde{n} - \bar{n}}{2(n-2)} \frac{\bar{\triangle} F}{F} \tilde{G}_{\alpha\beta\gamma\delta} - \frac{1}{n-2}\big(\tilde{g}_{\alpha\gamma}\tilde{R}_{\beta\delta} - \tilde{g}_{\alpha\delta}\tilde{R}_{\beta\gamma} + \tilde{g}_{\beta\delta}\tilde{R}_{\alpha\gamma} - \tilde{g}_{\beta\gamma}\tilde{R}_{\alpha\delta}\big).$$

However, the fiber manifold is Einstein in semi-conformally flat manifolds, that is,

$$\tilde{R}_{\alpha\beta\gamma\delta} = \left(-\frac{2FT + \tilde{n} - \bar{n}}{2(n-2)} \frac{\bar{\triangle} F}{F} - \frac{2\tilde{R}}{\tilde{n}(n-2)}\right)\tilde{G}_{\alpha\beta\gamma\delta}.$$

It is known that a manifold with pointwise constant sectional curvature has a global constant sectional curvature. Thus, using the above equation, the sectional curvature of the fiber manifold $\tilde{\kappa}$ is constant and is given by

$$\tilde{\kappa} = -\frac{2FT + \tilde{n} - \bar{n}}{2(n-2)} \frac{\bar{\triangle} F}{F} - \frac{2\tilde{R}}{\tilde{n}(n-2)}.$$

Theorem 8. *In a semi-conformally flat warped product manifold, the fiber manifold is of a constant sectional curvature.*

A manifold with a constant sectional curvature is a manifold with maximum symmetry, that is, the number of Killing vector fields attains its maximum.

Corollary 2. *In a semi-conformally flat warped product manifold, the number of Killing vector fields on the fiber manifold is* $\frac{1}{2}\tilde{n}(\tilde{n}+1)$. *Also, the number of Ricci collineations on the fiber manifold is* $\frac{1}{2}\tilde{n}(\tilde{n}+1)$.

In this section, it is shown that every Killing vector field on the fiber manifold is also a Killing vector field on the warped product manifold.

Corollary 3. *In a semi-conformally flat warped product manifold, the minimum number of Killing vector fields on the warped product manifold is* $\frac{1}{2}\tilde{n}(\tilde{n}+1)$. *Also, the minimum number of Ricci condemnations on the warped product manifold is* $\frac{1}{2}\tilde{n}(\tilde{n}+1)$.

5. Conclusions

Warped product manifolds are obvious and fruitful extensions of Cartesian product manifolds [36]. Bishop and O'Neill developed the use of this concept to construct manifolds with a negative sectional curvature [36–38]. It should also be noted that many types of space-times in general relativity are frequently shown as Lorentzian warped product manifolds. Generalized Robertson–Walker space-times and standard static space-times are examples of singly warped product manifolds. As a result, warped product manifolds are significant not only in differential geometry but also in general relativity. For this reason, warped product structures have been widely studied and extended. Multiply warped product manifolds, sequential warped product manifolds [39] and doubly warped product manifolds [40] are amazing generalizations of singly warped product manifolds. Warped product manifold geometry is intimately connected to factor manifolds' geometry. As a result, investigating curvature tensors [41] or geometric structures on warped product manifolds [42,43] aids in better understanding the warped product manifolds' geometry in relation to factor manifolds.

The semi-conformal curvature tensor is a conharmonic transformation invariant. The Weyl conformal curvature tensor and conharmonic curvature tensor are special cases of the semi-conformal curvature tensor. There are numerous fascinating studies of this tensor on relativistic space-times in the literature. Motivated by these findings, this paper investigates semi-conformal curvature on singly warped product manifolds. The Riemann tensor and the Ricci tensor of the factor manifolds of a semi-conformally flat singly warped product manifold are given. It is shown that the fiber manifold of a semi-conformally flat warped product manifold has a constant curvature. There are sufficient conditions on the warping function to ensure that the base manifold is a quasi-Einstein or an Einstein manifold. We can find more motivations of our work from the following papers [44–51].

Author Contributions: Conceptualization and methodology, S.S., A.R., U.C.D., C.M. and N.B.T.; investigation, S.S., A.R. and U.C.D.; writing—original draft preparation, S.S., A.R. and C.M.; writing—review and editing, S.S., A.R., U.C.D., C.M. and N.B.T. All authors have read and agreed to the published version of the manuscript.

Funding: This project was supported by the Researchers Supporting Project number (RSP2023R413), King Saud University, Riyadh, Saudi Arabia.

Institutional Review Board Statement: Not applicable.

Data Availability Statement: Data are contained within the article.

