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On the mean perimeter density of inhomogeneous random closed sets

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The computation of the mean perimeter density via the notion of mean covariogram for non-stationary Boolean models has been proposed as further work in Galerne (Image Anal. Stereol. 30 (2011) 39-51). We address this issue by considering here more general germ-grain models. Furthermore, we discuss similarities and differences with respect to the computation of the mean boundary density by means of the outer Minkowski content notion.

Keywords: germ-grain model; mean density; Minkowski content; perimeter; variation.

1. Introduction

Let us consider a full dimensional random closed set Θ in $\mathbb{R}^d$, and let $\partial\Theta$ be its topological boundary. Denote by $\mathcal{H}^{d-1}$ the $(d-1)$-dimensional Hausdorff measure, and assume that the mean surface measure $\mathbb{E}[\mathcal{H}^{d-1}(\partial\Theta\cap\cdot)]$ is locally finite. Whenever $\mathbb{E}[\mathcal{H}^{d-1}(\partial\Theta\cap\cdot)]$ is absolutely continuous with respect to the usual d-dimensional Lebesgue measure ν^d, its Radon-Nikodym density, say $\lambda_{\partial\Theta}(x)$, is also named *mean surface density* of Θ. A problem of interest in the literature concerns the determination and the estimation of $\lambda_{\partial\Theta}(x)$. In this regard, we mention the notion of *specific area of a random closed set* Θ *at a point* $x\in\mathbb{R}^d$, defined as the following limit

$$\sigma_\Theta(x) := \lim_{r\downarrow 0}\frac{\mathbb{P}(x\in\Theta_{\oplus r}\setminus\Theta)}{r}, \tag{1.1}$$

provided it exists. Here, $\Theta_{\oplus r}$ denotes the parallel set of Θ at distance r, i.e. $\Theta_{\oplus r} := \{x\in\mathbb{R}^d : \mathrm{dist}(x,\Theta)\le r\}$. The notion of specific area has been introduced in the seminal book on random closed sets by Matheron in (Matheron, 1975, p. 50), leaving the existence of the limit as an open problem; an answer to this has been given in Villa (2014) for a wide class of random closed sets, showing that, in general, $\sigma_\Theta(x)$ may differ from $\lambda_{\partial\Theta}(x)$. Nevertheless, under suitable regularity assumptions on Θ, for instance, if it is a stationary Boolean model with convex grains, it is well known that $\sigma_\Theta(x)=\lambda_{\partial\Theta}(x)=constant>0$. Actually, the stationarity of Θ implies that $\mathbb{E}[\mathcal{H}^{d-1}(\partial\Theta\cap\cdot)]$ is a multiple of ν^d, whereas the convexity of the grains implies that $\sigma_\Theta(x)$ equals the density of the mean measure of the topological boundary of Θ. It is worth noticing that the $\mathcal{H}^{d-1}$-measure of the topological boundary of a non-empty convex set, but more generally of a compact set with Lipschitz boundary, coincides with the $\mathcal{H}^{d-1}$-measure of the so-called *essential boundary* (denoted here by ∂^*) of the set. As we shall recall in the next section, the $\mathcal{H}^{d-1}$-measure of the essential boundary equals the *perimeter* (in the Geometric Measure Theory sense) of the set; the perimeter notion is strictly related to that of *total variation* of the indicator function of the set. Actually, the theory of sets of finite perimeter is a central topic in the broader framework of Geometric Measure Theory (e.g., see Ambrosio et al. (2000), Maggi (2012) for an exhaustive treatment); hence it is not surprising that, in the last decades, sets with finite perimeter appeared in various papers dealing with stochastic geometry topics (e.g., Galerne (2011, 2016), Galerne and Lachièze-Rey (2015), Kiderlen and Rataj (2018), Rataj (2015), Villa (2010, 2014)). Among these we point out the paper by Galerne (2011), where some useful formulas for the

computation of the perimeter are extended to any measurable set by means of the average directional derivative at the origin of the *covariogram* associated to the involved set. The author considers then random sets and their *mean covariogram*; in particular, he provides an explicit expression for the mean perimeter density of Boolean models in the stationary case.
Although stationarity is often a convenient condition to get simple and applicable formulas, in the literature several explicit results for non-stationary Boolean models are available (we refer to (Schneider and Weil, 2008, Ch. 11) for a more exhaustive discussion). Therefore the problem of possible generalization to the non-stationary case naturally arises, as mentioned also in Galerne (2011).

In the next section, we shall define properly all these notions only mentioned so far. Now, in order to clarify the aim of our paper, we anticipate that in Galerne (2011) it is proved that, for a stationary random closed set Θ, the density of the mean perimeter measure $\mathbb{E}[P(\Theta,\cdot)](=\mathbb{E}[\mathcal{H}^{d-1}(\partial^*\Theta\cap\cdot)])$, say $\lambda_{\partial^*\Theta}(x)$, is constant, given by

$$\lambda_{\partial^*\Theta}(x)\equiv\frac{1}{b_{d-1}}\int_{S^{d-1}}\lim_{r\to 0}\frac{\mathbb{P}(ru\in\Theta,0\notin\Theta)}{|r|}\mathcal{H}^{d-1}(\mathrm{d}u)\qquad\text{for } \nu^d\text{-a.e. } x\in\mathbb{R}^d,\tag{1.2}$$

where b_{d-1} is the volume of the unit ball in $\mathbb{R}^{d-1}$, and S^{d-1} the unit sphere in $\mathbb{R}^d$. Furthermore, we would like to point out that, if the boundary $\partial\Theta$ of Θ is sufficiently regular and such that $\mathbb{E}[\mathcal{H}^{d-1}(\partial^*\Theta\cap\cdot)]=\mathbb{E}[\mathcal{H}^{d-1}(\partial\Theta\cap\cdot)]$, by specializing to the stationary case some more general formulas for non-stationary random sets given in Theorem 7 and Theorem 18 in Villa (2014), we also know that

$$\lambda_{\partial^*\Theta}(x)=\sigma_\Theta(x)\equiv\lim_{r\downarrow 0}\frac{\mathbb{P}(0\in\Theta_{\oplus r}\setminus\Theta)}{r}=\lim_{r\downarrow 0}\frac{\mathbb{P}(0\in(\partial\Theta)_{\oplus r})}{2r}\qquad\text{for } \nu^d\text{-a.e. } x\in\mathbb{R}^d.\tag{1.3}$$

Therefore, on the one hand, the equations (1.2) and (1.3) provide equivalent ways for computing the density of the mean boudary measure of Θ whenever it is sufficiently regular; this might also bring to different approaches in possible estimations of $\lambda_{\partial^*\Theta}(x)$. On the other hand, we also observe that if Θ is lower dimensional with positive $\mathcal{H}^{d-1}$ measure, then $\lambda_{\partial^*\Theta}(x)\equiv 0$, whereas the limits in Eq. (1.3) are different from 0.

Hence, the main aim of the present paper is twofold. Firstly, to generalize some results in Galerne (2011) to the non-stationary case, left there as a further work; among these, Eq. (1.2) and related results for Boolean models. This provides a further step in extending the theory of sets with finite perimeter and related topics in a stochastic setting. Secondly, to clarify similarities, differences and links between the above limits in relation with the evaluation of the mean surface density of a random set. To do this, we shall compare the notion of *covariogram* on the one hand, and the notion of *(outer) Minkowski content* on the other hand.
We also point out here that, contrary to the Minkowski content notion, the covariogram and the perimeter notion do not require the closure of the involved set; for this reason some related recent results in the literature for random sets are stated in the more general setting of *random measurable sets (RAMS)* (e.g., see Galerne (2016), Galerne and Lachièze-Rey (2015)). Nevertheless, as we shall discuss in Section 4.3, the framework of random closed sets seems to be more suitable to our aims.

The paper is organized as follows. In Section 2, we recall some basic facts and definitions on sets with finite perimeter and on random sets useful for the sequel. In Section 3, we first introduce the assumptions, which are analogous to the corresponding assumptions in Villa (2010), where the role of the topological boundary of Θ is played here by its perimeter. Then we state and prove our main results; in particular we generalize the known results for homogeneous Boolean models to the non-stationary case, and then we address the problem for a more general germ-grain model. In particular, it will emerge why the general non-stationary case is much less tractable than the Boolean case. In Section 4, we collect some remarks and we discuss some similarities and differences in the evaluation of the (mean) boundary measure of a (random) set, via the covariogram notion and via the outer Minkowski content notion.

2. Preliminaries and notation

Throughout the paper we work in the Euclidean space $\mathbb{R}^d$, $d \geq 2$, equipped with the Borel σ-algebra $\mathcal{B}_{\mathbb{R}^d}$. Given a subset A of $\mathbb{R}^d$, $\partial A := \mathrm{cl}A \setminus \mathrm{int}A$ will be its topological boundary, A^c the complement set of A, $\mathrm{diam}(A)$ the diameter of A, $\mathrm{int}A$ and $\mathrm{cl}A$ the interior and the closure of A, respectively. For $r \geq 0$ and $x \in \mathbb{R}^d$, $B_r(x)$ is the closed ball with center x and radius r; finally, $b_n := \mathcal{H}^n(B_1(0))$ denotes the volume of the unit ball in $\mathbb{R}^n$.
We also recall the notion of *Minkowski addition* $A \oplus B$ (also named dilation), and the notion of *Minkowski difference* $A \ominus B$ (also named erosion) between two subsets A and B of $\mathbb{R}^d$ (we refer to Heijmans (1995) for further insights):

$$A \oplus B := \{a + b \,:\, a \in A,\ b \in B\} = \bigcup_{a \in A} a + B,$$

$$A \ominus B := \{x \in \mathbb{R}^d \,:\, B + x \subseteq A\} = \bigcap_{b \in B} A - b = (A^c \oplus (-B))^c.$$

To lighten the notation, in what follows we set $A_{\oplus r} := A \oplus B_r(0)$ for any $r > 0$; $A_{\oplus r}$ is also named the parallel set of A at distance r, or *Minkowski enlargement* of size r.

2.1. Sets with finite perimeter and related notions

In this section, we mainly refer to Ambrosio et al. (2000), Evans and Gariepy (1992) with regard to the theory of functions of bounded variations and sets with finite perimeter.

There exist several ways to measure and define the boundary of a subset of $\mathbb{R}^d$; one of them is given by the notion of *perimeter*, due to De Giorgi, which gives the measure of the so-called essential boundary of the set, a subset of the topological boundary defined in terms of the d-dimensional densities of the set. Let A be a ν^d-measurable subset of $\mathbb{R}^d$. The *d-dimensional density of A at x* is defined by

$$\delta_d(A, x) := \lim_{r \downarrow 0} \frac{\nu^d(A \cap B_r(x))}{b_d r^d},$$

provided that the limit exists. It is clear that $\delta_d(A, x)$ equals 1 for all $x \in \mathrm{int}A$, and 0 for all $x \in \mathrm{int}(A^c)$, while different values can be assumed at the boundary points of A. The set of points where the density is neither 0 nor 1 is called essential boundary. Namely, for every $t \in [0, 1]$ and every ν^d-measurable set $A \subset \mathbb{R}^d$, let $A^t := \{x \in \mathbb{R}^d : \delta_d(A, x) = t\}$; all the sets A^t are Borel sets, and in particular the set $\partial^* A := \mathbb{R}^d \setminus (A^0 \cup A^1)$ is called *essential boundary* of A.

A way to introduce a set of finite perimeter is through the definition of functions of bounded variations; in particular, a measurable set $A \subset \mathbb{R}^d$ is said to have *finite perimeter in an open set $U \subseteq \mathbb{R}^d$*, denoted by $P(A, U)$, if its characteristic function $\mathbf{1}_A$ has bounded variation in U:

$$P(A, U) := \sup\left\{ \int_U \mathbf{1}_A(x) \mathrm{div}\phi(x) \mathrm{d}x \,:\, \phi \in C_c^1(U, \mathbb{R}^d),\ \|\phi\|_\infty \leq 1 \right\} < \infty, \tag{2.1}$$

where $C_c^1(U,\mathbb{R}^d)$ is the set of continuously differentiable functions $\phi : U \to \mathbb{R}^d$ with compact support. It follows that the distributional derivative in U of $\mathbf{1}_A$ is representable by a $\mathbb{R}^d$-valued Radon measure in U, that we denote by $D\mathbf{1}_A$, such that

$$\int_A \mathrm{div}\phi \,\mathrm{d}x = -\sum_{i=1}^{d} \int_U \phi_i \,\mathrm{d}D_i\mathbf{1}_A \qquad \forall \phi \in C_c^1(U,\mathbb{R}^d).$$

The perimeter of A in U may be equivalently defined as the total variation $|D\mathbf{1}_A|$ in U, that is $P(A,U) := |D\mathbf{1}_A|(U)$. More generally, if A has finite perimeter in U, we define $P(A,B) := |D\mathbf{1}_A|(B)$ for any Borel set $B \subseteq U$. General theorems on sets with finite perimeter (see (Ambrosio et al., 2000, §3.5)) guarantee that, if A has finite perimeter in an open set $U \subseteq \mathbb{R}^d$, then the measures $|D\mathbf{1}_A|$ and $\mathcal{H}^{d-1}(\partial^* A \cap \cdot)$ coincide on the Borel subsets of U; as a consequence, the perimeter measure can be computed in terms of the $\mathcal{H}^{d-1}$ measure, and in particular the following equalities can be proved:

$$P(A,B) = \mathcal{H}^{d-1}(\partial^* A \cap B) = \mathcal{H}^{d-1}(A^{1/2} \cap B).$$

