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Almost Ricci–Bourguignon Solitons on Doubly Warped Product Manifolds

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Abstract: This study aims at examining the effects of an almost Ricci–Bourguignon soliton structure on the base and fiber factor manifolds of a doubly warped product manifold. First, a number of preconditions and sufficiency criteria for an almost Ricci–Bourguignon soliton doubly warped product are addressed. Additionally, an almost Ricci–Bourguignon soliton on doubly warped product manifolds admitting a conformal vector field is taken into consideration. Finally, how the almost Ricci–Bourguignon soliton behaves in doubly warped product space–times is examined.

Keywords: Einstein manifolds; Einstein soliton; almost Ricci–Bourguignon soliton; doubly warped product manifolds

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1. An Introduction

The Ricci flow treatment depends on the Ricci soliton. The evolution equation for metrics $\{h(t)\}$ of the Ricci flow on a Riemannian manifold (E, h) is of the form

$$\partial_t h(t) = -2\mathrm{Rc}, \tag{1}$$

where Rc is the tensor of the Ricci curvature [1,2]. A Ricci soliton refers to manifolds that allow for such a structure [3], where

$$\mathrm{Rc} + \frac{1}{2}\mathcal{L}_\xi h = \lambda h, \tag{2}$$

$\mathcal{L}_\xi$ stands for the Lie derivative in the direction of ξ on E, and λ is a constant. A Ricci soliton is a generalization of an Einstein manifold, where the Ricci tensor is proportional to the metric tensor. Hamilton initially concentrated on the study of Ricci solitons as fixed points of the Ricci flow in the space of metrics on E modulo diffeomorphisms and scaling [4]. If $\lambda > 0$ ($\lambda = 0$, or $\lambda < 0$), the Ricci soliton is considered shrinking (steady or expanding, respectively). The Ricci soliton is called a trivial Ricci soliton, if $\xi = 0$ or is Killing. If the Lie derivative of the metric tensor vanishes, a Ricci soliton is considered trivial, and the soliton constant changes to an Einstein constant, changing the metric g into an Einstein metric. The Ricci soliton is referred to as a gradient, ξ is known as the potential vector field, and f is known as the potential function if $\xi = \nabla f$. In this instance, Equation (2) becomes

$$\mathrm{Rc} + H^f = \lambda h, \tag{3}$$

where H^f represents the Hessian tensor. In the past, Ricci solitons have been thoroughly investigated for various purposes and in certain contexts [5–11]. It is demonstrated that a complete Ricci soliton is gradient in [12]. Basic generalizations of Einstein manifolds are gradient Ricci solitons [13]. Ricci solitons, which are self-similar solutions of the Ricci flow, are extremely important in differential geometry. Because Ricci solitons are a generalization of Einstein manifolds, acquiring conditions for a Ricci soliton is critical. Ricci solitons were used to answer the Poincare hypothesis, which has been contested for over a century. A compact soliton is always a gradient Ricci soliton. Ricci solitons' geometry is a highly desirable field due to its applications in a variety of fields, in addition to its attractive geometry. Myers-type theorems for Ricci solitons were established in [14]. A completely shrinking Ricci soliton was shown to have a finite fundamental group. Volume comparison theorems of the Bishop type were derived for non-compact shrinking Ricci solitons in [15]. Recently, some authors have discovered descriptions of trivial Ricci solitons. Characterizing such trivial solitons is a crucial difficulty in the geometry of Ricci solitons. For instance, there are several descriptions of a trivial Ricci soliton in [16]. It was demonstrated that a trivial Ricci soliton can be identified by a potential field of constant length satisfying an inequality. In both compact and noncompact examples, Deshmukh and Alsodaishey discovered necessary and sufficient criteria for Ricci solitons to be trivial [17]. They had a smooth function f, defined as half the squared length of the potential vector field, which they referred to as the Ricci soliton's energy function. They discovered that the energy function of the Ricci soliton is critical in characterizing a trivial Ricci soliton. By placing various limits on the energy function, they discovered three characterizations of connected trivial Ricci solitons. They demonstrated that the Laplacian Δf of the energy function, which is constrained by some geometrical number, yields a description of a connected trivial Ricci soliton on a connected Ricci soliton. Additionally, they demonstrated that for a specific Ricci soliton, the energy function is superharmonic, and the scalar curvature is constant along integral curves of the potential field ζ. Additionally, they noticed that a connected trivial Ricci soliton can be identified by the Ricci operator's invariance under the local flow of the potential vector field. Finally, they demonstrated that a compact Ricci soliton's energy function f is the solution to a Poisson equation if and only if the Ricci soliton is trivial. A study of a generalized soliton on a Riemannian manifold was conducted by the authors in [18], who characterized the Euclidean space and discovered a sufficient condition under which it reduced to a quasi-Einstein manifold in the compact case. They discovered a sufficient condition for its reduction to a quasi-Einstein manifold, as well as a set of prerequisites for the reduction from a compact generalized soliton to an Einstein manifold. Note that this topic is connected to the symmetry in the geometry of Riemannian manifolds since Ricci solitons are self-similar solutions of the heat flow. Furthermore, because they are generalizations of Ricci solitons, generalized solitons naturally relate to symmetry. The study of Ricci solitons has two facets: one examines how the Riemannian manifold's Ricci soliton structure affects its topology, and the other examines how it affects its geometry. A Ricci soliton on a Riemannian manifold is said to have a concurrent potential field if its potential field is a concurrent vector field. Recent research has focused on Ricci solitons formed by concurrent vector fields on Riemannian manifolds. The position vector field on Euclidean submanifolds is the most significant concurrent vector field. Some authors comprehensively categorize the Ricci solitons on Euclidean hypersurfaces that result from the hypersurfaces' position vector fields. The authors' goal in [19] was to offer some necessary conditions for the triviality of a generalized Ricci soliton on a Riemannian manifold. If a generalized Ricci soliton's vector field is a generalized geodesic or a 2-Killing vector field, the soliton is trivial. Important findings on Ricci solitons, which naturally exist on Riemannian submanifolds, were presented by Chen's survey in [20].

We say that (E, h) is a nearly Ricci soliton manifold if λ is a smooth function [21–23]. The Ricci–Bourguignon flows have been taken into consideration to derive a generalization of the Einstein soliton [24–26]:

$$\partial_t h(t) = -2(\mathrm{Rc} - \rho R h). \tag{4}$$

These manifolds are called almost Ricci–Bourguignon solitons and are defined as follows. Assume that (E, h) is a pseudo-Riemannian manifold, and let $\lambda, \rho \in \mathbb{R}$, $\rho \neq 0$, and $\xi \in \mathfrak{X}(E)$. Then, (E, h, ξ, λ) is called an almost Ricci–Bourguignon soliton if

$$\mathrm{Rc} + \frac{1}{2}\mathcal{L}_{\xi}h = \lambda h + \rho R h. \quad (5)$$

An almost Ricci–Bourguignon soliton (E, h, ξ, ρ) is gradient and denoted by (E, h, ϕ, ρ), if $\xi = \nabla\phi$. In this case, Equation (5) becomes

$$\mathrm{Rc} + H^{\phi} = \lambda h + \rho R h. \quad (6)$$

The almost Ricci–Bourguignon soliton is categorized as steady, shrinking, or expanding according to whether λ has zero, positive, or negative values, respectively. The function f is called the almost Ricci–Bourguignon potential of the gradient almost Ricci–Bourguignon soliton. Later, this idea was propagated in a variety of ways [27–31]. Recently, in [27], Dwivedi illustrated more gradient Ricci–Bourguignon soliton isometric theories. For Ricci–Bourguignon solitons and almost solitons with concurrent potential vector field, Soylu provided classification theorems in [32]. In [33], Ghosh investigated and demonstrated various triviality results for Ricci–Bourguignon solitons and almost Ricci–Bourguignon solitons on a Riemannian manifold.

