Research Paper

Separable MV-algebras and lattice-ordered groups

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A R T I C L E   I N F O

A B S T R A C T

The theory of extensive categories determines in particular the notion of separable MV-algebra (equivalently, of separable unital lattice-ordered Abelian group). We establish the following structure theorem: An MV-algebra is separable if, and only if, it is a finite product of algebras of rational numbers—i.e., of subalgebras of the MV-algebra $[0, 1] \cap \mathbb{Q}$. Beyond its intrinsic algebraic interest, this research is motivated by the long-term programme of developing the algebraic geometry of the opposite of the category of MV-algebras, in analogy with the classical case of commutative K-algebras over a field K.

1. Introduction

For any field K, a (commutative) K-algebra is separable if, and only if, it is a finite product of finite separable field extensions of K. See, for example, [15, Corollary 4.5.8]. The aim of the present paper is to establish the analogue of this fact for MV-algebras

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and lattice-ordered groups. We show as our main result that an MV-algebra is separable exactly when it is a finite product of algebras of rational numbers—the subalgebras of $[0,1] \cap \mathbb{Q}$ (Theorem 10.1). By a well-known theorem of Mundici [33], the category of MV-algebras is equivalent to the category of lattice-ordered Abelian groups with a unit. We frame our treatment in the language of MV-algebras, and postpone to the final Appendix A a synopsis of its translation to lattice-ordered groups.

Separable MV-algebras are an instance of the more general notion of separable object in a coextensive category. The comments that follow, leading to the definition of separable MV-algebra, are intended to motivate this concept.

As explained in [27], some of Grothendieck's algebro-geometric constructions may be abstracted to the context of extensive categories [25,8]. A category $\mathcal{C}$ with finite coproducts is *extensive* if the canonical functor

$$\mathcal{C}/X \times \mathcal{C}/Y \to \mathcal{C}/(X+Y)$$

is an equivalence for every pair of objects X, Y in $\mathcal{C}$. Extensivity attempts to make explicit a most basic property of (finite) coproducts in categories 'of spaces'. For instance, the category Top of topological spaces and continuous functions between them is extensive; the category of groups is not.

Experience indeed confirms that conceiving an extensive category as a category 'of spaces' is a useful conceptual guide. Essential to the development of Algebraic Geometry is the fact that $\mathsf{Ring}^{\mathrm{op}}$, the opposite of the category of (commutative unital) rings, is extensive. (It easily follows that, for any ring R, the opposite of the category R/Ring of R-algebras is extensive.) Extensivity naturally determines a notion of *complemented* subobject. So, in an extensive category with finite products, it is also natural to consider the objects with complemented diagonal. These are traditionally called *decidable objects*, and it is useful to think of them as the 'discrete spaces' inside the category 'of spaces' where they live. For instance, a topological space is decidable if, and only if, it is discrete. For any ring R, and any R-algebra A, let $\mathrm{Spec}\, A$ be the corresponding object in the extensive category $(R/\mathsf{Ring})^{\mathrm{op}}$. Then $\mathrm{Spec}\, A$ is decidable if, and only if, A is separable as an R-algebra. In other words, the separable R-algebras are precisely those for which the associated affine scheme is decidable.

Let us now say that a category is *coextensive* if its opposite is extensive. In light of the above comments, an object in a coextensive category $\mathcal{A}$ is called *separable* if the corresponding object in $\mathcal{A}^{\mathrm{op}}$ is decidable.

The category MV of MV-algebras is coextensive. This provides the notion of separable MV-algebra that is the topic of the present paper. Explicitly, the MV-algebra A is separable if, and only if, there is a homomorphism $f\colon A + A \to A$ such that the span

$$A \xleftarrow{\ \nabla\ } A + A \xrightarrow{\ f\ } A$$

is a product diagram, where $\nabla\colon A + A \to A$ denotes the codiagonal map.

The geometry of $\mathsf{MV}^{\mathrm{op}}$ has long been the subject of intensive hands-on study—sometimes in the language of lattice-ordered groups—because of its striking connections with several areas of classical mathematics, from piecewise-linear topology to the geometry of numbers. See e.g. [30,28,29,4,5] for some examples, and the bibliography in [35] for a more extensive set of references. The characterisation of decidable objects in $\mathsf{MV}^{\mathrm{op}}$ that we present here was motivated by our ongoing long-term project to study the 'gros Zariski' topos determined by the theory of MV-algebras as the domain of a pre-cohesive geometric morphism [26]. We postpone the topos-theoretic consequences of separability to further publications; no Topos Theory is required for the proof of the purely algebraic results in the present paper.

The plan of the paper is as follows. In Sections 2, 3, and 4 we introduce the necessary material to prove a sufficient condition for an extensive category with finite products to have the property that every decidable object is a finite coproduct of connected subterminals. In Section 5 we verify that MV is coextensive. In Theorem 6.9 we characterise the subterminal objects of $\mathsf{MV}^{\mathrm{op}}$ as, in MV, the subalgebras of $[0,1] \cap \mathbb{Q}$. In order to extend Theorem 6.9 to a characterisation of separable MV-algebras we need to introduce the Pierce functor for MV, an analogue of the standard ring-theoretic functor by the same name. The key fact is that the Pierce functor preserves coproducts. To prove it, in Section 7 we develop the required material on the connected-component functor π_0 in Top. Using the theory of spectra of MV-algebras recalled in Section 8 along with the topological π_0 functor, we are able to show in Theorem 9.9 that the Pierce functor does preserve all coproducts. Theorems 6.9 and 9.9 are combined in Section 10 to obtain our main result, the mentioned characterisation of separable MV-algebras. We conclude Section 10 with a discussion that points to further research aimed at enriching the connected-component functor on $\mathsf{MV}^{\mathrm{op}}$ to an 'arithmetic connected-component functor'; this functor, we submit, arises out of locally finite MV-algebras. Finally, in Appendix A we collect the translation of our main results to lattice-ordered groups.

2. Extensive categories and connected objects

In this section we recall the definition of extensive category and of connected object. For more details about extensive categories see, for example, [25,8] and references therein.

A category $\mathcal{C}$ with finite coproducts is called *extensive* if for every X and Y in $\mathcal{C}$ the canonical functor $\mathcal{C}/X \times \mathcal{C}/Y \to \mathcal{C}/(X+Y)$ is an equivalence. Examples of extensive categories are Set (sets and functions), fSet (finite sets and functions), any topos, Top, KHaus (compact Hausdorff spaces and continuous maps), Stone (Stone[1] spaces and continuous maps). The categories of rings, of Boolean algebras and of distributive lattices[2] are coextensive. See [27] and [9] for further examples.
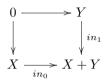

In extensive categories coproduct injections are regular monomorphisms, coproducts of monomorphisms are monomorphisms, and the initial object is *strict* in the sense that any map $X \to 0$ is an isomorphism. Also, extensive categories are closed under slicing.
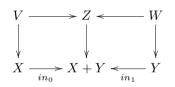
Definition 2.1. A coproduct $in_0 \colon X \to X + Y \leftarrow Y : in_1$ is

(1) *disjoint* if the coproduct injections are monic and the commutative square

$$
\begin{array}{ccc}
0 & \longrightarrow & Y \\
\downarrow & & \downarrow {\scriptstyle in_1} \\
X & \underset{in_0}{\longrightarrow} & X + Y
\end{array}
$$

is a pullback;

(2) *universal* if for every arrow $Z \to X + Y$ the two pullback squares below exist

$$
\begin{array}{ccccc}
V & \longrightarrow & Z & \longleftarrow & W \\
\downarrow & & \downarrow & & \downarrow \\
X & \underset{in_0}{\longrightarrow} & X + Y & \underset{in_1}{\longleftarrow} & Y
\end{array}
$$

and the top cospan is a coproduct diagram.

The following result is essentially [8, Proposition 2.14].

Proposition 2.2. *A category with finite coproducts is extensive if, and only if, coproducts are universal and disjoint.*

Assume from now on that $\mathcal{C}$ is an extensive category.

A monomorphism $u \colon U \to X$ in $\mathcal{C}$ is called *complemented* if there is a $v \colon V \to X$ such that the cospan $u \colon U \to X \leftarrow V : v$ is a coproduct diagram. In this case, v is the *complement* of u. Notice that complemented monomorphisms are regular monomorphisms because they are coproduct injections. In the next definition, and throughout, we identify monomorphisms and subobjects whenever convenient.

Definition 2.3. An object X in $\mathcal{C}$ is *connected* if it has exactly two complemented subobjects.

In KHaus or Top, an object is connected if and only if it has exactly two clopens. An object A in Ring is connected as an object in Ring^{op} if and only if A has exactly two idempotents. We remark that, in general, connected objects are not closed under finite products.

For each X in $\mathcal{C}$ we let $\mathrm{B}X$ denote the poset of complemented subobjects of X. We stress that if $u\colon U \to X$ and $v\colon V \to X$ are two complemented monomorphisms in $\mathcal{C}$ and $f\colon U \to V$ is such that $vf = u$ then f is complemented [17, Lemma 3.2]. So for any two complemented subobjects u, v of X, there is no ambiguity in writing $u \leqslant v$ since it means the same for u, v considered as subobjects, or as complemented subobjects.

Extensivity easily implies that the poset $\mathrm{B}X$ has finite infima, a bottom element, and an involution. This structure may be used to prove that $\mathrm{B}X$ is actually a Boolean algebra which interacts well with pullbacks in the sense that, for any map $f\colon X \to Y$ in $\mathcal{C}$, pulling back along f determines a Boolean algebra homomorphism $\mathrm{B}Y \to \mathrm{B}X$. So, assuming that $\mathcal{C}$ is well-powered, the assignment $X \mapsto \mathrm{B}X$ extends to a functor $\mathcal{C} \to \mathsf{BA}^{\mathrm{op}}$ between extensive categories that preserves finite coproducts.

We will use the following simple equivalences.

Lemma 2.4. *For any object X in $\mathcal{C}$ the following are equivalent.*

(1) *X is connected.*
(2) *X is not initial and, for every complemented subobject $u\colon U \to X$, U is initial or u is an isomorphism.*
(3) *X is not initial and, for every coproduct diagram $U \to X \leftarrow V$, U is initial or V is initial.*

3. Finite-coproduct preserving functors

Let $\mathcal{C}$ and $\mathcal{S}$ be extensive categories, and let $L\colon \mathcal{C} \to \mathcal{S}$ preserve finite coproducts. Such a functor preserves complemented monomorphisms so, for any X in $\mathcal{C}$, L induces a function $\mathrm{B}X \to \mathrm{B}(LX)$ which is actually a map in BA, natural in X. (It is relevant to remark such a functor also preserves pullbacks along coproduct injections. See [17, 3.8].)

We will say that L is *injective* (*surjective/bijective*) *on complemented subobjects* if and only if $\mathrm{B}X \to \mathrm{B}(LX)$ has the corresponding property for every X in $\mathcal{C}$.

Lemma 3.1. *The functor $L\colon \mathcal{C} \to \mathcal{S}$ is injective on complemented subobjects if and only if it reflects 0. In this case, L also reflects connected objects.*

Proof. Assume first that L is injective on complemented subobjects and let X in $\mathcal{C}$ be such that $LX = 0$. Then $\mathrm{B}(LX)$ is the terminal Boolean algebra and, as $\mathrm{B}X \to \mathrm{B}(LX)$ is injective by hypothesis, $\mathrm{B}X$ is also trivial. For the converse notice that if L reflects 0 then the map $\mathrm{B}X \to \mathrm{B}(LX)$ in BA has trivial kernel for every X in $\mathcal{C}$.

To prove the second part of the statement assume that X in $\mathcal{C}$ is such that LX is connected in $\mathcal{S}$. If X were initial then so would LX because L preserves finite coproducts and, in particular, the initial object. So X is not initial. Now assume that $U \to X \leftarrow V$ is a coproduct diagram. Then so is $LU \to LX \leftarrow LV$. Since LX is connected, either LU or LV is initial by Lemma 2.4. As L reflects 0, either U or V is initial, so X is connected

by the same lemma. (Alternatively, if $BX \to B(LX)$ is injective and its codomain is the initial Boolean algebra then so is the domain.) $\square$

We will be particularly interested in extensive categories wherein every object is a finite coproduct of connected objects. For example, fSet satisfies this property, but neither Set nor Stone does. If $\mathcal{A}$ is the category of finitely presentable K-algebras for a field K, then $\mathcal{A}^{\mathrm{op}}$ also satisfies this property.

