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Abstract. Usual presentations of Quantum Field Theory (QFT) start by postulating the existence of a classical field that obeys to the wave equation, derived from a conveniently chosen Lagrangian density. But while electromagnetic fields are given a proper physical meaning even before quantization, matter fields acquire it mostly only when quantized. Therefore, in this paper, we aim to give a pedagogical construction that allows us to assign a physical meaning also to non-quantized matter fields. This operation is particularly important since we believe that Quantum Field Theory is more suited than Quantum Mechanics (QM) to introduce quantum physics in secondary schools. The purpose of this paper is to show how to attribute a physical meaning to classical (that means, not yet quantized) matter fields. It is not intended for a high school presentation, but, besides being addressed to those who teach QFT at university, and to teacher educators, it is primarily addressed to researchers in physics education and to all those who work for an educational presentation of quantum physics based on QFT.

1. Introduction

Quantum physics has provided a so huge and important revolution in both of thought and of man-universe relationship that it has influenced and still influences the perception of who we are and of what is our place in the world. Our life is surrounded by quantum physics: electronic devices exploit our understanding of quantum physics; modern diagnosis tools – such as Computer Assisted Tomography, Positron Emission Tomography, and Magnetic Resonance Imaging – are also based on quantum physics; not to mention that life itself is a quantum phenomenon [1]. Since understanding the world in which we live is a key educational target, in a modern school it is inappropriate to avoid dealing with quantum physics in a coherent and meaningful way.

The understanding of modern physics in general, and of quantum physics in particular, is almost universally considered important for future citizens' education (see for instance [2]); in fact, it has become one of the hottest research topics in Physics Education, as evidenced, for instance, by the 2019 GIREP-ICPE-EPEC conference titled "Teaching-learning contemporary physics, from research to practice", and by the QTEdu project [3] (of the European Quantum flagship [4]) the purpose of which "is to assist the European Quantum Flagship with the creation of the learning ecosystem necessary to inform and educate society about quantum technologies", which has started in 2018 and will work until the end of 2027.

On the other hand, despite the large presence of modern physics topics in textbooks, the didactical quality of most presentations is clearly unsatisfactory. The results coming from Physics Education research are sufficiently unambiguous and have been known for at least twenty years: many students (even graduated and master's degree students in physics) show great difficulty in understanding the relevant aspects of quantum physics [5]. And this situation has become even worse during Covid-19 pandemic [6], which, however, led to a greater use of applets, smartphones and so on, which are more and more engaging and motivating for students [7].

The usual educational path toward quantum physics – in general, both at university and at school – starts from assigning a (continuous) wave nature to the electromagnetic field, the interpretation of which is generally given within the framework of classical Electromagnetism. It is only later, when the photoelectric and the Compton effects are discussed, that the granular aspects of radiation are presented. An inverse approach is made, instead, with regard to the matter: in this case, the corpuscular/particle aspects are examined first, while the wave aspects are discussed only later, interpreted from a quantum perspective, and starting from the de Broglie relations. In this way, we arrive at Quantum Mechanics (QM) within which the wave aspects of matter, the meaning of the wave function or, in a more abstract context, that of the state vector, has already been discussed in various research works in physics education [8] and, therefore, we will not deal with it in this paper.

In the previous approach leading to QM, however, apart from the above discussed inversion of presentation (not only in terms of contents but also from a logical point of view), the most remarkable thing to note is that, while from a certain point of view the matter/radiation corpuscular aspects resemble one another, the wave behaviours have, instead, different ontological natures. The electromagnetic field is considered a real "physical" object; on the contrary, the wave aspects of the matter are generally interpreted in terms of a wave function (that "lives" in the configuration space), and not of a field, (in our space and time). In this way the wave aspects of the matter are considered much less tangible and concrete than the electromagnetic ones. To be clearer, let us, in fact, consider interference experiments (in the general, high-intensity situation):