As far as we are aware, there has not been any research on a structure for doubly warped product manifolds. From the perspective of doubly warped product manifolds (DWPMs), the research issues can be divided into two categories in this regard:

1. What circumstances lead a doubly warped product manifold to become a Ricci–Bourguignon soliton?
2. What are the inheritable properties by a factor of the Ricci–Bourguignon soliton doubly warped product manifold?

In order to solve these issues, we examined the necessary and sufficient conditions on a doubly warped product manifold that has factors that are almost Ricci–Bourguignon soliton. On a doubly warped product manifold that admits a conformal vector field, we also explored the almost Ricci–Bourguignon soliton. Our findings were then applied to doubly warped product space–times.

2. Doubly Warped Product Manifolds

Warped product manifolds have substantial implications in both mathematics and physics. This notion, which was initially investigated as a way to simulate manifolds with negative curvature, has sparked tremendous scholarly interest. The factor manifolds of a warped product manifold are technically referred to as the base manifold and the fiber manifold.

When the warping function that governs the behavior of the warped product manifold remains constant, the resulting warped product manifold is called a Riemannian product manifold or a Cartesian product manifold. It is worth noting that relativistic space–time configurations can appear as Lorentzian warped product manifolds, wherein one of the factor manifolds is an open interval and the second factor is a Riemannian manifold.

A particular subclass of these structures, known as generalized Robertson–Walker space–times, has a warped product arrangement in which the base manifold is an open interval, and the fiber manifold is a Riemannian manifold. Given their immense applicability, it is critical to emphasize the importance of these generalized Robertson–Walker space–times.

A novel discovery revealed that a Lorentzian manifold can be classified as a generalized Robertson–Walker space–time if it admits a concircular vector field. This description has promoted great differential geometric interest and relativistic applications, serving as a focal point of scholarly endeavors throughout the previous two decades. The second warped product space–time is the standard static space–time. In such warped space–times, the fiber manifold is an open connected interval, whereas the base manifold is a Riemannian manifold.

The geometry of warped product manifolds is certainly connected to the geometry of the base manifold and the geometry of the fiber manifold. From a mathematical standpoint, the lifts of all tensors on the factor manifolds to the warped product manifold are studied in relation to the corresponding tensors on the warped product manifold. These relations constitute a pivotal focus. There are many fascinating outcomes from the studies of such relationships in the literature. The connections, Riemann curvature tensor, Ricci curvature tensor, and scalar curvature of the warped product manifold are given in terms of the connections, Riemann curvature tensors, Ricci curvature tensors, and scalar curvatures of the factor manifolds. After years of working on warped product manifolds, Chen introduced the first book on warped product manifolds. This book gathers different structures on warped product manifolds, such as Kahlerian warped product manifolds.

Einstein manifolds are distinguished by a fundamental characteristic: the proportionality between the Ricci tensor and the metric tensor. This defining property imbues these geometric spaces with remarkable symmetry features. Within Einstein manifolds, a particularly open question has been posed by the eminent mathematician Besse. This question revolves around the existence of nontrivial Einstein warped product manifolds, adding a layer of complexity to the exploration of such geometric structures. Surprisingly, despite the extensive exploration of this question, the survey of the literature is full of negative partial answers to Besse's inquiry. So far, neither a positive nor a completely negative response has been provided.

A doubly warped product manifold $(DWP)_n$ is the (pseudo-)Riemannian product manifold $E = E_1 \times E_2$ of two (pseudo-)Riemannian manifolds (E_i, h_i, D_i), $i = 1, 2$, furnished with the metric tensor

$$h = (f_2 \circ \pi_2)^2 \pi_1^*(h_1) \oplus (f_1 \circ \pi_1)^2 \pi_2^*(h_2),$$

where the functions $f_i : E_i \to (0, \infty)$, $i = 1, 2$ are the warping functions of the doubly warped product $E =_{f_2} E_1 \times_{f_1} E_2$ [34–38]. The maps $\pi_i : E_1 \times E_2 \to E_i$ are the natural projections of E onto E_i whereas * denotes the pull-back operator on the tensors. In particular, if, for example, $f_2 = 1$, then $E = E_1 \times_f E_2$ is called a singly warped product manifold. Several noteworthy investigations have focused on warped product manifolds. For instance, Gebarowski investigated the divergence-free and conformally recurrent doubly warped products in [39,40]. Lorentzian doubly warped product manifolds were examined by Beem and Powell [41].

The Levi–Civita connection D on $E =_{f_2} E_1 \times_{f_1} E_2$ is given by

$$\begin{aligned} D_{\varsigma_i}\varsigma_j &= \varsigma_i(\ln f_i)\varsigma_j + \varsigma_j(\ln f_j)\varsigma_i, \\ D_{\varsigma_i}\eta_i &= D^i_{\varsigma_i}\eta_i - \frac{f_j^2}{f_i^2} h_i(\varsigma_i, \eta_i)\nabla^j(\ln f_j), \end{aligned}$$

where $i \neq j$, and $X_i, Y_i \in \mathfrak{X}(E_i)$. Then, the Ricci curvature tensor Rc on D is given by

$$\begin{aligned} \mathrm{Rc}(\varsigma_i, \eta_i) &= \mathrm{Rc}^i(\varsigma_i, \eta_i) - \frac{n_j}{f_i} H^{f_i}(\varsigma_i, \eta_i) - \frac{f_j^\diamond}{f_i^2} h_i(\varsigma_i, \eta_i), \\ \mathrm{Rc}(\varsigma_i, \eta_j) &= (n-2)\varsigma_i(\ln f_i)\eta_j(\ln f_j), \end{aligned}$$

where $f_i^\diamond = f_i \triangle^i f_i + (n_j - 1)h_i(\nabla^i f_i, \nabla^i f_i)$, $i \neq j$ and $\varsigma_i, \eta_i \in \mathfrak{X}(E_i)$.

Lemma 1 ([42]). *In a $(DWP)_n$ manifold $E =_{f_2} E_1 \times_{f_1} E_2$, the Lie derivative with respect to a vector field $\xi = \xi_1 + \xi_2$ satisfies*

$$\mathcal{L}_{\xi}h(\varsigma,\eta) = f_2^2\left(\mathcal{L}^1_{\xi_1}h_1\right)(\varsigma_1,\eta_1)+f_1^2\left(\mathcal{L}^2_{\xi_2}h_2\right)(\varsigma_2,\eta_2)+2f_1\xi_1(f_1)h_2(\varsigma_2,\eta_2) +2f_2\xi_2(f_2)h_1(\varsigma_1,\eta_1), \tag{7}$$

for any vector fields $\varsigma=\varsigma_1+\varsigma_2, \eta=\eta_1+\eta_2$, where $\mathcal{L}^i_{\xi_i}$ is the Lie derivative on E_i with respect to ξ_i, for $i=1,2$.