It follows in particular that the perimeter of A is invariant under modification of a set of Lebesgue measure zero. For sets with finite perimeter in $U = \mathbb{R}^d$ we will write $P(A)$ instead of $P(A,\mathbb{R}^d)$, and we will say that A has finite perimeter.
Finally, noticing that $\partial^* A \subseteq \partial A$, it holds $P(A) \le \mathcal{H}^{d-1}(\partial A)$; such an inequality holds without any regularity or topological assumption on A. More generally, Theorem 3.61 in Ambrosio et al. (2000) states that any subset of $\mathbb{R}^d$ of finite perimeter has d-dimensional density either 0 or 1 or 1/2 at $\mathcal{H}^{d-1}$-almost every point of its boundary; therefore

$$\mathcal{H}^{d-1}(\partial A) = P(A) + \mathcal{H}^{d-1}(\partial A \cap A^0) + \mathcal{H}^{d-1}(\partial A \cap A^1). \tag{2.2}$$

Similarly to (2.1), for any unit vector $u \in S^{d-1}$ one may define the *directional variation in U in the direction u* of $\mathbf{1}_A$:

$$V_u(A,U) := \sup\left\{ \int_U \mathbf{1}_A(x)\langle \nabla\phi(x), u\rangle \mathrm{d}x \,:\, \phi \in C_c^1(U,\mathbb{R}),\ \|\phi\|_\infty \le 1 \right\},$$

where $\langle \nabla\phi(x), u\rangle$ is the classical directional derivative of ϕ in the direction u.
The directional distributional derivative in U of $\mathbf{1}_A$ in the direction $u \in S^{d-1}$ is representable by a signed Radon measure in U, that we denote by $D_u 1_A$, such that

$$\int_A \langle \nabla\phi(x), u\rangle \mathrm{d}x = -\int_U \phi \,\mathrm{d}D_u\mathbf{1}_A \qquad \forall \phi \in C_c^1(U,\mathbb{R});$$

if $P(A,U) < \infty$, then $V_u(A,U) = |D_u\mathbf{1}_A|(U) < \infty$.
As above, we write $V_u(A)$ for $V_u(A,\mathbb{R}^d)$. We refer to Galerne (2011, 2016), Kiderlen and Rataj (2018) for further insights; in particular, by (Galerne, 2011, Prop. 8) and Eq.s (3) and (4) in Galerne (2016), we recall here that $P(A,\cdot)$ is finite if and only if $V_u(A,\cdot)$ is finite for all $u \in S^{d-1}$, and that

$$P(A,B) = \frac{1}{2b_{d-1}} \int_{S^{d-1}} V_u(A,B)\mathcal{H}^{d-1}(\mathrm{d}u) \qquad \forall \text{ Borel B} \subseteq U, \tag{2.3}$$

with

$$\sup_{u \in S^{d-1}} V_u(A,B) \le P(A,B). \tag{2.4}$$

Hence, we may interpret the directional variation $V_u(A)$ as the projection of the perimeter of A along the direction u, and, vice versa, the perimeter as half the average of the directional variation along all the possible directions (half just because opposite unit vectors $u \in S^{d-1}$ give the same contribution).
A useful approximation of the directional variation of any measurable set $A \subset \mathbb{R}^d$ with $\nu^d(A) < \infty$ can be given in terms of the so-called *covariogram* $g_A : \mathbb{R}^d \to [0, +\infty)$, defined as $g_A(y) := \nu^d(A \cap (y + A))$. We refer to Galerne (2011) for a more detailed discussion and nice properties; here we recall that if A has finite perimeter, then

$$V_u(A) = 2 \lim_{r \to 0} \frac{g_A(0) - g_A(ru)}{|r|} = \lim_{r \to 0} \int_{\mathbb{R}^d} \frac{|\mathbf{1}_A(x + ru) - \mathbf{1}_A(x)|}{|r|} \mathrm{d}x \tag{2.5}$$

exists and is finite for all $u \in S^{d-1}$ (see (Galerne, 2011, Th. 13), and also Galerne (2016) for generalizations to functions with locally bounded variation).

In what follows, we shall also deal with sets with $\mathcal{H}^{d-1}$-rectifiable topological boundary. Namely, we say that a compact set $S \subset \mathbb{R}^d$ is $d-1$*-rectifiable* if it is representable as the image $f(K)$ of a compact set $K \subset \mathbb{R}^{d-1}$, with $f : \mathbb{R}^{d-1} \to \mathbb{R}^d$ Lipschitz; S is called *countably* $\mathcal{H}^{d-1}$*-rectifiable* if there exist countably many $d-1$-dimensional Lipschitz maps $f_i : \mathbb{R}^{d-1} \to \mathbb{R}^d$ such that $S \setminus \bigcup_i f_i(\mathbb{R}^{d-1})$ is $\mathcal{H}^{d-1}$-negligible; if, furthermore, $\mathcal{H}^{d-1}(S) < \infty$, then S is called $\mathcal{H}^{d-1}$*-rectifiable*.
Moreover, in our terminology, a compact set A has Lipschitz boundary if for every boundary point a there exists a neighborhood $\mathcal{U}$ of a, a rotation R in $\mathbb{R}^d$ and a Lipschitz function $f : \mathbb{R}^{d-1} \to \mathbb{R}$ such that $R(A \cap \mathcal{U}) = \{(x, y) \in (\mathbb{R}^{d-1} \times \mathbb{R}) \cap R(\mathcal{U}) \,:\, y \geq f(x)\}$, i.e. $A \cap \mathcal{U}$ is the epigraph of a Lipschitz function; in particular, A is d-dimensional, and ∂A is $d-1$-rectifiable.

Another useful notion from Geometric Measure theory related to the perimeter of a compact $A \subset \mathbb{R}^d$ is the so-called *outer Minkowski content* $\mathcal{SM}(A)$ of A, introduced in Ambrosio et al. (2008) and further investigated in Villa (2009). It is the quantity defined by

$$\mathcal{SM}(A) := \lim_{r \downarrow 0} \frac{\nu^d(A_{\oplus r} \setminus A)}{r}, \tag{2.6}$$

provided that the limit exists and is finite. A class of sets stable under finite unions, for which the outer Minkowski content exists and equals the perimeter of the involved sets, is provided in Ambrosio et al. (2008); such a class contains, for instance, all sets with Lipschitz boundary. More generally, it is proved in Villa (2009) that, if ∂A satisfies certain general regularity assumptions (satisfied, for instance, if ∂A is $(d-1)$-rectifiable and bounded), then

$$\mathcal{SM}(A) = P(A) + 2\mathcal{H}^{d-1}(\partial A \cap A^0). \tag{2.7}$$

2.2. Point processes and random sets

For an exhaustive treatment of point processes, and for a unified theory on germ-grain models, we refer to Daley and Vere-Jones (2003, 2008), and to Matheron (1975), Molchanov (2005), Schneider and Weil (2008), respectively. Here we only recall some basic facts and definitions.

A point process in $\mathbb{R}^d$, say $\widetilde{\Phi}$, is a locally finite collection $\{X_i\}_{i \in \mathbb{N}}$ of random points; more formally $\widetilde{\Phi}$ is a random counting measure, that is a measurable map from a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ into the space of locally finite counting measures on $\mathbb{R}^d$. The measure $\widetilde{\Lambda}(A) := \mathbb{E}[\widetilde{\Phi}(A)]$ on $\mathcal{B}_{\mathbb{R}^d}$ is called *intensity measure* of $\widetilde{\Phi}$. A *marked point process* in $\mathbb{R}^d$ with *marks* in a complete and separable metric space $\mathbf{K}$ is a collection $\Phi = \{(X_i, K_i)\}_{i \in \mathbb{N}}$ of random points X_i in $\mathbb{R}^d$, each one associated with a mark $K_i \in \mathbf{K}$, with the property that the unmarked process $\{\widetilde{\Phi}(B) : B \in \mathcal{B}_{\mathbb{R}^d}\} := \{\Phi(B \times \mathbf{K}) : B \in \mathcal{B}_{\mathbb{R}^d}\}$ is a point process in $\mathbb{R}^d$. The intensity measure of Φ, say Λ, is a σ-finite measure on $\mathcal{B}_{\mathbb{R}^d\times\mathbf{K}}$ defined as $\Lambda(B\times L):=\mathbb{E}[\Phi(B\times L)]$. A common assumption (e.g., see Hug and Last (2000) and (Schneider and Weil, 2008, Sec. 11.1)) is the existence of a measurable function $f:\mathbb{R}^d\times\mathbf{K}\to\mathbb{R}_+$ and of a probability measure Q on $\mathbf{K}$, such that $\Lambda(\mathrm{d}(x,k))=f(x,k)\mathrm{d}xQ(\mathrm{d}k)$. Another important measure associated to Φ is the *second factorial moment measure*, say $\nu_{[2]}$; it is the measure on $\mathcal{B}_{(\mathbb{R}^d\times\mathbf{K})^2}$ defined by

$$\int f(x_1,k_1,x_2,k_2)\nu_{[2]}(\mathrm{d}(x_1,k_1,x_2,k_2))=\mathbb{E}\Big[\sum_{\substack{(X_i,K_i),\,(X_j,K_j)\,\in\Phi,\\ X_i\neq X_j}} f(X_i,K_i,X_j,K_j)\Big],$$

for any non-negative measurable function f on $(\mathbb{R}^d\times\mathbf{K})^2$ (e.g., see Stoyan et al. (1995)). Similarly to Λ, we shall assume that there exist a measurable function $g:(\mathbb{R}^d\times\mathbf{K})^2\to\mathbb{R}_+$ and a probability measure $Q_{[2]}$ on $\mathbf{K}^2$ such that

$$\nu_{[2]}(\mathrm{d}(x_1,k_1,x_2,k_2))=g(x_1,k_1,x_2,k_2)\mathrm{d}x_1\mathrm{d}x_2Q_{[2]}(\mathrm{d}(k_1,k_2)). \tag{2.8}$$

A *random closed set* Θ in $\mathbb{R}^d$ is a measurable map $\Theta:(\Omega,\mathfrak{F},\mathbb{P})\longrightarrow(\mathbb{F},\sigma_{\mathbb{F}})$. Here $\mathbb{F}$ denotes the class of the closed subsets in $\mathbb{R}^d$, and $\sigma_{\mathbb{F}}$ is the σ-algebra generated by the so-called *Fell topology*, or *hit-or-miss topology*, that is the topology generated by the set system $\{\mathcal{F}_G:G\in\mathcal{G}\}\cup\{\mathcal{F}^C:C\in\mathcal{C}\}$, where $\mathcal{G}$ and $\mathcal{C}$ are the system of the open and compact subsets of $\mathbb{R}^d$, respectively, while $\mathcal{F}_G:=\{F\in\mathbb{F}:F\cap G\neq\emptyset\}$ and $\mathcal{F}^C:=\{F\in\mathbb{F}:F\cap C=\emptyset\}$ (e.g., see Matheron (1975)).
For the measurability of $\mathcal{H}^{d-1}(\partial\Theta)$ under rectifiability assumptions on $\partial\Theta$ we refer to Zähle (1982); for the measurability of $P(\Theta)$ and of $V_u(\Theta)$ we refer to (Galerne, 2011, p. 47). Whenever Θ is a random set with locally finite boundary measure, such that the measures $\mathbb{E}[\mathcal{H}^{d-1}(\partial\Theta\cap\cdot)]$, $\mathbb{E}[P(\Theta,\cdot)]$ and $\mathbb{E}[V_u(\Theta,\cdot)]$ are well defined and absolutely continuous w.r.t. ν^d, we denote by $\lambda_{\partial\Theta}(x)$, $\lambda_{\partial^*\Theta}(x)$ and $\lambda_{V_u,\Theta}(x)$ their respective (Radon-Nikodym) densities.
The random set Θ is said to be stationary if its probability law is invariant under translation; in such a case $\lambda_{\partial^*\Theta}(x)$, $\lambda_{V_u,\Theta}(x)$ and $\lambda_{\partial\Theta}(x)$ are constant and named *specific variation* (or specific perimeter), *specific directional variation* and *specific surface*, respectively.

Any random closed set Θ in $\mathbb{R}^d$ given by a random union of compact random sets (*particles*) can be represented as a *germ-grain model* driven by a marked point process in $\mathbb{R}^d$ with marks in the space $\mathcal{C}^d\setminus\{\emptyset\}$ of non-empty compact subsets of $\mathbb{R}^d$. As a matter of fact, chosen a suitable *center map* $c:\mathcal{C}^d\setminus\{\emptyset\}\to\mathbb{R}^d$, each particle C can be associated with a pair (x,C') such that $C=x+C'$; $x:=c(C)$ can be interpreted as the "centre" of C (*germ*), and $C':=C-x$ the "shape" (*grain*).
For instance, $c(C)$ might be choosen to be the circumcenter of C (e.g., see (Baddeley et al., 2007, p. 192) and (Schneider and Weil, 2008, Sec. 4.1)); as a consequence Θ may be described by a suitable marked point process Φ in $\mathbb{R}^d$ with marks in $\mathbf{K}:=\{C\in\mathcal{C}^d\setminus\{\emptyset\}\,:\,c(C)=0\}$, so that

$$\Theta=\bigcup_{(X_i,Z_i)\in\Phi}X_i+Z_i. \tag{2.9}$$

The intensity measure Λ of Φ is commonly assumed to be such that the mean number of grains hitting any compact subset of $\mathbb{R}^d$ is finite; this is equivalent to say that the mean number of grains hitting the ball $B_R(0)$ is finite for any $R>0$:

$$\mathbb{E}\Big[\sum_{(X_i,Z_i)\in\Phi}\mathbf{1}_{(-Z_i)_{\oplus R}}(X_i)\Big]=\int_{\mathbb{R}^d\times\mathbf{K}}\mathbf{1}_{(-z)_{\oplus R}}(x)\Lambda(\mathrm{d}(x,z))<\infty\qquad\forall R>0. \tag{2.10}$$

Such a condition guarantees that Θ is closed (see Schneider and Weil (2008)). The equality in the above equation follows by the well known Campbell formula for marked point processes (e.g., see Baddeley

et al. (2007)).
Whenever Φ is a marked Poisson point process, Θ is called *Boolean model*; whenever the grains Z_i are independent of the point process $\{X_i\}$ and i.i.d. as Z_0, the latter is called *typical grain*, the intensity measure Λ of Φ is then of the type $\Lambda(\mathrm{d}(x,z)) = f(x)\mathrm{d}xQ(\mathrm{d}z)$, and Q is called *mark distribution*.
Moreover, $\mathrm{disc} f$ will denote the set of all the points of discontinuity of f, whereas $\mathbb{E}_Q$ the expectation with respect to Q.