Acknowledgments: We would like to thank the referees for thoroughly reading our manuscript and providing such useful feedback, which significantly improved the paper's quality.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Pokhariyal, G.P.; Mishra, R.S. Curvature tensors and their relativistics significance. *Yokohama Math. J.* **1970**, *18*, 105–108.
2. Pokhariyal, G.P. Relativistic significance of curvature tensors. *Int. J. Math. Math.* **1982**, *5*, 133–139. [CrossRef]
3. Pokhariyal, G.P. Curvature tensors on A-Einstein Sasakian manifolds. *Balk. J. Geom. Appl.* **2001**, *6*, 45–50.
4. Hui, S.K.; Lemence, R.S. On generalized quasi Einstein manifold admitting W_2-curvature tensor. *Int. J. Math. Anal.* **2012**, *6*, 1115–1121.
5. Hui, S.K.; Sarkar, A. On the W_2-curvature tensor of generalized Sasakian-space-forms. *Math. Pannon.* **2012**, *23*, 113–124.
6. Mallick, S.; De, U.D. Space-times admitting W_2-curvature tensor. *Int. J. Geom. Methods Mod. Phys.* **2014**, *11*, 1450030. [CrossRef]
7. Shaikh, A.A.; Matsuyama, Y.; Jana, S.K. On a type of general relativistic spacetime with W_2-curvature tensor. *Indian J. Math.* **2008**, *50*, 53–62.
8. Singh, R.N.; Pandey, G. On the W_2-curvature tensor of the semi-symmetric non-metric connection in a Kenmotsu manifold. *Novi Sad J. Math.* **2013**, *43*, 91–105.
9. Singh, R.N.; Pandey, S.K.; Pandey, G. On W_2-curvature tensor in a Kenmotsu manifold. *Tamsui Oxf. J. Inf. Math. Sci.* **2013**, *29*, 129–141.
10. Zengin, F.O. On Riemannian manifolds admitting W_2-curvature. *Miskolc Math. Notes* **2011**, *12*, 289–296. [CrossRef]
11. Yano, K. Concircular geometry I. Concircular transformations. *Proc. Imp. Acad.* **1940**, *16*, 195–200.