Proposition 3.2. *If $L\colon \mathcal{C} \to \mathcal{S}$ is bijective on complemented subobjects then the following hold.*

(1) *The functor L preserves connected objects.*
(2) *For any object X in $\mathcal{C}$, if LX is a finite coproduct of connected objects then so is X.*
(3) *If every object in $\mathcal{S}$ is a finite coproduct of connected objects then so is the case in $\mathcal{C}$.*
(4) *Assume that $\mathcal{C}$ and $\mathcal{S}$ have finite products and that L preserves them. If $\mathcal{S}$ is such that finite products of connected objects are connected then so is the case in $\mathcal{C}$.*

Proof. To prove the first item just notice that, by hypothesis, $BX \to B(LX)$ is an isomorphism for each X in $\mathcal{C}$. Hence if X has exactly two complemented subobjects then so does LX.

Before proving the second item we establish an auxiliary fact. Let X be in $\mathcal{C}$ and let $u\colon U \to LX$ be a complemented subobject in $\mathcal{S}$ with connected U. Then, as L is surjective on complemented objects by hypothesis, there exists a complemented subobject $v\colon V \to X$ in $\mathcal{C}$ such that $Lv = u$ as subobjects of LX. Then $LV \cong U$ is connected, so V is connected by Lemma 3.1. Thus, we have lifted the 'connected component' u of LX to one of X.

To prove the second item let $(u_i \mid i \in I)$ be a finite family of pairwise-disjoint complemented subobjects of LX with connected domain whose join is the whole of LX. For each $i \in I$, let v_i be the complemented subobject of X induced by u_i as in the previous paragraph. As L reflects 0, the family $(v_i \mid i \in I)$ is pairwise disjoint. Also, $L \bigvee_{i \in I} v_i = \bigvee_{i \in I} Lv_i = \bigvee_{i \in I} u_i$ is the whole of LX. As L is injective on complemented subobjects, $\bigvee_{i \in I} v_i$ must be the whole of X. In summary, we have lifted the finite coproduct decomposition of LX to one of X.

The third item follows at once from the second.

For the fourth item, let X be the product of a finite family $(X_i \mid i \in I)$ of connected objects in $\mathcal{C}$. Then LX is the product of $(LX_i \mid i \in I)$ because L preserves finite products. Each LX_i is connected because L preserves connected objects by the first item, so LX is connected by our hypothesis on $\mathcal{S}$. Hence X is connected by Lemma 3.1. $\square$

We next prove a sufficient condition for a functor L as above to be bijective on complemented subobjects.

Lemma 3.3. *If $L\colon \mathcal{C} \to \mathcal{S}$ has a finite-coproduct preserving right adjoint, then L is bijective on complemented subobjects.*

Proof. Let R be the right adjoint to L and let σ and τ be the unit and counit of $L \dashv R$. We show that L is both injective and surjective on complemented subobjects.

To prove injectivity it is enough to show that L reflects 0 (Lemma 3.1). So let X be an object in $\mathcal{C}$ such that LX is initial. Then we may transpose the isomorphism $LX \to 0$ in $\mathcal{S}$ to a map $X \to R0$, but $R0 = 0$ because R is assumed to preserve finite coproducts. Since the initial object is strict, X is initial.

We next show that L is surjective on complemented subobjects. Let $u\colon U \to LX$ be a complemented monomorphism. Then Ru is complemented so the left pullback square below exists

$$
\begin{array}{ccc}
V & \longrightarrow & RU \\
{\scriptstyle v}\downarrow & & \downarrow{\scriptstyle Ru} \\
X & \xrightarrow{\ \sigma\ } & R(LX)
\end{array}
\qquad\qquad
\begin{array}{ccccc}
LV & \longrightarrow & L(RU) & \xrightarrow{\ \tau\ } & U \\
{\scriptstyle Lv}\downarrow & & \downarrow{\scriptstyle L(Ru)} & & \downarrow{\scriptstyle u} \\
LX & \xrightarrow{\ L\sigma\ } & L(R(LX)) & \xrightarrow{\ \tau\ } & LX
\end{array}
$$

by extensivity of $\mathcal{C}$. Then the two squares on the right above obviously commute, and the bottom composite is the identity. Moreover, [17, Lemma 3.7] implies that both squares are pullbacks, so u and Lv coincide as subobjects of LX. □

Combining Lemma 3.3 and Proposition 3.2 we obtain the following.

Corollary 3.4. *Assume that $L\colon \mathcal{C} \to \mathcal{S}$ has a finite-coproduct preserving right adjoint. If every object in $\mathcal{S}$ is a finite coproduct of connected objects then so is the case in $\mathcal{C}$.*

4. Decidable objects

Let $\mathcal{C}$ be an extensive category with finite products. In particular, $\mathcal{C}$ has a terminal object 1. An object X is called *subterminal* if the unique map $X \to 1$ is monic.

Lemma 4.1. *For any object X in $\mathcal{C}$, the following are equivalent.*

(1) *The object X is subterminal.*
(2) *The diagonal $\Delta\colon X \to X \times X$ is an isomorphism.*
(3) *The projections $\mathrm{pr}_0, \mathrm{pr}_1\colon X \times X \to X$ are equal.*

Proof. The first item implies the second because for any monomorphism $X \to 1$ the following diagram

$$X \xrightarrow{\ id\ } X$$

$$id \downarrow \qquad \downarrow \, !$$

$$X \xrightarrow{\ !\ } 1$$

is a pullback. The second item implies the third because any map has at most one inverse. To prove that the third item implies the first, let $f, g \colon Y \to X$. Then there exists a unique map $\langle f, g \rangle \colon Y \to X \times X$ such that $\mathrm{pr}_0 \langle f, g \rangle = f$ and $\mathrm{pr}_1 \langle f, g \rangle = g$. So $f = \mathrm{pr}_0 \langle f, g \rangle = \mathrm{pr}_1 \langle f, g \rangle = g$. That is, for any object Y there is a unique map $Y \to X$. This means that the unique map $X \to 1$ is monic. $\quad \square$

We stress that extensivity plays no rôle in Lemma 4.1, which is a general fact about categories with finite products.

Definition 4.2. An object X in $\mathcal{C}$ is *decidable* if the diagonal $\Delta \colon X \to X \times X$ is complemented.

Remark 4.3. Lemma 4.1 shows that subterminal objects in $\mathcal{C}$ are decidable, and that they may be characterised as those decidable objects X such that the diagonal $\Delta \colon X \to X \times X$ not only is complemented, but is actually an isomorphism.

The full subcategory of decidable objects will be denoted by $\mathrm{Dec}\,\mathcal{C} \to \mathcal{C}$. If $\mathcal{C}$ is *lextensive* (i.e. extensive and with finite limits) it follows from [6] that $\mathrm{Dec}\,\mathcal{C}$ is lextensive and that the inclusion $\mathrm{Dec}\,\mathcal{C} \to \mathcal{C}$ preserves finite limits, finite coproducts and that it is closed under subobjects. Moreover, for any X, Y in $\mathcal{C}$, $X + Y$ is decidable if, and only if, both X and Y are decidable. On the other hand, arbitrary coproducts of decidable objects need not be decidable—consider, for instance, an infinite copower of the terminal object in KHaus or Stone.

Proposition 4.4. *For any object X in $\mathcal{C}$ the following are equivalent:*

(1) *X is subterminal and connected.*
(2) *X is decidable and $X \times X$ is connected.*

Proof. If X is subterminal and connected then $\Delta \colon X \to X \times X$ is an isomorphism by Lemma 4.1. So X is decidable and $X \times X$ is as connected as X.

For the converse assume that X is decidable and that $X \times X$ is connected. Decidability means that the subobject $\Delta \colon X \to X \times X$ is complemented; as $X \times X$ is connected, X is initial or $\Delta \colon X \to X \times X$ is an isomorphism by Lemma 2.4. But X is not initial (because $X \times X$ is connected) so $\Delta \colon X \to X \times X$ is an isomorphism. Then X is as connected as $X \times X$, and X is subterminal by Lemma 4.1. $\quad \square$

Let $\mathcal{S}$ be another extensive category with finite products and let $L\colon \mathcal{C} \to \mathcal{S}$ preserve finite products and finite coproducts.

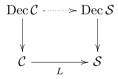**Lemma 4.5.** *Assume that L reflects 0 and that 1 is connected in $\mathcal{S}$. Then the following hold for every X in $\mathcal{C}$.*

(1) *If $LX = 1$ then X is connected.*
(2) *If X in $\mathcal{C}$ is decidable and $LX = 1$ then X is subterminal.*

Proof. The functor L reflects 0 so it reflects connected objects by Lemma 3.1. As 1 is connected in $\mathcal{S}$ by hypothesis, $LX = 1$ implies X connected.

If $LX = 1$ then $L(X \times X) = LX \times LX = 1$. So $X \times X$ is connected by the first item. Therefore X is subterminal by Proposition 4.4. $\square$

It easily follows from the definition of decidable object that L preserves decidable objects. In more detail, the preservation properties of L imply that the left-bottom composite below

$$
\begin{array}{ccc}
\mathrm{Dec}\,\mathcal{C} & \dashrightarrow & \mathrm{Dec}\,\mathcal{S} \\
\downarrow & & \downarrow \\
\mathcal{C} & \xrightarrow{\quad L \quad} & \mathcal{S}
\end{array}
$$

factors uniquely through the right inclusion and, moreover, $\mathrm{Dec}\,\mathcal{C} \to \mathrm{Dec}\,\mathcal{S}$ preserves finite products and finite coproducts. In fact, $\mathrm{Dec}\,\mathcal{C} \to \mathrm{Dec}\,\mathcal{S}$ preserves all the finite limits that L preserves (because the subcategories of decidable objects are closed under finite limits).

Additionally assume from now on that $L\colon \mathcal{C} \to \mathcal{S}$ has a finite-coproduct preserving right adjoint $R\colon \mathcal{S} \to \mathcal{C}$.

Notice that under the present hypotheses both L and R preserve finite products and finite coproducts. It follows that the adjunction $L \dashv R$ restricts to one between $\mathrm{Dec}\,\mathcal{S}$ and $\mathrm{Dec}\,\mathcal{C}$.

Corollary 4.6. *If every decidable object in $\mathcal{S}$ is a finite coproduct of connected objects then so is the case in $\mathcal{C}$.*

Proof. The adjunction $L \dashv R\colon \mathcal{S} \to \mathcal{C}$ restricts to one $L' \dashv R'\colon \mathrm{Dec}\,\mathcal{S} \to \mathrm{Dec}\,\mathcal{C}$, and every object in $\mathrm{Dec}\,\mathcal{S}$ is a finite coproduct of connected objects by hypothesis. So we may apply Corollary 3.4 to $L'\colon \mathrm{Dec}\,\mathcal{C} \to \mathrm{Dec}\,\mathcal{S}$ $\square$

Because $\mathcal{S}$ is lextensive, there exists an essentially unique coproduct preserving functor $\mathsf{fSet} \to \mathcal{S}$ that also preserves the terminal object. The functor sends a finite set I to the

copower $I \cdot 1$ in $\mathcal{S}$. The categories fSet, Stone, and other examples have the property that this functor fSet $\to \mathcal{S}$ coincides with $\operatorname{Dec} \mathcal{S} \to \mathcal{S}$. Notice that if this condition holds then 1 is connected in $\mathcal{S}$, because fSet $= \operatorname{Dec} \mathcal{S} \to \mathcal{S}$ is closed under subobjects and preserves 1.

