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ORIGINAL PAPER

Heat equation with an exponential nonlinear boundary condition in the half space

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Abstract
We consider the initial-boundary value problem for the heat equation in the half space with an exponential nonlinear boundary condition. We prove the existence of global-in-time solutions under the smallness condition on the initial data in the Orlicz space $\exp L^2(\mathbb{R}^N_+)$. Furthermore, we derive decay estimates and the asymptotic behavior for small global-in-time solutions.

Keywords Global existence · Asymptotic behavior · Initial-boundary value problem · Nonlinear boundary condition · Exponential nonlinearity · Orlicz space

Mathematics Subject Classification 35A01 · 35B40 · 35K20 · 35K60 · 46E30

1 Introduction

We consider the initial-boundary value problem for the heat equation in the half space $\mathbb{R}^N_+ = \{x = (x', x_N) \in \mathbb{R}^N : x_N > 0\}$ with a nonlinear boundary condition

$$\begin{cases} \partial_t u = \Delta u, & x \in \mathbb{R}^N_+, \quad t > 0, \\ u(x,0) = \varphi(x), & x \in \mathbb{R}^N_+, \\ \partial_\nu u = f(u), & x \in \partial\mathbb{R}^N_+, \quad t > 0, \end{cases} \tag{1.1}$$

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where $N \geq 1$, $\partial_t = \partial/\partial t$, $\partial_\nu = -\partial/\partial x_N$, and φ is the given initial data. Here $f(u)$ is the nonlinearity which has an exponential growth at infinity with $f(0) = 0$. More precisely, the condition for the nonlinearity (see (1.9)) covers certain limiting cases which are critical with respect to the growth of the nonlinearity and the regularity of the initial data. In this paper, under a smallness condition on the initial data, we prove the existence of global-in-time solutions to problem (1.1). Furthermore, we derive some decay estimates and the asymptotic behavior of small global-in-time solutions.

The nonlinear boundary value problem such as (1.1) can be physically interpreted as a nonlinear radiation law. The case of power nonlinearities $f(u) = |u|^{p-1}u$ with $p > 1$, that is,

$$\begin{cases} \partial_t u = \Delta u, & x \in \mathbb{R}^N_+, \quad t > 0, \\ u(x,0) = \varphi(x), & x \in \mathbb{R}^N_+, \\ \partial_\nu u = |u|^{p-1}u, & x \in \partial\mathbb{R}^N_+, \quad t > 0, \end{cases} \tag{1.2}$$

has been extensively studied in many papers (see e.g. [5, 6, 11, 13, 17–22, 25, 26] and the references therein). It is well-known that problem (1.2) satisfies a scale invariance property, namely, for $\lambda \in \mathbb{R}_+$, if u is a solution to problem (1.2), then

$$u_\lambda(x,t) := \lambda^{\frac{1}{p-1}} u(\lambda x, \lambda^2 t) \tag{1.3}$$

is also a solution to problem (1.2) with initial data $\varphi_\lambda(x) := \lambda^{1/(p-1)}\varphi(\lambda x)$. In the study of the nonlinear boundary value problem (1.2), it seems that all function spaces invariant with respect to the scaling transformation (1.3) play an important role. In fact, for Lebesgue spaces, we can easily show that the norm of the space $L^q(\mathbb{R}^N_+)$ is invariant with respect to (1.3) if and only if $q = q_c := N(p-1)$, and, for the given nonlinearity $|u|^{p-1}u$, the Lebesgue space $L^{q_c}(\mathbb{R}^N_+)$ plays the role of critical space for the local well-posedness and the existence of global-in-time solutions to problem (1.2) (see e.g. [13, 18, 20]).

On the other hand, the case of the Cauchy problem with the power nonlinearity, that is,

$$\partial_t u = \Delta u + |u|^{p-1}u, \quad x \in \mathbb{R}^N, \quad t > 0, \qquad u(x,0) = \varphi(x), \quad x \in \mathbb{R}^N, \tag{1.4}$$

also satisfies a scale invariance property, namely, for $\lambda \in \mathbb{R}_+$, if u is a solution to problem (1.4), then

$$u_\lambda(x,t) := \lambda^{\frac{2}{p-1}} u(\lambda x, \lambda^2 t) \tag{1.5}$$

is also a solution to problem (1.4) with the initial data $\varphi_\lambda(x) := \lambda^{2/(p-1)}\varphi(\lambda x)$. So we can easily show that the norm of the space $L^q(\mathbb{R}^N)$ is invariant with respect to (1.5) if and only if $q = \tilde{q}_c := N(p-1)/2$, and it is well-known that the Lebesgue space $L^{\tilde{q}_c}(\mathbb{R}^N)$ plays the role of critical space for the well-posedness of problem (1.4) (see e.g. [3, 12, 27, 30, 31] and references therein). Furthermore, the scaling property (1.5) also holds for the nonlinear Schrödinger equation

$$i\partial_t u + \Delta u = |u|^{p-1}u, \quad x \in \mathbb{R}^N, \quad t > 0, \qquad u(x,0) = \varphi(x), \quad x \in \mathbb{R}^N, \tag{1.6}$$

and it is well known that the Sobolev space $H^{s_c}(\mathbb{R}^N)$ with $s_c := N/2 - 2/(p-1)$ plays the role of critical space for the well-posedness of problem (1.6) (see e.g. [4]). From these results, we have two critical growth rates of the nonlinearity, that is, $p_h := 1 + (2q)/N$ and $p_s := 1 + 4/(N - 2s)$, and these two critical exponents are connected by the Sobolev embedding, $\dot{H}^s(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$, where s and q satisfy $0 \leq s < N/2$ and $1/q = 1/2 - s/N$. The case $s_c = N/2$ is a limiting case from the following points of view:

(i) for $s > N/2$, $H^s(\mathbb{R}^N)$ embeds into $L^\infty(\mathbb{R}^N)$;
(ii) any power nonlinearity is subcritical, since $H^{N/2}(\mathbb{R}^N)$ embeds into any $L^q(\mathbb{R}^N)$ space (for $q \geq 2$);
(iii) $H^{N/2}(\mathbb{R}^N)$ does not embed into $L^\infty(\mathbb{R}^N)$, and thanks to Trudinger's inequality [29] one knows that $H^{N/2}(\mathbb{R}^N)$ embeds into the Orlicz space $\exp L^2(\mathbb{R}^N)$.

For this limiting case, Nakamura and Ozawa [24] consider the nonlinear Schrödinger equation with an exponential nonlinearity of asymptotic growth $f(u) \sim e^{u^2}$ and with a vanishing behavior at the origin, and they show the existence of global-in-time solutions under a smallness assumption of the initial data in $H^{N/2}(\mathbb{R}^N)$.

As a natural analogy to the results of [24], the third author of this paper and Ruf [28] and Ioku [14] consider the Cauchy problem of the semilinear heat equation with exponential nonlinearity of the form

$$f(u) = |u|^{\frac{4}{N}} u e^{u^2} \tag{1.7}$$

and the initial data φ belonging to the Orlicz space $\exp L^2(\mathbb{R}^N)$ defined as

$$\exp L^2(\mathbb{R}^N) := \left\{ u \in L^1_{\mathrm{loc}}(\mathbb{R}^N);\ \int_{\mathbb{R}^N} \left(\exp\left(\frac{|u(x)|}{\lambda} \right)^2 - 1 \right) dx < \infty \quad \text{for some } \lambda > 0 \right\}$$

(see also Definition 2.1). They consider the corresponding integral equation

$$u(t) = e^{t\Delta}\varphi + \int_0^t e^{(t-s)\Delta} f(u(s))\, ds, \tag{1.8}$$

and prove the existence of local/global-in-time (mild) solutions to this equation (1.8) under the smallness assumption of initial data in $\exp L^2(\mathbb{R}^N)$. Furthermore, the authors of this paper and Ruf [10] show the equivalence between mild solutions (solution to the integral equation (1.8)) and weak solutions to the heat equation with the nonlinearity $f(u)$ as in (1.7), and derive some decay estimates and the asymptotic behavior for small global-in-time solutions. The growth rate of (1.7) at infinity seems to be optimal in the framework of the Orlicz space $\exp L^2(\mathbb{R}^N)$. In fact, if $f(u) \sim e^{|u|^r}$ with $r > 2$, there exist some positive initial data $\varphi \in \exp L^2(\mathbb{R}^N)$ such that problem (1.8) does not possess any classical local-in-time solutions (see [15]). For the fractional diffusion case and general power-exponential nonlinearities, see e.g. [8, 10, 23]. Furthermore, for $\varphi \in \exp L^2(\mathbb{R}^N)$, which implies $\varphi \in L^p(\mathbb{R}^N)$ for $p \in [2, \infty)$, the decay rate of (1.7) near origin, that is, $f(u) \sim |u|^{4/N} u$, is optimal in the framework of $L^2(\mathbb{R}^N)$. See e.g. [3, 30].

The above limiting case in $\mathbb{R}^N$ appears from the relationship between p_h and p_s by the Sobolev embedding. For problem (1.2), we can easily show that the norm of the space $H^s(\mathbb{R}^N_+)$ is invariant with respect to (1.3) if and only if $s = \tilde{s}_c := N/2 - 1/(p-1)$, and we have two critical growth rate of the nonlinearity, that is, $\tilde{p}_h = 1 + q/N$ and $\tilde{p}_s = 1 + 2/(N - 2s)$. These two exponents are also connected by the Sobolev embedding, $\dot{H}^s(\mathbb{R}^N_+) \hookrightarrow L^q(\mathbb{R}^N_+)$, where s and q satisfy the same conditions as in the case of $\mathbb{R}^N$. This means that the same limiting case appears for problem (1.2). On the other hand, as far as we know, there are no results which treat the exponential nonlinearity for the nonlinear boundary problem (1.1).

Based on the above, in this paper, we assume that the nonlinearity f satisfies the following: there exist $C_f > 0$ and $\lambda > 0$ such that

$$|f(u) - f(v)| \leq C_f |u - v| (|u|^{\frac{2}{N}} e^{\lambda u^2} + |v|^{\frac{2}{N}} e^{\lambda v^2})$$
$$\text{for every} \quad u, v \in \mathbb{R}, \qquad f(0) = 0. \tag{1.9}$$

This assumption covers the case

$$f(u) = \pm|u|^{\frac{2}{N}} u e^{u^2},$$

which is one of the candidates for the optimal growth rate of the nonlinearity in the framework of the Orlicz space $\exp L^2(\mathbb{R}^N_+)$ and the optimal decay rate near origin in the framework of $L^2(\mathbb{R}^N_+)$ (see e.g. [18]). Following [10, 14, 28], for problem (1.1) with (1.9), we consider the corresponding integral equation, and prove the existence of global-in-time (mild) solutions under some smallness assumption of the initial data in $\exp L^2(\mathbb{R}^N_+)$. Furthermore, we obtain some decay estimates for the solutions in the following two cases

$\varphi \in \exp L^2(\mathbb{R}^N_+)$ only (*slowly decaying case*), and $\varphi \in \exp L^2(\mathbb{R}^N_+) \cap L^p(\mathbb{R}^N_+)$ with $p \in [1, 2)$ (*rapidly decaying case*).

In particular, for the rapidly decaying case $p = 1$, we show that the global-in-time solutions with some suitable decay estimates behave asymptotically like suitable multiples of the Gauss kernel.

Before treating our main results, we introduce some notation and define a solution to problem (1.1). Throughout this paper we often identify $\mathbb{R}^{N-1}$ with $\partial\mathbb{R}^N_+$. Let $g_N = g_N(x, t)$ be the Gauss kernel on $\mathbb{R}^N$, that is,

$$g_N(x, t) := (4\pi t)^{-\frac{N}{2}} \exp\left(-\frac{|x|^2}{4t}\right), \quad x \in \mathbb{R}^N, \quad t > 0. \tag{1.10}$$

Let $G = G(x, y, t)$ be the Green function for the heat equation on $\mathbb{R}^N_+$ with the homogenous Neumann boundary condition, that is,

$$G(x, y, t) := g_N(x - y, t) + g_N(x - y_*, t), \quad x, y \in \overline{\mathbb{R}^N_+}, \quad t > 0, \tag{1.11}$$

where $y_* = (y', -y_N)$ for $y = (y', y_N) \in \overline{\mathbb{R}^N_+}$. Then, we define a (mild) solution to problem (1.1).

Definition 1.1 Let $\varphi \in \exp L^2(\mathbb{R}^N_+)$, $T \in (0, \infty]$, and $u \in C(\overline{\mathbb{R}^N_+} \times (0, T)) \cap L^\infty(0, T; \exp L^2(\mathbb{R}^N_+))$.

(i) In the case when $N \geq 2$, we call u a solution to problem (1.1) in $\mathbb{R}^N_+ \times (0, T)$ if u satisfies

$$u(x, t) = \int_{\mathbb{R}^N_+} G(x, y, t)\varphi(y)\, dy + \int_0^t \int_{\mathbb{R}^{N-1}} G(x, y', 0, t - s) f(u(y', 0, s))\, dy'\, ds \tag{1.12}$$

for $(x, t) \in \overline{\mathbb{R}^N_+} \times (0, T)$ and $u(t) \underset{t \to 0}{\longrightarrow} \varphi$ in the weak* topology.

(ii) In the case when $N = 1$, we call u a solution to problem (1.1) in $(0, \infty) \times (0, T)$ if u satisfies

$$u(x, t) = \int_0^\infty G(x, y, t)\varphi(y)\, dy + \int_0^t G(x, 0, t - s) f(u(0, s))\, ds \tag{1.13}$$

for $(x, t) \in [0, \infty) \times (0, T)$ and $u(t) \underset{t \to 0}{\longrightarrow} \varphi$ in the weak* topology.

In the case when $T = \infty$, we call u a global-in-time solution to problem (1.1).

We recall that $u(t) \underset{t\to 0}{\longrightarrow} \varphi$ in weak* topology if and only if

$$\lim_{t\to 0} \int_{\mathbb{R}^N_+} \Big(u(x,t)\psi(x) - \varphi(x)\psi(x) \Big)\, dx = 0$$

for any ψ belonging to the predual space of $\exp L^2(\mathbb{R}^N_+)$ (see Sect. 2).

In what follows, we denote by $\|\cdot\|_{\exp L^2}$ the norm of $\exp L^2 := \exp L^2(\mathbb{R}^N_+)$ defined by (2.14), for $r \in [1,\infty]$, we write $\|\cdot\|_{L^r} := \|\cdot\|_{L^r(\mathbb{R}^N_+)}$ and $|\cdot|_{L^r} := \|\cdot\|_{L^r(\mathbb{R}^{N-1})}$ for simplicity. Furthermore, for a function $\phi(x', x_N)$ with $x' \in \mathbb{R}^{N-1}$ and $x_N \in [0,\infty)$, we write $|\phi|_{L^r} := \|\phi(x', 0)\|_{L^r(\mathbb{R}^{N-1})}$.

Now we are ready to state the main results of this paper. First we show the existence of global-in-time solutions to problem (1.1) under the smallness assumption of the initial data in $\exp L^2$.

Theorem 1.1 *Let $N \geq 1$ and $\varphi \in \exp L^2$. Suppose that f satisfies (1.9). Then there exist positive constants $\varepsilon = \varepsilon(N) > 0$ and $C = C(N) > 0$ such that, if $\|\varphi\|_{\exp L^2} < \varepsilon$, then there exists a unique global-in-time solution u to problem (1.1) satisfying*

$$\sup_{t>0} \Big(\|u(t)\|_{\exp L^2} + h(t)\|u(t)\|_{L^\infty} \Big) \leq C\|\varphi\|_{\exp L^2}, \tag{1.14}$$

where $h(t) = \min\{t^{N/4}, 1\}$, and for any $q \in [2,\infty)$,

$$\begin{aligned} &\sup_{t>0} t^{\frac{1}{2q}} |u(t)|_{L^q} \leq C\left\{\Gamma\left(\frac{q}{2}+1\right)\right\}^{\frac{1}{q}} \|\varphi\|_{\exp L^2}, \qquad \text{if } N \geq 2,\\ &\sup_{t>0} t^{\frac{1}{2q}} |u(0,t)| \leq C\left\{\Gamma\left(\frac{q}{2}+1\right)\right\}^{\frac{1}{q}} \|\varphi\|_{\exp L^2}, \qquad \text{if } N = 1, \end{aligned} \tag{1.15}$$

where Γ is the gamma function

$$\Gamma(q) := \int_0^\infty \xi^{q-1} e^{-\xi}\, d\xi, \qquad q > 0.$$

Remark 1.1 (i) By the definition of $\mathbb{R}^N_+$, if $N \geq 2$, then the boundary of $\mathbb{R}^N_+$ is $\mathbb{R}^{N-1}$, namely, it is unbounded. On the other hand, if $N = 1$, then the boundary of $\mathbb{R}_+$ is $x = 0$, namely, it is only one point. From these differences, we need to divide the proof into two cases, $N \geq 2$ and $N = 1$, and we have two estimates as in (1.15).

(ii) Following [15], we denote by $\exp L^2_0(\mathbb{R}^N_+)$ the closure of $C_0^\infty(\mathbb{R}^N_+)$ in $\exp L^2(\mathbb{R}^N_+)$. Then, by an argument similar to that in the proof of [15, Theorem 1.2], it seems likely to obtain the existence of local-in-time solutions to problem (1.1) for any $\varphi \in \exp L^2_0(\mathbb{R}^N_+)$ under the weaker condition

$$|f(u) - f(v)| \leq C|u-v|(e^{\lambda u^2} + e^{\lambda v^2}) \quad \text{for every} \quad u, v \in \mathbb{R}, \qquad f(0) = 0,$$

where $\lambda > 0$ and $C > 0$. This has not been fully explored and it is left for further investigation.

From now, we focus on the unique solution u to problem (1.1) satisfying (1.14) and (1.15). The following result gives some decay estimates for the slowly decaying case, that is, $\varphi \in \exp L^2$ only.

Theorem 1.2 *Assume the same conditions as in Theorem 1.1. Furthermore, suppose that there exists a unique solution u to problem (1.1) satisfying (1.14) and (1.15). Then there exist some positive constants* $\varepsilon = \varepsilon(N) > 0$ *and* $C = C(N) > 0$ *such that, if* $\|\varphi\|_{\exp L^2} < \varepsilon$, *then the solution u satisfies*

$$\begin{aligned} &\sup_{t\geq 1} t^{\frac{N}{2}(\frac{1}{2}-\frac{1}{q})}\left(\|u(t)\|_{L^q} + t^{\frac{1}{2q}}|u(t)|_{L^q}\right) \leq C\|\varphi\|_{\exp L^2}, \qquad \text{if} \quad N \geq 2, \\ &\sup_{t\geq 1} t^{\frac{1}{2}(\frac{1}{2}-\frac{1}{q})}\left(\|u(t)\|_{L^q} + t^{\frac{1}{2q}}|u(0,t)|\right) \leq C\|\varphi\|_{\exp L^2}, \qquad \text{if} \quad N = 1, \end{aligned} \tag{1.16}$$

for all $q \in [2, \infty]$.

Remark 1.2 (i) By Theorem 1.1, if $\|\varphi\|_{\exp L^2}$ is small enough, then we can show that the assumption of Theorem 1.2 is not empty.
(ii) We obtain the same decay estimate as the solution to the heat equation in $\mathbb{R}^N_+$ with the homogeneous Neumann boundary condition and initial data in L^2. See (G_1) in Sect. 2.