All the results concerning the covariogram of deterministic measurable sets can be adapted to random closed sets (e.g., see (Galerne, 2011, Prop. 16), and (Galerne, 2016, Prop. 1) for the more general case of random fields with bounded variation); in particular, the *mean covariogram* of Θ is defined as $\gamma_\Theta(y) := \mathbb{E}[g_\Theta(y)]$, and the *mean version* of (2.3) and of (2.5) hold:

$$\mathbb{E}[P(\Theta,B)] = \frac{1}{2b_{d-1}} \int_{S^{d-1}} \mathbb{E}[V_u(\Theta,B)]\mathcal{H}^{d-1}(\mathrm{d}u) \qquad \forall B \in \mathcal{B}_{\mathbb{R}^d}, \tag{2.11}$$

and

$$\mathbb{E}[V_u(\Theta)] = 2\lim_{r\to 0} \frac{\gamma_\Theta(0) - \gamma_\Theta(ru)}{|r|}. \tag{2.12}$$

3. Generalization to the non-stationary case

As stated in the Introduction, the main aim of this paper is to generalize some results proved in Galerne (2011) for Boolean models to the non-stationary case. More precisely, from (Galerne, 2011, Th. 17) we know that, if Θ is a stationary random closed set with locally finite perimeter almost surely, then

$$\lambda_{\partial^*\Theta} \equiv \frac{1}{2b_{d-1}} \int_{S^{d-1}} \lambda_{V_u;\Theta}\mathcal{H}^{d-1}(\mathrm{d}u) \in \mathbb{R},$$

with

$$\lambda_{V_u;\Theta} \equiv 2\lim_{r\to 0} \frac{\mathbb{P}(ru\in\Theta, 0\notin\Theta)}{|r|} \in \mathbb{R} \qquad \forall u \in S^{d-1}.$$

In particular, if Θ is a stationary Boolean model with intensity α and typical grain Z_0 with distribution Q, from (Galerne, 2011, Prop. 19) we know that

$$\lambda_{V_u;\Theta} \equiv \alpha\mathbb{E}_Q[V_u(Z_0)]e^{-\alpha\mathbb{E}_Q[\nu^d(Z_0)]} \tag{3.1}$$

and

$$\lambda_{\partial^*\Theta} \equiv \alpha\mathbb{E}_Q[P(Z_0)]e^{-\alpha\mathbb{E}_Q[\nu^d(Z_0)]}. \tag{3.2}$$

Assumptions. Let Θ be a germ-grain model in $\mathbb{R}^d$ as in (2.9), where Φ has intensity measure $\Lambda(\mathrm{d}(x,z)) = f(x,z)\mathrm{d}xQ(\mathrm{d}z)$ satisfying (2.10), and such that

(A1) $\mathbb{E}_Q[P(Z)] := \int_{\mathbf{K}} P(z)Q(\mathrm{d}z) < \infty$;
(A2) for any $z \in \mathbf{K}$, $\mathcal{H}^{d-1}(\mathrm{disc}(f(\cdot,z))) = 0$, and $f(\cdot,z)$ is locally bounded such that for any relatively compact $B \subset \mathbb{R}^d$

$$\sup_{x\in B_{\oplus \mathrm{diam}(z)}} f(x,z) \le \tilde{\xi}_B(z) \tag{3.3}$$

for some $\tilde{\xi}_B(z)$ with $\int_{\mathbf{K}} \tilde{\xi}_B(z)P(z)Q(\mathrm{d}z) = c_B < \infty$.

Note that the assumption (A1) says, in particular, that any grain of Θ has finite perimeter Q-almost surely. By remembering that $P(A) \le \mathcal{H}^{d-1}(\partial A)$, a sufficient condition for (A1), sometimes easier to handle in practice, is $\mathbb{E}_Q[\mathcal{H}^{d-1}(\partial Z)] < \infty$.
The assumption (3.3) is a technical condition on f which will guarantee some uniformity results useful in the sequel. It is worth noting that it is trivially satisfied by (A1), whenever f is bounded, or $f(\cdot\, z)$ is locally bounded and $\text{diam}(z) \le D \in \mathbb{R}_+$ for Q-a.e. $z \in \mathbf{K}$.
Finally, by noticing that $\mathbf{1}_{-z_{\oplus R}}(x) \le \mathbf{1}_{B_{R+\text{diam}(z)}(0)}(x)$, a sufficient condition for the validity of (2.10) is $\int_{\mathbf{K}} \widetilde{\xi}_{B_R(0)}(z)(R+\text{diam}(z))^d Q(\text{d}z) < \infty$; in particular, (2.10) is trivially satisfied if f is bounded and $\int_{\mathbf{K}} \text{diam}(z)^d Q(\text{d}z) < \infty$.
In order to improve the readability of the proofs of our main theorems, we first state some preliminary results; among these, Theorem 3.1 below will provide the first step in finding out an explicit formula in the Boolean model case, and in treating more general germ-grain models.

3.1. Some preliminary results

In this section, we state some preliminary results useful for the sequel, referring to the Appendix for the proofs of some analytical statements.

First of all, let us remind that a local version of (2.5) is provided in (Galerne, 2016, Th. 1); in particular, together with (Galerne, 2016, Lemma 1), if A is a set of finite perimeter in an open set $U \subseteq \mathbb{R}^d$, it holds

$$\int_{U\ominus[0,ru]} \frac{|\mathbf{1}_A(x+ru)-\mathbf{1}_A(x)|}{|r|}\text{d}x \le V_u(A,U) = \lim_{r\to 0}\int_{U\ominus[0,ru]} \frac{|\mathbf{1}_A(x+ru)-\mathbf{1}_A(x)|}{|r|}\text{d}x \tag{3.4}$$

for all $u \in S^{d-1}$.
Note that the above integral is taken over $U \ominus [0, ru]$ in order to guarantee the well-posedness in case $\mathbf{1}_A$ has bounded variation in U, and not in U^c. We also point out that if $B_1 \supseteq B_2$, then $A \ominus B_1 \subseteq A \ominus B_2$; moreover, if $0 \in B$, then $A \ominus B = \{a \in A : a + B \subseteq A\} \subseteq A$. In particular, $A \ominus [0, ru] = \{a \in A : [a, a + ru] \subseteq A\}$.
The role of A here, will be played by the grains $Z_i(\omega)$ in the sequel. Thus, let A be a compact set in $\mathbb{R}^d$ with finite perimeter; then it is easy to prove (see Lemma A.6 in the Appendix) that, for any compact $A \in \mathcal{B}_{\mathbb{R}^d}$ and $u \in S^{d-1}$,

$$\lim_{r\to 0} \mathbf{1}_{A\ominus[0,ru]}(x) = \mathbf{1}_A(x) \qquad \forall x \notin \partial A, \tag{3.5}$$

and

$$|\mathbf{1}_A(x+ru)-\mathbf{1}_A(x)| \le \mathbf{1}_{(\partial A)_{\oplus|r|}}(x) \qquad \forall x \in \mathbb{R}^d. \tag{3.6}$$

Furthermore, for any $f : \mathbb{R}^d \to \mathbb{R}_+$ locally bounded with $\mathcal{H}^{d-1}(\text{disc} f) = 0$, it holds (see Proposition A.8 in the Appendix)

$$\int_{\mathbb{R}^d} f(x)V_u(A,\text{d}x) = \lim_{r\to 0}\int_{\mathbb{R}^d} \frac{|\mathbf{1}_A(x+ru)-\mathbf{1}_A(x)|}{|r|} f(x)\text{d}x. \tag{3.7}$$

(Note that, here, we require $\mathcal{H}^{d-1}(\text{disc} f) = 0$, so that $V_u(A, \text{disc} f) = 0$ for any A with finite perimeter).

Theorem 3.1. *Let Θ be a germ-grain model in $\mathbb{R}^d$ as in the above assumptions. Then, for ν^d-a.e. $x \in \mathbb{R}^d$ the densities $\lambda_{\partial^*\Theta}(x)$ and $\lambda_{V_u,\Theta}(x)$ exist for any $u \in S^{d-1}$, and it holds*

$$\lambda_{\partial^*\Theta}(x) = \frac{1}{2b_{d-1}} \int_{S^{d-1}} \lambda_{V_u,\Theta}(x)\mathcal{H}^{d-1}(\mathrm{d}u), \qquad \nu^d\text{-a.e. } x \in \mathbb{R}^d. \tag{3.8}$$

Moreover, by denoting $v_{\Theta;x}(y) := \mathbb{P}(x + y \in \Theta, x \notin \Theta) + \mathbb{P}(x + y \notin \Theta, x \in \Theta)$, for all $u \in S^{d-1}$ it holds

$$\liminf_{r\to 0} \frac{v_{\Theta;x}(ru)}{|r|} \le \lambda_{V_u;\Theta}(x) \le \limsup_{r\to 0} \frac{v_{\Theta;x}(ru)}{|r|} \qquad \text{for } \nu^d\text{-a.e. } x \in \mathbb{R}^d. \tag{3.9}$$

Proof. By the subadditivity property of the perimeter (e.g see (Ambrosio et al., 2000, p. 144)), for all $B \in \mathcal{B}_{\mathbb{R}^d}$, we have

$$\begin{aligned}
\mathbb{E}[P(\Theta, B)] &\le \mathbb{E}\Big[\sum_{(X_i,Z_i)\in\Phi} P(X_i + Z_i, B)\Big] \\
&= \int_{\mathbb{R}^d\times\mathbf{K}} P(x + z, B)\Lambda(\mathrm{d}(x, z)) \\
&\le \int_{\mathbb{R}^d\times\mathbf{K}} \mathcal{H}^{d-1}(\partial(x + z) \cap B)\Lambda(\mathrm{d}(x, z)) \\
&= \int_{\mathbb{R}^d\times\mathbf{K}} \Big(\int_{x+\partial z} \mathbf{1}_B(y)\mathcal{H}^{d-1}(\mathrm{d}y)\Big) f(x, z)\mathrm{d}xQ(\mathrm{d}z) \\
&= \int_B \Big(\int_{\mathbf{K}}\int_{\partial z} f(\xi - u, z)\mathcal{H}^{d-1}(\mathrm{d}u)Q(\mathrm{d}z)\Big)\mathrm{d}\xi,
\end{aligned}$$

where the last equality follows along the same lines as in the proof of (Villa, 2014, Prop. 5), by suitable changes of variables: $y \longrightarrow u = y - x$ first, and $x \longrightarrow \xi = x + u$ then. Therefore, $\mathbb{E}[P(\Theta, \cdot)]$ is absolutely continuous with respect to the Lebesgue measure, being $\mathbb{E}[P(\Theta, B)] = 0$ for any B such that $\nu^d(B) = 0$. By (2.4) we have $V_u(\Theta, \cdot) \le \sup_{u\in S^{d-1}} V_u(\Theta, \cdot) \le P(\Theta, \cdot)$, and so, for any $u \in S^{d-1}$, the measures $\mathbb{E}[V_u(\Theta, \cdot)]$ are absolutely continuous w.r.t. ν^d as well.
Let us denote the density of $\mathbb{E}[P(\Theta, \cdot)]$ and of $\mathbb{E}[V_u(\Theta, \cdot)]$ w.r.t ν^d by $\lambda_{\partial^*\Theta}$ and by $\lambda_{V_u,\Theta}$, respectively. Hence, the following chain of equalities holds for all $B \in \mathcal{B}_{\mathbb{R}^d}$:

$$\begin{aligned}
\int_B \lambda_{\partial^*\Theta}(x)\mathrm{d}x &= \mathbb{E}[P(\Theta, B)] \overset{(2.11)}{=} \frac{1}{2b_{d-1}} \int_{S^{d-1}} \mathbb{E}[V_u(\Theta, B)]\mathcal{H}^{d-1}(\mathrm{d}u) \\
&= \frac{1}{2b_{d-1}} \int_{S^{d-1}} \Big(\int_B \lambda_{V_u,\Theta}(x)\mathrm{d}x\Big)\mathcal{H}^{d-1}(\mathrm{d}u) = \int_B \Big(\frac{1}{2b_{d-1}} \int_{S^{d-1}} \lambda_{V_u,\Theta}(x)\mathcal{H}^{d-1}(\mathrm{d}u)\Big)\mathrm{d}x,
\end{aligned}$$

and so the Eq. (3.8).

