SYMPLECTIC AUTOMORPHISMS ON KUMMER SURFACES

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ABSTRACT. Nikulin proved that the isometries induced on the second cohomology group of a K3 surface X by a finite abelian group G of symplectic automorphisms are essentially unique. Moreover he computed the discriminant of the sublattice of $H^2(X,\mathbb{Z})$ which is fixed by the isometries induced by G. However for certain groups these discriminants are not the same of those found for explicit examples. Here we describe Kummer surfaces for which this phenomena happens and we explain the difference.

0. INTRODUCTION

An automorphism σ of finite order on a K3 surface X is symplectic if the desingularization of the quotient, denoted by $\widetilde{X/\sigma}$, is again a K3 surface. The finite abelian groups of automorphisms on a K3 surface were classified by Nikulin [N]. Later Mukai [Mu] classified all the finite groups of symplectic automorphisms. The main result of [N] is that the group of isometries of the second cohomology group induced by a finite abelian group of symplectic automorphisms of the K3 surfaces is essentially unique. So these isometries do not depend on the surface but only on the group. If the group G of symplectic automorphisms is generated by an involution, Morrison [Mo] found this isometry explicitly. A description of these isometries for the others finite abelian groups acting symplectically on a K3 surface is given in [GS1] and [GS2]. Since the isometries induced by G are essentially unique, the lattice $H^2(X,\mathbb{Z})^G$ (the sublattice of the G-invariant elements in $H^2(X,\mathbb{Z})$) depends on G but not on the K3 surface X, up to isometry. In particular the discriminant of $H^2(X,\mathbb{Z})^G$ depends only on G. In [N] this discriminant is found using the relation between the K3 surface X and the K3 surface Y which is the desingularization of X/G. This is an interesting idea essentially because Y seems to be easier to study then $H^2(X,\mathbb{Z})^G$. In fact $H^2(Y,\mathbb{Z})$ contains a sublattice M_G generated, at least over $\mathbb{Q}$, by rational curves with simple intersection properties, which is related to $H^2(X,\mathbb{Z})^G$ but is less complicated to analyze. The construction used by Nikulin to find the discriminant of $H^2(X,\mathbb{Z})^G$ is summarized here in *Section 1*.
Another way to compute the discriminant of $H^2(X,\mathbb{Z})^G$ is to construct an explicit example of a K3 surface X admitting G as group of symplectic automorphisms and to find the lattice $H^2(X,\mathbb{Z})^G$ explicitly (this method is the one adopted in [GS1], [GS2]). The discriminants given in [N] are not always equal to the ones computed on explicit examples. Here we present such example. Moreover we explain where the method described in [N] fails. In fact Nikulin asserts that if G is a symplectic group of automorphisms on X, then a certain map between a lattice related to Y and a lattice related to X would be a surjective map. In *Section 2* we will prove that if the group G is a cyclic group, that map is indeed surjective, but in *Section 3* we will exhibit some counterexamples in the case G is not cyclic. In these counterexamples X is a Kummer surface. The translations by 2-torsion points on an Abelian surface A induce symplectic involutions on the associated Kummer surface $Km(A)$. So the group $(\mathbb{Z}/2\mathbb{Z})^4$ is a group of symplectic automorphisms on each Kummer surface. For this example we will

explicitly describe the map introduced by Nikulin.

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1. Nikulin's construction

Let X be a K3 surface. The second cohomology group of X with the cup product is a lattice isometric to the lattice (which does not depend on X) $\Lambda_{K3} := U \oplus U \oplus U \oplus E_8(-1) \oplus E_8(-1)$, where U is the rank two lattice with pairing $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $E_8(-1)$ is the rank eight lattice associated to the Dynkin diagram E_8 with the pairing multiplied by -1 (cf. [BPV]).
Let σ be an automorphism of X. Then σ^* is an isometry of $H^2(X,\mathbb{Z})$ which preserves the set of the effective divisor and its $\mathbb{C}$-linear extension preserves the Hodge decomposition of $H^2(X,\mathbb{C})$.

Definition 1.1. *An automorphism $\sigma \in Aut(X)$ is* ***symplectic*** *if and only if $\sigma^*_{|H^{2,0}(X)} = Id_{|H^{2,0}(X)}$. A group of automorphisms acts symplectically on X if all the elements of the group are symplectic automorphisms.*

Remark 1.2. [N, Theorem 3.1 b)] An automorphism g of X is symplectic if and only if $g^*_{T_X} = Id_{T_X}$. □

Let G be a finite group acting symplectically on X. The quotient of X by G is a singular surface X/G. Let us call $Y = \widetilde{X/G}$ the minimal resolution of this surface. Then we have the following diagram

$$\begin{array}{ccc} X & \dashrightarrow & \widetilde{X/G} = Y \\ & {}^{\pi}\searrow \quad X/G \quad \swarrow {}^{\beta} & \end{array}$$

Here we recall one of the main results on the symplectic automorphisms of K3 surfaces.

Theorem 1.3. *Let X be a K3 surface and G be a finite group of symplectic automorphisms of X. Then $Y = \widetilde{X/G}$ is a K3 surface.*
Viceversa, let X be a K3 surface and G be a finite group of automorphisms of X. If $Y = \widetilde{X/G}$ is a K3 surface, then G is a group of symplectic automorphisms.

From now on we will assume that the group G is a finite abelian group.

Definition 1.4. *Let $X' = X \backslash \{\ldots, x_i, \ldots\}$ where $\{\ldots, x_i, \ldots\}$ is the set of points with a non-trivial stabilizer under the action of G. Let Y' be the surface $Y' = X'/G$ and $\theta : X' \to Y'$ be the quotient map.*

Remark 1.5. Since the action of G on X' is fixed points free, the surface Y' is smooth, and of course $Y' = Y \backslash \{\cup_j M_j\}$ where M_j are the curves which arise from the resolution of the singularities of X/G. Moreover the map $\theta : X' \to Y'$ is an unramified $|G|$-sheeted cover. □

Remark 1.6. Let ϑ be the rational map associated to the quotient X/G. So we have:

$$\begin{array}{rccc} & \widetilde{X} & & \\ & \downarrow & \searrow & \\ \vartheta: & X & \dashrightarrow & Y \\ & \cup & & \cup \\ \vartheta_{|X'} = \theta: & X' & \to & Y' \end{array} \tag{1}$$

□

Definition 1.7. [N, Definition 4.6] *We say that a group G **has a unique action on** Λ_{K3} if, given any two embeddings $i: G \hookrightarrow Aut(X)$ and $i': G \hookrightarrow Aut(X')$ under which G is a group of symplectic automorphisms of the K3 surfaces X and X', there exists an isometry $\varphi: H^2(X,\mathbb{Z}) \to H^2(X',\mathbb{Z})$ such that $i'(g)^* = \varphi \circ i(g)^* \circ \varphi^{-1}$ for any $g \in G$.*

Theorem 1.8. [N, Theorem 4.7] *Any finite abelian group has a unique action on Λ_{K3}.*

The natural question posed by this result is to find explicitly these actions on Λ_{K3}. If the group acting symplectically is $G = \mathbb{Z}/2\mathbb{Z}$, i.e. it is generated by a symplectic involution, the answer to this question was found by Morrison in [Mo]. The cases $G = \mathbb{Z}/p\mathbb{Z}$ where p is a prime are analyzed in [GS1] and the other cases are analized in [GS2]. To find these isometries the main problem is to find the sublattices $H^2(X,\mathbb{Z})^G$ and $(H^2(X,\mathbb{Z})^G)^\perp$. Nikulin does not determine them, but describes a technique to compute some invariants of these lattices, which we recall here.