Next we consider the rapidly decaying case, that is, $\varphi \in \exp L^2 \cap L^p$ with $p \in [1, 2)$. We can prove two kinds of results about decay estimates of solutions to problem (1.1). In Theorem 1.3, we only assume the smallness condition of the $\exp L^2$ norm of the initial data. This means that we can allow the L^p norm of the same data to be large. On the other hand, under this mild assumption, we have an additional restriction about the range of L^q spaces for the case $N \geq 3$. In Theorem 1.4, under a stronger assumption, that is, a smallness assumption not only for the $\exp L^2$ but also for the L^p norm of the initial data, we obtain better decay estimates, with no additional restrictions about the range of L^q spaces even for the case $N \geq 3$. In the following we denote for any $r \geq 1$

$$\|\cdot\|_{\exp L^2 \cap L^r} := \max\{\|\cdot\|_{\exp L^2}, \|\cdot\|_{L^r}\}. \tag{1.17}$$

Theorem 1.3 *Assume the same conditions as in Theorem 1.2. Furthermore, assume* $\varphi \in L^p(\mathbb{R}^N_+)$ *for some* $p \in [1, 2)$. *Put*

$$p_1 := \max\left\{p, \frac{2N}{N+2}\right\}. \tag{1.18}$$

Then there exist some positive constants $\varepsilon = \varepsilon(N) > 0$, $C = C(N) > 0$ *and a positive function* $F = F(N, p_1, \|\varphi\|_{L^{p_1}}, \lambda)$ *such that, if*

$$\|\varphi\|_{\exp L^2} < \min\{\varepsilon, F\}, \tag{1.19}$$

then the solution u satisfies

$$\begin{aligned} &\sup_{t>0} t^{\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})}\left(\|u(t)\|_{L^q} + t^{\frac{1}{2q}}|u(t)|_{L^q}\right) \leq C\|\varphi\|_{\exp L^2 \cap L^{p_1}}, \qquad \text{if} \quad N \geq 2, \\ &\sup_{t>0} t^{\frac{1}{2}(\frac{1}{p}-\frac{1}{q})}\left(\|u(t)\|_{L^q} + t^{\frac{1}{2q}}|u(0,t)|\right) \leq C\|\varphi\|_{\exp L^2 \cap L^p}, \qquad \text{if} \quad N = 1, \end{aligned} \tag{1.20}$$

for all $q \in [p_1, \infty]$. *In particular, if* $p_1 \in (1, 2)$, *then*

$$\|u(t)\|_{L^q} = o\left(t^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})}\right), \quad t \to \infty. \tag{1.21}$$

Theorem 1.4 *Assume the same conditions as in Theorem 1.3. Then there exists a positive constant $\varepsilon = \varepsilon(N)$ such that, if $\|\varphi\|_{\exp L^2 \cap L^p} < \varepsilon$, then (1.20) with $p_1 = p$ holds for all $q \in [p, \infty]$. In particular, for all $q \in [p, \infty)$,*

$$\begin{aligned} &\sup_{t>0} (1+t)^{\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\|u(t)\|_{L^q} + t^{\frac{1}{2q}} |u(t)|_{L^q} \right) \le C \|\varphi\|_{\exp L^2 \cap L^p}, \qquad \text{if} \quad N \ge 2, \\ &\sup_{t>0} (1+t)^{\frac{1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\|u(t)\|_{L^q} + t^{\frac{1}{2q}} |u(0,t)| \right) \le C \|\varphi\|_{\exp L^2 \cap L^p}, \qquad \text{if} \quad N = 1. \end{aligned} \tag{1.22}$$

Furthermore, if $p \in (1, 2)$ or $N \ge 3$, then (1.21) with $p_1 = p$ holds.

Remark 1.3 By (1.9) the nonlinearity $f(u)$ behaves like $|u|^{1+2/N}$ for $u \to 0$. So, for the case $N \ge 2$, since it follows from (1.15) that $u \in L^\infty_{\mathrm{loc}}(0, \infty; L^q(\partial \mathbb{R}^N_+))$ for $q \ge 2$, the nonlinear term $f(u)$ belongs to $L^p(\partial \mathbb{R}^N_+)$ for $p \ge (2N)/(N+2)$. For the case $N = 2$, this means that $f(u) \in L^p(\partial \mathbb{R}^N_+)$ for all $p \ge 1$, but this implies a true constraint for the case $N \ge 3$. This is the reason why in Theorem 1.3 we have to introduce some parameters p_1 (and p_2, p_3, and p_4 in Lemmata 2.2, 5.1, and 5.6, respectively) which are meaningful only for the case $N \ge 3$.

Finally we address the question of the asymptotic behavior of solutions to problem (1.1) when $\varphi \in \exp L^2 \cap L^1$. We show that global-in-time solutions with suitable decay properties behave asymptotically like suitable multiples of the Gauss kernel.

Theorem 1.5 *Let $N \ge 1$ and $\varphi \in \exp L^2 \cap L^1(\mathbb{R}^N_+)$. Furthermore, let u be the global-in-time solution to problem (1.1) satisfying (1.22). Then there exists the limit*

$$m_* := \lim_{t \to \infty} \int_{\mathbb{R}^N_+} u(x,t)\, dx$$

such that

$$\lim_{t \to \infty} t^{\frac{N}{2}(1-\frac{1}{q})} \|u(t) - 2m_* g_N(t)\|_{L^q} = 0, \qquad q \in [1, \infty]. \tag{1.23}$$

Remark 1.4 For the case $N \ge 2$, by (1.12) we see that

$$m_* = \int_{\mathbb{R}^N_+} \varphi(x)\, dx + \int_0^\infty \int_{\mathbb{R}^{N-1}} f(u(x', 0, t))\, dx'\, dt.$$

On the other hand, for the case $N = 1$, by (1.13) we have

$$m_* = \int_0^\infty \varphi(x)\, dx + \int_0^\infty f(u(0,t))\, dt.$$

The paper is organized as follows. In Sect. 2 we recall some properties of the kernel G and its associate semigroup. In Sect. 3, applying the Banach contraction mapping principle, we prove Theorem 1.1. In Sects. 4 and 5, modifying the arguments of [20], we derive decay estimates on the boundary, and prove Theorems 1.2, 1.3, and 1.4. In Sect. 6 we obtain the asymptotic behavior of solutions to problem (1.1).

2 Preliminaries

In this section we recall some properties of the kernel $G = G(x, y, t)$ and its associate semigroup. Throughout this paper, by the letter C we denote generic positive constants that may have different values also within the same line.

We first recall the following properties of the kernel G (see e.g [13, 20, 22]):

(i) $\displaystyle\int_{\mathbb{R}^N_+} G(x,y,t)dy = 1$ for any $x \in \overline{\mathbb{R}^N_+}$ and $t > 0$;

(ii) for any $(x,t), (z,s) \in \overline{\mathbb{R}^N_+} \times (0,\infty)$, it holds that

$$\int_{\mathbb{R}^N_+} G(x,y,t)G(y,z,s)\,dy = G(x,z,t+s). \tag{2.1}$$

By (1.11) we have

$$g_N(x-y,t) \le G(x,y,t) \le 2g_N(x-y,t), \qquad x,y \in \overline{\mathbb{R}^N_+}, \quad t>0. \tag{2.2}$$

Furthermore, it follows from (1.10) and (1.11) that

$$G(x',0,y,t) = 2g_1(y_N,t)g_{N-1}(x'-y',t), \qquad x' \in \mathbb{R}^{N-1}, \quad y \in \overline{\mathbb{R}^N_+}, \quad t>0. \tag{2.3}$$

We denote by $S_1(t)\varphi$ the unique bounded solution to the heat equation in $\mathbb{R}^N_+$ with the homogeneous Neumann boundary condition and the initial datum φ, that is,

$$[S_1(t)\varphi](x) := \int_{\mathbb{R}^N_+} G(x,y,t)\varphi(y)\,dy, \qquad x \in \overline{\mathbb{R}^N_+}, \quad t>0, \tag{2.4}$$

and denote by $e^{t\Delta'}\psi$ the unique bounded solution to the heat equation in $\mathbb{R}^{N-1}$ with the initial datum ψ, that is,

$$[e^{t\Delta'}\psi](x') := \int_{\mathbb{R}^{N-1}} g_{N-1}(x'-y',t)\psi(y')\,dy', \qquad x' \in \mathbb{R}^{N-1}, \quad t>0. \tag{2.5}$$

In the case where $N \ge 2$, we put

$$[S_2(t)\psi](x) := 2g_1(x_N,t)[e^{t\Delta'}\psi](x'), \qquad x \in \overline{\mathbb{R}^N_+}, \quad t>0, \tag{2.6}$$

for $\psi \in L^r(\mathbb{R}^{N-1})$ with some $r \in [1,\infty]$. Since it holds that, for any $r \in [1,\infty]$,

$$\|g_N(t)\|_{L^r} \le 4^{-\frac{1}{2r}}(4\pi t)^{-\frac{N}{2}(1-\frac{1}{r})}, \qquad t>0, \tag{2.7}$$

by (2.2), (2.3), and applying Young's inequality to (2.4) and (2.5) we have the following.

(G_1) There exists a constant c_1, which depends only on N, such that

$$\|S_1(t)\varphi\|_{L^r} \le c_1 t^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{r})}\|\varphi\|_{L^q}, \qquad t>0, \tag{2.8}$$

for $\varphi \in L^q(\mathbb{R}^N_+)$ and $1 \le q \le r \le \infty$. Furthermore, there exists a constant c_2, which depends only on N, such that, for the case $N \ge 2$,

$$|S_1(t)\varphi|_{L^r} \le c_2 t^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2r}}\|\varphi\|_{L^q}, \qquad t>0, \tag{2.9}$$

and, for the case $N=1$,

$$|[S_1(t)\varphi](0)| \le c_2 t^{-\frac{1}{2q}}\|\varphi\|_{L^q}, \qquad t>0. \tag{2.10}$$

(G_2) For any $\psi \in L^q(\mathbb{R}^{N-1})$ and $1 \le q \le r \le \infty$, it holds that

$$\|S_2(t)\psi\|_{L^r} \le Ct^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}(1-\frac{1}{q})}|\psi|_{L^q}, \qquad t>0, \tag{2.11}$$

$$|S_2(t)\psi|_{L^r} \le Ct^{-\frac{N-1}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}}|\psi|_{L^q}, \qquad t>0. \tag{2.12}$$

(G_3) Let $\varphi \in L^q(\mathbb{R}^N_+)$ with $1 \le q \le \infty$. Then, for any $T > 0$, $S_1(t)\varphi$ is bounded and smooth in $\overline{\mathbb{R}^N_+} \times (T, \infty)$.

We recall now the definition and the main properties of the Orlicz space $\exp L^2$.

Definition 2.1 We define the Orlicz space $\exp L^2$ as

$$\exp L^2 := \left\{ u \in L^1_{loc}(\mathbb{R}^N_+);\ \int_{\mathbb{R}^N_+} \left(\exp\left(\frac{|u(x)|}{\lambda} \right)^2 - 1 \right) dx < \infty \quad \text{for some } \lambda > 0 \right\}, \tag{2.13}$$

where the norm is given by the Luxemburg type

$$\|u\|_{\exp L^2} := \inf \left\{ \lambda > 0 \text{ such that } \int_{\mathbb{R}^N_+} \left(\exp\left(\frac{|u(x)|}{\lambda} \right)^2 - 1 \right) dx \le 1 \right\}. \tag{2.14}$$

The space $\exp L^2$ endowed with the norm $\|u\|_{\exp L^2}$ is a Banach space, and admits as predual the Orlicz space defined by the complementary function of $A(t) = e^{t^2} - 1$, denoted by $\tilde{A}(t)$. This complementary function is a convex function such that $\tilde{A}(t) \sim t^2$ as $t \to 0$ and $\tilde{A}(t) \sim t \log^{1/2} t$ as $t \to \infty$. (see e.g. [2, Section 8].) Furthermore, it follows from (2.13) that

$$L^2(\mathbb{R}^N_+) \cap L^\infty(\mathbb{R}^N_+) \subset \exp L^2, \tag{2.15}$$

and we have

$$\|u\|_{\exp L^2} \le \frac{1}{\sqrt{\log 2}} (\|u\|_{L^2} + \|u\|_{L^\infty}). \tag{2.16}$$

(In the case where $\Omega = \mathbb{R}^N$, see e.g. [15, 23].) On the other hand, it is well known that, for any $2 \le p < \infty$,

$$\|u\|_{L^p} \le \left[\Gamma\left(\frac{p}{2} + 1 \right) \right]^{\frac{1}{p}} \|u\|_{\exp L^2}. \tag{2.17}$$

(See e.g. [15, Proposition 2.1].) Then, applying the same argument as in the proof of [14, Lemma 2.2] with (2.17), we have

$$\|S_1(t)\varphi\|_{\exp L^2} \le \|\varphi\|_{\exp L^2}, \qquad t > 0. \tag{2.18}$$

Next we recall the following property of the Gamma function.

Lemma 2.1 [10, Lemma 3.3] *For any $q \ge 1$ and $r \ge 1$, there exists a positive constant $C > 0$, which is independent of q and r, such that*

$$\Gamma(rq + 1)^{\frac{1}{q}} \le C\Gamma(r + 1) q^r.$$

Applying this lemma, we prepare the following estimate for the nonlinear term f for the case $N \ge 2$.

Lemma 2.2 *Let $N \ge 2$ and $m > 0$. Suppose that, for any $q \in [2, \infty)$, the function $u \in C(\overline{\mathbb{R}^N_+} \times (0, \infty))$ satisfies the condition*

$$\sup_{t>0} t^{\frac{1}{2q}} |u(t)|_{L^q} \le \left\{ \Gamma\left(\frac{q}{2} + 1 \right) \right\}^{\frac{1}{q}} m. \tag{2.19}$$

Let f be the function satisfying the condition (1.9), and put

$$p_2 := \frac{2N}{N+2}. \tag{2.20}$$

Then, for all $r \in [p_2, \infty)$, there exists a positive constant $\varepsilon = \varepsilon(r, \lambda) > 0$ such that, if $m < \varepsilon$, then

$$\sup_{t>0} t^{\frac{1}{2r}} |f(u(t))|_{L^r} \le Crm^{1+\frac{2}{N}}, \tag{2.21}$$

where C is independent of r, N, and m.

Proof For any $k \in \mathbb{N} \cup \{0\}$, we put

$$\ell_k := 2k + 1 + \frac{2}{N}. \tag{2.22}$$

Then, since it holds from $N \ge 2$ and $r \ge p_2$ with (2.20) that

$$\ell_k r \ge \left(1 + \frac{2}{N}\right) p_2 = \left(1 + \frac{2}{N}\right) \frac{2N}{N+2} = 2, \qquad k \in \mathbb{N} \cup \{0\},$$

by (1.9) and (2.19) we have

$$\begin{aligned} |f(u(t))|_{L^r} &\le C \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} |u(t)|_{L^{\ell_k r}}^{\ell_k} \\ &\le C \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \left(\Gamma\left(\frac{\ell_k r}{2} + 1\right)^{\frac{1}{\ell_k r}} t^{-\frac{1}{2\ell_k r}} m \right)^{\ell_k}, t > 0. \end{aligned} \tag{2.23}$$

Applying Lemma 2.1 with the monotonicity property of the Gamma function $\Gamma(q)$ for $q \ge 3/2$ (see, e.g. [1]), we see that

$$\begin{aligned} \Gamma\left(\frac{\ell_k r}{2} + 1\right)^{\frac{1}{r}} &\le C\Gamma\left(\frac{\ell_k}{2} + 1\right) r^{\frac{\ell_k}{2}} \\ &= C\Gamma\left(k + \frac{3}{2} + \frac{1}{N}\right) r^{\frac{\ell_k}{2}} \\ &\le C\Gamma(k+2) r^{\frac{\ell_k}{2}} = C(k+1)! r^{\frac{\ell_k}{2}}. \end{aligned}$$

This together with (2.23) implies that

$$|f(u(t))|_{L^r} \le Ct^{-\frac{1}{2r}} \sum_{k=0}^{\infty} \frac{\lambda^k (k+1)!}{k!} (rm^2)^{\frac{\ell_k}{2}} = C(rm^2)^{\frac{1}{2}+\frac{1}{N}} t^{-\frac{1}{2r}} \sum_{k=0}^{\infty} (k+1)(\lambda rm^2)^k, \qquad t > 0.$$

Therefore, taking a sufficiently small $m < \varepsilon(r, \lambda)$ if necessary (e.g. $m^2 \le 1/(4\lambda r)$), we get

$$\sup_{t>0} t^{\frac{1}{2r}} |f(u(t))|_{L^r} \le C \frac{(rm^2)^{\frac{1}{2}+\frac{1}{N}}}{(1 - \lambda rm^2)^2} \le 2C(rm^2)^{\frac{1}{2}+\frac{1}{N}} \le 2Crm^{1+\frac{2}{N}}.$$

This implies (2.21), and the proof of Lemma 2.2 is complete. □

Similarly to the case $N \ge 2$, we prepare the following lemma, which is the one dimensional counterpart of Lemma 2.2.

Lemma 2.3 *Let $m > 0$. Suppose that, for any $q \in [2, \infty)$, the function $u \in C(0, \infty)$ satisfies the condition*

$$\sup_{t>0} t^{\frac{1}{2q}} |u(t)| \le \left\{\Gamma\left(\frac{q}{2}+1\right)\right\}^{\frac{1}{q}} m. \tag{2.24}$$

Let f be the function satisfying the condition (1.9). Then there exists a positive constant $\varepsilon = \varepsilon(\lambda) > 0$ such that, if $m < \varepsilon$, then

$$\sup_{t>0} t^{\frac{1}{2}} |f(u(t))| \le Cm^3, \tag{2.25}$$

and

$$\sup_{t>0} t^{\frac{1}{2r}+\frac{1}{2}} |f(u(t))| \le C\left\{\Gamma\left(\frac{r}{2}+1\right)\right\}^{\frac{1}{r}} m^3, \qquad r \in [2, \infty), \tag{2.26}$$

where C is independent of m and r.

Proof We first prove (2.25). For any $k \in \mathbb{N} \cup \{0\}$, let ℓ_k be the constant defined by (2.22) with $N = 1$, namely, $\ell_k = 2k + 3$. Then, by (1.9) and (2.24) with $q = \ell_k$ we have

$$|f(u(t))| \le C\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}|u(t)|^{\ell_k} \le C\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}\left(\Gamma\left(\frac{\ell_k}{2}+1\right)^{\frac{1}{\ell_k}} t^{-\frac{1}{2\ell_k}} m\right)^{\ell_k}, \quad t > 0. \tag{2.27}$$

Since it holds from the monotonicity property of the Gamma function that

$$\Gamma\left(\frac{\ell_k}{2}+1\right) = \Gamma\left(k+\frac{5}{2}\right) \le \Gamma(k+3) = (k+2)!,$$

by (2.27) we have

$$|f(u(t))| \le Ct^{-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{\lambda^k (k+2)!}{k!} m^{\ell_k} = Ct^{-\frac{1}{2}} m^3 \sum_{k=0}^{\infty} (k+2)(k+1)(\lambda m^2)^k, \qquad t > 0.$$

Therefore, taking a sufficiently small $m < \varepsilon(\lambda)$ if necessary, we get

$$\sup_{t>0} t^{\frac{1}{2}} |f(u(t))| \le C\frac{m^3}{(1-\lambda m^2)^3} \le 2Cm^3.$$

This implies (2.25).