Without loss of generality, let us consider $B \subset \mathbb{R}^d$ open and bounded; we get

$$\int_B \lambda_{V_u,\Theta}(x)\mathrm{d}x = \mathbb{E}[V_u(\Theta, B)] \overset{(3.4)}{=} \mathbb{E}\Big[\lim_{r\to 0} \int_{B\ominus[0,ru]} \frac{|\mathbf{1}_\Theta(x + ru) - \mathbf{1}_\Theta(x)|}{|r|}\mathrm{d}x\Big].$$

Thanks to the left hand side of (3.4), we can apply the dominated convergence theorem in order to exchange limit and expectation; then, by Fubini's theorem and the very definition of $v_{\Theta;x}(ru)$, we get

$$\int_B \lambda_{V_u,\Theta}(x)\mathrm{d}x = \lim_{r\to 0}\int_B \mathbf{1}_{B\ominus[0,ru]}(x)\frac{\mathbb{E}[|\mathbf{1}_\Theta(x+ru)-\mathbf{1}_\Theta(x)|]}{|r|}\mathrm{d}x$$
$$= \lim_{r\to 0}\int_B \mathbf{1}_{B\ominus[0,ru]}(x)\frac{\mathbb{P}(x\in\Theta-ru, x\notin\Theta)+\mathbb{P}(x\notin\Theta-ru, x\in\Theta)}{|r|}\mathrm{d}x$$
$$= \lim_{r\to 0}\int_B \mathbf{1}_{B\ominus[0,ru]}(x)\frac{v_{\Theta;x}(ru)}{|r|}\mathrm{d}x. \quad (3.10)$$

By Fatou lemma and by (3.5), we easily get

$$\lim_{r\to 0}\int_B \mathbf{1}_{B\ominus[0,ru]}(x)\frac{v_{\Theta;x}(ru)}{|r|}\mathrm{d}x \geq \int_B \liminf_{r\to 0}\frac{v_{\Theta;x}(ru)}{|r|}\mathrm{d}x;$$

therefore, the assertion (3.9) follows if we show that

$$\lim_{r\to 0}\int_B \mathbf{1}_{B\ominus[0,ru]}(x)\frac{v_{\Theta;x}(ru)}{|r|}\mathrm{d}x \leq \int_B \limsup_{r\to 0}\frac{v_{\Theta;x}(ru)}{|r|}\mathrm{d}x$$

holds as well.
To this aim, let us observe that $\Theta - ru = \bigcup_{(Y_i,Z_i)\in\Phi}(Y_i+Z_i-ru)$, and

$$\mathbb{P}(x\in\Theta-ru, x\notin\Theta) \leq \mathbb{P}\Big(\sum_{(Y_i,Z_i)\in\Phi}\mathbf{1}_{(Y_i+Z_i-ru)\setminus(Y_i+Z_i)}(x)\geq 1\Big)$$
$$\leq \mathbb{E}\Big[\sum_{(Y_i,Z_i)\in\Phi}\mathbf{1}_{(Y_i+Z_i-ru)\setminus(Y_i+Z_i)}(x)\Big] = \int_{\mathbb{R}^d\times\mathbf{K}}\mathbf{1}_{(y+z-ru)\setminus(y+z)}(x)\Lambda(\mathrm{d}(y,z)).$$

Similary, we have $\mathbb{P}(x\notin\Theta-ru, x\in\Theta) \leq \int_{\mathbb{R}^d\times\mathbf{K}}\mathbf{1}_{(y+z)\setminus(y+z-ru)}(x)\Lambda(\mathrm{d}(y,z))$.
Hence, by noticing that

$$\mathbf{1}_{(y+z-ru)\setminus(y+z)}(x)+\mathbf{1}_{(y+z)\setminus(y+z-ru)}(x) = |\mathbf{1}_{y+z-ru}(x)-\mathbf{1}_{y+z}(x)| = |\mathbf{1}_{x-z+ru}(y)-\mathbf{1}_{x-z}(y)|,$$

and that for any $x\in B$ and $|r|<1$

$$\sup_{y\in(x-\partial z)_{\oplus|r|}} f(y,z) \leq \sup_{y\in(B\oplus B_1(0))_{\oplus\mathrm{diam}(z)}} f(y,z) \overset{(3.3)}{\leq} \widetilde{\xi}_{B_{\oplus 1}}(z),$$

we get

$$\frac{v_{\Theta;x}(ru)}{|r|} \leq \int_{\mathbf{K}}\Big(\int_{\mathbb{R}^d}\frac{|\mathbf{1}_{x-z+ru}(y)-\mathbf{1}_{x-z}(y)|}{|r|}f(y,z)\mathrm{d}y\Big)Q(\mathrm{d}z)$$
$$\overset{(3.6)}{\leq} \int_{\mathbf{K}}\Big(\int_{\mathbb{R}^d}\big(\sup_{y\in(x-\partial z)_{\oplus|r|}} f(y,z)\big)\frac{|\mathbf{1}_{x-z+ru}(y)-\mathbf{1}_{x-z}(y)|}{|r|}\mathrm{d}y\Big)Q(\mathrm{d}z)$$
$$\leq \int_{\mathbf{K}}\widetilde{\xi}_{B_{\oplus 1}}(z)\Big(\int_{\mathbb{R}^d}\frac{|\mathbf{1}_{x-z+ru}(y)-\mathbf{1}_{x-z}(y)|}{|r|}\mathrm{d}y\Big)Q(\mathrm{d}z)$$
$$\overset{(3.4)}{\leq} \int_{\mathbf{K}}\widetilde{\xi}_{B_{\oplus 1}}(z)V_u(x-z)Q(\mathrm{d}z)$$

$$\leq \int_{\mathbf{K}} \widetilde{\xi}_{B_{\oplus 1}}(z) P(z) Q(\mathrm{d}z) \overset{(A2)}{=} c_{B_{\oplus 1}} < \infty.$$

Hence, for any $|r| < 1$, we have $\mathbf{1}_{B \ominus [0,ru]}(x) \frac{\nu_{\Theta;x}(ru)}{|r|} \leq \mathbf{1}_{B \ominus [0,ru]}(x) c_{B_{\oplus 1}} \leq c_{B_{\oplus 1}} \in \mathbb{R}$, which is trivially integrable on B. Therefore we may apply the reverse Fatou lemma:

$$\limsup_{r \to 0} \int_B \mathbf{1}_{B \ominus [0,ru]}(x) \frac{\nu_{\Theta;x}(ru)}{|r|} \mathrm{d}x \leq \int_B \limsup_{r \to 0} \mathbf{1}_{B \ominus [0,ru]}(x) \frac{\nu_{\Theta;x}(ru)}{|r|} \mathrm{d}x = \int_B \limsup_{r \to 0} \frac{\nu_{\Theta;x}(ru)}{|r|} \mathrm{d}x,$$

which establishes the assertion (3.9). □

We thus obtain easily the following result:

Corollary 3.2. *Under the assumptions of the above theorem, if the limit* $\lim_{r \to 0} \nu_{\Theta;x}(ru)/|r|$ *exists and is finite for* ν^d*-a.e.* $x \in \mathbb{R}^d$*, then it equals* $\lambda_{V_u;\Theta}(x)$.

Remark 3.3 (Stationary case). The stationary case is much easier to handle, and so it deserves a discussion in order to point out the main difference with respect to the general case. If Θ is stationary, then $\Lambda(\mathrm{d}(x, z)) = cQ(\mathrm{d}z)$ for some $c > 0$, and so the assumption (A2) is trivially satisfied. By the proof of the above theorem and by the subsequent corollary, it is evident that, in the general non-stationary case, the main problem is the existence of the $\lim_{r \to 0} \nu_{\Theta;x}(ru)/|r|$. Indeed, if it exists and is finite, Theorem 3.1 guarantees that we may exchange limit and integral in (3.10), and we may state that it equals $\lambda_{V_u;\Theta}(x)$ ν^d-a.e. In the stationary case such problem vanishes, in accordance with the results in Galerne (2011); indeed, the stationarity of Θ implies that $\Theta + x$ has the same distribution as Θ for any $x \in \mathbb{R}^d$, and $\mathbb{E}[V_u(\Theta, B)] = c\nu^d(B)$ with $c = c(u) > 0$ for any $B \in \mathcal{B}_{\mathbb{R}^d}$. Therefore $\lambda_{V_u,\Theta}(x) \equiv c$ and $\nu_{\Theta;x}(ru) = \nu_{\Theta;0}(ru)$ for all $x \in \mathbb{R}^d$; by this we get

$$c\nu^d(B) \overset{(3.10)}{=} \lim_{r \to 0} \frac{\nu_{\Theta;0}(ru)}{|r|} \nu^d(B \ominus [0, ru]) = \lim_{r \to 0} \frac{\nu_{\Theta;0}(ru)}{|r|} \nu^d(B).$$

Hence, we conclude that $\lim_{r \to 0} \nu_{\Theta;0}(ru)/|r|$ exists and is equal to $\lambda_{V_u,\Theta}(x)$, and, in particular, by observing that

$$\mathbb{P}(ru \notin \Theta, 0 \in \Theta) = \mathbb{P}(ru \notin \Theta) - \mathbb{P}(ru \notin \Theta, 0 \notin \Theta) = \mathbb{P}(0 \notin \Theta) - \mathbb{P}(ru \notin \Theta, 0 \notin \Theta) = \mathbb{P}(0 \notin \Theta, ru \in \Theta),$$

it holds

$$\lambda_{V_u,\Theta}(x) \equiv \lim_{r \to 0} \frac{\nu_{\Theta;0}(ru)}{|r|} = 2 \lim_{r \to 0} \frac{\mathbb{P}(ru \in \Theta, 0 \notin \Theta)}{|r|} \in \mathbb{R}_+ \qquad \forall x \in \mathbb{R}^d. \tag{3.11}$$

Actually, (3.11) coincides with Eq. (12) in Galerne (2016), where the variation intensity of more general stationary random fields is considered.

3.2. The Boolean model case

Let us pass to consider the Boolean model case. Such a particular germ-grain model plays a central role in Stochastic Geometry because it is the most tractable model, being its capacity functional $T_\Theta(K) := \mathbb{P}(\Theta \cap K \neq \emptyset)$ explicitly known. This, together with the independence property of the grains, will allow us to compute explicitly $\nu_{\Theta;x}(ru)$ in order to apply Corollary 3.2.

Theorem 3.4. *Let Θ be a Boolean model satisfying the assumptions above. Then, for all $u \in S^{d-1}$ and ν^d-a.e. $x \in \mathbb{R}^d$,*

$$\lambda_{V_u;\Theta}(x) = \lim_{r\to 0} \frac{v_{\Theta;x}(ru)}{|r|} = \exp\Big\{-\int_{\mathbf{K}}\int_{x-z} f(y,z)\mathrm{d}yQ(\mathrm{d}z)\Big\} \int_{\mathbf{K}}\int_{\mathbb{R}^d} f(y,z)V_u(x-z,\mathrm{d}y)Q(\mathrm{d}z), \tag{3.12}$$

and

$$\lambda_{\partial^*\Theta}(x) = \exp\Big\{-\int_{\mathbf{K}}\int_{x-z} f(y,z)\mathrm{d}yQ(\mathrm{d}z)\Big\}\cdot \\ \cdot \int_{\mathbf{K}}\Big(\frac{1}{2b_{d-1}}\int_{S^{d-1}}\int_{\mathbb{R}^d} f(y,z)V_u(x-z,\mathrm{d}y)\mathcal{H}^{d-1}(\mathrm{d}u)\Big)Q(\mathrm{d}z). \tag{3.13}$$

Proof. For any $x \in \mathbb{R}^d$, $u \in S^{d-1}$, and $r \geq 0$, let $\mathcal{Z}^{x,ru}$ be the subset of $\mathbb{R}^d \times \mathbf{K}$ defined by

$$\mathcal{Z}^{x,ru} := \{(y,z) \in \mathbb{R}^d \times \mathbf{K} : x \in y+z-ru\} = \{(y,z) \in \mathbb{R}^d \times \mathbf{K} : y \in x-z+ru\}.$$

Note that, by the independent increments property of Φ,

$$\mathbb{P}(x \in \Theta - ru, x \notin \Theta) = \mathbb{P}(\Phi(\mathcal{Z}^{x,ru}) > 0, \Phi(\mathcal{Z}^{x,0}) = 0) \\ = \mathbb{P}(\Phi(\mathcal{Z}^{x,ru} \setminus \mathcal{Z}^{x,0}) > 0, \Phi(\mathcal{Z}^{x,0}) = 0) = e^{-\Lambda(\mathcal{Z}^{x,0})}(1 - e^{-\Lambda(\mathcal{Z}^{x,ru}\setminus\mathcal{Z}^{x,0})}),$$

so that we have

$$v_{\Theta;x}(ru) = \mathbb{P}(\Phi(\mathcal{Z}^{x,ru}) > 0, \Phi(\mathcal{Z}^{x,0}) = 0) + \mathbb{P}(\Phi(\mathcal{Z}^{x,ru}) = 0, \Phi(\mathcal{Z}^{x,0}) > 0) \\ = e^{-\Lambda(\mathcal{Z}^{x,0})}(1 - e^{-\Lambda(\mathcal{Z}^{x,ru}\setminus\mathcal{Z}^{x,0})}) + e^{-\Lambda(\mathcal{Z}^{x,ru})}(1 - e^{-\Lambda(\mathcal{Z}^{x,0}\setminus\mathcal{Z}^{x,ru})}).$$

Let us observe that

$$e^{-\Lambda(\mathcal{Z}^{x,0})} = \exp\Big\{-\int_{\mathbf{K}}\int_{x-z} f(y,z)\mathrm{d}yQ(\mathrm{d}z)\Big\},$$

and

$$\lim_{r\to 0} e^{-\Lambda(\mathcal{Z}^{x,ru})} = \exp\Big\{-\lim_{r\to 0}\int_{\mathbf{K}}\int_{x-z+ru} f(y,z)\mathrm{d}yQ(\mathrm{d}z)\Big\} = e^{-\Lambda(\mathcal{Z}^{x,0})}.$$