Lemma 1.9. *The surfaces X, X', Y and Y' have the following properties:*
- $H^0(X,\mathbb{Z}) = H^0(X',\mathbb{Z}) = H^0(Y,\mathbb{Z}) = H^0(Y',\mathbb{Z}) = H_0(X,\mathbb{Z}) = H_0(X',\mathbb{Z}) = H_0(Y,\mathbb{Z}) = H_0(Y',\mathbb{Z}) = \mathbb{Z}$;
- $\pi_0(X')$ *is trivial;*
- $H_1(X,\mathbb{Z}) = H_1(X',\mathbb{Z}) = H_1(Y,\mathbb{Z}) = 0$ *and* $H_1(Y',\mathbb{Z}) = G$;
- $H^1(X,\mathbb{Z}) = H^1(X',\mathbb{Z}) = H^1(Y,\mathbb{Z}) = H^1(Y',\mathbb{Z}) = 0$.

Proof. All the surfaces X, X', Y, Y' are clearly path connected and so their homology and cohomology group in degree 0 is $\mathbb{Z}$ and $\pi_0(X')$ is trivial.
The surfaces X and Y are K3 surfaces, so by definition $H^1(X,\mathbb{Z}) = H^1(Y,\mathbb{Z}) = 0$. The complex surface X' is the surface X without some points. The four dimensional topological variety X is simply connected, this implies that X' is simply connected, i.e. $\pi_1(X') = 0$. So $H_1(X',\mathbb{Z}) = Ab(\pi_1(X')) = 0$. Since $H^n(X',\mathbb{Z}) \simeq Hom(H_n(X'),\mathbb{Z}) \oplus Ext(H_{n-1}(X'),\mathbb{Z})$ and $Ext(\mathbb{Z},\mathbb{Z}) = 0$, we have $H^1(X',\mathbb{Z}) = 0$.
Since X' is simply connected and G acts without fixed points on X', it follows that $\pi_1(Y') \simeq G$. Since G is an abelian group, $H_1(Y',\mathbb{Z}) = G$. Hence $H^1(Y',\mathbb{Z}) \simeq Hom(G,\mathbb{Z}) \oplus Ext(\mathbb{Z},\mathbb{Z}) = 0$. □

Lemma 1.10. *We have $H^2(Y',\mathbb{Z}) \simeq H^2(Y,\mathbb{Z})/\oplus_j \mathbb{Z}M_j$ and $H^2(X',\mathbb{Z}) \simeq H^2(X,\mathbb{Z})$* (see [N, Lemma 6.1], [X, Lemma 2]).

Definition 1.11. *Let M_G be the minimal primitive sublattice of Λ_{K3} containing the curves M_j arising from the resolution of the singularities of X/G. Let P_G be its orthogonal with respect to the bilinear form of Λ_{K3}.*

Remark 1.12. The lattice M_G is primitively embedded in $NS(Y)$, because it is generated over $\mathbb{Q}$ by the curves M_j on Y. □

Since $M_G \oplus P_G \hookrightarrow \Lambda_{K3}$ with a finite index and Λ_{K3} is a unimodular lattice, the discriminant group of M_G is the same as the discriminant group of P_G and $M_G^\vee/M_G = P_G^\vee/P_G$, where

$L^\vee = Hom_{\mathbb{Z}}(L,\mathbb{Z})$ is the dual of the lattice L. Moreover we have the following exact sequence

$$0 \to M_G \longrightarrow \Lambda_{K3} \xrightarrow{l_b} P_G^\vee \longrightarrow 0$$

where b is the bilinear form on Λ_{K3} and $l_b(y)(x) = b(y,x)$, for $y \in \Lambda_{K3}$ and $x \in P_G$. From this sequence we obtain

$$\Lambda_{K3}/M_G \simeq P_G^\vee. \tag{2}$$

The map $\theta : X' \to Y'$ induces a map $\theta^* : H^2(Y',\mathbb{Z}) \to H^2(X',\mathbb{Z})^G$. Thanks to Lemma 1.10 we have $\theta^* : H^2(Y,\mathbb{Z})/\oplus \mathbb{Z}M_j = H^2(Y',\mathbb{Z}) \to H^2(X',\mathbb{Z})^G = H^2(X,\mathbb{Z})^G$. Let σ_Y (resp. σ_X) be the isometry between $H^2(Y,\mathbb{Z})$ (resp. $H^2(X,\mathbb{Z})$) and Λ_{K3}. So we have the following exact sequences

$$\begin{array}{ccccccc} 0 & \to & tors(H^2(Y',\mathbb{Z})) & \to & H^2(Y',\mathbb{Z}) & \xrightarrow{\theta^*} & H^2(X',\mathbb{Z})^G \\ & & \downarrow\wr & & \alpha\downarrow\wr & & \beta\downarrow\wr \\ 0 & \to & tors(H^2(Y,\mathbb{Z})/\oplus \mathbb{Z}M_j) & \to & H^2(Y,\mathbb{Z})/\oplus\mathbb{Z}M_j & \longrightarrow & H^2(X,\mathbb{Z})^G \\ & & \downarrow\wr & & \sigma_Y\downarrow\wr & & \sigma_X\downarrow\wr \\ 0 & \to & M_G/\oplus\mathbb{Z}M_j & \to & \Lambda_{K3}/\oplus\mathbb{Z}M_j & \longrightarrow & \Lambda_{K3}^G. \end{array} \tag{3}$$

From these sequences we obtain

$$\Lambda_{K3}/M_G \simeq \mathrm{Im}(\Lambda_{K3}/\oplus_j \mathbb{Z}M_j \to \Lambda_{K3}^G) \simeq \theta^*(H^2(Y',\mathbb{Z})). \tag{4}$$

So $\Lambda_{K3}/M_G \simeq P_G^\vee$ is a submodule of Λ_{K3}^G.
If θ^* were surjective, this would implies $P_G^\vee \simeq \Lambda_{K3}/M_G \simeq \Lambda_{K3}^G$ (by formulas (2) and (4)).
We want to study the inclusion $P_G^\vee \subset \Lambda_{K3}^G$ and thus we need to study $\theta^* : H^2(Y',\mathbb{Z}) \to H^2(X',\mathbb{Z})$. More precisely since $P_G^\vee \subset H^2(Y',\mathbb{Z})$ and $\Lambda_{K3}^G \simeq H^2(X,\mathbb{Z})^G$ we need to consider $\vartheta^*(P_G^\vee) \subset H^2(X,\mathbb{Z})^G$. Since $P_G \subset H^2(Y',\mathbb{Z})$, $\vartheta^*(P_G^\vee) = \theta^*(P_G^\vee)$ (cf. (1)). Hence

$$\theta^* \text{ is surjective if and only if } H^2(X,\mathbb{Z})^G \simeq \theta^*(P_G^\vee). \tag{5}$$

As lattices, θ^* induces a scaling on $P_G^\vee : \theta^*(P_G^\vee) \simeq P_G^\vee(|G|)$ because $(\theta^*(x),\theta^*(y)) = |G|(x,y)$ for each $x,y \in (\oplus\mathbb{Z}M_j)^\perp$ as $\theta : X' \to Y'$ is of degree $|G|$.

Remark 1.13. In [N, Section 8,] the author states that θ^* is surjective. Assuming the surjectivity of the map the discriminant of the group Λ_{K3}^G is computed in [N, Lemma 10.2] . □

2. Spectral sequences and the map $\theta^* : H^2(Y',\mathbb{Z}) \to H^2(X',\mathbb{Z})^G$

We use a spectral sequence to analyze the properties of the map θ^*. The main result is the sequence (9).

