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Tesi di dottorato

Formal Hodge Structures

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where it will lead us from here?
(1973) The Rolling Stones, *Angie*

0.1 Introduction

The aim of this work is to develop the program proposed by S. Bloch, L. Barbieri-Viale and V. Srinivas ([12],[8]) of generalizing Deligne mixed Hodge structures providing a new cohomology theory for complex algebraic varieties. In other words to construct and study cohomological invariants of (proper) algebraic schemes over $\mathbb{C}$ which are finer than the associated mixed Hodge structures in the case of singular spaces.

Before stating our results we give a brief survey on Mixed Hodge Theory, 1-motives, extensions of mixed Hodge structures and the generalized Albanese variety. Then we will explain the guide lines of this work, report the main results and describe the possible future developments.

0.1.1 Background

Mixed Hodge Theory

Deligne defined a mixed Hodge structure as the data (H, W, F), where H is a finitely generated abelian group, W is an increasing filtration of sub-vector spaces of $H \otimes \mathbb{Q}$ and F is a decreasing filtration of $H \otimes \mathbb{C}$ such that F induces for each n a decomposition $\mathrm{gr}_n^W(H \otimes \mathbb{C}) = \oplus_{p+q=n} H^{pq}$ with $\overline{H^{pq}} = H^{qp}$. By classical Hodge theory, if X is a projective (or Kähler) manifold, then $H^i(X_{\mathrm{an}}, \mathbb{Z})$ carries a Hodge structure which is pure of weight i, i.e., with $W_j = 0$ for $j < i$, $W_j = H^i(X, Z)$ for $j \geq i$. More precisely we have the Hodge decomposition

$$H^i(X_{\mathrm{an}}, \mathbb{C}) = \bigoplus_{p+q=i} H^q(X_{\mathrm{an}}, \Omega_X^p) \ , \ H^q(X_{\mathrm{an}}, \Omega_X^p) = \overline{H^p(X_{\mathrm{an}}, \Omega_X^q)}$$

Moreover if $f : X \to Y$ is a morphism of smooth and projective varieties, then $f^* : H^n(Y_{\mathrm{an}}, \mathbb{C}) \to H^n(X_{\mathrm{an}}, \mathbb{C})$ is a morphism of Hodge structures.
Deligne proved that for every scheme X of finite type over $\mathbb{C}$, $H^i(X_{\mathrm{an}}, \mathbb{Z})$ carries a natural mixed Hodge structure and that there exists a family of functors

$$H^i : (\mathsf{Sch}/\mathbb{C})^\circ \to \mathsf{MHS} \qquad X \mapsto H^i(X_{\mathrm{an}}, \mathbb{Z})$$

where MHS is the category of mixed Hodge structures. (See [17], [18])

1-motives

Let $\boldsymbol{A}$ be complex abelian variety, then a result of Riemann says that the association $\boldsymbol{A} \mapsto H_1(\boldsymbol{A}_{\mathrm{an}}, \mathbb{Z})$ induces an equivalence of the category of abelian varieties with the category of torsion free polarizable Hodge structures of type

$(-1,0),(0,-1)$. This equivalence can be generalized.
In fact (see [18, §10]) Deligne introduced the notion of a 1-motive generalizing abelian varieties. A 1-motive over $\mathbb{C}$ is a morphism of abelian groups $u : \boldsymbol{X} \to \boldsymbol{G}_{\mathrm{an}}$ such that $\boldsymbol{X} \cong \mathbb{Z}^r$ is a free and finitely generated abelian group; $\boldsymbol{G}_{\mathrm{an}} = \boldsymbol{G}(\mathbb{C})$ is the group of $\mathbb{C}$-rational points of a semi-abelian algebraic group $\boldsymbol{G}$, i.e. $\boldsymbol{G}$ is an extension of an abelian variety by a torus.
Then he generalized the above equivalence by showing that there is a *Hodge realization* functor

$$T_{\mathrm{Hodge}} : \{\text{Deligne 1-motives}\} \to \{\mathsf{MHS}\ \text{level}\ \leq 1\}$$

yielding an equivalence between the category of Deligne 1-motives, and the category of mixed Hodge structures of level ≤ 1, i.e. mixed Hodge structures H of type $(0,0)$, $(-1,0)$, $(0,-1)$, $(-1,-1)$ such that $\mathrm{gr}^W_{-1} H$ is polarizable and $H_{\mathbb{Z}}$ is free.

In [8] Barbieri-Viale generalized the *Hodge realization* replacing Deligne 1-motives with effective 1-motives (first introduced by Laumon in [31]): an effective 1-motive (over $\mathbb{C}$) is a morphism of abelian groups $u : \boldsymbol{X} \times \boldsymbol{E} \to \boldsymbol{G}_{\mathrm{an}}$ such that $\boldsymbol{X}$ is a finitely generated abelian group; $\boldsymbol{E}$ is a finitely generated $\mathbb{C}$-vector space; $\boldsymbol{G}_{\mathrm{an}} = \boldsymbol{G}(\mathbb{C})$ is the group of $\mathbb{C}$-rational points of a connected commutative algebraic group $\boldsymbol{G}$.
He provided the category of formal Hodge structures of level ≤ 1 containing the category of mixed Hodge structures of level ≤ 1 (as a full sub-category) and defined the *formal Hodge realization* making commutative the following diagram

$$\begin{tikzcd} \{\text{Deligne 1-motives}\} \arrow[r, "T_{\mathrm{Hodge}}"] \arrow[d] & \{\mathsf{MHS}\ \text{level}\ \leq 1\} \arrow[d] \\ \{\text{Laumon 1-motives}\} \arrow[r, "T_{\oint}"'] & \{\mathsf{FHS}\ \text{level}\ \leq 1\} \end{tikzcd}$$

where the vertical arrows are full embeddings.

The equivalence $T_{\oint}$ has an important geometric counterpart which we are going to explain later.
The category of mixed Hodge structures of level ≤ 1 is the category of structures related to the first cohomology group, $H^1(X_{\mathrm{an}},\mathbb{Z})$, of an algebraic complex scheme X. This structure can be obtained via 1-motives: it is possible to construct algebraically (i.e. over any field) a 1-motive $M(X)$ starting from a scheme X; then, using the various realizations of 1-motives, it is possible to compute the various (i.e. Betti, étale, crystalline, de Rham) first cohomology groups of X. For instance if X is proper, by an important theorem of Grothendieck ([26] or [23, Part 5]), we can consider the connected algebraic

group $\mathrm{Pic}^0(X)$. By a theorem of Chevalley it follows that (canonically) there is an exact sequence of group schemes

$$0 \to \mathbb{G}_a^r \to \mathrm{Pic}^0(X) \to \boldsymbol{G} \to 0$$

where $\boldsymbol{G}$ is semi-abelian. Then we get

$$H^1(X_{\mathrm{an}}, \mathbb{Z}) = H_1(\boldsymbol{G}_{\mathrm{an}}, \mathbb{Z}) =: T_{\mathrm{Hodge}}([0 \to \boldsymbol{G}]) \ .$$

Note that via Hodge realization we can only detect the semi-abelian quotient of the Picard group associated to X: in fact only the semi-abelian quotient of $\mathrm{Pic}^0(X)$ can be viewed as a Deligne 1-motive.
We know that also the additive part carries a geometric information (related to the singularities as we will explain later).

This is a motivation to consider effective 1-motives (such as $[0 \to \mathrm{Pic}^0(X)]$) and to construct $T_{\oint}$. In fact this last equivalence of categories suggests the existence of a new cohomology theory, called sharp cohomology, such that the first cohomology group is in fact a formal Hodge structure of level ≤ 1. Roughly speaking we endow the first Betti cohomology group of a scheme with its mixed Hodge structure and some extra data strongly related to the kind of singularity of the scheme. In case of X proper we will have $H^1_\sharp(X) := T_{\oint}([0 \to \mathrm{Pic}^0(X)])$.

Extensions of Mixed Hodge Structures

In [13] Carlson studied mixed Hodge structures E that are extensions of one Hodge structure A by another B: $0 \to B \to E \to A \to 0$. Under the equivalence relation generated by congruences such extensions form a group, $\mathrm{Ext}^1_{\mathsf{MHS}}(A, B)$, which has the structure of a complex torus. When A has weight $< 2p$, each such extension determines a group homomorphism $f_E : A^{p,p}_{\mathbb{Z}} \to J^pB$ of the integral (p,p) classes of A to the p-th Jacobian of B, a complex torus that generalizes the Griffiths intermediate Jacobian. Hence it is possible to associate a 1-motive f_E to the extension E. His philosophy is that, when the extension comes from geometry, the associated 1-motive contains interesting geometric informations. He illustrated this with several examples, notably that in which E is the cohomology of a singular surface. As an applications he proved a Torelli theorem for such surfaces and showed how the associated 1-motive gives a Hodge-theoretic necessary and sufficient condition for a Weil divisor on a normal crossings surface to be a Cartier divisor.

Albanese Variety

Let X be a smooth irreducible projective variety of dimension d over an algebraically closed field k, and fix a base-point $x_0 \in X(k)$. The Albanese variety of X, $\mathrm{Alb}(X)$, is an abelian variety which is the universal regular quotient of $\mathrm{CH}^d(X)_{\deg=0}$ (i.e. the Chow group of 0-cycles of degree 0 of X). This means that:

i) there is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[dr, "\gamma"'] & \mathrm{Alb}(X) \\ & \mathrm{CH}^d(X)_{\deg=0} \arrow[u, "\phi"] \end{tikzcd}$$

where α is an algebraic map and $\gamma(x) = [x] - [x_0]$

ii) For any other algebraic group $\boldsymbol{G}$ satisfying (i) there is a map $\mathrm{Alb}(X) \to \boldsymbol{G}$.

Esnault, Srinivas and Viehweg generalized this result to any (non necessarily smooth) reduced projective scheme X (See [20] and [11] in the case of semi-abelian groups). In fact they constructed an algebraic group scheme $\mathrm{ESV}(X)$ satisfying (i) and (ii): they gave both an algebraic construction for projective schemes over an algebraically closed field, and an analytic version for projective schemes over $\mathbb{C}$.

For the analytic version they considered the Deligne cohomology groups:

$$H^p_{\mathrm{D}}(X, \mathbb{Z}(q)) := H^p(X_{\mathrm{an}}, (2\pi i)^q\mathbb{Z} \to \mathcal{O}_X \to \cdots \to \mathcal{A}_X^{q-1}) \ .$$

where $\mathcal{A}^\bullet$ is the analytic De Rham complex of X. Then $\mathrm{ESV}(X)$ (i.e. the generalized Albanese of [20]) is defined as the kernel of the natural map

$$H^{2n}_{\mathrm{D}}(X, \mathbb{Z}(n)) \to H^{2n}(X_{\mathrm{an}}, (2\pi i)^n\mathbb{Z}) \cong \mathbb{Z}; \quad n = \dim(X).$$

In the smooth case, this agrees with the classical analytic definition.

If one replaces the naive Deligne cohomology with the more standard Hodge-theoretic version, one has the semi-abelian variety

$$J^n(X) := \frac{H^{2n-1}(X_{\mathrm{an}}, \mathbb{C})}{F^n H^{2n-1}(X_{\mathrm{an}}, \mathbb{C}) + H^{2n-1}(X_{\mathrm{an}}, \mathbb{Z}(n))} \cong \mathrm{Ext}^1_{\mathsf{MHS}}(\mathbb{Z}(-n), H^{2n-1}(X_{\mathrm{an}}, \mathbb{Z})),$$

and a natural surjection $\mathrm{ESV}(X) \to J^n(X)$. This realizes $\mathrm{ESV}(X)$ as an extension of $J^n(X)$ by a product of additive groups. In general, the surjection $\mathrm{ESV}(X) \to J^n(X)$ is not an isomorphism (e.g. take X a curve with cusps).

The Main Problem

The general plan (according to [8],[7],[12]) is to construct an abelian category, FHS, and a family of functors

$$H^i_\sharp : (\mathsf{Sch}/\mathbb{C})^o \to \mathsf{FHS}$$

such that, at least:

1. The category of formal Hodge structures, FHS, fits in the following commutative diagram where the arrows are full embeddings

$$\begin{tikzcd} & \mathsf{FHS} & \\ \mathsf{MHS} \arrow[ur] & & \mathsf{FHS}_1 \arrow[ul] \\ & \mathsf{MHS}_1 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

2. There is a forgetful functor $f : \mathsf{FHS} \to \mathsf{MHS}$.

3. The following diagram is commutative

$$\begin{tikzcd} & \mathsf{FHS} \arrow[dd, "f"] \\ (\mathsf{Sch}/\mathbb{C})^\circ \arrow[ur, "H^i_\sharp"] \arrow[dr, "H^i"'] & \\ & \mathsf{MHS} \end{tikzcd}$$

where $H^i(X) := H^i(X_{\mathrm{an}}, \mathbb{Z})$.

Roughly speaking the sharp cohomology objects $H^i_\sharp(X)$ consist of the singular cohomology groups $H^i(X_{\mathrm{an}}, \mathbb{Z})$, with their mixed Hodge structure, plus some extra structure. Before giving an explicit definition we want to remark that there is no extra structure in the cohomology of smooth and projective varieties: in fact $H^i_\sharp(X)$ is different form the underling mixed Hodge structure only when X is singular or non-projective.

Let X a proper algebraic scheme over $\mathbb{C}$. Then there is a commutative diagram

$$\begin{tikzcd}
 & H^i(X_{\mathrm{an}},\mathbb{C})/F^i \arrow[r] & H^i(X_{\mathrm{an}},\mathbb{C})/F^{i-1} \arrow[r] & \cdots \arrow[r] & H^i(X_{\mathrm{an}},\mathbb{C})/F^1 \\
H^i(X_{\mathrm{an}},\mathbb{Z}) \arrow[ur] \arrow[dr] & & & & \\
 & H^i(X,\Omega^{<i}) \arrow[r] \arrow[uu, "\pi_i"] & H^i(X,\Omega^{<i-1}) \arrow[r] \arrow[uu, "\pi_{i-1}"] & \cdots \arrow[r] & H^i(X,\mathcal{O}_X) \arrow[uu, "\pi_1"]
\end{tikzcd}$$

where the $\mathbb{C}$-linear maps π are surjective. This diagram is the basic example of formal Hodge structure, and it is in fact $H^i_\sharp(X)$.
Note that this definition is compatible with the theory of 1-motives. In fact one can define $H^1_\sharp(X)$ as the generalized Hodge realization of $\mathrm{Pic}^0(X)$, i.e. $H^1_\sharp(X) := T_{\oint}(\mathrm{Pic}^0(X))$ which is in fact given by the diagram

$$\begin{tikzcd}
 & H^1(X_{\mathrm{an}},\mathbb{C})/F^1 \\
H^1(X_{\mathrm{an}},\mathbb{Z}) \arrow[ur] \arrow[dr] & \\
 & H^1(X,\mathcal{O}_X) \arrow[uu, "\pi_1"]
\end{tikzcd}$$

0.1.2 Results

Formal Hodge structures

We construct a family of categories $\mathsf{FHS_n}$, $n \geq 0$, of formal Hodge structures of level $\leq n$. The objects of $\mathsf{FHS_n}$ can be represented by commutative diagrams of the following type

$$\begin{tikzcd}
H_{\mathbb{Z}} \arrow[r, "c"] \arrow[dr, "h_{\mathbb{Z}}"] & H_{\mathbb{C}}/F^n \arrow[r] & H_{\mathbb{C}}/F^{n-1} \arrow[r] & \cdots \arrow[r] & H_{\mathbb{C}}/F^1 \\
H_{\mathrm{inf}} \arrow[r, "h_{\mathrm{inf}}"'] & V_n \arrow[r, "v_n"'] \arrow[u, "\pi_n"] & V_{n-1} \arrow[r, "v_{n-1}"'] \arrow[u, "\pi_{n-1}"] & \cdots \arrow[r] & V_1 \arrow[u, "\pi_1"]
\end{tikzcd}$$

where $H_{\mathbb{Z}}$ is a mixed Hodge structure of type $\{(i,j)|\ 0 \leq i,j \leq n\}$; H_{inf}, V_i are finite dimensional $\mathbb{C}$-vector spaces. We simply denote this object by the pair (H,V), where $H = H_{\mathbb{Z}} \times H_{\mathrm{inf}}$ can be viewed as a formal group over $\mathbb{C}$.

Each of the categories $\mathsf{FHS_n}$ satisfies the properties described in §0.1.1, but the embedding $\mathsf{FHS_1} \to \mathsf{FHS_n}$ is not full in general. Anyway if we restrict to the category of special formal Hodge structures we get a chain of full embeddings

$$\mathsf{FHS}_1^{\mathsf{spc}} \subset \mathsf{FHS}_2^{\mathsf{spc}} \subset \cdots \subset \mathsf{FHS}_\mathsf{n}^{\mathsf{spc}}$$

Extensions in FHS_n

We compute the group of classes of extensions, $\mathrm{Ext}^1_{\mathsf{FHS}_\mathsf{n}}((H,V),(H',V'))$, in several cases. As a corollary we can express the Albanese variety of Esnault, Srinivas and Viehweg using ext-groups. Precisely consider X proper, irreducible, algebraic over $\mathbb{C}$. Let $d = \dim X$ and denote by $H^{2d-1,d}(X)$ the formal Hodge structure represented by the following diagram

$$\begin{tikzcd}
H^{2d-1}(X_{\mathrm{an}},\mathbb{Z}) \arrow[r] \arrow[dr, "h"] & H^{2d-1}(X_{\mathrm{an}},\mathbb{C})/F^d \arrow[r] & \cdots H^{2d-1}(X_{\mathrm{an}},\mathbb{C})/F^1 \\
& H^{2d-1}(X,\Omega^{<d}) \arrow[u] \arrow[r] & \cdots H^{2d-1}(X,\mathcal{O}) \arrow[u]
\end{tikzcd} .$$

Then there is an isomorphism of complex Lie groups

$$\mathrm{ESV}(X)_{\mathrm{an}} \cong \mathrm{Ext}^1_{\mathsf{FHS}_\mathsf{d}}(\mathbb{Z}(-d), H^{2d-1,d}_\sharp(X))$$

where $\mathrm{ESV}(X)$ is the generalized Albanese of [20]. Note that this formula generalizes the classical one

$$\mathrm{Alb}(X)_{\mathrm{an}} \cong \mathrm{Ext}^1_{\mathsf{MHS}}(\mathbb{Z}(-d), H^{2d-1}(X_{\mathrm{an}},\mathbb{Z}))\ ,\ d := \dim X$$

which follows from the work of Carlson.

Higher extensions

It is well known that the groups $\mathrm{Ext}^i(A,B)$ vanish in category of mixed Hodge structures for any $i > 1$. A natural question is the following

Question *Do the groups* $\mathrm{Ext}^i_{\mathsf{FHS}_\mathsf{n}}((H,V),(H',V'))$ *vanish for $i > n$ (up to torsion) ?*

In particular Bloch and Srinivas raised a similar question for special formal Hodge structure (cf. [12]).

We answer this question for $n = 1$. More generally we prove that the category of Laumon k-1-motives up to isogeny is of cohomological dimension 1, for any k field of characteristic 0.
It follows that the category $\mathsf{FHS}_1 \otimes \mathbb{Q}$ of formal Hodge structures of level ≤ 1 modulo isogenies is of cohomological dimension 1, i.e. the higher extension groups are torsion groups only.

Sharp Cohomology

We prove that there exists a sharp cohomology theory (satisfying the axioms stated in the Main Problem) if we restrict to the category of proper schemes over $\mathbb{C}$. In fact via 1-motives we can define the formal Hodge structure $H^1_\sharp(X)$ for any algebraic scheme, but we have a good definition of $H^i_\sharp(X)$ only for proper schemes. Moreover we can define relative sharp cohomology groups $H^i_\sharp(X,Y)$ for $Y \subset X$ closed (and X proper) fitting in the long exact sequence

$$\cdots \to H^i_\sharp(X,Y) \to H^i_\sharp(X) \to H^i_\sharp(Y) \to \cdots$$

0.1.3 Perspectives

A Conjecture on 1-motives

In [18, §10.4.1] Deligne conjectured that the largest mixed sub-Hodge structure of a mixed Hodge structure (resp. the largest quotient mixed Hodge structure) of level 1 is algebraic. This means that there is a 1-motive in the sense of Deligne, defined algebraically, with Hodge realization the given mixed Hodge structure. He also showed, reinterpreting Picard's classical theorem, how it works for mixed Hodge structures arising from cohomology of curves.

In [6], the authors gave a complete answer to Deligne's conjecture for $H^i(X,\mathbb{Q}(1))$ (See also [39])

It is quite natural (see [9]) to consider the generalization of this conjecture in the framework of Laumon 1-motives and sharp cohomology objects (associated to proper schemes).

Conjecture *Given a proper scheme over* $\mathbb{C}$ *it is possible to construct algebraically a Laumon 1-motive* $\mathrm{Pic}^+_a(X,i)$ *such that*

$$T_{\oint}(\mathrm{Pic}^+_a(X,i)) = H^i_\sharp(X)_1 \quad \textit{in } \mathsf{FHS}_1 \otimes \mathbb{Q}$$

where $H^i_\sharp(X)_1$ *is the biggest sub-structure of* $H^i_\sharp(X)$ *in* FHS_1.

In fact we give a transcendental construction of $\mathrm{Pic}^+_a(X,i)$: the problem is to provide an algebraic construction of these 1-motives.

0.2 Summary

Chapter 1. We give a survey on the theory of 1-motives starting from the original work of Deligne [18, §10] (See also [40], [9]). Then we recall the theory of Laumon 1-motives developed in [31], [8], [5]. In §1.3.3 we prove that the cohomological dimension of the category of Laumon 1-motives up to isogenies is one following the proof of Orgogozo for Deligne 1-motives (see [37, Prop. 3.2.4]).

Chapter 2. This is the core of this work. We develop the theory of formal Hodge structures (of level $\leq n$) starting from [8]. In particular we extend to this setting some of the notions of the theory of 1-motives.
We give some results in order to compute the ext-groups of $\mathsf{FHS_n}$.
We recall the constructions of the algebraic group $\mathrm{ESV}(X)$ (by [20]) the generalized Albanese variety of Faltings and Wüstholz $\mathrm{FW}(K)$ (See [21]). Then we use the results on ext-groups to link the groups $\mathrm{ESV}(X)$ and $\mathrm{FW}(K)$ with sharp cohomology.

Chapter 3. Again starting from the definitions given in [8] we provide the sharp cohomology objects for a proper scheme and we prove the functoriality. We will use some result of cohomological descent: this topic can be found in [18] or more extensively in [15].

Appendix A. We recall some fact about the theory of algebraic and formal groups following [1, Exp. VII B]. Other good and more compact references are [19] and [14], specially for the theory of formal groups over a field.

Appendix B. This a survey on the category of mixed Hodge structures based on the original work of Deligne [17]. In particular we focus on the computations of the extensions groups. Many proofs are omitted. In fact all the results concerning MHS can be found in the book [38] or can be deduced by the reader adapting the arguments used in the abstract setting B.2. For this part we refer to [38]. The books [29] and [28] are good references for a general discussion on Ext functors.

0.3 Notations

mod_R is the category of R-modules, where R is a (commutative and unitary) ring; $\mathsf{Mod}_R \subset \mathsf{mod}_R$ is the full sub-category of finitely generated R-modules.

alg_R is the category of associative unitary algebras over R. $\mathsf{Alg}_R \subset \mathsf{alg}_R$ is the full sub-category of algebras of finite type over R.

sch_R is the category of schemes over $\mathrm{Spec}(R)$. $\mathsf{Sch}_R \subset \mathsf{sch}_R$ is the full sub-category of algebraic (i.e. of finite type) schemes over $\mathrm{Spec}(R)$.
aff_R is the category of affine schemes over $\mathrm{Spec}(R)$. $\mathsf{Aff}_R \subset \mathsf{aff}_R$ is the full sub-category of affine and algebraic (i.e. of finite type) schemes over $\mathrm{Spec}(R)$.

A is used to denote a small abelian category, in particular all the constructions we need will be applied to the case of finite dimensional vectors spaces.

We will write $\lim_n A_n$ (resp. $\operatorname{colim}_n A_n$) for the projective limit (resp. inductive limit) of a projective system (of an inductive system) $(A_n)_n$.

We assume the reader familiar with basics of algebraic geometry and sheaf theory. (See [27], [30])

Contents

Chapter 1

One motives

1.1 Deligne 1-motives

Definition 1.1.1. Let S be any scheme. A *Deligne 1-motive* over S (a *1-motif lisse* in [18, §10.1.10]; see also [40]) is the data of

i) An S-group scheme $\boldsymbol{X}$ which is locally isomorphic, for the étale topology over S, to a finitely-generated and free constant abelian group; an abelian S-scheme $\boldsymbol{A}$; an S-torus $\boldsymbol{T}$. (See [22, Ch.I] for a survey on abelian schemes and tori)

ii) An extension of S-group schemes

$$0 \to \boldsymbol{T} \to \boldsymbol{G} \to \boldsymbol{A} \to 0$$

iii) A morphism of S-group schemes $u : \boldsymbol{X} \to \boldsymbol{G}$.

Note that the above definition for $S = \operatorname{Spec} k$, where k is a perfect field, is equivalent to the data of an $\mathrm{Gal}(k^{\mathrm{sep}}|k)$-equivariant morphism $u : X \to \boldsymbol{G}(k^{\mathrm{sep}})$ where X is a free and finitely-generated $\mathrm{Gal}(k^{\mathrm{sep}}|k)$-module and $\boldsymbol{G}$ is a semi-abelian k-group scheme.[1]
In particular if k is an algebraically closed field one recover the definition given by Deligne in [18, §10.1.2] .

A Deligne 1-motive is endowed with an increasing filtration (of sub-1-motives) called the weight filtration ([18, §10.1.4]) defined as follows

$$W_i = W_i M := \begin{cases} [u : \boldsymbol{X} \to \boldsymbol{G}] & i \geq 0 \\ [0 \to \boldsymbol{G}] & i = -1 \\ [0 \to \boldsymbol{T}] & i = -2 \\ [0 \to 0] & i \leq -3 \end{cases}$$

[1] Following [22, Ch.I Def.2.3] we say that an S-group scheme $\boldsymbol{G}$ (where S is any base) is *semi-abelian* if any fiber is an extension of an abelian variety by a torus.

hence we get

$$\mathrm{gr}_i^W M = \begin{cases} [0 \to 0] & i \geq 1 \\ [\boldsymbol{X} \to 0] & i = 0 \\ [0 \to \boldsymbol{A}] & i = -1 \\ [0 \to \boldsymbol{T}] & i = -2 \\ [0 \to 0] & i \leq -3 \end{cases}$$

Example 1.1.2. Let $\boldsymbol{A}$ be an abelian scheme over a field k. Then according to the Barsotti-Weil formula (See [36] or [42, Ch.VII §3] for the classical case) we have $\mathrm{Ext}^1(\boldsymbol{A}, \mathbb{G}_m) \cong \boldsymbol{A}^\vee(\bar{k})$, i.e. the group of classes of extensions of $\boldsymbol{A}$ by $\mathbb{G}_m$ is canonically isomorphic to the group of $\bar{k}$-rational points of the dual abelian variety $\boldsymbol{A}^\vee$. It follows that for any closed point $P \in \boldsymbol{A}^\vee(\bar{k})$ different from the zero-section we find a non-trivial extension $0 \to \mathbb{G}_m \to \boldsymbol{G}_P \to \boldsymbol{A} \to 0$ of algebraic group schemes.

Definition 1.1.3. We define the category $\mathcal{M}_{1,S}^{\mathrm{fr}}$ whose objects are Deligne-1-motives over S and the morphisms are commutative squares

$$\begin{tikzcd} \boldsymbol{X} \arrow[r, "f"] \arrow[d, "u"'] & \boldsymbol{X}' \arrow[d, "u'"] \\ \boldsymbol{G} \arrow[r, "g"'] & \boldsymbol{G}' \end{tikzcd}$$

where f, g are morphism of S-group schemes.

Let $\tilde{S}_{\mathrm{fppf}}$ be the category of abelian sheaves on the category of S-schemes w.r.t. the fppf topology. Then (See [40]) $\mathcal{M}_{1,S}^{\mathrm{fr}}$ is the full sub-category of $C^b(\tilde{S}_{\mathrm{fppf}})$ whose objects are Deligne-1-motives $M = [u : \boldsymbol{X} \to \boldsymbol{G}]$ with $\boldsymbol{X}, \boldsymbol{G}$ in degree $0, 1$ respectively.[2]

1.1.1 Cartier Duality

Recall that the category $\tilde{S}_{\mathrm{fppf}}$, of abelian sheaves on the category of S-schemes w.r.t. the fppf topology, is naturally endowed with an internal Hom: we denote it by $\mathcal{H}om_{\mathrm{fppf}}(-,-)$.

Proposition 1.1.4. *i) Let $\boldsymbol{X}$ be an S-group scheme which is locally isomorphic for the étale topology over S to a finitely generated and free constant*

[2]This is the original convention of Deligne which is compatible with the theory of Voevodsky ([10]). Anyway is worth wile to mention that for some authors ([40], [37]) $\boldsymbol{X}$ is in degree -1 and $\boldsymbol{G}$ in degree 0.

abelian group. Then the sheaf $\boldsymbol{X}^\vee := \mathcal{H}om_{\mathrm{fppf}}(\boldsymbol{X}, \mathbb{G}_{m,S})$ *is represented by an* S*-torus.*

ii) Let $\boldsymbol{T}$ *be an* S*-torus. Then the sheaf* $\boldsymbol{T}^\vee := \mathcal{H}om_{\mathrm{fppf}}(\boldsymbol{T}, \mathbb{G}_{m,S})$ *is represented by an* S*-group scheme which is locally isomorphic for the étale topology over* S *to a finitely-generated and free constant abelian group.*

iii) The functor $\mathcal{H}om_{\mathrm{fppf}}(-, \mathbb{G}_{m,S})$ *induces an anti-equivalence between the category of* S*-tori and the category* S*-group schemes which are locally isomorphic for the étale topology over* S *to finitely-generated and free constant abelian groups.*

Proof. See [2, Exp. VIII §3]. □

Definition 1.1.5. Using the notations of the above proposition we call $\boldsymbol{X}^\vee$ (resp. $\boldsymbol{T}^\vee$) the *Cartier dual* of $\boldsymbol{X}$ (resp. $\boldsymbol{T}$). $\boldsymbol{T}^\vee$ is also called the *group of characters* of $\boldsymbol{T}$.

Cartier duality is naturally extended to 1-motives over a field(see [18, 10.2.11] and [11, §1.5]). For a locally noetherian base see [3]. To do that the yoga of biextensions is needed (see [2, Exp. VII]).