3. $(DWP)_n$ Manifolds Admitting an Almost Ricci–Bourguignon Soliton Structure

For the rest of this work, let $E={}_{f_2}E_1\times_{f_1}E_2$ be a $(DWP)_n$ manifold with warping functions f_i for factor manifolds E_i, and let $h=f_2^2h_1\oplus f_1^2h_2$. Also, let $\xi=\xi_1+\xi_2$ be a vector field on E.

Theorem 1. *Let (E,h,ξ,λ,ρ) be an almost Ricci–Bourguignon soliton, where $E={}_{f_2}E_1\times_{f_1}E_2$ is a $(DWP)_n$ manifold. Then, $\left(E_i,h_i,f_j^2\xi_i,\lambda_i,\rho_i\right)$ is an almost Ricci–Bourguignon soliton if $H^{f_i}=\psi_ih_i$, where*

$$\rho_iR_i+\lambda_i=\lambda f_i^2+\frac{n_j}{f_i}\psi_i+\frac{f_j^\diamond}{f_i^2}-f_j\xi_j(f_2)+\rho Rf_i^2,\ \text{whenever } i,j=1,2, \text{ and } i\neq j.$$

Moreover, $E={}_{f_2}E_1\times_{f_1}E_2$ reduces to a singly warped product manifold.

Proof. Let (E,h,ξ,λ,ρ) be an almost Ricci–Bourguignon soliton, that is,

$$\mathrm{Rc}(\varsigma,\eta)+\frac{1}{2}(\mathcal{L}_{\xi}h)(\varsigma,\eta)=\lambda h(\varsigma,\eta)+\rho Rh(\varsigma,\eta). \tag{8}$$

Thus, for any vector fields $\varsigma=\varsigma_1+\varsigma_2, \eta=\eta_1+\eta_2$, and $\xi=\xi_1+\xi_2$ on a $(DWP)_n$ manifold $E={}_{f_2}E_1\times_{f_1}E_2$, Lemma 1 implies that

$$\begin{aligned}
&\mathrm{Rc}^1(\varsigma_1,\eta_1)-\frac{n_2}{f_1}H^{f_1}(\varsigma_1,\eta_1)-\frac{f_2^\diamond}{f_1^2}h_1(\varsigma_1,\eta_1)+\mathrm{Rc}^2(\varsigma_2,\eta_2) \\
&-\frac{n_1}{f_2}H^{f_2}(\varsigma_2,\eta_2)-\frac{f_1^\diamond}{f_2^2}h_2(\varsigma_2,\eta_2)+(n-2)\varsigma_1(\ln f_1)\eta_2(\ln f_2) \\
&+(n-2)\varsigma_2(\ln f_2)\eta_1(\ln f_1)+\frac{1}{2}f_2^2\left(\mathcal{L}^1_{\xi_1}h_1\right)(\varsigma_1,\eta_1)+\frac{1}{2}f_1^2\left(\mathcal{L}^2_{\xi_2}h_2\right)(\varsigma_2,\eta_2) \\
&+f_1\xi_1(f_1)h_2(\varsigma_2,\eta_2)+f_2\xi_2(f_2)h_1(\varsigma_1,\eta_1) \\
=\ &\lambda f_1^2h_1(\varsigma_1,\eta_1)+\lambda f_2^2h_2(\varsigma_2,\eta_2)+\rho Rf_1^2h_1(\varsigma_1,\eta_1)+\rho Rf_2^2h_2(\varsigma_2,\eta_2).
\end{aligned} \tag{9}$$

Let $\varsigma=\varsigma_1$, $\eta=\eta_1$, and $H^{f_1}=\psi_1h_1$; then,

$$\begin{aligned}
&\mathrm{Rc}^1(\varsigma_1,\eta_1)+\frac{1}{2}f_2^2\left(\mathcal{L}^1_{\xi_1}h_1\right)(\varsigma_1,\eta_1) \\
=\ &\left[\lambda f_1^2+\frac{n_2}{f_1}\psi_1+\frac{f_2^\diamond}{f_1^2}-f_2\xi_2(f_2)+\rho Rf_1^2\right]h_1(\varsigma_1,\eta_1) \\
=\ &\lambda_1h_1(\varsigma_1,\eta_1)+\left[-\lambda_1+\lambda f_1^2+\frac{n_2}{f_1}\psi_1+\frac{f_2^\diamond}{f_1^2}-f_2\xi_2(f_2)+\rho Rf_1^2\right]h_1(\varsigma_1,\eta_1) \\
=\ &\lambda_1h_1(\varsigma_1,\eta_1)+\rho_1R_1h_1(\varsigma_1,\eta_1).
\end{aligned}$$

Let us define λ_1 and ρ_1 such that

$$\begin{aligned}
&\mathrm{Rc}^1(\varsigma_1,\eta_1)+\frac{1}{2}f_2^2\left(\mathcal{L}^1_{\xi_1}h_1\right)(\varsigma_1,\eta_1)\\
=\;&\lambda_1 h_1(\varsigma_1,\eta_1)+\left[-\lambda_1+\lambda f_1^2+\frac{n_2}{f_1}\psi_1+\frac{f_2^\diamond}{f_1^2}-f_2\xi_2(f_2)+\rho R f_1^2\right]h_1(\varsigma_1,\eta_1) \qquad (10)\\
=\;&\lambda_1 h_1(\varsigma_1,\eta_1)+\rho_1 R_1 h_1(\varsigma_1,\eta_1).
\end{aligned}$$

Then, $\left(E_1,h_1,f_2^2\xi_1,\lambda_1,\rho_1\right)$ is an almost Ricci–Bourguignon soliton, where

$$\rho_1 R_1+\lambda_1=\lambda f_1^2+\frac{n_2}{f_1}\psi_1+\frac{f_2^\diamond}{f_1^2}-f_2\xi_2(f_2)+\rho R f_1^2.$$

Similarly, one may obtain

$$\begin{aligned}
&\mathrm{Rc}^2(\varsigma_2,\eta_2)+\frac{1}{2}f_1^2\left(\mathcal{L}^2_{\xi_2}h_2\right)(\varsigma_2,\eta_2)\\
=\;&\lambda_2 h_2(\varsigma_2,\eta_2)+\left[-\lambda_2+\lambda f^2+\frac{n_1}{f_2}\psi_2+\frac{f_1^\diamond}{f_2^2}-f_1\xi_1(f_1)+\rho R f^2\right]h_2(\varsigma_2,\eta_2) \qquad (11)\\
=\;&\lambda_2 h_2(\varsigma_2,\eta_2)+\rho_2 R_2 h_2(\varsigma_2,\eta_2).
\end{aligned}$$