Theorem 2.1. [W, Theorem 6.10] *Let G be a group acting properly on a space S such that $\pi_0(S)$ is trivial. Then for every abelian group A there exists a spectral sequence and*

$$E_2^{p,q} = H^p(G, H^q(S,A)) \Rightarrow H^{p+q}(S/G,A).$$

Remark 2.2. This sequence is a first quadrant spectral sequence (i.e. $E_r^{p,q} = 0$ if $p<0$ or $q<0$). Moreover $E_2^{p,q} = 0$ for each $q > \dim(S)$. □

Under the hypothesis of the theorem, the spectral sequence converges, and so there are filtrations

$$E^2 = E_0^2 \supset E_1^2 \supset E_2^2 \supset E_3^2 = 0 \quad \text{and} \quad E_\infty^{0,2} = E_0^2/E_1^2,$$

so we have a exact sequence $0 \to E_1^2 \to E_0^2 \to E_\infty^{0,2} \to 0$. Analogously there is an exact sequence $0 \to E_2^2 \to E_1^2 \to E_\infty^{1,1} \to 0$. So we have

$$\begin{array}{ccccccccc} & & 0 & & & & & & \\ & & \downarrow & & & & & & \\ & & E_2^2 & & & & & & \\ & & \downarrow & & & & & & \\ 0 & \to & E_1^2 & \to & E^2 & \to & E_\infty^{0,2} & \to & 0 \\ & & \downarrow & & & & & & \\ & & E_\infty^{1,1} & & & & & & \\ & & \downarrow & & & & & & \\ & & 0 & & & & & & \end{array} \tag{6}$$

We now use the same notation as in the previous section.
The hypothesis of Theorem 2.1 are satisfied by the topological space $S = X'$, the group G acting properly on X' and $A = \mathbb{Z}$. We apply the Theorem 2.1 to the spectral sequence $E_2^{p,q} = H^p(G, H^q(X', \mathbb{Z}))$. Since $H^1(X', \mathbb{Z}) = 0$ (Lemma 1.9) we have

$$E_2^{p,1} = H^p(G, H^1(X', \mathbb{Z})) = H^p(G, 0) = 0. \tag{7}$$

We will compute the groups in (6) in this case.
Computation of $E_\infty^{0,2}$. Since the spectral sequence is a first quadrant spectral sequence $E_4^{0,2} = E_\infty^{0,2}$ ($E_r^{0,2} = E_\infty^{0,2}$ if $r > \max\{p, q+1\}$). By definition

$$E_\infty^{0,2} = E_4^{0,2} = \frac{\ker(d_3^{0,2} : E_3^{0,2} \to E_3^{3,0})}{\operatorname{Im}(d_3^{-3,4} : E_3^{-3,4} \to E_3^{0,2})} = \ker(d_3^{0,2} : E_3^{0,2} \to E_3^{3,0}) \tag{8}$$

where $d_3^{-3,4} = 0$ because the spectral sequence is in the first quadrant. But

$$E_3^{0,2} = \frac{\ker(d_2^{0,2} : E_2^{0,2} \to E_2^{2,1})}{\operatorname{Im}(d_2^{-2,3} : E_2^{-2,3} \to E_2^{0,2})} = \ker(d_2^{0,2} : E_2^{0,2} \to E_2^{2,1}) = E_2^{0,2}$$

where the last equality is a consequence of (7). Moreover

$$E_3^{3,0} = \frac{\ker(d_2^{3,0} : E_2^{3,0} \to E_2^{5,-1})}{\operatorname{Im}(d_2^{1,1} : E_2^{1,1} \to E_2^{3,0})} = \frac{\ker(d_2^{3,0} : E_2^{3,0} \to 0)}{\operatorname{Im}(d_2^{1,1} : 0 \to E_2^{3,0})} = E_2^{3,0}.$$

Substituting the last two formula in (8) we obtain

$$E_\infty^{0,2} = \ker(E_2^{0,2} \to E_2^{3,0}).$$

By the definition of the spectral sequence and by the equality $H^0(G, H^2(X', \mathbb{Z})) = H^2(X', \mathbb{Z})^G$ ([W, Definition 6.1.2]) we have

$$E_2^{0,2} = H^0(G, H^2(X', \mathbb{Z})) = H^2(X', \mathbb{Z})^G, \ E_2^{3,0} = H^3(G, H^0(X', \mathbb{Z})) = H^3(G, \mathbb{Z}), \text{ and so}$$

$$E_\infty^{0,2} = \ker(H^2(X', \mathbb{Z})^G \to H^3(G, \mathbb{Z})).$$

Computation of E^2. By the Theorem 2.1 applied to $S = X'$ we obtain that $E_2^{p,q} \Rightarrow H^2(X'/G, \mathbb{Z})$, so $E^2 = H^2(X'/G, \mathbb{Z}) = H^2(Y', \mathbb{Z})$.
Computation of E_1^2. To compute E_1^2 we consider the exact sequence $0 \to E_2^2 \to E_1^2 \to E_\infty^{1,1} \to 0$ from the diagram (6). By (7) $H^1(X', \mathbb{Z}) = 0$ and so $E_2^{1,1} = 0$ but then $E_\infty^{1,1} = 0$. Since $E_\infty^{2,0} = E_2^2/E_3^2$ and $E_3^2 = 0$, $E_2^2 \simeq E_\infty^{2,0}$. The sequence becomes $0 \to E_\infty^{2,0} \to E_1^2 \to 0$ and then $E_1^2 \simeq E_\infty^{2,0}$.

Since the spectral sequence is a first quadrant spectral sequence $E_3^{2,0}=E_\infty^{2,0}$ ($E_r^{2,0}=E_\infty^{2,0}$ if $r>\max\{p,q+1\}$). Moreover

$$E_3^{2,0}=\frac{\ker(d_2^{2,0}:E_2^{2,0}\to E_2^{4,-1})}{\operatorname{Im}(d_2^{0,1}:E_2^{0,1}\to E_2^{2,0})}=E_2^{2,0},$$

because $E_2^{0,1}=0$ by (7). So

$$E_1^2\simeq E_\infty^{2,0}\simeq E_2^{2,0}=H^2(G,\mathbb{Z}).$$

The horizontal sequence of (6) becomes

$$0\to H^2(G,\mathbb{Z})\to H^2(Y',\mathbb{Z})\stackrel{\theta^*}{\to}\ker(H^2(X',\mathbb{Z})^G\to H^3(G,\mathbb{Z}))\to 0. \tag{9}$$

The exact sequence (9) implies that the following sequence is exact:

$$0\to H^2(G,\mathbb{Z})\to H^2(Y',\mathbb{Z})\stackrel{\theta^*}{\to}H^2(X',\mathbb{Z})^G.$$

Remark 2.3. If $H^3(G,\mathbb{Z})=0$ then the sequence

$$0\to H^2(G,\mathbb{Z})\to H^2(Y',\mathbb{Z})\stackrel{\theta^*}{\to}H^2(X',\mathbb{Z})^G\to 0.$$

is exact and in particular the map

$$\theta^*:H^2(Y',\mathbb{Z})\to H^2(X',\mathbb{Z})^G$$

is onto.
If $G=\mathbb{Z}/n\mathbb{Z}$ is cyclic, then $H^3(G,\mathbb{Z})$ is given by [W, Example 6.2.3]:

$$H^n(\mathbb{Z}/n\mathbb{Z},\mathbb{Z})=\begin{cases}\mathbb{Z} & \text{if } n=0\\ 0 & \text{if } n=2k+1,\ k\in\mathbb{N}\\ \mathbb{Z}/n\mathbb{Z} & \text{if } n=2k, k\in\mathbb{N}.\end{cases}$$

Hence the map $H^2(Y',\mathbb{Z})\to H^2(X',\mathbb{Z})^G$ is surjective and the exact sequence is:

$$0\to\mathbb{Z}/n\mathbb{Z}\to H^2(Y',\mathbb{Z})\to H^2(X',\mathbb{Z})^G\to 0.$$

This proves the assertion of Nikulin that θ^* is surjective in case G is a cyclic group. □

More in general we want to compute the sequence (9) for all the finite abelian groups acting symplectically on a K3 surface (the complete list is given in [N, Proposition 1.7]).