1) In the case of Electromagnetism, the interference effects are explained as due to the oscillations of the electromagnetic field (as far as a future quantization is concerned, it would be more appropriate to consider them oscillations of the electromagnetic four-potential, but, from a heuristic point of view, we can easily think of them as oscillations of electric and magnetic fields).
2) In the case of electron beams, which quantity is, however, oscillating? A wave function in configuration space is the answer we should give in QM; that is, an "object" that "lives" in R^{2n} (where n is the number of electrons) whose squared modulus represents the probability density of finding electrons. This way, the oscillating quantity will even depend on how many electrons our source will simultaneously emit!

This asymmetry in the treatment of radiation and matter is exaggeratedly bizarre. A different situation occurs, however, in Quantum Field Theory (QFT). In QFT master's degree level courses [9-11], in fact, quantization process is, in some ways, more linear and symmetric: radiation and matter are both described by fields – each obeying a wave equation – and they are both quantized with similar techniques, that is by means of specific canonical commutation (or anticommutation) rules.

This last way of proceeding, which puts electromagnetic radiation and suitably prepared beams of matter on the same footing, seems to us more satisfactory from an epistemological point of view – and also more suitable as a starting point of a didactic high school path – than the quantum mechanical one.

In fact, when dealing with QM, even when referring to paths addressed to high school, the theoretical description is rather formal (perhaps not so difficult, but also not so much explanatory) [12]. This situation becomes particularly evident when concerning electromagnetic radiation, since in QM one cannot write a Schrödinger equation, nor an interaction Hamiltonian for photons. By reasoning in terms of the Schrödinger equation, when speaking about QM at high school, we do not suggest in any way that it should be presented in schools; however, we highlight that the impossibility of writing the Schrödinger equation for photons means that we could never find a simple way to explain the physics of photons within QM. Therefore, for example, one can reasonably describe the action of a calcite crystal, or that of a beam splitter and, even that of a Mach-Zehnder interferometer on photons, but one cannot ask what kind of interaction between photons and mirror, or between photons and electrons causes that behaviour. Simply put, the descriptive power of QM is astounding, but for photons its explanatory power is unsatisfactory. The situation is not the same as regarding matter; indeed, for example, the interaction between a charged particle and an electric field finds its clear formulation in QM and this leads us to understand very satisfactorily the structure of most things around us.

However, most of the historical and educational paths towards quantization – especially at school – start from the role of photons, and, in any case, we cannot overlook the role of the quanta of electromagnetic radiation in explaining the world around us; therefore, it is difficult to understand how we can neglect providing a reasonable complete theory of the behaviour of photons and matter.

Previous considerations push us to find a way to present the only theory in which radiation and matter interactions can be "explained" in quantum terms: QFT [13]. This is not the only reason to look at QFT as a referring theory, in fact we think that it has an epistemological content that, from a didactic point of view, is in many respects simpler and more convincing than QM especially in secondary school [14,15]

Indeed, the Physics Education Research Group of Milan has been working for years on the construction of an educational path for high school, based on QFT [14,16,17]. An important requisite for this path to be logically well structured, however, is that the non-quantized fields of matter and radiation, from which we start in QFT, be really placed on an equivalent level. To this purpose the problem of which meaning is to be given to the matter fields to be quantized (which, in a sense, we can call classical fields) arises.

Present work aims to show, in a didactically understandable way, how it is possible to construct the equivalence we just spoke about from a physical point of view, and therefore how to attribute a "classical" physical "reality" to matter fields. What follows is therefore not meant for a direct high school presentation, but is written for university QFT instructors, physics education researchers, and teacher educators, in order to provide them with solid conceptual basis for discussing the analogies of behaviour of the wave aspects of matter and radiation.