Thus, $\left(E_2,h_2,f_1^2\xi_2,\lambda_2,\rho_2\right)$ is an almost Ricci–Bourguignon soliton, where

$$\rho_2 R_2+\lambda_2=\rho R f_2^2+\lambda f_2^2+\frac{n_1}{f_2}\psi_2+\frac{f_1^\diamond}{f_2^2}-f_1\xi_1(f_1).$$

Finally, let $\varsigma=\varsigma_1$, $\eta=\eta_2$; then,

$$\begin{aligned}
\mathrm{Rc}(\varsigma_1,\eta_2)&=(n-2)\varsigma_1(\ln f_1)\eta_2(\ln f_2)\\
\left(\mathcal{L}_\xi h\right)(\varsigma_1,\eta_2)&=h(\varsigma_1,\eta_2)=0.
\end{aligned}$$

These equations with the defining equation of an almost Ricci–Bourguignon soliton infer

$$(n-2)\varsigma_1(\ln f_1)\eta_2(\ln f_2)=0.$$

Therefore, one of the warping functions is constant; that is, $E={}_{f_2}E_1\times_{f_1}E_2$ is a singly warped product manifold. □

A vector field ξ is concircular with factor φ if

$$D_\varsigma\xi=\varphi\varsigma$$

for any vector field ς [43]. Now, assume that the vector fields $\nabla^1 f_1$ and $\nabla^2 f_2$ are a concircular vector field with factor ψ_1 and ψ_2, respectively. Then, for any vector field $\varsigma_i\in\mathfrak{X}(E_i)$,

$$D_{\varsigma_i}\left(\nabla^i f_i\right)=\psi_i\varsigma_i.$$

This leads us to the following conclusion:

$$\begin{aligned}
h_i\left(D_{\varsigma_i}\left(\nabla^i f_i\right),\eta_i\right)&=\psi_i h_i(\varsigma_i,\eta_i)\\
H^{f_i}(\varsigma_i,\eta_i)&=\psi_i h_i(\varsigma_i,\eta_i).
\end{aligned}$$

Corollary 1. *In an almost Ricci–Bourguignon soliton (E,h,ξ,λ,ρ), where $E={}_{f_2}E_1\times_{f_1}E_2$ is a $(DWP)_n$ manifold, assume that $\nabla^1 f_1$ and $\nabla^2 f_2$ are concircular vector fields with factor ψ_1,ψ_2, respectively; then, $\left(E_i,h_i,f_j^2\xi_i,\lambda_i,\rho_i\right)$ is an almost Ricci–Bourguignon soliton, where*

$$\rho_i R_i + \lambda_i = \lambda f_i^2 + \frac{n_j}{f_i}\psi_i + \frac{f_j^\diamond}{f_i^2} - f_j\xi_j(f_j) + \rho R f_i^2.$$

Theorem 2. *Let $(E, h, \xi, \lambda, \rho)$ be an almost Ricci–Bourguignon soliton, where $E =_{f_2} E_1 \times_{f_1} E_2$ is a $(DWP)_n$ manifold. Assume that $H^{f_i} = \psi_i h_i$, then ξ_i is a conformal vector field on E_i, if and only if (E_i, h_i) is an Einstein manifold.*

Proof. Let (E_1, h_1) be an Einstein manifold with factor μ_1, and let $H^{f_1} = \psi_1 h_1$. The use of Equation (10) infers

$$\begin{aligned} &\mu_1 h_1(\varsigma_1, \eta_1) + \frac{1}{2} f_2^2 \left(\mathcal{L}^1_{\xi_1} h_1\right)(\varsigma_1, \eta_1) \\ = &\left[\lambda f_1^2 + \frac{n_2}{f_1}\psi_1 + \frac{f_2^\diamond}{f_1^2} - f_2\xi_2(f_2) + \rho R f_1^2\right] h_1(\varsigma_1, \eta_1). \end{aligned}$$

Thus,

$$\left(\mathcal{L}^1_{\xi_1} h_1\right)(\varsigma_1, \eta_1) = \frac{2}{f_2^2}\left[\frac{f_2^\diamond}{f_1^2} + \frac{n_2}{f_1}\psi_1 - \mu_1 - f_2\xi_2(f_2) + \lambda f_1^2 + \rho R f_1^2\right] h_1(\varsigma_1, \eta_1). \tag{12}$$

That is, ξ_1 is a conformal vector field on E_1. Now, let ξ_1 be a conformal vector field on E_1 with factor $2\tau_1$; then,

$$\begin{aligned} &\mathrm{Rc}^1(\varsigma_1, \eta_1) + f_2^2 \tau_1 h_1(\varsigma_1, \eta_1) \\ = &\left[\lambda f_1^2 + \frac{n_2}{f_1}\psi_1 + \frac{f_2^\diamond}{f_1^2} - f_2\xi_2(f_2) + \rho R f_1^2\right] h_1(\varsigma_1, \eta_1). \end{aligned}$$

Consequently,

$$\mathrm{Rc}^1(\varsigma_1, \eta_1) = \left[(\lambda + \rho R) f_1^2 + \frac{n_2}{f_1}\psi_1 + \frac{f_2^\diamond}{f_1^2} - f_2\xi_2(f_2) - f_2^2\tau_1\right] h_1(\varsigma_1, \eta_1). \tag{13}$$

Thus, (E_1, h_1) is an Einstein manifold with factor

$$\mu_1 = (\lambda + \rho R) f_1^2 + \frac{n_2}{f_1}\psi_1 + \frac{f_2^\diamond}{f_1^2} - f_2\xi_2(f_2) - f_2^2\tau_1.$$

This completes the proof. □

A contraction of Equation (13) gives

$$R_1 = \left[(\lambda + \rho R) f_1^2 + \frac{n_2}{f_1}\psi_1 + \frac{f_2^\diamond}{f_1^2} - f_2\xi_2(f_2) - f_2^2\tau_1\right] n_1.$$

This leads us to the following simple corollary.

Corollary 2. *Let $(E, h, \xi, \lambda, \rho)$ be an almost Ricci–Bourguignon soliton, where $E =_{f_2} E_1 \times_{f_1} E_2$ is a $(DWP)_n$ manifold. Assume that $H^{f_i} = \psi_i h_i$, and ξ_i is a conformal vector field on E_i with factor $2\tau_i$. Then, the scalar curvature R_i of E_i is given by*

$$R_i = \left[(\lambda + \rho R) f_i^2 + \frac{n_j}{f_i}\psi_i + \frac{f_j^\diamond}{f_i^2} - f_j\xi_j(f_j) - f_j^2\tau_i\right] n_i.$$

Theorem 3. *Let $(E, h, \xi, \lambda, \rho)$ be an almost Ricci–Bourguignon soliton, where $E =_{f_2} E_1 \times_{f_1} E_2$ is a $(DWP)_n$ manifold admitting a CVF $\xi = \xi_1 + \xi_2$; then, (E_i, h_i) is an Einstein manifold if $H^{f_i} = \psi_i h_i$.*