Proposition 2.4. *For each finite abelian group G acting sympletically on X we have the following exact sequences:*

$$0\to G\to H^2(Y',\mathbb{Z})\to ker\left(H^2(X',\mathbb{Z})^G\stackrel{d_3^{2,0}}{\to}V_G\right),\ \textit{where}$$

$$V_G=\begin{cases}0 & \textit{if} \quad G=\mathbb{Z}/n\mathbb{Z}\\ \mathbb{Z}/2\mathbb{Z} & \textit{if}\quad G=(\mathbb{Z}/2\mathbb{Z})^2\\ (\mathbb{Z}/2\mathbb{Z})^3 & \textit{if}\quad G=(\mathbb{Z}/2\mathbb{Z})^3\\ (\mathbb{Z}/2\mathbb{Z})^6 & \textit{if}\quad G=(\mathbb{Z}/2\mathbb{Z})^4\end{cases}\qquad V_G=\begin{cases}\mathbb{Z}/3\mathbb{Z} & \textit{if}\quad G=(\mathbb{Z}/3\mathbb{Z})^2\\ \mathbb{Z}/4\mathbb{Z} & \textit{if}\quad G=\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}\\ \mathbb{Z}/4\mathbb{Z} & \textit{if}\quad G=(\mathbb{Z}/4\mathbb{Z})^2\\ \mathbb{Z}/6\mathbb{Z} & \textit{if}\quad G=\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/6\mathbb{Z}.\end{cases}$$

Proof. The proof consists in the computation of the groups $H^2(G,\mathbb{Z})$ and $H^3(G,\mathbb{Z})$. This computation is based on the formulas (cf. [W, Exercise 6.1.8])

$$H^n(G_1\times G_2)\simeq\left(\bigoplus_{p+q=n}H^p(G_1,\mathbb{Z})\otimes H^q(G_2,\mathbb{Z})\right)\oplus\left(\bigoplus_{p+q=n+1}Tor(H^p(G_1,\mathbb{Z}),H^q(G_2,\mathbb{Z}))\right),$$

$Tor(\mathbb{Z},A)=0$ for each abelian group A and $Tor(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}/m\mathbb{Z})=\mathbb{Z}/(n,m)\mathbb{Z}$. □

Remark 2.5. Comparing the sequence (9) with the sequences in (3) one obtains that $M_G/\oplus_j \mathbb{Z}M_j \simeq G$ (cf. also [N, Theorem 6.3], [X, Lemma 2]). □

3. Symplectic automorphisms on $Km(A)$

Here we will construct an example of a K3 surface such that the group $(\mathbb{Z}/2\mathbb{Z})^4$ acts symplectically on it. This K3 surface is a Kummer surface, the symplectic automorphisms are induced by translation by points of order two on the Abelian surface.
We will compute the lattices Λ_{K3}^G for the subgroups $G \subseteq (\mathbb{Z}/2\mathbb{Z})^4$. In the case $G = \mathbb{Z}/2\mathbb{Z}$ we obtain the same result as Nikulin, in fact we have proved in Remark 2.3 that the map θ^* is surjective if G is cyclic. For the other cases, the sublattice $\theta^*(P_G^\vee)$ of Λ_{K3}^G is *not* equal to Λ_{K3}^G. Hence θ^* is not surjective (cf. (5)). As a consequence we find that the discriminant computed by Nikulin is not correct.

3.1. **Preliminaries on Kummer surfaces.** We recall some properties of Kummer surfaces and of Kummer lattices.

Definition 3.1. *Let A be an Abelian surface. Let $\iota : A \to A$ be the involution $\iota : a \mapsto -a$, $a \in A$. The quotient A/ι is a surface with sixteen singularities of type A_1 (the image of the sixteen 2-torsion points of A). The desingularization of this quotient is the K3 surface $Km(A)$, called **Kummer surface of** A.*
*The minimal primitive sublattice of $H^2(Km(A), \mathbb{Z})$ containing the sixteen rational curves resolving the singularities of A/ι is called **Kummer lattice**.*

Proposition 3.2. [PS, Appendix to section 5, Lemma 4] *The Kummer lattice is a rank sixteen negative definite lattice. Its discriminant lattice is isomorphic to the one of $U(2)^{\oplus 3}$ (where $U(2)$ is the lattice obtained from U by multiplying the bilinear form by two).*

Let $A[2]$ be the set of the 2-torsion points on A. We fix an isomorphism $A[2] \simeq (\mathbb{Z}/2\mathbb{Z})^4$ and we write $p_a \in A[2]$ for the point corresponding to $a \in (\mathbb{Z}/2\mathbb{Z})^4$. The surface $\widetilde{A}$ is the blow up of A in the 2-torsion points, so there are sixteen exceptional curves E_a on $\widetilde{A}$ which correspond to the points p_a. These sixteen curves are sent in the sixteen rational curves K_a of the Kummer lattice by the map $\pi : \widetilde{A} \to \widetilde{A}/\widetilde{\iota}$, where $\widetilde{\iota}$ is the involution induced on $\widetilde{A}$ by the involution ι on A described in Definition 3.1. We have the following commutative diagram:

$$\begin{array}{ccccc} \{p_a\} = A[2] \subset & A & \stackrel{\gamma}{\leftarrow} & \widetilde{A} & \supset \{E_a\} \\ & \downarrow & \circlearrowleft & \downarrow \pi & \\ Sing(A/\iota) \subset & A/\iota & \leftarrow & Km(A) & \supset \{K_a\} \end{array} \tag{10}$$

We can thus associate a 2-torsion point p_a on an Abelian surface to the corresponding curve K_a in the Kummer lattice of its Kummer surface. We use the following convention:
- if W is an affine subspace of the affine space $A[2]$, $\bar{K}_W$ is the class $\frac{1}{2}\sum_{a\in W} K_a$ and $\hat{K} = \frac{1}{2}\sum_{a\in(\mathbb{Z}/2\mathbb{Z})^4} K_a = \bar{K}_{A[2]}$;
- $W_i = \{a = (a_1, a_2, a_3, a_4) \in A[2]$ such that $a_i = 0\}$, $i = 1, 2, 3, 4$;
- $W_{i,j} = \{a = (a_1, a_2, a_3, a_4) \in A[2]$ such that $a_i + a_j = 0\}$, $1 \le i < j \le 4$;
- $V_{i,j} = \{0, \alpha_i, \alpha_j, \alpha_i + \alpha_j\}$, where $\alpha_1 = (1, 0, 0, 0)$, $\alpha_2 = (0, 1, 0, 0)$, $\alpha_3 = (0, 0, 1, 0)$, $\alpha_4 = (0, 0, 0, 1)$ and $1 \le i < j \le 4$.

Proposition 3.3. ([BPV, Chapter VIII, Section 5], [PS, Appendix 5]) *A set of generators, over $\mathbb{Z}$, of the Kummer lattice is made up of the sixteen classes: $\hat{K}$, $\bar{K}_{W_1}$, $\bar{K}_{W_2}$, $\bar{K}_{W_3}$, $\bar{K}_{W_4}$, K_a, with $a = (0,0,0,0)$, $(0,0,1,1)$, $(0,1,0,1)$, $(1,0,0,1)$, $(0,1,1,0)$, $(1,0,1,0)$, $(1,1,0,0)$, $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$.*

Definition 3.4. [Mo, Definition 5.3] *The* **Nikulin lattice** *is an even lattice N of rank eight generated by $\{N_i\}_{i=1}^8$ and $\hat{N}=\frac{1}{2}\sum N_i$, with bilinear form induced by $N_i\cdot N_j=-2\delta_{ij}$.*

Remark 3.5. The Nikulin lattice is the lattice M_G (cf. Definition 1.11) where G is generated by an involution, the N_i correspond to curves arising from the singularities of the quotient of a K3 surface by a symplectic involution. □

Remark 3.6. For each i the classes K_a, $a\in W_i$, and $\bar{K}_{W_i}$ generate a Nikulin lattice. □

Proposition 3.7. ([BPV, Chapter VIII, Section 5], [PS, Appendix 5]) *Let $A_K=K^\vee/K$ be the discriminant group of the lattice K and $q_{A_K}:A_K\rightarrow\mathbb{Q}/2\mathbb{Z}$ be the quadratic form on A_K induced by the pairing on K.*
The discriminant group A_K is made up of 63 elements and the zero 0_{A_K}: 35 of them are of type $\bar{K}_V$, for linear subspaces $V\simeq(\mathbb{Z}/2\mathbb{Z})^2\subset A[2]$, and $q_{A_K}(\bar{K}_V)=0$, the other 28 are of type $\bar{K}_{V+V'}$, for linear subspaces V and V', such that $V\cap V'=\{0\}$, and $q_{A_K}(\bar{K}_{V+V'})=1$. There are three orbits of $O(q_{A_K})$ on A_K: $\{0\}$, $\{v\in A_K: q_{A_K}(v)\equiv 0 \mod 2\mathbb{Z}\}$ and $\{v\in A_K: q_{A_K}(v)\equiv 1 \mod 2\mathbb{Z}\}$.