12. Gouli-Andreou, F.; Moutafi, E. On the concircular curvature of a (k,μ,ν)-manifold. *Pac. J. Math.* **2014**, *269*, 113–132.
13. Majhi, P.; De, U.C. Concircular Curvature Tensor on K-Contact Manifolds. *Acta Math. Acad. Nyregyhaziensis* **2013**, *29*, 89–99.
14. Majhi, P.; De, U.C. Classifications of $N(k)$-contact metric manifolds satisfying certain curvature conditions. *Acta Math. Univ. Comen.* **2015**, *84*, 167–178.
15. Youssef, N.L.; Soleiman, A. On concircularly recurrent Finsler manifolds. *Balk. J. Geom. Appl.* **2013**, *18*, 101–113.
16. Zlatanovica, M.; Hinterleitnerb, I.; Najdanovi, M. On Equitorsion Concircular Tensors of Generalized Riemannian Spaces. *Filomat* **2014**, *28*, 463–471.
17. Masao, H. On conformal transformations of Finsler metrics. *J. Math. Kyoto Univ.* **1976**, *16*, 25–50.
18. Morio, O. Conformal transformations of Riemannian manifolds. *J. Differ. Geom.* **1970**, *4*, 311–333.
19. Ishii, Y. On conharmonic transformations. *Tensor NS* **1957**, *7*, 73–80.
20. Jaeman, K. A type of conformal curvature tensor. *Far East J. Math. Sci.* **2016**, *99*, 61.
21. Kobayashi, S.; Nomizu, K. *Foundations of Differential Geometry*; Interscience Tracts in Pure and Applied Mathematics: New York, NY, USA, 1963; Volume 1.
22. Katsumi, N. On the spaces of generalized curvature tensor fields and second fundamental forms. *Osaka J. Math.* **1971**, *8*, 21–28.
23. Kim, J. On Pseudo Semiconformally Symmetric Manifolds. *Bull. Korean Math. Soc.* **2017**, *54*, 177–186. [CrossRef]
24. De, U.C.; Suh, Y.J. On weakly semiconformally symmetric manifolds. *Acta Math. Hung.* **2019**, *157*, 503–521. [CrossRef]
25. Hui, S.K.; Patra, A.; Patra, A. On Generalized Weakly Semi-Conformally Symmetric Manifolds. *Commun. Korean Math. Soc.* **2021**, *36*, 771–782.
26. Ali, M.; Pundeer, N.A.; Suh, Y.J. Proper semi-conformal symmetries of spacetimes with divergence-free semi-conformal curvature tensor. *Filomat* **2019**, *33*, 5191–5198. [CrossRef]
27. Ali, M.; Pundeer, N.A.; Ahsan, Z. Semi-conformal symmetry—A new symmetry of the spacetime manifold of the general relativity. *J. Math. Computer Sci.* **2020**, *20*, 241–254.
28. Ali, M.; Pundeer, N.A.; Ali, A. Semiconformal curvature tensor and perfect fluid spacetimes in general relativity. *J. Taibah Univ. Sci.* **2020**, *14*, 205–210. [CrossRef]
29. Singh, J.P.; Khatri, M. On Semi-conformal Curvature Tensor in (k,μ)-Contact Metric Manifold. *Conf. Proc. Sci. Technol.* **2021**, *4*, 2.
30. Pundeer, N.A.; Ali, M.; Bilal, M. A spacetime admitting semi-conformal curvature tensor. *Balk. J. Geom. Appl.* **2022**, *27*, 130–137.
31. Barman, A. Some properties of a semi-conformal curvature tensor on a Riemannian manifold. *Math. Stud.* **2022**, *91*, 201–208.
32. Pundeer, N.A.; Rahaman, F.; Ali, M.; Shenawy, S. Spacetime admitting semi-conformal curvature tensor in $f(R)$ modify gravity. *Int. J. Geom. Methods Mod. Phys.* **2023**, *20*, 2350176. [CrossRef]
33. De, C.U.; Dey, C. Lorentzian manifolds: A characterization with semi-conformal curvature tensor. *Commun. Korean Math. Soc.* **2019**, *34*, 911–920.
34. Mofarreh, F.; Ali, A.; Alluhaibi, N.; Belova, O. Ricci Curvature for Warped Product Submanifolds of Sasakian Space Forms and Its Applications to Differential Equations. *J. Math.* **2021**, *2021*, 1207646. [CrossRef]
35. Prvanović, M. On warped product manifolds. *Filomat* **1995**, *9*, 169–185.
36. Chen, B.Y. *Differential Geometry of Warped Product Manifolds and Submanifolds*; World Scientific: Hackensack, NJ, USA, 2017.
37. Bishop, R.L.; O'Neill, B. Manifolds of Negative Curvature. *Trans. Amer. Math. Soc.* **1969**, *145*, 1–49. [CrossRef]
38. O'Neill, B. *Semi-Riemannian Geometry with Applications to Relativity*; Academic Press Limited: London, UK, 1983.
39. De, C.U.; Shenawy, S.; Unal, B. Sequential warped products: Curvature and conformal vector fields. *Filomat* **2019**, *33*, 4071–4083. [CrossRef]
40. Shenawy, S.; Turki, N.B.; Syied, N.; Mantica, C. Almost Ricci–Bourguignon Solitons on Doubly Warped Product Manifolds. *Universe* **2023**, *9*, 396. [CrossRef]
41. De, C.U.; Shenawy, S.; Unal, B. Concircular curvature on warped product manifolds and applications. *Bull. Malays. Math. Sci. Soc.* **2020**, *43*, 3395–3409. [CrossRef]
42. De, C.U.; Mantica, C.A.; Shenawy, S.; Unal, B. Ricci solitons on singly warped product manifolds and applications. *J. Geom. Phys.* **2021**, *166*, 104257. [CrossRef]
43. Mantica, A.C.; Shenawy, S. Einstein-like warped product manifolds. *Int. J. Geom. Methods Mod. Phys.* **2017**, *14*, 1750166. [CrossRef]
44. Li, Y.; Gupta, M.K.; Sharma, S.; Chaubey, S.K. On Ricci Curvature of a Homogeneous Generalized Matsumoto Finsler Space. *Mathematics* **2023**, *11*, 3365. [CrossRef]
45. Li, Y.; Güler, E. A Hypersurfaces of Revolution Family in the Five-Dimensional Pseudo-Euclidean Space $\mathbb{E}^5_2$. *Mathematics* **2023**, *11*, 3427. [CrossRef]
46. Li, Y.; Mak, M. Framed Natural Mates of Framed Curves in Euclidean 3-Space. *Mathematics* **2023**, *11*, 3571. [CrossRef]
47. Li, Y.; Patra, D.; Alluhaibi, N.; Mofarreh, F.; Ali, A. Geometric classifications of k-almost Ricci solitons admitting paracontact metrices. *Open Math.* **2023**, *21*, 20220610. [CrossRef]
48. Li, Y.; Güler, E. Hypersurfaces of revolution family supplying in pseudo-Euclidean space. *AIMS Math.* **2023**, *8*, 24957–24970. [CrossRef]
49. Li, Y.; Mofarreh, F.; Abdel-Baky, R.A. Kinematic-geometry of a line trajectory and the invariants of the axodes. *Demonstratio Math.* **2023**, *56*, 20220252. [CrossRef]

50. Li, J.; Yang, Z.; Li, Y.; Abdel-Baky, R.A.; Saad, M.K. On the Curvatures of Timelike Circular Surfaces in Lorentz-Minkowski Space. *Filomat* **2023**, *38*, 1–15.
51. Li, Y.; Güler, E. Twisted Hypersurfaces in Euclidean 5-Space. *Mathematics* **2023**, *11*, 4612. [CrossRef]