Next we prove (2.26). For any $k \in \mathbb{N} \cup \{0\}$, put $\tilde{\ell}_k = 2k + 2$. Then, similarly to the proof of (2.25), we have

$$|f(u(t))| \le C|u(t)| \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} |u(t)|^{\tilde{\ell}_k}$$

and then, by (2.24) with $q = r$ and also $q = \tilde{\ell}_k$ and taking a sufficiently small $m < \varepsilon(\lambda)$ if necessary, we have

$$
\begin{aligned}
|u(t)| \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} |u(t)|^{\tilde{\ell}_k} &\le m \left\{ \Gamma\left(\frac{r}{2}+1\right) \right\}^{\frac{1}{r}} t^{-\frac{1}{2r}} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \left(\Gamma\left(\frac{\tilde{\ell}_k}{2}+1\right)^{\frac{1}{\tilde{\ell}_k}} t^{-\frac{1}{2\tilde{\ell}_k}} m \right)^{\tilde{\ell}_k} \\
&\le m \left\{ \Gamma\left(\frac{r}{2}+1\right) \right\}^{\frac{1}{r}} t^{-\frac{1}{2r}-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{\lambda^k (k+1)!}{k!} m^{\tilde{\ell}_k} \\
&\le C m^3 \left\{ \Gamma\left(\frac{r}{2}+1\right) \right\}^{\frac{1}{r}} t^{-\frac{1}{2r}-\frac{1}{2}}, \quad t > 0.
\end{aligned}
$$

This implies (2.26), and the proof of Lemma 2.3 is complete. □

3 Existence

In this section we prove Theorem 1.1. We first consider the case $N \ge 2$. We introduce some notation. Let $M > 0$. Set

$$
X_M := \left\{
\begin{array}{l}
u \in C(\overline{\mathbb{R}^N_+} \times (0,\infty)) \cap L^\infty(0,\infty; \exp L^2(\mathbb{R}^N_+)) : \\
\sup_{t>0} \|u(t)\|_{\exp L^2} \le M, \qquad \sup_{t>0} h(t)\|u(t)\|_{L^\infty} \le M \quad \text{with} \quad h(t) = \min\{t^{\frac{N}{4}}, 1\}, \\
\sup_{t>0} t^{\frac{1}{2q}} |u(t)|_{L^q} \le \left\{ \Gamma\left(\frac{q}{2}+1\right) \right\}^{\frac{1}{q}} M \quad \text{with} \quad q \in [2,\infty)
\end{array}
\right\},
$$

equipped with the metric

$$
d_X(u,v) := \sup_{t>0} \left(h(t)\|u(t)-v(t)\|_{L^\infty} + t^{\frac{1}{4N}} |u(t)-v(t)|_{L^{2N}} \right). \tag{3.1}
$$

Then (X_M, d_X) is a complete metric space. For the proof of Theorem 1.1 we apply the Banach contraction mapping principle in X_M to find a fixed point of

$$
\Phi[u](t) := S_1(t)\varphi + D[u](t), \tag{3.2}
$$

where $S_1(t)$ is as in (2.4) and

$$
D[u](t) := \int_0^t S_2(t-s) f(u(s))\, ds. \tag{3.3}
$$

Here $S_2(t)$ is as in (2.6) and f satisfies (1.9). We remark that, for $u \in X_M$, the function $f(u)$ belongs to $C(\overline{\mathbb{R}^N_+} \times (0,\infty))$. Therefore, by Lemma 2.2 we have that $f(u(\cdot,0,s)) \in L^r(\mathbb{R}^{N-1})$ with $r \in [p_2,\infty)$, and we can define $S_2(t-s)f(u(s))$ for $t > s > 0$. More precisely, with an abuse of notation we denote by $S_2(t-s)f(u(s))$ the operator $S_2(t-s)$ applied to the function $f(u(x',0,s))$. In particular, we have

$$
\begin{aligned}
D[u](t) &= \int_0^t S_2(t-s) f(u(s))\, ds \\
&= \int_0^t \int_{\mathbb{R}^{N-1}} 2g_1(x_N, t-s) g_{N-1}(x'-y', t-s) f(u(y',0,s))\, dy'\, ds \\
&= \int_0^t \int_{\mathbb{R}^{N-1}} G(x,y',0,t-s) f(u(y',0,s))\, dy'\, ds.
\end{aligned}
$$

Hence any fixed point of the integral operator Φ satisfies the equation (1.12).

Furthermore, we have the following estimates for the function $D[u]$.

Lemma 3.1 *Let $N \geq 2$ and $u \in X_M$. Then there exists a positive constant $\varepsilon_* = \varepsilon_*(N, \lambda) > 0$ such that, if $M < \varepsilon_*$, then, for any $q \in [2, \infty)$,*

$$\sup_{t>0} \left(\|D[u](t)\|_{L^2} + \|D[u](t)\|_{L^\infty} + t^{\frac{1}{2q}} |D[u](t)|_{L^q} \right) \leq C M^{1+\frac{2}{N}}, \tag{3.4}$$

where C is independent of q and M. Furthermore, $D[u]$ is continuous in $\overline{\mathbb{R}^N_+} \times (0, \infty)$.

Proof We first prove (3.4). Let p_2 be the constant given in (2.20). Then, it holds that

$$1 - \frac{N}{2}\left(\frac{1}{p_2} - \frac{1}{2}\right) - \frac{1}{2}\left(1 - \frac{1}{p_2}\right) - \frac{1}{2p_2} = \frac{N+2}{4} - \frac{N}{2p_2} = 0.$$

By (2.11) with $(q, r) = (p_2, 2)$ and (3.3) we have

$$\begin{aligned} \|D[u](t)\|_{L^2} &\leq \int_0^t \|S_2(t-s) f(u(s))\|_{L^2}\, ds \\ &\leq C \int_0^t (t-s)^{-\frac{N}{2}(\frac{1}{p_2} - \frac{1}{2}) - \frac{1}{2}(1 - \frac{1}{p_2})} |f(u(s))|_{L^{p_2}}\, ds, \qquad t > 0. \end{aligned} \tag{3.5}$$

Since $u \in X_M$, taking a sufficiently small $\varepsilon_1 = \varepsilon_1(p_2, \lambda) > 0$ such that, for $M < \varepsilon_1$, we can apply Lemma 2.2, and it holds that

$$|f(u(t))|_{L^{p_2}} \leq C p_2 M^{1+\frac{2}{N}} t^{-\frac{1}{2p_2}}, \qquad t > 0. \tag{3.6}$$

Substituting (3.6) to (3.5), we see that

$$\begin{aligned} \|D[u](t)\|_{L^2} &\leq C p_2 M^{1+\frac{2}{N}} \int_0^t (t-s)^{-\frac{N}{2}(\frac{1}{p_2} - \frac{1}{2}) - \frac{1}{2}(1 - \frac{1}{p_2})} s^{-\frac{1}{2p_2}}\, ds \\ &\leq C M^{1+\frac{2}{N}} B\left(\frac{1}{2p_2}, 1 - \frac{1}{2p_2}\right), \qquad t > 0, \end{aligned} \tag{3.7}$$

where B is the beta function, namely

$$B(p, q) = \Gamma(p)\Gamma(q)/\Gamma(p+q), \quad p, q > 0.$$

Furthermore, similarly to (3.5), by (2.11) with $(q, r) = (N, \infty)$ and (3.3) we have

$$\|D[u](t)\|_{L^\infty} \leq C \int_0^t (t-s)^{-\frac{N-1}{2N} - \frac{1}{2}} |f(u(s))|_{L^N}\, ds, \qquad t > 0. \tag{3.8}$$

Since $N \geq 2 \geq p_2$, similarly to (3.6), taking a sufficiently small $\varepsilon_2 = \varepsilon_2(N, \lambda) > 0$ such that, for $M < \varepsilon_2$, we get

$$|f(u(t))|_{L^N} \leq C N M^{1+\frac{2}{N}} t^{-\frac{1}{2N}}, \qquad t > 0. \tag{3.9}$$

Substituting (3.9) to (3.8), we see that

$$\begin{aligned} \|D[u](t)\|_{L^\infty} &\leq C N M^{1+\frac{2}{N}} \int_0^t (t-s)^{-1+\frac{1}{2N}} s^{-\frac{1}{2N}}\, ds \\ &\leq C M^{1+\frac{2}{N}} B\left(\frac{1}{2N}, 1 - \frac{1}{2N}\right), \qquad t > 0. \end{aligned} \tag{3.10}$$

On the other hand, for fixed $q \in [2, \infty)$, we put

$$q_* := \frac{Nq}{N+q}.$$

Then, it holds that $p_2 \le q_* < q$ and

$$-\frac{N-1}{2}\left(\frac{1}{q_*}-\frac{1}{q}\right)-\frac{1}{2}=-1+\frac{1}{2N}, \qquad \frac{1}{2N}-\frac{1}{2q_*}=-\frac{1}{2q}. \tag{3.11}$$

By (2.12) with $(q, r) = (q_*, q)$ and (3.3) we have

$$\begin{aligned} |D[u](t)|_{L^q} &\le \int_0^t |S_2(t-s)f(u(s))|_{L^q}\,ds \\ &\le C\int_0^t (t-s)^{-\frac{N-1}{2}(\frac{1}{q_*}-\frac{1}{q})-\frac{1}{2}}|f(u(s))|_{L^{q_*}}\,ds, \qquad t>0. \end{aligned} \tag{3.12}$$

Since $p_2 \le q_* \le N$, similarly to (3.6) again, taking a sufficiently small $\varepsilon_3 = \varepsilon_3(N, \lambda) > 0$ such that, for $M < \varepsilon_3$, we have

$$|f(u(t))|_{L^{q_*}} \le Cq_* M^{1+\frac{2}{N}} t^{-\frac{1}{2q_*}} \le CNM^{1+\frac{2}{N}} t^{-\frac{1}{2q_*}}, \qquad t>0.$$

This together with (3.11) and (3.12) yields that

$$\begin{aligned} |D[u](t)|_{L^q} &\le CM^{1+\frac{2}{N}}\int_0^t (t-s)^{-1+\frac{1}{2N}} s^{-\frac{1}{2q_*}}\,ds \\ &\le CM^{1+\frac{2}{N}} t^{-\frac{1}{2q}} B\left(\frac{1}{2N}, 1-\frac{1}{2q_*}\right) \le CM^{1+\frac{2}{N}} t^{-\frac{1}{2q}}, \qquad t>0, \end{aligned} \tag{3.13}$$

where the constant C depends only on N since $p_2 \le q_* \le N$. Thus, taking $\varepsilon_* = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ with (3.7), (3.10), and (3.13), we obtain (3.4).

Next we prove the continuity of $D[u](x, t)$. Let T be an arbitrary positive constant. Then, it follows from (2.1) that

$$\begin{aligned} D[u](x,t) &= \int_0^t [S_2(t-s)f(u(s))](x)\,ds \\ &= [S_1(t-T/2)D[u](T/2)](x) + \int_{T/2}^t [S_2(t-s)f(u(s))](x)\,ds \end{aligned}$$

for $x \in \overline{\mathbb{R}^N_+}$ and $0 < T < t < \infty$. Then, by (3.4) and (G_3) we see that

$$[S_1(t-T/2)D[u](T/2)](x)$$

is continuous in $\overline{\mathbb{R}^N_+} \times (T, \infty)$. Furthermore, since it follows from $u(t) \in L^\infty(\overline{\mathbb{R}^N_+})$ for $t \ge T/2$ that $f(u(t)) \in L^\infty(\partial\mathbb{R}^N_+)$ for $t \ge T/2$, we apply the same argument as in [9, Section 3, Chapter 1] to see that

$$\int_{T/2}^t [S_2(t-s)f(u(s))](x)\,ds$$

is also continuous in $\overline{\mathbb{R}^N_+} \times (T, \infty)$. (See also [7, Proposition 5.2] and [16, Lemma 2.1].) Therefore we deduce that $D[u]$ is continuous in $\overline{\mathbb{R}^N_+} \times (T, \infty)$. Thus Lemma 3.1 follows from arbitrariness of T. □

Lemma 3.2 *Let $N \geq 2$ and $u, v \in X_M$. Then there exist some positive constants $C = C(N)$ and $\varepsilon^* = \varepsilon^*(N, \lambda) > 0$ such that, if $M < \varepsilon^*$, then*

$$d_X(D[u], D[v]) \leq CM^{\frac{2}{N}} d_X(u, v). \tag{3.14}$$

Proof For any $k \in \mathbb{N} \cup \{0\}$, we put

$$\tilde{\ell}_k := 2k + \frac{2}{N}. \tag{3.15}$$

Then, by (1.9) we recall that

$$|f(u) - f(v)| \leq C|u - v| \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} (|u|^{\tilde{\ell}_k} + |v|^{\tilde{\ell}_k}). \tag{3.16}$$

Since $h(t) \leq 1$, by (2.11) with $(q, r) = (N, \infty)$, (3.3), and (3.16), for any $t > 0$, we have

$$\begin{aligned} &h(t)\|D[u](t) - D[v](t)\|_{L^\infty} \\ &\leq \int_0^t \|S_2(t-s)(f(u(s) - f(v(s)))\|_{L^\infty}\, ds \\ &\leq C \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \int_0^t (t-s)^{-1+\frac{1}{2N}} \left| |u(s) - v(s)| \left(|u(s)|^{\tilde{\ell}_k} + |v(s)|^{\tilde{\ell}_k} \right) \right|_{L^N} ds, \qquad t > 0. \end{aligned} \tag{3.17}$$

Since it follows from Hölder's inequality that

$$\left| |u(s) - v(s)| \left(|u(s)|^{\tilde{\ell}_k} + |v(s)|^{\tilde{\ell}_k} \right) \right|_{L^N} \leq |u(s) - v(s)|_{L^{2N}} \left(|u(s)|^{\tilde{\ell}_k}_{L^{2\tilde{\ell}_k N}} + |v(s)|^{\tilde{\ell}_k}_{L^{2\tilde{\ell}_k N}} \right),$$

by (3.1), (3.15), and (3.17) we see that, for $u, v \in X_M$,

$$\begin{aligned} &h(t)\|D[u](t) - D[v](t)\|_{L^\infty} \\ &\leq C \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \int_0^t (t-s)^{-1+\frac{1}{2N}} |u(s) - v(s)|_{L^{2N}} \left(|u(s)|^{\tilde{\ell}_k}_{L^{2\tilde{\ell}_k N}} + |v(s)|^{\tilde{\ell}_k}_{L^{2\tilde{\ell}_k N}} \right) ds \\ &\leq C \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \int_0^t (t-s)^{-1+\frac{1}{2N}} s^{-\frac{1}{4N} - \frac{\tilde{\ell}_k}{4\tilde{\ell}_k N}} \left(\sup_{s>0} s^{\frac{1}{4N}} |u(s) - v(s)|_{L^{2N}} \right) \times \\ &\qquad \times \left\{ \left(\sup_{s>0} s^{\frac{1}{4\tilde{\ell}_k N}} |u(s)|_{L^{2\tilde{\ell}_k N}} \right)^{\tilde{\ell}_k} + \left(\sup_{s>0} s^{\frac{1}{4\tilde{\ell}_k N}} |v(s)|_{L^{2\tilde{\ell}_k N}} \right)^{\tilde{\ell}_k} \right\} ds \\ &\leq C d_X(u, v) \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \left(\Gamma\left(\tilde{\ell}_k N + 1 \right)^{\frac{1}{2\tilde{\ell}_k N}} M \right)^{\tilde{\ell}_k} \int_0^t (t-s)^{-1+\frac{1}{2N}} s^{-\frac{1}{2N}}\, ds \\ &\leq CM^{\frac{2}{N}} d_X(u, v) B\left(\frac{1}{2N}, 1 - \frac{1}{2N} \right) \sum_{k=0}^{\infty} \frac{(\lambda M^2)^k}{k!} \Gamma\left(\tilde{\ell}_k N + 1 \right)^{\frac{1}{2N}}, \qquad t > 0. \end{aligned} \tag{3.18}$$

For $k = 0$, by (3.15) we have $\Gamma(\tilde{\ell}_0 N + 1) = \Gamma(3)$. Furthermore, applying Lemma 2.1 with (3.15) and by the monotonicity property of the Gamma function, for $k \geq 1$, we see that

$$\begin{aligned}\Gamma\left(\tilde{\ell}_k N + 1\right)^{\frac{1}{2N}} &\leq C\Gamma\left(\frac{\tilde{\ell}_k}{2} + 1\right)(2N)^{\frac{\tilde{\ell}_k}{2}} \\ &= C\Gamma\left(k + \frac{1}{N} + 1\right)(2N)^{\frac{\tilde{\ell}_k}{2}} \\ &\leq C\Gamma(k+2)(2N)^{\frac{\tilde{\ell}_k}{2}} = C(k+1)!(2N)^{\frac{\tilde{\ell}_k}{2}}.\end{aligned}$$

These together with (3.18) implies that

$$\begin{aligned}&h(t)\|D[u](t) - D[v](t)\|_{L^\infty} \\ &\leq CM^{\frac{2}{N}} d_X(u, v) \sum_{k=0}^{\infty} \frac{(\lambda M^2)^k}{k!}(k+1)!(2N)^{\frac{\tilde{\ell}_k}{2}} \\ &\leq C(NM^2)^{\frac{1}{N}} d_X(u, v) \sum_{k=0}^{\infty}(k+1)(2\lambda NM^2)^k, \qquad t > 0.\end{aligned}$$

Then, taking a sufficiently small $\varepsilon^* = \varepsilon^*(N, \lambda) > 0$ such that, for $M < \varepsilon^*$, in a similar way as in Lemma 2.2, it holds that

$$\begin{aligned}\sup_{t>0} h(t)\|D[u](t) - D[v](t)\|_{L^\infty} &\leq C\frac{(NM)^{\frac{2}{N}}}{(1 - 2\lambda NM^2)^2} d_X(u, v) \\ &\leq CM^{\frac{2}{N}} d_X(u, v). \qquad (3.19)\end{aligned}$$

On the other hand, similarly to (3.12), by (2.12) with $(q, r) = ((2N)/3, 2N)$, (3.3), and (3.16) we have

$$\begin{aligned}&t^{\frac{1}{4N}}|D[u](t) - D[v](t)|_{L^{2N}} \\ &\leq t^{\frac{1}{4N}} \int_0^t |S_2(t-s)(f(u(s)) - f(v(s)))|_{L^{2N}}\, ds \\ &\leq C\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} t^{\frac{1}{4N}} \int_0^t (t-s)^{-\frac{N-1}{2N} - \frac{1}{2}} \left| |u(s) - v(s)| \left(|u(s)|^{\tilde{\ell}_k} + |v(s)|^{\tilde{\ell}_k} \right) \right|_{L^{\frac{2N}{3}}} ds, \qquad t > 0.\end{aligned}$$

Therefore, applying the same argument as in the proof of (3.19), for $M < \varepsilon^*$, it holds that

$$\begin{aligned}&t^{\frac{1}{4N}}|D[u](t) - D[v](t)|_{L^{2N}} \\ &\leq Ct^{\frac{1}{4N}} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \int_0^t (t-s)^{-1+\frac{1}{2N}} |u(s) - v(s)|_{L^{2N}} \left(|u(s)|^{\tilde{\ell}_k}_{L^{\tilde{\ell}_k N}} + |v(s)|^{\tilde{\ell}_k}_{L^{\tilde{\ell}_k N}} \right) ds \\ &\leq Ct^{\frac{1}{4N}} d_X(u, v) \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \left(\Gamma\left(\frac{\tilde{\ell}_k N}{2} + 1\right)^{\frac{1}{\tilde{\ell}_k N}} M \right)^{\tilde{\ell}_k} \int_0^t (t-s)^{-1+\frac{1}{2N}} s^{-\frac{3}{4N}}\, ds \\ &\leq C(NM^2)^{\frac{1}{N}} d_X(u, v) B\left(\frac{1}{2N}, 1 - \frac{3}{4N}\right) \sum_{k=0}^{\infty}(k+1)(\lambda NM^2)^k \\ &\leq CM^{\frac{2}{N}} d_X(u, v), \qquad t > 0.\end{aligned}$$

This implies that

$$\sup_{t>0} t^{\frac{1}{4N}} |D[u](t) - D[v](t)|_{L^{2N}} \leq CM^{\frac{2}{N}} d_X(u, v). \tag{3.20}$$

Combining (3.19) and (3.20), we have (3.14), thus Lemma 3.2 follows. □

Remark 3.1 In the proof of Lemma 3.2, the estimate for $\sup_{t>0} t^{1/(4N)} |\cdot|_{L^{2N}}$ is closed by itself. We need the term $\sup_{t>0} h(t)\|\cdot\|_{L^\infty}$ in the definition of the metric d_X in order to ensure the uniform convergence of the Cauchy sequence so that the solution is continuous.