If we prove that

$$\lim_{r\to 0} \frac{\Lambda(\mathcal{Z}^{x,ru}\setminus\mathcal{Z}^{x,0}) + \Lambda(\mathcal{Z}^{x,0}\setminus\mathcal{Z}^{x,ru})}{|r|} = \int_{\mathbf{K}}\int_{\mathbb{R}^d} f(y,z)V_u(x-z,\mathrm{d}y)Q(\mathrm{d}z), \tag{3.14}$$

then we may conclude that

$$\lim_{r\to 0} \frac{v_{\Theta;x}(ru)}{|r|} = e^{-\Lambda(\mathcal{Z}^{x,0})} \lim_{r\to 0} \frac{(1 - e^{-\Lambda(\mathcal{Z}^{x,ru}\setminus\mathcal{Z}^{x,0})}) + (1 - e^{-\Lambda(\mathcal{Z}^{x,0}\setminus\mathcal{Z}^{x,ru})})}{|r|} \\ = e^{-\Lambda(\mathcal{Z}^{x,0})}\int_{\mathbf{K}}\int_{\mathbb{R}^d} f(y,z)V_u(x-z,\mathrm{d}y)Q(\mathrm{d}z),$$

which would provide (3.12) by Corollary 3.2.
In order to prove the limit in (3.14), let us notice that $\mathcal{Z}^{x,ru} \setminus \mathcal{Z}^{x,0} = \{(y,z) \in \mathbb{R}^d \times \mathbf{K} : y \in (x-z+ru) \setminus (x-z)\}$ (and similarly for $\mathcal{Z}^{x,0} \setminus \mathcal{Z}^{x,ru}$); therefore,

$$\begin{aligned}\frac{\Lambda(\mathcal{Z}^{x,ru} \setminus \mathcal{Z}^{x,0}) + \Lambda(\mathcal{Z}^{x,0} \setminus \mathcal{Z}^{x,ru})}{|r|} \\ = \frac{1}{|r|}\int_{\mathbf{K}}\Big(\int_{(x-z+ru)\setminus(x-z)} f(y,z)\mathrm{d}y + \int_{(x-z)\setminus(x-z+ru)} f(y,z)\mathrm{d}y\Big)Q(\mathrm{d}z) \\ = \int_{\mathbf{K}}\Big(\int_{\mathbb{R}^d} \frac{|\mathbf{1}_{x-z+ru}(y) - \mathbf{1}_{x-z}(y)|}{|r|} f(y,z)\mathrm{d}y\Big)Q(\mathrm{d}z). \end{aligned} \tag{3.15}$$

By the assumption (A2) we have that $f(\,\cdot\,,z)$ is locally bounded with $\mathcal{H}^{d-1}(\mathrm{disc}(f(\cdot,z))) = 0$, therefore

$$\lim_{r\to 0}\int_{\mathbb{R}^d} \frac{|\mathbf{1}_{x-z+ru}(y) - \mathbf{1}_{x-z}(y)|}{|r|} f(y,z)\mathrm{d}y \overset{(3.7)}{=} \int_{\mathbb{R}^d} f(y,z)V_u(x-z,\mathrm{d}y). \tag{3.16}$$

Along the same lines as in the end of the proof of Theorem 3.1, we may claim that

$$\int_{\mathbb{R}^d} \frac{|\mathbf{1}_{x-z+ru}(y) - \mathbf{1}_{x-z}(y)|}{|r|} f(y,z)\mathrm{d}y \leq \widetilde{\xi}_{B_1(x)}(z)P(z) \qquad \forall |r| < 1,$$

which is Q-integrable on $\mathbf{K}$ thanks to the assumption (A2). Therefore, by the dominated convergence theorem, we can exchange limit and integral below and conclude

$$\begin{aligned}\lim_{r\to 0}\frac{\Lambda(\mathcal{Z}^{x,ru} \setminus \mathcal{Z}^{x,0}) + \Lambda(\mathcal{Z}^{x,0} \setminus \mathcal{Z}^{x,ru})}{|r|} \\ \overset{(3.15)}{=} \lim_{r\to 0}\int_{\mathbf{K}}\Big(\int_{\mathbb{R}^d} \frac{|\mathbf{1}_{x-z+ru}(y) - \mathbf{1}_{x-z}(y)|}{|r|} f(y,z)\mathrm{d}y\Big)Q(\mathrm{d}z) \\ = \int_{\mathbf{K}}\Big(\lim_{r\to 0}\int_{\mathbb{R}^d} \frac{|\mathbf{1}_{x-z+ru}(y) - \mathbf{1}_{x-z}(y)|}{|r|} f(y,z)\mathrm{d}y\Big)Q(\mathrm{d}z) \\ \overset{(3.16)}{=} \int_{\mathbf{K}}\int_{\mathbb{R}^d} f(y,z)V_u(x-z,\mathrm{d}y),\end{aligned}$$

that is the limit in (3.14), which proves (3.12).
Finally, Eq. (3.13) easily follows by (3.8). □

Remark 3.5 (Stationary case). If Θ is a stationary Boolean model with typical grain Z_0 satisfying the assumption (A1), then Theorem 3.4 applies with $f(x,z) \equiv \alpha > 0$; thus Eq. (3.1) may be seen as a particular case of (3.12), whereas we recover Eq. (3.2) by (3.13) as follows:

$$\begin{aligned}\lambda_{\partial^*\Theta}(x) = e^{-\alpha\mathbb{E}_Q[\nu^d(Z_0)]}\int_{\mathbf{K}}\Big(\frac{1}{2b_{d-1}}\int_{S^{d-1}} \alpha V_u(z)\mathcal{H}^{d-1}(\mathrm{d}u)\Big)Q(\mathrm{d}z) \\ \overset{(2.3)}{=} \alpha e^{-\alpha\mathbb{E}_Q[\nu^d(Z_0)]}\mathbb{E}_Q[P(Z_0)].\end{aligned}$$

3.3. General germ-grain model

Before stating the main result of this section, we recall the notion of *reduced Palm version*, and of *generating functional* of a point process, which will appear in Theorem 3.6 below. (We refer to Daley and Vere-Jones (2008) and Coeurjolly et al. (2017) for further insights).

Let Ψ be a point process on a complete separable metric space $(E, \mathcal{B}_E)$ with σ-finite intensity measure Λ_Ψ, and $(\mathbf{N}_E, \mathcal{N}_E)$ be the measurable space of the locally finite counting measures on E. There exists a unique σ-finite measure C_Ψ on $E \times \mathbf{N}_E$ characterized by $C_\Psi(B \times L) = \mathbb{E}[\Psi(B)\mathbf{1}_L(\Psi)]$ for any $B \in \mathcal{B}_E$ and $L \in \mathcal{N}_E$. C_Ψ is called Campbell measure of Ψ, and it can be disintegrated as $C_\Psi(\mathrm{d}(x,\varphi)) = \mathcal{P}_x(\mathrm{d}\varphi)\Lambda_\Psi(\mathrm{d}x)$, where $\mathcal{P}_\cdot(\,\cdot\,)$ is a probability kernel from E to $\mathbf{N}_E$. Thus, in particular, $\mathcal{P}_x(\,\cdot\,)$ is a probability measure on $\mathbf{N}_E$, called *Palm distribution of* Ψ *at* x, and so it can be seen as the probability distribution of a point process, say Ψ_x, often called the *Palm version of* Ψ *at* x. It can be shown that the point process Ψ_x has almost surely an atom in x; then, the point process $\Psi^!_x := \Psi_x - \varepsilon_x$ (where ε_x is the Dirac measure at point x), equivalently denoted also by $\Psi_x \setminus \{x\}$, is called *reduced Palm version of* Ψ *at* x. The probability law of $\Psi^!_x$ is denoted by $\mathcal{P}^!_x$, and it is called *reduced Palm distribution of* Ψ *at* x. It can be shown that

$$\mathbb{E}\Big[\sum_{X \in \Psi} h(X, \Psi \setminus \{X\})\Big] = \int_{E \times \mathbf{N}_E} h(x,\varphi)\mathcal{P}^!_x(\mathrm{d}\varphi)\Lambda_\Psi(\mathrm{d}x) \tag{3.17}$$

for any measurable function $h : E \times \mathbf{N}_E \to \mathbb{R}_+$.

Finally, we recall that the *generating functional* G_Ψ *of* Ψ is defined by $G_\Psi[v] := \mathbb{E}[\prod_{X \in \Psi} v(X)]$ for any measurable function $v : E \to [0,1]$.

It is well known that $G_\Psi[v] = \exp\{\int_E (v(x) - 1)\Lambda(\mathrm{d}x)\}$, whenever Ψ is a Poisson point process.

With the aim of dealing with germ-grain models more general than the Boolean model, we are going to introduce a further assumption on the second moment density (2.8) of Φ; in order to avoid a too technical condition on g, analogous to (A2) on f (see Remark 3.8 below), we shall assume that g is bounded and the grains are uniformly bounded in $\mathbb{R}^d$ (that is $\mathrm{diam}(X + Z) \leq D \; \forall (X,Z) \in \Phi$). Such assumption is analogous to the assumption (A3) in Villa (2014), where the mean density of lower dimensional general germ-grain models is studied. (Note that in Villa (2014) the grains are assumed to be identified by some parameters in a suitable space, so that the role of $Z(s)$ there, is played by z here.)

In the following, $\mathbf{N}$ denotes the space of locally finite counting measures in $\mathbb{R}^d \times \mathbf{K}$; furthermore, to simplify the notation, we denote $\underline{x} := (x,z) \in \mathbb{R}^d \times \mathbf{K}$ and $\underline{y} := (y,w) \in \mathbb{R}^d \times \mathbf{K}$.

Theorem 3.6. *Let* Θ *be a germ-grain model satisfying the assumptions above and the following further condition:*

(A3) $\mathrm{diam}(X + Z) \leq D \in \mathbb{R}$ *for all* $(X,Z) \in \Phi$*, and the second factorial moment measure of* Φ *is of the type*

$$\nu_{[2]}(\mathrm{d}(x_1,z_1,x_2,z_2)) = g(x_1,z_1,x_2,z_2)\mathrm{d}x_1\mathrm{d}x_2 Q_{[2]}(\mathrm{d}(z_1,z_2)),$$

with g *bounded,* $\mathcal{H}^{d-1}(\mathrm{disc}(g(\,\cdot\,,z_1,x_2,z_2))) = 0$ *for any* $(z_1,x_2,z_2) \in \mathbf{K} \times \mathbb{R}^d \times \mathbf{K}$*, and* $\int_{\mathbf{K}^2} P(z_1)Q_{[2]}(\mathrm{d}(z_1,z_2)) < \infty$.

Let $G_{\Phi^!_{\underline{x}}}$ *be the generating functional of the reduced Palm version* $\Phi^!_{\underline{x}}$ *of* Φ*, and let* $\eta_{r,\xi,u} : \mathbb{R}^d \times \mathbf{K} \to \{0,1\}$ *be the function defined, for any* $(r,\xi) \in \mathbb{R} \times \mathbb{R}^d$*,* $u \in S^{d-1}$*, by*

$$\eta_{r,\xi,u}(y,w) := \mathbf{1}_{(\xi - w + ru)^c}(y)\mathbf{1}_{(\xi - w)^c}(y).$$

If there exists

$$\lim_{r\to 0}\int_{\mathbb{R}^d}\frac{|\mathbf{1}_{\xi-z+ru}(x)-\mathbf{1}_{\xi-z}(x)|}{|r|}G_{\Phi^!_{\underline{x}}}[\eta_{r,\xi,u}]f(x,z)\mathrm{d}x =: F_{\xi,u}(z) \quad \textit{for } Q\textit{-a.e. } z\in\mathbf{K},$$

then

$$\lim_{r\to 0}\frac{\nu_{\Theta;\xi}(ru)}{|r|}=\int_{\mathbf{K}}F_{\xi,u}(z)Q(\mathrm{d}z). \tag{3.18}$$

Remark 3.7. Let us notice that $\eta_{r,\xi,u}(\underline{y})\to\eta_{0,\xi}(\underline{y})$ as $r\to 0$, and that $\prod_{\underline{Y}\in\Phi^!_{\underline{x}}}\eta_{r,\xi,u}(\underline{Y})\le 1$; then $G_{\Phi^!_{\underline{x}}}[\eta_{r,\xi,u}]\to G_{\Phi^!_{\underline{x}}}[\eta_{0,\xi}]$ as $r\to 0$. Therefore, if the reduced Palm distribution $\mathcal{P}^!_{\underline{x}}$ of Φ does not depend on the point $\underline{x}$, say $\mathcal{P}^!_{\underline{x}}=P$ for some probability measure P on $\mathbf{N}$, then

$$F_{\xi,u}(z)=G_P[\eta_{0,\xi}]\lim_{r\to 0}\int_{\mathbb{R}^d}\frac{|\mathbf{1}_{\xi-z+ru}(x)-\mathbf{1}_{\xi-z}(x)|}{|r|}f(x,z)\mathrm{d}x=G_P[\eta_{0,\xi}]\int_{\mathbb{R}^d}f(x,z)V_u(\xi-z,\mathrm{d}x),$$

where G_P is the generating function associated to the point process with distribution P.