Definition 1.1.6. Let $M_i = [u_i : \boldsymbol{X}_i \to \boldsymbol{G}_i]$, $i = 1, 2$, be two 2-terms complexes of group schemes over S. A *biextension* $(\boldsymbol{P}, \tau, \sigma)$ of M_1, M_2 by an abelian sheaf $\boldsymbol{H}$ is given by

i) a Grothendieck biextension $\boldsymbol{P}$ of $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ by $\boldsymbol{H}$, i.e. an extension $0 \to \boldsymbol{H} \to \boldsymbol{P} \to \boldsymbol{G}_1 \times \boldsymbol{G}_2 \to 0$ along with a structure of compatible isomorphisms of torsors $\boldsymbol{P}_{x_1,x_2}\boldsymbol{P}_{y_1,x_2} \cong \boldsymbol{P}_{x_1y_1,x_2}$ and $\boldsymbol{P}_{x_1,x_2}\boldsymbol{P}_{x_1,y_2} \cong \boldsymbol{P}_{x_1,x_2y_2}$ (including associativity and commutativity) for all points $x_i, y_i \in \boldsymbol{G}_i$, $i = 1, 2$.

ii) a pair of compatible trivialization, i.e. a biadditive section τ_1 (resp. τ_2) of the biextension $(1 \times u_2)^*\boldsymbol{P}$ over $\boldsymbol{G}_1 \times \boldsymbol{X}_2$ (resp. $(u_1 \times 1)^*\boldsymbol{P}$ over $\boldsymbol{X}_1 \times \boldsymbol{G}_2$) such that τ_1 coincides with τ_2 when restricted to $\boldsymbol{X}_1 \times \boldsymbol{X}_2$.

Proposition 1.1.7. *Let* $M = [u : \boldsymbol{X} \to \boldsymbol{G}]$ *be a Deligne 1-motive over* S *(*S *locally noetherian). Then the functor*

$$M' \in \mathcal{M}^{\mathrm{fr}}_{1,S} \mapsto \mathrm{Biext}(M', M; \mathbb{G}_{m,S})$$

is representable, i.e. there exists a Deligne 1-motive $M^\vee$ *such that*

$$\mathrm{Hom}_{\mathcal{M}^{\mathrm{fr}}_{1,S}}(M', M^\vee) \cong \mathrm{Biext}(M', M; \mathbb{G}_{m,S})\ .$$

Proof. See [18, 10.2.11] and [11, 1.5] for the construction of $M^\vee$, [10, 4.1.1] for the representability. □

Remark 1.1.8. Cartier duality of 1-motives extends the duality of abelian varieties. In fact given an abelian S-scheme $\boldsymbol{A}$ and its dual $\boldsymbol{A}^\vee$, then the Poincaré $\mathbb{G}_{m,S}$-bundle on $\boldsymbol{A}^\vee \times \boldsymbol{A}$ induces a universal biextension $\boldsymbol{P}^{\text{univ}} \in \text{Biext}(\boldsymbol{A}^\vee, \boldsymbol{A}; \mathbb{G}_{m,S})$ (see [2, VIII.3.2])

Remark 1.1.9 (Explicit Cartier Duality). The Cartier dual $M^\vee = [u' : \boldsymbol{X'} \to \boldsymbol{G'}]$ of a Deligne 1-motive $M = [u : \boldsymbol{X} \to \boldsymbol{G}]$ can be described as follows: consider the canonical exact sequence

$$0 \to W_{-2}M = [0 \to \boldsymbol{T}] \to M \to M/W_{-2}M = [\boldsymbol{X} \to \boldsymbol{A}] \to 0$$

then u' is the boundary map of the long exact sequence generated by $\mathcal{H}om_{\text{fppf}}(-, [0 \to \mathbb{G}_{m,S}])$, i.e.

$$M^\vee = [u' : \underbrace{\mathcal{H}om_{\text{fppf}}(\boldsymbol{T}, \mathbb{G}_{m,S})}_{\boldsymbol{X'}} \to \underbrace{\mathcal{E}xt^1_{\text{fppf}}([\boldsymbol{X} \to \boldsymbol{A}], [0 \to \mathbb{G}_{m,S}])}_{\boldsymbol{G'}}]$$

recall that $\boldsymbol{T}$ is an S-torus, hence $\boldsymbol{X'} = \boldsymbol{T}^\vee$ is its group of characters; moreover using the exact sequence $0 \to \text{gr}^W_{-1} M \to M/W_{-2}M \to \text{gr}^W_0 M \to 0$ we get

$$\mathcal{E}xt^1_{\text{fppf}}(\text{gr}^W_0 M, [0 \to \mathbb{G}_{m,S}]) \to \mathcal{E}xt^1_{\text{fppf}}(M/W_{-2}M, [0 \to \mathbb{G}_{m,S}]) \to \mathcal{E}xt^1_{\text{fppf}}(\text{gr}^W_{-1} M, \mathbb{G}_{m,S})$$

where $\mathcal{E}xt^1_{\text{fppf}}(\text{gr}^W_{-1} M, [0 \to \mathbb{G}_{m,S}]) = \boldsymbol{A}^\vee$; $\mathcal{E}xt^1_{\text{fppf}}(\text{gr}^W_0 M, [0 \to \mathbb{G}_{m,S}]) = \mathcal{H}om_{\text{fppf}}(\boldsymbol{X}, \mathbb{G}_{m,S}) = \boldsymbol{X}^\vee$; the arrow on the left is injective because $\mathcal{H}om_{\text{fppf}}(\boldsymbol{A}, \mathbb{G}_{m,S}) = 0$; the arrow on the right is surjective because $\mathcal{E}xt^2_{\text{fppf}}(\text{gr}^W_0 M, [0 \to \mathbb{G}_{m,S}]) = \mathcal{E}xt^1_{\text{fppf}}(\boldsymbol{X}, \mathbb{G}_{m,S}) = 0$.

Hence there is an exact sequence

$$0 \to \boldsymbol{X}^\vee \to \boldsymbol{G'} \to \boldsymbol{A}^\vee \to 0$$

Example 1.1.10. Let $S = \operatorname{Spec} k$ where k is a field. Then it easy to compute $\mathcal{H}om(\mathbb{G}_m, \mathbb{G}_m)$ using the Hopf algebra characterization of affine group schemes. In fact $\mathbb{G}_m = \operatorname{Spec} k[x, 1/x]$ and a morphism of k-algebras $f : k[x, 1/x] \to k[x, 1/x]$ compatible w.r.t. the Hopf algebra structure is of the form $f(x) = x^n$ for some $n \in \mathbb{Z}$. This gives the natural isomorphism $\mathbb{G}_m^\vee = \mathbb{Z}$ as constant group schemes over $\operatorname{Spec} k$.

1.1.2 Universal Vector Extension

Let S be a scheme. The following results hold for any base, but we will only need the case $S = \operatorname{Spec}(k)$, where k is a field of characteristic 0. An extension of a Deligne 1-motive (over S) $M = [u : \boldsymbol{X} \to \boldsymbol{G}]$ by a connected commutative algebraic group scheme $\boldsymbol{H}$ is a commutative diagram of fppf sheaves

$$\begin{tikzcd} & & \boldsymbol{X} \arrow[r, "\mathrm{id}"] \arrow[d, "v"] & \boldsymbol{X} \arrow[d, "u"] & \\ 0 \arrow[r] & \boldsymbol{H} \arrow[r] & \boldsymbol{E} \arrow[r] & \boldsymbol{G} \arrow[r] & 0 \end{tikzcd}$$

with exact rows. If we denote such an extension by $(\boldsymbol{E}, v)$ we say that it is equivalent to $(\boldsymbol{E}', v')$ if there exists an isomorphism of extensions of $\boldsymbol{G}$ by $\boldsymbol{H}$, $\phi : \boldsymbol{E} \to \boldsymbol{E}'$, such that $v' = \phi \circ v$. Defining the Baer sum in the usual way we get a group of equivalence classes of extensions of M by $\boldsymbol{H}$, denoted by $\mathrm{Ext}_{\mathcal{M}^{\mathrm{fr}}_{1,S}}(M, \boldsymbol{H})$.

Note that this can be defined as the group of classes Yoneda extensions in the abelian category of complexes of fppf sheaves on S, i.e.

$$\mathrm{Ext}_{\mathcal{M}^{\mathrm{fr}}_{1,S}}(M, \boldsymbol{H}) := \mathrm{Ext}^1_{C^b(S_{\mathrm{fppf}})}(M, \boldsymbol{H}) \tag{1.1}$$

Definition 1.1.11. A *universal vector extension* of M is an extension $M^\natural = [u^\natural : \boldsymbol{X} \to \boldsymbol{G}(M^\natural)]$ (note that this is not a Deligne 1-motive!)

$$\begin{tikzcd} & & \boldsymbol{X} \arrow[r, "\mathrm{id}"] \arrow[d, "u^\natural"] & \boldsymbol{X} \arrow[d, "u"] & \\ 0 \arrow[r] & \boldsymbol{V}(M^\natural) \arrow[r] & \boldsymbol{G}(M^\natural) \arrow[r] & \boldsymbol{G} \arrow[r] & 0 \end{tikzcd}$$

where $\boldsymbol{V}(M^\natural)$ is a vector group over S and such that the push-out homomorphism

$$\epsilon : \mathrm{Hom}_{\mathcal{O}_S}(\boldsymbol{V}(M^\natural), \boldsymbol{W}) \longrightarrow \mathrm{Ext}_{\mathcal{M}^{\mathrm{fr}}_{1,S}}(M, [0 \to \boldsymbol{W}])$$

is an isomorphism for all vector groups $\boldsymbol{W}$ over S, i.e. the functor $\boldsymbol{W} \mapsto \mathrm{Ext}_{\mathcal{M}^{\mathrm{fr}}_{1,S}}(M, [0 \to \boldsymbol{W}])$ from the category of vector groups over S to the category of abelian groups is represented by $\boldsymbol{V}(M^\natural)$ and $M^\natural$ represents the class $\epsilon(\mathrm{id}_{\boldsymbol{V}(M^\natural)})$.

Remark 1.1.12. As explained in [33] if

i) $\mathcal{H}om_{\mathrm{fppf}}(M, \mathcal{O}_S) = 0$;

ii) $\mathcal{E}xt^1_{\mathrm{Zar}}(M, \mathcal{O}_S)$ is a locally free and of finite rank $\mathcal{O}_S$-module;

then a universal extension of M exists. Also in this case we have

$$\mathcal{E}xt^1_{\mathrm{Zar}}(M, \mathcal{W}) \cong \mathcal{E}xt^1_{\mathrm{Zar}}(M, \mathcal{O}_S) \otimes_{\mathcal{O}_S} \mathcal{W}$$

for any locally free $\mathcal{O}_S$-module of finite rank $\mathcal{W}$. Moreover in this case we have

$$\boldsymbol{V}(M^\natural) = \mathcal{H}om_{\mathcal{O}_S}(\mathcal{H}om_{\mathcal{O}_S}(\boldsymbol{V}(M^\natural), \mathcal{O}_S), \mathcal{O}_S) = \mathcal{H}om_{\mathcal{O}_S}(\mathcal{E}xt_{\mathrm{Zar}}(M, \mathcal{O}_S), \mathcal{O}_S) \tag{1.2}$$

Example 1.1.13 (Abelian schemes). Let $\boldsymbol{A}^\vee$ be the abelian scheme dual to $\boldsymbol{A}$. Let $\mathrm{Pic}^{0,\natural}$ be the functor of invertible sheaves in Pic^0 endowed with an integrable S-connection (see [33]). The map ϕ which forgets the connection gives to the functor $\mathrm{Pic}^{0,\natural}_{\boldsymbol{A}^\vee/S}$ the structure of a functor over $\boldsymbol{A} = \mathrm{Pic}^0_{\boldsymbol{A}^\vee/S}$. One proves (see [33, I 2.6 and 3.2.3]) that this functor is representable and

$$0 \to \underline{\omega}_{\boldsymbol{A}^\vee/S} \to \mathrm{Pic}^{0,\natural}_{\boldsymbol{A}^\vee/S} \xrightarrow{\phi} \boldsymbol{A} \to 0 \tag{1.3}$$

is the universal vector extension of $\boldsymbol{A}$. The kernel of the forgetful map ϕ classifies all possible connections on the structure sheaf of $\boldsymbol{A}^\vee$ and it is $\underline{\omega}_{\boldsymbol{A}^\vee/S}$, i.e. the $\mathcal{O}_S$-module of invariant differentials on $\boldsymbol{A}^\vee$.

Also using the log-De Rham complex (in characteristic zero) we get the following exact sequence

$$0 \to f_*\Omega^1_{\boldsymbol{A}^\vee/S} \to R^1 f_*(\mathcal{O}^*_{\boldsymbol{A}^\vee} \to \Omega^1_{\boldsymbol{A}^\vee/S}) \to R^1 f_*\mathcal{O}^*_{\boldsymbol{A}^\vee} \to 0$$

Then pulling-back via the inclusion $\mathrm{Pic}^0_{\boldsymbol{A}^\vee/S} \to \mathrm{Pic}_{\boldsymbol{A}^\vee/S} = R^1 f_*\mathcal{O}^*_{\boldsymbol{A}^\vee}$ we get the sequence (1.3).

Example 1.1.14 (Tori). By explicit computation it is easy to check that the conditions (i, ii) of remark 1.1.12 are satisfied when $M = [0 \to \boldsymbol{T}]$ and $\boldsymbol{T}$ is an S-torus. Precisely we get $\mathcal{E}xt^1_{\mathrm{Zar}}(M, \mathcal{O}_S) = 0$, hence $M^\natural = [0 \to \boldsymbol{T}]$.

Example 1.1.15 (Discrete groups). Consider $M = [\boldsymbol{X} \to 0]$. In this case we have $\mathrm{Ext}^1_{\mathrm{fppf}}(M, [0 \to \boldsymbol{W}]) = \mathrm{Hom}_{\mathrm{fppf}}(\boldsymbol{X}, \boldsymbol{W})$, hence $\boldsymbol{V}(M^\natural) \cong \boldsymbol{X} \otimes_{\mathbb{Z}_S} \mathcal{O}_S$ and the universal vector extension is

$$0 \to [0 \to \boldsymbol{X} \otimes_{\mathbb{Z}_S} \mathcal{O}_S] \to M^\natural = [\mathrm{id}_X \otimes 1 : \boldsymbol{X} \to \boldsymbol{X} \otimes_{\mathbb{Z}_S} \mathcal{O}_S] \to [\boldsymbol{X} \to 0] \to 0$$

Proposition 1.1.16. *For any Deligne 1-motive M over S, a universal vector extension exists.*

Proof. It follows by the general arguments in [33], [18]. See [3, 2.3] for an explicit construction. □

1.2 Laumon 1-motives

In this section k is a field of characteristic 0 and $\bar{k}$ is its algebraic closure. As explained in §A.3 we assume that the categories of formal and algebraic groups are full sub-category of Ab_k, i.e. the category of abelian sheaves on the category aff_k w.r.t. the fppf topology.

Definition 1.2.1. An *effective k-1-motive* (or an effective Laumon 1-motive over k cf.[5, 1.4.1]) is the data of

i) A (commutative) formal group $\boldsymbol{F}$ over k, such that $\mathrm{Lie}\,\boldsymbol{F}$ is finitely generated and $\boldsymbol{F}(\bar{k}) = \lim_{[k':k]<\infty} \boldsymbol{F}(k')$ is finitely generated $\mathrm{Gal}(\bar{k}|k)$-module (not necessarily torsion-free).

ii) A connected commutative algebraic group scheme $\boldsymbol{G}$ over k.

iii) A morphism $u : \boldsymbol{F} \to \boldsymbol{G}$ in the category $\mathsf{Ab_k}$ of abelian sheaves over $\mathsf{aff_k}$ for the fppf topology (see A.3.1).

As remarked for Deligne 1-motives we can consider an effective k-1-motive $M = [u : \boldsymbol{F} \to \boldsymbol{G}]$ as a complex of sheaves in $\mathsf{Ab_k}$ concentrated in degree $0, 1$. It is known that any formal k-group $\boldsymbol{F}$ splits canonically as product $\boldsymbol{F}_{\mathrm{inf}} \times \boldsymbol{F}_{\mathrm{et}}$ where $\boldsymbol{F}_{\mathrm{inf}}$ is the identity component of $\boldsymbol{F}$ and is a connected formal k-group, and $\boldsymbol{F}_{\mathrm{et}} = \boldsymbol{F}/\boldsymbol{F}_{\mathrm{inf}}$ is étale. Moreover, $\boldsymbol{F}_{\mathrm{et}}$ admits a maximal subgroup scheme $\boldsymbol{F}_{\mathrm{tor}}$, étale and finite, such that the quotient $\boldsymbol{F}_{\mathrm{et}}/\boldsymbol{F}_{\mathrm{tor}} = \boldsymbol{F}_{\mathrm{fr}}$ is constant of the type $\mathbb{Z}^r$ over $\bar{k}$. One says that $\boldsymbol{F}$ is without torsion if $\boldsymbol{F}_{\mathrm{tor}} = 0$.

By a theorem of Chevalley any connected group $\boldsymbol{G}$ is extension of an abelian variety $\boldsymbol{A}$ by a linear k-group $\boldsymbol{L}$ that is product of its maximal subtorus $\boldsymbol{T}$ with a vector k-group $\boldsymbol{V}$. (See Appendix A for a survey on algebraic and formal groups)

Definition 1.2.2. An *effective morphism* of effective k-1-motives is a commutative square in the category $\mathsf{Ab_k}$. We denote by ${}^t\mathcal{M}_1^{\mathrm{a,eff}} = {}^t\mathcal{M}_{1,k}^{\mathrm{a,eff}}$ the category of generalized k-1-motives with effective morphisms, i.e. the full sub-category of $C^b(\mathsf{Ab_k})$ whose objects are effective k-1-motives.

Let $M = [u : \boldsymbol{F} \to \boldsymbol{G}]$ be an effective k-1-motive. Then we have the following diagram in the category of abelian sheaves $\mathsf{Ab_k}$

$$\begin{tikzcd}
 & (\mathrm{Ker}\, u) \cap \boldsymbol{F}_{\mathrm{tor}} \arrow[r] \arrow[d] & \mathrm{Ker}\, u \arrow[d] & & \\
0 \arrow[r] & \boldsymbol{F}_{\mathrm{tor}} \arrow[r] \arrow[d] & \boldsymbol{F} \arrow[r] \arrow[d, "u"] & \boldsymbol{F}_{\mathrm{fr}} \times \boldsymbol{F}_{\mathrm{inf}} \arrow[r] \arrow[d, "\bar{u}"] & 0 \\
0 \arrow[r] & u(\boldsymbol{F}_{\mathrm{tor}}) \arrow[r] \arrow[d] & \boldsymbol{G} \arrow[r] \arrow[d] & \boldsymbol{G}/u(\boldsymbol{F}_{\mathrm{tor}}) \arrow[r] \arrow[d] & 0 \\
 & 0 & \mathrm{Coker}(u) \arrow[r] & \mathrm{Coker}(\bar{u}) &
\end{tikzcd}$$

where the rows are exact (in the category $\mathsf{Ab_k}$). We set

$M_{\mathrm{fr}} := [\bar{u} : \boldsymbol{F}_{\mathrm{fr}} \times \boldsymbol{F}_{\mathrm{inf}} \to \boldsymbol{G}/u(\boldsymbol{F}_{\mathrm{tor}})]$;

$M_{\text{tor}} := [\text{Ker}(u) \cap \boldsymbol{F}_{\text{tor}} \to 0]$;
$M_{\text{tf}} := [u : \boldsymbol{F}/(\text{Ker}(u) \cap \boldsymbol{F}_{\text{tor}}) \to \boldsymbol{G}]$.
Then there are canonical effective morphisms

$$M \to M_{\text{tf}} \qquad M_{\text{tor}} \to M \qquad M_{\text{tf}} \to M_{\text{fr}}$$

Definition 1.2.3. We say that an effective k-1-motive $M = [u : \boldsymbol{F} \to \boldsymbol{G}]$ is:
free if $\boldsymbol{F}_{\text{et}}$ is free, i.e. $M = M_{\text{fr}}$;
torsion if $\boldsymbol{F} = \boldsymbol{F}_{\text{tor}}$ and $\boldsymbol{G} = 0$, i.e. $M = M_{\text{tor}}$;
torsion-free if $\text{Ker}(u) \cap \boldsymbol{F}_{\text{tor}} = 0$, i.e. $M = M_{\text{tf}}$.

Denote by ${}^t\mathcal{M}_1^{\text{a,eff,fr}}$, ${}^t\mathcal{M}_1^{\text{a,eff,tor}}$, ${}^t\mathcal{M}_1^{\text{a,eff,tf}}$, the full sub-categories of ${}^t\mathcal{M}_1^{\text{a,eff}}$ given by free, torsion and torsion-free effective 1-motives (over k).

The category ${}^t\mathcal{M}_1^{\text{a,eff,fr}}$ is the category of *generalized* k-1-motives defined originally by Laumon (cf. [31]).

According to [10, C.11.1] we generalize the weight filtration of an effective 1-motive $M = [u : \boldsymbol{F} \to \boldsymbol{G}]$ in the following way

$$W_i = W_i M = \begin{cases} [u : \boldsymbol{F} \to \boldsymbol{G}] & i \geq 0 \\ [\boldsymbol{F}_{\text{tor}} \to \boldsymbol{G}] & i = -1 \\ [\boldsymbol{F}_{\text{tor}} \cap \operatorname{Ker} u_{\boldsymbol{A}} \to \boldsymbol{L}] & i = -2 \\ [\boldsymbol{F}_{\text{tor}} \cap \operatorname{Ker} u \to 0] & i = -3 \\ [0 \to 0] & i \leq -4 \end{cases} \tag{1.4}$$

The above filtration is obtained as follows: first consider a free effective 1-motive $M = [u : \boldsymbol{F} \to \boldsymbol{G}]$ and define the weight filtration by

$$W_{-3} = 0 \ \subset W_{-2} = [0 \to \boldsymbol{L}] \ \subset W_{-1} = [0 \to \boldsymbol{G}] \ \subset W_0 = M$$

Then the filtration in (1.4) is pull back along $M \to M_{\text{fr}}$ of the above filtration.

1.2.1 Localization

By definition the category of effective 1-motives (over k) is a full sub-category of the category of complexes of sheaves in $\mathsf{Ab_k}$. We can say that two effective 1-motives M, M' are quasi-isomorphism if there is an effective morphism $(f, g) : M \to M'$ inducing a quasi-isomorphism of complexes.

Lemma 1.2.4. *An effective morphism of 1-motives*

$$\begin{tikzcd} \boldsymbol{F} \arrow[r, "u"] \arrow[d, "f"'] & \boldsymbol{G} \arrow[d, "g"] \\ \boldsymbol{F}' \arrow[r, "u'"'] & \boldsymbol{G}' \end{tikzcd}$$

is a quasi-isomorphism $\iff$ it yields a pull-back diagram (in $\mathsf{Ab_k}$)

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{\Lambda} \arrow[r] \arrow[d, "\mathrm{id}"'] & \boldsymbol{F} \arrow[r] \arrow[d, "u"] & \boldsymbol{F}' \arrow[r] \arrow[d, "u'"] & 0 \\
0 \arrow[r] & \mathbf{\Lambda} \arrow[r] & \boldsymbol{G} \arrow[r] & \boldsymbol{G}' \arrow[r] & 0
\end{tikzcd}$$

where $\mathbf{\Lambda}$ is a finite étale group scheme.

Proof. See [10, C.2.2] for the classical case. The generalization is straightforward. $\square$

Definition 1.2.5. Denote by ${}^t\mathcal{M}_1^a$ the category of 1-motives *with torsion* obtained localizing the category of effective 1-motives at the multiplicative class of quasi-isomorphisms.

Proposition 1.2.6. *The category ${}^t\mathcal{M}_1^a$ of 1-motives with torsion is an abelian category.*

Proof. See [10, C.5.3]. $\square$

Corollary 1.2.7. *A short exact sequence of 1-motives in ${}^t\mathcal{M}_1^a$*

$$0 \to M' \to M \to M'' \to 0$$

can be represented up to isomorphism by an exact sequence of complexes (also called a strong exact sequence of effective 1-motives).

Proof. See [10, C.5.5]. $\square$

1.2.2 Cartier Duality for free Laumon-1-motives

Proposition 1.2.8. *i) Let $\boldsymbol{F}$ be a free formal group over k such that* $\mathrm{Lie}\,\boldsymbol{F}_{\mathrm{inf}}$ *(resp. $\boldsymbol{F}_{\mathrm{et}}(\bar{k})$) is finitely generated over k (resp. over $\mathbb{Z}$). Then the sheaf $\boldsymbol{F}^\vee := \mathcal{H}om_{\mathsf{Ab_k}}(\boldsymbol{F}, \mathbb{G}_{m,k})$ is represented by a connected affine (commutative) algebraic group over k.*

ii) Let $\boldsymbol{L}$ be a connected affine (commutative) algebraic group over k. Then the sheaf $\boldsymbol{L}^\vee := \mathcal{H}om_{\mathsf{Ab_k}}(\boldsymbol{L}, \mathbb{G}_{m,k})$ is represented by a free formal group over k such that $\mathrm{Lie}\,\boldsymbol{F}_{\mathrm{inf}}$ *(resp. $\boldsymbol{F}_{\mathrm{et}}(\bar{k})$) is finitely generated over k (resp. over $\mathbb{Z}$).*

Proof. See [1, VII B 2.2.2] $\square$

Remark 1.2.9. We know that any formal group $\boldsymbol{F}$ (resp. any connected affine algebraic group $\boldsymbol{L}$) is a direct product $\boldsymbol{F} = \boldsymbol{F}_{\mathrm{inf}} \times \boldsymbol{F}_{\mathrm{et}}$ (resp. $\boldsymbol{L} = \boldsymbol{V} \times \boldsymbol{T}$). The duality between étale formal groups of the above proposition was already cited, over any base, in 1.1.4. Note that there one consider $\mathcal{H}om_{\mathrm{fppf}}(-, \mathbb{G}_{m,S})$, which is the internal Hom in the category of abelian sheaves on the category sch_S w.r.t. the fppf topology. Here we consider $\mathcal{H}om_{\mathsf{Ab_k}}(-, \mathbb{G}_{m,k})$ where $\mathsf{Ab_k}$ is the category of sheaves on $\mathsf{aff_k}$ with respect to the fppf topology. Hence we are using two slightly different definitions even though we get the same result over k.

Example 1.2.10 (Dual of a vector group). Consider the additive group $\mathbb{G}_{a,k}$. In order to compute $\mathbb{G}_{a,k}^\vee$ it is sufficient to note that, for any k-algebra R, $\mathrm{Hom}_{\mathsf{Ab_R}}(\mathbb{G}_{a,R}, \mathbb{G}_{m,R})$ is the set of morphisms of R-algebras $\mathrm{Hom}_{\mathsf{Alg}_R}(R[t, 1/t], R[x])$ compatible w.r.t. the Hopf algebra structure. Hence any morphism of this type is uniquely determined by the image of t in $R[x]$. This is a polynomial in x, say $\phi(x)$, subject to the following conditions

i) (co-unit) $\phi(0) = 1$;

ii) (co-multiplication) $\phi(x) \otimes \phi(x) = \phi(x) \otimes 1 + 1 \otimes \phi(x)$;

iii) $\phi(x)$ is a unit of $R[x]$;

By explicit computation we get $\phi(x) = 1 + ax$ for some $a \in \mathrm{Nil}(R)$ nilpotent in R. Hence we get $\widehat{\mathbb{G}}_{a,k} = \mathbb{G}_{a,k}^\vee$.

Now let V be a k-vector space. Then one can define the vector group $\mathbf{V}$ such that $\mathbf{V}(R) = V \otimes_k R$ for any k-algebra R. It is easy to check that $\mathbf{V} = \mathrm{Spec}\,\mathrm{Sym}(V^*)$ where V^* is the dual vector space. For instance if $v_1, ..., v_s$ is a basis of V we get $\mathbf{V} = \mathrm{Spec}\, k[v_1^*, ..., v_s^*]$, where $(v_i^*)_i$ is the basis of V^* dual to $(v_i)_i$. Generalizing the above computation we get

$$\mathbf{V}^\vee(R) = V^* \otimes_k \mathrm{Nil}(R) \qquad \mathbf{V}^\vee = \widehat{\mathbf{V}^*} = \mathrm{Spf}\, k[[v_1, ..., v_s]]$$

where $\mathbf{V}^*$ is the vector group associated to V^*.

This duality can be extended to Laumon 1-motives (i.e. free effective 1-motives) using the same arguments of §1.1.1. In fact we have the following definition (see [31, §5]).

Definition 1.2.11. Let $M = [u : \boldsymbol{F} \to \boldsymbol{G}]$ be an effective free 1-motive over k. The *Cartier dual* of M is the 1-motive $M^\vee := [u' : \boldsymbol{F}' \to \boldsymbol{G}']$, where

i) $\boldsymbol{F}' := \boldsymbol{L}^\vee = \mathcal{H}om_{\mathsf{Ab_k}}(\boldsymbol{L}, \mathbb{G}_m)$, where $\boldsymbol{L} = W_{-2}M$;

ii) $\boldsymbol{G}' := \mathcal{E}xt^1_{\mathsf{Ab_k}}([\boldsymbol{F} \to \boldsymbol{A}], [0 \to \mathbb{G}_m])$;

iii) u' is the boundary map

$$\mathcal{H}om_{\mathsf{Ab_k}}(\boldsymbol{L}, \mathbb{G}_m) \to \mathcal{E}xt^1_{Ab/k}(M/W_{-2}M, [0 \to \mathbb{G}_m])$$

obtained by the short exact sequence $0 \to W_{-2}M \to M \to M/W_{-2}M \to 0$ via the functor $\mathcal{H}om_{\mathsf{Ab_k}}(-, [0 \to \mathbb{G}_m])$.

Example 1.2.12. We already know (1.1.13) that the universal vector extension of an abelian variety $\boldsymbol{A}$ fits in the following exact sequence

$$0 \to \underline{\omega}_{\boldsymbol{A}^\vee/k} \to \boldsymbol{A}^\natural \to \boldsymbol{A} \to 0$$

As observed in [31, 5.2.5] the Cartier dual of $\boldsymbol{A}^\natural$, which is a Laumon 1-motive, is $[c : \widehat{\boldsymbol{A}^\vee} \to \boldsymbol{A}^\vee]$, where $\widehat{\boldsymbol{A}^\vee}$ is the formal completion at the origin of $\boldsymbol{A}^\vee$ and c is the canonical map. Note that $\operatorname{Lie} \widehat{\boldsymbol{A}^\vee} = \operatorname{Lie} \boldsymbol{A}^\vee = \omega^*_{\boldsymbol{A}^\vee/k}$, where $\omega_{\boldsymbol{A}^\vee/k} = \underline{\omega}_{\boldsymbol{A}^\vee/k}(k)$.