Now we are ready to complete the proof of Theorem 1.1 for the case $N \geq 2$.

Proof of Theorem 1.1 $(N \geq 2)$. Let

$$M := 6 \max\{1, c_1, c_2\}\|\varphi\|_{\exp L^2},$$

where c_1 and c_2 are constant given in (G_1). Then, by (2.8), (2.9), (2.17), and (2.18) we see that

$$\begin{aligned} &\sup_{t>0} \|S_1(t)\varphi\|_{\exp L^2} \leq \frac{M}{2}, \qquad \sup_{t>0} t^{\frac{N}{4}} \|S_1(t)\varphi\|_{L^\infty} \leq \frac{M}{2}, \\ &\sup_{t>0} t^{\frac{1}{2q}} |S_1(t)\varphi|_{L^q} \leq \left\{\Gamma\left(\frac{q}{2}+1\right)\right\}^{\frac{1}{q}} \frac{M}{2}, \qquad q \in [2, \infty). \end{aligned} \tag{3.21}$$

Let $u \in X_M$. Then, by Lemma 3.1 with (2.16) we can take a sufficiently small $\varepsilon_4 = \varepsilon_4(N, \lambda) > 0$ such that, for $M < \varepsilon_4$, it holds $CM^{2/N} < 1/2$ and so

$$\begin{aligned} &\sup_{t>0} \|D[u](t)\|_{\exp L^2} \leq \frac{M}{2}, \qquad \sup_{t>0} \|D[u](t)\|_{L^\infty} \leq \frac{M}{2}, \\ &\sup_{t>0} t^{\frac{1}{2q}} |D[u](t)|_{L^q} \leq \left\{\Gamma\left(\frac{q}{2}+1\right)\right\}^{\frac{1}{q}} \frac{M}{2}, \qquad q \in [2, \infty). \end{aligned}$$

This together with property (G_3), Lemma 3.1, (3.2), and (3.21) yields that Φ is a map on X_M to itself. Furthermore, since it follows from (3.1) and (3.2) that

$$d_X(\Phi[u], \Phi[v]) = d_X(D[u], D[v])$$

for $u, v \in X_M$, taking a sufficiently small $\varepsilon_5 = \varepsilon_5(N) > 0$ if necessary, for $M < \varepsilon_5$, we can apply Lemma 3.2, and it holds that

$$d_X(\Phi[u], \Phi[v]) \leq \frac{1}{4} d_X(u, v).$$

Then, applying the contraction mapping theorem ensures that there exists a unique $u \in X_M$ with

$$u = \Phi[u](t) = S_1(t)\varphi + D[u](t) \quad \text{in} \quad X_M.$$

Thus we see that u is the unique global-in-time solution of problem (1.12) satisfying (1.14) and (1.15). Furthermore, by the same argument as in the proof of [14, (1.7)] with Lemma 3.1, we can prove that $u(t) \underset{t\to 0}{\longrightarrow} \varphi$ in the weak* topology, and the proof of Theorem 1.1 for the case $N \geq 2$ is complete. □

Next we consider the case $N = 1$. Similarly to the case $N \geq 2$, let $M > 0$, and we set

$$Y_M := \left\{ \begin{array}{l} u \in C([0,\infty)\times(0,\infty)) \cap L^\infty(0,\infty;\exp L^2(0,\infty)) : \\ \sup_{t>0} \|u(t)\|_{\exp L^2} \leq M, \quad \sup_{t>0} h(t)\|u(t)\|_{L^\infty} \leq M \quad \text{with} \quad h(t) = \min\{t^{\frac14}, 1\}, \\ \sup_{t>0} t^{\frac{1}{2q}} |u(0,t)| \leq \left\{\Gamma\left(\frac{q}{2}+1\right)\right\}^{\frac1q} M \quad \text{with} \quad q \in [2,\infty) \end{array} \right\},$$

equipped with the metric

$$d_Y(u,v) := \sup_{t>0} \left(h(t)\|u(t)-v(t)\|_{L^\infty} + t^{\frac14}|u(0,t)-v(0,t)| \right). \tag{3.22}$$

Then (Y_M, d_Y) is a complete metric space. Similarly to the proof of Theorem 1.1 for the case $N \geq 2$, we apply the Banach contraction mapping principle in Y_M to find a fixed point of

$$\Psi[u](t) := S_1(t)\varphi + \tilde{D}[u](t),$$

where

$$\tilde{D}[u](x,t) := 2\int_0^t g_1(x,t-s) f(u(0,s))\,ds, \quad x \in [0,\infty). \tag{3.23}$$

Here g_1 is as in (1.10) and f satisfies (1.9).

Applying Lemma 2.3, we have the following.

Lemma 3.3 *Let $u \in Y_M$. Then there exists a positive constant $\varepsilon_* = \varepsilon_*(\lambda) > 0$ such that, if $M < \varepsilon_*$, then, for any $q \in [2,\infty)$,*

$$\sup_{t>0}\left(\|\tilde{D}[u](t)\|_{L^2} + h(t)\|\tilde{D}[u](t)\|_{L^\infty} \right) \leq CM^3, \tag{3.24}$$

$$\sup_{t>0} t^{\frac{1}{2q}} |\tilde{D}[u](0,t)| \leq C\left\{\Gamma\left(\frac{q}{2}+1\right)\right\}^{\frac1q} M^3, \tag{3.25}$$

where C is independent of q and M. Furthermore, $\tilde{D}[u]$ is continuous in $[0,\infty)\times(0,\infty)$.

Proof By (2.7) with $(N,r) = (1,2)$ and (3.23) we have

$$\begin{aligned} \|\tilde{D}[u](t)\|_{L^2} &\leq 2\int_0^t \|g_1(t-s)\|_{L^2} |f(u(0,s))|\,ds \\ &\leq C\int_0^t (t-s)^{-\frac14}|f(u(0,s))|\,ds, \qquad t > 0. \end{aligned} \tag{3.26}$$

Since $u \in Y_M$, taking a sufficiently small $\varepsilon_* = \varepsilon_*(\lambda) > 0$ such that, for $M < \varepsilon_*$, we can apply Lemma 2.3, and it holds from (2.26) with $r = 2$ and (3.26) that

$$\|\tilde{D}[u](t)\|_{L^2} \le CM^3 \int_0^t (t-s)^{-\frac{1}{4}} s^{-\frac{3}{4}}\,ds \le CM^3 B\left(\frac{3}{4}, \frac{1}{4}\right), \qquad t > 0. \tag{3.27}$$

Similarly, by (2.7) with $(N, r) = (1, \infty)$, (2.26), and (3.23), for any $q \in [2, \infty)$, it holds that

$$\begin{aligned}
|\tilde{D}[u](x,t)| &\le C \int_0^t (t-s)^{-\frac{1}{2}} |f(u(0,s))|\,ds \\
&\le C \left\{\Gamma\left(\frac{q}{2}+1\right)\right\}^{\frac{1}{q}} M^3 \int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2q}-\frac{1}{2}}\,ds \\
&\le C \left\{\Gamma\left(\frac{q}{2}+1\right)\right\}^{\frac{1}{q}} M^3 \left\{\left(\frac{t}{2}\right)^{-\frac{1}{2}} \int_0^{t/2} s^{-\frac{1}{2q}-\frac{1}{2}}\,ds + \left(\frac{t}{2}\right)^{-\frac{1}{2q}-\frac{1}{2}} \int_{t/2}^t (t-s)^{-\frac{1}{2}}\,ds\right\} \\
&\le C \left\{\Gamma\left(\frac{q}{2}+1\right)\right\}^{\frac{1}{q}} M^3 t^{-\frac{1}{2q}} \left(\frac{2^{\frac{1}{2q}+1}}{1-\frac{1}{q}} + 2^{\frac{1}{2q}+1}\right) \\
&\le C \left\{\Gamma\left(\frac{q}{2}+1\right)\right\}^{\frac{1}{q}} M^3 t^{-\frac{1}{2q}}, \qquad x \in [0,\infty), \quad t > 0,
\end{aligned}$$

where C is independent of q and M. This implies that

$$\begin{aligned}
&h(t)\|\tilde{D}[u](t)\|_{L^\infty} \le CM^3, \\
&|\tilde{D}[u](0,t)| \le C \left\{\Gamma\left(\frac{q}{2}+1\right)\right\}^{\frac{1}{q}} M^3 t^{-\frac{1}{2q}}, \qquad t > 0.
\end{aligned} \tag{3.28}$$

Thus, by (3.27) and (3.28) we obtain (3.24) and (3.25). Furthermore, applying the same argument as in the proof of Lemma 3.1, we see that $\tilde{D}[u]$ is continuous in $[0, \infty) \times (0, \infty)$, and the proof of Lemma 3.3 is complete. □

Lemma 3.4 *Let $u, v \in Y_M$. Then there exists a positive constant $\varepsilon^* = \varepsilon^*(\lambda) > 0$ such that, if $M < \varepsilon^*$, then*

$$d_Y(\tilde{D}[u], \tilde{D}[v]) \le CM^2 d_Y(u, v), \tag{3.29}$$

where C is independent of M.

Proof For any $k \in \mathbb{N} \cup \{0\}$, let $\tilde{\ell}_k$ be the constant defined by (3.15) with $N = 1$. Then, similarly to (3.18), by (2.7) with $(N, r) = (1, \infty)$, (3.16), (3.22), and (3.23), for $u, v \in Y_M$, we have

$$\begin{aligned}
&|\tilde{D}[u](x,t) - \tilde{D}[v](x,t)| \\
&\le C \int_0^t (t-s)^{-\frac{1}{2}} |f(u(0,s)) - f(v(0,s))|\,ds \\
&\le C \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \int_0^t (t-s)^{-\frac{1}{2}} |u(0,s) - v(0,s)| \left(|u(0,s)|^{\tilde{\ell}_k} + |v(0,s)|^{\tilde{\ell}_k}\right) ds \\
&\le C \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{4}-\frac{\tilde{\ell}_k}{2\tilde{\ell}_k}} \left(\sup_{s>0} s^{\frac{1}{4}} |u(0,s) - v(0,s)|\right) \times \\
&\quad \times \left\{\left(\sup_{s>0} s^{\frac{1}{2\tilde{\ell}_k}} |u(0,s)|\right)^{\tilde{\ell}_k} + \left(\sup_{s>0} s^{\frac{1}{2\tilde{\ell}_k}} |v(0,s)|\right)^{\tilde{\ell}_k}\right\} ds
\end{aligned}$$

$$\leq C d_Y(u,v) \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \left(\Gamma\left(\frac{\tilde{\ell}_k}{2}+1\right)^{\frac{1}{\tilde{\ell}_k}} M \right)^{\tilde{\ell}_k} \int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{3}{4}}\, ds$$

$$\leq C M^2 t^{-\frac{1}{4}} d_Y(u,v) B\left(\frac{1}{2},\frac{1}{4}\right) \sum_{k=0}^{\infty} \frac{(\lambda M^2)^k}{k!} \Gamma(k+2)$$

$$\leq C M^2 t^{-\frac{1}{4}} d_Y(u,v) \sum_{k=0}^{\infty} (k+1)(\lambda M^2)^k, \qquad x \in [0,\infty), \quad t > 0.$$

Then, we can take a sufficiently small $\varepsilon^* = \varepsilon^*(\lambda) > 0$ such that, for $M < \varepsilon^*$, it holds that

$$\sup_{t>0} h(t) \|\tilde{D}[u](t) - \tilde{D}[v](t)\|_{L^\infty} \leq C M^2 d_Y(u,v),$$
$$\sup_{t>0} t^{\frac{1}{4}} |\tilde{D}[u](0,t) - \tilde{D}[v](0,t)| \leq C M^2 d_Y(u,v).$$

This implies (3.29), thus Lemma 3.4 follows. □

Proof of Theorem 1.1 ($N = 1$). By Lemmata 3.3, 3.4, and applying the same arguments as in the proof of Theorem 1.1 for the case $N \geq 2$, we can prove Theorem 1.1 for the case $N = 1$. □

4 Slowly decaying initial data

In this section we prove Theorem 1.2. Similarly to Sect. 3, we first consider the case $N \geq 2$. Let u be the unique solution to problem (1.1) satisfying (1.14) and (1.15). Put

$$v(x,t) := u(x,t+1). \tag{4.1}$$

Then, it follows from (1.12) and (2.1) that the function v satisfies

$$v(t) = S_1(t)u(1) + D[v](t), \qquad t > 0, \tag{4.2}$$

where $D[v]$ is the function defined by (3.3). Since it follows from (1.14) and (2.17) that

$$\|u(1)\|_{L^q} \leq c_* \|\varphi\|_{\exp L^2}, \qquad q \in [2,\infty], \tag{4.3}$$

by (2.9) with $q = r$, for any $q \in [2,\infty]$, we have

$$|S_1(t)u(1)|_{L^q} \leq c_2 t^{-\frac{1}{2q}} \|u(1)\|_{L^q} \leq c_2 c_* t^{-\frac{1}{2q}} \|\varphi\|_{\exp L^2}, \qquad t > 0. \tag{4.4}$$

Here c_* is a constant independent of q and $\|\varphi\|_{\exp L^2}$. Furthermore, since it follows from the continuity of the function $D[u](x,t)$ that $|D[u](t)|_{L^\infty} \leq \|D[u](t)\|_{L^\infty}$, applying the same argument as in the proof of Lemma 3.1 with (1.15) and (4.1), we see that, for any $q \in [2,\infty]$,

$$|D[v](t)|_{L^q} \leq C t^{-\frac{1}{2q}} \|\varphi\|_{\exp L^2}^{1+\frac{2}{N}}, \qquad t > 0. \tag{4.5}$$

Then, we can take a sufficiently small $\varepsilon > 0$ such that, for $\|\varphi\|_{\exp L^2} < \varepsilon$, it follows from (4.2), (4.4), and (4.5) that

$$|v(t)|_{L^q} \leq 2 c_2 c_* t^{-\frac{1}{2q}} \|\varphi\|_{\exp L^2}, \qquad t > 0. \tag{4.6}$$

On the other hand, we have the following.

Lemma 4.1 *Let $N \geq 2$, $T > 0$, and $A > 0$. Suppose that, for any $q \in [2, \infty]$, the function $v \in C(\overline{\mathbb{R}^N_+} \times (0, \infty))$ satisfies*

$$\sup_{0<t\leq T} (1+t)^{\frac{N}{2}(\frac{1}{2}-\frac{1}{q})} t^{\frac{1}{2q}} |v(t)|_{L^q} \leq A. \tag{4.7}$$

Let f be a function satisfying (1.9). Then, there exists $\varepsilon_* > 0$, independent of T, such that, if $A < \varepsilon_*$, then, for any $r \in [p_2, \infty]$,

$$\sup_{0<t\leq T} (1+t)^{\frac{N}{2}(\frac{1}{2}-\frac{1}{r})+\frac{1}{2}} t^{\frac{1}{2r}} |f(v(t))|_{L^r} \leq 2C_f A^{1+\frac{2}{N}}, \tag{4.8}$$

where C_f and p_2 are given in (1.9) and (2.20), respectively.

Proof Let $k \in \mathbb{N} \cup \{0\}$ and ℓ_k be the constant given in (2.22). Then, for any $r \in [p_2, \infty]$, by (1.9) and (4.7) we have

$$\begin{aligned} |f(v(t))|_{L^r} &\leq C_f \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} |v(t)|^{\ell_k}_{L^{\ell_k r}} \\ &\leq C_f \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \left((1+t)^{-\frac{N}{2}(\frac{1}{2}-\frac{1}{\ell_k r})} t^{-\frac{1}{2\ell_k r}} A \right)^{\ell_k} \\ &\leq C_f A^{1+\frac{2}{N}} (1+t)^{\frac{N}{2r}-\frac{N}{4}(1+\frac{2}{N})} t^{-\frac{1}{2r}} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \left((1+t)^{-\frac{N}{4}} A \right)^{2k} \\ &\leq C_f A^{1+\frac{2}{N}} (1+t)^{-\frac{N}{2}(\frac{1}{2}-\frac{1}{r})-\frac{1}{2}} t^{-\frac{1}{2r}} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} A^{2k}, \qquad t > 0. \end{aligned} \tag{4.9}$$

We can take a sufficiently small $\varepsilon_* = \varepsilon_*(\lambda) > 0$ so that, for $A < \varepsilon_*$, it holds that

$$\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} A^{2k} = e^{\lambda A^2} \leq 2. \tag{4.10}$$

This together with (4.9) implies (4.8). Thus Lemma 4.1 follows. □

Now we are in position to prove Theorem 1.2 for the case $N \geq 2$.

Proof of Theorem 1.2 ($N \geq 2$). Following the idea of the proof of [20, Lemma 2.4], we prove this theorem.

Let u be a unique solution to problem (1.1) satisfying (1.14) and (1.15), and let v be the function defined by (4.2). Then, applying arguments similar to that in the proof of [20, Lemma 2.1] with (4.6), we see that

$$v \in C((0, \infty); L^q(\partial\mathbb{R}^N_+)), \qquad q \in [2, \infty]. \tag{4.11}$$

Let $\|\varphi\|_{\exp L^2}$ be a sufficiently small to be chosen later. Put

$$\delta = 2^{\frac{N}{2}(\frac{1}{2}-\frac{1}{q})} c_* \|\varphi\|_{\exp L^2}, \tag{4.12}$$

and

$$T = \sup\left\{ 0 < s < \infty;\ |v(t)|_{L^q} \leq 2c_2\delta(1+t)^{-\frac{N}{2}(\frac{1}{2}-\frac{1}{q})} t^{-\frac{1}{2q}} \text{ for all} q \in [2, \infty] \text{and} 0 < t < s \right\},$$

where c_* and c_2 are given in (4.3) and (2.9), respectively. Then, by (4.6) and (4.12) we have $T \geq 1$.