Remark 3.8. The classes $\bar{K}_{V_{1,2}}$, $\bar{K}_{V_{3,4}}$, $\bar{K}_{V_{1,3}}$, $\bar{K}_{V_{2,4}}$, $\bar{K}_{V_{1,4}}$, $\bar{K}_{V_{2,3}}$ generate the discriminant group A_K.
Moreover if e_i, f_i, $i=1,2,3$ are the standard basis of $U(2)^{\oplus 3}$, then $\bar{K}_{V_{1,2}}+e_1/2$, $\bar{K}_{V_{1,3}}+e_2/2$, $\bar{K}_{V_{1,4}}+e_3/2$, $\bar{K}_{V_{3,4}}+f_1/2$, $\bar{K}_{V_{2,4}}+f_2/2$, $\bar{K}_{V_{2,3}}+f_3/2$ together with the classes generating the Kummer lattice (cf. Proposition 3.3) are 22 vectors in $(K\oplus U(2)^{\oplus 3})\otimes_{\mathbb{Z}}\mathbb{Q}$ which generate a unimodular even lattice with signature $(3,19)$, so a lattice isometric to Λ_{K3}. □

Definition 3.9. *Let x_i, $i=1,2,3,4$, be the real coordinates of an Abelian surface $A=(\mathbb{R}/\mathbb{Z})^4\simeq\mathbb{C}^2/\Lambda$. Let $\omega_{i,j}:=\pi_*(\gamma^*(dx_i\wedge dx_j))$, with the notation of the diagram* (10).

Remark 3.10. The classes $\omega_{i,j}$, $1\leq i<j\leq 4$ generate $U(2)^{\oplus 3}\simeq K^\perp\subset H^2(Km(A),\mathbb{Z})$.
In fact since $H^4(X,\mathbb{Z})\simeq\mathbb{Z}$, with $dx_1\wedge dx_2\wedge dx_3\wedge dx_4\mapsto 1$, and

$$(dx_i\wedge dx_j)\wedge(dx_h\wedge dx_k)\mapsto\left\{\begin{array}{l}\pm 1 \text{ if } \{i,j,h,k\}=\{1,2,3,4\}\\ 0 \text{ otherwise,}\end{array}\right.$$

the six forms $dx_1\wedge dx_2$, $dx_3\wedge dx_4$, $dx_1\wedge dx_3$, $dx_2\wedge dx_4$, $dx_1\wedge dx_4$, $dx_2\wedge dx_3$ generate three copies of the lattice U respectively. Hence they form a basis for $H^2(A,\mathbb{Z})\simeq U^{\oplus 3}$.
The forms $dx_i\wedge dx_j$ on A are invariant under ι^*, and so they generate $H^2(A,\mathbb{Z})^{\iota^*}=H^2(A,\mathbb{Z})$. Let us consider the lattice $H^2(A,\mathbb{Z})^{\iota^*}$ as sublattice of $H^2(\widetilde{A},\mathbb{Z})$ (it is exactly the sublattice orthogonal to the classes of the exceptional curves E_a). Since $\pi_*(H^2(A,\mathbb{Z}))^{\iota^*}\simeq H^2(A,\mathbb{Z})^{\iota^*}(2)$ (cf. [Mo, Lemma 3.1]), the lattice generated by $\omega_{i,j}=\pi_*(\gamma^*(dx_i\wedge dx_j))$, is isometric to $U(2)^{\oplus 3}$ and it is a sublattice of $H^2(Km(A),\mathbb{Z})$. Moreover the classes $\omega_{i,j}$, are orthogonal to the curves of the Kummer lattice in $H^2(Km(A),\mathbb{Z})$, and so they generate $K^\perp\subset H^2(Km(A),\mathbb{Z})$. □

Remark 3.11. We fix a complex structure on A such that the complex coordinates are $z_1=x_1+ix_2$ and $z_2=x_3+ix_4$, and we choose the lattice such that $\mathbb{C}^2/\Lambda\simeq\mathbb{C}/\Gamma\times\mathbb{C}/\Gamma'$ where Γ and Γ' are lattices in $\mathbb{C}$. This defines two elliptic curves $C:=\mathbb{C}/\Gamma$ and $C':=\mathbb{C}/\Gamma'$ (with real coordinates (x_1,x_2) and (x_3,x_4) respectively) and hence two classes $[C]$ and $[C']$ in $H_2(A,\mathbb{Z})$. By Poincaré duality $H_2(A,\mathbb{Z})\simeq H^2(A,\mathbb{Z})^*\simeq H^2(A,\mathbb{Z})$, so $[D]$ corresponds to a two form μ such that $\int_D\nu=\int_S\mu\wedge\nu$ for all $\nu\in H^2(S,\mathbb{R})$. As

$$\int_C dx_i\wedge dx_j=\left\{\begin{array}{rl}1 & \text{if } i=1,j=2,\\ -1 & \text{if } i=2,j=1,\\ 0 & \text{otherwise}\end{array}\right.$$

and $\int_A dx_h \wedge dx_k \wedge dx_1 \wedge dx_2 = 1$ if and only if $h = 3$ and $k = 4$, the class $[C]$ corresponds to $dx_3 \wedge dx_4$. Similarly $[C'] = dx_1 \wedge dx_2$. On each of these curves there are four 2-torsion points. With the notation of (10), $\gamma^*(C) = \widetilde{C} + \sum_{a \in C[2]} E_a$ where $\widetilde{C}$ is the strict transform of the C and E_a are the exceptional curves over the four 2-torsion points of $A[2]$ on the curve C (i.e. $a = (a_1, a_2, 0, 0)$). By Remark 3.8 we know that $(u/2 + \bar{K}_{V_{1,2}}) \in \Lambda_{K3}$ for a certain $u \in U(2)^{\oplus 3}$ with $u^2 = 0$. The curves $K_{a_1,a_2,0,0}$ correspond to the four 2-torsion points on the curve C, and the class $[C]$ is $dx_3 \wedge dx_4$, so $u = \omega_{3,4}$. This implies that $\omega_{3,4}/2 + \bar{K}_{V_{1,2}}$ is a class in $H^2(Km(A), \mathbb{Z})$. So we can restate Remark 3.8 in the following, more precise, way: a set of generators of the lattice $H^2(Km(A), \mathbb{Z})$ is given by the sixteen classes generating the Kummer lattice and by the six classes $\omega_{1,2}/2 + \bar{K}_{V_{3,4}}$, $\omega_{3,4}/2 + \bar{K}_{V_{1,2}}$, $\omega_{1,3}/2 + \bar{K}_{V_{2,4}}$, $\omega_{2,4}/2 + \bar{K}_{V_{1,3}}$, $\omega_{1,4}/2 + \bar{K}_{V_{2,3}}$, $\omega_{2,3}/2 + \bar{K}_{V_{1,4}}$. □

3.2. **The surfaces $Km(A)$ and $Km(A/\langle b \rangle)$.**
Let A be an Abelian surface. Let b be a point of order two on A and $\langle b \rangle = \{0, b\}$ be the group generated by b.