It is well known by the celebrated Slivnyak's theorem, that $\Phi^!_{\underline{x}}=\Phi$ for Λ-a.e. $\underline{x}\in\mathbb{R}^d\times\mathbf{K}$ if Φ is a Poisson point process. Therefore, if Θ is a Boolean model, we get

$$G_{\Phi^!_{\underline{x}}}[\eta_{0,\xi}]=G_\Phi[\eta_{0,\xi}]=e^{\int(\eta_{0,\xi}-1)\mathrm{d}\Lambda}=\exp\Big\{-\int_{\mathbb{R}^d\times\mathbf{K}}\mathbf{1}_{y+w}(\xi)f(y,w)\mathrm{d}yQ(\mathrm{d}w)\Big\},$$

and so $F_{\xi,u}(z)$ above exists and is finite; then, by (3.18), we recover

$$\lim_{r\to 0}\frac{\nu_{\Theta;\xi}(ru)}{|r|}=\exp\Big\{-\int_{\mathbf{K}}\int_{\xi-w}f(y,w)\mathrm{d}yQ(\mathrm{d}w)\Big\}\int_{\mathbf{K}}\int_{\mathbb{R}^d}f(x,z)V_u(\xi-z,\mathrm{d}x),$$

in accordance with the equation (3.12) in Theorem 3.4. (About this, we recall that $\nu_{[2]}=\Lambda\times\Lambda$ whenever Φ is a Poisson point process, and so

$$g(x_1,z_1,x_2,z_2)\mathrm{d}x_1\mathrm{d}x_2Q_{[2]}(\mathrm{d}(z_1,z_2))=f(x_1,z_1)f(x_2,z_2)\mathrm{d}x_1\mathrm{d}x_2Q(\mathrm{d}z_1)Q(\mathrm{d}z_2);$$

therefore the assumption (A3) is equivalent to the assumption (A2) if, for sake of simplificity, we assume f bounded and $\mathrm{diam}(X+Z)\le D\in\mathbb{R}$ for all $(X,Z)\in\Phi$.)

Among the processes whose Palm reduced distribution $\mathcal{P}^!_{\underline{x}}$ does not depend on $\underline{x}$, we mention the Binomial point process and the mixed Poisson point process (e.g., see (Baddeley et al., 2007, p. 47). Further examples of point processes with known reduced Palm distribution can be found in (Coeurjolly et al., 2017, Sec. 5).

Proof of Theorem 3.6. Let $\xi\in\mathbb{R}^d$ and $u\in S^{d-1}$.
First of all, let us prove that

$$\mathbb{E}\Big[\sum_{\substack{(X_i,Z_i),(X_j,Z_j)\in\Phi,\\ X_i\neq X_j}}|\mathbf{1}_{X_i+Z_i-ru}(\xi)-\mathbf{1}_{X_i+Z_i}(\xi)||\mathbf{1}_{X_j+Z_j-ru}(\xi)-\mathbf{1}_{X_j+Z_j}(\xi)|\Big]=o(|r|), \tag{3.19}$$

from which we shall deduce that the probability that the point $\xi\in\mathbb{R}^d\setminus\Theta$ belongs to more than one translated grain $X+Z+ru$ goes to 0 faster that $|r|$ for any $u\in S^{d-1}$.

To this aim, let us notice that

$$\frac{1}{|r|}\mathbb{E}\Big[\sum_{\substack{(X_i,Z_i),(X_j,Z_j)\in\Phi,\\ X_i\neq X_j}} |\mathbf{1}_{X_i+Z_i-ru}(\xi)-\mathbf{1}_{X_i+Z_i}(\xi)||\mathbf{1}_{X_j+Z_j-ru}(\xi)-\mathbf{1}_{X_j+Z_j}(\xi)|\Big]$$
$$=\frac{1}{|r|}\int_{(\mathbb{R}^d\times\mathbf{K})^2}|\mathbf{1}_{x_1+z_1-ru}(\xi)-\mathbf{1}_{x_1+z_1}(\xi)||\mathbf{1}_{x_2+z_2-ru}(\xi)-\mathbf{1}_{x_2+z_2}(\xi)|\nu_{[2]}(\mathrm{d}(x_1,z_1,x_2,z_2))$$
$$=\int_{\mathbb{R}^d\times\mathbf{K}\times\mathbf{K}}|\mathbf{1}_{x_2+z_2-ru}(\xi)-\mathbf{1}_{x_2+z_2}(\xi)|$$
$$\Big(\int_{\mathbb{R}^d} g(x_1,z_1,x_2,z_2)\frac{|\mathbf{1}_{x_1+z_1-ru}(\xi)-\mathbf{1}_{x_1+z_1}(\xi)|}{|r|}\mathrm{d}x_1\Big)\mathrm{d}x_2 Q_{[2]}(\mathrm{d}(z_1,z_2)).$$

The condition (A3) guarantees that the assumptions of Proposition A.8 are fulfilled; therefore,

$$\lim_{r\to 0}\int_{\mathbb{R}^d} g(x_1,z_1,x_2,z_2)\frac{|\mathbf{1}_{x_1+z_1-ru}(\xi)-\mathbf{1}_{x_1+z_1}(\xi)|}{|r|}\mathrm{d}x_1$$
$$=\lim_{r\to 0}\int_{\mathbb{R}^d} g(x_1,z_1,x_2,z_2)\frac{|\mathbf{1}_{\xi-z_1+ru}(x_1)-\mathbf{1}_{\xi-z_1}(x_1)|}{|r|}\mathrm{d}x_1$$
$$=\int_{\mathbb{R}^d} g(x_1,z_1,x_2,z_2)V_u(\xi-z_1,\mathrm{d}x_1),$$

where the last integral is finite being g bounded, say $g(x_1,z_1,x_2,z_2)\leq K$. Moreover, we have that

$$\lim_{r\to 0}|\mathbf{1}_{x_2+z_2-ru}(\xi)-\mathbf{1}_{x_2+z_2}(\xi)|\overset{(3.6)}{\leq}\lim_{r\to 0}\mathbf{1}_{\xi-(\partial z_2)_{\oplus|r|}}(x_2)=0\quad\text{for } \nu^d\text{-a.e.}\, x_2\in\mathbb{R}^d,\ \forall z_2\in\mathbf{K},$$

being ∂z_2 lower dimensional. Finally, for any $|r|<1$,

$$|\mathbf{1}_{x_2+z_2-ru}(\xi)-\mathbf{1}_{x_2+z_2}(\xi)|\Big(\int_{\mathbb{R}^d} g(x_1,z_1,x_2,z_2)\frac{|\mathbf{1}_{x_1+z_1-ru}(\xi)-\mathbf{1}_{x_1+z_1}(\xi)|}{|r|}\mathrm{d}x_1\Big)$$
$$\leq K\mathbf{1}_{\xi-(\partial z_2)_{\oplus 1}}(x_2)\int_{\mathbb{R}^d}\frac{|\mathbf{1}_{\xi-z_1+ru}(x_1)-\mathbf{1}_{\xi-z_1}(x_1)|}{|r|}\mathrm{d}x_1\leq K\mathbf{1}_{B_{1+D}(\xi)}(x_2)P(z_1),\quad (3.20)$$

with $\int_{\mathbb{R}^d\times\mathbf{K}\times\mathbf{K}}K\mathbf{1}_{B_{1+D}(\xi)}(x_2)P(z_1)\mathrm{d}x_2Q_{[2]}(\mathrm{d}(z_1,z_2))<\infty$ by the assumption (A3).
The dominated converge theorem allows us to exchange limit and integral in

$$\lim_{r\to 0}\int_{(\mathbb{R}^d\times\mathbf{K})^2}|\mathbf{1}_{x_1+z_1-ru}(\xi)-\mathbf{1}_{x_1+z_1}(\xi)|\frac{|\mathbf{1}_{x_2+z_2-ru}(\xi)-\mathbf{1}_{x_2+z_2}(\xi)|}{|r|}\nu_{[2]}(\mathrm{d}(x_1,z_1,x_2,z_2)),$$

and then we get (3.19).
From this we may state that $\mathbb{P}\Big(\sum_{\underline{X}\in\Phi}\mathbf{1}_{X+Z-ru}(\xi)\geq 2,\xi\notin\Theta\Big)=o(|r|)$; then,

$$\mathbb{P}(\xi\in\Theta-ru,\xi\notin\Theta)=\mathbb{P}\Big(\sum_{\underline{X}\in\Phi}\mathbf{1}_{X+Z-ru}(\xi)\geq 1,\prod_{\underline{X}\in\Phi}\mathbf{1}_{(X+Z)^c}(\xi)=1\Big)$$

$$= \mathbb{P}\Big(\sum_{\underline{X}\in\Phi} \mathbf{1}_{X+Z-ru}(\xi)=1, \prod_{\underline{X}\in\Phi} \mathbf{1}_{(X+Z)^c}(\xi)=1\Big) + o(|r|) = \mathbb{P}(Y_r(\xi)=1) + o(|r|),$$

where

$$Y_r(\xi) := \sum_{\underline{X}\in\Phi} \Big[\mathbf{1}_{X+Z-ru}(\xi)\mathbf{1}_{(X+Z)^c}(\xi) \prod_{\underline{Y}\in\Phi\setminus\{\underline{X}\}} \mathbf{1}_{(Y+W-ru)^c}(\xi)\mathbf{1}_{(Y+W)^c}(\xi)\Big].$$

Let $h : \mathbb{R}^d \times \mathbf{K} \times \mathbf{N} \to \{0,1\}$ be the function defined by

$$h(\underline{x},\varphi) := \mathbf{1}_{(x+z-ru)\setminus(x+z)}(\xi) \prod_{\underline{y}\in\varphi} \mathbf{1}_{(y+w-ru)^c}(\xi)\mathbf{1}_{(y+w)^c}(\xi);$$

by noticing that $Y_r(\xi) \in \{0,1\}$, the following chain of equalities holds:

$$\mathbb{P}(Y_r(\xi)=1) = \mathbb{E}[(Y_r(\xi)] = \mathbb{E}\Big[\sum_{\underline{X}\in\Phi} h(\underline{X}, \Phi\setminus\{\underline{Y}\})\Big] \overset{(3.17)}{=} \int_{\mathbb{R}^d\times\mathbf{K}} \int_{\mathbf{N}} h(\underline{x},\varphi)\mathcal{P}^!_{\underline{x}}(\mathrm{d}\varphi)\Lambda(\mathrm{d}\underline{x})$$

$$= \int_{\mathbb{R}^d\times\mathbf{K}} \mathbf{1}_{(x+z-ru)\setminus(x+z)}(\xi)\Big(\int_{\mathbf{N}} \prod_{\underline{y}\in\varphi} \mathbf{1}_{(y+w-ru)^c}(\xi)\mathbf{1}_{(y+w)^c}(\xi))\mathcal{P}^!_{\underline{x}}(\mathrm{d}\varphi)\Big)\Lambda(\mathrm{d}\underline{x})$$

$$= \int_{\mathbb{R}^d\times\mathbf{K}} \mathbf{1}_{(x+z-ru)\setminus(x+z)}(\xi) G_{\Phi^!_{\underline{x}}}[\eta_{r,\xi,u}]\Lambda(\mathrm{d}\underline{x}).$$

In analogous way one gets

$$\mathbb{P}(\xi\in\Theta, \xi\notin\Theta - ru) = \int_{\mathbb{R}^d\times\mathbf{K}} \mathbf{1}_{(x+z)\setminus(x+z-ru)}(\xi) G_{\Phi^!_{\underline{x}}}[\eta_{r,\xi,u}]\Lambda(\mathrm{d}\underline{x}) + o(|r|),$$

so that

$$\lim_{r\to 0} \frac{\nu_{\Theta;\xi}(ru)}{|r|} = \lim_{r\to 0} \frac{\mathbb{P}(\xi\in\Theta-ru, \xi\notin\Theta) + \mathbb{P}(\xi\in\Theta, \xi\notin\Theta-ru)}{|r|} =$$

$$\lim_{r\to 0} \int_{\mathbf{K}}\int_{\mathbb{R}^d} \frac{|\mathbf{1}_{\xi-z+ru}(x) - \mathbf{1}_{\xi-z}(x)|}{|r|} G_{\Phi^!_{\underline{x}}}[\eta_{r,\xi,u}] f(x,z)\mathrm{d}x Q(\mathrm{d}z).$$

Note that $G_{\Phi^!_{\underline{x}}}[\eta_{r,\xi,u}] \le 1$; then by proceeding along the same lines as in the final part of the proof of Theorem 3.1, we have that

$$\int_{\mathbb{R}^d} \frac{|\mathbf{1}_{\xi-z+ru}(x) - \mathbf{1}_{\xi-z}(x)|}{|r|} G_{\Phi^!_{\underline{x}}}[\eta_{r,\xi,u}] f(x,z)\mathrm{d}x \le \widetilde{\xi}_{B_1(\xi)}(z)P(z) \quad \forall |r|<1,$$

which is Q-integrable on $\mathbf{K}$ thanks to the assumption (A2), hence establishes the result. □

Remark 3.8. In the assumption (A3) above, we assumed g bounded, and the grain uniformly bounded in $\mathbb{R}^d$. Such assumption might be weakened by requiring a similar condition to (3.3), in order to get an integrability condition in (3.20), and to apply the dominated converge theorem subsequently.

4. Remarks

4.1. "One-grain" random set

As widely discussed in Villa (2014) for the computation of the mean boundary density of a full dimensional germ-grain model, it seems to be hard to find explicit expressions for $\lambda_{V_u;\Theta}$ and for $\lambda_{\partial^*\Theta}$ when Θ is a general germ-grain model (i.e., non-Boolean) in terms of its grains. This is evident in Theorem 3.6, unless the reduced Palm version of Φ is explicitly known and sufficiently tractable. The dependence between the grains is the main reason. Intuitively, such a problem might be addressed by considering Θ as an "one-grain" random set; namely, with the same argument that allows to describe particle processes as germ-grain processes, whenever Θ is compact (this is a reasonable assumption in practice) it can be written as $\Theta = X + Z$, where $X := c(\Theta)$ is the circumcenter of Θ, and $Z := \Theta - X$ is its random "shape". In such a case, the marked point process Φ in (2.9) is a very particular case of point process, given by the unique random point (X, Z) in $\mathbb{R}^d \times \mathbf{K}$; as a consequence $g \equiv 0$ in (2.8), and so only the Assumptions (A1) and (A2) have to be satisfied for the validity of Theorem 3.1 and its corollary. We refer the reader to (Villa, 2014, remark 10) and (Villa, 2014, Sec 4.2) for a more exhaustive discussion on the one-grain representation. We remind here that, still denoting by Λ the intensity measure of Φ, $\Lambda(\mathrm{d}(x,z))$ represents the probability that the unique point of Φ is in the infinitesimal region $\mathrm{d}x$ with mark in $\mathrm{d}z$, that is $\Lambda(\mathrm{d}(x,z))$ is the joint probability law of (X,Z). Moreover, being $\Theta = X + Z$, of course $\partial\Theta = X + \partial Z$, so that the regularity properties of $\partial\Theta$ and the perimeter of Θ coincide with the regularity properties of ∂Z and the perimeter of Z, respectively.