1.3 Extensions

1.3.1 The group of n-extensions

Let A be any abelian category (we don't suppose it has enough injective objects), then we can define its derived category $D(\mathsf{A})$ and the group of n-fold extension classes

$$\operatorname{Ext}^n_{\mathsf{A}}(A, B) := \operatorname{Hom}_{D(\mathsf{A})}(A, B[n]) \qquad A, B \in \mathsf{A}\ .$$

As usual we identify this group with the group of classes *Yoneda extensions*, i.e. the set of exact sequences

$$0 \to B \to E_1 \to \cdots \to E_n \to A \to 0$$

modulo congruences (See [29] or [25]).

A lemma on 2-fold extensions

Now consider a 2-fold extension $\gamma \in \operatorname{Ext}^2_{\mathsf{A}}(M, M')$, then it is represented by an exact sequence

$$0 \to M' \to E_1 \to E_2 \to M \to 0 \tag{1.5}$$

This can be written as the product of two 1-fold extensions as follows. Let $E := \operatorname{Ker}(E_2 \to M) = \operatorname{Coker}(M' \to E_1)$, then let $\gamma_1 \in \operatorname{Ext}^1_{\mathsf{A}}(E, M')$, $\gamma_2 \in \operatorname{Ext}^1_{\mathsf{A}}(M, E)$ be the classes represented by

$$0 \to M' \to E_1 \to E \to 0 \qquad 0 \to E \to E_2 \to M \to 0 \tag{1.6}$$

Then $\gamma = \gamma_1 \cdot \gamma_2$.

As a particular case, consider $W_{-2} \subset W_{-1} \subset W_0$ a sequence of objects of A. Then we have the following exact sequences

$$\gamma: \quad 0 \to W_{-2} \to W_{-1} \to W_0/W_{-2} \to W_0/W_{-1} \to 0$$

$$\gamma_1: \qquad 0 \to W_{-2} \to W_{-1} \to W_{-1}/W_{-2} \to 0$$

$$\gamma_2: \qquad 0 \to W_{-1}/W_{-2} \to W_0/W_{-2} \to W_0/W_{-1} \to 0$$

and $\gamma = \gamma_1 \cdot \gamma_2 \in \mathrm{Ext}^2_{\mathsf{A}}(W_0/W_{-1}, W_{-2})$. In this particular case we get

Lemma 1.3.1. $\gamma = 0$ *in* $\mathrm{Ext}^2_{\mathsf{A}}(W_0/W_{-1}, W_{-2})$.

Proof. See [37, Lemma 3.2.5], or [25, p. 184]. □

1.3.2 Universal and sharp vector extension

Definition 1.3.2. Let M be an effective 1-motive over k. A *universal vector extension* of M is an effective 1-motive $M^\natural = [u^\natural : \boldsymbol{F} \to \boldsymbol{G}(M^\natural)]$

$$\begin{tikzcd} & & \boldsymbol{F} \arrow[r, "\mathrm{id}"] \arrow[d, "u^\natural"] & \boldsymbol{F} \arrow[d, "u"] & \\ 0 \arrow[r] & \boldsymbol{V}(M^\natural) \arrow[r] & \boldsymbol{G}(M^\natural) \arrow[r] & \boldsymbol{G} \arrow[r] & 0 \end{tikzcd}$$

where $\boldsymbol{V}(M^\natural)$ is a vector group over k and such that the push-out homomorphism

$$\epsilon : \mathrm{Hom}_{\mathbb{G}_a}(\boldsymbol{V}(M^\natural), \boldsymbol{W}) \longrightarrow \mathrm{Ext}^1_{{}^t\mathcal{M}_1^{\mathrm{a,eff}}}(M, [0 \to \boldsymbol{W}]) := \mathrm{Ext}^1_{C(\mathsf{Ab_k})}(M, [0 \to \boldsymbol{W}])$$

is an isomorphism for all vector groups $\boldsymbol{W}$ over k, i.e. the functor

$$\boldsymbol{W} \mapsto \mathrm{Ext}_{C(\mathsf{Ab_k})}(M, [0 \to \boldsymbol{W}])$$

from the category of vector groups over k to the category of abelian groups is represented by $\boldsymbol{V}(M^\natural)$ and $M^\natural$ represents the class $\epsilon(\mathrm{id}_{\boldsymbol{V}(M^\natural)})$.

Remark 1.3.3. Assume M is a free effective 1-motive and that the universal vector extension of M exists (e.g. $M = [0 \to \boldsymbol{A}]$). Then we can consider its Cartier dual (exactness follows by [5, Prop. 1.3.3])

$$0 \to M^\vee \to (M^\natural)^\vee \to [V(M^\natural)^\vee \to 0] \to 0$$

which is the universal object for the functor

$$\widehat{\boldsymbol{W}} \mapsto \mathrm{Ext}^1_{{}^t\mathcal{M}_1^{\mathrm{a,eff}}}([\widehat{\boldsymbol{W}} \to 0], M^\vee)$$

from the category of infinitesimal formal groups to the category of abelian groups. See 1.3.5 for a general statement and the proof of this fact.

Lemma 1.3.4. *Let M be an effective k-1-motive and $\boldsymbol{W}$ a k vector group. Any isomorphism class of extension of M by $\boldsymbol{W}$ (resp. of $[\widehat{\boldsymbol{W}} \to 0]$ by M) in ${}^t\mathcal{M}_1^a$ can be represented by a strongly exact extension of M by $\boldsymbol{W}$ (resp. of $[\widehat{\boldsymbol{W}} \to 0]$ by M) and the canonical map*

$$\mathrm{Ext}^1_{C(\mathsf{Ab_k})}(M, [0 \to \boldsymbol{W}]) \to \mathrm{Ext}^1_{{}^t\mathcal{M}_1^a}(M, [0 \to \boldsymbol{W}])$$

(resp. $\mathrm{Ext}^1_{C(\mathsf{Ab_k})}(\widehat{\boldsymbol{W}}, M) \to \mathrm{Ext}^1_{{}^t\mathcal{M}_1^a}(\widehat{\boldsymbol{W}}, M)$) is an isomorphism.

Proof. See [5, A.4.2]. □

Proposition 1.3.5. *Let $M = [u : \boldsymbol{F} \to \boldsymbol{G}]$ be an effective 1-motive. Then the functor*

$$\widehat{\boldsymbol{W}} \longmapsto \mathrm{Ext}^1_{{}^t\mathcal{M}_1^{\mathrm{a,eff}}}([\widehat{\boldsymbol{W}} \to 0], M)$$

from the category of infinitesimal formal groups to the category of abelian groups is representable by $\widehat{\boldsymbol{G}}$, i.e. the formal completion at the origin of $\boldsymbol{G}$. Explicitly the universal object is the following extension

$$0 \to M \to M_\sharp := [(u, c) : \boldsymbol{F} \times \widehat{\boldsymbol{G}} \to \boldsymbol{G}] \to [\widehat{\boldsymbol{G}} \to 0] \to 0$$

where $c : \widehat{\boldsymbol{G}} \to \boldsymbol{G}$ is the canonical map.

Proof. Let $\widehat{\boldsymbol{W}}$ be an infinitesimal formal group. If there is an extension

$$0 \to M \to M' \to [\widehat{\boldsymbol{W}} \to 0] \to 0$$

then we can assume it is an exact sequence in the category of complexes of abelian sheaves (by definition 1.3.2 and lemma 1.3.4). Hence $M' = [u' : \boldsymbol{F}' \to \boldsymbol{G}]$ and $\boldsymbol{F}' = \boldsymbol{F} \times \widehat{\boldsymbol{W}}$. It follows that u' is completely determined by its restriction to $\widehat{\boldsymbol{W}}$, call it γ, and by u. Hence we have the following pull-back diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "\mathrm{id}"] & M' \arrow[r] \arrow[d] & \widehat{\boldsymbol{W}}[0] \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & M \arrow[r] & M_\sharp \arrow[r] & \widehat{\boldsymbol{G}}[0] \arrow[r] & 0
\end{tikzcd}$$

□

Definition 1.3.6. Let M be an effective 1-motive. We call $M_\sharp$ the *universal infinitesimal extension by M.*[3]

[3] $M_\sharp$ is denoted by $\vec{M}$ in [5].

Let M be an effective 1-motive over k, denote by $M_\times$ the quotient

$$0 \to [0 \to \boldsymbol{V}] \to M \to M_\times \to 0$$

where $M = [\boldsymbol{F} \to \boldsymbol{G}]$ and $0 \to \boldsymbol{T} \times \boldsymbol{V} \to \boldsymbol{G} \to \boldsymbol{A} \to 0$ is the canonical extension of $\boldsymbol{G}$.

Proposition 1.3.7. *For any effective 1-motive M, $M_\times$ has a universal vector extension $M_\times^\natural$.*

Proof. See [5, Prop. 2.2.3]. □

Definition 1.3.8. Let M be an effective 1-motive over k. We define the effective 1-motive $M^\sharp$, called *sharp vector-extension* of M, via the following fiber product

$$\begin{tikzcd}
 & & \boldsymbol{V} \arrow[r, "\mathrm{id}"] \arrow[d] & \boldsymbol{V} \arrow[d] & \\
0 \arrow[r] & \boldsymbol{V}(M_\times^\natural) \arrow[r] \arrow[d, "\mathrm{id}"'] & M^\sharp \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \boldsymbol{V}(M_\times^\natural) \arrow[r] & M_\times^\natural \arrow[r] & M_\times \arrow[r] & 0
\end{tikzcd}$$

Proposition 1.3.9. *Let M be a free effective 1-motive. Then*

$$(M^\sharp)^\vee = (M^\vee)_\sharp \ .$$

Proof. This is a direct consequence of the definitions, see [5, Prop. 2.2.10]. □

1.3.3 Ext of 1-motives up to isogenies

According to [37] we define the abelian category of Laumon k-1-motives (i.e. effective free 1-motives) modulo isogenies : the objects are the same of ${}^t\mathcal{M}_1^{\mathrm{a,fr}}$; the Hom groups are $\mathrm{Hom}_{{}^t\mathcal{M}_1^{\mathrm{a,fr}}}(M, M') \otimes_{\mathbb{Z}} \mathbb{Q}$. We denote this category by ${}^t\mathcal{M}_1^{\mathrm{a,fr}} \otimes \mathbb{Q}$. From now on we call 1-motive an effective free 1-motive over k and $\mathrm{Ext}^i_{\mathbb{Q}}(M, M')$ is the group of classes of i-fold extensions in ${}^t\mathcal{M}_1^{\mathrm{a,fr}} \otimes \mathbb{Q}$.

Moreover we adopt the weight convention used by Orgogozo: i.e. the weight filtration of an effective 1-motive (up to isogeny) $M = [\boldsymbol{F} \to \boldsymbol{G}]$ is

$$W_i = W_i M := \begin{cases} [\boldsymbol{F} \to 0] & i = 0 \\ [0 \to \boldsymbol{G}] & i = -1 \\ [0 \to \boldsymbol{L}] & i = -2 \\ [0 \to 0] & i \leq -3 \end{cases}$$

Note that this is compatible with the filtration given in (1.4) because the torsion part of $\boldsymbol{F}$ is isogenous to 0.

Theorem 1.3.10. *The category* ${}^t\mathcal{M}_1^{\mathrm{a,fr}} \otimes \mathbb{Q}$ *is of cohomological dimension 1.*

Proof. First note that we can restrict to consider pure motives M, M' (a 1-motive is pure if it is isomorphic to one of its graded pieces w.r.t. the weight filtration). In fact given M, M' 1-motives, not necessarily pure, we have the canonical exact sequences given by the weight filtration

$$0 \to W_{-1}M' \to M' \to \mathrm{gr}_0^W M' \to 0$$

$$0 \to W_{-2}M' \to W_{-1}M' \to \mathrm{gr}_{-1}^W M' \to 0$$

Hence applying $\mathrm{Hom}_{\mathbb{Q}}(M, -)$ we get two long exact sequences

$$\cdots \mathrm{Ext}^2_{\mathbb{Q}}(M, W_{-1}M') \to \mathrm{Ext}^2_{\mathbb{Q}}(M, M') \to \mathrm{Ext}^2_{\mathbb{Q}}(M, \mathrm{gr}_0^W M') \cdots$$

$$\cdots \mathrm{Ext}^2_{\mathbb{Q}}(M, W_{-2}M') \to \mathrm{Ext}^2_{\mathbb{Q}}(M, W_{-1}M') \to \mathrm{Ext}^2_{\mathbb{Q}}(M, \mathrm{gr}_{-1}^W M') \cdots$$

from this follows that we can reduce to prove $\mathrm{Ext}^2_{\mathbb{Q}}(M, M') = 0$ for M' pure. In the same way we reduce to consider M pure.

Step 1. We are going to prove the following: let M, M' pure of the same weight, then $\mathrm{Ext}^i_{\mathbb{Q}}(M, M') = 0$, for $i > 0$.

First consider $M = \boldsymbol{F}[1], M' = \boldsymbol{F}'[1]$ pure of weight 0 (i.e. formal groups). Let $0 \to \boldsymbol{F}'[1] \to E \to \boldsymbol{F}[1] \to 0$ an exact sequence of 1-motives modulo isogenies. Then E is also of weight 0 (this follows directly from the definitions). Hence $\mathrm{Ext}^1_{\mathbb{Q}}(\boldsymbol{F}[1], \boldsymbol{F}'[1])$ is isomorphic to the group of classes of extensions in the category of formal groups over k modulo isogenies. We know that Mod_k is semi-simple, and so is $\mathsf{Mod}^{\mathrm{free}}_{\mathrm{Gal}(\bar{k}|k)} \otimes \mathbb{Q}$ by the lemma of Maschke (See [41, p. 47], for the representations of finite groups; the case of pro-finite is a direct consequence). Hence the category of formal groups up to isogeny is of cohomological dimension 0.

The second case is that of abelian varieties (weight -1). Again using the definitions we get that $\mathrm{Ext}^1_{\mathbb{Q}}(\boldsymbol{A}', \boldsymbol{A})$ correspond to the group of extensions in the category of abelian varieties modulo isogenies. This group is zero (See [35, p. 173]).

The third case is that of linear groups (weight -2). This can be reduced to the first case by Cartier duality or proved explicitly using $\mathrm{Ext}^1_{\mathbb{Q}}(\boldsymbol{L}, \boldsymbol{L}') = 0$ if $\boldsymbol{L}, \boldsymbol{L}'$ are commutative linear group.

Step 2. From now on fix a 2-fold extension $\gamma \in \mathrm{Ext}^2_{\mathbb{Q}}(M, M')$ represented by

$$0 \to M' \to E_1 \to E_2 \to M \to 0$$

and take $\gamma_2 \in \operatorname{Ext}^1_{\mathbb{Q}}(M, E)$, $\gamma_1 \in \operatorname{Ext}^1_{\mathbb{Q}}(E, M')$ (as in 1.5, 1.6) such that $\gamma = \gamma_1 \cdot \gamma_2$. Moreover assume that M, M' are pure.

First we show that $\operatorname{Ext}^2_{\mathbb{Q}}(M, M') = 0$ if M, M' are pure with weights $w < w'$.

Suppose $-2 \leq w < w' \leq 0$. Then we have an exact sequence

$$\operatorname{Ext}^1_{\mathbb{Q}}(M, W_{-1}E) \to \operatorname{Ext}^1_{\mathbb{Q}}(M, E) \to \operatorname{Ext}^1_{\mathbb{Q}}(M, \operatorname{gr}_0 E)$$

with M pure of weight -1 or -2, then $\operatorname{Ext}^1_{\mathbb{Q}}(M, \operatorname{gr}_0 E) = 0$ and we can lift γ_2 to $\gamma_2' \in \operatorname{Ext}^1_{\mathbb{Q}}(M, W_{-1}E)$. Then let γ_1' be the image of γ_1 via $\operatorname{Ext}^1_{\mathbb{Q}}(E, M') \to \operatorname{Ext}^1_{\mathbb{Q}}(W_{-1}E, M')$. Now using

$$\operatorname{Ext}^1_{\mathbb{Q}}(\operatorname{gr}_{-1} E, M') \to \operatorname{Ext}^1_{\mathbb{Q}}(W_{-1}E, M') \to \operatorname{Ext}^1_{\mathbb{Q}}(W_{-2}E, M')$$

we can reduce to consider E pure of weight -1, in fact $\operatorname{Ext}^1_{\mathbb{Q}}(W_{-2}E, M') = 0$ because $w' > -2$. From this follows that $\gamma_1 = \gamma_2 = 0$

To conclude we have to prove $\operatorname{Ext}^2_{\mathbb{Q}}(M, M') = 0$ with M' of weight less than the weight of M. There are three cases

a) $M = \boldsymbol{F}[1]$, $M' = \boldsymbol{A}[0]$; *b*) $M = \boldsymbol{F}[1]$, $M' = \boldsymbol{L}[0]$; *c*) $M = \boldsymbol{A}[0]$, $M' = \boldsymbol{L}[0]$

where $\boldsymbol{F}$ is a formal group, $\boldsymbol{A}$ an abelian variety, $\boldsymbol{L}$ a linear group.

Case (a): now $\gamma_1 \in \operatorname{Ext}^1_{\mathbb{Q}}(E, \boldsymbol{A})$ $\gamma_2 \in \operatorname{Ext}^1_{\mathbb{Q}}(\boldsymbol{F}[1], E)$. Then $E = [\boldsymbol{F}' \to \boldsymbol{A}']$ is such that $W_{-2}E = 0$. Consider the exact sequence

$$0 \to \operatorname{gr}_{-1} E \to E \to \operatorname{gr}_0 E \to 0$$

applying $\operatorname{Hom}_{\mathbb{Q}}(\boldsymbol{F}[1], -)$ to it we get

$$\operatorname{Ext}^1_{\mathbb{Q}}(\boldsymbol{F}[1], \operatorname{gr}_{-1} E) \to \operatorname{Ext}^1_{\mathbb{Q}}(\boldsymbol{F}[1], E) \to \operatorname{Ext}^1_{\mathbb{Q}}(\boldsymbol{F}[1], \operatorname{gr}_0 E)$$

We proved that $\operatorname{Ext}^1_{\mathbb{Q}}(\boldsymbol{F}[1], \operatorname{gr}_0 E) = 0$ so we can lift γ_2 to a class $\gamma_2' \in \operatorname{Ext}^1_{\mathbb{Q}}(\boldsymbol{F}[1], \operatorname{gr}_{-1} E)$ (This lifting is not canonical). Similarly using $\operatorname{Hom}_{\mathbb{Q}}(-, \boldsymbol{A})$ to it we get an exact sequence

$$\operatorname{Ext}^1_{\mathbb{Q}}(\operatorname{gr}_0 E, \boldsymbol{A}) \to \operatorname{Ext}^1_{\mathbb{Q}}(E, \boldsymbol{A}) \to \operatorname{Ext}^1_{\mathbb{Q}}(\operatorname{gr}_{-1} E, \boldsymbol{A})$$

and we can map $\gamma_1 \mapsto \gamma_1' \in \operatorname{Ext}^1_{\mathbb{Q}}(\operatorname{gr}_{-1} E, \boldsymbol{A})$. By standard facts $\gamma_1' \cdot \gamma_2' = \gamma_1 \cdot \gamma_2 = \gamma$. But we know that $\operatorname{Ext}^1_{\mathbb{Q}}(\operatorname{gr}_{-1} E, \boldsymbol{A}) = 0$.

Case (c): Is similar to case (a).

Case (b): now $\gamma \in \operatorname{Ext}^2_{\mathbb{Q}}(\boldsymbol{F}[1], \boldsymbol{L})$. We want to reduce to the hypothesis of the lemma. Thus we have to show: we can take E pure of weight 1 (i.e. an abelian variety); there exists a 1-motive N such that $\gamma_1 \in \operatorname{Ext}^1_{\mathbb{Q}}(E, \boldsymbol{L})$ is represented by $0 \to W_{-2}N \to W_{-1}N \to \operatorname{gr}_{-1} N \to 0$; $\gamma_2 \in \operatorname{Ext}^1_{\mathbb{Q}}(\boldsymbol{F}[1], E)$ is

represented by $0 \to \mathrm{gr}_{-1} N \to W_0N/W_{-2} \to \mathrm{gr}_0 N \to 0$.
We know that $\mathrm{Ext}^1_{\mathbb{Q}}(\boldsymbol{F}[1], \mathrm{gr}_0 E) = 0$, so like in case (a) we can lift γ_2 to a class $\gamma_2' \in \mathrm{Ext}^1_{\mathbb{Q}}(\boldsymbol{F}[1], W_{-1}E)$. Let γ_1' the image of γ_1 via $\mathrm{Ext}^1_{\mathbb{Q}}(E, \boldsymbol{L}) \to \mathrm{Ext}^1_{\mathbb{Q}}(W_{-1}E, \boldsymbol{L})$. Hence $\gamma_1' \cdot \gamma_2' = \gamma_1 \cdot \gamma_2$.
Now we can suppose E of weight ≤ -1. Using the same argument we can lift γ_1' to $\gamma_1'' \in \mathrm{Ext}^1_{\mathbb{Q}}(\mathrm{gr}_{-1} E, \boldsymbol{L})$ (because $\mathrm{Ext}^1_{\mathbb{Q}}(\mathrm{gr}_{-2} E, \boldsymbol{L}) = 0$) and send $\gamma_2' \mapsto \gamma_2'' \in \mathrm{Ext}^1_{\mathbb{Q}}(\boldsymbol{F}[1], \mathrm{gr}_{-1} E)$. We proved that there exists an abelian variety $\boldsymbol{A}$, $\gamma_1 \in \mathrm{Ext}^1_{\mathbb{Q}}(\boldsymbol{A}, \boldsymbol{A})$, $\gamma_2 \in \mathrm{Ext}^1_{\mathbb{Q}}(\boldsymbol{F}[1], \boldsymbol{A})$, such that $\gamma_1 \cdot \gamma_2 = \gamma$. We claim that γ_1, γ_2 can be represented by extensions in the category Laumon-1-motives. In fact let

$$\gamma_1 : \quad 0 \to \boldsymbol{L} \stackrel{f\otimes n^{-1}}{\longrightarrow} \boldsymbol{G} \stackrel{g\otimes m^{-1}}{\longrightarrow} \boldsymbol{A} \to 0$$

be an extension in the category of 1-motives modulo isogenies: f, g are morphism of algebraic groups, $n, m \in \mathbb{Z}$. Then consider the push-forward by n^{-1} and the pull-back by m^{-1}, we get the following commutative diagram with exact rows in $\mathcal{M}_1^{\mathrm{a,fr}} \otimes \mathbb{Q}$

$$\begin{tikzcd}
0 \arrow[r] & \boldsymbol{L} \arrow[r, "f/n"] \arrow[d, "n^{-1}"] & \boldsymbol{G} \arrow[r, "g/m"] \arrow[d, "\mathrm{id}"] & \boldsymbol{A} \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & \boldsymbol{L} \arrow[r, "f"] & \boldsymbol{G} \arrow[r, "g/m"] & \boldsymbol{A} \arrow[r] & 0 \\
0 \arrow[r] & \boldsymbol{L} \arrow[r, "f"] \arrow[u, "\mathrm{id}"] & \boldsymbol{G} \arrow[r, "g"] \arrow[u, "\mathrm{id}"] & \boldsymbol{A} \arrow[r] \arrow[u, "m^{-1}"] & 0
\end{tikzcd}$$

The exactness of the last row is equivalent to the following: $\mathrm{Ker}\, f$ is finite; let $(\mathrm{Ker}\, g)^0$ be the connected component of $\mathrm{Ker}\, g$, then $\mathrm{Im} f \to (\mathrm{Ker}\, g)^0$ is surjective with finite kernel K; g is surjective. So after replacing $\boldsymbol{L}, \boldsymbol{A}$ with isogenous groups we have an exact sequence in $\mathcal{M}_1^{\mathrm{a,fr}}$

$$0 \to \boldsymbol{L} \to \boldsymbol{G} \to \boldsymbol{A} \to 0$$

Explicitly

$$\begin{tikzcd}
0 \arrow[r] & \boldsymbol{L} \arrow[r, "f"] \arrow[d] & \boldsymbol{G} \arrow[r, "g"] \arrow[d, "\mathrm{id}"] & \boldsymbol{A} \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & \boldsymbol{L}/\mathrm{Ker}\, f \arrow[r] \arrow[d] & \boldsymbol{G} \arrow[r, "g"] \arrow[d] & \boldsymbol{A} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Im} f/K \arrow[r] & \boldsymbol{G} \arrow[r, "g"] & \boldsymbol{A} \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Im} f/K \arrow[r] \arrow[u, "\mathrm{id}"] & \boldsymbol{G}' \arrow[r] \arrow[u] & \boldsymbol{G}/(\mathrm{Ker}\, g)^0 \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

With similar arguments we can prove that γ_2 is represented by an extension in the category $\mathcal{M}_1^{\mathrm{a,fr}}$

$$0 \to \boldsymbol{A} \to N \to \boldsymbol{F}[1] \to 0$$

with $N = [u : \boldsymbol{F} \to \boldsymbol{A}]$.

To apply the lemma we need to prove that there is lifting $u' : \boldsymbol{F} \to \boldsymbol{G}$. First suppose $\boldsymbol{F} = \boldsymbol{F}_{\mathrm{et}}$: consider the long exact sequence

$$\mathrm{Hom}_{\mathsf{Ab_k}}(\boldsymbol{F}, \boldsymbol{G}) \to \mathrm{Hom}_{\mathsf{Ab_k}}(\boldsymbol{F}, \boldsymbol{A}) \xrightarrow{\partial} \mathrm{Ext}^1_{\mathsf{Ab_k}}(\boldsymbol{F}, \boldsymbol{L})$$

We know ([34]) that $\mathrm{Ext}^1_{\mathsf{Ab_k}}(\boldsymbol{F}, \boldsymbol{L})$ is a torsion group. So modulo replacing $\boldsymbol{F}$ with an isogenous lattice we get $\partial u = 0$ and the lift exists.
In case $\boldsymbol{F} = \boldsymbol{F}_{\mathrm{inf}}$ is a connected formal group we have a commutative diagram in $\mathsf{Ab_k}$

$$\begin{tikzcd}
 & & \boldsymbol{F} \arrow[dl] \arrow[dd, "u"] \\
\widehat{\boldsymbol{G}} \arrow[r, "\widehat{\pi}"] \arrow[dr] & \widehat{\boldsymbol{A}} \arrow[dr] & \\
 & \boldsymbol{G} \arrow[r, "\pi"] & \boldsymbol{A}
\end{tikzcd}$$

where $\widehat{?}$ is the formal completion at the origin of $? = \boldsymbol{G}, \boldsymbol{A}$ (See A.2.3). The formal completion is an exact functor so $\widehat{\pi}$ is an epimorphism. The category of formal groups is of cohomological dimension 0, then we can choose a section of $\widehat{\pi}$ and lift u. □

1.4 Realizations

Let $\mathsf{FHS}_1^{\mathsf{pol}}(1)$ (resp. $\mathsf{MHS}_1^{\mathsf{pol}}(1)$) be the category of formal Hodge structures of level ≤ 1 (twisted by $\mathbb{Z}(1)$), (H, V), such that $\mathrm{gr}_{-1} H_{\mathbb{Z}}$ is polarized (cf. [8, Def. 1.1.1]). We denote by ${}^t\mathcal{M}_1$ the full sub-category of ${}^t\mathcal{M}_1^a$ whose objects are Deligne 1-motives (over k).

Proposition 1.4.1 (Formal Hodge realization)**.** *There is a commutative diagram of functors*

$$\begin{tikzcd}
{}^t\mathcal{M}_1 \arrow[r, "T_{\mathrm{Hodge}}"] \arrow[d, "\iota"] & \mathsf{MHS}_1^{\mathsf{pol}}(1) \arrow[d, "\iota'"] \\
{}^t\mathcal{M}_1^a \arrow[r, "T_{\oint}"'] & \mathsf{FHS}_1^{\mathsf{pol}}(1)
\end{tikzcd}$$

where the vertical arrows ι, ι' are the canonical inclusions and $T_\oint$, T_{Hodge} are equivalences of categories.

Proof. See [18, 10.1.3] for the original Hodge realization of free 1-motives; the formal Hodge realization first appears in [8]; see also [5, Prop. 4.3.1]. In the following remark we give a sketch of the construction. □

Remark 1.4.2 (Construction of $T_\oint$). Let $M = [u : \boldsymbol{F} \to \boldsymbol{G}]$ be an effective 1-motive. The associated formal Hodge structure $(H, V) := T_\oint(M)$ is defined as follows. First define $H_\mathbb{Z}$ to be the fiber product

$$\begin{tikzcd} H_\mathbb{Z} \arrow[r] \arrow[d, "h_\mathbb{Z}"'] & \boldsymbol{F}(\mathbb{C}) \arrow[d, "u"] \\ \operatorname{Lie}\boldsymbol{G}_{\mathrm{an}} \arrow[r, "\exp"'] & \boldsymbol{G}_{\mathrm{an}} \end{tikzcd}$$

then let $H_{\mathrm{inf}} := \operatorname{Lie}\boldsymbol{F}_{\mathrm{inf}}$ and $V = \operatorname{Lie}\boldsymbol{G}_{\mathrm{an}}$. From this one get the commutative diagram representing (H, V), i.e.

$$\begin{tikzcd} H_\mathbb{Z} \arrow[r, "c"] \arrow[dr, "h_\mathbb{Z}"'] & \frac{H_\mathbb{C}}{F^0} \\ H_{\mathrm{inf}} \arrow[r, "\operatorname{Lie}u"'] & V \arrow[u, "\pi"'] \end{tikzcd}$$

where $\pi : V = \operatorname{Lie}\boldsymbol{G} \to H_\mathbb{C}/F^0 = \operatorname{Lie}\boldsymbol{G}_\times$ is the canonical projection (note that $\operatorname{Lie}\boldsymbol{G} = \operatorname{Lie}\boldsymbol{G}_{\mathrm{an}}$).