2. Quantum Field Theory and second quantization approaches

The starting point of QFT is to postulate a classical field that obeys a wave equation obtained from a suitably chosen Lagrangian density. As already said, in the case of Electromagnetism, the meaning of this field is clear: it is the well-known four-potential field of the classical Maxwell's theory. In the case of matter fields, instead, a classic correspondent is missing, (for example, there is no discussion of what the "classical" field associated with charged particles is) and the field acquires a precise and clear meaning only when quantized, by promoting it to become an operator, choosing a particular Hilbert space, and imposing suitable commutation rules.

However, the quantization of a classical field is not the only way to arrive at a QFT – especially when the theory to be obtained is of a non-relativistic character, and, particularly in textbooks concerning many-body quantum systems – the approach called "second quantization" is, indeed, even found; in this case, the quantization is performed starting from the many-body wave function in the configuration space [18]. In short, for one reason or another, both in QM and in QFT, the wave aspect of the matter has an epistemological status that is very different from that attributed to radiation.

The purpose of this work is bridging the gap between these two epistemological statuses and to attribute, in analogy to what happens in Electromagnetism, a clear and "classical" physical meaning also to matter fields and to derive for them an equation of motion that is in no way linked to quantum or particle aspects. Besides its conceptual value, we think that this operation is essential if, as we do in the Physics Education Research Group of Milan, we believe that QFT might be more suited than QM to introduce quantum physics in secondary school [14,16,17] as it is able to provide a different and clearer conceptual framework in which quantum physics can be introduced.

The approach presented below is certainly not suitable for high school students, being essentially a university-level theoretical approach for the introduction of "classical" – in the sense of non-quantized – matter fields in view of their subsequent quantization within the QFT procedures. Nevertheless, we believe that it constitutes the conceptual basis that allows us to consider the wave aspects of matter and radiation as equivalent; and this is a fact that, in our opinion, does, indeed, have important implications, even at high school level.

3. The Klein-Gordon equation for a non-quantized matter field

In the last few decades, many experiments with suitably prepared non-self-interacting matter beams showing the typical behavior of wave fields have been performed. We quote, for example, the beautiful experiments of Tonomura on the interference of electronic beams [19], those of Rauch on neutron interferometry [20], those performed by Doak on the diffraction of helium beams [21], the fascinating experiments of Batelan on the Kapitza-Dirac effect [22] in which a matter beam is diffracted by light (that is by a grating made by stationary electromagnetic waves) or those on the diffraction of large molecules by Zeilinger and coworkers [23].

Most of those experiments have been performed with beams of very low intensity so that their interactions with the detectors of the apparatuses showed the very well space-time localization typical of quantum processes; in this sense, they were, as it is usually said, "single particle experiments". Some of them, however, were firstly done with higher intensity beams – such as those of Davisson and Germer – and, in most cases, nothing prevents, at least conceptually, conducting them in a sufficiently high-intensity regime so as to obtain results that – *mutatis mutandis* – are very close to those of Maxwell's Electromagnetism: in fact, in these cases, we can speak of "matter optics".

It is common to interpret these experiments in quantum terms, explaining also those performed with high-intensity beams by referring to the Broglie's relations and the so-called wave-particle duality. In our opinion, on the contrary, for what concerns interference phenomena, given the great analogy of behavior of suitably prepared matter beams (essentially considered as non-self-interacting) with electromagnetic ones, it is not necessary to introduce quantum concepts to interpret this phenomenology, in the same way as it is not necessary in the case of electromagnetic classical optics.

In analogy to what is done to describe elastic or electromagnetic waves, it seems somehow natural to us to introduce the idea that also for matter beams there is a field from which we can derive the properties characterizing the beam. That is, we want to introduce a field ψ from which to obtain, for instance, the density of mass, or of charge, or of momentum, or of energy, etc., for the specific beam under examination in a certain region of space and time by means of real expressions that in general (as it is customary) are quadratic in the field. Starting from phenomenology, we will let ourselves be guided by what we already know about the electromagnetic field, in order to come to a "classical" wave equation that will describe the propagation of the matter field ψ.