Proposition 4.7. *If the canonical functor* fSet $\to \mathcal{S}$ *coincides with* $\operatorname{Dec} \mathcal{S} \to \mathcal{S}$ *then every decidable object in* $\mathcal{C}$ *is a finite coproduct of connected subterminals.*

Proof. By Corollary 4.6 every decidable object in $\mathcal{C}$ is a finite coproduct of connected objects. So it is enough to prove that every connected decidable object in $\mathcal{C}$ is subterminal. For this, let X be connected and decidable. Then LX is decidable, because L preserves finite products and finite coproducts, and it is connected by Lemma 3.3 and Proposition 3.2. By hypothesis, the canonical fSet $\to \mathcal{S}$ coincides with $\operatorname{Dec} \mathcal{S} \to \mathcal{S}$ so $LX = 1$. Hence X is decidable and $LX = 1$. Therefore X is subterminal by Lemma 4.5. $\quad\square$

For a lextensive category $\mathcal{E}$ we have considered several conditions.

(1) Every decidable object is a finite coproduct of connected objects.
(2) Every decidable object is a finite coproduct of connected subterminals.
(3) The canonical functor fSet $\to \mathcal{E}$ coincides with the inclusion $\operatorname{Dec} \mathcal{E} \to \mathcal{E}$.

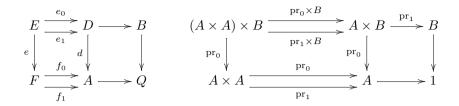For a field K, $(K/\mathsf{Ring})^{\mathrm{op}}$ satisfies the first condition but not the second. The categories Stone and KHaus satisfy the third condition. The third condition implies the second which, in turn, implies the first. Proposition 4.7 shows that for certain adjunctions $L \dashv R \colon \mathcal{S} \to \mathcal{C}$, if $\mathcal{S}$ satisfies the third condition then $\mathcal{C}$ satisfies the second. This will be used to prove that $\mathsf{MV}^{\mathrm{op}}$ satisfies the second condition (Theorem 10.1).

5. The coextensive category of MV-algebras

For background on MV-algebras we refer to the standard textbooks [10,35], of which we also follow the notation. In this section we show that MV is coextensive by proving that products are *co*disjoint and *co*universal (Proposition 2.2).

Lemma 5.1. *Let $\mathcal{A}$ be a regular category with finite colimits. If $0 \to 1$ is a regular epimorphism then products are codisjoint.*

Proof. Let A be an object in $\mathcal{A}$. As the composite $0 \to A \to 1$ is a regular epimorphism by hypothesis, so is $A \to 1$ by regularity of $\mathcal{A}$. That is, not only $0 \to 1$ but actually any $A \to 1$ is a regular epimorphism. As every regular epimorphism is the coequalizer of its kernel pair, $A \to 1$ is the coequalizer of the two projections $A \times A \to A$. Also, as products of regular epimorphisms are epimorphisms, the product of $id \colon A \to A$ and $B \to 1$ is a regular epimorphism $A \times B \to A \times 1$. That is, the projection $A \times B \to A$ is a regular epimorphism.

To complete the proof we recall a basic fact about colimits: for a commutative diagram as on the left below

such that $de_i = f_i e$ for $i \in \{0, 1\}$, the top and bottom forks are coequalizers and e is epic, the inner right square is a pushout. Applying this observation to the diagram on the right above we obtain that the inner right square in that diagram is a pushout. □
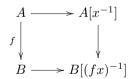
In particular, if $\mathcal{A}$ is the category of models for an algebraic theory with at least one constant then the initial object 0 is non-empty and so $0 \to 1$ is a regular epimorphism. This is the case, of course, for $\mathcal{A} = \mathsf{MV}$.

In Ring, couniversality of products is entailed by the intimate relationship between idempotents and product decompositions. The situation for MV is analogous. An element b of an MV-algebra A is called *Boolean* if it satisfies one of the following equivalent conditions (see [10, 1.5.3]):

$$b \oplus b = b \qquad b \odot b = b \qquad b \vee \neg b = 1 \qquad b \wedge \neg b = 0.$$

For $x \in A$ we let $A \to A[x^{-1}]$ be the quotient map induced by the congruence on A generated by the pair $(x, 1)$.

Lemma 5.2. *For any $f \colon A \to B$ in MV the following diagram is a pushout*

$$\begin{array}{ccc} A & \longrightarrow & A[x^{-1}] \\ f \downarrow & & \downarrow \\ B & \longrightarrow & B[(fx)^{-1}] \end{array}$$

where the right vertical map is the unique one making the square commute.

Proof. Standard, using the universal property of the (horizontal) quotient homomorphisms. □

Lemma 5.3. *For any MV-algebra A and every Boolean element $x \in A$, let $\langle \neg x \rangle$ be the ideal of A generated by $\{\neg x\}$. Then the quotient $q \colon A \to A/\langle \neg x \rangle$ has the universal property of $A \to A[x^{-1}]$.*

Proof. If $k\colon A \to B$ is such that $kx = 1$ then $\neg x \in \ker k$, so $\langle\neg x\rangle \subseteq \ker k$. By the universal property of quotients there is exactly one homomorphism $c\colon A/\langle\neg x\rangle \to C$ such that $cq = k$. $\square$

Lemma 5.4. *In* MV*, the diagram*

$$D \xleftarrow{\;\;q_0\;\;} C \xrightarrow{\;\;q_1\;\;} E$$

is a product precisely when there exists a Boolean element $x \in C$ such that q_0 has the universal property of $C \to C[(\neg x)^{-1}]$ and q_1 has the universal property of $C \to C[x^{-1}]$. When this is the case, the element $x \in C$ with the foregoing property is unique.

Proof. Assume the diagram is a product. Then there is a unique $x \in C$ such that $q_i x = i$, $i = 0, 1$. This x is Boolean because 0 and 1 are. Hence $\neg x$ is Boolean too, and thus $\oplus$-idempotent; therefore, $\langle\neg x\rangle = \{c \in C \mid c \leqslant \neg x\}$. If $c \leqslant \neg x$ then $q_1 c \leqslant q_1(\neg x) = 0$, so $q_1 c = 0$ and $c \in \ker q_1$. If $c \in \ker q_1$ then $q_1 c = 0 \leqslant q_1(\neg x)$ and $q_0 c \leqslant 1 = q_0(\neg x)$, so $c \leqslant \neg x$ by the definition of product order. We conclude $\ker q_1 = \langle\neg x\rangle$. The projection q_1 is surjective so Lemma 5.3 entails that q_1 has the universal property of $C \to C[x^{-1}]$. An entirely similar argument applies to q_0.
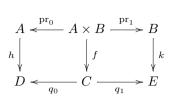
Conversely, assume q_0 and q_1 have the universal properties in the statement. By Lemma 5.3 we may identify q_0 with $C \to C/\langle x\rangle$ and q_1 with $C \to C/\langle\neg x\rangle$. So it is enough to show that the canonical $C \to C/\langle x\rangle \times C/\langle\neg x\rangle$ is bijective. Injectivity follows because if $c \leqslant x, \neg x$ then $c \leqslant x \wedge \neg x = 0$, so $\langle x\rangle \cap \langle\neg x\rangle = 0$. To prove surjectivity, let $(q_0 c_0, q_1 c_1) \in C/\langle x\rangle \times C/\langle\neg x\rangle$ with $c_0, c_1 \in C$ and consider $c = (c_0 \wedge \neg x) \vee (c_1 \wedge x) \in C$. It is easy to check that $C \to C/\langle x\rangle \times C/\langle\neg x\rangle$ sends c in the domain to $(q_0 c_0, q_1 c_1)$ in the codomain. $\square$

Remark 5.5. The content of Lemma 5.4 is far from new, cf. e.g. [10, Section 6.4] and [9, Proposition 3.9]. However, having expressed that content in the form that is most suitable for the sequel, we have included a proof for the reader's convenience.

Proposition 5.6. MV *is coextensive.*

Proof. Any algebraic category is complete and cocomplete, so in particular it has finite products and pushouts. We appeal to the characterization of extensive categories in Proposition 2.2. Codisjointness of products follows from Lemma 5.1 or from a direct calculation observing that the projections of a product $A \times B$ send $(0, 1)$ to 0 and 1 respectively, so $0 = 1$ must hold in the pushout.

It remains to show that products are couniversal. So we consider the pushout of a product diagram as below

$$A \xleftarrow{\text{pr}_0} A \times B \xrightarrow{\text{pr}_1} B$$

$$\downarrow{h} \qquad \downarrow{f} \qquad \downarrow{k}$$

$$D \xleftarrow{q_0} C \xrightarrow{q_1} E$$

and prove that the bottom span is product diagram. Indeed, observe that the Boolean element $(1, 0) \in A \times B$ is sent to the Boolean element $x := f(1, 0) \in C$ so, by Lemma 5.4, it is enough to check that q_0 inverts $\neg x$ and q_1 inverts x; but this follows from Lemma 5.2. $\quad \square$

Although it was not necessary to prove the main result of this section, it seems worthwhile to observe that, in the context of algebraic categories, Lemma 5.1 may be strengthened to a characterisation.

Proposition 5.7. *In any algebraic category, binary products are codisjoint if, and only if, the initial algebra has non-empty underlying set.*

Proof. If the initial algebra 0 is not empty then the unique map $0 \to 1$ is a regular epimorphism so we can apply Lemma 5.1. For the converse implication notice that the following square

$$\begin{array}{ccc} 0 \times 0 & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & 1 \end{array}$$

is a pushout by hypothesis. As any of the projections $0 \times 0 \to 0$ is split epic, its pushout $0 \to 1$ is a regular epimorphism, so 0 must be non-empty. $\quad \square$

6. Subterminals in $\mathsf{MV}^{\mathrm{op}}$, and rational algebras

The aim of this section is to characterize subterminal objects in $\mathsf{MV}^{\mathrm{op}}$. Perhaps unexpectedly, the following fact will play an important rôle.

Lemma 6.1. *Monomorphisms in MV are stable under pushout.*

Proof. It is well known [24] that, in algebraic categories, stability of monomorphisms under pushout is equivalent to the conjunction of the Amalgamation Property (AP) and of the Congruence Extension Property (CEP). Pierce proved the AP for Abelian lattice-ordered groups in [36], and Mundici [34, Proposition 1.1] observed that Pierce's result transfers through the functor Γ to MV-algebras. For a different proof of the AP for Abelian lattice-ordered groups and MV-algebras, see [32, Theorems 36 40]. The CEP for MV-algebras was proved in [18, Proposition 8.2]; for an alternative proof, see [35, p.

230]. For yet another proof in the more general context of residuated lattices, see [32, Corollary 44]. $\square$

Most of the work will be done on the algebraic side, so it is convenient to start with an arbitrary category $\mathcal{A}$ with finite coproducts whose initial object is denoted 0. As suggested above, we concentrate on the objects A such that the unique map $0 \to A$ is epic. Notice that such an object is exactly a subterminal object in $\mathcal{A}^{\mathrm{op}}$, but we prefer to avoid introducing new terminology such as 'cosubterminal' or 'supra-initial'. For convenience we state here the dual of Lemma 4.1.

Lemma 6.2. *For any object A in $\mathcal{A}$, the following are equivalent:*

(1) *The map $0 \to A$ is epic.*
(2) *The codiagonal $\nabla \colon A + A \to A$ is an isomorphism.*
(3) *The coproduct injections $in_0, in_1 \colon A \to A + A$ are equal.*

We shall also need a simple auxiliary fact.

Lemma 6.3. *Let $0 \to A$ be epic and $m \colon B \to A$ be a map. If the coproduct map $m + m \colon B + B \to A + A$ is monic then $0 \to B$ is epic.*

Proof. The following square commutes

$$
\begin{array}{ccc}
B + B & \xrightarrow{\ \nabla\ } & B \\
{\scriptstyle m+m}\big\downarrow & & \big\downarrow{\scriptstyle m} \\
A + A & \xrightarrow[\ \nabla\ ]{} & A
\end{array}
$$

by naturality of the codiagonal. The bottom map is an isomorphism by Lemma 6.2, and the left vertical map is monic by hypothesis. So the top map is also monic, as well as split epic. $\square$

Assume from now on that $\mathcal{A}$ has finite colimits and that monomorphisms are stable under pushout. We stress that this stability property is quite restrictive. For instance, it does not hold in Ring. On the other hand, we already know that it holds in MV by Lemma 6.1.
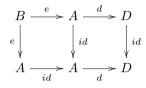
Lemma 6.4. *The map $0 \to A$ is epic if, and only if, for every monomorphism $B \to A$, $0 \to B$ is epic.*

Proof. One direction is trivial and does not need stability of monomorphisms. For the converse observe that, as monomorphisms are stable under pushout, finite coproducts of monomorphisms are monic. So we can apply Lemma 6.3. $\square$

The following is a further auxiliary fact.