Proposition 4.1. *Let $\Theta = X + Z$ be a random closed set described by the random point $(X, Z) \in \mathbb{R}^d \times \mathbf{K}$, and let (X, Z) have joint probability law $\Lambda(\mathrm{d}(x,z)) = f(x,z)\mathrm{d}xQ(\mathrm{d}z)$ satisfying the assumptions (A1) and (A2). Then Eq. (3.8) holds with*

$$\lambda_{V_u;\Theta}(x) = \lim_{r\to 0} \frac{v_{\Theta;x}(ru)}{|r|} = \int_{\mathbf{K}}\int_{\mathbb{R}^d} f(y,z)V_u(x-z,\mathrm{d}y)Q(\mathrm{d}z) \qquad \textit{for } \nu^d\textit{-a.e. } x \in \mathbb{R}^d. \tag{4.1}$$

Proof. Notice that Θ satisfies the hypotheses of Theorem 3.1; therefore, by Corollary 3.2, the assertion follows if we prove that

$$\lim_{r\to 0} \frac{v_{\Theta;x}(ru)}{|r|} = \int_{\mathbf{K}}\int_{\mathbb{R}^d} f(y,z)V_u(x-z,\mathrm{d}y)Q(\mathrm{d}z).$$

In order to do this, it is sufficient to notice that (with the same notation used in the proof of Theorem 3.4)

$$\begin{aligned}
&\mathbb{P}(x \in \Theta - ru, x \notin \Theta) + \mathbb{P}(x \notin \Theta - ru, x \in \Theta) \\
&\qquad = \mathbb{P}((X,Z) \in \mathcal{Z}^{x,ru} \setminus \mathcal{Z}^{x,0}) + \mathbb{P}((X,Z) \in \mathcal{Z}^{x,0} \setminus \mathcal{Z}^{x,ru}) \\
&\qquad\qquad = \Lambda(\mathcal{Z}^{x,ru} \setminus \mathcal{Z}^{x,0}) + \Lambda(\mathcal{Z}^{x,0} \setminus \mathcal{Z}^{x,ru}),
\end{aligned}$$

and then we conclude by (3.14). □

It is quite intuitive that it might be hard to determine the joint probability distribution of (X, Z) in order to evaluate the integral in Eq. (4.1); actually, the importance of the above proposition lies in the equality

$$\lambda_{V_u;\Theta}(x) = \lim_{r\to 0} \frac{\nu_{\Theta;x}(ru)}{|r|},$$

because it is quite simple to estimate $\nu_{\Theta;x}(ru)$ given a random sample $\Theta_1, \dots, \Theta_n$ for Θ. Indeed, in many real applications, the possible realizations of Θ vary continuously in space, so that it is reasonable to suppose that the assumptions of Proposition 4.1 are fulfilled. Hence, a consistent and unbiased estimator of $\nu_{\Theta;x}(ru)$ based on the random sample might be given by

$$W(x,r) := \frac{\sum_{i=1}^n \mathbf{1}_{(\Theta_i - ru)\setminus\Theta_i}(x) + \sum_{i=1}^n \mathbf{1}_{\Theta_i\setminus(\Theta_i - ru)}(x)}{n},$$

and so $W(x,r)/|r|$ would provide an approximate estimate of $\lambda_{V_u;\Theta}(x)$ for r "sufficiently small". From a statistical point of view, it would be then of interest to investigate on the *optimal bandwidth* r which, for instance, minimizes the mean squared error, in the spirit of what has been done in Camerlenghi and Villa (2015) for an analogous estimator. We leave this as an open problem for future works.

4.2. $\mathbb{P}$-continuity of Θ

Different notions of "continuity" of a random closed set are available in the literature; in particular the definitions of $\mathbb{P}$-*continuity* and *a.s. continuity* are given in (Matheron, 1975, Sec. 2.5):

Definition 4.2. A random closed set Θ in $\mathbb{R}^d$ is said to be

- $\mathbb{P}$-*continuous at a point* $x \in \mathbb{R}^d$ if

$$\lim_{y\to x} \mathbb{P}(x \in \Theta, y \notin \Theta) = \lim_{y\to x} \mathbb{P}(x \notin \Theta, y \in \Theta) = 0;$$

- *a.s. continuous at a point* $x \in \mathbb{R}^d$ if $\mathbb{P}(x \in \partial\Theta) = 0$.

The a.s. continuity at a point x implies the $\mathbb{P}$-continuity (see (Matheron, 1975, Prop. 2.5.2)); if we assume that $dim_{\mathcal{H}}(\partial\Theta) = d-1$ $\mathbb{P}$.a.s., having denoted by $dim_{\mathcal{H}}$ the Hausdorff dimension, then we have $0 = \mathbb{E}[\nu^d(\partial\Theta)] = \int_{\mathbb{R}^d} \mathbb{P}(x \in \partial\Theta)\mathrm{d}x$, and so we deduce that Θ is a.s. continuous and $\mathbb{P}$-continuous at ν^d-a.e. $x \in \mathbb{R}^d$. Actually, among these notions of continuity, the $\mathbb{P}$-continuity seems to be the most related with $\nu_{\Theta;x}(ru)$, and it does not involve any regularity assumption on $\partial\Theta$. Indeed, Θ is $\mathbb{P}$-continuous at a point x if $\nu_{\Theta;x}(ru) = o(1)$ for $r \to 0$ (that is $\lim_{r\to 0} \nu_{\Theta;x}(ru) = 0$) for any $u \in S^{d-1}$. Actually, by $\lim_{r\to 0} \frac{\nu_{\Theta;x}(ru)}{|r|} = \lambda_{V_u;\Theta}(x) \in \mathbb{R}$, we have $\nu_{\Theta;x}(ru) = O(|r|)$ for $r \to 0$. Thus, in particular, any Boolean model as in our assumptions is $\mathbb{P}$-continuous at ν^d-a.e. $x \in \mathbb{R}^d$.

4.3. Random sets with finite perimeter and random measurable sets

The notion of *random* (not-necessarily closed) *set with finite perimeter* has been introduced and discussed in Rataj (2015). Namely, a random set with (locally) finite perimeter is a measurable map from a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ into the space of sets with (locally) finite perimeter; the latter, as subspace of the Lebesgue space L^1, is the family of equivalence classes on the family of sets with (locally) finite perimeter, modulo symmetric difference of volume zero. Therefore, two sets whose symmetric difference has ν^d-measure zero are indistinguishable.

Related to the framework of random set with finite perimeter, the notion of *random measurable set* (RAMS) has been introduced in Galerne and Lachièze-Rey (2015), to which we refer for further insights. Roughly speaking, a RAMS is a measurable map from a probability space to the class of Lebesgue measurable subsets of $\mathbb{R}^d$, endowed with the Borel σ-algebra induced by the local convergence in measure, which correspond to the $L^1_{\rm loc}(\mathbb{R}^d)$-topology for the indicator functions. Note that RAMS do not necessarily have finite perimeter; thus they include random closed sets and random sets with finite perimeter as special cases.

Even if the framework of RAMS (and so that of random sets with finite perimeter as well) seems to be the appropriate setting when dealing with the perimeter, we point out that, as discussed in (Kiderlen and Rataj, 2018, Sect. 5), without assuming the closure of the involved random sets, the sets of the type $\{\omega \in \Omega : x \in \Theta(\omega)\}$ (and similarly $\{\omega \in \Omega : \Theta(\omega) \cap K \neq \emptyset\}$, with $K \subset \mathbb{R}^d$ compact) do not make sense if Θ is a RAMS, because $\{x \in \Theta\}$ is not a measurable subset of Ω any more (since $\Theta(\omega)$ is given only up to measure 0). Therefore, the main object of the present paper, $v_{\Theta;x}(y)$, would not make sense. Furthermore, as we shall discuss in the subsequent section, considering random closed sets allows us to compare the notion of (mean) covariogram and that of (mean) outer Minkowski content, which requires the closure of the involved set.
However, it is worth mentioning that for stationary RAMS it might be possible to give sense to $v_{\Theta;0}(y)$ by means of the so-called *shift randomization procedure* described in Kiderlen and Rataj (2018).

4.4. (Mean) covariogram and (mean) outer Minkowski content

We discuss now some similarities and differences between the notion of (mean) covariogram and the notion of (mean) outer Minkowski content in the evaluation of the (mean) boundary measure and of the (mean) perimeter of the involved set.

Let us consider a compact subset A of $\mathbb{R}^d$ with ∂A sufficiently regular so that its outer Minkowski content $\mathcal{SM}(A)$ exists; from (2.6) and (2.7) we have

$$\lim_{r\to 0} \frac{\nu^d(A_{\oplus|r|} \setminus A)}{|r|} = P(A) + 2\mathcal{H}^{d-1}(A^0 \cap \partial A). \tag{4.2}$$

On the other hand, by means of the covariogram notion, from (2.3) and (2.5) we have

$$\frac{1}{b_{d-1}} \int_{S^{d-1}} \lim_{r\to 0} \frac{\nu^d(A \setminus (A + ru))}{|r|} \mathcal{H}^{d-1}(\mathrm{d}u) = P(A). \tag{4.3}$$

Note that $\nu^d(A \setminus (A + ru)) = 0$ for any $u \in S^{d-1}$ if $\nu^d(A) = 0$; whereas $\nu^d(A_{\oplus r} \setminus A)$ is always greater than 0 for all $r > 0$. Hence, the main difference is that $V_u(A)$ does not "see" the boundary points of A with d-dimensional density 0; instead, from (4.2), such points "count twice" in the computation of the outer Minkowski content, with respect to the points of the essential boundary.
In particular:

- If $P(A) = \mathcal{H}^{d-1}(\partial A)$, by taking into account (2.2), we get $\mathcal{H}^{d-1}(A^0 \cap \partial A) = 0$, and so the above equations provide equivalent ways to compute the boundary measure of the set; in this regard we remind that any compact set $A \subset \mathbb{R}^d$ with Lipschitz boundary satisfies $P(A) = \mathcal{H}^{d-1}(\partial A) < \infty$ (e.g., see (Ambrosio et al., 2000, p. 159)). Moreover, limit and integral can be exchanged in (4.3), so that

$$\lim_{r\to 0} \frac{\nu^d(A_{\oplus|r|} \setminus A)}{|r|} = \lim_{r\to 0} \frac{1}{b_{d-1}} \int_{S^{d-1}} \frac{\nu^d(A \setminus (A + ru))}{|r|} \mathcal{H}^{d-1}(\mathrm{d}u) = P(A).$$

- If every point of a closed subset A of $\mathbb{R}^d$ has d-dimensional density equal to 0, then $A = \partial A$; in this case Eq. (4.3) equals 0, while we get the boundary measure of A by (4.2) $= 2\mathcal{H}^{d-1}(A)$.
- Since the perimeter $P(A)$ of A is invariant under modifications by a set of ν^d-measure zero, sometimes it is useful to consider the *good representative* of A, that is the set so defined

$$[A] := A^1 \cup \partial^* A.$$

Then $[A]$ may be considered the class of equivalence of sets with finite perimeter which differ only in points x with d-dimensional density $\delta_d(A,x) = 0$, and it is clear that (4.2) and (4.3) give the same value $P([A])$.

Of course, the above discussion for a deterministic set A can be rephrased also for a random set Θ sufficiently regular, such that there exists its *mean* outer Minkowski content:

$$\lim_{r\downarrow 0} \frac{\mathbb{E}[\nu^d(\Theta_{\oplus r} \setminus \Theta)]}{r} = \mathbb{E}[P(\Theta)] + 2\mathbb{E}[\mathcal{H}^{d-1}(\Theta^0 \cap \partial\Theta)].$$

By (2.11) and (2.12) we have

$$\frac{1}{b_{d-1}} \int_{S^{d-1}} \lim_{r\to 0} \frac{\mathbb{E}[\nu^d(\Theta \setminus (\Theta + ru))]}{|r|} \mathcal{H}^{d-1}(\mathrm{d}u) = \mathbb{E}[P(\Theta)];$$

therefore, if either $\mathbb{E}[P(\Theta)] = \mathbb{E}[\mathcal{H}^{d-1}(\partial\Theta)]$ or Θ is a good representative, the above equations equal the same value.
It is evident that the notion of good representative $[\Theta]$ of Θ might be related to the notion of RAMS recalled in the previous section, as a RAMS a random element in the space of Lebesgue subsets of $\mathbb{R}^d$ modulo differences of Lebesgue measure zero. (We also refer the interested reader to Galerne and Lachièze-Rey (2015) for further insights on stationary RAMS and their related covariogram.)