A general electromagnetic perturbation in a vacuum, in the absence of charges and currents (for simplicity, we will here neglect all the issues relating to polarization), is described by the four-potential A^ν which satisfies the wave equation:

$$0 = \left(\frac{\partial^2}{\partial t^2} - \nabla^2\right) A^\nu(\underline{x}, t), \tag{3.1}$$

where we have set $c = 1$ and adopted the standard convention in relativity in which the component A^0 represents the electric scalar potential, while $\underline{A} \equiv (A^1, A^2, A^3)$ is the magnetic vector potential. Each component of the four-vector can be represented as a Fourier integral (plane waves expansion):

$$\begin{aligned} &A^\nu(\underline{x},t) = \frac{1}{(2\pi)^{3/2}} \int d^3k \left[a_+^\nu(\underline{k}) e^{i(\underline{k}\cdot\underline{x} - k_0 t)} + a_-^\nu(\underline{k}) e^{i(\underline{k}\cdot\underline{x} + k_0 t)}\right] \\ &a_-^\nu(\underline{k}) = a_+^{\nu\,*}(\underline{k}) \end{aligned} \tag{3.2}$$

(the condition on a^ν arises from the request that the four potential be a real function).

The integrand of (3.2) consists of complex plane waves, with $a_+^\nu(\underline{k})$ and $a_-^\nu(\underline{k})$ that are respectively the amplitude of the progressive and regressive waves, of wavelength and period given by:

$$\lambda = \frac{2\pi}{|\underline{k}|} \quad ; \quad T = \frac{2\pi}{k_0}\,. \tag{3.3}$$

Therefore (3.2) represents an electromagnetic wave packet. For completeness and clarity, we observe that the expression of the electromagnetic fields can be easily obtained from the four-potential; in fact, the well-known relations hold:

$$\underline{E} = -\underline{\nabla} A^0 - \frac{\partial \underline{A}}{\partial t} \quad ; \quad \underline{B} = \underline{\nabla} \times \underline{A}\,. \tag{3.4}$$

In order that (3.2) really represents a general electromagnetic perturbation, it has to satisfy equation (3.1) of the electromagnetic waves in a vacuum. It is important to note that the generic plane wave, of which (3.2) is a superposition, satisfies (3.1) if and only if the well-known dispersion relation holds:

$$k_\nu k^\nu = 0. \tag{3.5}$$

In fact, when (3.5) holds, the two coefficients of the exponential terms, corresponding to the progressive and regressive wave, cancel both out. In (3.5) we used the summation convention on repeated indices, with the Minkowski metric $(1, -1, -1, -1)$ that is:

$$k_\nu k^\nu \equiv k_0 k^0 - k_1 k^1 - k_2 k^2 - k_3 k^3; \tag{3.6}$$

(3.6) represents the square of the modulus of the four-vector k^ν and is, therefore, a Lorentz scalar.

Given the analogy of behaviour between matter and electromagnetic beams, we assume also for the matter field ψ a plane waves expansion similar to (3.2). If, for simplicity, we consider ψ as a scalar field, we can therefore write:

$$\psi(\underline{x}, t) = \frac{1}{(2\pi)^{3/2}} \int d^3k\, e^{i\underline{k}\cdot\underline{x}} \left[c_+(\underline{k}) e^{-ik_0 t} + c_-(\underline{k}) e^{ik_0 t}\right]. \tag{3.7}$$

We might wonder why looking for an equation that our field – for which we have already the general expression – has to obey. The underlying reasons are twofold. The first is that if we find this equation, which describes a non-self-interacting free field, we have a reasonable hope to be able to generalize it in a not too complicated way, taking also interactions into account. On the contrary, starting by considering only the field, without having the equation it obeys, it could be very difficult to understand how to introduce interactions. The second reason is that, as we will see in sections 4 and 5, the search for this equation will give us the possibility to attribute a physical meaning to this field and to make us understand aspects of our matter field that we would have not probably thought of.