We prove $T = \infty$. The proof is by contradiction. We assume that $T < \infty$. Then, by (4.11) we see that

$$|v(T)|_{L^q} = 2c_2\delta(1+T)^{-\frac{N}{2}(\frac{1}{2}-\frac{1}{q})}T^{-\frac{1}{2q}}. \tag{4.13}$$

On the other hand, by (2.9) with $(q, r) = (2, q)$, (4.3) and (4.12) we have

$$\begin{aligned} |S_1(T)u(1)|_{L^q} &\leq c_2 T^{-\frac{N}{2}(\frac{1}{2}-\frac{1}{q})-\frac{1}{2q}}\|u(1)\|_{L^2} \\ &\leq 2^{\frac{N}{2}(\frac{1}{2}-\frac{1}{q})}c_2(1+T)^{-\frac{N}{2}(\frac{1}{2}-\frac{1}{q})}T^{-\frac{1}{2q}}c_*\|\varphi\|_{\exp L^2} \\ &\leq c_2\delta(1+T)^{-\frac{N}{2}(\frac{1}{2}-\frac{1}{q})}T^{-\frac{1}{2q}}. \end{aligned} \tag{4.14}$$

Furthermore, by the definition of T, taking a sufficiently small $\|\varphi\|_{\exp L^2}$ if necessary, we can apply Lemma 4.1, and it holds that, for any $r \in [p_2, \infty]$,

$$\sup_{0<t\leq T}(1+t)^{\frac{N}{2}(\frac{1}{2}-\frac{1}{r})+\frac{1}{2}}t^{\frac{1}{2r}}|f(v(t))|_{L^r} \leq 2C_f(2c_2\delta)^{1+\frac{2}{N}}, \tag{4.15}$$

where C_f and p_2 are given in (1.9) and (2.20), respectively. Let $D[v]$ be the function defined by (3.3). Then, we put

$$\begin{aligned} |D[v](T)|_{L^q} &\leq \left(\int_0^{T/2}+\int_{T/2}^{T}\right)|S_2(T-s)f(v(s))|_{L^q}\,ds \\ &=: I_1(T)+I_2(T). \end{aligned} \tag{4.16}$$

For the term I_1, since $T \geq 1$ and $N(1/2 - 1/p_2) = -1$, by (2.12) with $(q, r) = (p_2, q)$ and (4.15) we obtain

$$\begin{aligned} I_1(T) &\leq C\int_0^{T/2}(T-s)^{-\frac{N-1}{2}(\frac{1}{p_2}-\frac{1}{q})-\frac{1}{2}}|f(v(s))|_{L^{p_2}}\,ds \\ &\leq C\delta^{1+\frac{2}{N}}T^{-\frac{N-1}{2}(\frac{1}{p_2}-\frac{1}{q})-\frac{1}{2}}\int_0^{T/2}(1+s)^{-\frac{N}{2}(\frac{1}{2}-\frac{1}{p_2})-\frac{1}{2}}s^{-\frac{1}{2p_2}}\,ds \\ &\leq C\delta^{1+\frac{2}{N}}T^{-\frac{N-1}{2}(\frac{1}{p_2}-\frac{1}{q})-\frac{1}{2}}\int_0^{T/2}s^{-\frac{1}{2p_2}}\,ds \\ &\leq C\delta^{1+\frac{2}{N}}T^{-\frac{N}{2}(\frac{1}{2}-\frac{1}{q})-\frac{1}{2q}} \\ &\leq D_1\delta^{1+\frac{2}{N}}(1+T)^{-\frac{N}{2}(\frac{1}{2}-\frac{1}{q})}T^{-\frac{1}{2q}}, \end{aligned} \tag{4.17}$$

where D_1 is a positive constant independent of q and δ. Furthermore, for the term I_2, since $T \geq 1$, by (2.12) with $q = r$ and (4.15) we have

$$\begin{aligned} I_2(T) &\leq C\int_{T/2}^{T}(T-s)^{-\frac{1}{2}}|f(v(s))|_{L^q}\,ds \\ &\leq C\delta^{1+\frac{2}{N}}\int_{T/2}^{T}(T-s)^{-\frac{1}{2}}(1+s)^{-\frac{N}{2}(\frac{1}{2}-\frac{1}{q})-\frac{1}{2}}s^{-\frac{1}{2q}}\,ds \\ &\leq C\delta^{1+\frac{2}{N}}(1+T)^{-\frac{N}{2}(\frac{1}{2}-\frac{1}{q})}T^{-\frac{1}{2q}}\int_0^T(T-s)^{-\frac{1}{2}}s^{-\frac{1}{2}}\,ds \\ &\leq D_2\delta^{1+\frac{2}{N}}(1+T)^{-\frac{N}{2}(\frac{1}{2}-\frac{1}{q})}T^{-\frac{1}{2q}}, \end{aligned} \tag{4.18}$$

where D_2 is a positive constant independent of q and δ. Then, combining (4.17) and (4.18), we see that

$$|D[v](T)|_{L^q} \le (D_1 + D_2)\delta^{1+\frac{2}{N}}(1+T)^{-\frac{N}{2}(\frac{1}{2}-\frac{1}{q})}T^{-\frac{1}{2q}}. \tag{4.19}$$

Taking a sufficiently small $\|\varphi\|_{\exp L^2}$ if necessary, we have

$$(D_1 + D_2)\delta^{\frac{2}{N}} < c_2.$$

This together with (4.2), (4.14), and (4.19) implies that

$$|v(T)|_{L^q} \le |S_1(T)u(1)|_{L^q} + |D[v(T)]|_{L^q} < 2c_2\delta(1+T)^{-\frac{N}{2}(\frac{1}{2}-\frac{1}{q})}T^{-\frac{1}{2q}}.$$

This contradicts (4.13), and we see $T = \infty$. Therefore, for any $q \in [2, \infty]$, it holds that

$$|v(t)|_{L^q} \le 2c_2\delta(1+t)^{-\frac{N}{2}(\frac{1}{2}-\frac{1}{q})}t^{-\frac{1}{2q}}, \qquad t > 0. \tag{4.20}$$

It remains to show that, for any $q \in [2, \infty]$,

$$\|v(t)\|_{L^q} \le C\delta t^{-\frac{N}{2}(\frac{1}{2}-\frac{1}{q})}, \qquad t > 0. \tag{4.21}$$

By (2.8), (4.3), and (4.12) we see that

$$\|S_1(t)u(1)\|_{L^q} \le Ct^{-\frac{N}{2}(\frac{1}{2}-\frac{1}{q})}\|u(1)\|_{L^2} \le C\delta t^{-\frac{N}{2}(\frac{1}{2}-\frac{1}{q})}, \qquad t > 0. \tag{4.22}$$

On the other hand, by (4.20), similarly to (4.15), it holds that, for any $r \in [p_2, \infty]$,

$$|f(v(t))|_{L^r} \le C\delta(1+t)^{-\frac{N}{2}(\frac{1}{2}-\frac{1}{r})-\frac{1}{2}}t^{-\frac{1}{2r}}, \qquad t > 0. \tag{4.23}$$

Similarly to (4.16), by (3.3) we put

$$\begin{aligned} \|D[v](t)\|_{L^q} &\le \int_0^{t/2} \|S_2(t-s)f(v(s))\|_{L^q}\,ds + \int_{t/2}^t \|S_2(t-s)f(v(s))\|_{L^q}\,ds \\ &=: J_1(t) + J_2(t), \qquad t > 0. \end{aligned} \tag{4.24}$$

Then, for the term J_1, it holds from (2.11) with $(q, r) = (p_2, q)$ and (4.23) that

$$\begin{aligned} J_1(t) &\le C\int_0^{t/2}(t-s)^{-\frac{N}{2}(\frac{1}{p_2}-\frac{1}{q})-\frac{1}{2}(1-\frac{1}{p_2})}|f(v(s))|_{L^{p_2}}\,ds \\ &\le C\delta t^{-\frac{N}{2}(\frac{1}{p_2}-\frac{1}{q})-\frac{1}{2}(1-\frac{1}{p_2})}\int_0^{t/2}(1+s)^{-\frac{N}{2}(\frac{1}{2}-\frac{1}{p_2})-\frac{1}{2}}s^{-\frac{1}{2p_2}}\,ds \\ &\le C\delta t^{-\frac{N}{2}(\frac{1}{2}-\frac{1}{q})-1+\frac{1}{2p_2}}\int_0^t s^{-\frac{1}{2p_2}}\,ds \le C\delta t^{-\frac{N}{2}(\frac{1}{2}-\frac{1}{q})}, \qquad t > 0. \end{aligned} \tag{4.25}$$

Furthermore, for the term J_2, by (2.11) with $q = r$ and (4.23) we have

$$\begin{aligned} J_2(t) &\le C\int_{t/2}^t (t-s)^{-\frac{1}{2}(1-\frac{1}{q})}|f(v(s))|_{L^q}\,ds \\ &\le C\delta\int_{t/2}^t (t-s)^{-\frac{1}{2}(1-\frac{1}{q})}(1+s)^{-\frac{N}{2}(\frac{1}{2}-\frac{1}{q})-\frac{1}{2}}s^{-\frac{1}{2q}}\,ds \\ &\le C\delta(1+t)^{-\frac{N}{2}(\frac{1}{2}-\frac{1}{q})}t^{-\frac{1}{2}-\frac{1}{2q}}\int_0^t s^{-\frac{1}{2}(1-\frac{1}{q})}\,ds \le C\delta(1+t)^{-\frac{N}{2}(\frac{1}{2}-\frac{1}{q})}, \qquad t > 0. \end{aligned} \tag{4.26}$$

Then, combining (4.24), (4.25), and (4.26), we see that

$$\|D[v]\|_{L^q} \le C\delta t^{-\frac{N}{2}(\frac{1}{2}-\frac{1}{q})}, \qquad t > 0.$$

This together with (4.2) and (4.22) yields (4.21). Therefore, by (4.1), (4.20), and (4.21) we have (1.16), and the proof of Theorem 1.2 for the case $N \ge 2$ is complete. □

We next consider the case $N = 1$. Let v be the function defined by (4.1). Then, it follows from (1.13) and (2.1) that the function v satisfies

$$v(t) = S_1(t)u(1) + \tilde{D}[v](t), \qquad t > 0, \tag{4.27}$$

where $\tilde{D}[v]$ is the function defined by (3.23). Then, by (2.10) and (4.3) we have

$$|[S_1(t)u(1)](0)| \le c_2 t^{-\frac{1}{4}}\|u(1)\|_{L^2} \le d_* t^{-\frac{1}{4}}\|\varphi\|_{\exp L^2}, \qquad t > 0. \tag{4.28}$$

Here d_* is a constant independent of $\|\varphi\|_{\exp L^2}$. Furthermore, similarly to (4.5), applying the same argument as in the proof of Lemma 3.3 with (1.15) and (4.1), we see that

$$|\tilde{D}[v](0,t)| \le C t^{-\frac{1}{4}}\|\varphi\|^3_{\exp L^2}, \qquad t > 0. \tag{4.29}$$

Then, we can take a sufficiently small $\varepsilon > 0$ such that, for $\|\varphi\|_{\exp L^2} < \varepsilon$, it follows from (4.27), (4.28), and (4.29) that

$$|v(0,t)| \le 2d_* t^{-\frac{1}{4}}\|\varphi\|_{\exp L^2}, \qquad t > 0. \tag{4.30}$$

On the other hand, we have the following, which is the one dimensional counterpart of Lemma 4.1.

Lemma 4.2 *Let $N = 1$, $T > 0$, and $A > 0$. Suppose that the function $v \in C(0,\infty)$ satisfying*

$$\sup_{0<t\le T} (1+t)^{\frac{1}{4}}|v(t)| \le A. \tag{4.31}$$

Let f be a function satisfying (1.9). Then, there exists $\varepsilon_* > 0$, independent of T, such that, if $A < \varepsilon_*$, then

$$\sup_{0<t\le T} (1+t)^{\frac{3}{4}}|f(v(t))| \le 2C_f A^3, \tag{4.32}$$

where C_f is constant given in (1.9).

Proof Let $k \in \mathbb{N}\cup\{0\}$ and ℓ_k be the constant given in (2.22) with $N = 1$, namely, $\ell_k = 2k+3$. Then, by (1.9) and (4.31) we have

$$\begin{aligned} |f(v(t))| &\le C_f \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}|v(t)|^{\ell_k} \le C_f \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}\left((1+t)^{-\frac{1}{4}}A\right)^{\ell_k} \\ &\le C_f A^3 (1+t)^{-\frac{3}{4}} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}\left((1+t)^{-\frac{1}{4}}A\right)^{2k} \\ &\le C_f A^3 (1+t)^{-\frac{3}{4}} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} A^{2k}, \qquad t > 0. \end{aligned} \tag{4.33}$$

This together with (4.10) implies (4.32). Thus Lemma 4.2 follows. □

Proof of Theorem 1.2 ($N = 1$). Let v be the function defined by (4.27). Then, since $\|\varphi\|_{\exp L^2}$ is sufficiently small, by (4.30), Lemma 4.2, and applying the same argument as in the proof of Theorem 1.2 for the case $N \geq 2$, we can prove that

$$|v(0,t)| \leq C(1+t)^{-\frac{1}{4}}\|\varphi\|_{\exp L^2}, \qquad t > 0, \tag{4.34}$$

and

$$|f(v(0,t))| \leq C(1+t)^{-\frac{3}{4}}\|\varphi\|_{\exp L^2}, \qquad t > 0. \tag{4.35}$$

Let $q \in [2, \infty]$. Then, by (2.7) with $(N, r) = (1, q)$, (3.23), and (4.35) we have

$$\begin{aligned}
\|\tilde{D}[v](t)\|_{L^q} &\leq 2\int_0^t \|g_1(t-s)\|_{L^q}|f(v(0,s))|\,ds \\
&\leq C\|\varphi\|_{\exp L^2}\int_0^t (t-s)^{-\frac{1}{2}(1-\frac{1}{q})}s^{-\frac{3}{4}}\,ds \\
&\leq Ct^{-\frac{1}{2}(\frac{1}{2}-\frac{1}{q})}B\left(\frac{1}{2}+\frac{1}{2q}, \frac{1}{4}\right)\|\varphi\|_{\exp L^2} \\
&\leq Ct^{-\frac{1}{2}(\frac{1}{2}-\frac{1}{q})}\|\varphi\|_{\exp L^2}, \qquad t > 0.
\end{aligned}$$

This together with (4.22) and (4.27) implies

$$\|v(t)\|_{L^q} \leq Ct^{-\frac{1}{2}(\frac{1}{2}-\frac{1}{q})}\|\varphi\|_{\exp L^2}, \qquad t > 0. \tag{4.36}$$

Therefore, by (4.1), (4.34), and (4.36) we have (1.16), and the proof of Theorem 1.2 for the case $N = 1$ is complete. □

5 Rapidly decaying initial data

In this section we prove Theorems 1.3 and 1.4. Let

$$L := \|\varphi\|_{\exp L^2} \tag{5.1}$$

We can assume, without loss of generality, that $L < 1$. Let p_1 be the constant given in (1.18). For $\|\varphi\|_{L^{p_1}} > 0$, we denote

$$K := 2\max\{1, c_1, c_2\}\|\varphi\|_{\exp L^2 \cap L^{p_1}} \tag{5.2}$$

and

$$\tilde{K} := 2\max\{1, c_1, c_2\}\max\{1, \|\varphi\|_{L^{p_1}}\}, \tag{5.3}$$

where c_1 and c_2 are given in (G_1). Since we assume $L < 1$ and thanks to (1.17) we have

$$L \leq K \leq \tilde{K}. \tag{5.4}$$

Then we first show the following lemma, which is analogous to Lemma 4.1.

Lemma 5.1 *Let $N \geq 2$, $T > 0$, and $p \in [1, 2)$. Furthermore let p_1 be the constant given in* (1.18)*. Suppose that, for any $q \in [p_1, \infty]$, the function $u \in C(\overline{\mathbb{R}^N_+} \times (0, \infty))$ satisfies*

$$\sup_{0<t\leq T} t^{\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})+\frac{1}{2q}}|u(t)|_{L^q} \leq DK, \tag{5.5}$$

where D is independent of q and K is the constant given in (5.2). Let f be a function satisfying (1.9). Then, for $\tilde{K}$ as in (5.3), there exists a sufficiently large constant $T_1 = T_1(\tilde{K}, p_1, \lambda, D) \geq 1$ such that if $T \geq T_1$ it follows that, for any $r \in [p_3, \infty]$,

$$\sup_{T_1 \leq t \leq T} t^{\frac{N}{2}(\frac{1}{p_1}-\frac{1}{r})+\frac{1}{p_1}+\frac{1}{2r}} |f(u(t))|_{L^r} \leq 2C_f(DK)^{1+\frac{2}{N}}, \tag{5.6}$$

where C_f is given in (1.9) and

$$p_3 := \max\left\{1, \frac{p_1 N}{N+2}\right\}. \tag{5.7}$$

Proof Let $k \in \mathbb{N} \cup \{0\}$ and ℓ_k be the constant given in (2.22). Since

$$\ell_k r \geq \left(1 + \frac{2}{N}\right) p_3 \geq p_1,$$

for any $r \in [p_3, \infty]$, by (1.9) and (5.5) we have

$$\begin{aligned} |f(u(t))|_{L^r} &\leq C_f \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} |u(t)|_{L^{\ell_k r}}^{\ell_k} \\ &\leq C_f \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \left(DKt^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{\ell_k r})-\frac{1}{2\ell_k r}} \right)^{\ell_k} \\ &\leq C_f (DK)^{1+\frac{2}{N}} t^{\frac{N}{2r}-\frac{N}{2p_1}(1+\frac{2}{N})-\frac{1}{2r}} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \left(DKt^{-\frac{N}{2p_1}} \right)^{2k} \\ &\leq C_f (DK)^{1+\frac{2}{N}} t^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{r})-\frac{1}{p_1}-\frac{1}{2r}} \exp\left(\lambda (DK)^2 t^{-\frac{N}{p_1}} \right), \qquad t > 0. \end{aligned} \tag{5.8}$$

for all $t > 0$. We can take a sufficiently large constant $T_1 \geq 1$ such that, for all $t > T_1$, it holds that

$$\exp\left(\lambda (DK)^2 t^{-\frac{N}{p_1}} \right) \leq 2.$$

It is enough to choose

$$T_1 \geq \left(\frac{\lambda (D\tilde{K})^2}{\log 2} \right)^{\frac{p_1}{N}} \geq \left(\frac{\lambda (DK)^2}{\log 2} \right)^{\frac{p_1}{N}}. \tag{5.9}$$

This together with (5.8) implies (5.6). Thus Lemma 5.1 follows. □

Similarly, for the case $N = 1$, we have the following.

Lemma 5.2 *Let $N = 1$, $T > 0$, and $p \in [1, 2)$. Suppose that the function $u \in C(0, \infty)$ satisfies*

$$\sup_{0 < t \leq T} t^{\frac{1}{2p}} |u(t)| \leq DK, \tag{5.10}$$

where $D > 0$ and K is the constant given in (5.2). Let f be a function satisfying (1.9). Then, for $\tilde{K}$ as in (5.3), there exists a sufficiently large constant $\tilde{T}_1 = \tilde{T}_1(\tilde{K}, p, \lambda, D)$ such that, if $T \geq \tilde{T}_1$, then it follows that

$$\sup_{\tilde{T}_1 \leq t \leq T} t^{\frac{3}{2p}} |f(u(t))| \leq 2C_f(DK)^3, \tag{5.11}$$

where C_f is given in (1.9).

Proof Let $k \in \mathbb{N}\cup\{0\}$ and ℓ_k be the constant given in (2.22) with $N = 1$, namely, $\ell_k = 2k+3$. Furthermore, let $\tilde{T}_1$ be a sufficiently large constant satisfying (5.9) with $(N, p_1) = (1, p)$. Then, by (1.9) and (5.10) we have

$$\begin{aligned}|f(u(t))| &\le C_f \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}|u(t)|^{\ell_k} \le C_f \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}\left(DKt^{-\frac{1}{2p}}\right)^{\ell_k} \\ &\le C_f(DK)^3 t^{-\frac{3}{2p}} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}\left(DKt^{-\frac{1}{2p}}\right)^{2k} \\ &\le C_f(DK)^3 t^{-\frac{3}{2p}} \exp\left(\lambda(DK)^2 t^{-\frac{1}{p}}\right) \\ &\le 2C_f(DK)^3 t^{-\frac{3}{2p}}, \qquad t \ge \tilde{T}_1.\end{aligned}$$

This implies (5.11). Thus Lemma 5.2 follows. □

Next we prove (1.20) for small times.