Remark 3.12. The surface $A/\langle b \rangle$ is again an Abelian surface. The 2-torsion points on $A/\langle b \rangle$ are the images of the points r on A such that $2r \in \langle b \rangle$. We observe that the points r such that $2r = b$ are 4-torsion points on A. Hence the sixteen points in $(A/\langle b \rangle)[2]$ are the eight points in the image of the 2-torsion points of A and the eight points in the image of the 4-torsion points r such that $2r = b$. □

The translation by the point b of order two on A induces an involution on $Km(A)$ and on $\widetilde{Km(A)}$. Since the translation by the point b acts as the identity on the holomorphic two forms on A and since the holomorphic two form on $Km(A)$ is induced by the one on A, the involution induced on $Km(A)$ by the translation by b fixes the holomorphic two form on $Km(A)$ and hence is symplectic. We have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & A & & \longrightarrow & & A/\langle b \rangle & & & \\
\swarrow & & \searrow & & \swarrow & & & & \searrow \\
A/\iota & & \widetilde{A} & & (A/\langle b \rangle)/\iota & & & & \widetilde{A/\langle b \rangle} \\
\searrow & & \swarrow & & \searrow & & & & \swarrow \\
 & Km(A) & \longleftarrow & \widetilde{Km(A)} & \longrightarrow & Km(A/\langle b \rangle) = \widetilde{Km(A)}/\langle b \rangle & & &
\end{array}$$

where $\widetilde{Km(A)}$ is the blow up of $Km(A)$ in the points of $Km(A)$ with a non-trivial stabilizer for the action of the involution induced by the translation by b and $\widetilde{Km(A)}/\langle b \rangle$ is the smooth quotient of $\widetilde{Km(A)}$ by the action this involution on $Km(A)$.

Remark 3.13. Let $x \in A$ such that its image $\overline{x} \in A/\iota$ is a point with a non trivial stabilizer (i.e. a fixed point) with respect to the translation by b. Then we have $\overline{x + b} = \overline{x}$ and so $x + b = \pm x$. Since b is of order two, $2x = b$ and so x has order four. There are 16 such points x on A and so on A/ι we find $16/2 = 8$ fixed points. After blow up of the images of the 2-torsion points in A/ι we obtain $Km(A)$, in particular there are 8 fixed points in $Km(A)$ (with respect to the involution induced on $Km(A)$ by the translation on A). Note that none of the curves of the Kummer lattice on $Km(A)$ are fixed. Thus the branch locus of $\widetilde{Km(A)} \to Km(A/\langle b \rangle)$ are the eight curves on $Km(A/\langle b \rangle)$ corresponding to the eight points in $(A/\langle b \rangle)[2]$ which are images of x with $2x = b$. □

3.3. **The group $G=(\mathbb{Z}/2\mathbb{Z})^4$.**
Let $G=(\mathbb{Z}/2\mathbb{Z})^4$ be the group of symplectic automorphisms on $Km(A)$ induced by the group of the translation by points of order two on A. As $A/A[2]\simeq A$ the desingularization of the quotient of $Km(A)$ by G is again $Km(A)$. The image of the point p_a on A under the map $\varphi: A\rightarrow A/A[2]$ is the point p_{2a}, so in particular the image of p_a with $a\in(\mathbb{Z}/2\mathbb{Z})^4=A[2]$ under the map $A\rightarrow A/A[2]$ is the point p_0.

The lattice $M_{(\mathbb{Z}/2\mathbb{Z})^4}$. Since the 2-torsion points p_a are sent to p_0, the curves K_a on $Km(A)$ are sent to the curve K_0 on $\widetilde{Km(A)/G}=Km(A)$. The branch locus of the cover $Km(A)\rightarrow \widetilde{Km(A)/G}$ can be found as in Remark 3.13. In fact the stabilizer, with respect to the action of G, of a point $\overline{x}\in A/\iota$ is either trivial or the group $\{0,b\}$, (and then $2x=b$) for a certain $b\in A[2]\backslash\{0\}$. Since there are fifteen 2-torsion points, there are fifteen different stabilizer groups. For each of these stabilizer groups there are eight points with that group as stabilizer (cf. Remark 3.13). Hence there are $8\cdot 15=120$ points with a non trivial stabilizer on A/ι. The quotient by the group G identifies the eight points with the same stabilizer group, hence the branch locus of the cover $Km(A)\rightarrow\widetilde{Km(A)/G}$ is made up of the fifteen curves of the Kummer lattice K_a, $a\neq 0$. By definition $M_{(\mathbb{Z}/2\mathbb{Z})^4}$ is the minimal primitive sublattice of $H^2(Y,\mathbb{Z})$ containing these fifteen curves,

$$\langle K_a, a\in(\mathbb{Z}/2\mathbb{Z})^4\backslash\{0\}\rangle\stackrel{f.\ i.}{\hookrightarrow}M_{(\mathbb{Z}/2\mathbb{Z})^4},$$

where the inclusion has a finite index.

The lattice $P_{(\mathbb{Z}/2\mathbb{Z})^4}=M^{\perp}_{(\mathbb{Z}/2\mathbb{Z})^4}$. Since $M_{(\mathbb{Z}/2\mathbb{Z})^4}\subset K$, the lattice $K^{\perp}=U(2)^{\oplus 3}$ is contained in $M^{\perp}_{(\mathbb{Z}/2\mathbb{Z})^4}=P_{(\mathbb{Z}/2\mathbb{Z})^4}$. Moreover the curve K_0 is contained in $M^{\perp}_{(\mathbb{Z}/2\mathbb{Z})^4}$. So $\mathbb{Z}K_0\oplus U(2)^{\oplus 3}$ is contained, with finite index, in $P_{(\mathbb{Z}/2\mathbb{Z})^4}$.
There are no classes which are in $P_{(\mathbb{Z}/2\mathbb{Z})^4}$ but not in $\mathbb{Z}K_0\oplus U(2)^{\oplus 3}$, in fact there are no classes in the dual of the lattice $K\oplus U(2)^{\oplus 3}$ involving only one curve of the Kummer lattice (cf. Remark 3.8 and Remark 3.11), so

$$\begin{array}{ll}P_{(\mathbb{Z}/2\mathbb{Z})^4} & =\langle K_0,\omega_{1,2},\omega_{3,4},\omega_{1,3},\omega_{2,4},\omega_{1,4},\omega_{2,3}\rangle\\ & \simeq\langle -2\rangle\oplus U(2)\oplus U(2)\oplus U(2).\end{array}$$

The lattice $\theta^*(P^{\vee}_{(\mathbb{Z}/2\mathbb{Z})^4})$. The dual lattice of $P_{(\mathbb{Z}/2\mathbb{Z})^4}$ is made up of the classes $c\in P_{(\mathbb{Z}/2\mathbb{Z})^4}\otimes\mathbb{Q}$ such that the intersection product $c\cdot r\in\mathbb{Z}$ for each $r\in P_{(\mathbb{Z}/2\mathbb{Z})^4}$. Then

$$P^{\vee}_{(\mathbb{Z}/2\mathbb{Z})^4}=\langle K_0/2,\omega_{1,2}/2,\omega_{3,4}/2,\omega_{1,3}/2,\omega_{2,4}/2,\omega_{1,4}/2,\omega_{2,3}/2\rangle.$$