As can be seen, (3.7) is an expression formally identical to (3.2), except for having put in evidence the term $e^{i\underline{k}\cdot\underline{x}}$. Condition (3.6) characterizes the electromagnetic waves; therefore, it cannot be assumed valid also for the matter waves described by (3.7). As $k_\nu k^\nu$ is a Lorentz scalar, the simplest generalisation of (3.7) leads us to assume that:

$$k_\nu k^\nu = \mu^2, \tag{3.8}$$

with μ^2 a generic constant, different from zero [24].

Due to (3.8), the field expressed by (3.7) can no longer satisfy equation (3.1). Instead, it satisfies the following equation:

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + \mu^2\right) \psi(\underline{x}, t) = 0; \tag{3.9}$$

as can be verified with a direct check, by substituting (3.7) in (3.9), and by taking into account (3.8). According with the value of μ^2, we can think of (3.9) as a very general equation satisfied by all scalar fields, electromagnetic or material, which propagate freely and without self-interaction. We observe that (3.9) is formally identical to the Klein-Gordon equation in natural units. Usually, the Klein-Gordon equation is arrived at starting from the relativistic expression for the energy of a particle (with obvious symbology):

$$E^2 = m^2 + p^2, \tag{3.10}$$

with the standard substitution

$$E \to i\hbar\frac{\partial}{\partial t}; \quad \underline{p} \to i\hbar\underline{\nabla}. \tag{3.11}$$

In our approach, on the contrary, we arrived at the Klein-Gordon equation without going through any corpuscular aspect and without using the relativistic expression of the energy. So we did not even have to choose $\hbar = 1$; indeed, we remained within a context in which the corpuscular aspects of matter play no role at all, and, therefore, also the constant $\hbar$ has no role, in fact, it finds no place inside (3.9).

4. The Schrödinger equation for a non-quantized matter field

From the dispersion relations (3.5) and (3.8), for the electromagnetic field, we have:

$$k_0 = |\underline{k}|; \tag{4.1}$$

while for the matter one we get:

$$k_0 = \sqrt{\mu^2 + |\underline{k}|^2}. \tag{4.2}$$

Thus, we find an expression which is formally completely analogous to that of the relativistic energy of a point-like particle. So far, the constant μ has not received any physical interpretation. To start understanding its meaning, we observe that, from a dimensional point of view, μ is the inverse of a length and that, therefore, it fixes a characteristic length with respect to which consider a gauge for the space variation of the field. We can therefore say, in an intrinsic way, that the field varies slowly if the "central" wavelengths of the packet (3.7) are much greater than the inverse of μ; that is (see (3.3)) if $\frac{|\underline{k}|}{\mu} \ll 1$.

When this condition holds, we say that we are in the slowly varying fields approximation. With this approximation, (4.2) becomes:

$$k_0 = \sqrt{\mu^2\left[1 + \left(\frac{|\underline{k}|}{\mu}\right)^2\right]} \approx \mu + \frac{|\underline{k}|^2}{2\mu}. \tag{4.3}$$

Substituting (4.3) in (3.7), we obtain

$$\psi(\underline{x}, t) = \frac{1}{(2\pi)^{3/2}}\int d^3k\, e^{i\underline{k}\cdot\underline{x}}\left[c_+(\underline{k})e^{-i\left(\mu + \frac{|\underline{k}|^2}{2\mu}\right)t} + c_-(\underline{k})e^{i\left(\mu + \frac{|\underline{k}|^2}{2\mu}\right)t}\right]. \tag{4.4}$$