Lemma 5.3 *Let $N \ge 1$ and u be the unique solution to problem* (1.1) *satisfying* (1.14) *and* (1.15)*. Suppose $\varphi \in L^p$ for $p \in [1, 2)$. Let p_1 and K be the constants given in* (1.18) *and* (5.2)*, respectively. Then, for any fixed $T_* \ge 1$, there exists a constant $\varepsilon = \varepsilon(p_1, T_*) > 0$ such that, if $L < \varepsilon$ (where L is the constant given in* (5.1)*) then, for any $q \in [p_1, \infty]$,*

$$\sup_{0<t\le 2T_*} t^{\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})}\left(\|u(t)\|_{L^q} + t^{\frac{1}{2q}}|u(t)|_{L^q}\right) \le C_* K, \qquad \text{if} \quad N \ge 2, \tag{5.12}$$

$$\sup_{0<t\le 2T_*} t^{\frac{1}{2}(\frac{1}{p}-\frac{1}{q})}\left(\|u(t)\|_{L^q} + t^{\frac{1}{2q}}|u(0,t)|\right) \le C_* K, \qquad \text{if} \quad N = 1, \tag{5.13}$$

where C_* is independent of q, K, and T_*.

Proof We first prove (5.12). Let $N \ge 2$. By (1.12) we consider

$$u(t) = S_1(t)\varphi + D[u](t), \tag{5.14}$$

where $D[u]$ is the function defined by (3.3). For the linear part, by (2.8), (2.9), and (5.2), for any $q \in [p_1, \infty]$, we have

$$\begin{aligned}\|S_1(t)\varphi\|_{L^q} + t^{\frac{1}{2q}}|S_1(t)\varphi|_{L^q} &\le (c_1 + c_2)t^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})}\|\varphi\|_{L^{p_1}} \\ &\le Kt^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})}, \quad t > 0.\end{aligned} \tag{5.15}$$

For the nonlinear part $D[u]$, let p_2 be the constant given in (2.20). Then, by (2.11) with $(q, r) = (2N, \infty)$ and (3.3) we see that

$$\|D[u](t)\|_{L^\infty} \le C\int_0^t (t-s)^{-\frac{3}{4}+\frac{1}{4N}}|f(u(s))|_{L^{2N}}\, ds, \qquad t > 0. \tag{5.16}$$

On the other hand, for $r \in [p_2, \infty)$, by (1.15) and taking a sufficiently small $\varepsilon = \varepsilon(r) > 0$, for $L < \varepsilon$, we can apply Lemma 2.2, and it holds that

$$|f(u(t))|_{L^r} \le CrL^{1+\frac{2}{N}}t^{-\frac{1}{2r}}, \qquad t > 0, \tag{5.17}$$

where $C > 0$ is independent of r, N, and L. Since $2N \geq p_2$, by (5.16) and (5.17) we obtain

$$\|D[u](t)\|_{L^\infty} \leq CL^{1+\frac{2}{N}} \int_0^t (t-s)^{-\frac{3}{4}+\frac{1}{4N}} s^{-\frac{1}{4N}}\, ds \leq CL^{1+\frac{2}{N}} T_*^{\frac{1}{4}}, \qquad t \leq 2T_*. \tag{5.18}$$

Furthermore, since it follows from $p \in [1, 2)$ with (1.18) and (2.20) that $N(1/p_2 - 1/p_1) < 1$, by (2.11) with $(q, r) = (p_2, p_1)$, (3.3), and (5.17) we have

$$\begin{aligned}
\|D[u](t)\|_{L^{p_1}} &\leq C \int_0^t (t-s)^{-\frac{N}{2}(\frac{1}{p_2}-\frac{1}{p_1})-\frac{1}{2}(1-\frac{1}{p_2})} |f(u(s))|_{L^{p_2}}\, ds \\
&\leq CL^{1+\frac{2}{N}} \int_0^t (t-s)^{-\frac{N}{2}(\frac{1}{p_2}-\frac{1}{p_1})-\frac{1}{2}(1-\frac{1}{p_2})} s^{-\frac{1}{2p_2}}\, ds \\
&\leq CL^{1+\frac{2}{N}} t^{\frac{1}{2}-\frac{N}{2}(\frac{1}{p_2}-\frac{1}{p_1})} \\
&\leq CL^{1+\frac{2}{N}} T_*^{\frac{1}{2}-\frac{N}{2}(\frac{1}{p_2}-\frac{1}{p_1})}, \qquad t \leq 2T_*.
\end{aligned} \tag{5.19}$$

Similarly, by (2.12) with $(q, r) = (p_2, p_1)$ we obtain

$$\begin{aligned}
|D[u](t)|_{L^{p_1}} &\leq \int_0^t (t-s)^{-\frac{N-1}{2}(\frac{1}{p_2}-\frac{1}{p_1})-\frac{1}{2}} |f(u(s))|_{L^{p_2}}\, ds \\
&\leq CL^{1+\frac{2}{N}} \int_0^t (t-s)^{-\frac{N-1}{2}(\frac{1}{p_2}-\frac{1}{p_1})-\frac{1}{2}} s^{-\frac{1}{2p_2}}\, ds \\
&\leq CL^{1+\frac{2}{N}} t^{\frac{1}{2}-\frac{N-1}{2}(\frac{1}{p_2}-\frac{1}{p_1})-\frac{1}{2p_1}} \\
&\leq CL^{1+\frac{2}{N}} t^{-\frac{1}{2p_1}} T_*^{\frac{1}{2}-\frac{N-1}{2}(\frac{1}{p_2}-\frac{1}{p_1})}, \qquad t \leq 2T_*.
\end{aligned} \tag{5.20}$$

If we choose L small enough such that

$$\max\left(T_*^{\frac{1}{4}}, T_*^{\frac{1}{2}-\frac{N-1}{2}(\frac{1}{p_2}-\frac{1}{p_1})}\right) L^{\frac{2}{N}} < T_*^{-\frac{N}{2p_1}},$$

then, by (5.4), (5.18), (5.19), and (5.20), for any $q \in [p_1, \infty]$, we get

$$\|D[u](t)\|_{L^q} + t^{\frac{1}{2q}} |D[u](t)|_{L^q} \leq CLT_*^{-\frac{N}{2p_1}} \leq CKT_*^{-\frac{N}{2p_1}}, \qquad t \leq 2T_*. \tag{5.21}$$

Since $T_* \geq 1$, by (5.15) and (5.21), for any $q \in [p_1, \infty]$, we obtain

$$\begin{aligned}
\|u(t)\|_{L^q} + t^{\frac{1}{2q}} |u(t)|_{L^q} &\leq C_* K \left(t^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})} + T_*^{-\frac{N}{2p_1}}\right) \\
&\leq C_* K \left(t^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})} + T_*^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})}\right) \\
&\leq C_* K t^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})} \quad t \leq 2T_*.
\end{aligned}$$

where C_* is independent of q, K, and T_*. This implies (5.12).

Next we prove (5.13). Let $N = 1$. Then, we recall that $p_1 = p$. By (1.13) we consider

$$u(t) = S_1(t)\varphi + \tilde{D}[u](t), \tag{5.22}$$

where $\tilde{D}[u]$ is the function defined by (3.23). For the linear part, by (2.8), (2.10), and (5.2), for any $q \in [p, \infty]$, we have

$$\|S_1(t)\varphi\|_{L^q} \leq c_1 t^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{q})} \|\varphi\|_{L^p} \leq Kt^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{q})}, \quad t > 0 \tag{5.23}$$

and

$$|[S_1(t)\varphi](0)| \le c_2 t^{-\frac{1}{2p}} \|\varphi\|_{L^p} \le K t^{-\frac{1}{2p}}, \quad t > 0. \tag{5.24}$$

On the other hand, by (1.15) and taking a sufficiently small $\varepsilon > 0$, for $L < \varepsilon$, we can apply Lemma 2.3, and we have

$$|f(u(0,t))| \le C L^3 t^{-\frac{1}{2}}, \qquad t > 0. \tag{5.25}$$

Then, for the nonlinear part $\tilde{D}[u]$, it holds from (2.7), (3.23), and (5.25) that, for any $q \in [p, \infty]$,

$$\begin{aligned} \|\tilde{D}[u](t)\|_{L^q} &\le C \int_0^t (t-s)^{-\frac{1}{2}(1-\frac{1}{q})} |f(u(0,s))|\, ds \\ &\le C L^3 \int_0^t (t-s)^{-\frac{1}{2}(1-\frac{1}{q})} s^{-\frac{1}{2}}\, ds \\ &\le C L^3 t^{\frac{1}{2q}} B\left(\frac{1}{2} + \frac{1}{2q}, \frac{1}{2}\right) \le C L^3 T_*^{\frac{1}{2q}} \qquad t \le 2T_*. \end{aligned} \tag{5.26}$$

Similarly, we have

$$\begin{aligned} |\tilde{D}[u](0,t)| &\le C \int_0^t (t-s)^{-\frac{1}{2}} |f(u(0,s))|\, ds \\ &\le C L^3 \int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}}\, ds \le C L^3, \qquad t \le 2T_*. \end{aligned} \tag{5.27}$$

If we choose $L < T_*^{-1/(4p)}$, then, by (5.4) and (5.26), for any $q \in [p, \infty]$, we get

$$\|\tilde{D}[u](t)\|_{L^q} \le C L T_*^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{q})} \le C K T_*^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{q})}, \qquad t \le 2T_*. \tag{5.28}$$

Furthermore, by (5.4) and (5.27), it holds that

$$|\tilde{D}[u](0,t)| \le C L T_*^{-\frac{1}{2p}} \le C K T_*^{-\frac{1}{2p}}, \qquad t \le 2T_*. \tag{5.29}$$

Combining (5.23) and (5.28), we have

$$\|u(t)\|_{L^q} \le C K \left(t^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{q})} + T_*^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{q})} \right) \le C K t^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{q})}, \qquad t \le 2T_*.$$

Similarly, by (5.24) and (5.29), we obtain

$$|u(0,t)| \le C K \left(t^{-\frac{1}{2p}} + T_*^{-\frac{1}{2p}} \right) \le C K t^{-\frac{1}{2p}}, \qquad t \le 2T_*.$$

These imply (5.13), thus Lemma 5.3 follows. □

For the case $N \ge 2$, applying Lemmata 5.1 and 5.3, we show the decay estimate of $|u(t)|_{L^q}$.

Lemma 5.4 *Assume the same conditions as in Lemma 5.3 for the case $N \ge 2$. Then, for $\tilde{K}$ as in (5.3), there exists a positive function $F = F(N, p_1, \tilde{K}, \lambda)$ such that, if $L < F$ and L is small enough, then, for any $q \in [p_1, \infty]$,*

$$\sup_{t>0} t^{\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})+\frac{1}{2q}} |u(t)|_{L^q} \le C K, \tag{5.30}$$

where C depends only on N.

Proof Let u be a unique solution to problem (1.1) satisfying (1.14) and (1.15). Then, similarly to (4.11), applying arguments similar to that in the proof of [20, Lemma 2.1] with (1.15), we see that

$$u \in C((0,\infty); L^q(\partial\mathbb{R}^N_+)), \qquad q \in [p_1, \infty]. \tag{5.31}$$

By Lemma 5.3, for any $T_* \geq 1$, there exists $\varepsilon = \varepsilon(p_1, T_*)$ such that, if $L < \varepsilon$, then

$$|u(t)|_{L^q} \leq C_* K t^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})-\frac{1}{2q}}, \qquad 0 < t \leq 2T_*, \tag{5.32}$$

where $C_* \geq 1$ is independent of q, K and T_*. Let us fix T_* large enough to be chosen later, put

$$T = \sup\left\{0 < s < \infty;\ |u(t)|_{L^q} \leq 2C_* K t^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})-\frac{1}{2q}} \quad \text{for all } q \in [p_1,\infty] \text{ and } 0 < t < s\right\}.$$

Then, since $T_* \geq 1$, by (5.32) we have $T \geq 2T_* \geq 2$.

We prove $T = \infty$. The proof is by contradiction. We assume that $T < \infty$. Then, by (5.31) we see that

$$|u(T)|_{L^q} = 2C_* K T^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})-\frac{1}{2q}}. \tag{5.33}$$

On the other hand, by (2.9) with $(q, r) = (p_1, q)$ and (5.2) we have

$$|S_1(T)\varphi|_{L^q} \leq c_2 T^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})-\frac{1}{2q}} \|\varphi\|_{L^{p_1}} \leq C_* K T^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})-\frac{1}{2q}}. \tag{5.34}$$

Let T_1 be the constant given in Lemma 5.1 with $D = 2C_*$, and let us assume that

$$T_* \geq T_1 \geq 1. \tag{5.35}$$

Furthermore, let I_1 and I_2 be functions given in (4.16), and let p_2 be the constant given in (2.20). Then, for the term I_1, since $T \geq 2T_*$, by (2.12) with $(q, r) = (p_2, q)$ we get

$$\begin{aligned} I_1(T) &\leq C\left(\int_0^{T_*} + \int_{T_*}^{T/2}\right)(T-s)^{-\frac{N-1}{2}(\frac{1}{p_2}-\frac{1}{q})-\frac{1}{2}} |f(u(s))|_{L^{p_2}}\, ds \\ &=: A(T) + B(T). \end{aligned} \tag{5.36}$$

Since $p_1 \geq p_2 \geq 1$ and $T \geq 1$, due to (1.15) and taking a sufficiently small L if necessary, we can apply Lemma 2.2 to the term $A(T)$, and we obtain

$$\begin{aligned} A(T) &\leq C T^{-\frac{N-1}{2}(\frac{1}{p_2}-\frac{1}{q})-\frac{1}{2}} \int_0^{T_*} L^{1+\frac{2}{N}} s^{-\frac{1}{2p_2}}\, ds \\ &\leq C L^{1+\frac{2}{N}} T^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})-\frac{1}{2q}} T^{-\frac{1}{2}(1-\frac{1}{p_2})-\frac{N}{2}(\frac{1}{p_2}-\frac{1}{p_1})} T_*^{1-\frac{1}{2p_2}} \\ &\leq C L^{1+\frac{2}{N}} T^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})-\frac{1}{2q}} T_*^{1-\frac{1}{2p_2}}. \end{aligned} \tag{5.37}$$

Furthermore, let p_3 be the constant given in (5.7). Then, since $T_* \geq T_1$ and it follows from $p_1 < 2$ that $p_2 \geq p_3$ we can apply Lemma 5.1 to the term $B(T)$, and we have

$$\begin{aligned} B(T) &\leq CT^{-\frac{N-1}{2}(\frac{1}{p_2}-\frac{1}{q})-\frac{1}{2}} \int_{T_*}^{T/2} K^{1+\frac{2}{N}} s^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{p_2})-\frac{1}{p_1}-\frac{1}{2p_2}}\, ds \\ &\leq CK^{1+\frac{2}{N}} T^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})-\frac{1}{2q}} \int_{T_*}^{T/2} T^{-\frac{1}{2}(1-\frac{1}{p_2})-\frac{N}{2}(\frac{1}{p_2}-\frac{1}{p_1})} s^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{p_2})-\frac{1}{p_1}-\frac{1}{2p_2}}\, ds \\ &\leq CK^{1+\frac{2}{N}} T^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})-\frac{1}{2q}} \int_{T_*}^{T/2} s^{-\frac{1}{2}-\frac{1}{p_1}}\, ds \\ &\leq CK^{1+\frac{2}{N}} T^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})-\frac{1}{2q}} \int_{T_*}^{\infty} s^{-\frac{1}{2}-\frac{1}{p_1}}\, ds \\ &\leq CK^{1+\frac{2}{N}} T^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})-\frac{1}{2q}} T_*^{-\frac{1}{p_1}+\frac{1}{2}}, \end{aligned} \tag{5.38}$$

where C is independent of q, L, K, and T_*. Moreover, for the term I_2, since $q \geq p_1 \geq p_3$ and $p_1 < 2$, by (2.12) with $q = r$ and (5.6) we see that

$$\begin{aligned} I_2(T) &\leq \int_{T/2}^{T} (T-s)^{-\frac{1}{2}} |f(u(s))|_{L^q}\, ds \\ &\leq CK^{1+\frac{2}{N}} \int_{T/2}^{T} (T-s)^{-\frac{1}{2}} s^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})-\frac{1}{p_1}-\frac{1}{2q}}\, ds \\ &\leq CK^{1+\frac{2}{N}} T^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})-\frac{1}{p_1}-\frac{1}{2q}} \int_{T/2}^{T} (T-s)^{-\frac{1}{2}}\, ds \\ &\leq CK^{1+\frac{2}{N}} T^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})-\frac{1}{2q}} T_*^{-\frac{1}{p_1}+\frac{1}{2}}, \end{aligned}$$

where C is independent of q, L, K, and T_*. This together with (4.16), (5.36), (5.37), and (5.38) implies that

$$\begin{aligned} |D[u](T)|_{L^q} &\leq I_1(T) + I_2(T) \\ &\leq D_* T^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})-\frac{1}{2q}} \left(L^{1+\frac{2}{N}} T_*^{1-\frac{1}{2p_2}} + K^{1+\frac{2}{N}} T_*^{-\frac{1}{p_1}+\frac{1}{2}} \right), \end{aligned} \tag{5.39}$$

where D_* is a constant independent of L, K, and T_*. Since $p_1 < 2$, we can take a sufficiently large constant $T_* \geq 1$ so that

$$D_* T_*^{-\frac{1}{p_1}+\frac{1}{2}} K^{\frac{2}{N}} \leq D_* T_*^{-\frac{1}{p_1}+\frac{1}{2}} \tilde{K}^{\frac{2}{N}} \leq \frac{C_*}{4} \tag{5.40}$$

which means

$$T_* \geq \left(\frac{4D_* \tilde{K}^{\frac{2}{N}}}{C_*} \right)^{\frac{1}{\frac{1}{p_1}-\frac{1}{2}}}. \tag{5.41}$$

This together with (5.35) implies that T_* depends on λ, $\tilde{K}$, and p_1 but not on L. Then we can also take a sufficiently small constant L so that

$$D_* T_*^{1-\frac{1}{2p_2}} L^{\frac{2}{N}} \leq \frac{C_*}{4} \tag{5.42}$$

and this means

$$L \le \left(\frac{4D_* T_*^{1-\frac{1}{2p_2}}}{C_*} \right)^{-\frac{N}{2}}. \tag{5.43}$$

By (5.4), (5.39), (5.40), and (5.42) we have

$$|D[u](T)|_{L^q} \le \frac{1}{2} C_* K T^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})-\frac{1}{2q}}. \tag{5.44}$$

Combining (5.14), (5.34), and (5.44), we see that

$$\begin{aligned} |u(T)|_{L^q} &\le |S_1(T)\varphi|_{L^q} + |D[u](t)|_{L^q} \\ &\le \left(C_* + \frac{C_*}{2}\right) K T^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})-\frac{1}{2q}} < 2C_* K T^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})-\frac{1}{2q}}. \end{aligned}$$

This contradicts (5.33), and we see $T = \infty$. In order to make clear the dependence of the choice we made on T_* and L, we collect below all the conditions (5.35), (5.41), and (5.43)

$$T_* \ge T_1, \quad T_* \ge \left(\frac{4D_* \tilde{K}^{\frac{2}{N}}}{C_*} \right)^{\frac{1}{\frac{1}{p_1}-\frac{1}{2}}}, \quad L \le \left(\frac{4D_* T_*^{1-\frac{1}{2p_2}}}{C_*} \right)^{-\frac{N}{2}},$$

where T_1 satisfies (5.9) with $D = 2C_*$, namely

$$T_1 \ge \left(\frac{\lambda (2C_* \tilde{K})^2}{\log 2} \right)^{\frac{p_1}{N}}.$$

Here C_* and D_* are constants depending at most on N and p_1. Then we can find a function F depending on N, p_1, $\tilde{K}$, and λ such that the conditions on L can be written as $L < F(N, p_1, \tilde{K}, \lambda)$ and L small enough. Thus Lemma 5.4 follows. □

Similarly, for the case $N = 1$, applying Lemmata 5.2 and 5.3, we have the following.