To construct a quasi-inverse of the formal Hodge realization let (H, V) be a formal Hodge structure and consider the canonical map of analytic groups

$$\left[(h_\mathbb{Z}, h_{\mathrm{inf}}) : \frac{H_\mathbb{Z}}{W_{-1}H_\mathbb{Z}} \times H_{\mathrm{inf}} \to \frac{V}{W_{-1}H_\mathbb{Z}}\right]$$

Then let $\boldsymbol{F}$ be the formal group such that $\boldsymbol{F}(\mathbb{C}) = H_\mathbb{Z}/W_{-1}H_\mathbb{Z}$ and $\operatorname{Lie}\boldsymbol{F} = H_{\mathrm{inf}}$; note that $V/W_{-1}H_\mathbb{Z} = \boldsymbol{G}_{\mathrm{an}}$ for an algebraic group $\boldsymbol{G}$, in fact there is an exact sequence

$$0 \to V^0 \times \operatorname{Hom}_\mathbb{Z}(\operatorname{gr}_{-2} H_\mathbb{Z}, \mathbb{C}^*) \to \frac{V}{W_{-1}H_\mathbb{Z}} \to \frac{\operatorname{gr}_{-1} H_\mathbb{C}}{W_{-1}H_\mathbb{Z} + F^0} \to 0$$

the last term on the right is an abelian variety because $\operatorname{gr}_{-1} H_\mathbb{Z}$ is assumed to be polarized and the group on the left corresponds to a linear group.

Hence the quasi-inverse of $T_\oint$ is induced by the association $(H, V) \mapsto [\boldsymbol{F} \to \boldsymbol{G}]$.

Example 1.4.3. i) $T_{\oint}([\mathbb{Z} \to 0]) = (\mathbb{Z}(0), 0)$ is an étale formal Hodge structure.

ii) $T_{\oint}([\widehat{\mathbb{G}}_a \to 0]) = (H_{\mathrm{inf}} = \mathbb{C}, 0)$ is a strictly non homotopic formal Hodge structure.

iii) $T_{\oint}([0 \to \boldsymbol{A}]) = (H_1(\boldsymbol{A}_{\mathrm{an}}, \mathbb{Z}), H^0(\boldsymbol{A}, \Omega^1))$.

iv) $T_{\oint}([0 \to \mathbb{G}_m]) = (\mathbb{Z}(1), \mathbb{C} = \mathrm{Lie}\,\mathbb{G}_m)$.

v) $T_{\oint}([0 \to \mathbb{G}_a]) = (0, \mathbb{C})$.

Proposition 1.4.4 (Sharp De Rham)**.** *The following diagram of functors commutes up to isomorphism*

$$\begin{tikzcd} {}^t\mathcal{M}_1^a \arrow[r, "T_{\oint}"] \arrow[d, "(-)^\sharp"'] & \mathsf{FHS}_1^{\mathsf{pol}}(1) \arrow[d, "(-)^\sharp"] \\ {}^t\mathcal{M}_1^a \arrow[r, "T_{\oint}"'] & \mathsf{FHS}_1^{\mathsf{pol}}(1) \end{tikzcd}$$

Proof. See [5, 4.4.8] □

Remark 1.4.5. The classical De Rham theorem gives the comparison isomorphism

$$H^1(X_{\mathrm{an}}, \mathbb{Z}) \otimes \mathbb{C} \cong H^1_{\mathrm{DR}}(X_{\mathrm{an}})$$

In the setting of Deligne 1-motives this result corresponds to the following isomorphism of complex vector spaces (cf. [18, §10.1.8])

$$T_{\mathbb{C}}(M) \cong T_{\mathrm{DR}}(M) \qquad T_{\mathrm{Hodge}}(M) := (T_{\mathbb{Z}}(M), W, F)$$

where $T_{\mathrm{DR}}(M) = \mathrm{Lie}(W_{-1}(M^\sharp))$. Hence the above proposition is a generalization of this fact. Explicitly let M be a Deligne 1-motive (over $\mathbb{C}$) than

$$\begin{aligned} (T_{\oint}(M))^\sharp =& (T_{\mathbb{Z}}(M), \mathrm{Lie}(W_{-1}M))^\sharp = (T_{\mathbb{Z}}(M), T_{\mathbb{C}}(M)) \\ (T_{\oint}(M^\sharp)) =& (T_{\mathbb{Z}}(M^\sharp), \mathrm{Lie}(W_{-1}M^\sharp))^\sharp = (T_{\mathbb{Z}}(M), T_{\mathrm{DR}}(M)) \ . \end{aligned}$$

For this reason we call the functor $M \mapsto T_{\sharp-\mathrm{DR}}(M) := T_{\oint}(M^\sharp)$ the *sharp De Rham realization* of $M \in {}^t\mathcal{M}_1^a$.

Chapter 2

Formal Hodge structures

2.1 Category of sequences of maps $\mathsf{A_n}$

Let A be an abelian category and $n > 0$ an integer. We define the category $\mathsf{A_n}$, as follows. The objects are diagrams of $n-1$ composable arrows of A denoted by

$$V:\ V_n \xrightarrow{v_n} V_{n-1} \xrightarrow{v_{n-1}} V_{n-2} \to \cdots \to V_1\ .$$

Let V, $V' \in \mathsf{A_n}$, a *morphism* $f : V \to V'$ is a family $f_i : V_i \to V'_i$ of morphisms in A such that

$$\begin{tikzcd} V_{i+1} \arrow[r] \arrow[d, "f_{i+1}"] & V_i \arrow[d, "f_i"] \\ V'_{i+1} \arrow[r] & V'_i \end{tikzcd}$$

is commutative for all $1 \leq i \leq n$.

Proposition 2.1.1. *The category* $\mathsf{A_n}$ *is abelian.*

Proof. (Zero object) Consider the trivial object 0, then it is initial and final by construction.

(Group Hom) Let $f, g : V \to V'$ be two morphisms in $\mathsf{A_n}$, then we can define $f_i + g_i : V_i \to V'_i$ because A is an abelian category. Moreover we have

$$v'_i \circ (f_i + g_i) = (v'_i \circ f_i) + (v'_i \circ g_i) = (f_{i-1} \circ v_i) + (g_{i-1} \circ v_i) = (f_{i-1} + g_{i-1}) \circ v_i$$

Hence the sum $f + g$, defined component-wise, is a morphism in $\mathsf{A_n}$. It easy to check that the set $\mathrm{Hom}_{\mathsf{A_n}}(V, V')$ is a group w.r.t. this operation.

(Direct Sum) Let V, $V' \in \mathsf{A_n}$. Then the direct sum is defined component-wise

$$(V \oplus V')_i := V_i \oplus V'_i \to (V \oplus V')_{i-1} := V_{i-1} \oplus V'_{i-1} \quad (x, x') \mapsto (v_i(x), v'_i(x'))\ .$$

(Ker/Coker) Given a morphism $f : V \to V'$ in $\mathsf{A_n}$, then $\operatorname{Ker} f$ exists and it is the following object: $(\operatorname{Ker} f)_i = \operatorname{Ker} f_i$; the structural maps are defined in the obvious way, in fact by definition of morphisms in $\mathsf{A_n}$ we have $f_i \circ v_{i-1}|_{\operatorname{Ker} f_{i-1}} = v'_{i-1} \circ f_{i-1}|_{\operatorname{Ker} f_{i-1}} = 0$. Hence $v_{i-1}|_{\operatorname{Ker} f_{i-1}}$ factors through $\operatorname{Ker} f_i$. Dually we can show the existence of cokernels.

(Normal mono/epi) Everything is defined component-wise, hence every monomorphism (resp. epimorphism) is normal because A is an abelian category. □

Consider the following functors

$$\iota, \eta : \mathsf{A_n} \to \mathsf{A_{n+1}}$$

defined as follows

$$\begin{aligned} \iota(V) : &\quad \iota(V)_{n+1} = V_n \xrightarrow{\mathrm{id}} \iota(V)_n = V_n \xrightarrow{v_n} \cdots \to V_1 \\ \eta(V) : &\quad \eta(V)_{n+1} = 0 \xrightarrow{0} \iota(V)_n = V_n \xrightarrow{v_n} \cdots \to V_1 \end{aligned}$$

Proposition 2.1.2. *The functors ι, η are full and faithful. Moreover the essential image of ι (resp. η) is a thick sub-category.*

Proof. To check that ι, η are embeddings it is straightforward. We prove that the essential image of ι (resp. η) is closed under extensions only in case $n = 2$ just to simplify the notations.

First consider an extension of ηV by $\eta V'$ in $\mathsf{Vec_3}$

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & V''_3 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & V'_2 \arrow[r] \arrow[d] & V''_2 \arrow[r] \arrow[d] & V_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & V'_1 \arrow[r] & V''_1 \arrow[r] & V_1 \arrow[r] & 0
\end{tikzcd}$$

then it follows that $V''_3 = 0$.

Now consider an extension of ιV by $\iota V'$ in $\mathsf{Vec_3}$

$$\begin{tikzcd}
0 \arrow[r] & V'_2 \arrow[r] \arrow[d, "\mathrm{id}"] & V''_3 \arrow[r] \arrow[d, "v"] & V_2 \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & V'_2 \arrow[r] \arrow[d] & V''_2 \arrow[r] \arrow[d] & V_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & V'_1 \arrow[r] & V''_1 \arrow[r] & V_1 \arrow[r] & 0
\end{tikzcd}$$

Then v is an isomorphism (by the snake lemma). It follows that V'' is isomorphic, in $\mathsf{Vec_3}$, to an object of $\iota\mathsf{Vec_2}$. To check that the essential image of ι (resp. η) is closed under kernels and cokernels is straightforward. □

Remark 2.1.3. The category of complexes of objects of A concentrated in degrees $1, ..., n$ is a full sub-category of $\mathsf{A_n}$, but it is not a thick sub-category in general. Only for $n = 1, 2$ there is an equivalence of categories.

Proposition 2.1.4. *Let* A *be an abelian category of cohomological dimension* 0 *(e.g. the category of finite dimensional vector spaces over a field). Then there is a canonical isomorphism of groups*

$$\phi : \mathrm{Ext}^1_{\mathsf{A_2}}(V, V') \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{A}}(\mathrm{Ker}\, v, \mathrm{Coker}\, v')$$

where $V = \{v : V_2 \to V_1\}$ *(resp.* $V' = \{v' : V_2' \to V_1'\}$*).*

Explicitly ϕ *associate to any extension class the* Ker-Coker *boundary map of the snake lemma.*

Proof. Note that $\mathsf{A_2}$ is an abelian category which is equivalent to the full sub-category C' of $C^b(\mathsf{A})$ of complexes concentrated in degree $0, 1$. Hence the group of classes of extensions is isomorphic. Now let $a : A^0 \to A^1$, $b : B^0 \to B^1$ be two complexes of objects of A. Then we have

$$\mathrm{Ext}^1_{C'}(A^\bullet, B^\bullet) = \mathrm{Ext}^1_{C^b(\mathsf{A})}(A^\bullet, B^\bullet) = \mathrm{Hom}_{D^b(\mathsf{A})}(A^\bullet, B^\bullet[1])$$

because C' is a thick sub-category of $C^b(\mathsf{A})$.

By hypothesis A is of cohomological dimension 0, then $a : A^0 \to A^1$ is quasi-isomorphic to $\mathrm{Ker}\, a \xrightarrow{0} \mathrm{Coker}\, a$, similarly for $B^\bullet$. It follows that

$$\begin{aligned}\mathrm{Hom}_{D^b(\mathsf{A})}(A^\bullet, B^\bullet[1]) &= \mathrm{Hom}_{D^b(\mathsf{A})}(\mathrm{Ker}\, a[0] \oplus \mathrm{Coker}\, a[-1], \mathrm{Ker}\, b[1] \oplus \mathrm{Coker}\, b[0]) \\ &= \mathrm{Hom}_{\mathsf{A}}(\mathrm{Ker}\, a, \mathrm{Coker}\, b)\ .\end{aligned}$$

□

Corollary 2.1.5. *Let* A *be an abelian category of cohomological dimension* 0*. Then the category* $\mathsf{A_2}$ *is of cohomological dimension* 1*.*

Proof. By the proposition it follows that $\mathrm{Ext}^1_{\mathsf{A_2}}(V, -)$ is a right exact functor and this is a sufficient condition (see B.2.5). □

Example 2.1.6. Let A be the category of finite dimensional complex vector spaces. Denote $\mathsf{Vec_n} = \mathsf{A_n}$. Then $\mathsf{Vec_n}$ is an abelian category and $\mathsf{Vec_2}$ is a category of cohomological dimension 1.

Remark 2.1.7 (Twist). Let $k \in \mathbb{Z}$. We can define the twisted category $\mathsf{A_n(k)}$ starting form the category $\mathsf{A_n}$ and shifting the indexes by $-k$. This is motivated by the following situation. Le $(H_\mathbb{Z}, W, F)$ be a mixed Hodge structure and consider the following diagram

$$H_\mathbb{C}/F : \; H_\mathbb{C}/F^n \to H_\mathbb{C}/F^{n-1} \to \cdots \to H_\mathbb{C}/F^1$$

as an object of $\mathsf{Vec_n}$. Then the association $H \mapsto H \otimes_{mhs} \mathbb{Z}(k)$ induces a shift of indexes in the above diagram, i.e.

$$H_\mathbb{C}(k)/F : \; H_\mathbb{C}/F^{n-k} \to H_\mathbb{C}/F^{n-1-k} \to \cdots \to H_\mathbb{C}/F^{1-k} \quad .$$

2.2 Generalities

Given a formal group (see A.2.1,A.2.6) $H = H_{\mathrm{et}} \times H_{\mathrm{inf}}$ over $\mathbb{C}$ we identify the étale component with the abelian group $H_\mathbb{Z} := H_{\mathrm{et}}(\mathbb{C})$, the infinitesimal component with its Lie algebra $\mathrm{Lie}(H_{\mathrm{inf}})$.

Let $(H_\mathbb{Z}, F, W)$ be a mixed Hodge structure such that $F^{b+1}H_\mathbb{C} = 0$ and $F^{a-1}H_\mathbb{C} = H_\mathbb{C}$. This is equivalent to say that H is of type $\{(n,m)|\ a-1 \le n, m \le b\}$ (B.3). Note that under these assumptions $H_\mathbb{Z}$ is a mixed Hodge structure of level $\le b+1-a$.

We denote the category of mixed Hodge structures of level $\le l$ ($l \ge 0$) and type $\{(n,m)|\ 0 \le n, m \le l\}$ by $\mathsf{MHS_l} = \mathsf{MHS_l(0)}$. Also we define the category $\mathsf{MHS_l(n)}$ to be the full sub-category of MHS whose objects are $H \in \mathsf{MHS}$ such that $H \otimes \mathbb{Z}(-n)$ is in $\mathsf{MHS_l(0)}$.

Using this notation we get that the category defined by Deligne in [18, §10.1.3] is the full sub-category of $\mathsf{MHS_1(1)}$ whose objects are polarized in degree -1 and $H_\mathbb{Z}$ is free.

Definition 2.2.1 (level = 0)**.** We define the category of *formal Hodge structures of level* 0 (twisted by k), $\mathsf{FHS_0(k)}$ as follows: the objects are formal groups H such that $H_\mathbb{Z}$ is a pure Hodge structure of type $(-k,-k)$; morphism are maps of formal groups.

Equivalently $\mathsf{FHS_0(k)}$ is the product category $\mathsf{MHS_0(k)} \times \mathsf{Mod}_\mathbb{C}$.

Definition 2.2.2 (level $\le n$)**.** Fix $n > 0$ an integer. We define a *formal Hodge structure of level* $\le n$ (or a *n-formal Hodge structure*) to be the data of

i) A (commutative) formal group H (over $\mathbb{C}$) carrying a mixed Hodge structure on the étale component, $(H_\mathbb{Z}, F, W)$, of level $\le n$. Hence we get $F^{n+1}H_\mathbb{C} = 0$ and $F^0H_\mathbb{C} = H_\mathbb{C}$.

ii) A family of fin. gen. $\mathbb{C}$-vector spaces V_i, for $1 \le i \le n$.

iii) A commutative diagram of abelian groups

$$
\begin{tikzcd}
H_{\mathbb{Z}} \arrow[r, "c"] \arrow[dr, "h_{\mathbb{Z}}"] & H_{\mathbb{C}}/F^n \arrow[r] & H_{\mathbb{C}}/F^{n-1} \arrow[r] & \cdots \arrow[r] & H_{\mathbb{C}}/F^1 \\
H_{\text{inf}} \arrow[r, "h_{\text{inf}}"'] & V_n \arrow[u, "\pi_n"] \arrow[r, "v_n"'] & V_{n-1} \arrow[u, "\pi_{n-1}"] \arrow[r, "v_{n-1}"'] & \cdots \arrow[r] & V_1 \arrow[u, "\pi_1"]
\end{tikzcd}
$$

such that π_i, h_{inf} are $\mathbb{C}$-linear maps.

We denote this object by (H, V) or (H, V, h, π). Note that $V = \{V_n \to \cdots \to V_1\}$ can be viewed as an object of $\mathsf{Vec_n}$ (see 2.1.6).
The map $h = (h_{\mathbb{Z}}, h_{\text{inf}}) : H \to V_n$ is called *augmentation* of the given formal Hodge structure.

A *morphism* of n-formal Hodge structures is a pair (f, ϕ) such that: $f : H \to H'$ is a morphism of formal groups; f induces a morphism of mixed Hodge structures $f_{\mathbb{Z}}$; $\phi_i : V_i \to V_i'$ is a family of $\mathbb{C}$-linear maps; $\phi : V \to V'$ is a morphism in $\mathsf{Vec_n}$; (f, ϕ) are compatible with the above structure, i.e. such that the following diagram commutes

$$
\begin{tikzcd}
 & & H'_{\mathbb{Z}} \arrow[r] \arrow[dr, "h'_{\mathbb{Z}}"] & H'_{\mathbb{C}}/F \\
H_{\mathbb{Z}} \arrow[r] \arrow[urr, "f_{\mathbb{Z}}"] \arrow[dr, "h_{\mathbb{Z}}"] & H'_{\mathbb{C}}/F \arrow[urr, "\bar{f}_{\mathbb{C}}"] & H'_{\text{inf}} \arrow[r, "h'_{\text{inf}}"] & V' \arrow[u, "\pi'"'] \\
H_{\text{inf}} \arrow[r, "h_{\text{inf}}"'] \arrow[urr, "f_{\text{inf}}"] & V \arrow[u, "\pi"] \arrow[urr, "\phi"'] & &
\end{tikzcd}
$$

We denote this category by $\mathsf{FHS_n} = \mathsf{FHS_n}(0)$.

Remark 2.2.3. Note that the commutativity of the diagram (iii) of the above definitions implies that the maps π_i are surjective. In fact after tensor by $\mathbb{C}$ we get that the composition $\pi_n \circ h_{\mathbb{C}}$ is the canonical projection $H_{\mathbb{C}} \to H_{\mathbb{C}}/F^n$: hence π_n is surjective. Similarly we obtain the surjectivity of π_i for all i.

Example 2.2.4 (Sharp cohomology of a curve). Let $U = X \setminus D$ be a complex projective curve minus a finite number of points. Then we get the following commutative diagram

$$
\begin{tikzcd}
H^1(U_{\text{an}}, \mathbb{Z}) \arrow[r] \arrow[dr, "h_{\mathbb{Z}}"] & H^1(U_{\text{an}}, \mathbb{C})/F^1 \\
\operatorname{Ker}(H^1(X_{\text{an}}, \mathcal{O}) \to H^1(U_{\text{an}}, \mathcal{O})) \arrow[r] & H^1(X_{\text{an}}, \mathcal{O}) \arrow[u, "\pi_1"']
\end{tikzcd}
$$

representing a formal Hodge structure of level ≤ 1.

Remark 2.2.5 (Twisted fhs). In a similar way one can define the category $\mathsf{FHS_n(k)}$ whose object are represented by diagrams

$$\begin{tikzcd} H_{\mathbb{Z}} \arrow[r] \arrow[dr, "h_{\mathbb{Z}}"] & H_{\mathbb{C}}/F^{n-k} \arrow[r] & H_{\mathbb{C}}/F^{n-1-k} \arrow[r] & \cdots \arrow[r] & H_{\mathbb{C}}/F^{1-k} \\ H_{\inf} \arrow[r, "h_{\inf}"'] & V_{n-k} \arrow[u, "\pi_{n-k}"] \arrow[r, "v_{n-k}"'] & V_{n-k-1} \arrow[u, "\pi_{n-k-1}"] \arrow[r, "v_{n-k-1}"'] & \cdots \arrow[r] & V_{1-k} \arrow[u, "\pi_{1-k}"] \end{tikzcd}$$

where $H_{\mathbb{Z}}$ is an object of $\mathsf{MHS_n(k)}$.

Hence the Tate twist $H_{\mathbb{Z}} \mapsto H_{\mathbb{Z}} \otimes \mathbb{Z}(k)$ induces an equivalence of categories

$$\mathsf{FHS_n(0)} \to \mathsf{FHS_n(k)} \qquad (H, V) \mapsto (H(k), V(k))$$

where $H(k) = H_{\mathbb{Z}} \otimes \mathbb{Z}(k) \times H_{\inf}$ and $V(k)$ is obtained by V shifting the degrees, i.e. $V(k)_i = V_{i+k}$, for $1 - k \le i \le n - k$.

Example 2.2.6 (Level $\le$ 1). According to the above definition a 1-formal Hodge structure twisted by 1 is represented by a diagram

$$\begin{tikzcd} H_{\mathbb{Z}} \arrow[r] \arrow[dr, "h_{\mathbb{Z}}"] & H_{\mathbb{C}}/F^{0} \\ H_{\inf} \arrow[r, "h_{\inf}"'] & V_0 \arrow[u, "\pi_0"] \end{tikzcd}$$

where is $(H_{\mathbb{Z}}, F, W)$ be a mixed Hodge structure of level ≤ 1 (twisted by $\mathbb{Z}(1)$), i.e. of type $[-1, 0] \times [-1, 0] \subset \mathbb{Z}^2$ (recall that this implies $F^1 H_{\mathbb{C}} = 0$ and $F^{-1} H_{\mathbb{C}} = H_{\mathbb{C}}$).

In particular given a mixed Hodge structure $(H_{\mathbb{Z}}, F, W)$ of level ≤ 1 we can consider the diagram

$$\begin{tikzcd} H_{\mathbb{Z}} \arrow[r] \arrow[dr, "h_{\mathbb{Z}}"] & H_{\mathbb{C}}/F^{0} \\ & H_{\mathbb{C}}/F^{0} \arrow[u, "\mathrm{id}"] \end{tikzcd}$$

Hence any mixed Hodge structure satisfying $F^1 H_{\mathbb{C}} = 0$ and $F^{-1} H_{\mathbb{C}} = H_{\mathbb{C}}$ can be viewed as an object of $\mathsf{FHS_1(1)}$.

This is in fact the category first defined in [8] if we forget the polarization on the -1-graded sub-quotient of $H_{\mathbb{Z}}$.

Note that we can also consider $(H_{\mathbb{Z}}, F, W)$ as a n-formal Hodge structure for any $n > 1$ and $k \in \mathbb{Z}$ such that $\mathsf{MHS_1(1)} \subset \mathsf{MHS_n(k)}$: for instance if $a < 0 < b$ we can consider it as an object of $\mathsf{FHS_{b+1-a}(1-a)}$ represented by the diagram

$$\begin{tikzcd} H_{\mathbb{Z}} \arrow[r] \arrow[dr, "h_{\mathbb{Z}}"] & H_{\mathbb{C}} = \frac{H_{\mathbb{C}}}{F^b} \arrow[r] & \cdots \arrow[r] & H_{\mathbb{C}}/F^0 \arrow[r] & \cdots \arrow[r] & 0 = \frac{H_{\mathbb{C}}}{F^a} \\ & H_{\mathbb{C}} \arrow[u, "\mathrm{id}"] \arrow[rr] & \cdots \arrow[r] & H_{\mathbb{C}}/F^0 \arrow[u, "\mathrm{id}"] \arrow[r] & \cdots \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

Proposition 2.2.7 (Properties of FHS). *i) The category* $\mathsf{FHS_n}$ *is an abelian category.*

ii) The forgetful functor $(H, V) \mapsto H$ *(resp.* $(H, V) \mapsto V$*) is an exact functor with values in the category of formal groups (resp. the category* $\mathsf{Vec_n}$*).*

iii) There exists a full and thick embedding $\mathsf{MHS_l}(0) \to \mathsf{FHS_l}(0)$ *induced by* $(H_{\mathbb{Z}}, F, W) \mapsto (H = H_{\mathbb{Z}}, V_i = H_{\mathbb{C}}/F^i)$.

iv) There exists a full and thick embedding $\mathsf{Vec_l}(0) \to \mathsf{FHS_l}(0)$ *induced by* $V \mapsto (0, V)$.

Proof. i) This follows from the fact that we can compute kernels, co-kernels and direct sum component-wise. Explicitly we have:

(zero object) The object $(0, 0)$ is the zero object.

(kernels/co-kernels) Let $(f, \phi) : (H, V) \to (H', V')$ be a morphism of formal Hodge structures of level $\leq n$. Then we can consider the pair $(\operatorname{Ker} f, \operatorname{Ker} \phi)$ (resp. $(\operatorname{Coker} f, \operatorname{Coker} \phi)$) with the induced augmentation map. By the properties of mixed Hodge structures we get $F^i(\operatorname{Ker} f)_{\mathbb{C}} = F^i H_{\mathbb{C}} \cap \operatorname{Ker} f_{\mathbb{C}}$ (resp. $F^i(\operatorname{Coker} f)_{\mathbb{C}} = F^i H'_{\mathbb{C}}/f_{\mathbb{C}}(H_{\mathbb{C}})$). It follows that $(\operatorname{Ker} f, \operatorname{Ker} \phi)$ (resp. $(\operatorname{Coker} f, \operatorname{Coker} \phi)$) is a formal Hodge structure of level $\leq n$ and satisfies the universal property of the kernel (resp. cokernel).

Hence kernels and co-kernels can be calculated in the abelian category $\mathsf{FGr} \times \mathsf{Vec_n}$. From this follows that the canonical morphism Coim $\to$ Im is an isomorphism.

Finally we have to prove the existence of finite direct sums. Again it is easy to check that $(H, V) \oplus (H', V') := (H \oplus H', V \oplus V')$ with the obvious augmentation is the direct sum in $\mathsf{FHS_n}$.

ii) This follows by (i).

iii) Let $(f, \phi) : (H_{\mathbb{Z}}, H_{\mathbb{C}}/F) \to (H'_{\mathbb{Z}}, H'_{\mathbb{C}}/F)$ be a morphism in $\mathsf{FHS_n}$. Then by definition for any $1 \leq i \leq n$ there is a commutative diagram

$$\begin{tikzcd} H_{\mathbb{C}}/F^i \arrow[r, "\phi_i"] \arrow[d, "\mathrm{id}"'] & H'_{\mathbb{C}}/F^i \arrow[d, "\mathrm{id}"] \\ H_{\mathbb{C}}/F^i \arrow[r, "\bar{f}_i"'] & H'_{\mathbb{C}}/F^i \end{tikzcd}$$

where $\bar{f}_i(x + F^i H_{\mathbb{C}}) = f(x) + F^i H'_{\mathbb{C}}$ is the map induce by f: it is well defined because the morphism of mixed Hodge structures are strictly compatible w.r.t. the Hodge filtration. Hence ϕ is completely determined by f.

iv) Is a direct consequence of the definition of $\mathsf{FHS_n}$. $\square$

Lemma 2.2.8. *Fix $n \in \mathbb{Z}$. The following functor*

$$\mathsf{MHS} \to \mathsf{Mod}_{\mathbb{C}} \ , \quad (H_{\mathbb{Z}}, W, F) \mapsto H_{\mathbb{C}}/F^n$$

is an exact functor.

Proof. This follows from (v) of B.3.2. $\square$

2.3 Sub-categories of $\mathsf{FHS_n}$

Recall that a formal Hodge structure of level $\leq n$ can be visualized as a diagram

$$\begin{tikzcd}
H_{\mathbb{Z}} \arrow[r] \arrow[dr, "h_{\mathbb{Z}}"] & H_{\mathbb{C}}/F^n \arrow[r] & H_{\mathbb{C}}/F^{n-1} \arrow[r] & \cdots \arrow[r] & H_{\mathbb{C}}/F^1 \\
H_{\mathrm{inf}} \arrow[r, "h_{\mathrm{inf}}"'] & V_n \arrow[u, "\pi_n"] \arrow[r, "v_n"'] & V_{n-1} \arrow[u, "\pi_{n-1}"] \arrow[r, "v_{n-1}"'] & \cdots \arrow[r] & V_1 \arrow[u, "\pi_1"] \\
 & V_n^0 \arrow[u] \arrow[r] & V_{n-1}^0 \arrow[u] \arrow[r] & \cdots \arrow[r] & V_1^0 \arrow[u]
\end{tikzcd}$$

where $V_i^0 := \mathrm{Ker}(\pi_i : V_i \to H_{\mathbb{C}}/F^i)$.

Definition 2.3.1. Fix $n > 0$. Given $(H, V) \in \mathsf{FHS_n}$ we can define the following objects of the same category

i) $(H, V)_{\mathrm{et}} := (H_{\mathbb{Z}}, H_{\mathbb{C}}/F^i)$, called the *étale part* of (H, V).

ii) $(H, V)_{\times} := (H, V/V^0)$, where the augmentation $H \to H_{\mathbb{C}}/F^n = V_n/V_n^0$ is the composite $\pi_n \circ h$.

We say that (H, V) is *étale* (resp. *strictly non-homotopic*) if $(H, V) = (H, V)_{\mathrm{et}}$ (resp. $(H, V)_{\mathrm{et}} = 0$).

Also we say that (H, V) is *special* if $h_{\mathrm{inf}} : H_{\mathrm{inf}} \to V_n$ factors through V_n^0.

Proposition 2.3.2. *i) Let $(H, V) \in \mathsf{FHS_n}$ $(n > 0)$, then there are two canonical exact sequences*

$$0 \to (0, V^0) \to (H, V) \to (H, V)_{\times} \to 0 \quad ; 0 \to (H, V)_{\mathrm{et}} \to (H, V)_{\times} \to (H_{\mathrm{inf}}, 0) \to 0$$

ii) The augmentation $h_{\mathrm{inf}} : H_{\mathrm{inf}} \to V_n$ factors trough $V_n^0 \iff$ there is a canonical exact sequence

$$0 \to (H, V)_{\mathrm{snh}} := (H_{\mathrm{inf}}, V^0) \to (H, V) \to (H, V)_{\mathrm{et}} \to 0$$

Proof. i) Let $(0,\theta) : (0, V^0) \to (H, V)$ be the canonical inclusion. By 2.2.7 $\mathrm{Coker}(0,\theta)$ can be calculated in the product category $\mathsf{FGr} \times \mathsf{Vec_n}$, i.e. $\mathrm{Coker}(0,\theta) = \mathrm{Coker}\, 0 \times \mathrm{Coker}\, \theta = H \times H_{\mathbb{C}}/F$ and the augmentation $H \to H_{\mathbb{C}}/F^n$ is the composition $H \xrightarrow{h} V_n \xrightarrow{\pi_n} H_{\mathbb{C}}/F^n$.