Proof. Assume that ξ is a conformal vector field on E, i.e., $\mathcal{L}_\xi h = 2\omega h$ for some scalar function ω; then,

$$\mathrm{Rc}(\varsigma, \eta) = (\lambda - \omega + \rho R) h(\varsigma, \eta). \tag{14}$$

Then,

$$\mathrm{Rc}^1(\varsigma_1, \eta_1) = \left[(\lambda - \omega + \rho R) f_2^2 + \frac{f_2^\diamond}{f_1^2}\right] h_1(\varsigma_1, \eta_1) + \frac{n_2}{f_1} H^{f_1}(\varsigma_1, \eta_1), \tag{15}$$

$$\mathrm{Rc}^2(\varsigma_2, \eta_2) = \left[(\lambda - \omega + \rho R) f_1^2 + \frac{f_1^\diamond}{f_2^2}\right] h_2(\varsigma_2, \eta_2) + \frac{n_1}{f_2} H^{f_2}(\varsigma_2, \eta_2). \tag{16}$$

Now, let $H^{f_1} = \psi_1 h_1$ and $H^{f_2} = \psi_2 h_2$. The above equations become

$$\mathrm{Rc}^1(\varsigma_1, \eta_1) = \left[(\lambda - \omega + \rho R) f_2^2 + \frac{n_2}{f_1}\psi_1 + \frac{f_2^\diamond}{f_1^2}\right] h_1(\varsigma_1, \eta_1), \tag{17}$$

$$\mathrm{Rc}^2(\varsigma_2, \eta_2) = \left[(\lambda - \omega + \rho R) f_1^2 + \frac{n_1}{f_2}\psi_2 + \frac{f_1^\diamond}{f_2^2}\right] h_2(\varsigma_2, \eta_2). \tag{18}$$

That is, both the base and fiber manifolds are Einstein manifolds. □

Corollary 3. *Let $(E, h, \xi, \lambda, \rho)$ be an almost Ricci–Bourguignon soliton, where $E =_{f_2} E_1 \times_{f_1} E_2$ is a $(DWP)_n$ manifold admitting a CVF $\xi = \xi_1 + \xi_2$; then, (E_i, h_i) is an Einstein manifold, if $\nabla^i f_i$ is a concircular vector field.*

Theorem 4. *Let $(E, h, \xi, \lambda, \rho)$ be an almost Ricci–Bourguignon soliton, where $E =_{f_2} E_1 \times_{f_1} E_2$ is a $(DWP)_n$ manifold admitting a CVF $\xi = \xi_1 + \xi_2$, and assume that $H^{f_i} = \psi_i h_i$; then,*

$$\mathcal{L}^i_{\zeta_i} \mathrm{Rc}^i(\varsigma_i, \eta_i) = \varphi_i h_i(\varsigma_i, \eta_i),$$

where

$$\varphi_i = \left[(\lambda - \omega + \rho R) f_j^2 + \frac{n_j}{f_i}\psi_i + \frac{f_j^\diamond}{f_i^2}\right](\omega_i + \zeta_i).$$

Proof. From Lemma 1, it is clear that ζ_1, ζ_2 are CVFs on E_1, E_2 with conformal factors ω_1, ω_2, respectively. Then, by employing Equations (17) and (18) we obtain

$$\begin{aligned}
\mathcal{L}^1_{\zeta_1} \mathrm{Rc}^1(\varsigma_1, \eta_1) &= \left[(\lambda - \omega + \rho R) f_2^2 + \frac{n_2}{f_1}\psi_1 + \frac{f_2^\diamond}{f_1^2}\right] \mathcal{L}^1_{\zeta_1} h_1(\varsigma_1, \eta_1) \\
&\quad + \zeta_1 \left[(\lambda - \omega + \rho R) f_2^2 + \frac{n_2}{f_1}\psi_1 + \frac{f_2^\diamond}{f_1^2}\right] h_1(\varsigma_1, \eta_1). \\
\mathcal{L}^1_{\zeta_1} \mathrm{Rc}^1(\varsigma_1, \eta_1) &= \left[\left((\lambda - \omega + \rho R) f_2^2 + \frac{n_2}{f_1}\psi_1 + \frac{f_2^\diamond}{f_1^2}\right)(\omega_1 + \zeta_1)\right] h_1(\varsigma_1, \eta_1) \\
&= \varphi_1 h_1(\varsigma_1, \eta_1),
\end{aligned}$$

where

$$\varphi_1 = \left((\lambda - \omega + \rho R) f_2^2 + \frac{n_2}{f_1}\psi_1 + \frac{f_2^\diamond}{f_1^2}\right)(\omega_1 + \zeta_1).$$

Similarly,

$$\begin{aligned}\mathcal{L}^2_{\zeta_2}\mathrm{Rc}^2(\varsigma_2,\eta_2) &= \left[\left((\lambda-\omega+\rho R)f_1^2+\frac{n_1}{f_2}\psi_2+\frac{f_1^\diamond}{f_2^2}\right)(\omega_2+\zeta_2)\right]h_2(\varsigma_2,\eta_2)\\ &= \varphi_2 h_2(\varsigma_2,\eta_2),\end{aligned}$$

where

$$\varphi_2=\left((\lambda-\omega+\rho R)f_1^2+\frac{n_1}{f_2}\psi_2+\frac{f_1^\diamond}{f_2^2}\right)(\omega_2+\zeta_2).$$

This completes the proof. □

4. An Almost Ricci–Bourguignon Soliton on $(DWST)_n$

Let (E,h) be a Riemannian manifold and $f:E\to(0,\infty)$ and $\sigma:I\to(0,\infty)$ be two smooth functions. The doubly warped product manifold $\bar{E}={}_f I\times_\sigma E$ furnished with the metric tensor $\bar{h}=-f^2dt^2\oplus\sigma^2 h$ is called a doubly warped space–time $(DWST)_n$. Then, the Ricci curvature tensor $\bar{\mathrm{R}}\mathrm{c}$ on $\bar{E}$ is

$$\begin{aligned}\bar{\mathrm{R}}\mathrm{c}(\partial_t,\partial_t) &= \frac{n}{\sigma}\ddot{\sigma}+\frac{f^\diamond}{\sigma^2},\\ \bar{\mathrm{R}}\mathrm{c}(\varsigma,\eta) &= \mathrm{Rc}(\varsigma,\eta)-\frac{1}{f}H^f(\varsigma,\eta)-\frac{\sigma^\diamond}{f^2}h(\varsigma,\eta),\\ \bar{\mathrm{R}}\mathrm{c}(\partial_t,\varsigma) &= (n-1)\frac{\dot{\sigma}}{\sigma}\varsigma(\ln f).\end{aligned}$$

E is a GRW space–time if f is constant and a standard static space–time if σ is constant.

Lemma 2. *Suppose that $\kappa\partial_t, u\partial_t, v\partial_t\in\mathfrak{X}(I)$ and $\xi,\varsigma,\eta\in\mathfrak{X}(E)$; then,*

$$\mathcal{L}_\xi\bar{h}(\bar{\varsigma},\bar{\eta})=-2uvf^2[\kappa+\xi(\ln f)]+\sigma^2\mathcal{L}_\xi h(\varsigma,\eta)+2\kappa\sigma\dot{\sigma}h(\varsigma,\eta),$$

where $\bar{\varsigma}=u\partial_t+\varsigma$, $\bar{\eta}=v\partial_t+\eta$, and $\bar{\xi}=\kappa\partial_t+\xi$.