Lemma 6.5. *For any $d\colon A \to D$ and $e\colon B \to A$ in $\mathcal{A}$, if e is epic and the composite $de\colon B \to D$ is monic then d is an monic.*

Proof. The right square below is trivially a pushout and, since $e\colon B \to A$ is epic, the left square is also a pushout

$$
\begin{array}{ccccc}
B & \xrightarrow{\ e\ } & A & \xrightarrow{\ d\ } & D \\
{\scriptstyle e}\downarrow & & {\scriptstyle id}\downarrow & & {\scriptstyle id}\downarrow \\
A & \xrightarrow[\ id\ ]{} & A & \xrightarrow[\ d\ ]{} & D
\end{array}
$$

so the rectangle is a pushout too. As the top composite is monic, and these are stable under pushout by hypothesis, the bottom map is monic. $\quad\square$

We emphasise the next particular case of Lemma 6.5.

Lemma 6.6. *Let $d\colon A \to D$ be a regular epimorphism in $\mathcal{A}$. If $0 \to A$ is epic and $0 \to D$ is monic then d is an isomorphism.*

Assume now that our category $\mathcal{A}$ with finite colimits and stable monomorphisms has a terminal object 1 such that for any object A in $\mathcal{A}$ the unique $A \to 1$ is a regular epimorphism. This is common in algebraic categories.

A *quotient* of A in $\mathcal{A}$ is an equivalence class of regular epimorphisms with domain A, where two such are equivalent if they are isomorphic as objects of $A/\mathcal{A}$.

An object A is *simple* if it has exactly two quotients, namely, those represented by $A \to 1$ and $id\colon A \to A$. So, if $\mathcal{A}$ is an algebraic category, then an object is simple if and only if it has exactly two congruences.

To motivate the hypotheses of the following lemma observe that for every object A in BA, A is terminal or $0 \to A$ is monic. Similarly for MV and for K/Ring with K a field. In contrast, that is not the case in Ring.

Lemma 6.7. *If for every object D of $\mathcal{A}$, D is terminal or $0 \to D$ is monic, then for every epic $0 \to A$ the following hold.*

(1) *A is simple or terminal.*
(2) *If $m\colon B \to A$ is monic then $B + B$ is simple or terminal.*

Proof. To prove the first item let $d\colon A \to D$ be a regular epimorphism. Then D is terminal or $0 \to D$ is monic by hypothesis. If $0 \to D$ is monic then d is an isomorphism by Lemma 6.6. So the only possible quotients of A are $A \to 1$ or $id\colon A \to A$. So A is terminal or simple.

To prove the second item first recall that epimorphisms are closed under coproduct. Then recall that, as monomorphisms are stable by hypotheses, they are closed under finite coproducts. Therefore, $m + m \colon B + B \to A + A$ is a monomorphism and $0 = 0 + 0 \to A + A$ is epic. So, by Lemma 6.4, $0 \to B + B$ is also epic. The first item implies that $B + B$ is simple or terminal. $\square$

The material in this section applies to the case $\mathcal{A} = \mathsf{MV}$, so we may now prove our first MV-algebraic result. For the proof we require a standard fact from the theory of MV-algebras and lattice-ordered groups, which will also find further application later in this paper. An ideal $\mathfrak{m}$ of the MV-algebra A is *maximal* if it is proper, and inclusion-maximal amongst proper ideals of A; equivalently, the quotient $A/\mathfrak{m}$ is a simple algebra.

Lemma 6.8 *(Hölder's Theorem [22] for MV-algebras [10, 3.5.1]). For every MV-algebra A, and for every maximal ideal $\mathfrak{m}$ of A, there is exactly one homomorphism of MV-algebras*

$$\mathfrak{h}_\mathfrak{m} \colon \tfrac{A}{\mathfrak{m}} \longrightarrow [0, 1],$$

and this homomorphism is injective.

In connection with the result that follows, let us explicitly recall that the initial object 0 in MV is the two-element Boolean algebra $\{0, 1\}$.

Theorem 6.9. *For any MV-algebra A the following are equivalent.*

(i) *A is a subalgebra of $[0, 1] \cap \mathbb{Q}$.*
(ii) *A is non-trivial and the unique map $0 \to A$ is epic.*
(iii) *The unique map $0 \to A$ is monic and epic.*
(iv) *A is simple and $0 \to A$ is epic.*

Proof. If $A \subseteq [0, 1] \cap \mathbb{Q}$ then A is certainly non-trivial, and [35, Proposition 7.2] shows that the coproduct inclusions $in_0, in_1 \colon A \to A + A$ are equal. So $0 \to A$ is epic by Lemma 6.2.

The second and third items are clearly equivalent, and they imply the fourth by Lemma 6.7.

Finally, assume that A is simple and that $0 \to A$ is epic. By Hölder's Theorem (Lemma 6.8) together with simplicity, there is exactly one monomorphism $A \to [0, 1]$. Now let $r \in A$ and write $\iota \colon A_r \to A$ for the subalgebra of A generated by r. As A_r is not trivial (and $0 \to A$ is epic) Lemma 6.7 implies that $A_r + A_r$ is simple. Hence, by the computation in [35, Proposition 7.3], r must be rational. $\square$

7. The π_0 functor for topological spaces

In this section we show that the full inclusion Stone $\to$ KHaus of the category of Stone spaces into that of compact Hausdorff spaces has a left adjoint $\pi_0\colon$ KHaus $\to$ Stone that preserves set-indexed products. The result just stated may be concisely referenced as follows. That the inclusion at hand is reflective is well known and flows readily from the universal property of the quotient topology. As shown in [7, Section 7], the reflection has "stable units"; we need not discuss this property here, except to recall that it easily implies that the left adjoint π_0 preserves finite products. Since Gabriel and Ulmer in [16, p. 67] show that π_0 preserves cofiltered limits, π_0 preserves all products.[3]

We give here a different proof that emphasises the key rôle of totally disconnected spaces in the general case. We first obtain a product-preserving left adjoint to the full inclusion of the category TD of totally disconnected topological spaces into Top. We then show how to restrict this left adjoint to the categories of interest to us in the present paper.

A topological space X is *connected* if it is so in the sense of Definition 2.3. A subset of a space is *clopen* if it is both closed and open. Then, a space X is connected if and only if it contains exactly two clopen sets, which are then necessarily $\emptyset$ and X. Equivalently [14, Theorem 6.1.1], X is connected if whenever $X = A \cup B$ with $A \cap B = \emptyset$ and A and B closed subsets of X, then exactly one of A and B is empty. If X is a space and $x \in X$, the *component* of x in X, written C_x (with X understood), is defined as

$$C_x := \bigcup \{C \subseteq X \mid x \in C \text{ and } C \text{ is connected}\} \subseteq X.$$

It can be shown that C_x is a connected subspace of X [14, Corollary 6.1.10], and it therefore is the inclusion-largest such to which x belongs. Also, C_x is closed in X [14, Corollary 6.1.11]. A topological space X is *totally disconnected* if for each $x \in X$ we have $C_x = \{x\}$.

Consider the equivalence relation on X given by

$$x \sim y \text{ if, and only if, } C_x = C_y, \tag{1}$$
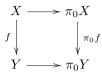
and define

$$\pi_0 X := \frac{X}{\sim}.$$

We equip $\pi_0 X$ with the quotient topology, and call it the *space of components* of X. We write

$$q\colon X \longrightarrow \pi_0 X \tag{2}$$

---

for the quotient map.

Lemma 7.1. *For every continuous map* $f: X \to Y$ *between topological spaces there is exactly one map such that the square below commutes.*

$$
\begin{array}{ccc}
X & \longrightarrow & \pi_0 X \\
f \downarrow & & \downarrow \pi_0 f \\
Y & \longrightarrow & \pi_0 Y
\end{array}
$$

Proof. A straightforward verification. □

Lemma 7.1 implies that the assignment that sends f to $\pi_0 f$ extends to an endofunctor

$$\pi_0 \colon \mathsf{Top} \longrightarrow \mathsf{Top}. \tag{3}$$

This endofunctor determines the full subcategory TD, as we now show.

Lemma 7.2. *If* $C \subseteq \pi_0 X$ *is a connected subspace then so is* $q^{-1}[C] \subseteq X$.

Proof. Let $q^{-1}[C] = F_1 \cup F_2$ with F_1 and F_2 disjoint closed subsets of X. For any $y \in C$ we can write the fibre $q^{-1}[\{y\}]$ as C_x for any $x \in q^{-1}[\{y\}]$. Further, we can express C_x as the disjoint union $C_x = (F_1 \cap C_x) \cup (F_2 \cap C_x)$. And C_x is closed and connected, because it is a component. Hence exactly one of $q^{-1}[\{y\}] = C_x \subseteq F_1$ or $q^{-1}[\{y\}] = C_x \subseteq F_2$ holds, for each $y \in C$. We can then define

$$S_i := \{y \in C \mid q^{-1}[\{y\}] \subseteq F_i\}, \ i = 1, 2,$$

to the effect that $C = S_1 \cup S_2$ and $S_1 \cap S_2 = \emptyset$. By construction we have $F_i = q^{-1}[S_i]$, $i = 1, 2$. The definition of quotient topology then entails that S_i is closed because F_i is. Since C is connected, exactly one of S_1 and S_2 is empty, and hence so is exactly one of F_1 and F_2. □

Lemma 7.3. *For any space* X, *the quotient map* $q: X \to \pi_0 X$ *in* (2) *is universal from* X *to the full inclusion* $\mathsf{TD} \to \mathsf{Top}$.

Proof. We first show that $\pi_0 X$ is totally disconnected. Let C_y be the component of $y \in \pi_0 X$, with the intent of showing it is a singleton. By Lemma 7.2, since C_y is connected in $\pi_0 X$, so is $q^{-1}[C_y]$ connected in X. Therefore $q^{-1}[C_y]$ is contained in the component C_x of any $x \in X$ with $x \in q^{-1}[C_y]$; and thus, the direct image $q[q^{-1}[C_y]]$ is contained in $q[C_x] = \{y\}$. Since $q[q^{-1}[C_y]] = C_y$, because q is surjective, we conclude $C_y \subseteq \{y\}$, as was to be shown.

  Let $f: X \to Y$ be a continuous map, with Y totally disconnected. We already know from the proof of Lemma 7.1 that f preserves $\sim$ so, as Y is totally disconnected, $x \sim x'$

in X implies $fx = fx'$ in Y. The universal property of the quotient $q\colon X \to \pi_0 X$ implies the existence of a unique $g\colon \pi_0 X \to Y$ such that $gq = f$. $\square$

We conclude that the full inclusion $\mathsf{TD} \to \mathsf{Top}$ has a left adjoint that, with no risk of confusion, will again be denoted by $\pi_0\colon \mathsf{Top} \to \mathsf{TD}$. A reformulation of [14, Theorem 6.1.21] now yields:

Proposition 7.4. *The functor $\pi_0\colon \mathsf{Top} \to \mathsf{TD}$ preserves all set-indexed products.* $\square$

We next identify a related construction which will provide a useful alternative description of π_0 when restricted to KHaus. Let us write $\mathrm{C}\,(X, 2)$ for the set of continuous maps from the space X to the discrete two-point space $2 \coloneqq \{0, 1\}$. There is a canonical continuous function

$$E = \langle f \mid f \in \mathrm{C}\,(X, 2) \rangle \colon X \longrightarrow 2^{\mathrm{C}\,(X, 2)}, \tag{4}$$
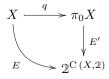
$$x \longmapsto (fx \mid f \in \mathrm{C}\,(X, 2)).$$

For any subset $S \subseteq X$, write $\chi_S \colon X \to 2$ for the characteristic function defined by $\chi_S x = 1$ if, and only if, $x \in S$. Then S is clopen precisely when $\chi_S \in \mathrm{C}\,(X, 2)$. Thus, E in (4) can equivalently be described as the function that sends each point $x \in X$ to the set of clopen subsets of X that contain x.