For the second exact sequence consider the natural projection $p_{\mathrm{inf}} : H \to H_{\mathrm{inf}}$. This induces a morphism $(p_{\mathrm{inf}}, 0) : (H, V)_\times \to (H_{\mathrm{inf}}, 0)$. Using the same argument as above we get $\mathrm{Ker}(p_{\mathrm{inf}}, 0) = \mathrm{Ker}\, p_{\mathrm{inf}} \times \mathrm{Ker}\, 0 = H_{\mathbb{Z}} \times H/F$ as an object of $\mathsf{FGr} \times \mathsf{Vec_n}$. From this follows the second exact sequence.

ii) By the definition of a morphism of formal Hodge structures (of level $\leq n$) we get that the canonical map, in the category $\mathsf{FGr} \times \mathsf{Vec_n}$, $(p_{\mathbb{Z}}, \pi) : H \times V \to H_{\mathbb{Z}} \times H_{\mathbb{C}}/F$ induces a morphism of formal Hodge structures $\iff$ the following diagram commutes

$$\begin{tikzcd} H \arrow[r, "p_{\mathbb{Z}}"] \arrow[d, "h"'] & H_{\mathbb{Z}} \arrow[d] \\ V_n \arrow[r, "\pi_n"'] & H_{\mathbb{C}}/F^n \end{tikzcd}$$

i.e. $\pi_n h(x, y) = y \mod F^n H_{\mathbb{C}}$ for all $x \in H_{\mathrm{inf}},\ y \in H_{\mathbb{Z}} \iff h_{\mathrm{inf}}(x) = 0$. □

Remark 2.3.3. Using the notation of the proof one can consider the map $(p_{\mathrm{inf}}, 0) : H \times V \to H_{\mathrm{inf}} \times 0$. Note that this is a morphism of formal Hodge structure $\iff V^0 = 0 \iff (H, V) = (H, V)_\times$.

Remark 2.3.4. For $n = 0$ we can also use the same definitions, but the situation is much more easier. In fact a formal structure of level 0 is just a formal group H, hence there is a split exact sequence

$$0 \to H_{\mathrm{inf}} \to H \to H_{\mathrm{et}} \to 0$$

in $\mathsf{FHS_0(0)}$.

2.3.1 Adjunctions

Proposition 2.3.5. *The following adjunction formulas hold*

i) $\mathrm{Hom}_{\mathsf{MHS}}(H_{\mathbb{Z}}, H'_{\mathbb{Z}}) \cong \mathrm{Hom}_{\mathsf{FHS_n}}((H, V), (H'_{\mathbb{Z}}, H'_{\mathbb{C}}/F))$ *for all* $(H, V) \in \mathsf{FHS_n^{spc}}$, $H'_{\mathbb{Z}} \in \mathsf{MHS_n}$.

ii) $\mathrm{Hom}_{\mathsf{FHS_n}}((H_{\mathrm{inf}}, V), (H', V')) \cong \mathrm{Hom}_{\mathsf{FHS_n}}((H_{\mathrm{inf}}, V), (H'_{\mathrm{inf}}, (V')^0))$ *for all* $(H_{\mathrm{inf}}, V) \in \mathsf{FHS_n^{snh}}$, $(H', V') \in \mathsf{FHS_n^{spc}}$.

Proof. The proof is straightforward. Explicitly: i) Let $(H, V) \in \mathsf{FHS_n^{spc}}$, $H'_{\mathbb{Z}} \in \mathsf{MHS_n}$. By definition a morphism $(f, \phi) \in \mathrm{Hom}_{\mathsf{FHS_n}}((H, V), (H'_{\mathbb{Z}}, H'_{\mathbb{C}}/F))$ is a morphism of formal groups $f : H \to H'$ such that $f_{\mathbb{Z}}$ is a morphism of mixed Hodge structures, hence $f = f_{\mathbb{Z}}$, and $\phi : V \to H'_{\mathbb{C}}/F$ is subject to the condition $f/F \circ \pi = \phi$. Then the association $(f, \phi) \mapsto f_{\mathbb{Z}} \in \mathrm{Hom}_{\mathsf{MHS}}(H_{\mathbb{Z}}, H'_{\mathbb{Z}})$ is an isomorphism.

ii) Let $(H_{\inf}, V) \in \mathsf{FHS}_{\mathsf{n}}^{\mathsf{snh}}$, $(H', V') \in \mathsf{FHS}_{\mathsf{n}}^{\mathsf{spc}}$. A morphism $(f, \phi) \in \mathrm{Hom}_{\mathsf{FHS}_{\mathsf{n}}}((H_{\inf}, V), (H', V'))$ is of the form $f = f_{\inf} : H_{\inf} \to H'_{\inf}$, $\phi : V \to V'$ must factor through $(V')^0$ because $\pi' \circ \phi = \pi \circ f/F = 0$. □

2.4 Different levels

Any mixed Hodge structure of level $\leq n$ (say in $\mathsf{MHS}_{\mathsf{n}}(0)$) can also be viewed as an object of $\mathsf{MHS}_{\mathsf{m}}(0)$ for any $m > n$. This give a sequence of full embeddings

$$\mathsf{MHS}_0 \subset \mathsf{MHS}_1 \subset \cdots \subset \mathsf{MHS}$$

Now we want to investigate the analogous situation in the case of formal Hodge structures.

Example 2.4.1 ($\mathsf{FHS}_1 \subset \mathsf{FHS}_2$). The basic construction is the following: let (H, V) be a 1-fhs, we can associate a 2-fhs (H', V') represented by a diagram of the following type

$$\begin{tikzcd}
H'_{\mathbb{Z}} \arrow[r] \arrow[dr, "h'_{\mathbb{Z}}"'] & H'_{\mathbb{C}}/F^2 \arrow[r] & H'_{\mathbb{C}}/F^1 \\
H'_{\inf} \arrow[r, "h'_{\inf}"'] & V'_2 \arrow[u, "\pi'_2"'] \arrow[r, "v'_2"'] & V'_1 \arrow[u, "\pi'_1"']
\end{tikzcd}$$

Take $H'_{\mathbb{Z}} := H_{\mathbb{Z}}$, then $H'_{\mathbb{C}}/F^2 = H_{\mathbb{C}}$, $H'_{\mathbb{C}}/F^1 = H_{\mathbb{C}}/F^1$ and the augmentation $h'_{\mathbb{Z}}$ is the canonical inclusion; let $V'_1 := V_1$, $\pi'_1 := \pi_1$ and define V'_2, π'_2, v'_2 via fiber product

$$\begin{tikzcd}
V'_2 \arrow[r, "\pi'_2"] \arrow[d, "v'_2"'] & H_{\mathbb{C}} \arrow[d] \\
V_1 \arrow[r, "\pi_1"'] & H_{\mathbb{C}}/F^1
\end{tikzcd}$$

Hence V'_2 fits in the following exact sequences

$$0 \to F^1 H_{\mathbb{C}} \to V'_2 \to V_1 \to 0 \quad ; \quad 0 \to V_1^0 \to V'_2 \to H_{\mathbb{C}} \to 0 \; .$$

Finally we define $h'_{\inf} : H'_{\inf} \to V'_2$ again via fiber product

$$\begin{tikzcd}
H'_{\inf} \arrow[r, "h'_{\inf}"] \arrow[d] & V'_2 \arrow[d, "v'_2"] \\
H_{\inf} \arrow[r, "h_{\inf}"'] & V_1
\end{tikzcd}$$

hence we get the following exact sequence

$$0 \to F^1 H_{\mathbb{C}} \to H'_{\text{inf}} \to H_{\text{inf}} \to 0 \ .$$

By induction is easy to extend this construction. We have the following result.

Proposition 2.4.2. *Let $n, k > 0$. Then there exists a faithful functor*

$$\iota = \iota_k : \mathsf{FHS_n} \to \mathsf{FHS_{n+k}}$$

Moreover ι induces an equivalence between $\mathsf{FHS_n}$ and the sub-category of $\mathsf{FHS_{n+k}}$ whose objects are (H, V) such that

a) $H_{\mathbb{Z}}$ is of level $\leq n$. Hence $F^{n+1}H_{\mathbb{C}} = 0$ and $F^0 H_{\mathbb{C}} = H_{\mathbb{C}}$.

b) $V_{n+i} = V_{n+1}$ for $1 \leq i \leq k$.

c) There exists a commutative diagram with exact rows

$$\begin{tikzcd}
F^n \arrow[r] & H_{\mathbb{C}} \arrow[r] & H_{\mathbb{C}}/F^n \\
F^n \arrow[u, "\mathrm{id}"] \arrow[r] \arrow[dr, "\alpha"'] & V_{n+1} \arrow[u, "\pi_{n+1}"] \arrow[r, "v_{n+1}"] & V_n \arrow[u, "\pi_n"] \\
& H_{\text{inf}} \arrow[u, "h_{\text{inf}}"] &
\end{tikzcd}$$

where α is a $\mathbb{C}$-linear map.

And morphisms are those in $\mathsf{FHS_{n+k}}$ compatible w.r.t. the diagram in (c).

Proof. The construction of ι_k is a generalization of that in 2.4.1. We have $\iota_k = \iota_1 \circ \iota_{k-1}$, hence it is enough to define ι_1 which is the same as in 2.4.1 up to a change of subscripts: $n = 1$, $n + 1 = 2$.

To prove the equivalence we define a quasi-inverse: Let $(H', V') \in \mathsf{FHS_{n+1}}$ and satisfying a, b, c and $\alpha : F^n H'_{\mathbb{C}} \to H'_{\text{inf}}$ as in the proposition.
Define $(H, V) \in \mathsf{FHS_n}$ in the following way: $H = H'/\alpha(F^n H'_{\mathbb{C}})$; $V_i = V'_i$ for all $1 \leq i \leq n$; $h : H'/\alpha(F^n H'_{\mathbb{C}}) \xrightarrow{\bar{h}'} V'_{n+1} \xrightarrow{v'_{n+1}} V'_n = V_n$, where $\bar{h}' = (h'_{\mathbb{Z}}, h'_{\text{inf}} \bmod F^n)$. □

Proposition 2.4.3. *Let $n, k > 0$ and denote by $\iota_k \mathsf{FHS_n} \subset \mathsf{FHS_{n+k}}$ the essential image of $\mathsf{FHS_n}$ (See the previous proposition). Then $\iota_k \mathsf{FHS_n} \subset \mathsf{FHS_{n+k}}$ is an abelian (not full) sub-category closed under kernels, cokernels and extensions.*

Proof. Note that it is sufficient to prove the result for $k = 1$, the other cases follow by induction. In order to simplify the notations we just consider $n = 1$.

Let (H, V), (H', V') be two objects of $\iota_1\mathsf{FHS}_1$ and $(f, \phi) : (H, V) \to (H', V')$ a morphism in FHS_2. Then $\operatorname{Ker} f_{\mathbb{Z}}$ (resp. $\operatorname{Coker} f_{\mathbb{Z}}$) is a mixed Hodge structure of level ≤ 1, because gr^W is an exact functor and the type just depend on the bi-grading of $\mathrm{gr}^W H_{\mathbb{C}}$. By the exactness of $(H, V) \mapsto V$, with values in the category Vec_2, it follows that $(H, V) \mapsto V^0$ is also an exact functor. Hence $\operatorname{Ker} \phi^0$ (resp. $\operatorname{Coker} \phi^0$) is of the form $\mathrm{id} : K \to K$ as expected.
It remains to prove the condition that $F^1 \operatorname{Ker} f_{\mathbb{C}} \subset \operatorname{Ker} f_{\inf}$ (resp. $F^1 \operatorname{Coker} f_{\mathbb{C}} \subset \operatorname{Coker} f_{\inf}$) and that the augmentation induces the identity on this sub-object. This follows by the commutativity of the following diagram of $\mathbb{C}$-vector spaces

$$\begin{tikzcd}
F^1 H_{\mathbb{C}} \arrow[rrr, "f_{\mathbb{C}}"] \arrow[dr] \arrow[dd, "\alpha"'] & & & F^1 H'_{\mathbb{C}} \arrow[dl] \arrow[dd, "\alpha'"] \\
& V_2 \arrow[r] & V'_2 & \\
H_{\inf} \arrow[ur, "h_{\inf}"] \arrow[rrr, "f_{\inf}"'] & & & H'_{\inf} \arrow[ul, "h'_{\inf}"']
\end{tikzcd}$$

Finally we have to show that $\iota_k\mathsf{FHS}_1$ is closed under extensions, i.e. for any exact sequence in FHS_1

$$0 \to (H', V') \to (H, V) \to (H'', V'') \to 0$$

with (H', V'), (H'', V'') in $\iota_1\mathsf{FHS}_1$, then (H, V) is also an object of $\iota_1\mathsf{FHS}_1$. Again the conditions on $H_{\mathbb{Z}}$ and V follow by the exactness of gr^W and $(H, V) \mapsto V$ (See 2.1.2). Moreover recall that if $H_{\mathbb{Z}} \in \mathrm{Ext}^1_{\mathsf{MHS}}(H''_{\mathbb{Z}}, H'_{\mathbb{Z}})$ we can suppose $H_{\mathbb{C}} = H'_{\mathbb{C}} \oplus H''_{\mathbb{C}}$ and that the weight filtration is also given component-wise; but the Hodge filtration of $H_{\mathbb{Z}}$ is of the form $F' + \phi(F'') \oplus F''$ where: $\phi : H''_{\mathbb{C}} \to H'_{\mathbb{C}}$ is a $\mathbb{C}$-linear map compatible w.r.t. the weight filtrations; F' (resp. F'') is the Hodge filtration of $H'_{\mathbb{C}}$ (resp. $H''_{\mathbb{C}}$). Hence we have a commutative diagram with exact rows and columns

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & F^1H'_{\mathbb{C}} \arrow[r] \arrow[d] & F^1H'_{\mathbb{C}} + \phi(F^1H''_{\mathbb{C}}) \oplus F^1H''_{\mathbb{C}} \arrow[r] \arrow[d] & F^1H''_{\mathbb{C}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & V'_2 \arrow[r] \arrow[d] & V_2 \arrow[r] \arrow[d] & V''_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H'_{\mathbb{C}}/F^1H'_{\mathbb{C}} \arrow[r] \arrow[d] & H_{\mathbb{C}}/F^1H_{\mathbb{C}} \arrow[r] \arrow[d] & H''_C/F^1H''_{\mathbb{C}} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

Then after choosing (non canonically) a splitting of the exact sequence

$$0 \to H'_{\mathrm{inf}} \to H_{\mathrm{inf}} \to H''_{\mathrm{inf}} \to 0$$

we can define a map $F^1H_{\mathbb{C}} \to H_{\mathrm{inf}}$ compatible with the given extension. $\square$

Remark 2.4.4. Note that $\iota_k\mathsf{FHS_n} \subset \mathsf{FHS_{n+k}}$ it is not closed under sub-objects.

Remark 2.4.5. Let $\mathsf{FHS}_\mathsf{n}^\mathsf{prp}$ be the full sub-category of $\mathsf{FHS_n}$ whose objects are formal Hodge structures (H, V) with $H_{\mathrm{inf}} = 0$[1]. Then ι_k induces a full and faithful functor

$$\iota = \iota_k : \mathsf{FHS}_\mathsf{n}^\mathsf{prp} \to \mathsf{FHS}_\mathsf{n+k}^\mathsf{prp}$$

Moreover $\iota_k\mathsf{FHS}_\mathsf{n}^\mathsf{prp} \subset \mathsf{FHS}_\mathsf{n+k}^\mathsf{prp}$ is an abelian thick sub-category.

Example 2.4.6 (Special structures). For special structures it is natural to consider the following construction, similar to ι_k (Compare with 2.4.1). Let (H, V) be a formal Hodge structures of level ≤ 1. Define $\tau(H, V) = (H, V')$ to be the formal Hodge structure of level ≤ 2 represented by the following diagram

$$
\begin{tikzcd}
H_{\mathbb{Z}} \arrow[r] \arrow[dr] & H_{\mathbb{C}} \arrow[r, "h_{\mathbb{C}}"] & H_{\mathbb{C}}/F^1 \\
H_{\mathrm{inf}} \arrow[r, "h'_{\mathrm{inf}}"'] & V'_2 \arrow[r, "v'_2"'] \arrow[u, "\pi'_2"] & V_1 \arrow[u, "\pi_1"]
\end{tikzcd}
$$

[1]The superscript prp stands for proper. In fact the sharp cohomology objects (3.1.3) of a proper variety have this property.

where $V_2',\ v_2',\ h_{\mathrm{inf}}'$ are defined via fiber product as follows

$$\begin{tikzcd} H_{\mathrm{inf}} \arrow[dr, dotted, "h_{\mathrm{inf}}'"] \arrow[drr, bend left, "0"] \arrow[ddr, bend right, "h_{\mathrm{inf}}"'] & & \\ & V_2' \arrow[r, "\pi_2'"] \arrow[d, "v_2'"] & H_{\mathbb{C}} \arrow[d] \\ & V_1 \arrow[r, "\pi_1"'] & H_{\mathbb{C}}/F^1 \end{tikzcd}$$

Note that the commutativity of the external square is equivalent to say that (H, V) is special. Hence this construction cannot be used for general formal Hodge structures.

Proposition 2.4.7. *Let $n, k > 0$ integers. Then there exists a full and faithful functor*

$$\tau = \tau_k : \mathsf{FHS}_{\mathsf{n}}^{\mathsf{spc}} \to \mathsf{FHS}_{\mathsf{n+k}}^{\mathsf{spc}}$$

Moreover the essential image of τ_k, $\tau_k \mathsf{FHS}_{\mathsf{n}}^{\mathsf{spc}}$, is the full and thick abelian sub-category of $\mathsf{FHS}_{\mathsf{n+k}}^{\mathsf{spc}}$ with objects (H, V) such that

a) $H_{\mathbb{Z}}$ is of level $\leq n$. Hence $F^{n+1}H_{\mathbb{C}} = 0$ and $F^0 H_{\mathbb{C}} = H_{\mathbb{C}}$.

b) $V_{n+i} = V_{n+1}$ for $1 \leq i \leq k$.

c) $V_{n+1} = H_{\mathbb{C}} \times_{H_{\mathbb{C}}/F^n} V_n$.

Proof. Note that $\tau_k = \tau_1 \circ \tau_{k-1}$, hence is enough to construct τ_1. Let (H, V) be a special formal Hodge structure of level $\leq n$, then $\tau_1(H, V)$ is defined as in 2.4.6 up to change the sub-scripts $n = 1,\ n + 1 = 2$.

To prove the equivalence it is enough to construct a quasi-inverse of τ_1. Let (H', V') be a special formal Hodge structure of level $\leq n$ satisfying the conditions a, b, c of the proposition, then define $(H, V) \in \mathsf{FHS}_{\mathsf{n}}$ as follows: $H := H'$; $V_i := V_i'$ for all $1 \leq i \leq n$; $h = v_{n+1}' \circ h'$.

Thickness follows directly from the exactness of the functors

$$(H, V) \mapsto H_{\mathbb{Z}} , \quad (H, V) \mapsto V^0 .$$

□

Remark 2.4.8. The functors τ_k, ι_k agree on the full sub-category of $\mathsf{FHS}_{\mathsf{n}}$ formed by (H, V) with $H_{\mathrm{inf}} = 0$.

2.5 Extensions of FHS

Example 2.5.1. Let $\boldsymbol{A}$ be an abelian variety over $\mathbb{C}$. Then we can consider the formal Hodge structure of level ≤ 1 $(H_1(\boldsymbol{A}_{\mathrm{an}}, \mathbb{Z})(-1), \mathrm{Lie}\,\boldsymbol{A})$ (See 1.4.3). Then there is an extension in $\mathsf{FHS_1}$

$$0 \to (0, H^0(\boldsymbol{A}^\vee_{\mathrm{an}}, \Omega^1)) \to (H_1(\boldsymbol{A}_{\mathrm{an}}, \mathbb{Z})(-1), H_1(\boldsymbol{A}_{\mathrm{an}}, \mathbb{C})) \to (H_1(\boldsymbol{A}_{\mathrm{an}}, \mathbb{Z})(-1), \mathrm{Lie}\,\boldsymbol{A}) \to 0$$

corresponding to the universal vector extension

$$0 \to \underline{\omega}_{\boldsymbol{A}^\vee} \to \boldsymbol{A}^\natural \to \boldsymbol{A} \to 0$$

via the formal Hodge realization $T_{\oint}$.

The above construction has been generalized to Laumon 1-motives and formal Hodge structures of level ≤ 1 in [5]. This motivates the following definition.

Definition 2.5.2 (Sharp envelope)**.** Let $(H, V) \in \mathsf{FHS_n}$. We define a new sharp structure $(H, V)^\natural_\times := (H, V')$ where $V'_i = H_\mathbb{C} \times H_{\mathrm{inf}}$ and the augmentation $h : H \to V'_n = H_\mathbb{C} \times H_{\mathrm{inf}}$ is induced by the identity. Moreover we define the *sharp envelope* $(H, V)^\sharp$ of (H, V) via the following fiber product

$$\begin{tikzcd}
(H,V)^\sharp \arrow[r] \arrow[d] & (H,V) \arrow[d, "p"] \\
(H,V)^\natural_\times \arrow[r, "q"'] & (H,V)_\times
\end{tikzcd}$$

where p is the canonical projection (see 2.3.2) and $q : (H, V)^\natural_\times \to (H, V)_\times$ is induced by the canonical epimorphisms : $H \to H_\mathbb{C}/F^i$, for $1 \leq i \leq n$.

Proposition 2.5.3. *The association* $(H, V) \mapsto (H, V)^\sharp$ *induces a covariant functor*

$$(-)^\sharp : \mathsf{FHS_n} \to \mathsf{FHS_n}$$

Proof. We already know that the association $(H, V) \mapsto (H, V)_\times$ is functorial (see 2.3.2). Hence given a morphism of formal Hodge structures of level $\leq n$ we get a commutative diagram

$$\begin{tikzcd}
(H,V) \arrow[r, "(f,\phi)"] \arrow[d] & (H',V') \arrow[d] \\
(H,V)_\times \arrow[r, "(f,\phi)_\times"'] & (H',V')_\times
\end{tikzcd}$$

To conclude the proof note that $(H, H_\mathbb{C}/F) \mapsto (H, H_\mathbb{C})$ induces a functor form $\mathsf{FHS_{n,\times}}$ to FHS: here $\mathsf{FHS_{n,\times}}$ is the full sub-category of $\mathsf{FHS_n}$ with objects (H, V) such that $V^0 = 0$. □

Definition 2.5.4. Let $(H,V) \in \mathsf{FHS_n}$. We define $(H,V)_\sharp := (H \times V_n, V)$ with augmentation $(h_{\mathrm{inf}}, \mathrm{id}) : H_{\mathrm{inf}} \times V_n \to V_n$. We call it the *universal extension of an infinitesimal structure by* (H,V): in fact it is characterized by the following universal property.

Proposition 2.5.5 (Universal property of $(H,V)_\sharp$)**.** *Let E be a finite dimensional $\mathbb{C}$-vector space and $(H,V) \in \mathsf{FHS_n}$. Then there is a isomorphism*

$$\epsilon : \mathrm{Hom}_{\mathbb{C}}(E, V_n) \to \mathrm{Ext}^1_{\mathsf{FHS_n}}((E,0),(H,V))$$

functorial in E.

Explicitly ϵ is obtained via the pull-back of the following extension

$$0 \to (H,V) \to (H,V)_\sharp \to (V_n, 0) \to 0 \ .$$

Proof. Let $0 \to (H,V) \to (H',V') \to (E,0) \to 0$ be an extension of formal Hodge structures of level $\leq n$. By the exactness of the forgetful functors (see 2.2.7) we get $H' = H \times E$, $V' = V$. Hence the (H',V') is determined by the augmentation $h' : H' \to V_n$. By definition of morphism of formal Hodge structures we get $h'(x+y,z) = h(x,z) + e(y)$ for $x, \in H_{\mathrm{inf}}$, $y \in E$, $z \in H_{\mathbb{Z}}$ for some $e : E \to V_n$. It easy to check that for different choices of e we get different equivalence classes. □

It follows by the proposition above that the association $(H,V) \mapsto (H,V)_\sharp$ is functorial.

Remark 2.5.6. In $\mathsf{FHS_1}(1)^{\mathrm{free}}$ we get the following formula

$$((H,V)^\sharp)^\vee = ((H,V)^\vee)_\sharp$$

where $(H,V) \mapsto (H,V)^\vee$ is the functor induced by Cartier duality on free 1-motives. This can be proven directly or using the formal Hodge realization 1.4.1 and 1.3.9.

Note that the universal property of $(H,V)_\sharp$ generalizes to any level the result known for 1-motives (see 1.3.5).

Proposition 2.5.7. *Let $H_{\mathbb{Z}}$ be a mixed Hodge structure of level $\leq n$: we consider it as an étale formal Hodge structure. Let (H',V') be be a formal Hodge structure of level $\leq n$ (for $n > 0$). Then*

i) There is a canonical isomorphism of abelian groups

$$\mathrm{Ext}^1_{\mathsf{MHS}}(H_{\mathbb{Z}}, H'_{\mathbb{Z}}) \cong \mathrm{Ext}^1_{\mathsf{FHS_n}}(H_{\mathbb{Z}}, (H', V'/V'^0)) \ .$$

ii) For any $i \geq 2$ there is a canonical isomorphism

$$\mathrm{Ext}^i_{\mathsf{FHS_n}}(H_{\mathbb{Z}}, (H', V'/V'^0)) \cong \mathrm{Ext}^i_{\mathsf{FHS_n}}(H_{\mathbb{Z}}, (H'_{\mathrm{inf}}, 0)) \ .$$

Proof. This follows by the computation of the long exact sequence obtained applying $\mathrm{Hom}_{\mathsf{FHS_n}}(H_{\mathbb{Z}}, -)$ to the short exact sequence

$$0 \to (H', V')_{\mathrm{et}} \to (H', V')_{\times} \to (H'_{\mathrm{inf}}, 0) \to 0$$

In fact the associated long exact sequence is the following

$$0 \to \mathrm{Hom}(H_{\mathbb{Z}}, H'_{\mathbb{Z}}) \to \mathrm{Hom}(H_{\mathbb{Z}}, (H', V'/V'^0)) \to \mathrm{Hom}(H_{\mathbb{Z}}, (H'_{\mathrm{inf}}, 0)) \xrightarrow{\partial}$$

$$\xrightarrow{\partial} \mathrm{Ext}^1(H_{\mathbb{Z}}, H'_{\mathbb{Z}}) \to \mathrm{Ext}^1(H_{\mathbb{Z}}, (H', V'/V'0)) \to \mathrm{Ext}^1(H_{\mathbb{Z}}, (H'_{\mathrm{inf}}, 0))$$

First note that $\mathrm{Hom}(H_{\mathbb{Z}}, (H'_{\mathrm{inf}}, 0)) = 0$ and $\mathrm{Ext}^1_{\mathsf{FHS_n}}(H_{\mathbb{Z}}, H'_{\mathbb{Z}}) = \mathrm{Ext}^1_{\mathsf{MHS}}(H_{\mathbb{Z}}, H'_{\mathbb{Z}})$ by construction. Then we have to show that $\mathrm{Ext}^1(H_{\mathbb{Z}}, (H'_{\mathrm{inf}}, 0)) = 0$. Let $(\tilde{H}, \tilde{V})$ be an extension of $H_{\mathbb{Z}}$ by $(H'_{\mathrm{inf}}, 0)$ in $\mathsf{FHS_n}$. It is easy to see that $(\tilde{H}, \tilde{V})$ is represented by a diagram of the following type

$$\begin{tikzcd}
H_{\mathbb{Z}} \arrow[r] \arrow[dr, "h_{\mathbb{Z}}"] & H_{\mathbb{C}}/F^n \arrow[r] & H_{\mathbb{C}}/F^{n-1} \arrow[r] & \cdots \arrow[r] & H_{\mathbb{C}}/F^1 \\
H'_{\mathrm{inf}} \arrow[r, "h_{\mathrm{inf}}"'] & H_{\mathbb{C}}/F^n \arrow[u, "\mathrm{id}"] \arrow[r] & H_{\mathbb{C}}/F^{n-1} \arrow[u, "\mathrm{id}"] \arrow[r] & \cdots \arrow[r] & H_{\mathbb{C}}/F^1 \arrow[u, "\mathrm{id}"]
\end{tikzcd}$$

and $(H'_{\mathrm{inf}}, 0) \subset (\tilde{H}, \tilde{V})$, hence the augmentation $h'_{\mathrm{inf}} = 0$. We can conclude that $(\tilde{H}, \tilde{V}) = (H'_{\mathrm{inf}}, 0) \oplus H_{\mathrm{et}}$ is the trivial extension.
From the above discussion we get (i). To prove (ii) just continue the long exact sequence and use that $\mathrm{Ext}^i_{\mathsf{MHS}}(H_{\mathbb{Z}}, H'_{\mathbb{Z}}) = 0$ for $i \geq 2$ (B.3.5). □

Remark 2.5.8. With similar arguments we can show that there exists an exact sequence

$$0 \to \mathrm{Ext}^1_{\mathsf{FHS_n}}(H_{\mathbb{Z}}, (0, V'^0)) \to \mathrm{Ext}^1_{\mathsf{FHS_n}}(H_{\mathbb{Z}}, (H', V')) \to \mathrm{Ext}^1_{\mathsf{MHS}}(H_{\mathbb{Z}}, H'_{\mathbb{Z}}) \to \mathrm{Ext}^2_{\mathsf{FHS_n}}(H_{\mathbb{Z}}, (0, V'^0))\ .$$

In fact just apply $\mathrm{Hom}(H_{\mathbb{Z}}, -)$ to the short exact sequence

$$0 \to (0, V'^0) \to (H', V') \to (H', V'/V'^0) \to 0$$

and use the previous proposition.