It is quite obvious that the matter field ψ of (4.4) no longer satisfies equation (3.9). In fact, the spatial part of (4.3) contains $|\underline{k}|$ while the temporal part contains $|\underline{k}|$ squared. If we differentiated ψ twice with respect to time and also twice with respect to space, as in (3.9), we would obtain a term with a factor $|\underline{k}|$ squared and a term with a factor $|\underline{k}|$ to the fourth power. This would obviously lead to terms that certainly could not cancel each other out, as, instead, should happen if (4.4) satisfied (3.9). In an attempt to find a new equation, of which we want ψ to be the solution, it then seems reasonable to look for an expression that contains the first derivative with respect to time and the second derivative with respect to space components. Unfortunately, such an equation does not exist – at least not in an easy form –

since the two addends containing the c_+ or the c_-, into which (4.3) can be divided, give contributions of different sign when differentiated only once with respect to time. Therefore, the condition which makes null the coefficient of the exponential term corresponding to the traveling wave is different from the one which cancels the coefficient of the exponential term corresponding to the regressive wave. And, thus, we are not able to write a "unique" dispersion relation valid for (4.4). We can, however, write two separate equations, which will differ from each other precisely by a sign: one for the part containing the c_+ (that we will continue to call ψ) and the other one for that with the c_- (that we will call $\tilde{\psi}$):

$$\psi(\underline{x},t) = \frac{1}{(2\pi)^{3/2}} e^{-i\mu t} \int d^3k\, e^{i\underline{k}\cdot\underline{x}} c_+(\underline{k}) e^{-i\frac{|\underline{k}|^2}{2\mu}t}; \tag{4.5}$$

$$\tilde{\psi}(\underline{x},t) = \frac{1}{(2\pi)^{3/2}} e^{i\mu t} \int d^3k\, e^{i\underline{k}\cdot\underline{x}} c_-(\underline{k}) e^{i\frac{|\underline{k}|^2}{2\mu}t}. \tag{4.6}$$

Recalling, as already observed in section ~~2~~ 3, that physically meaningful quantities have to contain real expressions that are in general quadratic in the fields, the equation of which (4.5) is the solution, and the equation whose solution is (4.5) once the oscillating factor $e^{-i\mu t}$ has been cancelled, although in general different, would provide the same expressions as regards the quadratic quantities built from the fields (identical considerations are applied to (4.6)). We will, therefore, from now on, neglect the common factors $e^{-i\mu t}$ and $e^{i\mu t}$ respectively in (4.5) and (4.6) and write:

$$\psi(\underline{x},t) = \frac{1}{(2\pi)^{3/2}} \int d^3k\, e^{i\underline{k}\cdot\underline{x}} c_+(\underline{k}) e^{-i\frac{|\underline{k}|^2}{2\mu}t} \tag{4.7}$$

$$\tilde{\psi}(\underline{x},t) = \frac{1}{(2\pi)^{3/2}} \int d^3k\, e^{i\underline{k}\cdot\underline{x}} c_-(\underline{k}) e^{i\frac{|\underline{k}|^2}{2\mu}t}. \tag{4.8}$$

We have no reason to prefer either expression (4.5) or (4.6) to describe our matter field; we are therefore forced to consider matter as composed of two fields ψ and $\tilde{\psi}$, each obedient to a precise equation. It is not difficult to find that the first one, related to the field ψ, is

$$\left(\frac{1}{2\mu}\nabla^2 + i\frac{\partial}{\partial t}\right)\psi(\underline{x},t) = 0. \tag{4.9}$$

Equation (4.9) is the Schrödinger equation (with the choice $\hbar = 1$) for a non-quantized matter field and has, of course, the general solution (4.7).

The other equation, concerning $\tilde{\psi}$, is, instead:

$$\left(\frac{1}{2\mu}\nabla^2 - i\frac{\partial}{\partial t}\right)\tilde{\psi}(\underline{x},t) = 0 \tag{4.10}$$

and its general solution is (4.8). Its physical meaning will be discussed here below.