To determine $\theta^*(P^{\vee}_G)$, note that $\theta^*(K_0)=\sum_{a\in(\mathbb{Z}/2\mathbb{Z})^4}K_a$. The map ϑ, and the hence also map θ (cf. (1)), is induced by the map $A\stackrel{\cdot 2}{\rightarrow}A$, so its action on $\omega_{i,j}$ is induced by the action of $\cdot 2$ on $dx_i\wedge dx_j$. The map $\cdot 2: dx_i\wedge dx_j\rightarrow d(2x_i)\wedge d(2x_j)=4dx_i\wedge dx_j$, and so $\theta^*(\omega_{i,j})=4\omega_{i,j}$. We observe that $(\theta^*(\omega_{i,j}),\theta^*(\omega_{h,k}))=(4\omega_{i,j},4\omega_{h,k})=|G|(\omega_{i,j},\omega_{h,k})$ as predicted.
By the descriptions of $P^{\vee}_{(\mathbb{Z}/2\mathbb{Z})^4}$ and θ^* we obtain:

$$\begin{array}{rl}\theta^*(P^{\vee}_{(\mathbb{Z}/2\mathbb{Z})^4}) & =\left\langle \hat{K},2\omega_{1,2},2\omega_{3,4},2\omega_{1,3},2\omega_{2,4},2\omega_{1,4},2\omega_{2,3}\right\rangle\\ & \simeq\langle -8\rangle\oplus U(8)\oplus U(8)\oplus U(8).\end{array}$$

The lattice $\Lambda_{K3}^{(\mathbb{Z}/2\mathbb{Z})^4}$. The group G acts trivially on the $\omega_{i,j}$ and so the $\omega_{i,j}$ are contained in $\Lambda_{K3}^{(\mathbb{Z}/2\mathbb{Z})^4}$. Moreover each class K_a of the Kummer lattice is sent into $K_{a+a'}$ by $a'\in G$, so no class K_a is fixed, but the sum of all these classes is fixed. So $\hat{K}$ is contained in $\Lambda_{K3}^{(\mathbb{Z}/2\mathbb{Z})^4}$. The classes $\omega_{i,j}$ generate the lattice $U(2)^{\oplus 3}$ which is primitive in Λ_{K3}, and so

$$\begin{aligned}\Lambda_{K3}^{(\mathbb{Z}/2\mathbb{Z})^4} &= \left\langle \hat{K}, \omega_{1,2}, \omega_{3,4}, \omega_{1,3}, \omega_{2,4}, \omega_{1,4}, \omega_{2,3} \right\rangle \\ &\simeq \langle -8 \rangle \oplus U(2) \oplus U(2) \oplus U(2).\end{aligned}$$

Comparing the discriminant or observing the classes $\omega_{i,j}$ it is clear that $\theta^*(P^\vee_{(\mathbb{Z}/2\mathbb{Z})^4})$ is not isomorphic to $\Lambda_{K3}^{(\mathbb{Z}/2\mathbb{Z})^4}$ but it is contained with a finite index in $\Lambda_{K3}^{(\mathbb{Z}/2\mathbb{Z})^4}$,

$$\theta^*(P^\vee_{(\mathbb{Z}/2\mathbb{Z})^4}) \overset{f.i}{\hookrightarrow} \Lambda_{K3}^{(\mathbb{Z}/2\mathbb{Z})^4} \qquad \text{and} \qquad [\Lambda_{K3}^{(\mathbb{Z}/2\mathbb{Z})^4} : \theta^*(P^\vee_{(\mathbb{Z}/2\mathbb{Z})^4})] = 2^6.$$

In particular θ^* is not surjective in this case.

3.4. **The group $G = (\mathbb{Z}/2\mathbb{Z})^2$.**
Let us consider the subgroup $G_{2,2}$ of $(\mathbb{Z}/2\mathbb{Z})^4$ induced by the translation by two 2-torsion points. The quotient $A/G_{2,2}$ is an Abelian surface. With $A = (\mathbb{R}/\mathbb{Z})^4$, we choose the map $A \to A/G_{2,2}$ to correspond to multiplication by two on the first two coordinates, x_1 and x_2, on A. We observe that $A/G_{2,2}$ is not isomorphic to A in general.
The subgroup $G_{2,2}$ induces the group $G = (\mathbb{Z}/2\mathbb{Z})^2$ of symplectic automorphisms on $Km(A)$.

The lattice $M_{(\mathbb{Z}/2\mathbb{Z})^2}$. It is the minimal primitive sublattice of $H^2(Y,\mathbb{Z})$ containing the twelve curves $K_a, a = (\underbrace{a_1, a_2}, \underbrace{a_3, a_4}) \in (\mathbb{Z}/2\mathbb{Z})^4, (a_1, a_2) \in (\mathbb{Z}/2\mathbb{Z})^2 \backslash \{0\}$ which are the branch locus of the cover $Km(A) \to Km(A)/G$.

The lattice $P_{(\mathbb{Z}/2\mathbb{Z})^2} = M^\perp_{(\mathbb{Z}/2\mathbb{Z})^2}$. By the computation of $M_{(\mathbb{Z}/2\mathbb{Z})^2}$, the $16-12=4$ remaining curves of K are in $P_{(\mathbb{Z}/2\mathbb{Z})^2}$, in fact:

$$\langle \omega_{1,2}, \omega_{3,4}, \omega_{1,3}, \omega_{2,4}, \omega_{1,4}, \omega_{2,3}, K_0, K_{0,0,0,1}, K_{0,0,1,0}, K_{0,0,1,1} \rangle \overset{f.i.}{\hookrightarrow} P_{(\mathbb{Z}/2\mathbb{Z})^2},$$

with a finite index. Fixing a complex structure on A as in Remark 3.11, we know that $\bar{K}_{V_{3,4}} + \omega_{1,2}/2 \in H^2(Y,\mathbb{Z})$. Then:

$$P_{(\mathbb{Z}/2\mathbb{Z})^2} = \left\langle \bar{K}_{V_{3,4}} + \omega_{1,2}/2, \omega_{3,4}, \omega_{1,3}, \omega_{2,4}, \omega_{1,4}, \omega_{2,3}, K_0, K_{0,0,0,1}, K_{0,0,1,0}, K_{0,0,1,1} \right\rangle.$$

The lattice $\theta^*(P^\vee_{(\mathbb{Z}/2\mathbb{Z})^2})$. The dual lattice of $P_{(\mathbb{Z}/2\mathbb{Z})^2}$ is

$$\begin{aligned}P^\vee_{(\mathbb{Z}/2\mathbb{Z})^2} = \langle &\omega_{1,2}/2, \omega_{1,3}/2, \omega_{1,4}/2, \omega_{2,3}/2, \omega_{2,4}/2, (\omega_{3,4} + K_0)/2, \\ &(K_0 + K_{0,0,0,1})/2, (K_0 + K_{0,0,1,0})/2, (K_0 + K_{0,0,1,1})/2, K_0 \rangle, \text{ and}\end{aligned}$$

$$\theta^*(P^\vee_{(\mathbb{Z}/2\mathbb{Z})^2}) = \left\langle 2\omega_{1,2}, \omega_{1,3}, \omega_{1,4}, \omega_{2,3}, \omega_{2,4}, \left(\omega_{3,4}/2 + \bar{K}_{V_{1,2}}\right), \bar{K}_{W_3}, \bar{K}_{W_4}, \bar{K}_{W_{3,4}}, 2\bar{K}_{V_{1,2}} \right\rangle.$$