Proposition 2.5.9. *The forgetful functor $(H, V) \mapsto H_{\mathbb{Z}}$ induces a natural and surjective morphism of abelian groups*

$$\gamma : \mathrm{Ext}^1_{\mathsf{FHS_n}}((H, V), (H', V')) \to \mathrm{Ext}^1_{\mathsf{MHS}}(H_{\mathbb{Z}}, H'_{\mathbb{Z}})\ .$$

Proof. Recall the extension formula for mixed Hodge structures is (see B.3.4)

$$\mathrm{Ext}^1_{\mathsf{MHS}}(H_{\mathbb{Z}}, H'_{\mathbb{Z}}) \cong \frac{W_0\mathcal{H}om(H_{\mathbb{Z}}, H'_{\mathbb{Z}})_{\mathbb{C}}}{F^0 \cap W_0(\mathcal{H}om(H_{\mathbb{Z}}, H'_{\mathbb{Z}})_{\mathbb{C}}) + W_0\mathcal{H}om(H_{\mathbb{Z}}, H'_{\mathbb{Z}})_{\mathbb{Z}}}$$

more precisely we get that any extension class can be represented by $\tilde{H}_{\mathbb{Z}} = (H'_{\mathbb{Z}} \oplus H_{\mathbb{Z}}, W, F_\theta)$ where the weight filtration is the direct sum $W_iH'_{\mathbb{Z}} \oplus W_iH_{\mathbb{Z}}$ and $F^i_\theta := F^iH'_{\mathbb{Z}} + \theta(F^iH_{\mathbb{Z}}) \oplus F^iH_{\mathbb{Z}}$, for some $\theta \in W_0\mathcal{H}om(H_{\mathbb{Z}}, H'_{\mathbb{Z}})_{\mathbb{C}}$. It follows that $\tilde{H}_{\mathbb{C}}/F^i_\theta = H'_{\mathbb{C}}/F^i \oplus H_{\mathbb{C}}/F^i$. Then we can consider the formal Hodge structure of level $\leq n$ $(\tilde{H}, \tilde{V})$ defined as follows: $\tilde{H}_{\mathbb{Z}} = (H'_{\mathbb{Z}} \oplus H_{\mathbb{Z}}, W, F_\theta)$ as above; $\tilde{H}_{\mathrm{inf}} := H'_{\mathrm{inf}} \oplus H_{\mathrm{inf}}$; $\tilde{V}_i := V'_i \oplus V_i$, $\tilde{v}_i := (v'_i, v_i)$; $\tilde{h} = (h', h)$. Then it easy to check that $(\tilde{H}, \tilde{V}) \in \mathrm{Ext}^1_{\mathsf{FHS_n}}((H', V'), (H, V))$ and $\gamma(\tilde{H}, \tilde{V}) = (H'_{\mathbb{Z}} \oplus H_{\mathbb{Z}}, W, F_\theta)$. ☐

Example 2.5.10 (Infinitesimal deformation). Let $f : \widehat{X} \to \operatorname{Spec} \mathbb{C}[\epsilon]/(\epsilon^2)$ a smooth and projective morphism. Write $X/\mathbb{C}$ for the smooth and projective variety corresponding to the special fiber, i.e. the fiber product

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & \widehat{X} \arrow[d, "f"] \\
\operatorname{Spec} \mathbb{C} \arrow[r] & \operatorname{Spec} \mathbb{C}[\epsilon]/(\epsilon^2)
\end{tikzcd}$$

then (see [12, 2.4]) for any i, n there is a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & H^{n-i+1}(X_{\mathrm{an}}, \Omega^{i-1}) \arrow[r] \arrow[d, "0"] & H^n(\widehat{X}_{\mathrm{an}}, \Omega^{<i}) \arrow[r] \arrow[d] & H^n(X_{\mathrm{an}}, \mathbb{C})/F^i \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^{n-i+2}(X_{\mathrm{an}}, \Omega^{i-2}) \arrow[r] & H^n(\widehat{X}_{\mathrm{an}}, \Omega^{<i-1}) \arrow[r] & H^n(X_{\mathrm{an}}, \mathbb{C})/F^{i-1} \arrow[r] & 0
\end{tikzcd}$$

Hence there is an extension of formal Hodge structures of level $\leq n$

$$0 \to (0, V) \to (H^n(X_{\mathrm{an}}, \mathbb{Z}), H^n(\widehat{X}_{\mathrm{an}}, \Omega^{<*})) \to (H^n(X_{\mathrm{an}}, \mathbb{Z}), H^n(X_{\mathrm{an}}, \mathbb{C})/F) \to 0$$

with $V_i = H^{n-i+1}(X_{\mathrm{an}}, \Omega^{i-1})$ and $v_i = 0$.

2.5.1 Formal Carlson theory

Proposition 2.5.11. *Let A, B torsion-free mixed Hodge structures. Suppose B pure of weight $2p$ and A of weights $\leq 2p - 1$. There is a commutative* *diagram of complex Lie group*

$$\begin{tikzcd} \mathrm{Ext}^1_{\mathsf{MHS}}(B,A) \arrow[r, "\gamma"] \arrow[dr, "i^*"'] & \mathrm{Hom}_{\mathbb{Z}}(B^{p,p}_{\mathbb{Z}}, J^p(A)) \\ & \mathrm{Ext}^1_{\mathsf{MHS}}(B^{p,p}_{\mathbb{Z}}, A) \arrow[u, "\bar{\gamma}"] \end{tikzcd}$$

where $\bar{\gamma}$ is an isomorphism; i^* is the surjection induced by the inclusion $i : B^{p,p}_{\mathbb{Z}} \to B$.

Proof. This follows easily from B.3.5. The construction of γ, $\bar{\gamma}$ is given in the following remark. Then choosing a basis of $B^{p,p}_{\mathbb{Z}}$ it is easy to check that $\bar{\gamma}$ is an isomorphism. $\square$

Remark 2.5.12. i) Let $\{b_1, ..., b_n\}$ a $\mathbb{Z}$-basis of $B^{p,p}_{\mathbb{Z}}$, then $\mathrm{Hom}_{\mathbb{Z}}(B^{p,p}_{\mathbb{Z}}, J^p(A)) \cong \oplus_{i=1}^n J^p(A)$ which is a complex Lie group.

ii) Explicitly γ can be constructed as follows. Let $x \in \mathrm{Ext}^1_{\mathsf{MHS}}(B,A)$ represented by the extension

$$0 \to A \to H \to B \to 0$$

then apply $\mathrm{Hom}_{\mathsf{MHS}}(\mathbb{Z}(-p), -)$ to the above exact sequence and consider the boundary of the associated long exact sequence

$$\cdots \to \mathrm{Hom}_{\mathsf{MHS}}(\mathbb{Z}(-p), B) \xrightarrow{\partial_x} \mathrm{Ext}^1_{\mathsf{MHS}}(\mathbb{Z}(-p), A) \to \cdots$$

Note that ∂_x does not depend on the choice of the representative of x; $\mathrm{Hom}_{\mathsf{MHS}}(\mathbb{Z}(-p), B) = B^{p,p}_{\mathbb{Z}}$; $J^p(A) = \mathrm{Ext}^1_{\mathsf{MHS}}(\mathbb{Z}(-p), A)$.

Hence we can define $\gamma(x) := \partial_x \in \mathrm{Hom}_{\mathbb{Z}}(B^{p,p}_{\mathbb{Z}}, J^p(A))$.

iii) If the complex Lie group $J^p(A)$ is algebraic then $\mathrm{Hom}_{\mathbb{Z}}(B^{p,p}_{\mathbb{Z}}, J^p(A))$ can be identified with set of one motives of type

$$u : B^{p,p}_{\mathbb{Z}} \to J^p(A)$$

Definition 2.5.13 (formal-p-Jacobian)**.** Let (H, V) be a formal Hodge structure of level $\leq n$. Assume $H_{\mathbb{Z}}$ is a torsion free mixed Hodge structure. For $1 \leq p \leq n$ the *p-th formal Jacobian of* (H, V) is defined as

$$J^p_\sharp(H, V) := V_p/H_{\mathbb{Z}}.$$

where $H_{\mathbb{Z}}$ acts on V_p via the augmentation h. By construction there is an extension of abelian groups

$$0 \to V^0_p \to J^p_\sharp(H, V) \to J^p(H, V) \to 0$$

where we define $J^p(H, V) := J^p(H_{\mathbb{Z}}) = H_{\mathbb{C}}/(F^p + H_{\mathbb{Z}})$.

By lemma B.3.7 it follows that $J^p_\sharp(H,V)$ is a complex Lie group if the weights of $H_\mathbb{Z}$ are $\leq 2p-1$.

Proposition 2.5.14. *There is an extension of abelian groups*

$$0 \to V_p^0 \to \mathrm{Ext}^1_{\mathsf{FHS_p}}(\mathbb{Z}(-p),(H,V)) \to \mathrm{Ext}^1_{\mathsf{MHS}}(\mathbb{Z}(-p),H_\mathbb{Z}) \to 0$$

for any (H,V) formal Hodge structure of level $\leq p+1$. In particular if $H_\mathbb{Z}$ has weights $\leq 2p-1$ there is an extension

$$0 \to V_p^0 \to \mathrm{Ext}^1_{\mathsf{FHS_p}}(\mathbb{Z}(-p),(H,V)) \to J^p(H_\mathbb{Z}) \to 0 \ . \tag{2.1}$$

Proof. By 2.5.9 there is a surjective map

$$\gamma : \mathrm{Ext}^1_{\mathsf{FHS_p}}(\mathbb{Z}(-p),(H,V)) \to \mathrm{Ext}^1_{\mathsf{MHS}}(\mathbb{Z}(-p),H_\mathbb{Z}) \ .$$

Recall that $\mathbb{Z}(-p)$ is a mixed Hodge structure and here is considered as a formal Hodge structure of level $\leq p$ represented by the following diagram

$$\begin{tikzcd}
\mathbb{Z} \arrow[r] \arrow[dr, "h_\mathbb{Z}"] & 0 \arrow[r] & \cdots \\
0 \arrow[r, "h_{\inf}"'] & 0 \arrow[u] \arrow[r] & \cdots
\end{tikzcd}$$

It follows directly from the definition of a morphism of formal Hodge structures that an element of $\operatorname{Ker}\gamma$ is a formal Hodge structure of the form $(H \times \mathbb{Z}(-p), H/F)$ represented by

$$\begin{tikzcd}
H_\mathbb{Z} \times \mathbb{Z} \arrow[r] \arrow[dr, "h'_\mathbb{Z}"] & H_\mathbb{C}/F^n \arrow[r] & H_\mathbb{C}/F^{n-1} \arrow[r] & \cdots \arrow[r] & H_\mathbb{C}/F^1 \\
H_{\inf} \arrow[r, "h_{\inf}"'] & V_n \arrow[u, "\pi_n"] \arrow[r, "v_n"'] & V_{n-1} \arrow[u, "\pi_{n-1}"] \arrow[r, "v_{n-1}"'] & \cdots \arrow[r] & V_1 \arrow[u, "\pi_1"]
\end{tikzcd}$$

where the augmentation $h'_\mathbb{Z}(x,z) = h_\mathbb{Z}(x) + \theta(z)$ for some $\theta : \mathbb{Z} \to V_p^0$. The map θ does not depend on the representative of the class of the extension because V_p and $\mathbb{Z}(-p)$ are fixed. □

Example 2.5.15. By the previous proposition for $p=1$ we get

$$0 \to (V_1)^0 \to \mathrm{Ext}^1_{\mathsf{FHS_1}}(\mathbb{Z}(-1),(H,V)) \to \mathrm{Ext}^1_{\mathsf{MHS}}(\mathbb{Z}(-1),H_\mathbb{Z}) \to 0 \ .$$

2.6 Albanese varieties

2.6.1 The generalized Albanese of Esnault-Srinivas-Viehweg

Let X be proper and irreducible algebraic scheme of dimension d over $\mathbb{C}$. Then there exists an algebraic group, say $\mathrm{ESV}(X)$, such that $\mathrm{ESV}(X)_{\mathrm{an}} = H^{2d-1}(X, \Omega^{<d})/H^{2d-1}(X_{\mathrm{an}}, \mathbb{Z})$ and it fits in the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}\, c \arrow[r] \arrow[d, "\rho"] & \frac{H^{2d-1}(X_{\mathrm{an}},\mathbb{C})}{H^{2d-1}(X_{\mathrm{an}},\mathbb{Z})} \arrow[r, "c"] \arrow[d, "\alpha"] & J^d(H^{2d-1}(X_{\mathrm{an}}, \mathbb{Z})) \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & \mathrm{Ker}\, \theta \arrow[r] & \frac{H^{2d-1}(X_{\mathrm{an}},\Omega^{<d})}{H^{2d-1}(X_{\mathrm{an}},\mathbb{Z})} \arrow[r, "\theta"] & J^d(H^{2d-1}(X_{\mathrm{an}}, \mathbb{Z})) \arrow[r] & 0
\end{tikzcd}$$

where α is induced by de canonical map of complexes of sheaves $\mathbb{C} \to \Omega^{<d}$. (See [20, Theorem 1, Lemma 3.1])

Recall that the formal Hodge structure of level $\leq 2d-1$ $H_{\sharp}^{2d-1,d}(X)$ can be viewed as fhs of level $\leq d$ (see 3.1.4) represented by the following diagram

$$\begin{tikzcd}
H^{2d-1}(X_{\mathrm{an}}, \mathbb{Z}) \arrow[r] \arrow[dr, "h"] & H^{2d-1}(X_{\mathrm{an}}, \mathbb{C})/F^d \arrow[r] & \cdots H^{2d-1}(X_{\mathrm{an}}, \mathbb{C})/F^1 \\
& H^{2d-1}(X, \Omega^{<d}) \arrow[r] \arrow[u] & \cdots H^{2d-1}(X, \mathcal{O}) \arrow[u]
\end{tikzcd}$$

Proposition 2.6.1. *There is an isomorphism of complex connected Lie groups (not only of abelian groups!)*

$$\mathrm{ESV}(X)_{\mathrm{an}} \cong \mathrm{Ext}^1_{\mathsf{FHS_d}}(\mathbb{Z}(-d), H_{\sharp}^{2d-1,d}(X))$$

where $\mathbb{Z}(-d)$ is the Tate structure of type (d,d) viewed as an étale formal Hodge structure.

Proof. Step 1. By [8] there is a canonical isomorphism of Lie groups

$$\mathrm{ESV}_{\mathrm{an}}(X) \cong \mathrm{Ext}^1_{{}^t\mathcal{M}_1^{\mathrm{a}}}([\mathbb{Z} \to 0], [0 \to \mathrm{ESV}(X)]) \cong \mathrm{Ext}^1_{\mathsf{FHS_1(1)}}(\mathbb{Z}(0), T_{\oint}(\mathrm{ESV}(X)))$$

(recall that in [8] $\mathsf{FHS_1}(1)$ is simply denote by $\mathsf{FHS_1}$) where $T_{\oint}(\mathrm{ESV}(X))$ is the formal Hodge structure represented by

$$\begin{tikzcd}
H^{2d-1}(X_{\mathrm{an}}, \mathbb{Z}(d)) \arrow[r] \arrow[dr] & H^{2d-1}(X_{\mathrm{an}}, \mathbb{C}(d))/F^0 \\
& H^{2d-1}(X_{\mathrm{an}}, \Omega^{<d}) \arrow[u]
\end{tikzcd}$$

Step 2. Up to a twist by $-d$ we can view $T_{\oint}(\mathrm{ESV}(X))$ diagram as an object of $\mathsf{FHS_d}$, say (H,V) with $H = H^{2d-1}(X_{\mathrm{an}}, \mathbb{Z})$, $V_d = H^{2d-1}(X_{\mathrm{an}}, \Omega^{<d})$, $V_i = 0$ for $1 \le i < d$. This is a sub-object of $H_{\sharp}^{2d-1,d}(X)$ and $\mathrm{Ext}^1_{\mathsf{FHS_1(1)}}(\mathbb{Z}(0), T_{\oint}(\mathrm{ESV}(X))) = \mathrm{Ext}^1_{\mathsf{FHS_d}}(\mathbb{Z}(-d), (H,V))$. Then applying $\mathrm{Ext}^1_{\mathsf{FHS_d}}(\mathbb{Z}(-d), -)$ to the canonical inclusion $(H,V) \subset H_{\sharp}^{2d-1,d}(X)$ we get a natural map

$$\mathrm{Ext}^1_{\mathsf{FHS_1(1)}}(\mathbb{Z}(0), T_{\oint}(\mathrm{ESV}(X))) \to \mathrm{Ext}^1_{\mathsf{FHS_d}}(\mathbb{Z}(-d), H_{\sharp}^{2d-1,d}(X))$$

which is an isomorphism by (2.1). □

2.6.2 The generalized Albanese of Faltings and Wüstholz

Let U be a smooth algebraic scheme over $\mathbb{C}$. Then it is possible to construct a smooth compactification, i.e. $\exists\, j : U \to X$ open embedding with X proper and smooth. Moreover we can suppose that the complement $Y := X \setminus U$ is a normal crossing divisor.[2]

Remark 2.6.2. There is a commutative diagram (See [32, §3])

$$\begin{tikzcd}
0 \arrow[r] & H^0(X_{\mathrm{an}}, \Omega^1(\log Y)) \arrow[r] \arrow[d, "a"] & H^1(U_{\mathrm{an}}, \mathbb{C}) \arrow[r] \arrow[d, "\mathrm{id}"] & H^1(X, \mathcal{O}) \arrow[r] \arrow[d, "b"] & 0 \\
0 \arrow[r] & H^1(\Gamma(U_{an}, \Omega^\bullet)) \arrow[r] & H^1(U_{\mathrm{an}}, \mathbb{C}) \arrow[r] & H^1(U_{\mathrm{an}}, \mathcal{O}) &
\end{tikzcd}$$

hence, by the snake lemma, $\mathrm{Ker}\, b \cong \mathrm{Coker}\, a$. We identify these two $\mathbb{C}$-vector spaces and we denote both by K.

For any $Z \subset K$ sub-vector space we define the $\mathbb{C}$-linear map $\alpha_Z : H^1(X, \mathcal{O})^* \to Z^*$ as the dual of the canonical inclusion $Z \subset H^1(X, \mathcal{O})$.

Definition 2.6.3 (The generalized Albanese of Serre)**.** We know that

$$H^1(U_{\mathrm{an}}, \mathbb{Z})(1) = T_{Hodge}([\mathrm{Div}^0_Y(X) \to \mathrm{Pic}^0(X)])$$

and that the generalized Albanese of Serre is the Cartier dual of the above 1-motive, i.e.

$$[0 \to \mathrm{Ser}(U)] = [\mathrm{Div}^0_Y(X) \to \mathrm{Pic}^0(X)]^\vee$$

Note that by construction $\mathrm{Ser}(U)$ is a semi-abelian group scheme corresponding to the mixed Hodge structure $H^1(U_{\mathrm{an}}, \mathbb{Z})(1)^* := \mathcal{H}om_{\mathsf{MHS}}(H^1(U_{\mathrm{an}}, \mathbb{Z})(1), \mathbb{Z}(1))$.

The universal vector extension of $\mathrm{Ser}(U)$ is

$$0 \to \underline{\omega}_{\mathrm{Pic}^0(X)} \to \mathrm{Ser}(U)^\natural \to \mathrm{Ser}(U) \to 0$$

[2] It is possible to replace $\mathbb{C}$ with a field k of characteristic zero. In that case we must assume that there exists a k rational point in order to have $\mathrm{FW}(Z)$ defined over k.

this follows by the construction of $\mathrm{Ser}(U)$ as the Cartier dual of $[\mathrm{Div}^0_Y(X) \to \mathrm{Pic}^0(X)]$ and [5] lemma 2.2.4.
Recall that $\mathrm{Lie}(\mathrm{Pic}^0(X)) = H^1(X, \mathcal{O})$, then $\underline{\omega}_{\mathrm{Pic}^0(X)}(\mathbb{C}) = H^1(X, \mathcal{O})^*$.

Definition 2.6.4 (The gen. Albanese of Faltings and Wüstholz)**.** We define an algebraic group $\mathrm{FW}(Z)$ (depending on U and the choice of the vector space Z) to be the vector extension of $\mathrm{Ser}(U)$ by Z^* defined by

$$\alpha_Z \in \mathrm{Hom}_{\mathbb{C}}(H^1(X, \mathcal{O})^*, Z^*) \cong \mathrm{Hom}_{\mathbb{C}}(\omega_{\mathrm{Pic}^0(X)}, Z^*) \cong \mathrm{Ext}^1(\mathrm{Ser}(U), Z^*)$$

i.e. $\mathrm{FW}(Z)$ is the following push-forward

$$\begin{tikzcd}
0 \arrow[r] & H^1(X, \mathcal{O})^* \arrow[r] \arrow[d, "\alpha_Z"] & \mathrm{Ser}(U)^\natural \arrow[r] \arrow[d] & \mathrm{Ser}(U) \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & Z^* \arrow[r] & \mathrm{FW}(Z) \arrow[r] & \mathrm{Ser}(U) \arrow[r] & 0
\end{tikzcd}$$

Proposition 2.6.5. *With the above notation consider the formal Hodge structure* $(H_{\mathbb{Z}}, V) \in \mathsf{FHS}_1$ *represented by*

$$\begin{tikzcd}
H^1(U_{\mathrm{an}}, \mathbb{Z})(1)^* \arrow[r] \arrow[dr, "h"] & H^0(X_{\mathrm{an}}, \Omega^1(\log Y))^* \\
& H^1(\Gamma(U_{\mathrm{an}}, \Omega^\bullet))^* \arrow[u, "a^*"]
\end{tikzcd}$$

(This diagram is the dual of the left square in remark 2.6.2). Recall that $K = \mathrm{Ker}\, a$*. Then*

$$\mathrm{FW}(K)_{\mathrm{an}} \cong \mathrm{Ext}^1_{\mathsf{FHS}_1}(\mathbb{Z}(-1), (H_{\mathbb{Z}}, V))$$

Proof. It is a direct consequence of 2.5.14. □

Chapter 3

Sharp Cohomology

3.1 Generalities

Let X be a scheme over $\mathbb{C}$, then we denoted by $\mathcal{A}^\bullet$ (resp. $\Omega^\bullet$) the holomorphic (resp. algebraic) De Rham complex, i.e. $\mathcal{A}^0$ (resp. Ω^0) is the structural sheaf of the complex analytic space X_{an} (resp. of the scheme X); and $\mathcal{A}^1$ (resp. Ω^1) is the sheaf of holomorphic (resp. algebraic) 1-forms and $\mathcal{A}^p := \wedge^p_{\mathcal{A}^0}\mathcal{A}^1$ (resp. $\Omega^p = \wedge^p_{\Omega^0}\Omega^1$).

Remark 3.1.1 (GAGA). Let X be a proper scheme over $\mathbb{C}$. Then the sheaf Ω^p is coherent $\mathcal{O}_X$-module and the associated analytic sheaf is $\mathcal{A}^p$.

Example 3.1.2 (Motivating ex.). [1] Let X be a proper but possibly singular complex variety. According to the basic construction of [18], we may find $\pi : X_\bullet \to X$ a proper and smooth hypercover (See [15]) such that π^* is an isomorphism on Betti cohomology and such that the Hodge structure coming from $X_\bullet$ is canonically defined depending only on X . On the other hand we can consider the naive holomorphic De Rham complex $\mathcal{A}^\bullet$ associated to the singular variety X . It is contravariant functorial and receives the constant sheaf $\mathbb{C}$ on X, so we get a splitting in cohomology

$$H^i(X_{\text{an}}, \mathbb{C}) \to H^i(X_{\text{an}}, \mathcal{A}^\bullet) \xrightarrow{\pi^*} H^i((X_{\bullet,\text{an}}), \mathcal{A}^\bullet) \cong H^i(X_{\text{an}}, \mathbb{C})$$

The Hodge filtration on $H = H^i(X_{\text{an}}, \mathbb{C})$ is the image of the stupid filtration: i.e. $F^pH = \text{im}(H^i(X_{\bullet,\text{an}}, \sigma^{\geq p}\mathcal{A}^\bullet) \to H^i(X_{\text{an}}, \mathbb{C}))$. So we define the Hodge filtration on $H^i(X_{\text{an}}, \mathcal{A}^\bullet)$ in the same way

$$F^pH^i(X_{\text{an}}, \mathcal{A}^\bullet) := \text{im}(H^i(X_{\text{an}}, \sigma^{\geq p}\mathcal{A}^\bullet) \to H^i(X_{\text{an}}, \mathcal{A}^\bullet))$$

[1] See the introduction of [12].

there is a commutative diagram

$$\begin{tikzcd}
H \arrow[r] & H/F^iH \arrow[r] & H/F^{i-1}H \arrow[r] & \cdots \arrow[r] & H/F^1H \\
H^i(X_{\mathrm{an}},\mathcal{A}^\bullet) \arrow[r] \arrow[u, "\pi^*"] & H^i(X_{\mathrm{an}},\sigma^{<i}\mathcal{A}^\bullet) \arrow[r] \arrow[u, "\pi^*"] & H^i(X_{\mathrm{an}},\sigma^{<i-1}\mathcal{A}^\bullet) \arrow[r] \arrow[u, "\pi^*"] & \cdots \arrow[r] & H^i(X,\mathcal{O}) \arrow[u, "\pi^*"] \\
H^i(X_{\mathrm{an}},\mathbb{C}) \arrow[u] & & & &
\end{tikzcd}$$

Definition 3.1.3. Let X be a proper scheme over $\mathbb{C}$, $n>0$ and $1\leq k\leq n$. We define the *sharp cohomology object* $H^{n,k}_\sharp(X)$ to be the n-formal Hodge structure represented by the following diagram

$$\begin{tikzcd}
H^n(X_{\mathrm{an}},\mathbb{Z}) \arrow[r] \arrow[dr] & H^n(X_{\mathrm{an}},\mathbb{C})/F^n \arrow[r] & \cdots \arrow[r] & H^n(X_{\mathrm{an}},\mathbb{C})/F^1 \\
& V^{n,k}_n(X) \arrow[r] \arrow[u] & \cdots \arrow[r] & V^{n,k}_1(X) \arrow[u]
\end{tikzcd}$$

where

$$V^{n,k}_i(X) := \begin{cases} H^n(X_{\mathrm{an}},\sigma^{<i}\mathcal{A}^\bullet) & \text{if } 1\leq i\leq k \\ H^n(X_{\mathrm{an}},\mathbb{C})/F^i \times_{H^n(X_{\mathrm{an}},\mathbb{C})/F^k} H^n(X_{\mathrm{an}},\sigma^{<k}\mathcal{A}^\bullet) & \text{if } k<i\leq n \end{cases}$$

In the case $n=k$ we will simply write $H^n_\sharp(X)=H^{n,n}_\sharp(X)$. This object is represented by

$$\begin{tikzcd}
H^n(X_{\mathrm{an}},\mathbb{Z}) \arrow[r] \arrow[dr] & H^n(X_{\mathrm{an}},\mathbb{C})/F^n \arrow[r] & H^n(X_{\mathrm{an}},\mathbb{C})/F^{n-1} \arrow[r] & \cdots \arrow[r] & H^n(X_{\mathrm{an}},\mathbb{C})/F^1 \\
& H^n(X,\sigma^{<n}\Omega^\bullet) \arrow[r] \arrow[u] & H^n(X,\sigma^{<n-1}\Omega^\bullet) \arrow[r] \arrow[u] & \cdots \arrow[r] & H^n(X,\Omega^0) \arrow[u]
\end{tikzcd}$$

Example 3.1.4. Let X be a proper scheme of dimension d (over $\mathbb{C}$). Then $H^{2d-1}(X_{\mathrm{an}},\mathbb{Z})$ is a mixed Hodge structure satisfying $F^{d+1}=0$ and the sharp cohomology object $H^{2d-1,d}_\sharp(X)$ is represented by

$$\begin{tikzcd}
H^{2d-1}(X_{\mathrm{an}},\mathbb{Z}) \arrow[r] \arrow[dr] & H^{2d-1}(X_{\mathrm{an}},\mathbb{C}) \arrow[r, "\mathrm{id}"] & \cdots H^{2d-1}(X_{\mathrm{an}},\mathbb{C}) \arrow[r] & H^{2d-1}(X_{\mathrm{an}},\mathbb{C})/F^d \cdots \\
& V^{2d-1,k}_n(X) \arrow[r, "\mathrm{id}"] \arrow[u] & \cdots V^{2d-1,k}_{k+1}(X) \arrow[r] \arrow[u] & H^{2d-1}(X,\sigma^{<d}\Omega^\bullet)\cdots \arrow[u]
\end{tikzcd}$$

and

$$F^{d+1}H^{2d-1}(X_{\mathrm{an}},\mathbb{C}) \subset V^{2d-1,k}_n(X) = V^{2d-1,k}_{n-1}(X) = \cdots = V^{2d-1,k}_{k+1}(X)$$

Hence, according to Proposition 2.4.2, $H_\sharp^{2d-1,d}(X)$ can be viewed as a formal Hodge structure of level $\leq d$.

Proposition 3.1.5. *For any n and $1 \leq p \leq n$, the association $X \mapsto H_\sharp^{n,p}(X)$ induces a contravariant functor $H_\sharp^{n,p}$: $\mathsf{Proper} \to \mathsf{FHS_n}$.*

Proof. It is enough to prove the claim for $p = n$. We know that $H^n(X) := H^n(X_{\mathrm{an}}, \mathbb{Z})$ along with its mixed Hodge structures is functorial in X, so for any $f : X \to Y$ we have $H^n(f) : H^n(Y) \to H^n(X)$. Also by the theory of Kähler differentials there exist a map of complexes of sheaves over X, $\phi_\bullet : f^*\Omega_Y^\bullet \to \Omega_X^\bullet$, inducing

$$\alpha : H^n(X, \tau^{<r} f^*\Omega_Y^\bullet) \longrightarrow H^n(X, \tau^{<r}\Omega_X^\bullet)$$

Moreover there exists $\beta : H^n(Y, \tau^r\Omega_Y^\bullet) \to H^n(X, \tau^{<r} f^*\Omega_Y^\bullet)$. For it is sufficient to construct a map $\beta' : H^n(Y, \tau^r\Omega_Y^\bullet) \to H^n(X, \tau^{<r} f^{-1}\Omega_Y^\bullet)$. So let $I^\bullet$ (resp. $J^\bullet$) an injective resolution[2] of $\tau^{<r}\Omega_Y^\bullet$ (resp. $\tau^{<r} f^{-1}\Omega_Y^\bullet$). Using that f^{-1} preserves quasi-isomorphisms, we have the commutative diagram

$$\begin{tikzcd} f^{-1}\tau^{<r}\Omega_Y^\bullet \arrow[r, "quis"] \arrow[d, "quis"] & J^\bullet \\ f^{-1}I^\bullet \arrow[ur, "\exists\gamma"'] & \end{tikzcd}$$

where the existence of γ follows from the fact that $J^\bullet$ is injective. So we have defined a map $\psi_r : H^n(Y, \tau^{<r}\Omega^\bullet) \to H^n(X, \tau^{<r}\Omega^\bullet)$.
Now choosing $I_r^\bullet, J_r^\bullet$ for any r it's easy to see that the maps ψ_r fit in the commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & H^n(Y, \tau^{<r}\Omega^\bullet) \arrow[r] \arrow[d, "\psi_r"] & H^n(Y, \tau^{<r-1}\Omega^\bullet) \arrow[r] \arrow[d, "\psi_{r-1}"] & \cdots \\ \cdots \arrow[r] & H^n(X, \tau^{<r}\Omega^\bullet) \arrow[r] & H^n(X, \tau^{<r}\Omega^\bullet) \arrow[r] & \cdots \end{tikzcd}$$

Now it is straightforward to check that $H_\sharp^{n,n}(g \circ f) = H_\sharp^{n,n}(f) \circ H_\sharp^{n,n}(g)$, for any $f : X \to Y$, $g : Y \to Z$. $\square$

Example 3.1.6 (No Künneth). Let X, Y be complete, connected, complex varieties. Then by Künneth formula follows

$$H^1((X \times Y)_{\mathrm{an}}, ?) = H^1(X_{\mathrm{an}}, ?) \oplus H^1(Y_{\mathrm{an}}, ?) \qquad ? = \mathbb{Z},\ \mathcal{O}$$

so that $H_\sharp^1(X \times Y, \mathbb{Z}) = H_\sharp^1(X, \mathbb{Z}) \oplus H_\sharp^1(Y, \mathbb{Z})$.