5. Physical interpretation

It's quite obvious to observe that, since space and time are no longer treated on the same footing – there are second derivatives for spatial coordinates and first derivative for time –, (4.9) and (4.10) are no longer relativistic equations. Giving also a look at the form of relation (4.3) – formally identical to the relativistic energy approximation in the case of low velocities –, it seems, therefore, natural to think that the fields described by (4.7) and (4.8) propagate at low speed and "have" low energy. This fact is due to the approximation of slowly variable fields made above, that is, fields characterized by a sufficiently large wavelength, that means low frequency fields. This prompts us to quite naturally link speed and frequency together by observing that "low-frequency" packets are also to be "low-speed" ones. We are, thus, able to heuristically find that, for matter fields, energy is an increasing function of frequency without having yet introduced quantization.

The choice of the sign of the constant μ allows us to pass from (4.9) to (4.10) and *vice versa*. We can, therefore, choose any of the two equations as the fundamental one, allowing μ to assume both values (positive and negative) of the $\sqrt{\mu^2}$. For definitiveness, we will make the choice of equation (4.9) as fundamental. In other words, our discussion prompts us to consider, for each value of $|\mu|$, two distinct fields, characterized by the sign of positive or negative μ. We will call the first ones "matter fields" and the second ones "antimatter fields". What has been said so far deserves a pause for reflection; in fact, we are used to consider the idea of antimatter as a prediction that arises from purely quantum, moreover relativistic considerations. We see here, instead, that our discussion leads in an almost natural way to consider correlated pairs of matter and antimatter fields, both of which, moreover, are described by a non-relativistic equation.

As already said, in order to obtain physically meaningful expressions from the ψ of (4.7), we must construct real quadratic expressions. The first that comes to mind is $\left|\psi(\underline{x},t)\right|^2$.

To understand its meaning, we note, that $\psi(\underline{x},t)$ is the (anti) Fourier transform of $c_+(\underline{k})e^{-i\frac{|\underline{k}|^2}{2\mu}t}$. Therefore, setting the question in $L^2(\mathbb{R}^3)$ and remembering – paying little attention to mathematical precision – that the Fourier transform is isometric, we can say that the squared norm of $\psi(\underline{x},t)$ is equal to the squared norm of $c_+(\underline{k})e^{-i\frac{|\underline{k}|^2}{2\mu}t}$, that is:

$$\int d^3x\left|\psi(\underline{x},t)\right|^2 = \int d^3k\left|c_+(\underline{k})e^{-i\frac{|\underline{k}|^2}{2\mu}t}\right|^2 = \int d^3k\left|c_+(\underline{k})\right|^2. \tag{5.1}$$

Equation (5.1) tells us that, while the square modulus of $\psi(\underline{x},t)$ varies in space and time, its integral over the whole space does not depend on t, and therefore it remains constant. This leads to the interpretation of this square modulus as describing a quantity which, although varying locally, is conserved globally. It is therefore natural to think that an equation of continuity has to hold, in which the square modulus of $\psi(\underline{x},t)$ represents a density $\rho(\underline{x},t)$ of a locally conserved physical quantity, for which we can write the continuity equation:

$$\frac{\partial}{\partial t}\left|\psi(\underline{x},t)\right|^2 + \underline{\nabla}\cdot\underline{j}(\underline{x},t) = 0 \tag{5.2}$$

where $\underline{j}$ is the current density of the quantity of which $\rho(\underline{x},t) \equiv \left|\psi(\underline{x},t)\right|^2$ is the density. In fact, (5.2) is satisfied if we define:

$$\underline{j}(\underline{x},t) \equiv \frac{1}{2\mu}\left[\psi(\underline{x},t)\underline{\nabla}\psi(\underline{x},t)^* - \psi(\underline{x},t)^*\underline{\nabla}\psi(\underline{x},t)\right]. \tag{5.3}$$