Recall [14, p. 356] that the *quasi-component* of $x \in X$ is defined as

$$\widetilde{C}_x \coloneqq \bigcap \{S \subseteq X \mid S \text{ is clopen, and } x \in S\}.$$

It is clear that the quasi-components of a space X partition X into closed non-empty sets.[4] Quasi-components can be straightforwardly used to prove:
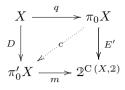
Lemma 7.5. *For any X there exists a unique $E'\colon \pi_0 X \to 2^{\mathrm{C}\,(X,2)}$ such that the following diagram*

$$
\begin{array}{ccc}
X & \xrightarrow{\ q\ } & \pi_0 X \\
 & \searrow{\scriptstyle E} & \downarrow{\scriptstyle E'} \\
 & & 2^{\mathrm{C}\,(X,2)}
\end{array}
$$

commutes. $\square$

Let $X \xrightarrow{\ D\ } \pi_0' X \xrightarrow{\ m\ } 2^{\mathrm{C}\,(X,2)}$ be the epi/regular-mono factorization of the canonical map E in (4). Then the following square commutes

---

[4] The quasi-component $\widetilde{C}_x$ of $x \in X$ need not be connected. Indeed, the inclusion $C_x \subseteq \widetilde{C}_x$ always holds for each $x \in X$ [14, Theorem 6.1.22], and may be proper [14, Example 6.1.24].
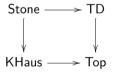
$$\begin{array}{ccc} X & \xrightarrow{\;q\;} & \pi_0 X \\ {\scriptstyle D}\downarrow & \searrow{\scriptstyle c} & \downarrow{\scriptstyle E'} \\ \pi_0' X & \xrightarrow{\;m\;} & 2^{\mathrm{C}\,(X,2)} \end{array}$$

by Lemma 7.5 and, as q is regular-epi and m is monic, there is exactly one continuous map $c\colon \pi_0(X) \to \pi_0'(X)$ making the inner triangles commute. Since D is epic, so is c. Also, since m is a regular mono, $\pi_0'X$ carries the subspace topology inherited from the product $2^{\mathrm{C}\,(X,2)}$ and, as the latter is a Stone space, $\pi_0'X$ is Hausdorff.

Lemma 7.6. *If X is compact Hausdorff then $c\colon \pi_0 X \to \pi_0'X$ is a homeomorphism and these homeomorphic spaces are Stone spaces.*

Proof. First recall [14, Theorem 6.1.23] that, in any compact Hausdorff space X, the equality $C_x = \widetilde{C}_x$ holds for each $x \in X$. In other words, in this case, the function $\pi_0 X \to \pi_0'X$ is bijective. Also, since X is compact, so is $\pi_0 X$ because q is surjective. Hence, as we already know that $\pi_0'X$ is Hausdorff, the Closed Map Lemma implies that c is a homeomorphism.

Similarly, compactness of X implies compactness of $\pi_0'X$ and hence, the Closed Map Lemma implies that m is closed. Therefore, $\pi_0'X$ is a closed subspace of the Stone space $2^{\mathrm{C}\,(X,2)}$. $\quad\square$

It is classical that each Stone space is totally disconnected, so there is a full inclusion $\mathsf{Stone} \to \mathsf{TD}$ such that the following diagram

$$\begin{array}{ccc} \mathsf{Stone} & \longrightarrow & \mathsf{TD} \\ \downarrow & & \downarrow \\ \mathsf{KHaus} & \longrightarrow & \mathsf{Top} \end{array}$$

commutes. Lemma 7.6 implies that the composite $\mathsf{KHaus} \longrightarrow \mathsf{Top} \xrightarrow{\;\pi_0\;} \mathsf{TD}$ factors through the full inclusion $\mathsf{Stone} \to \mathsf{TD}$. The factorization will be conveniently denoted by $\pi_0\colon \mathsf{KHaus} \to \mathsf{Stone}$.

Theorem 7.7. *The functor $\pi_0\colon \mathsf{KHaus} \to \mathsf{Stone}$ is left adjoint to the full inclusion $\mathsf{Stone} \to \mathsf{KHaus}$, and preserves all set-indexed products.* $\quad\square$

Proof. Since, as observed above, $\pi_0\colon \mathsf{Top} \to \mathsf{TD}$ restricts to $\pi_0\colon \mathsf{KHaus} \to \mathsf{Stone}$, the fact that the former is a left adjoint to $\mathsf{TD} \to \mathsf{Top}$ (Lemma 7.6) restricts to the fact that $\pi_0\colon \mathsf{KHaus} \to \mathsf{Stone}$ is left adjoint to $\mathsf{Stone} \to \mathsf{KHaus}$. It is standard that products in KHaus and in Stone agree with products in Top (using, in particular, Tychonoff's

Theorem that any product of compact spaces is compact), so Proposition 7.4 entails that $\pi_0 \colon \mathsf{KHaus} \to \mathsf{Stone}$ preserves all set-indexed products.   □

8. Spectra of MV-algebras

In this section we recall the material about spectra of MV-algebras that is needed in the sequel.

Recall that an ideal $\mathfrak{p}$ of an MV-algebra A is *prime* if it is proper, and the quotient $A/\mathfrak{p}$ is totally ordered. The (*prime*) *spectrum* of an MV-algebra A is

$$\operatorname{Spec} A := \{\mathfrak{p} \subseteq A \mid \mathfrak{p} \text{ is a prime ideal of } A\}$$

topologised into the *spectral space* of A, as follows. For a subset $S \subseteq A$, define

$$\mathbb{V}(S) := \{\mathfrak{p} \in \operatorname{Spec} A \mid S \subseteq \mathfrak{p}\},$$
$$\mathbb{S}(S) := \operatorname{Spec} A \setminus \mathbb{V}(S) = \{\mathfrak{p} \in \operatorname{Spec} A \mid S \not\subseteq \mathfrak{p}\}.$$

The set $\mathbb{V}(S)$ is called the *vanishing locus*, or *zero set*, of S, while $\mathbb{S}(S)$ is called its *support*. If $a \in A$, write $\mathbb{V}(a)$ as a shorthand for $\mathbb{V}(\{a\})$, and similarly for $\mathbb{S}(a)$. Then the collection

$$\{\mathbb{V}(I) \mid I \text{ is an ideal of } A\}$$

is the set of closed sets for a topology on $\operatorname{Spec} A$ that makes the latter a spectral space in the sense of Hochster [21]. The collection

$$\{\mathbb{S}(a) \mid a \in A\}$$

is a basis of compact open sets for this topology; see [3, Chapitre 10] and [35, Chapter 4]. The topology is variously known as the *Stone*, *Zariski*, or *hull-kernel* topology of A.

The assignment $A \mapsto \operatorname{Spec} A$ extends to a functor $\mathsf{MV}^{\mathrm{op}} \to \mathsf{Top}$, because inverse images of primes ideals along homomorphisms are prime. Although it is common to take the codomain of Spec as the category of spectral spaces and spectral maps, for our purposes in this paper it is expedient to regard Spec as taking values in Top.

The *maximal spectrum* of an MV-algebra A is

$$\operatorname{Max} A := \{\mathfrak{m} \subseteq A \mid \mathfrak{m} \text{ is a maximal ideal of } A\}.$$

We have $\operatorname{Max} A \subseteq \operatorname{Spec} A$, or equivalently, any simple MV-algebra is totally ordered (see e.g. [10, 3.5.1]). The *maximal spectral space* of A is the set $\operatorname{Max} A$ equipped with the subspace topology it inherits from $\operatorname{Spec} A$. Then $\operatorname{Max} A$ is a compact Hausdorff space [35, Proposition 4.15], and every compact Hausdorff space arises in this manner from some MV-algebra A [35, Theorem 4.16(iv)].

The standard example of MV-algebra, the interval $[0,1]$ equipped with the constant 0 and the operations $\oplus$, $\neg$, generalises as follows. If X is any set, the collection $[0,1]^X$ of all functions from X to $[0,1]$ inherits the structure of an MV-algebra upon defining operations pointwise. If, additionally, X is a topological space, since $\oplus\colon [0,1]^2 \to [0,1]$, $\neg\colon [0,1] \to [0,1]$, and 0 are continuous with respect to the Euclidean topology of $[0,1]$, the subset

$$\mathrm{C}(X) \coloneqq \{f\colon X \to [0,1] \mid f \text{ is continuous}\} \tag{5}$$

is a subalgebra of the MV-algebra $[0,1]^X$. We shall describe a natural MV-homomorphism $\eta_A\colon A \longrightarrow \mathrm{C}\,(\mathrm{Max}\,A)$, for each MV-algebra A. Its existence descends from Hölder's Theorem (Lemma 6.8), which allows us to define a close analogue to the Gelfand transform in functional analysis. Indeed, in light of that result, to $a \in A$ and $\mathfrak{m} \in \mathrm{Max}\,A$ we associate the real number $\mathfrak{h}_\mathfrak{m}(a/\mathfrak{m}) \in [0,1]$, obtaining the function

$$\widehat{a}\colon \mathrm{Max}\,A \longrightarrow [0,1] \tag{6}$$
$$\mathfrak{m} \longmapsto h_\mathfrak{m}(\tfrac{a}{\mathfrak{m}}).$$

It can be shown [35, 4.16.iii] that the function (6) is continuous with respect to the Stone topology of $\mathrm{Max}\,A$ and the Euclidean topology of $[0,1]$. We thereby arrive at the announced homomorphism

$$\eta_A\colon A \longrightarrow \mathrm{C}(\mathrm{Max}\,A) \tag{7}$$
$$a \longmapsto \widehat{a}$$

for each MV-algebra A.

Lemma 8.1. *For any MV-homomorphism $h\colon A \to B$ and any $\mathfrak{m} \in \mathrm{Max}\,B$ we have $h^{-1}(\mathfrak{m}) \in \mathrm{Max}\,A$. Moreover, the inverse-image map $h^{-1}\colon \mathrm{Max}\,B \to \mathrm{Max}\,A$ is continuous with respect to the Stone topology.*

Proof. The first assertion is proved in [10, 1.2.16]. The second assertion is a straightforward verification using the definition of Stone topology. $\square$

In light of Lemma 8.1 we henceforth regard Max as a functor:

$$\mathrm{Max}\colon \mathsf{MV} \longrightarrow \mathsf{KHaus}^{\mathrm{op}}, \tag{8}$$

where KHaus denotes the category of compact Hausdorff spaces and their continuous maps.

Given a continuous map $f\colon X \to Y$ in KHaus, it is elementary that the induced function

$$\mathrm{C}(f)\colon \mathrm{C}(Y) \longrightarrow \mathrm{C}(X),$$

$$g \in \mathrm{C}(Y) \longmapsto g \circ f \in \mathrm{C}(X)$$

is a morphism in MV. We therefore regard C as a functor:

$$\mathrm{C}\colon \mathsf{KHaus}^{\mathrm{op}} \longrightarrow \mathsf{MV}.$$

There is an adjunction

$$\mathrm{Max} \dashv \mathrm{C}\colon \mathsf{KHaus}^{\mathrm{op}} \to \mathsf{MV} \tag{9}$$

known as the *Cignoli-Dubuc-Mundici* adjunction [11]; see [31, Section 3] for further references and details not mentioned below. Dually to (7), for any space X in KHaus there is a continuous map

$$\epsilon_X \colon X \longrightarrow \mathrm{Max}\,\mathrm{C}(X) \tag{10}$$

$$x \longmapsto \{f \in \mathrm{C}(X) \mid f(x) = 0\},$$

and it is a standard fact that ϵ_X is a homeomorphism. (Compare [35, 4.16].) Writing Id_C for the identity functor on a category C, we can summarise the adjunction as follows.