Let us now turn to the mean boundary densities associated to the random set. We point out that the existence of the Minkowski content of a (compact) set requires rectifiability assumptions; moreover, related results in the stochastic case require stronger assumptions than those introduced so far. Therefore in this section we shall assume the following
Stronger Assumptions: let Θ be a germ-grain model in $\mathbb{R}^d$ as in (2.9), where Φ has intensity measure $\Lambda(\mathrm{d}(x,z)) = f(x,z)\mathrm{d}xQ(\mathrm{d}z)$ satisfying (2.10), and such that

(B1) for Q-a.e $z \in \mathbf{K}$, z is a $\mathcal{H}^{d-1}$-rectifiable and compact subset of $\mathbb{R}^d$, such that there exists a closed set $\Xi(z) \supseteq \partial z$ with $\int_{\mathbf{K}} \mathcal{H}^{d-1}(\Xi(z))Q(\mathrm{d}z) < \infty$ and

$$\mathcal{H}^{d-1}(\Xi(z) \cap B_r(x)) \geq \gamma r^{d-1} \qquad \forall x \in \partial z,\ \forall r \in (0,1)$$

for some $\gamma > 0$ independent of z;

(B2) for any $z \in \mathbf{K}$, $\mathcal{H}^{d-1}(\mathrm{disc}(f(\cdot,z))) = 0$ and $f(\cdot,z)$ is locally bounded such that for any relatively compact $B \subset \mathbb{R}^d$

$$\sup_{x\in B_{\oplus \mathrm{diam}(z)}} f(x,z) \leq \tilde{\xi}_B(z) \tag{4.4}$$

for some $\tilde{\xi}_B(z)$ with $\int_{\mathbf{K}} \tilde{\xi}_B(z)\mathcal{H}^{d-1}(\Xi(z))Q(\mathrm{d}z) < \infty$.

Such assumptions are basically the same as those introduced in Villa (2010) and related works on the mean density approximation and estimation of random closed sets (see also Villa (2014)), to which

we refer for a more exhaustive discussion. In a nutshell, we point out here that the assumption (B1) guarantees that the grains admit mean outer Minkowski content, and it is often fulfilled with $\Xi = \partial Z$ or $\Xi = \partial Z \cup \widetilde{A}$ for some sufficiently regular random closed set $\widetilde{A}$; as a matter of fact, it can be seen as the stochastic version of a common assumption in Geometric Measure theory (see (Ambrosio et al., 2000, p. 111)) which guarantees the existence of the Minkowski content of the involved set, extending a classical result by Federer (Federer, 1969, p. 275) to $\mathcal{H}^n$-rectifiable sets. The condition (4.4) is trivially satisfied by (B1) whenever f is bounded, or $f(\cdot z)$ is locally bounded and the $\mathrm{diam}(z) \leq D \in \mathbb{R}_+$ for Q-a.e. $z \in \mathbf{K}$.
It is evident that (B1) and (B2) play here the same role of the assumptions (A1) and (A2) in Section 3; actually, by remembering that $P(Z) \leq \mathcal{H}^{d-1}(\partial Z)$, (A1) and (A2) are fulfilled for any germ grain model satisfying (B1) and (B2).

Let Θ be a Boolean model satisfying the Stronger Assumptions above; then, by (Villa, 2010, Theorem 3.9) and (Villa, 2010, Prop. 3.12) (stated there for Boolean models with f independent of z, but easily generalizable), we have, respectively, that

- denoted by $\lambda_{\Theta^0\cap\partial\Theta}(x)$ the density of the measure $\mathbb{E}[\mathcal{H}^{d-1}(\Theta^0 \cap \partial\Theta \cap \cdot)]$ with respect to ν^d,

$$\sigma_\Theta(x) = \lambda_{\partial^*\Theta}(x) + 2\lambda_{\Theta^0\cap\partial\Theta}(x) \qquad \text{for } \nu^d\text{-a.e. } x \in \mathbb{R}^d,$$

  where σ_Θ is the specifica area of Θ defined in (1.1) and recalled in the Introduction;
- under the further assumption $\mathbb{E}_Q[P(Z)] = \mathbb{E}_Q[\mathcal{H}^{d-1}(\partial Z)]$,

$$\begin{aligned}\sigma_\Theta(x) &= \lambda_{\partial^*\Theta}(x) = \lambda_{\partial\Theta}(x)\\ &= \exp\left\{-\int_{\mathbf{K}}\int_{x-z} f(y,z)\mathrm{d}y Q(\mathrm{d}z)\right\}\int_{\mathbf{K}}\int_{x-\partial^* z} f(y,z)\mathcal{H}^{d-1}(\mathrm{d}y)Q(\mathrm{d}z) \qquad (4.5)\end{aligned}$$

  for ν^d-a.e. $x \in \mathbb{R}^d$.

Therefore, by this and by Theorem 3.4 we can state the following result.

Proposition 4.3. *Let Θ be a Boolean model satisfying the Stronger Assumptions above with $\mathbb{E}_Q[P(Z)] = \mathbb{E}_Q[\mathcal{H}^{d-1}(\partial Z)]$; then, for ν^d-a.e. $x \in \mathbb{R}^d$,*

$$\begin{aligned}\lim_{r\to 0}\frac{\mathbb{P}(x \in \Theta_{\oplus|r|}\setminus\Theta)}{|r|} &= \lambda_{\partial^*\Theta}(x) = \lambda_{\partial\Theta}(x)\\ &= \frac{1}{b_{d-1}}\int_{S^{d-1}}\lim_{r\to 0}\frac{\mathbb{P}(x\in\Theta - ru, 0\notin\Theta) + \mathbb{P}(x\notin\Theta - ru, 0\in\Theta)}{2|r|}\mathcal{H}^{d-1}(\mathrm{d}u); \qquad (4.6)\end{aligned}$$

in particular, it holds

$$\int_{\mathbf{K}}\int_{x-\partial^* z} f(y,z)\mathcal{H}^{d-1}(\mathrm{d}y)Q(\mathrm{d}z) = \int_{\mathbf{K}}\Big(\frac{1}{2b_{d-1}}\int_{S^{d-1}}\int_{\mathbb{R}^d} f(y,z)V_u(x-z,\mathrm{d}y)\mathcal{H}^{d-1}(\mathrm{d}u)\Big)Q(\mathrm{d}z). \qquad (4.7)$$

Remark 4.4. As mentioned in the Introduction, several mean value formulas for inhomogeneous Boolean models are available in the literature. We like to notice the formal similarity between Eq. (3.12) for $\lambda_{V_u;\Theta}(x)$. and Eq. (4.5) for $\lambda_{\partial^*\Theta}(x) = \lambda_{\partial\Theta}(x)$. The latter is in accordance with the formula in (Schneider and Weil, 2008, Theorem 11.1.3) where an inhomogeneous Boolean model of convex grain is considered; indeed, in such case, the condition $\mathbb{E}_Q[P(Z)] = \mathbb{E}_Q[\mathcal{H}^{d-1}(\partial Z)]$ is fulfilled because the topological boundary of a full dimensional convex set coincides with its essential boundary.

The same arguments apply to the case of "one-grain" random set discussed in Section 4.1, as well; namely, by Proposition 4.1 and by (Villa, 2014, Theorem 18), we may state the following result.

Proposition 4.5. *Let $\Theta = X + Z$ be a random closed set described by the random point $(X, Z) \in \mathbb{R}^d \times \mathbf{K}$, and let (X, Z) have joint probability law $\Lambda(\mathrm{d}(x, z)) = f(x, z)\mathrm{d}xQ(\mathrm{d}z)$ satisfying the assumptions (B1) and (B2). Then (4.6) and (4.7) are still valid, with*

$$\lambda_{\partial\Theta}(x) = \lambda_{\partial^*\Theta}(x) = \int_{\mathbf{K}} \int_{x-\partial^* z} f(y, z)\mathcal{H}^{d-1}(\mathrm{d}y)Q(\mathrm{d}z).$$

To conclude, we point out that by means of the covariogram notion (in particular, by means of the evaluation of the limit $\lim_{r\to 0} \frac{v_{\Theta;x}(ru)}{|r|}$), it is possible to get the mean density of the essential boundary $\partial^*\Theta$ of Θ, disregarding any subsets of $\partial\Theta$ with d-dimensional density equal to 0; whereas by means of the Minkowski content notion (in particular, by means of the evaluation of the limit $\lim_{r\to 0} \frac{\mathbb{P}(x\in\Theta_{\oplus|r|}\setminus\Theta)}{|r|}$), it is possible to detect the mean density of $\Theta^0 \cap \partial\Theta$. Whenever $\partial^*\Theta = \partial\Theta$ (for instance, when the grains of Θ have Lipschitz boundary), the two procedures are equivalent.

Appendix

Lemma A.6. *For any compact set $A \subset \mathbb{R}^d$ and $u \in S^{d-1}$ it holds $\lim_{r\to 0} \mathbf{1}_{A\ominus[0,ru]}(x) = \mathbf{1}_A(x)$ for all $x \notin \partial A$, and $|\mathbf{1}_A(x + ru) - \mathbf{1}_A(x)| \le \mathbf{1}_{(\partial A)_{\oplus|r|}}(x)$ for all $x \in \mathbb{R}^d$.*

Proof. Let $x \in A \setminus \partial A$; then $\mathbf{1}_A(x) = 1$ and $x \in \mathrm{int}(A)$. Therefore there exists $R > 0$ such that $B_R(x) \subset A$, and so $[x, x + Ru] \subset A$ for all $u \in S^{d-1}$. From this we can state that $\mathbf{1}_{A\ominus[0,ru]}(x) = 1$ for all $r < R$, and so $\lim_{r\to 0} \mathbf{1}_{A\ominus[0,ru]}(x) = 1$.
Let $x \in A^c$; then $\mathbf{1}_A(x) = 0$. Since $A \ominus [0, ru] \subseteq A$ for all $r > 0$, we have $\mathbf{1}_{A\ominus[0,ru]}(x) \le \mathbf{1}_A(x) = 0$ for all $r > 0$, and so $\lim_{r\to 0} \mathbf{1}_{A\ominus[0,ru]}(x) = 0$.

With reference to the second statement, it is sufficient to notice that $|\mathbf{1}_A(x + ru) - \mathbf{1}_A(x)| = 1$ if and only if $x \in (A - ru) \setminus A$ or $x \in A \setminus (A - ru)$.
Let $x \in A \setminus (A - ru)$. By contradiction, let us suppose that $x \notin (\partial A)_{\oplus|r|}$; then x is an interior point of A with $B_{|r|}(x) \cap \partial A = \emptyset$, so that there exist $y \in B_{|r|}(x)$ interior point of A such that $y = x + ru$. This is equivalent to say that $x = y - ru \in A - ru$, which is absurd. The case $x \in (A - ru) \setminus A$ follows similarly. □

From (Galerne and Lachièze-Rey, 2015, Theorem 1) it follows that the sequence of the signed measures $\{\mu_r\}_r$ defined by

$$\mu_r(\mathrm{d}x) := \frac{\mathbf{1}_A(x + ru) - \mathbf{1}_A(x)}{r}\mathrm{d}x, \qquad r \neq 0,$$

weakly* converges to $D_u\mathbf{1}_A$, if A is a measurable subset of $\mathbb{R}^d$ with finite perimeter.
In particular, we have the following result.

Lemma A.7. *Let $A \subset \mathbb{R}^d$ be a measurable set with finite perimeter. Then the sequence of measures*

$$M_r(\mathrm{d}x) := \frac{|\mathbf{1}_A(x + ru) - \mathbf{1}_A(x)|}{|r|}\mathrm{d}x, \quad r \neq 0,$$

weakly converge to $V_u(A, \cdot) := |D_u\mathbf{1}_A|(\cdot)$ in $\mathbb{R}^d$ for $r \to 0$.*

Proof. Let us notice that, A being of finite perimeter in $\mathbb{R}^d$, and so of finite perimeter in any open $U \subset \mathbb{R}^d$, it holds

$$\liminf_{r\to 0} M_r(U) \geq \liminf_{r\to 0} \int_{U\ominus[0,ru]} \frac{|\mathbf{1}_A(x+ru) - \mathbf{1}_A(x)|}{|r|} \mathrm{d}x$$

$$= \lim_{r\to 0} \int_{U\ominus[0,ru]} \frac{|\mathbf{1}_A(x+ru) - \mathbf{1}_A(x)|}{|r|} \mathrm{d}x \overset{(3.4)}{=} V_u(A,U)$$

for any open $U \subset \mathbb{R}^d$. By (2.5), we have $\lim_{r\to 0} M_r(\mathbb{R}^d) = V_u(A)$. Then the weak* convergence of $\{M_r\}$ to $V_u(A,\cdot)$ follows as a direct application of (Ambrosio et al., 2000, Proposition 1.80), by replacing X with $\mathbb{R}^d$, μ_h with M_r, and μ with $V_u(A\,\cdot)$. □

Thanks to the above convergence result, we get the following proposition; it may be seen as a particular generalization of Eq. (2.5), which is a result interesting in its own right.

Proposition A.8. *Let $A \subset \mathbb{R}^d$ be a relatively compact set with finite perimeter; then for all $u \in S^{d-1}$*

$$\int_{\mathbb{R}^d} f(x) V_u(A, \mathrm{d}x) = \lim_{r\to 0} \int_{\mathbb{R}^d} \frac{|\mathbf{1}_A(x+ru) - \mathbf{1}_A(x)|}{|r|} f(x)\mathrm{d}x \tag{A.8}$$

for any $f : \mathbb{R}^d \to \mathbb{R}_+$ locally bounded with $V_u(A, \mathrm{disc} f) = 0$.

We omit here the proof because it may be done by mimicking that of Theorem 35 in Villa (2010), by taking into account that, as a byproduct of the above lemma,

$$V_u(A,B) = \lim_{r\to 0} \int_B \frac{|\mathbf{1}_A(x+ru) - \mathbf{1}_A(x)|}{|r|} \mathrm{d}x$$

for any continuity set B for $V_u(A,\cdot)$, that is $V_u(A, \partial B) = 0$.
Alternatively, by (Ambrosio et al., 2000, Prop. 1.62 b)) we get that (A.8) holds for any bounded Borel function $f : \mathbb{R}^d \to \mathbb{R}$ with compact support, such that $V_u(A, \mathrm{disc} f) = 0$. Since A is assumed to be relatively compact, for r sufficiently small such condition on f may be weakened in the integrals in (A.8), and f taken locally bounded.

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