Lemma 5.5 *Assume the same conditions as in Lemma 5.3 for the case $N = 1$. Then, for $\tilde{K}$ as in (5.3), there exists a positive function $F = F(p, \tilde{K}, \lambda)$ such that, if $L < F$ and L is small enough, then,*

$$\sup_{t>0} t^{\frac{1}{2p}} |u(0,t)| \le CK,$$

where C is independent of p and K.

Proof Applying the same argument as in the proof of Lemma 5.4, we can prove this lemma. For reader's convenience, we give it here.

Let u be a unique solution to problem (1.1) satisfying (1.14) and (1.15). Then, similarly to (5.31), we can easily show that

$$u(0,t) \in C((0,\infty)). \tag{5.45}$$

By Lemma 5.3, for any $T_* \ge 1$, there exists $\varepsilon = \varepsilon(p, T_*)$ such that, if $L < \varepsilon$, then

$$|u(0,t)| \le C_* K t^{-\frac{1}{2p}}, \qquad 0 < t \le 2T_*, \tag{5.46}$$

where $C_* \ge 1$ is independent of K and T_*. Let us fix T_* large enough to be chosen later, put

$$T = \sup \left\{ 0 < s < \infty;\ |u(0,t)| \le 2C_* K t^{-\frac{1}{2p}} \quad \text{for all } 0 < t < s \right\}.$$

Then, since $T_* \geq 1$, by (5.46) we have $T \geq 2T_* \geq 2$.

We prove $T = \infty$. The proof is by contradiction. We assume that $T < \infty$. Then, by (5.45) we see that

$$|u(0,T)| = 2C_* K T^{-\frac{1}{2p}}. \tag{5.47}$$

On the other hand, by (2.10) with $q = p$ and (5.2) we have

$$|[S_1(T)\varphi](0)| \leq c_2 T^{-\frac{1}{2p}} \|\varphi\|_{L^p} \leq C_* K T^{-\frac{1}{2p}}. \tag{5.48}$$

Let $\tilde{T}_1$ be the constant given in Lemma 5.2 with $D = 2C_*$, and let us assume

$$T_* \geq \tilde{T}_1 \geq 1. \tag{5.49}$$

Furthermore, since $T \geq 2T_*$, by (3.23) we put

$$\begin{aligned} |\tilde{D}[u](0,T)| &\leq C\left(\int_0^{T_*} + \int_{T_*}^{T/2} + \int_{T/2}^{T}\right)(T-s)^{-\frac{1}{2}}|f(u(0,s))|\,ds \\ &=: \tilde{I}_1(T) + \tilde{I}_2(T) + \tilde{I}_3(T). \end{aligned} \tag{5.50}$$

Since $p \geq 1$ and $T \geq 2T_*$, due to (1.15) and taking a sufficiently small L if necessary, we can apply Lemma 2.3 to the term $\tilde{I}_1(T)$, and we obtain

$$\begin{aligned} \tilde{I}_1(T) &\leq C T^{-\frac{1}{2}} \int_0^{T_*} L^3 s^{-\frac{1}{2}}\,ds \\ &\leq C L^3 T^{-\frac{1}{2p}} T^{-\frac{1}{2}(1-\frac{1}{p})} T_*^{\frac{1}{2}} \leq C L^3 T^{-\frac{1}{2p}} T_*^{\frac{1}{2p}}. \end{aligned} \tag{5.51}$$

Furthermore, since $T_* \geq \tilde{T}_1$ and $p < 2$, for the terms $\tilde{I}_2(T)$ and $\tilde{I}_3(T)$, we can apply Lemma 5.2, and we have

$$\begin{aligned} \tilde{I}_2(T) &\leq C T^{-\frac{1}{2}} \int_{T_*}^{T/2} K^3 s^{-\frac{3}{2p}}\,ds \\ &\leq C K^3 T^{-\frac{1}{2p}} \int_{T_*}^{T/2} T^{-\frac{1}{2}(1-\frac{1}{p})} s^{-\frac{3}{2p}}\,ds \\ &\leq C K^3 T^{-\frac{1}{2p}} \int_{T_*}^{T/2} s^{-\frac{1}{2}-\frac{1}{p}}\,ds \leq C K^3 T^{-\frac{1}{2p}} T_*^{-\frac{1}{p}+\frac{1}{2}}, \end{aligned}$$

and

$$\begin{aligned} \tilde{I}_3(T) &\leq C K^3 \int_{T/2}^{T} (T-s)^{-\frac{1}{2}} s^{-\frac{3}{2p}}\,ds \\ &\leq C K^3 T^{-\frac{3}{2p}} \int_{T/2}^{T} (T-s)^{-\frac{1}{2}}\,ds \leq C K^3 T^{-\frac{1}{2p}} T_*^{-\frac{1}{p}+\frac{1}{2}}, \end{aligned}$$

where C is a constant independent of p, L, K, and T_*. These together with (5.50) and (5.51) imply that

$$|\tilde{D}[u](0,T)| \leq D_* T^{-\frac{1}{2p}} \left(L^3 T_*^{\frac{1}{2p}} + K^3 T_*^{-\frac{1}{p}+\frac{1}{2}}\right), \tag{5.52}$$

where D_* is a constant independent of L, K, and T_*. Since $p < 2$, we can take a sufficiently large constant $T_* \geq 1$ so that

$$D_* T_*^{-\frac{1}{p}+\frac{1}{2}} K^2 \leq D_* T_*^{-\frac{1}{p}+\frac{1}{2}} \tilde{K}^2 \leq \frac{C_*}{4} \tag{5.53}$$

which means

$$T_* \geq \left(\frac{4D_*\tilde{K}^2}{C_*}\right)^{\frac{1}{\frac{1}{p}-\frac{1}{2}}} \geq \left(\frac{4D_*K^2}{C_*}\right)^{\frac{1}{\frac{1}{p}-\frac{1}{2}}}. \tag{5.54}$$

This together with (5.49) implies that T_* depends on λ, $\tilde{K}$, and p but not on L. Then we can also take a sufficiently small constant L so that

$$D_* T_*^{\frac{1}{2p}} L^2 \leq \frac{C_*}{4} \tag{5.55}$$

and this means

$$L \leq \left(\frac{4D_* T_*^{\frac{1}{2p}}}{C_*}\right)^{-\frac{1}{2}}. \tag{5.56}$$

By (5.4), (5.52), (5.53), and (5.55) we have

$$|\tilde{D}[u](0,T)| \leq \frac{1}{2} C_* K T^{-\frac{1}{2p}}.$$

This together with (5.22) and (5.48) implies

$$|u(0,T)| \leq |[S_1(T)\varphi](0)| + |\tilde{D}[u](0,T)| \leq \left(C_* + \frac{C_*}{2}\right) K T^{-\frac{1}{2p}} < 2C_* K T^{-\frac{1}{2p}}.$$

This contradicts (5.47), and we see $T = \infty$. In order to make clear the dependence of the choice we made on T_* and L, we collect below all the conditions (5.49), (5.54), and (5.56)

$$T_* \geq \tilde{T}_1, \quad T_* \geq \left(\frac{4D_*\tilde{K}^2}{C_*}\right)^{\frac{1}{\frac{1}{p}-\frac{1}{2}}}, \quad L \leq \left(\frac{4D_* T_*^{\frac{1}{2p}}}{C_*}\right)^{-\frac{1}{2}},$$

where $\tilde{T}_1$ satisfies (5.9) with $(N, p_1) = (1, p)$ and $D = 2C_*$, namely

$$\tilde{T}_1 \geq \left(\frac{\lambda(2C_*\tilde{K})^2}{\log 2}\right)^p.$$

Here C_* and D_* are constants depending at most on p. Then we can find a function F depending on p, $\tilde{K}$, and λ such that the condition L can be written as $L < F(p, \tilde{K}, \lambda)$ and L small enough. Thus Lemma 5.5 follows. □

Now we ready to prove Theorem 1.3. We first prove it for the case $N \geq 2$.

Proof of Theorem 1.3 $(N \geq 2)$. Let u be a unique solution to problem (1.1) satisfying (1.14) and (1.15). Let T be a sufficiently large constant to be chosen later, which satisfies $T \geq T_1$, where T_1 is the constant given in Lemma 5.1 with $D = C_*$. Suppose that L is small enough such that Lemmata 5.3 and 5.4 hold. Then, by (5.12) and (5.30), in order to prove (1.20), it suffices to prove the decay estimate of $\|u(t)\|_{L^q}$ for $t \geq 2T$.

Let p_1 be the constant given in (1.18) and $q \in [p_1, \infty]$. For the linear part, by (2.8) with $(q, r) = (p_1, q)$ and (5.2) we have

$$\|S_1(t)\varphi\|_{L^q} \leq c_1 t^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})} \|\varphi\|_{L^{p_1}} \leq K t^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})}, \qquad t > 0. \tag{5.57}$$

For the nonlinear part, let J_1 and J_2 be functions given in (4.24), and let p_2 be the constant given in (2.20). Then, for the term J_1, similarly to (5.36), by (2.11) with $(q, r) = (p_2, q)$ we put

$$\begin{aligned} J_1(t) &\le C\left(\int_0^T + \int_T^{t/2}\right)(t-s)^{-\frac{N}{2}(\frac{1}{p_2}-\frac{1}{q})-\frac{1}{2}(1-\frac{1}{p_2})}|f(u(s))|_{L^{p_2}}\,ds \\ &=: \tilde{A}(t) + \tilde{B}(t), \qquad t \ge 2T. \end{aligned} \tag{5.58}$$

For the term $\tilde{A}(t)$, since $p_1 \ge p_2 \ge 1$, by (1.15) and taking a sufficiently small L if necessary, we can apply Lemma 2.2, and we have

$$\begin{aligned} \tilde{A}(t) &\le CL^{1+\frac{2}{N}} t^{-\frac{N}{2}(\frac{1}{p_2}-\frac{1}{q})-\frac{1}{2}(1-\frac{1}{p_2})} \int_0^T s^{-\frac{1}{2p_2}}\,ds \\ &\le CL^{1+\frac{2}{N}} t^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})} t^{-\frac{N}{2}(\frac{1}{p_2}-\frac{1}{p_1})-\frac{1}{2}(1-\frac{1}{p_2})} T^{1-\frac{1}{2p_2}}, \qquad t \ge 2T. \end{aligned} \tag{5.59}$$

Furthermore, let p_3 be the constant given in (5.7). Then, for the term $\tilde{B}(t)$, since $T \ge T_1$, and $p_2 \ge p_3$, we can apply Lemma 5.1, and it follows from $p_2 \ge 1$ that

$$\begin{aligned} \tilde{B}(t) &\le CK^{1+\frac{2}{N}} t^{-\frac{N}{2}(\frac{1}{p_2}-\frac{1}{q})-\frac{1}{2}(1-\frac{1}{p_2})} \int_T^{t/2} s^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{p_2})-\frac{1}{p_1}-\frac{1}{2p_2}}\,ds \\ &\le CK^{1+\frac{2}{N}} t^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})} t^{-\frac{N}{2}(\frac{1}{p_2}-\frac{1}{p_1})-\frac{1}{2}(1-\frac{1}{p_2})} \int_T^{t/2} s^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{p_2})-\frac{1}{p_1}-\frac{1}{2p_2}}\,ds, \quad t \ge 2T. \end{aligned} \tag{5.60}$$

For $p \in (p_2, 2)$ (which implies $p_1 = p$), since $p_1 < 2$, we can choose $\sigma_1 \in (0, 1)$ satisfying

$$0 < \sigma_1 < \min\left\{\frac{1}{p_1} - \frac{1}{2}, \frac{N}{2}\left(\frac{1}{p_2} - \frac{1}{p_1}\right)\right\}.$$

Then, by (5.59) and (5.60) we have

$$\tilde{A}(t) \le CL^{1+\frac{2}{N}} t^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})-\frac{N}{2}(\frac{1}{p_2}-\frac{1}{p_1})} T^{1-\frac{1}{2p_2}} \le CL^{1+\frac{2}{N}} t^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})-\sigma_1} T^{1-\frac{1}{2p_2}}, \qquad t \ge 2T,$$

and

$$\begin{aligned} \tilde{B}(t) &\le CK^{1+\frac{2}{N}} t^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})-\sigma_1} \int_T^{t/2} t^{-\frac{N}{2}(\frac{1}{p_2}-\frac{1}{p_1})-\frac{1}{2}(1-\frac{1}{p_2})+\sigma_1} s^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{p_2})-\frac{1}{p_1}-\frac{1}{2p_2}}\,ds \\ &\le CK^{1+\frac{2}{N}} t^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})-\sigma_1} \int_T^{t/2} s^{-\frac{1}{p_1}-\frac{1}{2}+\sigma_1}\,ds \\ &\le CK^{1+\frac{2}{N}} t^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})-\sigma_1} \int_T^{\infty} s^{-\frac{1}{p_1}-\frac{1}{2}+\sigma_1}\,ds \\ &\le CK^{1+\frac{2}{N}} t^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})-\sigma_1} T^{\frac{1}{2}-\frac{1}{p_1}+\sigma_1}, \qquad t \ge 2T. \end{aligned}$$

This together with (5.4) and (5.58) implies that

$$J_1(t) \le CL^{1+\frac{2}{N}} t^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})-\sigma_1} T^{1-\frac{1}{2p_2}} + CK^{1+\frac{2}{N}} t^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})-\sigma_1} T^{\frac{1}{2}-\frac{1}{p_1}+\sigma_1}, \qquad t \ge 2T.$$

Choosing T large enough such that

$$K^{\frac{2}{N}} T^{\frac{1}{2}-\frac{1}{p_1}+\sigma_1} \le \tilde{K}^{\frac{2}{N}} T^{\frac{1}{2}-\frac{1}{p_1}+\sigma_1} \le 1$$

namely

$$T \geq (\tilde{K}^{\frac{2}{N}})^{\frac{1}{\frac{1}{p_1}-\frac{1}{2}-\sigma_1}}$$

and L small enough such that

$$L^{\frac{2}{N}} T^{1-\frac{1}{2p_2}} \leq 1,$$

thanks to (5.4) we get

$$\begin{aligned} J_1(t) &\leq CLt^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})-\sigma_1} + CKt^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})-\sigma_1} \\ &\leq CKt^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})-\sigma_1}, \qquad t \geq 2T. \end{aligned} \tag{5.61}$$

On the other hand, for $p \leq p_2$, namely $p_1 = p_2$, we consider two cases, $N = 2$ and $N \geq 3$. For the case $N \geq 3$, since $p_1 \in (1, 2)$, we can choose $\sigma_2 \in (0, 1)$ satisfying

$$0 < \sigma_2 < \min\left\{\frac{1}{p_1} - \frac{1}{2}, \frac{1}{2}\left(1 - \frac{1}{p_1}\right)\right\}.$$

Then, by (5.59) and (5.60) we see that

$$\tilde{A}(t) \leq CL^{1+\frac{2}{N}} t^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})-\sigma_2} T^{1-\frac{1}{2p_2}}, \qquad t \geq 2T,$$

and

$$\begin{aligned} \tilde{B}(t) &\leq CK^{1+\frac{2}{N}} t^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})-\sigma_2} \int_T^{t/2} t^{-\frac{1}{2}(1-\frac{1}{p_1})+\sigma_2} s^{-\frac{3}{2p_1}}\, ds \\ &\leq CK^{1+\frac{2}{N}} t^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})-\sigma_2} \int_T^{t/2} s^{-\frac{1}{p_1}-\frac{1}{2}+\sigma_2}\, ds \\ &\leq CK^{1+\frac{2}{N}} t^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})-\sigma_2} \int_T^{\infty} s^{-\frac{1}{p_1}-\frac{1}{2}+\sigma_2}\, ds \\ &\leq CK^{1+\frac{2}{N}} t^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})-\sigma_2} T^{\frac{1}{2}-\frac{1}{p_1}+\sigma_2}, \qquad t \geq 2T. \end{aligned}$$

This together with (5.4) and (5.58) implies that

$$J_1(t) \leq CL^{1+\frac{2}{N}} t^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})-\sigma_2} T^{1-\frac{1}{2p_2}} + CK^{1+\frac{2}{N}} t^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})-\sigma_2} T^{\frac{1}{2}-\frac{1}{p_1}+\sigma_2}, \qquad t \geq 2T.$$

Choosing T large enough such that

$$K^{\frac{2}{N}} T^{\frac{1}{2}-\frac{1}{p_1}+\sigma_2} \leq \tilde{K}^{\frac{2}{N}} T^{\frac{1}{2}-\frac{1}{p_1}+\sigma_2} \leq 1$$

and L small enough such that

$$L^{\frac{2}{N}} T^{1-\frac{1}{2p_2}} \leq 1,$$

thanks to (5.4) we get

$$\begin{aligned} J_1(t) &\leq CLt^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})-\sigma_2} + CKt^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})-\sigma_2} \\ &\leq CKt^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})-\sigma_2}, \qquad t \geq 2T. \end{aligned} \tag{5.62}$$

For the case $N = 2$, since $p_1 = p_2 = 1$ (which implies $p = 1$), by (5.59) and (5.60) again we see that

$$\tilde{A}(t) \leq CL^2 t^{-(\frac{1}{p_1}-\frac{1}{q})} T^{\frac{1}{2}}, \qquad t \geq 2T,$$

and

$$\tilde{B}(t) \le CK^2 t^{-(\frac{1}{p_1}-\frac{1}{q})} \int_T^{t/2} s^{-\frac{3}{2}}\, ds \le CK^2 t^{-(\frac{1}{p_1}-\frac{1}{q})} T^{-\frac{1}{2}}, \qquad t \ge 2T.$$

This together with (5.4) and (5.58) implies that

$$J_1(t) \le CL^2 t^{-(\frac{1}{p_1}-\frac{1}{q})} T^{\frac{1}{2}} + CK^2 t^{-(\frac{1}{p_1}-\frac{1}{q})} T^{-\frac{1}{2}}, \qquad t \ge 2T.$$

Choosing T large enough such that

$$KT^{-\frac{1}{2}} \le \tilde{K} T^{-\frac{1}{2}} \le 1$$

and L small enough such that

$$LT^{\frac{1}{2}} \le 1,$$

thanks to (5.4) we get

$$J_1(t) \le CLt^{-(\frac{1}{p_1}-\frac{1}{q})} + CKt^{-(\frac{1}{p_1}-\frac{1}{q})} \le CKt^{-(\frac{1}{p_1}-\frac{1}{q})}, \qquad t \ge 2T. \tag{5.63}$$

Therefore, by (5.61), (5.62), and (5.63), for $N \ge 2$, we have

$$J_1(t) \le CKt^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})}, \qquad t \ge 2T. \tag{5.64}$$

Let us come back to the $J_2(t)$ term. Since $T \ge T_1$ and $q \ge p_1 \ge p_3$, we can apply Lemma 5.1, and by (2.11) with $q = r$ and (5.6) we have

$$\begin{aligned} J_2(t) &\le CK^{1+\frac{2}{N}} \int_{t/2}^{t} (t-s)^{-\frac{1}{2}(1-\frac{1}{q})} s^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})-\frac{1}{p_1}-\frac{1}{2q}}\, ds \\ &\le CK^{1+\frac{2}{N}} t^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})-\frac{1}{p_1}-\frac{1}{2q}} \int_{t/2}^{t} (t-s)^{-\frac{1}{2}(1-\frac{1}{q})}\, ds \\ &\le CK^{1+\frac{2}{N}} t^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})-\frac{1}{p_1}+\frac{1}{2}}, \qquad t \ge 2T. \end{aligned}$$

Since $p_1 < 2$, we can choose $\sigma_3 > 0$ satisfying $0 < \sigma_3 < 1/p_1 - 1/2$, and we get

$$J_2(t) \le CK^{1+\frac{2}{N}} t^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})-\sigma_3} \le CK^{1+\frac{2}{N}} t^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})-\frac{\sigma_3}{2}} T^{-\frac{\sigma_3}{2}}, \qquad t \ge 2T.$$

Choosing T large enough such that

$$K^{\frac{2}{N}} T^{-\frac{\sigma_3}{2}} \le \tilde{K}^{\frac{2}{N}} T^{-\frac{\sigma_3}{2}} \le 1,$$

we have

$$J_2(t) \le CKt^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})-\frac{\sigma_3}{2}}, \qquad t \ge 2T. \tag{5.65}$$

Combining (5.57), (5.64), and (5.65), we obtain

$$\|u(t)\|_{L^q} \le CKt^{-\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})}, \qquad t \ge 2T,$$

thus (1.20) follows.