Theorem 8.2 *([10, Propositions 4.1 and 4.2]). The functor* Max *is left adjoint to the fully faithful functor* C, *i.e.* $\mathrm{Max} \dashv \mathrm{C}\colon \mathsf{KHaus}^{\mathrm{op}} \to \mathsf{MV}$. *The unit and the counit of the adjunction are the natural transformations* $\eta\colon \mathrm{Id}_\mathsf{MV} \to \mathrm{C}\,\mathrm{Max}$ *and* $\epsilon\colon \mathrm{Max}\,\mathrm{C} \to \mathrm{Id}_{\mathsf{KHaus}^{\mathrm{op}}}$ *whose components are given by* (7) *and* (10), *respectively.* $\square$

9. The Pierce functor preserves coproducts

The category BA of Boolean algebras may be identified with the domain of the full subcategory I: BA → MV determined by the MV-algebras whose operation $\oplus$ is idempotent. It is then clear that I: BA → MV is a variety so, in particular, it has a left adjoint. It also has a right adjoint that we now describe.

We write $\mathrm{P}\,A$ for the collection of all Boolean elements of the MV-algebra A. By [10, 1.5.4], $\mathrm{P}\,A$ is the largest subalgebra of A that is a Boolean algebra. A homomorphism $h\colon A \to B$ preserves Boolean elements, because the latter are defined by equational conditions. Therefore, h induces by restriction a function $\mathrm{P}\,h\colon \mathrm{P}\,A \to \mathrm{P}\,B$ that is evidently a homomorphism of Boolean algebras. We thus obtain a functor

$$\mathrm{P}\colon \mathsf{MV} \longrightarrow \mathsf{BA}$$

from the category of MV-algebras to that of Boolean algebras; we call it the *Pierce functor* because of the close analogy with the theory developed in [37] for rings.

Lemma 9.1. *The functor* P *is right adjoint to the functor* I.

Proof. This is a direct consequence of the fact that $P\,A$ is the largest Boolean subalgebra of A, for any MV-algebra A. $\square$

The proof of Proposition 5.6—in particular, Lemma 5.4—makes it clears that $P\colon \mathsf{MV} \to \mathsf{BA}$ is essentially the 'complemented subobjects' functor B determined by the extensive category $\mathsf{MV}^{\mathrm{op}}$.

We now embark on the proof of the central fact that $P\colon \mathsf{MV} \to \mathsf{BA}$ preserves coproducts. Our aim is to reduce the problem to a situation where we can apply the topological results in Section 7.

Lemma 9.2. *For any MV-algebra A and any element $a \in A$, a is Boolean if, and only if, for each prime ideal $\mathfrak{p}$ of A, we have $a/\mathfrak{p} \in \{0,1\} \subseteq A/\mathfrak{p}$.*

Proof. Let C be any totally ordered MV-algebra. For $x \in C$, either $x \leqslant \neg x$ or $\neg x \leqslant x$. If the former holds then $x \wedge \neg x = x$, so that if x is Boolean then $x = 0$. If the latter holds then $x \vee \neg x = x$, and thus $x = 1$ if x is Boolean. In summary, if $x \in C$ is Boolean then $x \in \{0,1\}$. The converse implication is clear. Summing up, the Boolean elements of C are precisely 0 and 1.

Boolean elements, being definable by equational conditions, are preserved by homomorphisms. Hence if a is Boolean then $a/\mathfrak{p} \in A/\mathfrak{p}$ is Boolean, and therefore, since $A/\mathfrak{p}$ is totally ordered, $a/\mathfrak{p} \in \{0,1\}$ by the argument in the preceding paragraph. This proves the left-to-right implication in the statement of the lemma. For the converse implication, we recall that in any MV-algebra A we have $\bigcap \operatorname{Spec} A = \{0\}$ [10, 1.3.3]. Hence, the function $\iota\colon A \longrightarrow \prod_{\mathfrak{p} \in \operatorname{Spec} A} A/\mathfrak{p}$ defined by $a \in A \longmapsto (a/\mathfrak{p})_{\mathfrak{p} \in \operatorname{Spec}} \in \prod_{\mathfrak{p} \in \operatorname{Spec} A} A/\mathfrak{p}$ is an injective homomorphism. Assume that for each $\mathfrak{p} \in \operatorname{Spec} A$ we have $a/\mathfrak{p} \in \{0,1\}$. Since operations in $\prod_{\mathfrak{p} \in \operatorname{Spec} A} A/\mathfrak{p}$ are computed pointwise, we infer $\iota(a) \vee \neg \iota(a) = (a/\mathfrak{p})_{\mathfrak{p} \in \operatorname{Spec}} \vee \neg (a/\mathfrak{p})_{\mathfrak{p} \in \operatorname{Spec}} = 1$, and therefore, since ι is an isomorphism onto its range, $a \vee \neg a = 1$. This completes the proof. $\square$

Lemma 9.3. *Let A be an MV-algebra, and suppose there exist (possibly empty) closed subsets $X_0, X_1 \subseteq \operatorname{Spec} A$ with $\operatorname{Spec} A = X_0 \cup X_1$ and $X_0 \cap X_1 = \emptyset$. Then there exists exactly one Boolean element $b \in A$ such that $b/\mathfrak{p} = 0$ for each $\mathfrak{p} \in X_0$ and $b/\mathfrak{p} = 1$ for each $\mathfrak{p} \in X_1$.*

Proof. By [3, 10.1.7], there is exactly one ideal I_i of A such that $\mathbb{V}(I_i) = X_i$, $i = 0,1$. Consider the elements $0, 1 \in A$. The fact that $\operatorname{Spec} A$ is partitioned into X_0 and X_i entails $I_0 \vee I_1 = A$ and $I_0 \cap I_1 = \{0\}$, so that the Chinese Remainder Theorem [3, Lemme 10.6.3] applied to 0 and X_0, and to 1 and X_1, yields one element $b \in A$ such that $b/I_0 = 0$ and $b/I_1 = 1$. Using the Third Isomorphism Theorem, the latter conditions imply $b/\mathfrak{p} \in \{0,1\}$ for each $\mathfrak{p} \in \operatorname{Spec} A$ so that by Lemma 9.2 we conclude that b is

Boolean. If $b' \in A$ also satisfies $b'/\mathfrak{p} = 0$ for each $\mathfrak{p} \in X_0$ and $b'/\mathfrak{p} = 1$ for each $\mathfrak{p} \in X_1$, then $b/\mathfrak{p} = b'/\mathfrak{p}$ for $\mathfrak{p} \in \operatorname{Spec} A$, so that $b = b'$ because $\bigcap \operatorname{Spec} A = \{0\}$ [10, 1.3.3]. $\quad \square$

We record a corollary that will have further use in the paper. It is the exact analogue for MV-algebras of a standard result for the category Ring, see e.g. [23, Theorem 7.3]. In order to state it, let us write $\operatorname{Cp} X$ for the Boolean algebra of clopen sets of any topological space X. Let us then observe that the uniqueness assertion about the Boolean element b in Lemma 9.3 allows us to define, for any MV-algebra A, a function

$$\chi_A \colon \operatorname{Cp}(\operatorname{Spec} A) \longrightarrow \mathrm{P}A \tag{11}$$

that assigns to each $X_0 \in \operatorname{Cp}(\operatorname{Spec} A)$ the unique element $b \in \mathrm{P}A$ with the properties stated in that lemma with respect to X_0 and $X_1 := \operatorname{Spec} A \setminus X_0$. It is then elementary to verify that χ_A is a homomorphism of Boolean algebras.

Corollary 9.4. *For any MV-algebra A, the function*

$$\phi_A \colon \mathrm{P}A \longrightarrow \operatorname{Cp}(\operatorname{Spec} A)$$

that sends $b \in \mathrm{P}A$ to $\mathbb{V}(b) \subseteq \operatorname{Cp}(\operatorname{Spec} A)$ is an isomorphism of Boolean algebras whose inverse is the homomorphism χ_A in (11). In particular, A is indecomposable if, and only if, $\operatorname{Spec} A$ is connected.

Proof. By Lemma 9.2 it is clear that $\mathbb{V}(b)$ for each $b \in \mathrm{P}A$ is clopen and that ϕ_A is a homomorphism. Let us consider $b' := \chi_A \phi_A b$. For each $\mathfrak{p} \in \mathbb{V}(b)$ we have $b/\mathfrak{p} = 0$ by definition of $\mathbb{V}$, and $b'/\mathfrak{p} = 0$ by the defining property of b'. Similarly, for each $\mathfrak{p} \in \operatorname{Spec} A \setminus \mathbb{V}(A)$ we have $b/\mathfrak{p} = b'/\mathfrak{p} = 0$. Thus, b and b' agree at each prime and thus $b = b'$ because $\bigcap \operatorname{Spec} A = \{0\}$ [10, 1.3.3]. Conversely, for $X_0 \in \operatorname{Cp}(\operatorname{Spec} A)$, consider the clopen $\phi_A \chi_A X_0$. For $\mathfrak{p} \in \operatorname{Spec} A$, by definition of χ_A we have $\mathfrak{p} \in X_0$ if, and only if, $(\chi_A X_0)/\mathfrak{p} = 0$. Hence $\phi_A(\chi_A X_0) = X_0$, and the proof is complete. $\quad \square$

The *radical* of A is the ideal

$$\operatorname{Rad} A := \bigcap \operatorname{Max} A.$$

In accordance with standard terminology in general algebra, one says A is *semisimple* precisely when $\operatorname{Rad} A = \{0\}$. We note in passing that, unless A is semisimple, the statement in Lemma 9.2 cannot be strenghtened to "a is Boolean if, and only if, for each $\mathfrak{m} \in \operatorname{Max} A$ we have $a/\mathfrak{m} \in \{0, 1\} \subseteq A/\mathfrak{m}$".

Lemma 9.5. *Let A be an MV-algebra, and suppose there exist (possibly empty) closed subsets $X_0, X_1 \subseteq \operatorname{Max} A$ with $\operatorname{Max} A = X_0 \cup X_1$ and $X_0 \cap X_1 = \emptyset$. Then there exists exactly one Boolean element $b \in A$ such that $b/\mathfrak{m} = 0$ for each $\mathfrak{m} \in X_0$ and $b/\mathfrak{m} = 1$ for each $\mathfrak{m} \in X_1$.*

Proof. By [10, 1.2.12], each $\mathfrak{p} \in \operatorname{Spec} A$ is contained in exactly one $\lambda\mathfrak{p} \in \operatorname{Max} A$, so that we can define a function

$$\lambda \colon \operatorname{Spec} A \longrightarrow \operatorname{Max} A, \tag{12}$$

$$\mathfrak{p} \longmapsto \lambda\mathfrak{p}.$$

By [3, 10.2.3], this function is continuous, and it is a retraction for the inclusion $\operatorname{Max} A \subseteq \operatorname{Spec} A$. Therefore, $X_0' := \lambda^{-1}[X_0]$ and $X_1' := \lambda^{-1}[X_1]$ are closed subsets of $\operatorname{Spec} A$ satisfying $\operatorname{Spec} A = X_0' \cup X_1'$ and $X_0' \cap X_1' = \emptyset$. Now Lemma 9.3 provides a unique Boolean element b such that $b/\mathfrak{p} = 0$ for each $\mathfrak{p} \in X_0'$, and $b/\mathfrak{p} = 1$ for each $\mathfrak{p} \in X_1'$. As $X_i \subseteq X_i'$, $i = 0, 1$, b satisfies the condition in the statement. Concerning uniqueness, suppose a is a Boolean element of A such that $a/\mathfrak{m} = 0$ for each $\mathfrak{m} \in X_0$, and $a/\mathfrak{m} = 1$ for each $\mathfrak{m} \in X_1$. We claim $a = b$. Indeed, let $\mathfrak{p} \in X_i'$, $i = 0, 1$. Then $a/\lambda\mathfrak{p} = i$ because $\lambda\mathfrak{p} \in X_i$. The inclusion $\mathfrak{p} \subseteq \lambda\mathfrak{p}$ induces a quotient map $q \colon A/\mathfrak{p} \to A/\lambda\mathfrak{p}$. By Lemma 9.2 we have $a/\mathfrak{p} \in \{0, 1\}$. Also, $A/\lambda\mathfrak{p}$ is nontrivial. Therefore since $q(a/\mathfrak{p}) = a/\lambda\mathfrak{p} = i$ it follows that $a/\mathfrak{p} = i$. By the uniqueness assertion in Lemma 9.3 we conclude $a = b$. $\square$

Remark 9.6. We observe that the analogue of Lemma 9.5 about coproduct decompositions of $\operatorname{Max} A$ being indexed by idempotent elements does not hold in general for rings. Indeed, spectra of MV-algebras always are completely normal—which affords the existence of the map λ used in the proof above—whereas spectra of rings are not, in general. For more on the important rôle that the continuous retraction λ in (12) plays in the theory of lattice-ordered groups and MV-algebras, see [2] and the references therein.