[2] By injective resolution of a complex of sheaves $A^\bullet$ we mean a quasi isomorphism $A^\bullet \to I^\bullet$, where $I^\bullet$ is a complex of injective objects.

Now we consider the cohomology groups in degree 2. With the same notation we get

$$H^2((X\times Y)_{\mathrm{an}},\mathbb{Q}) = H^2(X_{\mathrm{an}},\mathbb{Q})\oplus H^1(X_{\mathrm{an}},\mathbb{Q})\otimes H^1(Y_{\mathrm{an}},\mathbb{Q})\oplus H^2(Y_{\mathrm{an}},\mathbb{Q})$$

which is the usual decomposition of singular cohomology. Let $p : X\times Y\to X$, $q : X\times Y\to Y$ the two projections; note that

$$\mathcal{O}_{X\times Y}\to\Omega^1_{X\times Y} = \sigma^{<2}\left(p^*(\mathcal{O}_X\to\Omega^1_X)\otimes q^*(\mathcal{O}_Y\to\Omega^1_Y)\right)$$

hence there is a canonical map

$$H^2(X\times Y, p^*(\sigma^{<2}\Omega^\bullet_X)\otimes q^*(\sigma^{<2}\Omega^\bullet_Y)) = \oplus_{i=0}^2 H^{2-i}(X,\sigma^{<2}\Omega^\bullet_X)\otimes H^i(Y,\sigma^{<2}\Omega^\bullet_Y)\to H^2(X\times Y,\sigma^{<2}\Omega^\bullet)$$

which is not necessarily an isomorphism. From this follows that we cannot have a Kunneth formula for $H^{2,2}_\sharp(X\times Y)$.

Definition 3.1.7 (relative cohomology)**.** Let $f : X\to Y$ be a morphism of proper algebraic schemes over $\mathbb{C}$. Let $Hdg(-)$ be the $\mathbb{C}$-mixed Hodge complex of $(-)$ (See [4, §2]). Then we can consider the following diagram of complexes

$$\begin{tikzcd}
f^{-1}\mathbb{Z}_Y \arrow[r, "\alpha"] \arrow[d] & f^{-1}\Omega^\bullet_Y \arrow[r, "\beta"] \arrow[d, "f^*"] & f^{-1}Hdg(Y) \arrow[d] \\
\mathbb{Z}_X \arrow[r, "\alpha'"] \arrow[d] & \Omega^\bullet_X \arrow[r, "\beta'"] \arrow[d] & Hdg(X)_{\mathbb{C}} \arrow[d] \\
\mathbb{Z}_{X;Y} \arrow[r, "Cone(\alpha,\alpha')"] & \Omega^\bullet_{X;Y} \arrow[r, "Cone(\beta,\beta')"] & Cone(Hdg(f))
\end{tikzcd}$$

where $\mathbb{Z}_{X;Y} := Cone(f^{-1}\mathbb{Z}_Y\to\mathbb{Z}_X)$, $\Omega^\bullet_{X;Y} := Cone(f^*)$.
We define $H^{n,n}_\sharp(X;Y)$ to be the formal Hodge structure represented by

$$\begin{tikzcd}
H^n(X_{\mathrm{an}};Y_{\mathrm{an}},\mathbb{Z}) \arrow[r] \arrow[dr] & H^n(X_{\mathrm{an}};Y_{\mathrm{an}},\mathbb{C})/F^n \arrow[r] & \cdots \arrow[r] & H^n(X_{\mathrm{an}};Y_{\mathrm{an}},\mathbb{C})/F^1 \\
& H^n(X,\sigma^{<n}\Omega^\bullet_{X;Y}) \arrow[u] \arrow[r] & \cdots \arrow[r] & H^n(X,\sigma^{<1}\Omega^\bullet_{X;Y}) \arrow[u]
\end{tikzcd}$$

Moreover let $1\le k\le n$. We define

$$V_i^{n,k}(X;Y) := \begin{cases} H^n(X,\sigma^{<i}\Omega^\bullet_{X;Y}) & \text{if } 1\le i\le k \\ H^n(X_{\mathrm{an}},\mathbb{C}_{X,Y})/F^i\times_{H^n(X_{\mathrm{an}},\mathbb{C}_{X;Y})/F^k} H^n(X,\sigma^{<k}\Omega^\bullet_{X;Y}) & \text{if } k<i\le n\end{cases}$$

3.1.1 Sharp cohomology for curves

Definition 3.1.8. Let $U/\mathbb{C}$ be a curve and X be a compactification such that $X \setminus U = Y$ is a finite set of smooth points. Then we define the first $\sharp$-*cohomology object* of U as the sharp-structure $H^1_\sharp(U) \in \mathsf{FHS_1}$ represented by the diagram

$$\begin{tikzcd} H^1(U_{\mathrm{an}}, \mathbb{Z}) \arrow[r] \arrow[dr] & H^1(U_{\mathrm{an}}, \mathbb{C})/F^1 \\ E(U) \arrow[r] & H^1(X_{\mathrm{an}}, \mathcal{O}_{X_{\mathrm{an}}}) \arrow[u] \end{tikzcd}$$

where $E(U) := \mathrm{Ker}(H^1(X_{\mathrm{an}}, \mathcal{O}_{X_{\mathrm{an}}}) \to H^1(U_{\mathrm{an}}, \mathcal{O}_{U_{\mathrm{an}}}))$.

This is in fact defined in order to agree with the $\sharp$-Hodge realization of the 1-motive $\mathrm{Pic}^+_a(U)$ defined in [32]: we denote it by $M^1_\sharp(U)$ and it is defined as follows

$$M^1_\sharp(U) := \left[\widehat{E(U)} \times \mathrm{Div}^0_Y(X) \xrightarrow{u} \mathrm{Pic}^0(X)\right]$$

where $\widehat{E(U)}$ is the infinitesimal formal group with Lie algebra $E(U)$; $\mathrm{Div}^0_Y(X)$ is defined via fiber product as

$$\begin{tikzcd} \mathrm{Div}^0_Y(X) \arrow[r] \arrow[d] & H^0_Y(X, \mathcal{K}^*/\mathcal{O}^*) \arrow[d] \\ \mathrm{Pic}^0(X) \arrow[r] & H^1(X, \mathcal{O}^*) \end{tikzcd}$$

and the morphism u is defined in the obvious way by this diagram and the canonical inclusion $E(U) \subset H^1(X_{\mathrm{an}}, \mathcal{O})$.

Appendix A

Algebraic and formal groups

A.1 Algebraic groups

Definition A.1.1. A (commutative) *algebraic group* over k is a scheme $\boldsymbol{G}$, separated and of finite type over k, which is a group object in the category of schemes over k, i.e. the associated functor of points factors through the category of abelian groups

$$h_{\boldsymbol{G}} : \mathsf{sch}_k \to \mathsf{mod}_{\mathbb{Z}}, \qquad h_{\boldsymbol{G}}(T) := \mathrm{Hom}_{\mathsf{sch}_k}(T, \boldsymbol{G})$$

A *morphism of algebraic group* is a morphism of schemes which induces a morphism of group functor.

Remark A.1.2. Equivalently an algebraic group scheme is a quadruple $(\boldsymbol{G}, e, m, i)$, where $\boldsymbol{G} \in \mathsf{Sch_k}$ and

$$\mathrm{Spec}(k) \xrightarrow{e} \boldsymbol{G} \qquad \boldsymbol{G} \times_k \boldsymbol{G} \xrightarrow{m} \boldsymbol{G} \qquad \boldsymbol{G} \xrightarrow{i} \boldsymbol{G}$$

are morphisms in Sch_k such that the following diagrams commute

Go1
$$\begin{tikzcd}
\boldsymbol{G} \times_k \boldsymbol{G} \times_k \boldsymbol{G} \arrow[r, "m \times \mathrm{id}_{\boldsymbol{G}}"] \arrow[d, "\mathrm{id}_{\boldsymbol{G}} \times m"'] & \boldsymbol{G} \times_k \boldsymbol{G} \arrow[d, "m"] \\
\boldsymbol{G} \times_k \boldsymbol{G} \arrow[r, "m"'] & \boldsymbol{G}
\end{tikzcd}$$

Go2
$$\begin{tikzcd}
\boldsymbol{G} \times_k \mathrm{Spec}(k) \arrow[d, "pr_1"'] \arrow[dr, "\mathrm{id}_{\boldsymbol{G}} \times e"] & \\
\boldsymbol{G} & \boldsymbol{G} \times_k \boldsymbol{G} \arrow[l, "m"'] \\
\mathrm{Spec}(k) \times_k \boldsymbol{G} \arrow[u, "pr_2"] \arrow[ur, "e \times \mathrm{id}_{\boldsymbol{G}}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \boldsymbol{G} \times_k \boldsymbol{G} \arrow[dr, "m"] & \\
\boldsymbol{G} \arrow[ur, "(\mathrm{id}_{\boldsymbol{G}}, i)"] \arrow[r] \arrow[dr, "(i, \mathrm{id}_{\boldsymbol{G}})"'] & \mathrm{Spec}(k) \arrow[r, "e"] & \boldsymbol{G} \\
& \boldsymbol{G} \times_k \boldsymbol{G} \arrow[ur, "m"'] &
\end{tikzcd}$$
Go3

where pr_i are the canonical projections and $\boldsymbol{G} \to \mathrm{Spec}(k)$ the canonical map.

Example A.1.3 (Constant group). Let H be a finite (abstract) group and consider the k-algebra $B := k^H$ of all maps $f : H \to k$; there is a natural identification

$$B \otimes_k B \xrightarrow{\sim} k^{H \times H} \qquad f \otimes g \mapsto \{H \times H \ni (x, y) \mapsto f(x)g(y) \in k\}$$

One defines a group structure on $\boldsymbol{H} = \mathrm{Spec}(B)$ by

$$B \xrightarrow{e^\#} k \quad e^\#(f) := f(1_H) \qquad B \xrightarrow{m^\#} B \otimes_k B \quad m^\#(f)(x, y) := f(xy)\}$$

$$B \xrightarrow{i^\#} B \qquad i^\#(f)(x) := f(x^{-1})$$

These affine group schemes are called *constant finite groups* over k.

Example A.1.4 (The *additive group scheme*). Let $\mathbb{G}_a = \mathrm{Spec}(k[t])$: we have

$$\mathrm{Hom}_{\mathsf{sch}_k}(T, \mathbb{G}_a) = \mathrm{Hom}_{\mathsf{alg}_k}(k[t], \mathcal{O}_T(T)) = \mathcal{O}_T(T) \quad \forall T \in \mathsf{sch}_k$$

which has naturally the additive group structure of $\mathcal{O}_T(T)$. It's easily verified that

$$e^\#(t) = 0 \qquad m^\#(t) = t \otimes 1 + 1 \otimes t \quad i^\#(t) = -t$$

Example A.1.5 (*Multiplicative group schemes*). Let M be an arbitrary commutative (abstract) group; consider its *group algebra* $B := k[M]$[1] which is commutative. Then

$$\mathrm{Hom}_{\mathsf{alg}_k}(B, A) = \mathrm{Hom}_{\mathbb{Z}}(A^*, M)$$

which is again an abelian group. So $\mathbb{D}_k(M) := \mathrm{Spec}(k[M])$ is a commutative affine group scheme. Explicitly we have $\forall x \in M$

$$\eta(x) = 0 \qquad \mu(x) = x \otimes x \quad \iota(x) = x^{-1}$$

There are two interesting groups of this kind

[1] $k[M]$ is the algebra of polynomials where the variables are the elements of M.

a) $M = \mathbb{Z}$. $\mathbb{D}_k(\mathbb{Z}) = \mathrm{Spec}(k[t, t^{-1}])$, usually written $\mathbb{G}_m$, and

$$\mathrm{Hom}_{\mathsf{sch}_k}(T, \mathrm{Spec}(k[t, t^{-1}])) = \mathrm{Hom}_{\mathsf{alg}_k}(k[t, t^{-1}], \mathcal{O}_T(T)) = \mathcal{O}_T(T)^*$$

b) $M = \mathbb{Z}/n\mathbb{Z}$. We have the *group scheme of n-roots of unity* over k and

$$\boldsymbol{\mu}_n = \mathbb{D}_k(\mathbb{Z}/n\mathbb{Z}) = \mathrm{Spec}(k[t]/(t^n - 1))$$

If the characteristic p of k not divides n, the k-algebra $k[\mathbb{Z}/n\mathbb{Z}]$ is isomorphic to a direct sum of n copies of k; if $n = p$, $\boldsymbol{\mu}_n$ is a scheme with only one point, but the local ring at that point has nilpotent elements.

Example A.1.6 (*Vector groups*). Let V be a finite dimensional k-vector space. Let $\mathrm{Sym}(V^*)$ be the symmetric algebra on the dual vector space of V. Explicitly, if $v_1, ..., v_n$ is a basis of V and v_i^* is the associated dual basis, we have

$$\mathrm{Sym}(V^*) \cong k[v_1^*, ..., v_n^*] \quad \text{and} \quad m^{\#}(v_i) = v_i \otimes 1 + 1 \otimes v_i$$

Then $\boldsymbol{V} := \mathrm{Spec}(\mathrm{Sym}(V^*))$ is an affine algebraic group and

$$h_{\boldsymbol{V}}(T) = \mathrm{Hom}_{\mathsf{sch}_k}(T, \boldsymbol{V}) = \mathcal{O}_T(T) \otimes_k V$$

Proposition A.1.7. *Let k be a field of characteristic* 0.

i) Let $\boldsymbol{G}$ be an algebraic group over k, then $\boldsymbol{G}$ is smooth and equidimensional over k.

ii) The category of (commutative) algebraic group schemes is an abelian category.

iii) (Chevalley's Theorem) Let $\boldsymbol{G}$ be an algebraic group over k, the there are two exact sequences

$$0 \to \boldsymbol{G}_{\mathrm{con}} \to \boldsymbol{G} \to \boldsymbol{G}_{\mathrm{et}} \to 0 \qquad 0 \to \boldsymbol{L} \to \boldsymbol{G}_{\mathrm{con}} \to \boldsymbol{A} \to 0$$

where $\boldsymbol{G}_{\mathrm{con}}$ is the connected component of the unit section $e : \mathrm{Spec}(k) \to \boldsymbol{G}$; $\boldsymbol{G}_{\mathrm{et}} = \boldsymbol{G}/\boldsymbol{G}_{\mathrm{con}}$ is étale over k; $\boldsymbol{L}$ is the smallest algebraic subgroup of $\boldsymbol{G}_{\mathrm{con}}$ such that $\boldsymbol{G}_{\mathrm{con}}/\boldsymbol{L}$ is proper over k.
Moreover $\boldsymbol{L}$ is canonical isomorphic to a product $\boldsymbol{T} \times_k \boldsymbol{V}$, where $\boldsymbol{V}$ is a vector group and $\boldsymbol{T}$ a torus.

Proof. See [24, Ch. I 6.6, §7]. See also [16] for a modern proof of the Chevalley theorem. □

A.2 Formal groups

Recall that according to [1, VII B] a formal scheme over k is the formal spectrum $\mathrm{Spf}(A)$ of a pro-finite k-algebra A, i.e. $A = \lim_\alpha A_\alpha$ is a projective limit of finite dimensional algebras A_α over k.

Definition A.2.1. A (commutative) *formal group* over k is a formal scheme $\boldsymbol{F} = \mathrm{Spf}(A)$ which is a group object in the category of formal schemes.

Remark A.2.2. Equivalently $\boldsymbol{F} = \mathrm{Spf}(A)$ is a formal group if the functor of points (from the category of affine schemes)

$$h_{\boldsymbol{F}} : \mathsf{aff}_k \to \mathsf{Set} \quad h_{\boldsymbol{F}}(\mathrm{Spec}(R)) := \mathrm{Hom}^{\mathrm{cont}}_{\mathsf{alg_k}}(A, R) \quad (= \text{continuous homomorphisms})$$

factors through the category of abelian groups.

Example A.2.3 (Formal Lie group). The *formal Lie group* $\widehat{\boldsymbol{G}}$ *associated to* $\boldsymbol{G}$ (also called the formal completion at the origin). Let $\boldsymbol{G}$ be an algebraic group. Let $m : \boldsymbol{G} \times \boldsymbol{G} \to \boldsymbol{G}$ be the multiplication morphism and e be the closed point of $\boldsymbol{G}$ given by the unit section. Then consider the local homomorphism $m_e^\# : \mathcal{O}_{\boldsymbol{G},e} \to \mathcal{O}_{\boldsymbol{G}\times\boldsymbol{G},(e,e)}$ and, completing w.r.t. the maximal ideals, the continuous extension of $m_e^\#$

$$\begin{array}{ccc} \mathcal{O}_{\boldsymbol{G},e} & \xrightarrow{m_e^\#} & \mathcal{O}_{\boldsymbol{G}\times\boldsymbol{G},(e,e)} \\ \downarrow & & \downarrow \\ \widehat{\mathcal{O}}_{\boldsymbol{G},e} & \xrightarrow{\widehat{m}_e} & \widehat{\mathcal{O}}_{\boldsymbol{G}\times\boldsymbol{G},(e,e)} \end{array}$$

By A.1.7 an algebraic group over a field is smooth, so

$$\widehat{\mathcal{O}}_{\boldsymbol{G},e} \cong k[[t_1, ..., t_n]] \qquad \widehat{\mathcal{O}}_{\boldsymbol{G}\times\boldsymbol{G},(e,e)} \cong k[[t_1, .., t_n, u_1, .., u_n]] .$$

From the associativity property of m follows that $\widehat{m}_e$ is a co-multiplication satisfying the (dual) axioms for a group object.

Example A.2.4. Let $\boldsymbol{G} = \mathbb{G}_a, \mathbb{G}_m$. Of course we have $\widehat{\mathcal{O}}_{\boldsymbol{G},e} = k[[t]]$ and the co-multiplication $m^\#$ is

$$m^\#(x) = \begin{cases} t + u & \boldsymbol{G} = \mathbb{G}_a \\ t + u + tu & \boldsymbol{G} = \mathbb{G}_m \end{cases}$$

Example A.2.5. Let V be a finite dimensional k-vector space, then $\boldsymbol{F}(\mathrm{Spec}\, R) := \mathrm{Nil}(R) \otimes_k V$ is a vector group functor represented by the formal group $\mathrm{Spf}(\widehat{\mathrm{Sym}}(V^*))$, where

$$\widehat{\mathrm{Sym}}(V^*) = \varprojlim \frac{\mathrm{Sym}\, V^*}{I^n} \qquad I := V \cdot \mathrm{Sym}\, V^*$$

Explicitly, if $v_1, ..., v_n$ is a basis of V and v_i^* is the associated dual basis, we have

$$\widehat{\mathrm{Sym}}(V^*) \cong k[[v_1^*, ..., v_n^*]] \quad \text{and} \quad m^\#(v_i) = v_i \otimes 1 + 1 \otimes v_i$$

Note that this formal group is the completion at the origin (or the formal Lie group) of the vector group $\boldsymbol{V} = \mathrm{Spec}(\mathrm{Sym}(V^*))$.

Let $\Gamma := \mathrm{Gal}(\bar{k}|k)$ be the absolute Galois group of k. A Γ-*module* is a pair (M, α), where M is an abelian group and $\alpha : \Gamma \to \mathrm{End}_{\mathbb{Z}}(M)$ is a continuous action[2]. We write Mod_Γ for the category of Γ-modules, where a morphism from (M, α) to (M', α') is a morphism of abelian groups $f : M \to M'$ such that $\alpha' = f \circ \alpha$.

A formal group $\boldsymbol{F} = \mathrm{Spf}(A)$ is *étale* if for every open maximal ideal of A we have $A_{\mathfrak{m}}$ is a separable finite extensions of k.[3] $\boldsymbol{F}$ is *infinitesimal* if $\boldsymbol{F}(k) = 0$.

We will write $\mathsf{FGr}_{\mathsf{et}}$ (resp. $\mathsf{FGr}_{\mathsf{inf}}$) for the full subcategory of FGr with objects the étale formal groups (resp. infinitesimal formal groups).

Proposition A.2.6. *i) The category of formal groups (over k) is an abelian category.*

ii) Let k be a perfect field. If $\boldsymbol{F}$ is a formal group (over k) there is a split exact sequence

$$0 \to \boldsymbol{F}_{\mathrm{inf}} \to \boldsymbol{F} \to \boldsymbol{F}_{\mathrm{et}} \to 0$$

where the $\boldsymbol{F}_{\mathrm{inf}}$ is the connected component of the unit section $e : \mathrm{Spec}(k) \to \boldsymbol{F}$; $\boldsymbol{F}_{\mathrm{et}}$ is étale.

Proof. See [1, Exp. VII B §2.4.2 - 2.5.2]. □

A.2.1 Étale formal groups

Proposition A.2.7. *i) The category of étale formal groups over a field k is equivalent to the category Mod_Γ via*

$$\mathsf{FGr}_{\mathsf{et}} \ni \boldsymbol{F} \mapsto \boldsymbol{F}(\bar{k}) := \operatorname*{colim}_{[k':k]<\infty,\ k' \subset \bar{k}} \boldsymbol{F}(k') \in \mathsf{Mod}_\Gamma$$

where $\bar{k}$ is an algebraic closure of k.

ii) Any étale formal group $\boldsymbol{F}$ decomposes canonically as $\boldsymbol{F}_{\mathrm{fr}} \times \boldsymbol{F}_{\mathrm{tor}}$ where $\boldsymbol{F}_{\mathrm{fr}}(\bar{k})$ is a free $\mathbb{Z}$-module and $\boldsymbol{F}_{\mathrm{tor}}(\bar{k})$ is torsion.

Proof. See [1, Exp. VII B §2.5]. □

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[2] i.e. α is continuos if it factorizes through $\mathrm{Gal}(k'|k)$ for some $k' \subset \bar{k}$ finite over k.

[3] By the structure theorem A is étale $\iff$ $A = \prod_I k_i$ product of finite separable extensions of k.

A.2.2 Infinitesimal formal groups

Theorem A.2.8 (Cartier). *Let $\boldsymbol{F} = \mathrm{Spf}(A)$ be a formal group over k, $\mathrm{char} k = 0$. The followings are equivalent*

i) There exists $V \in \mathsf{Mod}_k$ such that $\boldsymbol{F} \cong \mathrm{Spf}(\widehat{\mathrm{Sym}}(V^*))$ (See A.2.5).

ii) $\boldsymbol{F}$ is infinitesimal.

In this case $\mathrm{Lie}(\boldsymbol{F}) = V$ (where the Lie brackets are 0).

Proof. See [1, p. 548]. □

Corollary A.2.9. *Let k be a field of characteristic 0. The* Lie *functor induces an equivalence from the category of infinitesimal formal group to the category of vector spaces over k*

$$\mathrm{Lie} : \mathsf{FGr_{inf}} \to \mathsf{Mod}_k \ .$$

A.3 fppf sheaves

Let $\mathsf{sch_k}$ be the category of schemes over k and $\mathsf{aff_k}$ be the full sub-category of affine schemes. According to [1, Exp. IV §6.3] the fppf topology on $\mathsf{sch_k}$ is the one generated by: the families of jointly surjective open immersions in $\mathsf{sch_k}$; the finite families of jointly surjective, flat, of finite presentation and quasi-finite morphisms in $\mathsf{aff_k}$.

Recall that $\mathsf{Ab_k}$ is the category of abelian sheaves on $\mathsf{aff_k}$ w.r.t. the fppf topology.

Proposition A.3.1. *i) The category of commutative group schemes over k is a full sub-category of $\mathsf{Ab_k}$ via the functor of points $\boldsymbol{G} \mapsto h_{\boldsymbol{G}} := \mathrm{Hom}_{\mathsf{sch_k}}(-, \boldsymbol{G})$.*

ii) Let $\mathrm{char} k = 0$. *The category of formal group schemes (see definition A.2.1) is a full sub-category of of $\mathsf{Ab_k}$ via the functor of points $\boldsymbol{F} = \mathrm{Spf}(A) \mapsto h_{\boldsymbol{F}} := \mathrm{Hom}^{\mathrm{cont}}_{\mathsf{alg_k}}(A, -)$ (i.e. the set of continuous homomorphisms of algebras).*

Proof. By a result of Grothendieck ([23, Part I, §2.3.6]) every scheme (over k) is a sheaf (on sets) w.r.t. the fppf topology on $\mathsf{sch_k}$. Hence it is also a fppf-sheaf on $\mathsf{aff_k} \subset \mathsf{sch_k}$. From this follows (i) and (ii) for étale formal groups.

By A.2.6 it remains to prove that any infinitesimal formal group is a sheaf. It is sufficient to note that

$$\widehat{\mathbb{G}}_a \cong \operatorname*{colim}_n \mathrm{Spec}(k[t]/(t^{n+1}))$$

which is a direct limit of affine schemes, hence a direct limit of sheaves on sets w.r.t. the fppf topology. □

Appendix B

Mixed Hodge structures

B.1 Opposed Filtrations and †-structures

In this section A is a small abelian category. In fact we are interested in the case when A is category $\mathsf{Mod}_{\mathbb{C}}$ of finitely generated vector spaces over $\mathbb{C}$. We define two equivalent categories.

i) Let $n \in \mathbb{Z}$. An object $A \in \mathsf{A}$ is bigraded of weight n if there exists a finite family of object $A^{p,q} \in \mathsf{A}$ such that

$$A = \bigoplus_{p,q} A^{p,q} \in \mathsf{A} \ , \quad A^{p,q} = 0 \text{ if } p+q \neq n$$

(e.g. the complex cohomology groups of a compact Kähler manifolds). We denote by A' the category of pairs (A, n) where A is bigraded of weight n; the morphism are morphisms in A compatible with the bigrading, i.e.

$$\mathrm{Hom}_{\mathsf{A}'}((A,n),(B,m)) = \{f \in \mathrm{Hom}_{\mathsf{A}}(A,B) |\ f(A^{p,q}) \subset B^{p,q} \ \forall\ p,q\}$$

which is empty if $n \neq m$.

ii) Let A'' the category whose objects are systems $(A, F, \bar{F}, n)$, where $n \in \mathbb{Z}$; $A \in \mathsf{A}$; $F, \bar{F}$ are two n-opposite filtrations of A, i.e.

$$\mathrm{gr}^p_{\bar{F}} \mathrm{gr}^q_F (A) = 0 \qquad \text{if } p+q \neq n$$

A morphism in A'' is a morphism in A compatible w.r.t. both filtrations, i.e.

$$\begin{aligned} \mathrm{Hom}_{\mathsf{A}''}((A,n),(B,m)) = \ & \{f \in \mathrm{Hom}_{\mathsf{A}}(A,B) | \\ & \quad f(F^j(A)) \subset F^j(B),\ f(\bar{F}^j(A)) \subset \bar{F}^j(B) \forall j\} \end{aligned}$$

Proposition B.1.1. *There is an equivalence of categories*

$$\Phi : \mathsf{A}' \to \mathsf{A}'' \ , \quad \left(A = \bigoplus_{p+q=n} A^{p,q}, n \right) \mapsto \begin{cases} A \\ F^p := \bigoplus_{t \geq p} A^{t,q} \\ \bar{F}^q := \bigoplus_{t \geq q} A^{p,t} \end{cases}$$

and a quasi-inverse is given by

$$(A, F, \bar{F}, n) \mapsto (A = \oplus A^{p,q}, n), \quad A^{p,q} := \mathrm{gr}_F^p \, \mathrm{gr}_{\bar{F}}^q (A)$$

Proof. See the following remark or [17, 1.2.6.] for a complete proof. □

Remark B.1.2. i) An important ingredient of the proof is the Zassenhaus lemma: let $F, \bar{F}$ two (finite) filtrations of $A \in \mathsf{A}$, then there is a canonical isomorphism

$$\mathrm{gr}_{\bar{F}}^q \, \mathrm{gr}_F^p (A) \cong \mathrm{gr}_F^p \, \mathrm{gr}_{\bar{F}}^q (A)$$

ii) It is worth noting that the category of filtered objects of A is not abelian. In fact this category has kernels and cokernels, but the canonical morphism Coim → Im is not an isomorphism in general.

iii) An useful characterization of n-opposed filtration is the following:

$$\mathrm{gr}_F^p \, \mathrm{gr}_{\bar{F}}^q (A) = 0 \quad \forall p + q \neq n \iff F^p \oplus \bar{F}^q \cong A \text{ if } p + q = n + 1$$

So in that case $\mathrm{gr}_F^p \, \mathrm{gr}_{\bar{F}}^q (A) = F^p \cap \bar{F}^q$, if $p + q = n$.

Definition B.1.3. We define now a category $\mathsf{A}^\dagger$ whose objects are systems $(A, F, \overline{F}, W)$ such that $A \in \mathsf{A}$ and $F, \overline{F}, W$ are opposed filtrations, i.e.

$$\mathrm{gr}_F^p \, \mathrm{gr}_{\bar{F}}^q \, \mathrm{gr}_n^W (A) = 0 \text{ if } p + q \neq n \tag{B.1}$$

where $F, \overline{F}$ are finite decreasing filtrations and W is an increasing filtration of A. A morphism is an A-morphism $f : A \to B$ compatible with respect to all the three filtrations.