Theorem 5. *In an almost Ricci–Bourguignon soliton $(\bar{E},\bar{h},\bar{\xi},\bar{\lambda},\bar{\rho})$, where $\bar{E}={}_f I\times_\sigma E$ is a $(DWST)_n$, it is*

$$n\ddot{\sigma}=-\frac{f^\diamond}{\sigma}+f^2\sigma\left[\kappa+\xi(\ln f)-\bar{\lambda}-\bar{\rho}\bar{R}\right].$$

Also, $(E,h,f^2\xi,\lambda,\rho)$ is an almost Ricci–Bourguignon soliton given that $H^f=\gamma h$, where

$$\rho R+\lambda=\frac{\gamma}{f}+\frac{\sigma^\diamond}{f^2}+\kappa\sigma\dot{\sigma}+\bar{\lambda}\sigma^2+\bar{\rho}\bar{R}\sigma^2.$$

Proof. Let $(\bar{E},h,\xi,\bar{\lambda},\bar{\rho})$ be an almost Ricci–Bourguignon soliton, where $\bar{E}={}_f I\times_\sigma E$ is a $(DWST)_n$. Then,

$$\bar{\mathrm{R}}\mathrm{c}(\bar{\varsigma},\bar{\eta})+\frac{1}{2}\mathcal{L}_\xi h(\bar{\varsigma},\bar{\eta})=\bar{\lambda}h(\bar{\varsigma},\bar{\eta})+\bar{\rho}\bar{R}h(\bar{\varsigma},\bar{\eta}),$$

where $\bar{\varsigma}=u\partial_t+\varsigma$, $\bar{\eta}=v\partial_t+\eta$ and $\bar{\xi}=\kappa\partial_t+\xi$ are vector fields on $\bar{E}$. Thus,

$$\begin{aligned}&\left(\frac{n}{\sigma}\ddot{\sigma}+\frac{f^\diamond}{\sigma^2}\right)uv+\mathrm{Rc}(\varsigma,\eta)-\frac{1}{f}H^f(\varsigma,\eta)-\frac{\sigma^\diamond}{f^2}h(\varsigma,\eta)\\ &(n-1)\frac{v\dot{\sigma}}{\sigma}\varsigma(\ln f)+(n-1)\frac{u\dot{\sigma}}{\sigma}\eta(\ln f)-uvf^2[\kappa+\xi(\ln f)]\\ &+\frac{1}{2}\sigma^2\mathcal{L}_\xi h(\varsigma,\eta)+\kappa\sigma\dot{\sigma}h(\varsigma,\eta)\\ =\;&-\bar{\lambda}f^2uv+\bar{\lambda}\sigma^2h(\varsigma,\eta)-\bar{\rho}\bar{R}f^2uv+\bar{\rho}\bar{R}\sigma^2h(\varsigma,\eta).\end{aligned}$$

Thus, in the instance where $\bar{\eta} = \bar{\eta} = \partial_t$ lands on the first factor, this yields

$$\begin{aligned} \frac{n}{\sigma}\ddot{\sigma} + \frac{f^\diamond}{\sigma^2} - f^2[\kappa + \xi(\ln f)] &= -\bar{\lambda} f^2 - \bar{\rho}\bar{R} f^2 \\ n\ddot{\sigma} &= -\frac{f^\diamond}{\sigma} + f^2\sigma\big[\kappa + \xi(\ln f) - \bar{\lambda} - \bar{\rho}\bar{R}\big]. \end{aligned}$$

Now, in the instance where $\eta = \eta, \bar{\varsigma} = \varsigma$ lands on the second factor, one may obtain

$$\begin{aligned} &\mathrm{Rc}(\varsigma,\eta) - \frac{1}{f}H^f(\varsigma,\eta) - \frac{\sigma^\diamond}{f^2}h(\varsigma,\eta) + \frac{1}{2}\sigma^2\mathcal{L}_\xi h(\varsigma,\eta) + \kappa\sigma\dot{\sigma}h(\varsigma,\eta) \\ = \;& \bar{\lambda}\sigma^2 h(\varsigma,\eta) + \bar{\rho}\bar{R}\sigma^2 h(\varsigma,\eta). \end{aligned}$$

Assuming that $H^f = \gamma h$, we obtain

$$\begin{aligned} &\mathrm{Rc}(\varsigma,\eta) + \frac{1}{2}\sigma^2\mathcal{L}_\xi h(\varsigma,\eta) = \\ &\frac{\gamma}{f}h(\varsigma,\eta) + \frac{\sigma^\diamond}{f^2}h(\varsigma,\eta) + \kappa\sigma\dot{\sigma}h(\varsigma,\eta) \\ &+\bar{\lambda}\sigma^2 h(\varsigma,\eta) + \bar{\rho}\bar{R}\sigma^2 h(\varsigma,\eta) \\ =\;& \left[\frac{\gamma}{f} + \frac{\sigma^\diamond}{f^2} + \kappa\sigma\dot{\sigma} + \bar{\lambda}\sigma^2 + \bar{\rho}\bar{R}\sigma^2\right]h(\varsigma,\eta) \\ =\;& \lambda h(\varsigma,\eta) + \left[-\lambda + \frac{\gamma}{f} + \frac{\sigma^\diamond}{f^2} + \kappa\sigma\dot{\sigma} + \bar{\lambda}\sigma^2 + \bar{\rho}\bar{R}\sigma^2\right]h(\varsigma,\eta) \\ =\;& \lambda h(\varsigma,\eta) + \rho R h(\varsigma,\eta). \end{aligned}$$

Thus, $(E, h, \sigma^2\xi, \lambda, \rho)$ is an almost Ricci–Bourguignon soliton, where

$$\rho R + \lambda = \frac{\gamma}{f} + \frac{\sigma^\diamond}{f^2} + \kappa\sigma\dot{\sigma} + \bar{\lambda}\sigma^2 + \bar{\rho}\bar{R}\sigma^2.$$

This completes the proof. □

Theorem 6. *Let $(\bar{E}, \bar{h}, \bar{\xi}, \bar{\lambda}, \bar{\rho})$ be an almost Ricci–Bourguignon soliton, where $\bar{E} =_f I \times_\sigma E$ is a $(DWST)_n$ admitting a CVF $\bar{\xi} = \kappa\partial_t + \xi$. Assume that $H^f = \gamma h$; then, (E, h) is an Einstein manifold with factor*

$$\mu = \frac{\gamma}{f} + \frac{\sigma^\diamond}{f^2} - \left(\frac{n}{\sigma}\ddot{\sigma} + \frac{f^\diamond}{\sigma^2}\right)f^2.$$