Finally we prove (1.21) by the same arguments as in the proof of [10, Theorem 2.2]. Indeed, let $p_1 \in (1, 2)$. By (5.61), (5.62), and (5.65) we have

$$t^{\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})} \|u(t) - S_1(t)\varphi\|_{L^q} = o(1), \qquad t \to \infty.$$

Now, by density, let $\{\varphi_n\} \subset C_0^\infty$ such that $\varphi_n \to \varphi$ in L^{p_1}. Then, by (2.8), it holds that

$$\begin{aligned} t^{\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})}\|S_1(t)\varphi\|_{L^q} &\le t^{\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})}\|S_1(t)(\varphi-\varphi_n)\|_{L^q} + t^{\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})}\|S_1(t)\varphi_n\|_{L^q} \\ &\le C\|\varphi-\varphi_n\|_{L^{p_1}} + Ct^{\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})}t^{-\frac{N}{2}(1-\frac{1}{q})}\|\varphi_n\|_{L^1} \\ &\le C\|\varphi-\varphi_n\|_{L^{p_1}} + Ct^{-\frac{N}{2}(1-\frac{1}{p_1})}\|\varphi_n\|_{L^1}, \qquad t>0. \end{aligned}$$

Since $p_1 > 1$, this proves that

$$t^{\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})}\|S_1(t)\varphi\|_{L^q} = o(1), \quad t\to\infty,$$

and so

$$t^{\frac{N}{2}(\frac{1}{p_1}-\frac{1}{q})}\|u(t)\|_{L^q} = o(1), \quad t\to\infty.$$

Thus the proof of Theorem 1.3 for the case $N \ge 2$ is complete. □

Next, applying the same argument as in the prof of Theorem 1.3 for the case $N \ge 2$, we prove Theorem 1.3 for the case $N = 1$.

Proof of Theorem 1.3 ($N = 1$). Let u be a unique solution to problem (1.1) satisfying (1.14) and (1.15). Let T be a sufficiently large constant to be chosen later, which satisfies $T \ge \tilde{T}_1$, where $\tilde{T}_1$ is the constant given in Lemma 5.2 with $D = C_*$. Suppose that L is sufficiently small so that Lemmata 5.3 and 5.5 hold. Then, it is enough to prove the decay estimate of $\|\tilde{D}(t)\|_{L^q}$ for $t \ge 2T$ in order to obtain (1.20).

Let $q \in [p,\infty]$. Then, similarly to (5.50), by (2.7) and (3.23) we put

$$\begin{aligned} \|\tilde{D}[u](t)\|_{L^q} &\le C\left(\int_0^T + \int_T^{t/2} + \int_{t/2}^t\right)(t-s)^{-\frac{1}{2}(1-\frac{1}{q})}|f(u(0,s))|\,ds \\ &=: \tilde{J}_1(t) + \tilde{J}_2(t) + \tilde{J}_3(t), \qquad t \ge 2T. \end{aligned} \tag{5.66}$$

For the term $\tilde{J}_1$, by (1.15) and taking a sufficiently small L if necessary, we can apply Lemma 2.3, and we have

$$\begin{aligned} \tilde{J}_1(t) &\le CL^3 t^{-\frac{1}{2}(1-\frac{1}{q})}\int_0^T s^{-\frac{1}{2}}\,ds \\ &\le CL^3 t^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{q})}t^{-\frac{1}{2}(1-\frac{1}{p})}T^{\frac{1}{2}}, \qquad t \ge 2T. \end{aligned} \tag{5.67}$$

Furthermore, for the term $\tilde{J}_2(t)$, since $T \ge \tilde{T}_1$, we can apply Lemma 5.2, and it holds that

$$\begin{aligned} \tilde{J}_2(t) &\le CK^3 t^{-\frac{1}{2}(1-\frac{1}{q})}\int_T^{t/2} s^{-\frac{3}{2p}}\,ds, \\ &\le CK^3 t^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{q})}t^{-\frac{1}{2}(1-\frac{1}{p})}\int_T^{t/2} s^{-\frac{3}{2p}}\,ds, \qquad t \ge 2T. \end{aligned}$$

For $p \in (1,2)$, we can choose $\tilde{\sigma}_1 \in (0,1)$ satisfying

$$0 < \tilde{\sigma}_1 < \min\left(\frac{1}{p}-\frac{1}{2}, \frac{1}{2}\left(1-\frac{1}{p}\right)\right).$$

Then, for $t \geq 2T$ we have

$$\begin{aligned} & CK^3 t^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{q})} t^{-\frac{1}{2}(1-\frac{1}{p})} \int_T^{t/2} s^{-\frac{3}{2p}}\, ds \\ & = CK^3 t^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{q})-\tilde{\sigma}_1} \int_T^{t/2} t^{-\frac{1}{2}(1-\frac{1}{p})+\tilde{\sigma}_1} s^{-\frac{3}{2p}}\, ds \\ & \leq CK^3 t^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{q})-\tilde{\sigma}_1} \int_T^{t/2} s^{-\frac{3}{2p}-\frac{1}{2}(1-\frac{1}{p})+\tilde{\sigma}_1}\, ds \\ & \leq CK^3 t^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{q})-\tilde{\sigma}_1} T^{\frac{1}{2}-\frac{1}{p}+\tilde{\sigma}_1}. \end{aligned} \tag{5.68}$$

Then, by (5.67) and (5.68) we have

$$\begin{aligned} \tilde{J}_1(t) + \tilde{J}_2(t) & \leq CL^3 t^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{q})} t^{-\frac{1}{2}(1-\frac{1}{p})} T^{\frac{1}{2}} + CK^3 t^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{q})-\tilde{\sigma}_1} T^{\frac{1}{2}-\frac{1}{p}+\tilde{\sigma}_1} \\ & \leq C t^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{q})-\tilde{\sigma}_1} \left(L^3 T^{\frac{1}{2}} + K^3 T^{\frac{1}{2}-\frac{1}{p}+\tilde{\sigma}_1} \right). \end{aligned}$$

Now, choosing T large enough such that

$$K^2 T^{\frac{1}{2}-\frac{1}{p}+\tilde{\sigma}_1} \leq \tilde{K}^2 T^{\frac{1}{2}-\frac{1}{p}+\tilde{\sigma}_1} \leq 1$$

and then L small enough so that

$$L^2 T^{\frac{1}{2}} \leq 1,$$

thanks to (5.4) we get

$$\tilde{J}_1(t) + \tilde{J}_2(t) \leq CK t^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{q})-\tilde{\sigma}_1}. \tag{5.69}$$

On the other hand, for $p = 1$, by (5.67) and (5.68) again we see that

$$\tilde{J}_1(t) + \tilde{J}_2(t) \leq CK t^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{q})}, \qquad t \geq 2T.$$

This together with (5.69) implies for all $p \in [1, 2)$

$$\tilde{J}_1(t) + \tilde{J}_2(t) \leq CK t^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{q})}, \qquad t \geq 2T. \tag{5.70}$$

For the $\tilde{J}_3(t)$ term, since $T \geq \tilde{T}_1$, we can apply Lemma 5.2, and it holds that

$$\begin{aligned} \tilde{J}_3(t) & \leq CK^3 \int_{t/2}^t (t-s)^{-\frac{1}{2}(1-\frac{1}{q})} s^{-\frac{3}{2p}}\, ds \\ & \leq CK^3 t^{-\frac{3}{2p}} \int_{t/2}^t (t-s)^{-\frac{1}{2}(1-\frac{1}{q})}\, ds \leq CK^3 t^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{p}+\frac{1}{2}}, \qquad t \geq 2T. \end{aligned}$$

Since $p < 2$, we can choose $\tilde{\sigma}_2 > 0$ satisfying $0 < \tilde{\sigma}_2 < 1/p - 1/2$, and we get

$$\tilde{J}_3(t) \leq CK^3 t^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{q})-\tilde{\sigma}_2} \leq CK^3 t^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{q})-\frac{\tilde{\sigma}_2}{2}} T^{-\frac{\tilde{\sigma}_2}{2}}, \qquad t \geq 2T.$$

Now, choosing T large enough such that

$$K^2 T^{-\frac{\tilde{\sigma}_2}{2}} \leq \tilde{K}^2 T^{-\frac{\tilde{\sigma}_2}{2}} \leq 1,$$

we get

$$\tilde{J}_3(t) \leq CK t^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{q})-\frac{\tilde{\sigma}_2}{2}}. \tag{5.71}$$

Combining (5.66), (5.70), and (5.71), we see that

$$\|\tilde{D}[u](t)\|_{L^q} \le CKt^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{q})}, \qquad t \ge 2T,$$

thus (1.20) follows. Furthermore, applying the same arguments as in the proof of Theorem 1.3 for the case $N \ge 2$ with (5.69) and (5.71), we obtain (1.21). Thus the proof of Theorem 1.3 for the case $N = 1$ is complete. □

Remark 5.1 Similarly to the case of the Cauchy problem for the semilinear heat equation with (1.7), the nonlinear boundary problem (1.1) with (1.9) has no scaling invariance and the L^p and $\exp L^2$ norms have no relationship between each other. In order to have initial data which fulfill condition (1.19), let us choose a function $\varphi \in L^p(\mathbb{R}^N_+) \cap L^\infty(\mathbb{R}^N_+)$ with $p \in [1, 2)$. Then, by (2.15) we see that $\varphi \in \exp L^2$. Then, let us consider a dilation $\varphi_\lambda(x) = \lambda^{N/p}\varphi(\lambda x)$ so that $\|\varphi_\lambda\|_{L^p} = \|\varphi\|_{L^p}$. Since $\|\varphi_\lambda\|_{L^2} = \lambda^{N(1/p-1/2)}\|\varphi\|_{L^2}$ and $\|\varphi_\lambda\|_{L^\infty} = \lambda^{N/p}\|\varphi\|_{L^\infty}$, it follows

$$\limsup_{\lambda\to 0} \|\varphi_\lambda\|_{\exp L^2} \le \lim_{\lambda\to 0} \left(\|\varphi_\lambda\|_{L^2} + \|\varphi_\lambda\|_{L^\infty}\right) = 0.$$

This implies that there is $\lambda > 0$ so that φ_λ fulfills condition (2.4), even though its L^p norm might be large.

In the end of this section we prove Theorem 1.4. In the following Lemmata, we assume $\|u(t)\|_{L^q}$ bounded at the origin and decaying at infinity, and we can deduce that also $\|f(u(t))\|_{L^r}$ is bounded and decays at infinity for $r \ge p_3$, where p_3 is given in (5.7).

Lemma 5.6 *Let $N \ge 2$, $p \in [1, 2)$, and $K > 0$. Suppose that $u \in C(\overline{\mathbb{R}^N_+} \times (0, \infty))$ and for any $q \in [p, \infty]$,*

$$\sup_{t>0} (1+t)^{\frac{N}{2}(\frac{1}{p}-\frac{1}{q})} t^{\frac{1}{2q}} |u(t)|_{L^q} \le CK, \tag{5.72}$$

where C is independent of q and K. Let f be a function satisfying (1.9). Then, there is $\varepsilon > 0$ depending only on λ such that, if $K < \varepsilon$, then, for any $r \in [p_4, \infty]$,

$$\sup_{t>0} (1+t)^{\frac{N}{2}(\frac{1}{p}-\frac{1}{r})+\frac{1}{p}} t^{\frac{1}{2r}} |f(u(t))|_{L^r} \le 2C_f (CK)^{1+\frac{2}{N}}, \tag{5.73}$$

where C_f is given in (1.9) and

$$p_4 := \max\left\{1, \frac{pN}{N+2}\right\}. \tag{5.74}$$

Proof Let $k \in \mathbb{N} \cup \{0\}$ and ℓ_k be the constant given in (2.22). Then, since it follows from (5.74) that

$$\ell_k r \ge \left(1 + \frac{2}{N}\right) p_4 \ge p,$$

similarly to (4.9), for any $r \in [p_4, \infty]$, it follows from (1.9) and (5.72) that

$$
\begin{aligned}
|f(u(t))|_{L^r} &\le C_f \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} |u(t)|_{L^{\ell_k r}}^{\ell_k} \\
&\le C_f \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \left((1+t)^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{\ell_k r})} t^{-\frac{1}{2\ell_k r}} (CK) \right)^{\ell_k} \\
&\le C_f (CK)^{1+\frac{2}{N}} (1+t)^{\frac{N}{2r}-\frac{N}{2p}(1+\frac{2}{N})} t^{-\frac{1}{2r}} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \left((1+t)^{-\frac{N}{2p}} (CK) \right)^{2k} \\
&\le C_f (CK)^{1+\frac{2}{N}} (1+t)^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{r})-\frac{1}{p}} t^{-\frac{1}{2r}} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} (CK)^{2k}, \qquad t > 0.
\end{aligned}
\tag{5.75}
$$

We can take a sufficiently small $\varepsilon = \varepsilon(\lambda) > 0$ so that, for $K \le \varepsilon$, it holds that

$$
\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} (CK)^{2k} = e^{\lambda (CK)^2} \le 2. \tag{5.76}
$$

This together with (5.75) implies (5.73). Thus Lemma 5.6 follows. □

Lemma 5.7 *Let $N = 1$, $p \in [1, 2)$, and $K > 0$. Suppose $u \in C((0, \infty))$ and*

$$
\sup_{t>0} (1+t)^{\frac{1}{2p}} |u(t)| \le CK, \tag{5.77}
$$

where C is independent of K. Let f be a function satisfying (1.9). Then, there is $\varepsilon > 0$ such that, if $K < \varepsilon$, then,

$$
\sup_{t>0} (1+t)^{\frac{3}{2p}} |f(u(t))| \le 2C_f (CK)^3, \tag{5.78}
$$

where C_f is given in (1.9).

Proof Let $k \in \mathbb{N} \cup \{0\}$ and ℓ_k be the constant given in (2.22) with $N = 1$, namely, $\ell_k = 2k+3$. Furthermore, let ε be a sufficiently small constant given in Lemma 5.6. Then, similarly to (4.33), it follows from (1.9), (5.76), and (5.77) that

$$
\begin{aligned}
|f(u(t))| &\le C_f \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} |u(t)|^{\ell_k} \\
&\le C_f \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \left((1+t)^{-\frac{1}{2p}} (CK) \right)^{\ell_k} \\
&\le C_f (CK)^3 (1+t)^{-\frac{3}{2p}} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} (CK)^{2k} \le 2C_f (CK)^3 (1+t)^{-\frac{3}{2p}}, \qquad t > 0.
\end{aligned}
$$

This implies (5.78), thus Lemma 5.7 follows. □

Proof of Theorem 1.4. Put $K = \|\varphi\|_{\exp L^2 \cap L^p}$. Applying the same arguments as in the proofs of Theorems 1.2 and 1.3 with Lemmata 5.6 and 5.7, we can prove Theorem 1.4. □

6 Asymptotic behavior

Let us come to the asymptotic behavior of the solution u as stated in Theorem 1.5.

Proof of Theorem 1.5. Let u be the global-in-time solution to problem (1.1) satisfying (1.22). Furthermore, let $\varepsilon > 0$ be a sufficiently small constant chosen later. Then, by (1.22) and (2.16) we can take a sufficiently large $T = T(\varepsilon, N) > 0$ so that

$$\|u(T)\|_{\exp L^2} \le C(\|u(T)\|_{L^2} + \|u(T)\|_{L^\infty}) \le C(1+T)^{-\frac{N}{4}} < \varepsilon.$$

Therefore, applying the semigroup property of the kernel G, namely (2.1), we can assume, without loss of generality, that $\|\varphi\|_{\exp L^2 \cap L^1} < \varepsilon$.

We first consider the case $N \ge 2$. By (1.22), taking a sufficiently small $\varepsilon > 0$ if necessary, and applying the same argument as in the proof of Lemmata 2.2 and 5.1 with $p_1 = p_2 = p_3 = 1$, we have

$$\sup_{t>0} t^{\frac{1}{2}}(1+t)|f(u(t))|_{L^1} < \infty.$$

Therefore we can define a mass of $u(t)$ denote by $m(t)$, that is,

$$m(t) := \int_{\mathbb{R}^N_+} \varphi(x)\,dx + \int_0^t \int_{\mathbb{R}^{N-1}} f(u(x',0,s))\,dx'\,ds, \qquad t \ge 0.$$

Furthermore, it holds that

$$\begin{aligned} \int_0^t \int_{\mathbb{R}^{N-1}} f(u(x',0,s))\,dx'\,ds &= \left(\int_0^1 + \int_1^t\right)|f(u(s))|_{L^1}\,ds \\ &\le C\int_0^1 s^{-\frac{1}{2}}\,ds + C\int_1^\infty s^{-\frac{3}{2}}\,ds \le C, \qquad t \ge 1. \end{aligned} \tag{6.1}$$

This implies that there exists the limit of $m(t)$, which we denote by m_*, such that

$$m_* := \lim_{t\to\infty} m(t) = \int_{\mathbb{R}^N_+} \varphi(x)\,dx + \int_0^\infty \int_{\mathbb{R}^{N-1}} f(u(x',0,s))\,dx'\,ds.$$

Furthermore, similarly to (6.1), we obtain

$$m_* - m(t) \le C\int_t^\infty \int_{\mathbb{R}^{N-1}} f(u(x',0,s))\,dx'\,ds \le Ct^{-\frac{1}{2}}, \qquad t \ge 1.$$

Therefore, applying an argument similar to the proof of [20, Theorem 1.1] (see also [22]) with (1.22), we have (1.23) for the case $N \ge 2$.

Next we consider the case $N = 1$. By (1.22) and taking a sufficiently small $\varepsilon > 0$ if necessary, we can apply Lemmata 2.3 and 5.2, and we have

$$\sup_{t>0} t^{\frac{1}{2}}(1+t)|f(u(0,t))| < \infty.$$

Therefore we can define a mass of $u(t)$ denote by $m(t)$, that is,

$$m(t) := \int_0^\infty \varphi(x)\,dx + \int_0^t f(u(0,s))\,ds, \qquad t \ge 0.$$

Furthermore, it holds that

$$\int_0^t f(u(0,s))\,ds = \left(\int_0^1 + \int_1^t\right)|f(u(0,s))|\,ds \le C\int_0^1 s^{-\frac{1}{2}}\,ds + C\int_1^t s^{-\frac{3}{2}}\,ds \le C, \quad t \ge 1.$$

This implies that there exists the limit of $m(t)$, which we denote by m_*, such that

$$m_* := \lim_{t\to\infty} m(t) = \int_0^\infty \varphi(x)\,dx + \int_0^\infty f(u(0,s))\,ds,$$

and it holds that

$$m_* - m(t) \le C\int_t^\infty f(u(0,s))\,ds \le Ct^{-\frac{1}{2}}, \qquad t \ge 1.$$

Therefore, applying the same argument as in the proof of (1.23) for the case $N \ge 2$, we have (1.23) for the case $N = 1$. Thus the proof of Theorem 1.5 is complete. □

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