Given $(A, F, \bar{F}, W)$ in $\mathsf{A}^\dagger$ we have the following

$$A \cong \bigoplus A^{p,q} \qquad A^{p,q} := [\mathrm{gr}_{p+q}^W A]^{p,q} \ . \tag{B.2}$$

As the components on the right hand side are only sub-quotients, this is not a bigrading.

Let $I \subset \mathbb{Z}^2$, we say that A is of *type* I if the following holds

$$A^{p,q} \neq 0 \iff (p, q) \in I \tag{B.3}$$

Moreover we define the *level* of $(A, F, \bar{F}, W)$ by

$$\mathrm{level}(\mathsf{A}) := \max\{|p - q| \ : \ A^{p,q} \neq 0\}$$

Example B.1.4. i) The category of bigraded objects (B.1.1) is fully embedded in $\mathsf{A}^\dagger$.

ii) Let $Z \in \mathsf{A}$. We can always consider it as bigraded of bidgree (p,p). We will denote it by $Z(-2p)$.

Lemma B.1.5. *If $f : A \to B$ is a morphism in $\mathsf{A}^\dagger$, then it is strict with respect to all filtrations.*

Proof. The result follows if we can construct, for any $(A, F, \overline{F}, W)$, two family $I^{p,q}, \bar{I}^{p,q} \subset W_{p+q}$ such that

- The canonical projection $W_{p+q}(A) \to \mathrm{gr}^W_n(A)$ induces an isomorphism of $I^{p,q}$ and $\bar{I}^{p,q}$ with $A^{p,q}$, i.e.
$$A^{p,q} = I^{p,q} = \bar{I}^{p,q} \qquad \mod W_{p+q-1}(A)$$
- We can recover the filtrations of A from $I^{p,q}, \bar{I}^{p,q}$ in the following way
$$W_n(A) = \oplus_{p+q\leq n} I^{p,q} = \oplus_{p+q\leq n} \bar{I}^{p,q}, \quad F^p = \oplus_{p'\geq p} I^{p',q'} \quad \overline{F}^q = \oplus_{q'\geq q} \bar{I}^{p',q'}$$
- If $f : A \to B$ is a morphism in $\mathsf{A}^\dagger$, then
$$f(I^{p,q}(A)) \subset I^{p,q}(B) \qquad f(\bar{I}^{p,q}(A)) \subset \bar{I}^{p,q}(B)$$

We claim that we can take
$$I^{p,q}(A) := \left(\sum_{i\geq 0} W_{p+q-i} \cap F^{x_i}\right) \cap \left(\sum_{i\geq 0} W_{p+q-i} \cap \bar{F}^{y_i}\right)$$
where $(x_i, y_i)_{i\geq 0}$ is the following sequence of double indexes
$$(x_0, y_0) = (p, q), \quad (x_i, y_i) = (p, q - i + 1) \ i > 0$$
and $\bar{I}^{p,q}$ is defined by the same formula after replacing the indexes with
$$(x_0, y_0) = (p, q), \quad (x_i, y_i) = (p - i + 1, q) \ i > 0 \ .$$
□

Theorem B.1.6. *i) The kernel (resp. cokernel) of a morphism $f : A \to B$ in $\mathsf{A}^\dagger$ is the kernel (resp. cokernel) of f in A endowed with the induced (resp. quotient) filtrations from A (resp. from B).*

ii) The morphism $\mathrm{gr}^n_W(f) : \mathrm{gr}^n_W(A) \to \mathrm{gr}^n_W(B)$ is a morphism of pure A-structures.

iii) The category $\mathsf{A}^\dagger$, defined above, is abelian.

iv) All the following functors from $\mathsf{A}^\dagger$ to A are exact: the forgetful functor $(A, F, \overline{F}, W) \mapsto A$; the graded functors $\mathrm{gr}_F, \mathrm{gr}_{\overline{F}}, \mathrm{gr}_W$; the bigraded functors
$$\mathrm{gr}_W \mathrm{gr}_F \cong \mathrm{gr}_F \mathrm{gr}_W \cong \mathrm{gr}_{\overline{F}} \mathrm{gr}_F \mathrm{gr}_W \cong \mathrm{gr}_{\overline{F}} \mathrm{gr}_W \cong \mathrm{gr}_W \mathrm{gr}_{\overline{F}}$$

Proof. It follows easily by the previous lemma. □

Let $\mathsf{A}, \mathsf{B}, \mathsf{C}$ be abelian categories and

$$r, l : \mathsf{A} \times \mathsf{B} \to \mathsf{C}$$

two additive functors. Assume that r (resp. l) is right (resp. left) exact. Let $A \in \mathsf{A}, B \in \mathsf{B}$ be filtered objects, then we can define a filtration on $r(A, B)$ (resp. $l(A, B)$)

$$F^k r(A, B) := \oplus_m \operatorname{Im}(r(F^m(A), F^{k-m}(B)) \to r(A, B))$$
$$F^k l(A, B) := \cap_m \operatorname{Ker}(l(A, B) \to l(A/F^m, B/F^{k-m})))$$

Example B.1.7. For any abelian category A we have that $\operatorname{Hom}_{\mathsf{A}}(-,-)$ is a left exact functor. Hence if $A, B \in \mathsf{A}$ are filtered we can define the following filtration on the abelian group $\operatorname{Hom}_{\mathsf{A}}(A, B)$

$$\begin{aligned} F^k(\operatorname{Hom}_{\mathsf{A}}(A, B)) &= \cap_m \operatorname{Ker}(\operatorname{Hom}_{\mathsf{A}}(A, B) \to \operatorname{Hom}(F^m A, B/F^{k+m} B)) \\ &= \{f : A \to B \mid f(F^m_A) \subset F^{m+k}_B,\ \forall m \in \mathbb{Z}\} \end{aligned}$$

Fixed level/support

Let $[a, b] \subset \mathbb{Z}$. We denote by $\mathsf{A}^\dagger_{[\mathsf{a},\mathsf{b}]}$ the full subcategory of $\mathsf{A}^\dagger$ whose objects are of type $[a, b]^2$: i.e. $A \in \mathsf{A}^\dagger$ such that $A^{p,q} \neq 0 \iff (p, q) \in [a, b]^2$.

Proposition B.1.8. *i)* $\mathsf{A}^\dagger_{[\mathsf{a},\mathsf{b}]}$ *is an abelian category and is a thick subcategory of* $\mathsf{A}^\dagger$.

ii) Let $[a', b'] \subset [a, b]$. *There exist two covariant functors*

$$\begin{tikzcd} & \mathsf{A}^\dagger_{[\mathsf{a},\mathsf{b}]} \arrow[dl, "Q"'] \arrow[dr, "S"] & \\ \mathsf{A}^\dagger_{[\mathsf{a}',\mathsf{b}]} & & \mathsf{A}^\dagger_{[\mathsf{a},\mathsf{b}']} \end{tikzcd}$$

such that $S(A) = S^{ab}_{ab'}(A)$ *(resp.* $Q(A) = Q^{ab}_{a'b}(A)$*) is the maximal* †*-substructure (resp. the maximal quotient) of* A *of type* $[a, b']^2$ *(resp.* $[a', b]^2$*).*

iii) There is a commutative diagram of functors

$$\begin{tikzcd} \mathsf{A}^\dagger_{[\mathsf{a},\mathsf{b}]} \arrow[r, "S"] \arrow[d, "Q"'] & \mathsf{A}^\dagger_{[\mathsf{a},\mathsf{b}']} \arrow[d, "Q"] \\ \mathsf{A}^\dagger_{[\mathsf{a}',\mathsf{b}]} \arrow[r, "S"'] & \mathsf{A}^\dagger_{[\mathsf{a}',\mathsf{b}']} \end{tikzcd}$$

Proof. i) By B.1.6 the graded functor gr^W are exact. The type of a †-structure depends only on the associated bigraded object $\mathrm{gr}^W(A)$. Hence $\mathsf{A}^\dagger_{[\mathsf{a},\mathsf{b}]}$ is closed under kernels, cokernels and extensions.

ii)[1] Let $c \in [a, b]$. We want to construct $B = S^{ab}_{ac}(A)$ (resp. $Q^{ab}_{cb}(A)$), i.e. the maximal sub-structure of $W_c A$ of type $[a, c]^2$ (resp. the maximal quotient of $A/W_{c-1}A$ of type $[c, b]^2$).

Let $\phi : A \to \mathrm{gr}^W(A)$ the canonical isomorphism as objects of A. We can define

$$B := \phi^{-1}\left(\bigoplus_{p,q\in[a,c]} A^{p,q}\right) \subset \mathrm{gr}^W(A) = \bigoplus_{p,q\in[a,b]} A^{p,q}$$

$B \subset A$ as objects of A. We can consider B as a tri-filtered object w.r.t. the filtrations induced by those of A. From B.1 it follows that B is in fact an object of $\mathsf{A}^\dagger$ and the canonical inclusion is a morphism of †-structures.
The case $B = Q^{ab}_{cb}(A)$ so we just limit ourselves to define B

$$B := A/\phi^{-1}\left(\bigoplus_{p,q\in I} A^{p,q}\right) \qquad I := [a, c-1] \times [a, b] \cup [a, b] \times [a, c-1]$$

The functoriality follows from the above construction and B.1.6.

iii) Follows by the constructions and the fact that there is a unique way to induce a filtration on a sub-object. □

Definition B.1.9. It follows from the proposition that there is a functor $SQ : \mathsf{A}^\dagger_{[\mathsf{a},\mathsf{b}]} \to \mathsf{A}^\dagger_{[\mathsf{a}',\mathsf{b}']}$ such that $SQ(A) = (SQ)^{ab}_{a'b'}(A)$ is the maximal sub-quotient of A of type $[a', b']^2$.

Adjunction's formulas w.r.t. Q, S Let $a < a' \le b' < b$ integers. By construction the following adjunctions follow

$$\mathrm{Hom}_{\mathsf{A}^\dagger}(A, B) \cong \mathrm{Hom}_{\mathsf{A}^\dagger}(A, S^{ab}_{ab'}B) \qquad A \in \mathsf{A}^\dagger_{[\mathsf{a},\mathsf{b}']},\ \mathsf{B} \in \mathsf{A}^\dagger_{[\mathsf{a},\mathsf{b}]}$$

$$\mathrm{Hom}_{\mathsf{A}^\dagger}(Q^{ab}_{a'b}A, B) \cong \mathrm{Hom}_{\mathsf{A}^\dagger}(A, B) \qquad A \in \mathsf{A}^\dagger_{[\mathsf{a},\mathsf{b}]},\ \mathsf{B} \in \mathsf{A}^\dagger_{[\mathsf{a}',\mathsf{b}]}$$

Note that, in the situation of the first row, we have in general $\mathrm{Hom}_{\mathsf{A}^\dagger}(B, A) \ne \mathrm{Hom}_{\mathsf{A}^\dagger}(S^{ab}_{ab'}B, A)$: in fact take $A := S^{ab}_{ab'}B$, then on right side we have the identity morphism. If the equality holds we will have an element of $\mathrm{Hom}_{\mathsf{A}^\dagger}(B, A)$ which is a section of the canonical inclusion $A \subset B$. But in general this section does not exists as we shall see in the next section where we classify the extensions in $\mathsf{A}^\dagger$.

Dually we find the same obstruction in the case of Q.

[1] A meno di mettersi in una categoria di moduli, la dimostrazione é diretta, ma va controllata

Level ≤ 1 Consider the case $b - a = 1$, for instance take $b = 1,\ a = 0$. We call $\mathsf{A}_1^\dagger := \mathsf{A}_{[0,1]}^\dagger$ the category of †-structures of level ≤ 1. Such structures can be characterized as follows: $(A, F, \bar{F}, W)$ is an object of $\mathsf{A}_1^\dagger$ if

$$0 = W_{-1}A \subset W_0 A \subset W_1 A \subset W_2 A = A$$

and

$$0 = F^2 A \subset F^1 A \subset F^0 A = A \qquad 0 = \bar{F}^2 A \subset \bar{F}^2 A \subset \bar{F}^0 A = A$$

where all the above considered inclusions are not supposed to be strict.

In the same way we can define the category of †-structures of level $\leq n$ for any $n \leq 0$ by

$$\mathsf{A}_n^\dagger := \mathsf{A}_{[0,n]}^\dagger \tag{B.4}$$

Note that the category of †-structures of level 0 is just the category A.

B.2 Extensions in $\mathsf{A}^\dagger$

Let A be any abelian category (we don't suppose it has enough injective objects), then we can define its derived category $D(\mathsf{A})$ and the group of classes of n-fold extensions

$$\mathrm{Ext}^n_{\mathsf{A}}(A, B) := \mathrm{Hom}_{D(\mathsf{A})}(A, B[n]) \qquad A, B \in \mathsf{A}$$

As usual we identify this group with the group of classes of *Yoneda extensions*, i.e. the set of exact sequences

$$0 \to B \to E^{-n} \to \cdots \to E^{-1} \to A \to 0$$

modulo congruences, where the group law is the Baer sum.

Proposition B.2.1 (Extension Formula)**.** *Let A, B objects of $\mathsf{A}^\dagger$. If A has cohomological dimension 0 (e.g. A is the category of vector spaces over a field), then there is a canonical isomorphism of groups*

$$\mathrm{Ext}^1_{\mathsf{A}^\dagger}(A, B) \cong \frac{W_0 C}{(W_0 \cap F^0)(C) + (W_0 \cap \bar{F}^0)(C)} \qquad C = \mathrm{Hom}_{\mathsf{A}}(A, B) \tag{B.5}$$

(See B.1.7 for the filtrations defined on C).

Proof. Consider an exact sequence in $\mathsf{A}^\dagger$

$$(*) \qquad 0 \longrightarrow B \xrightarrow{\beta} E \xrightarrow{\alpha} A \longrightarrow 0$$

by definition it is in the class of the trivial extension $\iff$ there exist a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r, "\beta"] \arrow[dr, "i_1"'] & E \arrow[r, "\alpha"] & A \arrow[r] & 0 \\ & & B \oplus A \arrow[u, "\phi"] \arrow[ur, "\pi_2"'] & & \end{tikzcd}$$

with ϕ compatible w.r.t. all the 3 filtrations. This also is equivalent to have a section of α: just take $\sigma = i_2 \circ \phi$, where $i_2 : A \to B \oplus A$ is the canonical inclusion.

Suppose that the extension $(*)$ is not trivial, hence we cannot find a section of α compatible w.r.t. all filtration but we can always find (non canonically) a section σ_1 (resp. $\bar{\sigma}_1$) strictly compatible w.r.t. W, F (resp. $W, \bar{F}$). This follows by the proof of lemma B.1.5. Note that if we make a different choice, say σ_2 (resp. $\bar{\sigma}_2$) we find that $\sigma_1 - \sigma_2 \in \mathrm{Hom}_{\mathsf{A}}(A, B)$ and it is compatible with respect to W, F i.e.

$$\sigma_1 - \sigma_2 \in (W_0 \cap F^0)(\mathrm{Hom}(A, B)) \qquad \bar{\sigma}_1 - \bar{\sigma}_2 \in (W_0 \cap \bar{F}^0)(\mathrm{Hom}(A, B))$$

so the difference $\sigma_1 - \bar{\sigma}_1$ is an element of $W_0 \mathrm{Hom}(A, B)$ and its equivalence class

$$[\sigma_1 - \sigma_2] \in \frac{W_0 \mathrm{Hom}(A, B)}{W_0 \cap F^0 + \bar{F}^0}$$

is independent by the choices of the sections.

From the above discussion we can define a map from $\mathrm{Ext}^1_{\mathsf{A}^\dagger}(A, B)$ to $W_0 \mathrm{Hom}(A, B)/(W_0 \cap F^0 + W_0 \cap \bar{F}^0)$. To conclude the proof we need to show that this map is surjective. Consider $\sigma \in W_0 \mathrm{Hom}(A, B)$ and define the map

$$g_\sigma : B \oplus A \to B \oplus A \qquad (b, a) \mapsto (b + \sigma(a), a)$$

then g_σ is compatible w.r.t. W. Define the following filtrations on $B \oplus A$

$$F^k_\sigma(B \oplus A) := g_\sigma(F^k B \oplus F^k B) \qquad \bar{F}^k_\sigma(B \oplus A) := g_\sigma(\bar{F}^k B \oplus \bar{F}^k B)$$

and write E_σ for the object of A given by the $B \oplus A$ and the filtrations W (i.e. $W_k = W_k B \oplus W_k A$), F_σ, $\bar{F}_\sigma$. □

Corollary B.2.2. *Suppose furthermore that for some m we have $W_m B = B$ while $W_m A = 0$ (i.e. the weights of B are less than the weights of A). There is a natural isomorphism of groups*

$$\mu : \mathrm{Ext}^1_{\mathsf{A}}(A, B) \xrightarrow{\sim} \frac{\mathrm{Hom}_{\mathsf{A}}(A, B)}{F^0 + \bar{F}^0}$$

Proof. By the hypothesis on the distribution of weights we have $W_0 \operatorname{Hom}(A, B) = \operatorname{Hom}(A, B)$. $\square$

Corollary B.2.3. *With the same hypothesis of B.2.1, the functor* $\operatorname{Ext}^1_{\mathsf{A}^\dagger} : \mathsf{A}^\dagger \to \mathsf{Mod}_{\mathbb{Z}}$ *is right exact for any* $A \in \mathsf{A}^\dagger$.

Proof. We know that $\operatorname{Hom}_{\mathsf{A}}(A, -)$ is right exact. From this follows that $W_0 \operatorname{Hom}(A, -)$ is right exact, in fact we can use the definition

$$W_0 \operatorname{Hom}(A, B) = \{f : A \to B |\ f(W_n(A)) \subset W_n(B),\ \forall n\}$$

Then the result follows from the previous theorem and B.1.6. $\square$

Corollary B.2.4. *If* A *is of cohomological dimension 0, then* $\mathsf{A}^\dagger$ *is a category of cohomological dimension 1, i.e.* $\operatorname{Ext}^n_{\mathsf{A}^\dagger}(A, B) = 0$ *if* $n \leq 2$.

Proof. Just use the following lemma and the previous corollary. $\square$

Lemma B.2.5. *Fix* $A \in \mathsf{A}$. *If* $\operatorname{Ext}^k_{\mathsf{A}}(A, -)$ *is right exact for all* $A \in \mathsf{A} \Rightarrow \operatorname{Ext}^n_{\mathsf{A}}(A, B) = 0$ *for all* $n > k$ *and for all* $A, B \in \mathsf{A}$.

Proof. To calculate the Yoneda-class of an n-fold extension E of A by B

$$[0 \to B \to E_n \to E_{n-1} \to \cdots \to E_1 \to A \to 0]$$

we may splice E from an k-fold extension of X by B and an $(n-k)$-fold extension of A by X

$$[0 \to B \to E_n \to E_{n-1} \to \cdots \to E_{n-k+1} \to X \to 0]$$

$$[0 \to X \to E_{n-k} \to E_{n-k-1} \to \cdots \to E_1 \to A \to 0]\ .$$

It suffices therefore to prove that $\operatorname{Ext}^{k+1}(A, B) = 0$. Now we view a $(k+1)$-fold extension of A by B as spliced from a simple extension

$$0 \to B \to H \to C \to 0$$

and a k-fold extension of A by C. We consider the connecting homomorphism $\operatorname{Ext}^k(A, C) \to \operatorname{Ext}^{k+1}(A, B)$ from the long exact sequence for $\operatorname{Hom}(A, -)$ with respect to the preceding short exact sequence. Since $\operatorname{Ext}^k(A, -)$ is right exact, this connecting homomorphism is zero. Now we apply this to the Yoneda class $f \in \operatorname{Ext}^k(A, C)$ of the second extension. If the Yoneda-class of the short exact sequence is e, the connecting homomorphism is given by taking the composition product with e. But this gives the Yoneda class $e \cdot f$ of the extension we started with. This class is therefore zero. $\square$

B.3 Hodge structures

Definition B.3.1. Let $R \subset \mathbb{R}$ be a noetherian sub-ring and let K be its fraction field (e.g. $R = \mathbb{Z}, \mathbb{Q}, \mathbb{R}$). A *(mixed) Hodge structure* over R is given by: a finitely generated R-module A_R, called the *R-residue*; an object $(A, F, \overline{F}, W) \in \mathsf{Mod}_{\mathbb{C}}^{\dagger}$ (See B.1.3). Such that

1. $A_{\mathbb{C}} := A_R \otimes_R \mathbb{C} \cong A$.
2. The weight filtration W is defined over K, i.e. $W_n = (W_n \cap A_K) \otimes_K \mathbb{C}$. Where $A_K := A_R \otimes_R K$
3. The filtration $\overline{F}$ is the complex conjugate of F, i.e. $\overline{F}^p = c(F^p)$ where $c : A = A_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \to A = A_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ is induced by the complex conjugation on $\mathbb{C}$.

A morphism of mixed Hodge structure over R is an R-linear map $f : A_R \to B_R$ such that $f_K := f \otimes \mathrm{id}_K$ is compatible w.r.t. W and $f_{\mathbb{C}} := f \otimes \mathrm{id}_{\mathbb{C}}$ is compatible w.r.t. F.

We denote by MHS_R the category of mixed Hodge structure over R. In case $R = \mathbb{Z}$ we simply write $\mathsf{MHS} = \mathsf{MHS}_{\mathbb{Z}}$.

Proposition B.3.2 (Properties of MHS)**.** *i) If $f : A \to B$ is a morphism in* MHS*, then it is strict with respect to all filtrations.*

ii) The kernel (resp. cokernel) of a morphism $f : A \to B$ in MHS *is the kernel (resp. cokernel) of f in* $\mathsf{Mod}_{\mathbb{Z}}$ *endowed with the induced (resp. quotient) filtrations from A (resp. from B).*

iii) The morphism $\mathrm{gr}^n_W(f) : \mathrm{gr}^n_W(A) \to \mathrm{gr}^n_W(B)$ is a morphism of pure MHS *(i.e. a morphism of bigraded objects).*

iv) The category MHS *is abelian.*

v) All the following functors from MHS *are exact: the forgetful functor $(A, F, \overline{F}, W) \mapsto A_{\mathbb{Z}} \in \mathsf{Mod}_{\mathbb{Z}}$; the graded functors $\mathrm{gr}_F, \mathrm{gr}_{\overline{F}}, \mathrm{gr}_W$; the bigraded functor*

$$\mathrm{gr}_W \mathrm{gr}_F \cong \mathrm{gr}_F \mathrm{gr}_W \cong \mathrm{gr}_{\overline{F}} \mathrm{gr}_F \mathrm{gr}_W \cong \mathrm{gr}_{\overline{F}} \mathrm{gr}_W \cong \mathrm{gr}_W \mathrm{gr}_{\overline{F}}$$

Proof. Part (i) is a consequence of B.1.5. From this follows the other properties as in B.1.6. □

B.3.1 Extensions in MHS

Remark B.3.3 (Torsion Part). A mixed Hodge structure on $A_{\mathbb{Z}}$ is completely determined specifying a mixed Hodge structure on the free quotient $A_{\mathbb{Z}}/t := A_{\mathbb{Z}}/tA_{\mathbb{Z}}$, where $tA_{\mathbb{Z}}$ is the torsion part of $A_{\mathbb{Z}}$. In particular, first of all, the forgetful functor induces an isomorphism $\mathrm{Ext}_{\mathsf{MHS}}(tA_{\mathbb{Z}}, t(B_{\mathbb{Z}})) \simeq \mathrm{Ext}_{\mathbb{Z}}(tA_{\mathbb{Z}}, tB_{\mathbb{Z}})$.

Secondly, there is a forgetful functor $\mathrm{Ext}_{\mathsf{MHS}}(A_{\mathbb{Z}}, B_{\mathbb{Z}}) \to \mathrm{Ext}_{\mathbb{Z}}(A_{\mathbb{Z}}, B_{\mathbb{Z}}) = \mathrm{Ext}_{\mathbb{Z}}(tA_{\mathbb{Z}}, tB_{\mathbb{Z}})$ which can be shown to be a retraction for the natural exact sequence

$$0 \to \mathrm{Ext}_{\mathbb{Z}}(tA_{\mathbb{Z}}, tB_{\mathbb{Z}}) \to \mathrm{Ext}_{\mathsf{MHS}}(A_{\mathbb{Z}}, B_{\mathbb{Z}}) \to \mathrm{Ext}_{\mathsf{MHS}}(A_{\mathbb{Z}}/t, B_{\mathbb{Z}}/t) \to 0.$$

So this sequence is split and there is no loss of information if we work with mixed Hodge structures on torsion free modules.

Proposition B.3.4. *Let $A, B \in \mathsf{MHS}_R$ such that A_R, B_R are free. Then there is a canonical isomorphism*

$$\mu : \mathrm{Ext}^1_{\mathsf{MHS}_R}(A, B) \xrightarrow{\sim} \mathrm{Hom}^W_{\mathbb{C}}(A_{\mathbb{C}}, B_{\mathbb{C}})/(F^0 \mathrm{Hom}^W_{\mathbb{C}}(A_{\mathbb{C}}, B_{\mathbb{C}}) + \mathrm{Hom}^W_R(A_R, B_R))$$

Proof. The proof is similar to the one of B.2.1, in particular one proves that that if $H \in \mathrm{Ext}^1_{\mathsf{MHS}}(A, B)$ we can suppose $H_{\mathbb{C}} = B_{\mathbb{C}} \oplus A_{\mathbb{C}}$ and that the weight filtration is also given component-wise; but the Hodge filtration of H is of the form $F_B + \phi(F_A) \oplus F_A$ where: $\phi : A \to B$ is a $\mathbb{C}$-linear map compatible w.r.t. the weight filtrations; F_A (resp. F_B) is the Hodge filtration of A (resp. B). The details are given in [38]. $\square$

Corollary B.3.5. *i)* $\mathrm{Ext}^1_{\mathsf{MHS}_R}(A, -)$ *is a right exact functor for any* $A \in \mathsf{MHS}_R$.

ii) $\mathrm{Ext}^i_{\mathsf{MHS}_R}(A, B) = 0$ *for any* $i \geq 2$.

iii) Let A and B be mixed Hodge structures with A_R and B_R torsion free. Suppose that for some m we have $W_m B = B$ while $W_m A = 0$ (i.e. the weights of B are less than the weights of A, one says that A and B are separated mixed Hodge structures*). There is a natural isomorphism of groups*

$$\mu : \mathrm{Ext}^1_{\mathsf{MHS}_R}(A, B) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}}(A_{\mathbb{C}}, B_{\mathbb{C}})/(F^0 \mathrm{Hom}_{\mathbb{C}}(A_{\mathbb{C}}, B_{\mathbb{C}}) + \mathrm{Hom}_R(A_R, B_R))$$

Proof. This follows by the previous proposition as the corollaries of B.2.1 (see there for details). $\square$

Definition B.3.6 (p-Jacobian)**.** Let A be a mixed Hodge structure over $\mathbb{Z}$ with $A_{\mathbb{Z}}$ torsion free. For $p \in \mathbb{Z}$ the *p-th Jacobian of A* is defined as

$$J^p(A) := A_{\mathbb{C}}/(F^p + A_{\mathbb{Z}}).$$

Since $F^0 A(p) = F^p A$ we have

$$J^p(A) \simeq J^0 \mathcal{H}om_{\mathsf{MHS}}(\mathbb{Z}(0), A(p)) = J^0 \mathcal{H}om_{\mathsf{MHS}}(\mathbb{Z}(-p), A).$$

Lemma B.3.7. *If $W_{2p-1}A_{\mathbb{Q}} = A_{\mathbb{Q}}$ the group*

$$J^p(A) \cong \mathrm{Ext}^1_{\mathsf{MHS}}(\mathbb{Z}(-p), A) \cong \mathrm{Ext}^1_{\mathsf{MHS}}(\mathbb{Z}, A(p))$$

is a complex Lie group. If moreover A is polarized $J^p(A)$ can be viewed as a semi-abelian complex algebraic group.

Proof. The condition implies that $F^pA_{\mathbb{C}} \cap \overline{F^pA_{\mathbb{C}}} = 0$ and hence $F^pH_{\mathbb{C}}$ does not meet the image of $A_{\mathbb{Z}}$ in $A_{\mathbb{C}}$. In particularly, $A_{\mathbb{Z}}$ embeds discretely in $A_{\mathbb{C}}/F^p$. □

Example B.3.8. i) If A and B are separated, the group $\mathrm{Ext}^1(A, B)$ has the structure of a complex Lie group. Indeed, separateness is equivalent to saying that $\mathcal{H}om_{\mathsf{MHS}}(A, B)$ has only negative weights, i.e.

$$W_{-1}\mathcal{H}om_{\mathsf{MHS}}(A, B)_{\mathbb{Q}} = \mathcal{H}om_{\mathsf{MHS}}(A, B)_{\mathbb{Q}}$$

and the result follows upon applying Lemma B.3.7 to the mixed Hodge structure $\mathcal{H}om_{\mathsf{MHS}}(A, B)$.

ii) For $m < n$ the group $\mathrm{Ext}_{\mathsf{MHS}}(\mathbb{Z}(m), \mathbb{Z}(n))$ is isomorphic to $\mathbb{C}/(2\pi i)^{n-m}\mathbb{Z}$, a twist of $\mathbb{C}^{\times}$.

iii) Let H be a pure Hodge structure of weight $2m-1$. Then $J^m(H)$ is a compact complex torus. Indeed, we have a direct sum decomposition

$$H_{\mathbb{C}} := H_{\mathbb{Z}} \otimes \mathbb{C} = F^m \oplus \overline{F^m}$$

and any real element $x \in H_{\mathbb{Z}} \otimes \mathbb{R}$ belonging to one of these summands also belongs to the other one and so must be zero. Moreover $J^mH \simeq J^0\mathcal{H}om_{\mathsf{MHS}}(\mathbb{Z}(-m), H)$ and hence

$$J^mH = \mathrm{Ext}_{\mathsf{MHS}}(\mathbb{Z}(-m), H),$$

this is useful for an algebraic description of the Abel-Jacobi map.

iv) Let X be any smooth projective manifold. Take $A = \mathbb{Z}$ and $B = H^k(X_{\mathrm{an}}, \mathbb{Z})(d)$ where d is chosen so that $k < 2d$ (for instance $k = 2d-1$). Then the weights are separated and by B.3.6 we have

$$\begin{aligned}\mathrm{Ext}_{\mathsf{MHS}}(\mathbb{Z}, H^k(X, \mathbb{Z})(d)) &= J^0\,\mathrm{Hom}(\mathbb{Z}, H^k(X, \mathbb{Z})(d)) \\ &\simeq J^dH^k((X, \mathbb{Z})) = H^k(X; \mathbb{C})/H^k(X) \oplus F^dH^k(X).\end{aligned}$$

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