Our next objective is to show that P sends the unit η of $\mathrm{C} \dashv \operatorname{Max}$ in (7) to an isomorphism.

Lemma 9.7. *For any MV-algebra A, the morphism $\mathrm{P}\,\eta_A \colon \mathrm{P}\,A \to (\mathrm{P}\,\mathrm{C}\operatorname{Max})A$ is an isomorphism.*

Proof. Let $b' \in \mathrm{C}\,(\operatorname{Max} A)$ be Boolean, with the aim of exhibiting $b \in \mathrm{P}\,A$ such that $\eta_A(b) = b'$. Evaluating the defining equality $b' \oplus b' = b'$ at each $\mathfrak{m} \in \operatorname{Max} A$ we see that $b'(\mathfrak{m}) \in \{0, 1\}$ holds. Therefore, the two closed subsets $X_0 := b'^{-1}[\{0\}]$ and $X_1 := b'^{-1}[\{1\}]$ of $\operatorname{Max} A$ satisfy the hypotheses of Lemma 9.5. We conclude that there exists one Boolean element $b \in A$ with $b/\mathfrak{m} = 0$ for $\mathfrak{m} \in X_0$ and $b/\mathfrak{m} = 1$ for $\mathfrak{m} \in X_1$. By the definition of η_A this entails at once $\eta_A(b) = b'$, so η_A is surjective. By the uniqueness statement in Lemma 9.5, η_A is also injective. $\square$

Our next step will be to factor P into a manner that is useful to our purposes.

Lemma 9.7 implies that the functors $\mathsf{MV} \to \mathsf{BA}$ in the diagram below

$$
\begin{array}{ccc}
\mathsf{MV} & \xrightarrow{\ \mathrm{P}\ } & \mathsf{BA} \\
{\scriptstyle \mathrm{Max}}\big\downarrow & & \big\uparrow{\scriptstyle \mathrm{P}} \\
\mathsf{KHaus}^{\mathrm{op}} & \xrightarrow{\ \mathrm{C}\ } & \mathsf{MV}
\end{array}
\tag{13}
$$

are naturally isomorphic.

Lemma 9.8. *The functor* $\mathrm{PC} \colon \mathsf{KHaus}^{\mathrm{op}} \to \mathsf{BA}$ *preserves all set-indexed coproducts.*

Proof. Using Stone duality, it is an exercise to verify that the composite functor $\mathrm{PC} \colon \mathsf{KHaus}^{\mathrm{op}} \to \mathsf{BA}$ induces, by taking opposite categories on each side, a functor naturally isomorphic to the functor $\pi_0 \colon \mathsf{KHaus} \to \mathsf{Stone}$ of Section 7. The lemma then follows from Theorem 7.7. □

We finally obtain the main result of this section.

Theorem 9.9. *The Pierce functor* $\mathrm{P} \colon \mathsf{MV} \to \mathsf{BA}$ *preserves all set-indexed coproducts.*

Proof. As we saw above, the triangle (13) commutes up to a natural isomorphism. Further, Max preserves arbitrary set-indexed colimits because it is left adjoint by Theorem 8.2; and PC preserves set-indexed coproducts by Lemma 9.8. Hence P preserves set-indexed coproducts. □

10. Main result, and final remarks

Let $\mathcal{A}$ be a coextensive category. Recall from the introduction that an object A in $\mathcal{A}$ is *separable* if A is decidable as an object in the extensive $\mathcal{A}^{\mathrm{op}}$. Thus, A is separable if, and only if, there is a morphism $f \colon A + A \to A$ such that the span

$$
A \xleftarrow{\ \nabla\ } A + A \xrightarrow{\ f\ } A
$$

is a product diagram.

Theorem 10.1. *Separable MV-algebras coincide with finite products of subalgebras of* $[0,1] \cap \mathbb{Q}$.

Proof. By Theorem 9.9 we have a reflection $\pi_0 \dashv \mathrm{I}^{\mathrm{op}} \colon \mathsf{Stone} \to \mathsf{MV}^{\mathrm{op}}$ such that both adjoints preserve finite products and finite coproducts, so Proposition 4.7 implies that every decidable object in $\mathsf{MV}^{\mathrm{op}}$ is a finite coproduct of subterminal objects. Theorem 6.9 completes the proof. □

We conclude the paper with some final remarks that point to further research aimed at developing an 'arithmetic connected-component functor'. The guiding result from Algebraic Geometry is this: the category $\mathcal{E}$ of étale schemes over K is reflective as a subcategory of that of locally algebraic schemes over K [13, Proposition I, §4, 6.5]. The left adjoint there is denoted by π_0, and $\pi_0 X$ is called the *k-schéma des composantes connexes de X* in [13, Definition I, §4, 6.6]. Moreover, it is then proved that π_0 preserves finite coproducts. In terms of extensive categories, this says that for $\mathcal{C} = \mathcal{E}^{\mathrm{op}}$, the sub-category $\mathrm{Dec}\,\mathcal{C} \to \mathcal{C}$ has a finite-product preserving left adjoint. We announce that the same holds for $\mathcal{C} = \mathsf{MV}_{\mathrm{fp}}^{\mathrm{op}}$, where $\mathsf{MV}_{\mathrm{fp}}$ is category of finitely presentable MV-algebras. The proof will be published elsewhere, but it is appropriate to indicate here the rôle of locally finite MV-algebras in connection with that result.

An MV-algebra A is *locally finite* if each finitely generated subalgebra of A is finite. Finite MV-algebras are evidently locally finite; $[0,1] \cap \mathbb{Q}$ is an example of a locally finite MV-algebra that is not finite. Locally finite MV-algebras were studied in [11]; see also [12] for a generalisation of the results in [11], and [35, Section 8.3] for further material and [1] for recent progress on the topic. The connection with Theorem 6.9 is the following characterisation of rational algebras.

Lemma 10.2. *For any MV-algebra A the following are equivalent.*

(i) *A is simple and locally finite.*
(ii) *A is a subalgebra of $[0,1] \cap \mathbb{Q}$.*

Proof. (i) $\Rightarrow$ (ii). By Hölder's Theorem (Lemma 6.8), since A is simple there is exactly one monomorphism $A \to [0,1]$; let us therefore identify A with a subalgebra of $[0,1]$. If A contains an irrational number $\rho \in [0,1]$ then the subalgebra generated by ρ is infinite. Indeed, the Euclidean algorithm of successive subtractions applied to $\rho, 1 \in \mathbb{R}$ does not terminate (because ρ and 1 are incommensurable) and produces an infinite descending sequence of distinct, non-zero elements of A. Thus, $A \subseteq [0,1] \cap \mathbb{Q}$ by local finiteness.
(ii) $\Rightarrow$ (i). Any subalgebra of $[0,1]$ evidently has no proper non-trivial ideal, by the Archimedean property of the real numbers, and is therefore simple. If, moreover, $A \subseteq [0,1] \cap \mathbb{Q}$, the subgroup of $\mathbb{R}$ generated by finitely many $a_1, \ldots, a_n \in A$ together with 1 is discrete, and therefore by [10, 3.5.3] the subalgebra generated by $a_1, \ldots, a_n$ is a finite chain. Thus A is locally finite. $\square$

Corollary 10.3. *An MV-algebra A is separable if, and only if, A is locally finite and $\mathrm{P}A$ is finite.*

Proof. If A is separable then, by Theorem 10.1, $A = \prod_{i \in I} A_i$ with I finite and $A_i \subseteq [0,1] \cap \mathbb{Q}$ for each $i \in I$. In particular, $\mathrm{P}\,A$ is finite. Also, each A_i is locally finite by Lemma 10.2. As finite products of locally finite algebras are locally finite, A is locally finite. Conversely, assume that A is locally finite and $\mathrm{P}\,A$ is finite. Then, $A = \prod_{i \in I} A_i$

with I finite and A_i directly indecomposable for each $i \in I$. As locally finite algebras are closed under quotients, each A_i is locally finite. Hence, each A_i is locally finite and indecomposable. But then A must be simple. Indeed, Corollary 9.4 entails that Spec A is connected, and Spec $A =$ Max A by [11, Theorem 5.1]. Then the spectral space Spec A is Hausdorff, and thus has a base of clopen sets—hence, being compact, it is a Stone space. Since Stone spaces are totally disconnected, connectedness of Spec A entails that Spec A is a singleton, so A has exactly two ideals, and so is simple. By Lemma 10.2, A is then a subalgebra of $[0, 1] \cap \mathbb{Q}$. Therefore, A is separable by Theorem 10.1. $\quad\square$

Now, let $\mathsf{LF} \to \mathsf{MV}$ be the full subcategory determined by locally finite MV-algebras. Let us prove that this subcategory is coreflective.

An element a of an MV-algebra A is *of finite order-rank*[5] if the subalgebra B it generates in A is finite. If B is terminal, we say the order-rank of a is zero. Otherwise, there exists exactly one $n \in \{1, 2, \ldots\}$ such that $B = C_1 \times \cdots \times C_n$ with each C_i directly indecomposable and non-terminal, and we then say the order-rank of a is n. We set

$$\mathrm{R}A := \{a \in A \mid a \text{ is of finite order-rank}\}.$$

Note that $\mathrm{P}A \subseteq \mathrm{R}A$, because any Boolean algebra is locally finite. For any MV-algebra A and subset $G \subseteq A$, let us write $\mathrm{S}G$ for the subalgebra of A generated by G. When $G = \{g\}$ we write $\mathrm{S}g$ for $\mathrm{S}\{g\}$.

Lemma 10.4. *Any homomorphism of MV-algebras sends elements of finite order-rank to elements of finite order-rank.*

Proof. Let $h: A \to B$ be a homomorphism and let $a \in \mathrm{R}A$. Since h commutes with operations, a routine argument in general algebra shows that $h[\mathrm{S}a] = \mathrm{S}(ha)$; since $\mathrm{S}a$ is finite, so is $\mathrm{S}(ha)$. $\quad\square$

Lemma 10.5. *For any MV-algebra A, $\mathrm{R}A$ is a locally finite subalgebra of A. Further, $\mathrm{R}A$ is the inclusion-largest locally finite subalgebra of A.*

Proof. Let $F := \{a_1, \ldots, a_n\} \subseteq A$ be a finite subset of elements of finite order-rank, $n \geqslant 0$ an integer. We need to show that the subalgebra $\mathrm{S}F$ of A generated by F is finite. Induction on n. If $n = 0$ then $\mathrm{S}\emptyset$ is either the terminal one-element algebra or the initial two-element algebra. Now suppose $G := \{a_1, \ldots, a_{n-1}\}$ is such that $\mathrm{S}G$ is finite. The subalgebra $\mathrm{S}a_n$ is also finite, because a_n is of finite order-rank by hypothesis. The subalgebra $\mathrm{S}F$ is the least upper bound of $\mathrm{S}G$ and of $\mathrm{S}a_n$ in the lattice of subalgebras of A, and therefore can be written as a quotient of the coproduct $\mathrm{S}G + \mathrm{S}a_n$. In more detail, by the universal property of the coproduct, the inclusion maps $\mathrm{S}G \subseteq \mathrm{S}F$ and

---

[5] The terminology we introduce here is best motivated using lattice-ordered groups—please see Appendix A.