Proof. Let $(\bar{E}, \bar{h}, \bar{\xi}, \bar{\lambda}, \bar{\rho})$ be an almost Ricci–Bourguignon soliton, where $\bar{E} =_f I \times_\sigma E$ is a $(DWST)_n$ admitting a CVF $\bar{\xi} = \kappa\partial_t + \xi$, i.e., $\mathcal{L}_{\bar{\xi}}\bar{h} = \bar{\omega}\bar{h}$; then,

$$\bar{\mathrm{R}}\mathrm{c}(\bar{\varsigma},\bar{\eta}) = \big(\bar{\lambda} - \bar{\omega} + \bar{\rho}\bar{R}\big)h(\bar{\varsigma},\bar{\eta}).$$

Thus,

$$\begin{aligned} &\left(\frac{n}{\sigma}\ddot{\sigma} + \frac{f^\diamond}{\sigma^2}\right)uv + \mathrm{Rc}(\varsigma,\eta) - \frac{1}{f}H^f(\varsigma,\eta) - \frac{\sigma^\diamond}{f^2}h(\varsigma,\eta) \\ &(n-1)\frac{v\dot{\sigma}}{\sigma}\varsigma(\ln f) + (n-1)\frac{u\dot{\sigma}}{\sigma}\eta(\ln f) \\ =\;& -\big(\bar{\lambda} - \bar{\omega} + \bar{\rho}\bar{R}\big)f^2 uv + \big(\bar{\lambda} - \bar{\omega} + \bar{\rho}\bar{R}\big)\sigma^2 h(\varsigma,\eta). \end{aligned}$$

Assuming that $H^f = \gamma h$, we obtain,

$$\begin{aligned}
\frac{n}{\sigma}\ddot{\sigma} + \frac{f^\diamond}{\sigma^2} &= -(\bar{\lambda} - \bar{\omega} + \bar{\rho}\bar{R})f^2, &(19)\\
\mathrm{Rc}(\varsigma,\eta) &= \left[\frac{\gamma}{f} + \frac{\sigma^\diamond}{f^2} + (\bar{\lambda} - \bar{\omega} + \bar{\rho}\bar{R})f^2\right]h(\varsigma,\eta). &(20)
\end{aligned}$$

By using Equation (19), it is

$$\mathrm{Rc}(\varsigma,\eta) = \left[\frac{\gamma}{f} + \frac{\sigma^\diamond}{f^2} - \left(\frac{n}{\sigma}\ddot{\sigma} + \frac{f^\diamond}{\sigma^2}\right)f^2\right]h(\varsigma,\eta). \tag{21}$$

Therefore, (E,h) is an Einstein manifold with factor

$$\mu = \frac{\gamma}{f} + \frac{\sigma^\diamond}{f^2} - \left(\frac{n}{\sigma}\ddot{\sigma} + \frac{f^\diamond}{\sigma^2}\right)f^2.$$

This completes the proof. □

Theorem 7. *Let $(\bar{E},\bar{h},\bar{\xi},\bar{\lambda},\bar{\rho})$ be an almost Ricci–Bourguignon soliton, where $\bar{E} =_f I \times_\sigma E$ is a $(DWST)_n$ admitting a CVF $\bar{\xi} = \kappa\partial_t + \xi$, and assume that $H^f = \gamma h$; then,*

$$\mathcal{L}_\zeta \mathrm{Rc}(\varsigma,\eta) = \varphi h(\varsigma,\eta),$$

where

$$\varphi = \left(\frac{\gamma}{f} + \frac{\sigma^\diamond}{f^2} - \left(\frac{n}{\sigma}\ddot{\sigma} + \frac{f^\diamond}{\sigma^2}\right)f^2\right)(\mu+\zeta).$$

Proof. From Lemma 2, it is clear that ζ is CVF on E with conformal factor μ. Then, by employing Equation (21) we obtain

$$\begin{aligned}
\mathcal{L}_\zeta \mathrm{Rc}(\varsigma,\eta) &= \left[\frac{\gamma}{f} + \frac{\sigma^\diamond}{f^2} - \left(\frac{n}{\sigma}\ddot{\sigma} + \frac{f^\diamond}{\sigma^2}\right)f^2\right]\mathcal{L}_\zeta h(\varsigma,\eta)\\
&\quad +\zeta\left[\frac{\gamma}{f} + \frac{\sigma^\diamond}{f^2} - \left(\frac{n}{\sigma}\ddot{\sigma} + \frac{f^\diamond}{\sigma^2}\right)f^2\right]h(\varsigma,\eta).\\
\mathcal{L}_\zeta \mathrm{Rc}(\varsigma,\eta) &= \left[\left(\frac{\gamma}{f} + \frac{\sigma^\diamond}{f^2} - \left(\frac{n}{\sigma}\ddot{\sigma} + \frac{f^\diamond}{\sigma^2}\right)f^2\right)(\mu+\zeta)\right]h(\varsigma,\eta)\\
&= \varphi h(\varsigma,\eta),
\end{aligned}$$

where

$$\varphi = \left(\frac{\gamma}{f} + \frac{\sigma^\diamond}{f^2} - \left(\frac{n}{\sigma}\ddot{\sigma} + \frac{f^\diamond}{\sigma^2}\right)f^2\right)(\mu+\zeta).$$

This completes the proof. □

Theorem 8. *Let $(\bar{E},\bar{h},\bar{\xi},\bar{\lambda},\bar{\rho})$ be an almost Ricci–Bourguignon soliton, where $\bar{E} =_f I \times_\sigma E$ is a $(DWST)_n$. Assume that $H^f = \gamma h$; then, ξ is a CVF on E if (E,h) is an Einstein manifold with conformal factor μ, where*

$$\mu = 2\left[\frac{1}{\sigma^2}\left(-\mu + \frac{\gamma}{f} + \frac{\sigma^\diamond}{f^2} - \kappa\sigma\dot{\sigma}\right) + \bar{\lambda} + \bar{\rho}\bar{R}\right].$$

Proof. Let $(\bar{E},\bar{h},\bar{\xi},\bar{\lambda},\bar{\rho})$ be an almost Ricci–Bourguignon soliton, where $\bar{E} =_f I \times_\sigma E$ is a $(DWST)_n$. It is

$$\bar{\mathrm{R}}\mathrm{c}(\bar{\varsigma},\bar{\eta}) + \frac{1}{2}\mathcal{L}_{\bar{\xi}}\bar{h}(\bar{\varsigma},\eta) = \bar{\lambda}\bar{h}(\bar{\varsigma},\bar{\eta}) + \bar{\rho}\bar{R}\bar{h}(\bar{\varsigma},\bar{\eta}).$$

Assume that (E,h) is an Einstein manifold with conformal factor μ and $H^f=\gamma h$; then, for any vector fields $\bar{\varsigma}=\varsigma,\bar{\eta}=\eta$ and $\bar{\bar{\xi}}=\kappa\partial_t+\xi$, we obtain

$$\begin{aligned} & \mathrm{Rc}(\varsigma,\eta)-\frac{1}{f}H^f(\varsigma,\eta)-\frac{\sigma^\diamond}{f^2}h(\varsigma,\eta)+\frac{1}{2}\sigma^2\mathcal{L}_\xi h(\varsigma,\eta)+\kappa\sigma\dot{\sigma}h(\varsigma,\eta) \\ = & \ \bar{\lambda}\sigma^2 h(\varsigma,\eta)+\bar{\rho}\bar{R}\sigma^2 h(\varsigma,\eta). \end{aligned}$$