The lattice $\Lambda_{K3}^{(\mathbb{Z}/2\mathbb{Z})^2}$. As before the classes $\omega_{i,j}$ are in $\Lambda_{K3}^{(\mathbb{Z}/2\mathbb{Z})^2}$. The group G acts as the identity on the third and fourth coordinates of A, so for $(a_3,a_4) \in (\mathbb{Z}/2\mathbb{Z})^2$ classes $K_{0,0,a_3,a_4} + K_{0,1,a_3,a_4} + K_{1,0,a_3,a_4} + K_{1,1,a_3,a_4}$ are in $\Lambda_{K3}^{(\mathbb{Z}/2\mathbb{Z})^2}$. Over $\mathbb{Q}$, $\Lambda_{K3}^{(\mathbb{Z}/2\mathbb{Z})^2}$ is generated by $\omega_{i,j}$ and $\sum_{(a_1,a_2)\in(\mathbb{Z}/2\mathbb{Z})^2} K_{a_1,a_2,a_3,a_4}$, $(a_3,a_4) \in (\mathbb{Z}/2\mathbb{Z})^2$. The sum of eight curves K_a which correspond to a hyperplane are divisible by two in the Kummer lattice (cf. Proposition 3.3), so in particular they are elements of Λ_{K3}. Moreover $\bar{K}_{V_{1,2}} + \omega_{3,4}/2$ is contained in Λ_{K3} (cf. Remark 3.11). Then the primitive lattice in Λ_{K3} generated by $\omega_{i,j}$ and $\sum_{(a_1,a_2)\in(\mathbb{Z}/2\mathbb{Z})^2} K_{a_1,a_2,a_3,a_4}$, $a_3, a_4 \in (\mathbb{Z}/2\mathbb{Z})^2$ is

$$\Lambda_{K3}^{(\mathbb{Z}/2\mathbb{Z})^2} = \left\langle \omega_{1,2}, \omega_{1,3}, \omega_{2,4}, \omega_{1,4}, \omega_{2,3}, \left(\omega_{3,4}/2 + \bar{K}_{V_{1,2}}\right), \bar{K}_{W_3}, \bar{K}_{W_4}, \bar{K}_{W_{3,4}}, 2\bar{K}_{V_{1,2}} \right\rangle.$$

Comparing this lattice with $\theta^*(P^{\vee}_{(\mathbb{Z}/2\mathbb{Z})^2})$ it is clear that they are not isomorphic and

$$\theta^*(P^{\vee}_{(\mathbb{Z}/2\mathbb{Z})^2}) \overset{f.i}{\hookrightarrow} \Lambda_{K3}^{(\mathbb{Z}/2\mathbb{Z})^2} \qquad \text{and} \qquad [\Lambda_{K3}^{(\mathbb{Z}/2\mathbb{Z})^2} : \theta^*(P^{\vee}_{(\mathbb{Z}/2\mathbb{Z})^2})] = 2.$$

3.5. $G = (\mathbb{Z}/2\mathbb{Z})^3$**.**
Similar computations can be done in case $G = (\mathbb{Z}/2\mathbb{Z})^3$. Let $G_{2,2,2}$ be the group of translations by three independent points of order two on A, inducing the group G of symplectic automorphisms on $Km(A)$. We may assume that the quotient map $A \to A/G_{2,2,2}$ corresponds to a multiplication by two on the first three real coordinates of A.
In this case one obtains:

$$P_G = \langle \omega_{1,2}, \omega_{3,4}, \omega_{1,3}, \omega_{2,4}, \omega_{1,4}, \omega_{2,3}, K_0, K_{0,0,0,1} \rangle\,,$$

$$\begin{array}{ll} \theta^*(P^{\vee}_{(\mathbb{Z}/2\mathbb{Z})^3}) & = \langle\ 2\omega_{1,2}, \omega_{3,4}, 2\omega_{1,3}, \omega_{2,4}, \omega_{1,4}, 2\omega_{2,3}, \bar{K}_{W_4}, \bar{K}_{(\mathbb{Z}/2\mathbb{Z})^4 - W_4}\ \rangle \\ & \simeq U(4) \oplus U(4) \oplus U(4) \oplus \langle -4 \rangle \oplus \langle -4 \rangle, \end{array}$$

$$\begin{array}{ll} \Lambda_{K3}^{(\mathbb{Z}/2\mathbb{Z})^3} & = \langle\ \omega_{1,2}, \omega_{3,4}, \omega_{1,3}, \omega_{2,4}, \omega_{1,4}, \omega_{2,3}, \bar{K}_{W_4}, \bar{K}_{(\mathbb{Z}/2\mathbb{Z})^4 - W_4}\ \rangle \\ & \simeq U(2) \oplus U(2) \oplus U(2) \oplus \langle -4 \rangle \oplus \langle -4 \rangle. \end{array}$$

Comparing $\Lambda_{K3}^{(\mathbb{Z}/2\mathbb{Z})^3}$ and $\theta^*(P^{\vee}_{(\mathbb{Z}/2\mathbb{Z})^3})$, one obtains $[\Lambda_{K3}^{(\mathbb{Z}/2\mathbb{Z})^3} : \theta^*(P^{\vee}_{(\mathbb{Z}/2\mathbb{Z})^3})] = 2^3$.

3.6. $G = \mathbb{Z}/2\mathbb{Z}$**.**
In this case G is generated by a symplectic involution (which is induced by the translation by a point of order two). In Remark 2.3 we proved that in this case the map $H^2(Y', \mathbb{Z}) \to H^2(X, \mathbb{Z})^G$ is surjective, and then $\Lambda_{K3}^{\mathbb{Z}/2\mathbb{Z}} = \theta^*(P^{\vee}_{\mathbb{Z}/2\mathbb{Z}})$. This result can also be obtained with the method used above. In fact let G_2 be the translation by a point of order two on A such that the map $A \to A/G_2$ corresponds to a multiplication by two on the first real coordinate of A. Let G be the group generated by the involution induced by G_2 on $Km(A)$. Then

$$\begin{array}{lcl} P_{\mathbb{Z}/2\mathbb{Z}} & = & \langle \bar{K}_{V_{3,4}} + \omega_{1,2}/2, \omega_{3,4}, \bar{K}_{V_{2,4}} + \omega_{1,3}/2, \omega_{2,4}, \bar{K}_{V_{2,3}} + \omega_{1,4}/2, \omega_{2,3}, \bar{K}_{W_1}, \\ & & K_{0,0,0,1}, K_{0,0,1,0}, K_{0,1,0,0}, K_{0,0,1,1}, K_{0,1,0,1}, K_{0,1,1,0}, K_{0,1,1,1} \rangle, \end{array}$$

$$\begin{array}{lcl} \theta^*(P^{\vee}_{\mathbb{Z}/2\mathbb{Z}}) & = & \langle \omega_{1,2}, \omega_{3,4}, \omega_{1,3}, \omega_{2,4}, \omega_{1,4}, \omega_{2,3}, \bar{K}_{W_2}, \bar{K}_{W_3}, \bar{K}_{W_4}, \hat{K}, K_{0,0,1,1} + K_{1,0,1,1}, \\ & & K_{0,1,0,1} + K_{1,1,0,1}, K_{0,1,1,0} + K_{1,1,1,0}, K_{0,1,1,1} + K_{1,1,1,1} \rangle, \end{array}$$

$$\begin{array}{lcl} \Lambda_{K3}^{\mathbb{Z}/2\mathbb{Z}} & = & \langle \omega_{1,2}, \omega_{3,4}, \omega_{1,3}, \omega_{2,4}, \omega_{1,4}, \omega_{2,3}, \bar{K}_{W_2}, \bar{K}_{W_3}, \bar{K}_{W_4}, \hat{K}, K_{0,0,1,1} + K_{1,0,1,1}, \\ & & K_{0,1,0,1} + K_{1,1,0,1}, K_{0,1,1,0} + K_{1,1,1,0}, K_{0,1,1,1} + K_{1,1,1,1} \rangle. \end{array}$$

The last two lattices are the same, as proved before.

Remark 3.14. We observe that for each group $G = (\mathbb{Z}/2\mathbb{Z})^i$, $i = 2, 3, 4$ the lattice $\theta^*(P^{\vee}_G)$ has a finite index in Λ^G_{K3} and $\Lambda^G_{K3}/\theta^*(P^{\vee}_G) \simeq V_G$, where V_G is the group described in Proposition 2.4. □

Remark 3.15. In [GS2] the discriminant of the lattice Λ^G_{K3} is computed for each finite abelian group acting symplectically on a K3 surface. For each non cyclic group it is different from the one computed by Nikulin. Comparing the two discriminant we always obtain that the index of the inclusion of the lattice computed by Nikulin in our lattice is exactly the order of V_G, with V_G as in Proposition 2.4. □

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