$\mathrm{S}a_n \subseteq \mathrm{S}F$ induce a unique homomorphism $h\colon \mathrm{S}G + \mathrm{S}a_n \to A$ whose regular-epi/mono factorization $h = mq$ is such that $m\colon S \to A$ exhibits the subobject of A that is the join of the subobjects $\mathrm{S}G$ and $\mathrm{S}a_n$—in particular, S is isomorphic to $\mathrm{S}F$. So $\mathrm{S}F$ is a quotient of the algebra $\mathrm{S}G + \mathrm{S}a_n$. Since finite coproducts of finite MV-algebras are finite by [35, Corollary 7.9(iii)], $\mathrm{S}G + \mathrm{S}a_n$ is finite and therefore so is $\mathrm{S}F$.

To show that $\mathrm{R}A$ is a subalgebra of A, first note that clearly $0 \in \mathrm{R}A$. If $a \in \mathrm{R}A$ then $\neg a$ lies in the subalgebra generated by a, which is finite; hence $\neg a$ is of finite order-rank. If $a, b \in \mathrm{R}A$, then $a \oplus b$ lies in the subalgebra generated by $\{a, b\}$, which is finite by the argument in the preceding paragraph; hence $a \oplus b$ is of finite order-rank.

For the last assertion in the statement, let B be a locally finite subalgebra of A. Given any $b \in B$, the subalgebra generated by b in A is finite, by our assumption about B; hence b is of finite order-rank, and $b \in \mathrm{R}\, A$. This completes the proof. □

Lemmas 10.4 and 10.5 allow us to regard R as a functor

$$\mathrm{R}\colon \mathsf{MV} \longrightarrow \mathsf{LF}.$$

Corollary 10.6. *The functor* $\mathrm{R}\colon \mathsf{MV} \longrightarrow \mathsf{LF}$ *is right adjoint to the full inclusion* $\mathsf{LF} \longrightarrow \mathsf{MV}$.

Proof. This is an immediate consequence of the fact that $\mathrm{R}A$ is the largest locally finite subalgebra of the MV-algebra A, as proved in Lemma 10.5. □

Remark 10.7. It is proved in [35, Theorem 8.10] that LF has all set-indexed products. This follows at once from Corollary 10.6: indeed, for any set-indexed family $\{A_i\}_{i \in I}$ of locally finite MV-algebras the product of $\{A_i\}_{i \in I}$ in LF is the coreflection $\mathrm{R}\left(\prod_{i \in I} A_i\right)$ of the product $\prod_{i \in I} A_i$ in MV.

We have been unable to prove that $\mathrm{R}^{\mathrm{op}}\colon \mathsf{MV}^{\mathrm{op}} \to \mathsf{LF}^{\mathrm{op}}$ preserves finite products. However, writing $\mathcal{C}$ for $\mathsf{MV}_{\mathrm{fp}}^{\mathrm{op}}$, we can show that the functor R^{op} restricts to a left adjoint $\pi_0\colon \mathcal{C} \to \mathrm{Dec}\,\mathcal{C}$ to the inclusion $\mathrm{Dec}\,\mathcal{C} \to \mathcal{C}$ and, moreover, it preserves finite products. As mentioned, the proof will appear elsewhere.

Data availability

No data was used for the research described in the article.

Appendix A. Separable unital lattice-ordered Abelian groups

For background on lattice-ordered groups we refer to [3]. We recall that a *lattice-ordered group*, or *ℓ-group* for short, is a group that is also a lattice[6] such that the group operation distributes over binary meets and joins. We only consider Abelian ℓ-groups, and thus adopt additive notation. The underlying group of an Abelian ℓ-group is torsion-free, and its underlying lattice is distributive. Write $\ell\mathsf{A}$ for the category of Abelian ℓ-groups and of their homomorphisms preserving both the group and the lattice structure. An element $1 \in G$ in an Abelian ℓ-group is a (*strong order*) *unit* if for each $g \in G$ there is a natural number n such that $n1 \geqslant g$. An Abelian ℓ-group G equipped with a distinguished unit 1 is called *unital*, and denoted $(G, 1)$. Write $\ell\mathsf{A}_1$ for the category of unital Abelian ℓ-groups and of unit-preserving homomorphisms.

There is a functor $\Gamma\colon \ell\mathsf{A}_1 \to \mathsf{MV}$ that acts on objects by sending $(G, 1)$ to its unit interval $[0, 1] \coloneqq \{x \in G \mid 0 \leqslant x \leqslant 1\}$, and on morphisms by restriction; here, $[0, 1]$ is regarded as an MV-algebra under the operations $x \oplus y \coloneqq (x + y) \wedge 1$, $\neg x \coloneqq 1 - x$, and 0. This functor has an adjoint $\Xi\colon \mathsf{MV} \to \ell\mathsf{A}_1$, and Mundici proved in [33] that Γ and Ξ constitute an equivalence of categories.

The initial object in $\ell\mathsf{A}_1$ is $(\mathbb{Z}, 1)$, and the terminal object is the trivial unital ℓ-group $(\{0 = 1\}, 0)$. In analogy with the relationship between non-unital and unital rings, the category $\ell\mathsf{A}$ has a zero object and is not coextensive, while the category $\ell\mathsf{A}_1$ is. Separable unital Abelian ℓ-groups are defined as for any coextensive category, cf. the beginning of Section 10.

An object G of $\ell\mathsf{A}$ is *Archimedean* if whenever $nx \leqslant y$ holds in G for each positive integer n, then $x \leqslant 0$; and an object $(G, 1)$ of $\ell\mathsf{A}_1$ is called Archimedean if G is. The following characterisations hold: $(G, 1)$ is Archimedean precisely when $\Gamma(G, 1)$ is semisimple; and $(G, 1)$ is totally ordered and Archimedean precisely when $\Gamma(G, 1)$ is simple. Hölder's Theorem for the category $\ell\mathsf{A}_1$ may be stated as follows: *Any $(G, 1)$ that is Archimedean and totally ordered has exactly one morphism to $(\mathbb{R}, 1)$, and that morphism is monic* (equivalently, its underlying function is injective).

Let us say that an object $(G, 1)$ of $\ell\mathsf{A}_1$ is *rational* if it is isomorphic to an ordered subgroup of the additive group $\mathbb{Q}$ containing 1, where the order of G is inherited from the natural order of the rationals. Theorem 6.9 may be then formulated for the category $\ell\mathsf{A}_1$ as follows.

Theorem A.1. *For any unital Abelian ℓ-group $(G, 1)$ the following are equivalent.*

(i) *$(G, 1)$ is rational.*
(ii) *$(G, 1)$ is non-trivial, and the unique map $(\mathbb{Z}, 1) \to (G, 1)$ is epic.*
(iii) *The unique map $(\mathbb{Z}, 1) \to (G, 1)$ is monic and epic.*

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[6] In this appendix, lattices are only required to have binary meets and joins, but not top or bottom elements.

(iv) $(G, 1)$ *is totally ordered and Archimedean, and the unique map* $(\mathbb{Z}, 1) \to (G, 1)$ *is epic.*

An object $(G, 1)$ of $\ell\mathsf{A}_1$ is *Specker* if its unit-interval MV-algebra $\Gamma(G, 1)$ is a Boolean algebra. Write Sp_1 for the full subcategory of $\ell\mathsf{A}_1$ on the Specker objects. The inclusion functor $\mathsf{Sp}_1 \to \ell\mathsf{A}_1$ has a right adjoint $\mathrm{P} \colon \ell\mathsf{A}_1 \to \mathsf{Sp}_1$, the *Pierce functor* for $\ell\mathsf{A}_1$, and P preserves arbitrary coproducts (Theorem 9.9). Our main result, Theorem 10.1, would be proved for the category $\ell\mathsf{A}_1$ using this Pierce functor; it can be phrased as follows.

Theorem A.2. *Separable unital Abelian ℓ-groups coincide with finite products of rational unital Abelian ℓ-groups.*

Remark A.3. Products in the category $\ell\mathsf{A}$ are Cartesian products, because $\ell\mathsf{A}$ is a variety of algebras. On the other hand, while $\ell\mathsf{A}_1$ is equivalent to a variety by Mundici's cited theorem, its underlying-set functor is not right adjoint. Indeed, products in $\ell\mathsf{A}_1$ are not, in general, Cartesian products. However, finite products in $\ell\mathsf{A}_1$ *are* Cartesian—the product of $(G, 1)$ and $(H, 1)$ is $(G \times H, (1, 1))$ with the Cartesian projections.

An Abelian ℓ-group is called a *simplicial group* if it is isomorphic in $\ell\mathsf{A}$ to a free Abelian group of finite rank $\mathbb{Z}^r$ equipped with the coordinatewise order. A unit in such a simplicial group is then any element $1 \in \mathbb{Z}^r$ whose each coordinate is strictly positive; the pair $(\mathbb{Z}^r, 1)$ is called a *unital simplicial group*. These lattice-ordered groups play a key rôle in the representation theory of dimension groups, see e.g. [20].

An object $(G, 1)$ in $\ell\mathsf{A}_1$ is a unital simplicial group exactly when its unit-interval MV-algebra $\Gamma(G, 1)$ is finite. An object $(G, 1)$ is *locally simplicial* if each sublattice subgroup generated by finitely many elements along with 1 is a unital simplicial group. An object $(G, 1)$ in $\ell\mathsf{A}_1$ is locally simplicial exactly when its unit-interval MV-algebra $\Gamma(G, 1)$ is locally finite. Then: *An object $(G, 1)$ of $\ell\mathsf{A}_1$ is separable just when it is locally simplicial, and* $\mathrm{P}(G, 1)$ *has finite ($\mathbb{Z}$-module) rank*[7] (Corollary 10.3).

Write LS_1 for the full subcategory of $\ell\mathsf{A}_1$ on the locally simplicial objects. *The inclusion functor* $\mathsf{LS}_1 \to \ell\mathsf{A}_1$ *has a right adjoint* $\mathrm{R} \colon \ell\mathsf{A}_1 \to \mathsf{LS}_1$ (Corollary 10.6); that is, every $(G, 1)$ has an inclusion-largest locally simplicial unital sublattice subgroup. To prove this in the category $\ell\mathsf{A}_1$ one would introduce the notion of element of 'finite-order rank' of a unital Abelian ℓ-group. It is this notion that motivates the terminology we adopted in the context of MV-algebras in Section 10; by way of conclusion of this appendix, we offer a short discussion.

Let $(G, 1)$ be a unital Abelian ℓ-group, let $g \in G$, and let H be the sublattice subgroup of G generated by g and by 1. If $(H, 1)$ is a unital simplicial group $(\mathbb{Z}^r, 1)$—equivalently, if the MV-algebra $\Gamma(H, 1)$ is finite—then we call g an element of *finite order-rank r*.

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[7] In the literature on lattice-ordered groups, the condition that $\mathrm{P}(G, 1)$ has finite rank is expressed in the following traditional manner: the unit of G has finitely many components.

This notion of rank crucially depends on the interplay between the lattice and the group structure, and is not reducible to the linear notion of rank. To explain why, let us preliminarily observe that a simplicial group $\mathbb{Z}^r$ enjoys the finiteness property that its positive cone $(\mathbb{Z}^r)^+$—that is, the monoid of non-negative elements of $\mathbb{Z}^r$—is finitely generated as a monoid. Next, let us point out that the underlying group of the Abelian ℓ-group H generated by g and 1 in G is necessarily free: indeed, any finitely generated object of $\ell\mathsf{A}$ has free underlying group, as was proved in [19]. The $\mathbb{Z}$-module rank of H is at most countably infinite, because H is countable. But even if we assume the rank of H is finite, the unit-interval $\Gamma(H, 1)$ may be infinite, and in that case the lattice order of $\mathbb{Z}^r \cong H$ cannot be simplicial—and indeed, one can prove that the monoid H^+ cannot be finitely generated. Hence, the condition that the sublattice subgroup H of G generated by g and 1 is simplicial is strictly stronger than the condition that H has finite $\mathbb{Z}$-module rank. To illustrate, consider the subgroup H of $\mathbb{R}$ generated by an irrational number $\rho \in \mathbb{R}$ together with 1; then $H \cong \mathbb{Z}^2$ as groups, the total order inherited by $\mathbb{Z}^2$ from $\mathbb{R}$ is palpably not simplicial, the positive cone H^+ can be shown not to be finitely generated by an easy direct argument, and $\Gamma(H, 1)$ is an infinite simple MV-algebra.

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