Then,

$$\begin{aligned} \mathcal{L}_\xi h(\varsigma,\eta) &= 2\left[\frac{1}{\sigma^2}\left(-\mu+\frac{\gamma}{f}+\frac{\sigma^\diamond}{f^2}-\kappa\sigma\dot{\sigma}\right)+\bar{\lambda}+\bar{\rho}\bar{R}\right]h(\varsigma,\eta) \\ &= \mu h(\varsigma,\eta). \end{aligned}$$

Then, ξ is a CVF on E with conformal factor μ, where

$$\mu=2\left[\frac{1}{\sigma^2}\left(-\mu+\frac{\gamma}{f}+\frac{\sigma^\diamond}{f^2}-\kappa\sigma\dot{\sigma}\right)+\bar{\lambda}+\bar{\rho}\bar{R}\right].$$

This completes the proof. □

Theorem 9. *Let $(\bar{E},\bar{h},\bar{\xi},\bar{\lambda},\bar{\rho})$ be an almost Ricci–Bourguignon soliton, where $\bar{E}={}_f\ I\times_\sigma E$ is a $(DWST)_n$. Assume that $H^f=\gamma h$. If ξ is a CVF on E with conformal factor 2φ, then (E,h) is an Einstein manifold with factor*

$$\mu=\frac{\gamma}{f}-\varphi\sigma^2+\frac{\sigma^\diamond}{f^2}+\kappa\sigma\dot{\sigma}+\bar{\lambda}\sigma^2+\bar{\rho}\bar{R}\sigma^2.$$

Proof. Let $(\bar{E},h,\xi,\bar{\lambda})$ be an almost Ricci–Bourguignon soliton, where $\bar{E}={}_f\ I\times_\sigma E$ is a $(DWST)_n$; then,

$$\bar{\mathrm{R}}\mathrm{c}(\bar{\varsigma},\eta)+\frac{1}{2}\mathcal{L}_{\bar{\xi}}h(\bar{\varsigma},\eta)=\bar{\lambda}h(\bar{\varsigma},\eta)+\bar{\rho}\bar{R}h(\bar{\varsigma},\eta),$$

where $\bar{\varsigma}=u\partial_t+\varsigma,\eta=v\partial_t+\eta$ and $\bar{\xi}=\kappa\partial_t+\xi$ are vector fields on $\bar{E}$. Thus,

$$\begin{aligned} & \left(\frac{n}{\sigma}\ddot{\sigma}+\frac{f^\diamond}{\sigma^2}\right)uv+\mathrm{Rc}(\varsigma,\eta)-\frac{1}{f}H^f(\varsigma,\eta)-\frac{\sigma^\diamond}{f^2}h(\varsigma,\eta) \\ & (n-1)\frac{v\dot{\sigma}}{\sigma}\varsigma(\ln f)+(n-1)\frac{u\dot{\sigma}}{\sigma}\eta(\ln f)-uvf^2[\kappa+\xi(\ln f)] \\ & +\frac{1}{2}\sigma^2\mathcal{L}_\xi h(\varsigma,\eta)+\kappa\sigma\dot{\sigma}h(\varsigma,\eta) \\ = & \ -\bar{\lambda}f^2uv+\bar{\lambda}\sigma^2h(\varsigma,\eta)-\bar{\rho}\bar{R}f^2uv+\bar{\rho}\bar{R}\sigma^2h(\varsigma,\eta). \end{aligned}$$

Let $\bar{\eta}=\eta,\bar{\varsigma}=\varsigma$; then,

$$\begin{aligned} & \mathrm{Rc}(\varsigma,\eta)-\frac{1}{f}H^f(\varsigma,\eta)-\frac{\sigma^\diamond}{f^2}h(\varsigma,\eta)+\frac{1}{2}\sigma^2\mathcal{L}_\xi h(\varsigma,\eta)+\kappa\sigma\dot{\sigma}h(\varsigma,\eta) \\ = & \ \bar{\lambda}\sigma^2 h(\varsigma,\eta)+\bar{\rho}\bar{R}\sigma^2 h(\varsigma,\eta). \end{aligned}$$

Assume that $H^f=\gamma h$, and let ξ be a CVF on E with conformal factor 2φ; then,

$$\begin{aligned} & \mathrm{Rc}(\varsigma,\eta)=-\varphi\sigma^2h(\varsigma,\eta)+\frac{\gamma}{f}h(\varsigma,\eta)+\frac{\sigma^\diamond}{f^2}h(\varsigma,\eta) \\ & +\kappa\sigma\dot{\sigma}h(\varsigma,\eta)+\bar{\lambda}\sigma^2h(\varsigma,\eta)+\bar{\rho}\bar{R}\sigma^2h(\varsigma,\eta) \\ = & \left[\frac{\gamma}{f}-\varphi\sigma^2+\frac{\sigma^\diamond}{f^2}+\kappa\sigma\dot{\sigma}+\bar{\lambda}\sigma^2+\bar{\rho}\bar{R}\sigma^2\right]h(\varsigma,\eta). \\ = & \ \mu h(\varsigma,\eta). \end{aligned}$$

Then, (E, h) is Einstein manifold, with factor

$$\mu = \frac{\gamma}{f} - \varphi\sigma^2 + \frac{\sigma^\diamond}{f^2} + \kappa\sigma\dot{\sigma} + \bar{\lambda}\sigma^2 + \bar{\rho}\bar{R}\sigma^2.$$

This completes the proof. □

5. Conclusions

Ricci solitons and their generalizations merit careful consideration. Several Ricci soliton investigations on the setup of warped product manifolds can be found in the literature. The study focused on the characterization of the geometry of the warped product Ricci soliton utilizing factor manifold geometry. However, no research has been conducted on the generalization of Ricci solitons on doubly warped product manifolds. In [44], the authors deeply investigated spherically symmetric doubly warped space–times. These are tractable settings for models of stellar collapse, inhomogeneous cosmology, and wormholes. The conditions for isotropy accommodate various doubly warped space–time models in the literature (see Table 1 [44] and references therein). Among them, we mention the model of Banerjee and Chatterjee [45]. This model of doubly warped space–times describes the gravitational collapse of a star, starting at $t = -\infty$ (Example 5.6 [44]). Wagh et al. [46] developed a doubly warped space–time model that describes the collapse of a radiating star with an equation of state $p = w\mu$. Thus, doubly warped space–times play a crucial role in describing radiating or collapsing stars. Other examples of doubly warped space–times may be found in [47].

Motivated by these studies, we investigated almost Ricci–Bourguignon soliton structures on doubly warped product manifolds as well as doubly warped space–times. In an almost Ricci–Bourguignon doubly warped product soliton and under certain conditions on warping functions, it is shown that the factor manifolds are almost Ricci–Bourguignon solitons. In this case, one of the warping functions will be constant, i.e., the doubly warped product manifold reduces to a singly warped product manifold. Similar results were obtained by imposing conditions on the potential vector field of an almost Ricci–Bourguignon doubly warped product soliton. Finally, many interesting results were obtained on doubly warped space–times.

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