Finally, what we have seen so far allows us to give a physical interpretation of the "classical" field $\psi(\underline{x},t)$. Indeed, $\rho(\underline{x},t)$ can be considered a pure number and, therefore $\mu\rho(\underline{x},t)$ can be easily interpreted as the mass density of the matter beam, so that $p(\underline{x},t) \equiv \mu\underline{j}(\underline{x},t)$ becomes its momentum density. At this point it is quite easy to derive expressions for other significant physical quantities; for example, the energy density linked to the convection of the material continuum is given by:

$$w(\underline{x},t) \equiv \frac{1}{2}\mu\frac{\left|\underline{j}(\underline{x},t)\right|^2}{\rho(\underline{x},t)} \tag{5.4}$$

6. Conclusions

In general, there are two starting points for usual presentations of QFT. The first one is to postulate a classical field that obeys the wave equation derived from a suitably chosen Lagrangian density. In the case of Electromagnetism, the meaning of this field is clear: it is the well-known four-vector field of the classical electromagnetic potential, that of Maxwell's theory. In the case of matter fields, instead, a classic correspondent is missing, and the field is given a precise meaning by means of its quantization. The second one consists in taking the N-particles quantum mechanical wave function in configuration space to arrive at QFT of matter through the so-called second quantization procedure. In the first case, a "classic" field of matter does not receive an adequate physical interpretation, while in the second it does not even exist.

Given the central importance of identifying within precise theoretical aspects the fundamental conceptual knots on which an adequate didactic presentation has to be built [14], in this paper, we have shown a way to attribute – in analogy to what happens in Electromagnetism – a physical meaning also to "classical" matter fields and to derive for ~~it~~ them a wave equation that is in no way linked to quantum or particle aspects. This physical meaning is at the basis of our idea that, for the construction of an educational path about quantum physics, it is more effective to start from a non-necessary historical approach; in particular, by considering first, and in parallel, the wave aspects of both electromagnetic and matter beams, as they emerge from high-intensity interference experiments (as described in other papers, *i.e.*, [1,14,16,17]); without having to wait for quantum physical considerations, and without having to introduce the constant h.

In general, on the contrary, textbooks proceed in a different pseudo-historical way by first presenting the wave aspects of Electromagnetism together with the particle aspects of matter, and only subsequently reverse their perspective by considering the particle aspect of electromagnetic radiation – for example to explain the photoelectric effect –, together with the wave aspects of matter – for example to explain the quantization in the atom. In this way, implicitly transmitting the idea that the wave aspects of matter are of purely quantum origin.

From the point of view of a phenomenological description, interference experiments of electromagnetic and matter beams are similar. But, from a didactic and also a physical point of view, this similarity is greatly reinforced by the knowledge that in addition to the phenomenological description there is also a mathematical description, given in terms of "classical" fields, which treats matter and radiation in a similar way. This fact has also implications for teaching classical physics; indeed, we could, thus, for instance, discuss a "classical" optics of "high" intensity matter beams together with that of light beams, and talk about matter waves, also at school, well before, and independently of any quantum aspect. Moreover, back to the teaching of quantum aspects, with the awareness of the existence of classical matter fields, we could also approach more gently the introduction of the idea of a possible quantization. Which would be much more difficult if radiation were described by a classical field, and matter beams by a function in configuration space.

Besides having shown that a non-quantum description of matter fields is possible, in this work we have also given an interpretation of the meaning of this field in the non-relativistic case. In fact, for the different matter fields there exists a family of equations parametrized by the parameter μ such that $\mu|\psi(\underline{x},t)|^2$ represents the density of the type of the considered matter beam, of which one can also introduce the convection kinetic energy density, the momentum density, and so on (see for example equations (5.3) and (5.4)). Furthermore, our discussion naturally led to the conclusion that for each value of the characteristic mass μ we have to consider two distinct fields, obeying two different equations; we call one field of matter, the other of antimatter. From a conceptual, and also a didactical, point of view, it seems interesting to us to find that the existence of antimatter can also emerge from non-quantum considerations in a non-relativistic regime.

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